SQUARE FUNCTIONS AND UNIFORM RECTIFIABILITY

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ABSTRACT. In this paper it is shown that an Ahlfors-David n-dimensional measure μ on $\mathbb{R}^d$ is uniformly n-rectifiable if and only if for any ball $B(x_0, R)$ centered at $\operatorname{supp}(\mu)$,

$$\int_0^R \int_{x\in B(x_0,R)} \left| \frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n} \right|^2 d\mu(x)\, \frac{dr}{r} \le c\, R^n.$$

Other characterizations of uniform n-rectifiability in terms of smoother square functions are also obtained.

1. Introduction

Given integers $0 < n < d$, a Borel set $E \subset \mathbb{R}^d$ is said to be *n-rectifiable* if it is contained in a countable union of n-dimensional C^1 manifolds and a set of zero n-dimensional Hausdorff measure $\mathcal{H}^n$. On the other hand, a Borel measure μ in $\mathbb{R}^d$ is called *n-rectifiable* if it is of the form $\mu = g\, \mathcal{H}^n|_E$, where E is a Borel n-rectifiable set and g is positive and $\mathcal{H}^n$ integrable on E. Rectifiability is a qualitative notion, but David and Semmes in their landmark works [DS1] and [DS2] introduced the more quantitative notion of uniform rectifiability. To define uniform rectifiability we need first to recall the notion of Ahlfors-David regularity.

We say a Radon measure μ in $\mathbb{R}^d$ is *n-dimensional Ahlfors-David regular with constant c_0* if

$$c_0^{-1} r^n \le \mu(B(x,r)) \le c_0\, r^n \quad \text{for all } x \in \operatorname{supp}(\mu),\ 0 < r \le \operatorname{diam}(\operatorname{supp}(\mu)). \tag{1.1}$$

For short, we sometimes omit the constant c_0 and call μ n-AD-regular. It follows easily that such a measure μ must be of the form $\mu = h\, \mathcal{H}^n|_{\operatorname{supp}(\mu)}$, where h is a positive function bounded from above and from below.

An n-AD-regular measure μ is *uniformly n-rectifiable* if there exist $\theta, M > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and all $r > 0$ there exists a Lipschitz mapping ρ from the ball $B_n(0, r)$ in $\mathbb{R}^n$ to $\mathbb{R}^d$ with $\operatorname{Lip}(\rho) \le M$ such that

$$\mu(B(x,r) \cap \rho(B_n(0,r))) \ge \theta r^n.$$

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When $n=1$, μ is uniformly 1-rectifiable if and only if $\operatorname{supp}(\mu)$ is contained in a rectifiable curve in $\mathbb{R}^d$ on which the arc length measure satisfies (1.1). A Borel set $E\subset\mathbb{R}^d$ is n-AD-*regular* if $\mu=\mathcal{H}^n|_E$ is n-AD-regular, and it is called *uniformly n-rectifiable* if, further, $\mathcal{H}^n|_E$ is uniformly n-rectifiable. Thus μ is an uniformly n-rectifiable measure if and only if $\mu=h\,\mathcal{H}^n|_E$ where $h>0$ is bounded above and below and E is an uniformly n-rectifiable closed set.

Uniform rectifiability is closely connected to the geometric study of singular integrals. In [Da1] David proved that if $E\subset\mathbb{R}^d$ is uniformly n-rectifiable, then for any convolution kernel $K:\mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ satisfying

$$K(-x)=-K(x) \quad\text{and}\quad \left|\nabla^j K(x)\right|\le c_j\,|x|^{-n-j}\,,\ \text{for } x\in\mathbb{R}^d\setminus\{0\},\ j=0,1,2,\dots, \tag{1.2}$$

the associated singular integral operator $T_K f(x)=\int K(x-y)\,f(y)\,d\mathcal{H}^n|_E(y)$ is bounded in $L^2(\mathcal{H}^n|_E)$. David and Semmes in [DS1] proved conversely that the $L^2(\mathcal{H}^n|_E)$-boundedness of all singular integrals T_K with kernels satisfying (1.2) implies that E is uniformly n-rectifiable. However if one only assumes the boundedness of some particular singular integral operators satisfying (1.2), then the situation becomes much more delicate.

In [MMV] Mattila, Melnikov and Verdera proved that if E is an 1-AD regular set, the Cauchy transform is bounded in $L^2(\mathcal{H}^n|_E)$ if and only if E is uniformly 1-rectifiable. It is remarkable that their proof depends crucially on a special subtle positivity property of the Cauchy kernel related to the so-called Menger curvature. See [CMPT] for other examples of 1-dimensional homogeneous convolution kernels whose L^2-boundedness is equivalent to uniform rectifiability, again because of Menger curvature. Recently in [NToV] it was shown that in the codimension 1 case, that is, for $n=d-1$, if E is n-AD-regular, then the vector valued Riesz kernel $x/|x|^{n+1}$ defines a bounded operator on $L^2(\mathcal{H}^n|_E)$ if and only if E is uniformly n-rectifiable. In this case, the notion of Menger curvature is not applicable and the proof relies instead on the harmonicity of the kernel $x/|x|^{n+1}$. It is an open problem if the analogous result holds for $1<n<d-1$.

In this paper we prove several characterizations of uniform n-rectifiability in terms of square functions. Our first characterization involves the following difference of densities

$$\Delta_\mu(x,r):=\frac{\mu(B(x,r))}{r^n}-\frac{\mu(B(x,2r))}{(2r)^n}$$

and reads as follows.

Theorem 1.1. *Let μ be an n-AD-regular measure. Then μ is uniformly n-rectifiable if and only if there exists a constant c such that, for any ball $B(x_0,R)$ centered at* $\operatorname{supp}(\mu)$,

$$\int_0^R\int_{x\in B(x_0,R)}|\Delta_\mu(x,r)|^2\,d\mu(x)\,\frac{dr}{r}\le c\,R^n. \tag{1.3}$$

Recall that a celebrated theorem of Preiss [Pr] asserts that a Borel measure μ in $\mathbb{R}^d$ is n-rectifiable if and only if the density $\lim_{r\to0}\dfrac{\mu(B(x,r))}{r^n}$ exists and is positive for μ-a.e. $x\in\mathbb{R}^d$. In a sense, Theorem 1.1 can be considered as a square function version of Preiss' theorem for uniform rectifiability. On the other hand, let us mention that the "if" implication in our theorem relies on some of the deep results by Preiss in [Pr].

It is also worth comparing Theorem 1.1 to some earlier results from Kenig and Toro [KT], David, Kenig and Toro [DKT] and Preiss, Tolsa and Toro [PTT]. In these works it is shown among other things that, given $\alpha > 0$, there exists $\beta(\alpha) > 0$ such that if μ is n-AD-regular and for each compact set K there exists some constant c_K such that

$$\left|\frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,tr))}{(tr)^n}\right| \leq c_K\, r^\alpha \quad \text{for } 1 < t \leq 2,\ x \in K \cap \operatorname{supp}(\mu),\ 0 < r \leq 1,$$

then μ is supported on an $C^{1+\beta}$ n-dimensional manifold union a closed set with zero μ-measure. This result can be thought of as the Hölder version of one of the implications in Theorem 1.1.

We also want to mention the forthcoming work [ADT] by Azzam, David and Toro for some other conditions on a doubling measure which imply rectifiability. One of the conditions in [ADT] quantifies the difference of the measure at different close scales in terms of the Wasserstein distance W_1. In our case, the square function in Theorem 1.1 just involves the difference of the n-dimensional densities of two concentric balls such that the largest radius doubles the smallest one.

Motivated by the recent work [LM] studying local scales on curves and surfaces, which was the starting point of this paper's research, we also prove smooth versions of Theorem 1.1. For any Borel function $\varphi : \mathbb{R}^d \to \mathbb{R}$ let

$$\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right),\ t > 0$$

and define

$$\Delta_{\mu,\varphi}(x,t) := \int \big(\varphi_t(y-x) - \varphi_{2t}(y-x)\big)\, d\mu(y),$$

whenever the integral makes sense. If φ is smooth, let

$$\partial_\varphi(x,t) = t\partial_t\, \varphi_t(x)$$

and define

$$\widetilde{\Delta}_{\mu,\varphi}(x,t) := \int \partial_\varphi(y-x,t)\, d\mu(y),$$

again whenever the integral makes sense. Our second theorem characterizes uniform n-rectifiable n-AD-regular measures using the square functions associated with $\Delta_{\mu,\varphi}$ and $\widetilde{\Delta}_{\mu,\varphi}$.

Theorem 1.2. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be of the form $e^{-|x|^{2N}}$, with $N \in \mathbb{N}$, or $(1+|x|^2)^{-a}$, with $a > n/2$. Let μ be an n-AD-regular measure in $\mathbb{R}^d$. The following are equivalent:*

(a) *μ is uniformly n-rectifiable.*

(b) *There exists a constant c such that for any ball $B(x_0,R)$ centered at* $\operatorname{supp}(\mu)$,

$$\int_0^R \int_{x\in B(x_0,R)} |\Delta_{\mu,\varphi}(x,r)|^2\, d\mu(x)\, \frac{dr}{r} \leq c\, R^n. \tag{1.4}$$

(c) *There exists a constant c such that for any ball $B(x_0,R)$ centered at* $\operatorname{supp}(\mu)$,

$$\int_0^R \int_{x\in B(x_0,R)} |\widetilde{\Delta}_{\mu,\varphi}(x,r)|^2\, d\mu(x)\, \frac{dr}{r} \leq c\, R^n. \tag{1.5}$$

The functions φ_t above are radially symmetric and (constant multiples of) approximate identities on any n-plane containing the origin. The definitions of $\Delta_{\mu,\varphi}(x,t)$ and $\widetilde{\Delta}_{\mu,\varphi}(x,t)$

arise from convolving the measure μ with the kernels $\varphi_t(x) - \varphi_{2t}(x)$ and $\partial_\varphi(x,t)$, respectively. Note that $\varphi_t(x) - \varphi_{2t}(x)$ is a discrete approximation to $\partial_\varphi(x,t)$. Note also that the quantities $\Delta_\mu(x,t)$, $\Delta_{\mu,\varphi}(x,t)$ and $\widetilde{\Delta}_{\mu,\varphi}(x,r)$ are identically zero whenever $\mu = \mathcal{H}^n|_L$, L is an n-plane, and $x \in L$.

For each integer $k > 0$, let

$$\widetilde{\Delta}^k_{\mu,\varphi}(x,t) = \int \partial^k_\varphi(y-x,t)\; d\mu(y), \text{ where } \partial^k_\varphi(x,t) = t^k \partial_t^k \varphi_t(x).$$

Similarly, let

$$\Delta^k_{\mu,\varphi}(x,t) = \int D^k\left[\varphi_t\right](y-x)\; d\mu(y),$$

where

$$D^k[\varphi_t](x) = D^{k-1}[D\varphi_t](x), \quad \text{and} \quad D\varphi_t(x) = \varphi_t(x) - \varphi_{2t}(x).$$

By arguments analogous to the ones of Theorem 1.2, we obtain the following equivalent square function conditions for uniform rectifiability.

Proposition 1.3. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be of the form $e^{-|x|^{2N}}$, with $N \in \mathbb{N}$, or $(1+|x|^2)^{-a}$, with $a > n/2$. Let μ be an n-AD-regular measure in $\mathbb{R}^d$ and $k > 0$. The following are equivalent:*

(a) *μ is uniformly n-rectifiable.*

(b) *There exists a constant c_k such that for any ball $B(x_0,R)$ centered at* $\operatorname{supp}\mu$,

$$\int_0^R \int_{x\in B(x_0,R)} |\Delta^k_{\mu,\varphi}(x,r)|^2\, d\mu(x)\, \frac{dr}{r} \le c_k\, R^n. \tag{1.6}$$

(c) *There exists a constant c_k such that for any ball $B(x_0,R)$ centered at* $\operatorname{supp}\mu$,

$$\int_0^R \int_{x\in B(x_0,R)} |\widetilde{\Delta}^k_{\mu,\varphi}(x,r)|^2\, d\mu(x)\, \frac{dr}{r} \le c_k\, R^n. \tag{1.7}$$

Proposition 1.3 is in the same spirit as the characterization of Lipschitz function spaces in Chapter V, Section 4 of [St].

There are other characterizations of uniform n-rectifiability via square functions in the literature. Among the most relevant of these is a condition in terms of the β-numbers of Peter Jones. For $x \in \operatorname{supp}(\mu)$ and $r > 0$, consider the coefficient

$$\beta_1^\mu(x,r) = \inf_L \int_{B(x,r)} \frac{\operatorname{dist}(y,L)}{r^{n+1}} d\mu(y),$$

where the infimum is taken over all n-planes L. Like $\Delta_\mu(x,r)$, $\beta_1^\mu(x,r)$ is a dimensional coefficient, but while $\beta_1^\mu(x,r)$ measures how close $\operatorname{supp}(\mu)$ is to some n-plane, $\Delta_\mu(x,r)$ measures the oscillations of μ. In [DS1], David and Semmes proved that μ is uniformly n-rectifiable if and only if $\beta_1^\mu(x,r)^2 dx \frac{dr}{r}$ is a Carleson measure on $\operatorname{supp}(\mu) \times (0,\infty)$, that is, (1.3) is satisfied with $\Delta_\mu(x,r)$ replaced by $\beta_1^\mu(x,r)$.

The paper is organized as follows. In Section 2 we provide the preliminaries for the proofs of Theorems 1.1 and 1.2. In Section 3 we show first that the boundedness of the smooth square functions in (1.4) implies uniform rectifiability. We then show, using convex combinations, that (1.3) implies (1.4), and thereby establish one of the implications in Theorem 1.1. Then we show by a simple argument that (1.5) implies (1.4) and thereby establish another of the implications in Theorem 1.2. In Section 4 we prove that uniform n-rectifiability implies (1.4) and (1.5), and thus complete the proof of Theorem 1.2. In Section 5 we prove that (1.3) holds if μ is uniform n-rectifiable; this is the most delicate part of the paper because of complications which arise from the non-smoothness of the function $r^{-n}\chi_{B(0,r)} - (2r)^{-n}\chi_{B(0,2r)}$. Finally, in Section 6 we outline the proof for Proposition 1.3.

Throughout the paper the letter C stands for some constant which may change its value at different occurrences. The notation $A \lesssim B$ means that there is some fixed constant C such that $A \leq CB$, with C as above. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

2. Preliminaries

2.1. **The David cubes.** Below we will need to use the David lattice $\mathcal{D}$ of "cubes" associated with μ (see [Da2, Appendix 1], for example). Suppose for simplicity that $\mu(\mathbb{R}^d) = \infty$. In this case, $\mathcal{D} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$ and each set $Q \in \mathcal{D}_j$, which is called a cube, satisfies $\mu(Q) \approx 2^{-jn}$ and $\operatorname{diam}(Q) \approx 2^{-j}$. In fact, we will assume that

$$c^{-1}2^{-j} \leq \operatorname{diam}(Q) \leq 2^{-j}.$$

We set $\ell(Q) := 2^{-j}$. For $R \in \mathcal{D}$, we denote by $\mathcal{D}(R)$ the family of all cubes $Q \in \mathcal{D}$ which are contained in R. In the case when $\mu(\mathbb{R}^d) < \infty$ and $\operatorname{diam}(\operatorname{supp}(\mu)) \approx 2^{-j_0}$, then $\mathcal{D} = \bigcup_{j\geq j_0} \mathcal{D}_j$. The other properties of the lattice $\mathcal{D}$ are the same as in the previous case.

2.2. **The α coefficients.** The so called α coefficients from [To1] play a crucial role in our proofs. They are defined as follows. Given a closed ball $B \subset \mathbb{R}^d$ which intersects $\operatorname{supp}(\mu)$, and two finite Borel measures σ and ν in $\mathbb{R}^d$, we set

$$\operatorname{dist}_B(\sigma, \nu) := \sup\Big\{\Big|\textstyle\int f\,d\sigma - \int f\,d\nu\Big| : \operatorname{Lip}(f) \leq 1, \operatorname{supp} f \subset B\Big\},$$

where $\operatorname{Lip}(f)$ stands for the Lipschitz constant of f. It is easy to check that this is indeed a distance in the space of finite Borel measures supported in the interior of B. See [Ma, Chapter 14] for other properties of this distance. Given a subset $\mathcal{A}$ of Borel measures, we set

$$\operatorname{dist}_B(\mu, \mathcal{A}) := \inf_{\sigma\in\mathcal{A}} \operatorname{dist}_B(\mu, \sigma).$$

We define

$$\alpha^n_\mu(B) := \frac{1}{r(B)^{n+1}} \inf_{c\geq 0, L} \operatorname{dist}_B(\mu, c\mathcal{H}^n_{|L}),$$

where $r(B)$ stands for the radius of B and the infimum is taken over all the constants $c \geq 0$ and all the n-planes L such that $L \cap \frac{1}{2}B \neq \varnothing$. To simplify notation, we will write $\alpha(B)$ instead of $\alpha^n_\mu(B)$.

Given a cube $Q \in \mathcal{D}$, let B_Q be a ball with radius $10\ell(Q)$ with the same center as Q. We denote

$$\alpha(Q) := \alpha(B_Q).$$

We also denote by c_Q and L_Q a constant and an n-plane minimizing $\alpha(Q)$.

The following is shown in [To1].

Theorem 2.1. *Let μ be an n-AD-regular measure in $\mathbb{R}^d$. If μ is uniformly n-rectifiable, then there exists a constant c such that*

$$\sum_{Q\subset R} \alpha(Q)^2\,\mu(Q) \leq c\,\mu(R) \qquad \text{for all } R\in\mathcal{D}. \tag{2.1}$$

2.3. **The weak constant density condition.** Given μ satisfying (1.1), we denote by $G(C,\varepsilon)$ the subset of those $(x,r)\in \operatorname{supp}(\mu)\times(0,\infty)$ for which there exists a Borel measure $\sigma=\sigma_{x,r}$ satisfying

(1) $\operatorname{supp}(\sigma)=\operatorname{supp}(\mu)$,
(2) the AD-regularity condition (1.1) with constant C,
(3) $|\sigma(B(y,t))-t^n|\leq \varepsilon r^n$ for all $y\in\operatorname{supp}(\mu)\cap B(x,r)$ and all $0<t<r$.

We remark that the error term in (3) is in terms of r^n and not of t^n.

Definition 2.2. A Borel measure μ satisfies the *weak constant density condition* (WCD) if there exists a positive constant C such that the set

$$G(C,\varepsilon)^c := [\operatorname{supp}(\mu)\times(0,\infty)]\setminus G(C,\varepsilon)$$

is a Carleson set for every $\varepsilon>0$, that is, for every $\varepsilon>0$ there exists a constant $C(\varepsilon)$ such that

$$\int_0^R\int_{B(x_0,R)} \chi_{G(C,\varepsilon)^c}(x,r)\,d\mu(x)\frac{dr}{r}\leq C(\varepsilon)R^n \tag{2.2}$$

for all $x_0\in\operatorname{supp}(\mu)$ and $R>0$.

Theorem 2.3. *Let $n\in(0,d)$ be an integer. An n-AD-regular measure μ in $\mathbb{R}^d$ is uniformly n-rectifiable if and only if it satisfies the weak constant density condition.*

David and Semmes in [DS1, Chapter 6] showed that if μ is uniformly n-rectifiable, then it satisfies the WCD. In [DS2, Chapter III.5], they also proved the converse in the cases when $n=1,2,d-1$. The proof of the converse for all codimensions was obtained very recently in [To2]. The arguments rely on two essential and deep ingredients: the so called bilateral weak geometric lemma of David and Semmes [DS2], and the (partial) characterization of uniform measures by Preiss [Pr].

3. Boundedness of square functions implies uniform rectifiability

In this section we assume that either $\varphi(x)=e^{-|x|^{2N}}$, with $N\in\mathbb{N}$, or $\varphi(x)=(1+|x|^2)^{-a}$, with $a>n/2$, as in Theorem 1.2. We will show that if (1.3), (1.4) or (1.5) holds, then μ is uniformly n-rectifiable. We work first with the case of (1.4) and afterward derive the other two cases from it.

We denote by $\mathcal{U}(\varphi,c_0)$ the family of n-AD-regular measures with constant c_0 in $\mathbb{R}^d$ such that

$$\Delta_{\mu,\varphi}(x,r)=0 \quad \text{for all } r>0 \text{ and all } x\in\operatorname{supp}(\mu).$$

Lemma 3.1. *For all $\varepsilon>0$ there exists $\delta>0$ such that all n-AD-regular measures μ with constant c_0 and $0\in\operatorname{supp}(\mu)$ such that*

$$\int_\delta^{\delta^{-1}}\int_{x\in\bar{B}(0,\delta^{-1})} |\Delta_{\mu,\varphi}(x,r)|\,d\mu(x)\,dr\leq\delta,$$

satisfy

$$\operatorname{dist}_{B(0,1)}(\mu,\mathcal{U}(\varphi,c_0))<\varepsilon.$$

Proof. Suppose that there exists an $\varepsilon>0$, and for each $m\geq 1$ there exists an n-AD-regular measure μ_m with constant c_0 such that $0\in\operatorname{supp}(\mu_m)$,

$$\int_{1/m}^{m}\int_{x\in\bar{B}(0,m)}|\Delta_{\mu_m,\varphi}(x,r)|\,d\mu_m(x)\,dr\leq\frac{1}{m},\tag{3.1}$$

and

$$\operatorname{dist}_{B(0,1)}(\mu_m,\mathcal{U}(\varphi,c_0))\geq\varepsilon.\tag{3.2}$$

By (1.1), we can replace $\{\mu_m\}$ by a subsequence converging weak * (i.e. when tested against compactly supported continuous functions) to a measure μ and it is easy to check that $0\in\operatorname{supp}(\mu)$ and that μ is also n-dimensional AD-regular with constant c_0. We claim that

$$\int_{0}^{\infty}\int_{x\in\mathbb{R}^d}|\Delta_{\mu,\varphi}(x,r)|\,d\mu(x)\,dr=0.\tag{3.3}$$

The proof of (3.3) is elementary. Fix m_0 and let $\eta>0$. Because of (1.1) and the decay conditions assumed for φ there exists $A>2m_0$ so that

$$\sup_{1/m_0\leq t\leq m_0}\int_{\bar{B}(0,2m_0)}\int_{|x-y|>A}|\varphi_t(x-y)-\varphi_{2t}(x-y)|d\nu(y)d\nu(x)<\frac{\eta}{m_0}\tag{3.4}$$

whenever ν satisfies (1.1) with constant c_0. Set $K=[1/m_0,\,m_0]\times\bar{B}(0,2m_0)$ and let $\widetilde{\chi}$ be a continuous function with compact support such that $\chi_{B(0,A)}\leq\widetilde{\chi}\leq 1$. Then, writing $\psi_t(x)=\varphi_t(x)-\varphi_{2t}(x)$ we have by (3.4)

$$\iint_K|((1-\widetilde{\chi})\psi_t)*\mu(x)|d\mu(x)dt<\eta,$$

and by (3.1)

$$\iint_K|(\widetilde{\chi}\psi_t)*\mu_m(x)|d\mu_m(x)dt<\eta+\frac{1}{m}.$$

Now $\{y\to\widetilde{\chi}(x-y)\psi_t(x-y),\ (t,x)\in K\}$ is an equicontinuous family of continuous functions supported inside a fixed compact set, which implies that $(\widetilde{\chi}\psi_t)*\mu_m(x)$ converges to $(\widetilde{\chi}\psi_t)*\mu(x)$ uniformly on K. It therefore follows that

$$\iint_K|\psi_t*\mu(x)|d\mu(x)dt\leq\eta+\limsup_m\int_{1/m_0}^{m_0}\int_{x\in\bar{B}(0,m_0)}|(\widetilde{\chi}\psi_t)*\mu_m(x)|d\mu_m(x)dt\leq 2\eta.\tag{3.5}$$

Since η is arbitrary the left side of (3.5) vanishes, and since this holds for any $m_0\geq 1$, our claim (3.3) proved.

By continuity it follows that $\varphi_r*\mu(x)$ is constant on $(0,\infty)\times\operatorname{supp}(\mu)$. In other words, $\mu\in\mathcal{U}(\varphi,c_0)$. However, by condition (3.2), letting $m\to\infty$, we have

$$\operatorname{dist}_{B(0,1)}(\mu,\mathcal{U}(\varphi,c_0))\geq\varepsilon,$$

because $\operatorname{dist}_{B(0,1)}(\cdot,\mathcal{U}(\varphi,c_0))$ is continuous under the weak * topology, see [Ma, Lemma 14.13]. So $\mu\notin\mathcal{U}(\varphi,c_0)$, which is a contradiction. □

By renormalizing the preceding lemma we get:

Lemma 3.2. *Let μ be an n-AD-regular measure such that $x_0 \in \operatorname{supp}(\mu)$. For all $\varepsilon > 0$ and $r > 0$ there exists a constant $\delta > 0$ such that if*

$$\int_{\delta r}^{\delta^{-1} r} \int_{x \in \bar{B}(x_0,\delta^{-1} r)} |\Delta_{\mu,\varphi}(x,t)| \, d\mu(x) \, dt \leq \delta \, r^{n+1},$$

then

$$\operatorname{dist}_{B(x_0,r)}(\mu, \mathcal{U}(\varphi, c_0)) < \varepsilon \, r^{n+1}.$$

Proof. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an affine map which maps $B(x_0, r)$ to $B(0, 1)$. Consider the image measure $\sigma = \frac{1}{r^n} T\#\mu$, where as usual $T\#\mu(E) := \mu(T^{-1}(E))$, and apply the preceding lemma to σ. $\square$

Definition 3.3. Given $n > 0$, a Borel measure μ in $\mathbb{R}^d$ is called *n-uniform* if there exists a constant $c > 0$ such that

$$\mu(B(x,r)) = c\, r^n \quad \text{for all } x \in \operatorname{supp}(\mu) \text{ and } r > 0.$$

We will denote by $\mathcal{U}(c)$ the collection of all n-uniform measures with constant c. By the following lemma, it turns out that $\mathcal{U}(\varphi, \cdot)$ and $\mathcal{U}(\cdot)$ coincide.

Lemma 3.4. *Let $f : [0,\infty) \to [0,\infty)$ be defined either by $f(x) = e^{-x^N}$, for some $N \in \mathbb{N}$, or by $f(x) = (1+x)^{-a}$, for $a > 1$. Let μ be a n-dimensional AD-regular Borel measure in $\mathbb{R}^d$. Then μ is n-uniform if and only if there exists some constant $c > 0$ such that*

$$\int f\left(\frac{|x-y|^2}{t^2}\right) d\mu(y) = c\, t^n \quad \textit{for all } x \in \operatorname{supp}(\mu) \textit{ and } t > 0. \tag{3.6}$$

Proof. For $f(x) = e^{-x}$ this lemma is due to De Lellis (see [DeL, pp. 60-61]) and an identical proof works for the functions of the form $f(x) = e^{-x^N}$. We provide a similar proof for the case $f(x) \equiv f_a(x) = (1+x)^{-a}$. It is clear that (3.6) holds if μ is n-uniform. Now assume (3.6) and set $Df(x) = x\, f'(x)$. We claim that

$$\operatorname{span}\{D^m f : m \geq 0\} \text{ is dense in } L^1((0,\infty)). \tag{3.7}$$

To verify (3.7) we note that

$$f_a(x) + \frac{1}{a} D f_a(x) = (1+x)^{-1-a} = f_{a+1}(x),$$

so that $f_{a+1} \in \operatorname{span}\{D^m f_a : m \geq 0\}$. Hence by induction we see that whenever P is a polynomial with $P(0) = 0$,

$$(1+x)^{-1-a} P((1+x)^{-1}) \in \operatorname{span}\{D^m f_a : m \geq 0\}. \tag{3.8}$$

Moreover if $\mathcal{P}$ is the algebra of functions of the form $P((1+x)^{-1})$ where P is a polynomial with $P(0) = 0$, an application of the Weierstrass approximation theorem as in [DeL, Lemma 6.14] shows that $\mathcal{P}$ is dense in $C_0((0,\infty))$. Therefore for any $\varepsilon > 0$ and any $h \in C_0((0,\infty))$ there exists some $P \in \mathcal{P}$ such that

$$\|P - (1+x)^{a+1}\, h\|_\infty < \varepsilon.$$

Hence we deduce that the functions on the left side of (3.8) form a dense subset of $L^1((0,\infty))$, and (3.7) follows.

Now let $\mathcal{B}$ be the set of $g \in L^1((0,\infty))$ for which there is a constant c_g such that

$$\int g\left(\frac{|x-y|^2}{t^2}\right) d\mu(y) = c_g t^n.$$

Then $f \in \mathcal{B}$, by the hypothesis (3.6). Differentiating (3.6) with respect to t shows that $Df(x) = x\,f'(x) \in \mathcal{B}$ with constant $-2cn$ independent of x. Then by induction and (3.7) $\mathcal{B}$ contains a dense subset of $L^1((0,\infty))$. Since $\mathcal{B}$ is closed in $L^1((0,\infty))$, it follows that $\chi_{(0,1)} \in \mathcal{B}$ and the lemma is proved. □

Lemma 3.5. *If $\mu \in \mathcal{U}(\varphi, c_0)$ then μ is supported on an n-rectifiable set.*

Proof. Since $\mu \in \mathcal{U}(\varphi, c_0)$ we have

$$\varphi_{2^{-k}} * \mu(x) - \varphi_{2^k} * \mu(x) = 0 \qquad \text{for all } k > 0 \text{ and all } x \in \operatorname{supp}(\mu). \tag{3.9}$$

Now consider the function $F : \mathbb{R}^d \to \mathbb{R}$ defined by

$$F(x) = \sum_{k>0} 2^{-k} \Big(\varphi_{2^{-k}} * \mu(x) - \varphi_{2^k} * \mu(x)\Big)^2.$$

Taking into account that $|\varphi_{2^{-k}} * \mu(x) - \varphi_{2^k} * \mu(x)| \leq c$ for all $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$, we see that $F(x) < \infty$ for all $x \in \mathbb{R}^d$, and so F is well defined. Moreover, by (3.9) we have $F = 0$ on $\operatorname{supp}(\mu)$.

Now we claim that $F(x) > 0$ for all $x \in \mathbb{R}^d \setminus \operatorname{supp}(\mu)$. Indeed, it follows easily that

$$\lim_{k\to\infty} \varphi_{2^{-k}} * \mu(x) = 0 \qquad \text{for all } x \in \mathbb{R}^d \setminus \operatorname{supp}(\mu),$$

while, by the n-AD-regularity of μ,

$$\liminf_{k\to\infty} \varphi_{2^k} * \mu(x) \geq c\, c_0^{-1} \qquad \text{for all } x \in \mathbb{R}^d.$$

Thus if $x \in \mathbb{R}^d \setminus \operatorname{supp}(\mu)$ we have $\varphi_{2^{-k}} * \mu(x) - \varphi_{2^k} * \mu(x) \neq 0$ for all large $k > 0$, which implies that $F(x) > 0$ and proves our claim.

We have shown that for $\mu \in \mathcal{U}(\varphi, c_0)$, $\operatorname{supp}(\mu) = F^{-1}(0)$. Next we will show $F^{-1}(0)$ is a real analytic variety. Notice that the lemma will follow from this assertion because $\operatorname{supp}(\mu)$ has locally finite $\mathcal{H}^n$ measure, so that the analytic variety $F^{-1}(0)$ is n-dimensional and any n-dimensional real analytic variety is n-rectifiable.

To prove that the zero set of F is a real analytic variety it is enough to check that $\varphi_{2^{-k}} * \mu - \varphi_{2^k} * \mu$ is a real analytic function for each $k > 0$, because the zero set of a real analytic function is a real analytic variety and the intersection of any family of real analytic varieties is again a real analytic variety; see [Na]. So it is enough to show that $\varphi_r * \mu$ is a real analytic function for every $r > 0$.

In the case $\varphi(x) = e^{-|x|^{2N}}$, consider the function $f : \mathbb{C}^d \to \mathbb{C}$ defined by

$$f(z_1, \ldots, z_d) = \frac{1}{r^n} \int \exp\left(-r^{-2N} \left(\sum_{i=1}^d (y_i - z_i)^2\right)^N\right) d\mu(y).$$

It is easy to check that f is well defined and holomorphic in the whole $\mathbb{C}^d$, and thus $\varphi_r * \mu = f|_{\mathbb{R}^d}$ is real analytic.

In the case $\varphi(x) = (1+|x|^2)^{-a}$, $a > n/2$, for $(z_1,\dots,z_d) \in \mathbb{C}^d$ we take

$$f(z_1,\dots,z_d) = \frac{1}{r^n}\int \left(1 + r^{-2}\sum_{i=1}^d (y_i - z_i)^2\right)^{-a} d\mu(y).$$

This is a holomorphic function in the open set

$$V = \left\{ z \in \mathbb{C}^d : |\mathrm{Im}\, z_i| < \frac{r}{2d^{1/2}} \text{ for } 1 \le i \le d \right\}.$$

Indeed, for $z \in V$, we have

$$\mathrm{Re}\left(1+r^{-2}\sum_{i=1}^d (y_i-z_i)^2\right) = 1+r^{-2}\sum_{i=1}^d \left((y_i-\mathrm{Re}\, z_i)^2 - (\mathrm{Im}\, z_i)^2\right) \ge 1-r^{-2}\sum_{i=1}^d (\mathrm{Im}\, z_i)^2 > \frac{3}{4}.$$

Thus f is well defined and holomorphic in V, and so $\varphi_r * \mu = f|_{\mathbb{R}^d}$ is real analytic. □

Theorem 3.6. *If $\mu \in \mathcal{U}(\varphi, c_0)$ then μ is n-uniform.*

Proof. If $\mu \in \mathcal{U}(\varphi, c_0)$, then $\varphi_r * \mu(x) = \varphi_{2r} * \mu(x)$ for all $x \in \mathrm{supp}(\mu)$ and all $r > 0$, and consequently

$$\varphi_{2^k r} * \mu(x) = \varphi_r * \mu(x) \quad \text{for all } 1 \le r < 2, \text{ all } k \in \mathbb{Z}, \text{ and all } x \in \mathrm{supp}(\mu). \tag{3.10}$$

By the preceding lemma μ is of the form

$$\mu = \rho\,\mathcal{H}^n \lfloor E,$$

where ρ is some positive function on E bounded from above and below and $E \subset \mathbb{R}^d$ is an n-rectifiable set. This implies that the density

$$\Theta^n(x,\mu) = \lim_{\varepsilon\to 0} \frac{\mu(B(x,\varepsilon))}{(2\varepsilon)^n}$$

exists at μ-a.e. $x \in \mathbb{R}^d$; see [Ma, Theorem 16.2]. It then follows easily that

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon * \mu(x) \quad \text{exists at } \mu\text{-a.e. } x \in \mathbb{R}^d$$

and with (3.10) this implies that

$$\varphi_{R_1} * \mu(x) = \varphi_{R_2} * \mu(x) \quad \text{for all } R_1, R_2 > 0 \text{ and } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{3.11}$$

Notice that

$$\nabla(\varphi_R * \mu)(x) = \int \nabla\varphi_R(x-y)\, d\mu(y),$$

and by decomposing this integral into annuli centered at x, using the fast decay of $\nabla\varphi_R$ at ∞ and the fact that $\mu(B(x,r)) \le c_0\, r^n$ for all $r > 0$, we easily see that

$$\|\nabla(\varphi_R * \mu)\|_\infty \le \frac{c}{R}, \tag{3.12}$$

with c depending on c_0. Thus as $R \to \infty$ the right side of (3.12) tends to 0 and we conclude that from (3.11) that

$$\varphi_{R_1} * \mu(x) = \varphi_{R_2} * \mu(y) \quad \text{for all } R_1, R_2 > 0 \text{ and all } x, y \in \mathrm{supp}(\mu).$$

Therefore, by Lemma 3.4, μ is n-uniform. □

We return to Lemma 3.2 and the proof that (1.4) implies uniform n-rectifiability.

Lemma 3.7. *Let μ be an n-AD-regular measure in $\mathbb{R}^d$ such that $x_0 \in \operatorname{supp}(\mu)$. For all $\varepsilon > 0$, there exists a constant $\delta := \delta(\varepsilon) > 0$ such that if, for some $r > 0$,*

$$\int_{\delta r}^{\delta^{-1} r} \int_{x \in \bar{B}(x_0, \delta^{-1} r)} |\Delta_{\mu,\varphi}(x,t)|^2 \, d\mu(x) \, \frac{dt}{t} \leq \delta^{n+4} \, r^n,$$

then there exists some constant $c_1 > 0$ such that

$$|\mu(B(y,t)) - c_1 t^n| < \varepsilon r^n \tag{3.13}$$

for all $y \in B(x_0, r) \cap \operatorname{supp}(\mu)$ and $0 < t \leq r$.

Proof. Let $\varepsilon > 0$. By Cauchy-Schwarz, we have

$$\begin{aligned}
&\int_{\delta r}^{\delta^{-1} r} \int_{x \in \bar{B}(x_0, \delta^{-1} r)} |\Delta_{\mu,\varphi}(x,t)| \, d\mu(x) \, dt \\
&\quad \leq \left[\int_{\delta r}^{\delta^{-1} r} \int_{x \in \bar{B}(x_0, \delta^{-1} r)} |\Delta_{\mu,\varphi}(x,t)|^2 \, d\mu(x) \, \frac{dt}{t} \right]^{1/2} \left[\int_{\delta r}^{\delta^{-1} r} \int_{x \in \bar{B}(x_0, \delta^{-1} r)} t \, d\mu(x) \, dt \right]^{1/2} \\
&\quad \leq c \big[\delta^{n+4} \, r^n \big]^{1/2} \big[\delta^{-2} \, r^2 \, \mu(B(x_0, \delta^{-1} r)) \big]^{1/2} \\
&\quad \leq c \left[\delta^{(n+4)/2} \, r^{n/2} \right] \left[\delta^{-(n+2)/2} \, r^{(n+2)/2} \right] = c \, \delta \, r^{n+1}.
\end{aligned}$$

Hence for any $\varepsilon_1 > 0$ we see that if δ is small enough then by Lemma 3.2,

$$\operatorname{dist}_{B(x_0,3r)}(\mu, \mathcal{U}(\varphi, c_0)) < \varepsilon_1 \, r^{n+1}$$

and there exists $\sigma \in \mathcal{U}(c_1)$ such that $\operatorname{dist}_{B(x_0,3r)}(\mu, \sigma) < \varepsilon_1 \, r^{n+1}$ for a suitable constant c_1.

Let $y \in B(x_0, r)$ and for $0 < t \leq r$ consider a smooth bump function $\widetilde{\chi}_{y,t}$ such that $\chi_{B(y,t)} \leq \widetilde{\chi}_{y,t} \leq \chi_{B(y,t(1+\eta))}$ and $\|\nabla \widetilde{\chi}_{y,t}\|_\infty \leq \frac{c}{t\eta}$, where η is some small constant to be determined later. For $y \in B(x_0, r)$ and for $0 < t \leq r$, we have

$$\begin{aligned}
&\left| \int \widetilde{\chi}_{y,t}(x) d\mu(x) - \int \widetilde{\chi}_{y,t}(x) d\sigma(x) \right| \\
&\quad \leq \|\nabla \widetilde{\chi}_{y,t}\|_\infty \operatorname{dist}_{B(x_0,3r)}(\mu, \sigma) \leq c \frac{\varepsilon_1 \, r^{n+1}}{\eta \, t}.
\end{aligned} \tag{3.14}$$

Therefore by (3.14) and Lemma 3.4, for $0 < t \leq r$,

$$\begin{aligned}
\mu(B(y,t)) &\leq \int \widetilde{\chi}_{y,t}(x) \, d\mu(x) \leq \int \widetilde{\chi}_{y,t}(x) \, d\sigma(x) + c \frac{\varepsilon_1 \, r^{n+1}}{\eta \, t} \\
&\leq c_1 t^n (1+\eta)^n + c \frac{\varepsilon_1 \, r^{n+1}}{\eta \, t},
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\mu(B(y,t)) &\geq \int \widetilde{\chi}_{y,\frac{t}{1+\eta}}(x) \, d\mu(x) \geq \int \widetilde{\chi}_{y,\frac{t}{1+\eta}}(x) \, d\sigma(x) - c \frac{\varepsilon_1 \, r^{n+1}}{\eta \, t} \\
&\geq c_1 \frac{t^n}{(1+\eta)^n} - c \frac{\varepsilon_1 \, r^{n+1}}{\eta \, t}.
\end{aligned} \tag{3.16}$$

Choosing η and ε_1 appropriately, we get that for some small $\varepsilon_2 := \varepsilon_2(\varepsilon_1, \eta)$,

$$|\mu(B(y,t)) - c_1 t^n| \leq \varepsilon_2 \left(\frac{r^{n+1}}{t} + t^n \right). \tag{3.17}$$

Hence if $t > {\varepsilon_2}^{1/2} r$, then because $t^n \leq r^{n+1}/t$,

$$|\mu(B(y,t)) - c_1 t^n| \leq c\,\varepsilon_2 \frac{r^{n+1}}{{\varepsilon_2}^{1/2} r} \leq c\,\varepsilon_2^{1/2} r^n.$$

On the other hand, if $t \leq {\varepsilon_2}^{1/2} r$, then by the AD-regularity of μ,

$$|\mu(B(y,t)) - c_1 t^n| \leq \mu(B(y,t)) + c_1 t^n \leq c({\varepsilon_2}^{1/2})^n \, r^n.$$

Therefore, since $\lim_{\varepsilon_1 \to 0, \eta \to 0} \varepsilon_2 = 0$, (3.13) holds if ε_1 and η are sufficiently small. $\square$

Lemma 3.8. *Let μ be an n-AD-regular measure. Assume that $|\Delta_{\mu,\varphi}(x,r)|^2 \, d\mu(x) \, \frac{dr}{r}$ is a Carleson measure on $\operatorname{supp}(\mu) \times (0, \infty)$. Then the weak constant density condition holds for μ.*

Proof. Let $\varepsilon > 0$ and let $A := A_\varepsilon \subset \mathbb{R}^d \times \mathbb{R}$ consist of those pairs (x,r) such that (3.13) does not hold. We have to show that

$$\int_0^R \int_{x \in B(z,R)} \chi_A(x,r) \, d\mu(x) \, \frac{dr}{r} \leq c(\varepsilon) \, R^n \qquad \text{for all } z \in \operatorname{supp}(\mu), \ r > 0.$$

To this end, notice that if $(x,r) \in A$, then

$$\int_{\delta r}^{\delta^{-1} r} \int_{y \in \bar{B}(x, \delta^{-1} r)} |\Delta_{\mu,\varphi}(y,t)|^2 \, d\mu(y) \, \frac{dt}{t} \geq \delta^{n+4} \, r^n,$$

where $\delta = \delta(\varepsilon)$ is as in Lemma 3.7. Then by Chebychev's inequality,

$$\begin{aligned}
&\int_0^R \int_{x \in B(z,R)} \chi_A(x,r) \, d\mu(x) \, \frac{dr}{r} \\
&\qquad \leq \int_0^R \int_{x \in B(z,R)} \frac{1}{\delta^{n+4} \, r^n} \left(\int_{\delta r}^{\delta^{-1} r} \int_{y \in \bar{B}(x, \delta^{-1} r)} |\Delta_{\mu,\varphi}(y,t)|^2 \, d\mu(y) \, \frac{dt}{t} \right) d\mu(x) \, \frac{dr}{r} \\
&\qquad \leq \int_0^{\delta^{-1} R} \int_{|y-z| \leq (1+\delta^{-1})R} |\Delta_{\mu,\varphi}(y,t)|^2 \int_{\delta t}^{\delta^{-1} t} \frac{\mu(\bar{B}(y, \delta^{-1} r))}{\delta^{n+4} \, r^{n+1}} dr \ d\mu(y) \, \frac{dt}{t}.
\end{aligned}$$

But since

$$\int_{\delta t}^{\delta^{-1} t} \frac{\mu(B(y, \delta^{-1} r))}{\delta^{n+4} \, r^{n+1}} dr \leq c_0 \, \delta^{-2(n+2)} \int_{\delta t}^{\delta^{-1} t} \frac{dr}{r} \leq c_0 \, \delta^{-2(n+3)},$$

we then get

$$\begin{aligned}
&\int_0^R \int_{x \in B(z,R)} \chi_A(x,r) \, d\mu(x) \, \frac{dr}{r} \\
&\qquad \leq c_0 \, \delta^{-2(n+3)} \int_0^{\delta^{-1} R} \int_{|y-z| \leq (1+\delta^{-1})R} |\Delta_{\mu,\varphi}(y,t)|^2 \, d\mu(y) \, \frac{dt}{t} \leq c \, \delta^{-3n-7} R^n,
\end{aligned}$$

which is what we needed to show. $\square$

As an immediate corollary of Theorem 2.3 and Lemma 3.8 we obtain the following.

Theorem 3.9. *If μ is an n-AD-regular measure in $\mathbb{R}^d$ and if c is a constant such that for any ball $B(x_0,R)$ with center $x_0\in\operatorname{supp}(\mu)$,*

$$\int_0^R\int_{x\in B(x_0,R)}|\Delta_{\mu,\varphi}(x,r)|^2\,d\mu(x)\,\frac{dr}{r}\le c\,R^n,$$

then μ is uniformly n-rectifiable.

Corollary 3.10. *Suppose that for any ball $B(x_0,R)$ centered at* $\operatorname{supp}(\mu)$

$$\int_0^R\int_{x\in B(x_0,R)}|\Delta_{\mu}(x,r)|^2\,d\mu(x)\,\frac{dr}{r}\le c\,R^n.$$

Then μ is uniformly n-rectifiable.

Proof. We will show that (1.3) implies (1.4), by taking a suitable convex combination, and then apply Theorem 3.9.

For $R>0$ we seek a function $\widetilde{\varphi}_R:(0,\infty)\to(0,\infty)$ such that

$$\frac{1}{R^n}e^{\frac{-s^2}{R^2}}=\int_0^\infty\frac{1}{r^n}\chi_{[0,r]}(s)\widetilde{\varphi}_R(r)\,dr=\int_s^\infty\frac{\widetilde{\varphi}_R(r)}{r^n}\,dr,\qquad\text{for } s>0. \tag{3.18}$$

Differentiating with respect to s we get

$$-\frac{2s}{R^{n+2}}\,e^{\frac{-s^2}{R^2}}=-\frac{\widetilde{\varphi}_R(s)}{s^n}.$$

Hence (3.18) is solved for $R>0$ and $s>0$ by

$$\widetilde{\varphi}_R(s)=\frac{2s^{n+1}}{R^{n+2}}e^{\frac{-s^2}{R^2}}.$$

Using (3.18) we can now write, for $x\in\operatorname{supp}(\mu)$, and any $R_1>0$,

$$\int_0^\infty|\Delta_{\mu,\varphi}(x,R)|^2\,\frac{dR}{R}=\int_0^\infty|(\varphi_R-\varphi_{2R})*\mu(x)|^2\,\frac{dR}{R}$$
$$=\int_0^\infty\Big|\left(\int_0^\infty\frac{1}{r^n}\chi_{[0,r]}(|\cdot|)\widetilde{\varphi}_R(r)\,dr\right)*\mu(x)-\left(\int_0^\infty\frac{1}{r^n}\chi_{[0,r]}(|\cdot|)\widetilde{\varphi}_{2R}(r)\,dr\right)*\mu(x)\Big|^2\,\frac{dR}{R}.$$

By a change of variables we get

$$\int_0^\infty\frac{1}{r^n}\chi_{[0,r]}(|y-x|)\widetilde{\varphi}_{2R}(r)\,dr=\int_0^\infty\frac{1}{(2r)^n}\chi_{[0,2r]}(|y-x|)\widetilde{\varphi}_R(r)\,dr.$$

Therefore, using Cauchy-Schwarz and the fact that $\int_0^\infty\widetilde{\varphi}_R(r)\,dr\lesssim 1$, we obtain

$$\begin{aligned}\int_0^\infty|\Delta_{\mu,\varphi}(x,R)|^2\,\frac{dR}{R}&=\int_0^\infty\left|\int_0^\infty\left(\frac{1}{r^n}\chi_{B(0,r)}(\cdot)-\frac{1}{(2r)^n}\chi_{B(0,2r)}(\cdot)\right)*\mu(x)\,\widetilde{\varphi}_R(r)\,dr\right|^2\,\frac{dR}{R}\\&\lesssim\int_0^\infty\int_0^\infty|\Delta_\mu(x,r)|^2\widetilde{\varphi}_R(r)\,dr\,\frac{dR}{R}\\&\lesssim\int_0^\infty\left(\int_0^\infty\widetilde{\varphi}_R(r)\,\frac{dR}{R}\right)|\Delta_\mu(x,r)|^2\,dr.\end{aligned}$$

Moreover,

$$\int_0^\infty\widetilde{\varphi}_R(r)\,\frac{dR}{R}=2\int_0^\infty\left(\frac{r}{R}\right)^{n+1}e^{\frac{-r^2}{R^2}}\,\frac{dR}{R^2}=\frac{2}{r}\int_0^\infty t^{n+1}e^{-t^2}\,dt\lesssim\frac{1}{r}.$$

Hence we infer that

$$\int_0^\infty |\Delta_{\mu,\varphi}(x,r)|^2\,\frac{dr}{r} \lesssim \int_0^\infty |\Delta_\mu(x,r)|^2\,\frac{dr}{r},$$

which shows that (1.3) implies (1.4). $\square$

Corollary 3.11. *Suppose that for any ball $B(x_0,R)$ centered at* $\operatorname{supp}(\mu)$

$$\int_0^R \int_{x\in B(x_0,R)} |\widetilde{\Delta}_\mu(x,r)|^2\,d\mu(x)\,\frac{dr}{r} \leq c\,R^n.$$

Then μ is uniformly n-rectifiable.

Proof. We will show that (1.5) implies (1.4) by calculus and Cauchy-Schwarz and Theorem 3.9 again.

By calculus

$$\Delta_{\mu,\varphi}(x,r) = \int_r^{2r} \widetilde{\Delta}_{\mu,\varphi}(x,t)\frac{dt}{t}$$

so that by Cauchy-Schwarz

$$|\Delta_{\mu,\varphi}(x,r)|^2 \leq \int_r^{2r}\frac{dt}{t}\int_r^{2r} |\widetilde{\Delta}_{\mu,\varphi}(x,t)|^2\,\frac{dt}{t} \leq \log 2\int_r^{2r} |\widetilde{\Delta}_{\mu,\varphi}(x,t)|^2\,\frac{dt}{t}.$$

Therefore

$$\begin{aligned}\int_0^R\int_{B(x_0,R)} |\Delta_{\mu,\varphi}(x,r)|^2 d\mu(x)\frac{dr}{r} &\leq \log 2\int_{B(x_0,R)}\int_0^R\int_r^{2r} |\widetilde{\Delta}_{\mu,\varphi}(x,t)|^2\frac{dt}{t}\frac{dr}{r}d\mu(x)\\ &\leq \log 2\int_{B(x_0,R)}\int_0^{2R}\Big(\int_t^{2t}\frac{dr}{r}\Big)|\widetilde{\Delta}_{\mu,\varphi}(x,t)|^2\frac{dt}{t}d\mu(x).\\ &\leq (\log 2)^2\int_{B(x_0,2R)}\int_0^{2R}|\widetilde{\Delta}_{\mu,\varphi}(x,t)|^2\frac{dt}{t}d\mu(x),\end{aligned}$$

so that (1.5) implies (1.4). $\square$

4. Uniform rectifiabilty implies boundedness of smooth square functions

Let $h:\mathbb{R}^d\to\mathbb{R}$ be a smooth function for which there exist positive constants c and ε such that

$$|h(x)| \leq \frac{c}{(1+|x|)^{n+\varepsilon}} \quad\text{and}\quad |\nabla h(x)| \leq \frac{c}{(1+|x|)^{n+1+\varepsilon}}, \tag{4.1}$$

for all $x\in\mathbb{R}^d$. Furthermore assume that

$$\int h(y-x)d\mathcal{H}^n_{|L}(y) = 0 \tag{4.2}$$

for every n-plane L and every $x\in L$. For $r>0$, denote

$$h_r(x) = \frac{1}{r^n}\,h\left(\frac{x}{r}\right).$$

Theorem 4.1. *Let μ be an n-AD-regular measure in $\mathbb{R}^d$. If μ is uniformly n-rectifiable, then there exists a constant c such that*

$$\int_0^R \int_{x\in B(x_0,R)} |h_r * \mu(x)|^2\, d\mu(x)\, \frac{dr}{r} \leq c\, R^n, \tag{4.3}$$

for all $x_0 \in \operatorname{supp}(\mu)$, $R>0$.

Because the functions $\varphi_t(x) - \varphi_{2t}(x)$ and $\partial_\varphi(x,t)$ have the form h_t for functions with h satisfying (4.1) and (4.2), Theorem 4.1 establishes the remaining parts of Theorem 1.2.

Proof. It is immediate to check that the estimate (4.3) holds if and only if for all $R_0 \in \mathcal{D}$

$$\sum_{Q\in\mathcal{D}:Q\subset R_0} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |h_r * \mu(x)|^2\, \frac{dr}{\ell(Q)}\, d\mu(x) \leq c\, \mu(R_0). \tag{4.4}$$

Let $x \in \frac{1}{2}B_Q$ and $\ell(Q) \leq r \leq 2\ell(Q)$. If $x \in \frac{1}{2}B_Q \cap L_Q$ (recall that L_Q is the n-plane minimizing $\alpha(Q)$), we have

$$\int h_r(y-x)\, d\mathcal{H}^n_{|L_Q}(y) = 0.$$

Hence

$$\begin{aligned}
\left|\int h_r(y-x)\, d\mu(y)\right| &= \left|\int h_r(y-x)\, d(\mu - c_Q\mathcal{H}^n_{|L_Q})(y)\right| \\
&= \left|\int \sum_{k\geq 0} \widetilde{\chi}_k(y)\, h_r(y-x)\, d(\mu - c_Q\mathcal{H}^n_{|L_Q})(y)\right| \\
&\leq \sum_{k\geq 0} \left|\int \widetilde{\chi}_k(y)\, h_r(y-x)\, d(\mu - c_Q\mathcal{H}^n_{|L_Q})(y)\right|,
\end{aligned}$$

where $\widetilde{\chi}_k$, $k \geq 0$, are bump smooth functions such that

- $\sum_{k\geq 0} \widetilde{\chi}_k = 1$
- $\|\nabla \widetilde{\chi}_k\|_\infty \leq \ell(Q^k)^{-1}$,
- $\chi_{A(x,2^k r, 2^{k+1} r)} \leq \widetilde{\chi}_k \leq \chi_{A(x,2^{k-1} r, 2^{k+2} r)}$ for $k \geq 1$, and
- $\chi_{B(x,r)} \leq \widetilde{\chi}_0 \leq \chi_{B(x,2r)}$.

As usual $A(x, r_1, r_2) = \{y : r_1 \leq |y-x| < r_2\}$. Moreover for $m \in \mathbb{N}$, Q^m denotes the ancestor of Q such that $\ell(Q^m) = 2^m \ell(Q)$.

Set $F_k(y) = h_r(x-y)\widetilde{\chi}_k(y)$, and notice that $\operatorname{supp} F_k \subset B_{Q^{k+2}}$. Then

$$
\begin{aligned}
\left|\int h_r(y-x)\,d\mu(y)\right| &\leq \sum_{k\geq 0}\left|\int F_k(y)d(\mu - c_{Q^{k+2}}\mathcal{H}^n_{|L_{Q^{k+2}}})(y)\right| \\
&\qquad + \sum_{k\geq 0}\left|\int F_k(y)d(c_Q\mathcal{H}^n_{|L_Q} - c_{Q^{k+2}}\mathcal{H}^n_{|L_{Q^{k+2}}})(y)\right| \\
&\leq \sum_{k\geq 0}\|\nabla F_k\|_\infty\,\alpha(Q^{k+2})\,\ell(Q^{k+2})^{n+1} \\
&\qquad + \sum_{k\geq 0}\|\nabla F_k\|_\infty\,\operatorname{dist}_{B_{Q^{k+2}}}(c_Q\mathcal{H}^n_{|L_Q}, c_{Q^{k+2}}\mathcal{H}^n_{|L_{Q^{k+2}}}) \\
&:= I_1 + I_2
\end{aligned}
\tag{4.5}
$$

For $y \in \operatorname{supp} F_k$ using (4.1) it follows easily that

$$
|h_r(y-x)| \lesssim \frac{1}{\ell(Q)^n}\left(\frac{\ell(Q)}{\ell(Q^k)}\right)^{n+\varepsilon} \quad \text{and} \quad |\nabla h_r(y-x)| \lesssim \frac{1}{\ell(Q)^{n+1}}\left(\frac{\ell(Q)}{\ell(Q^k)}\right)^{n+1+\varepsilon}.
$$

Hence

$$
\|\nabla F_k\|_\infty \lesssim \frac{1}{\ell(Q^k)}\frac{1}{\ell(Q)^n}\left(\frac{\ell(Q)}{\ell(Q^k)}\right)^{n+\varepsilon} + \frac{1}{\ell(Q)^{n+1}}\left(\frac{\ell(Q)}{\ell(Q^k)}\right)^{n+1+\varepsilon} \lesssim \frac{\ell(Q)^\varepsilon}{\ell(Q^k)^{n+1+\varepsilon}}.
\tag{4.6}
$$

We can now estimate I_1:

$$
\begin{aligned}
I_1 &\lesssim \sum_{k\geq 0}\alpha(Q^{k+2})\ell(Q^k)^{n+1}\frac{\ell(Q)^\varepsilon}{\ell(Q^k)^{n+1+\varepsilon}} = \sum_{k\geq 0}\alpha(Q^{k+2})\left(\frac{\ell(Q)}{\ell(Q^k)}\right)^\varepsilon \\
&\lesssim \sum_{P\in\mathcal{D}:P\supset Q}\alpha(P)\left(\frac{\ell(Q)}{\ell(P)}\right)^\varepsilon.
\end{aligned}
\tag{4.7}
$$

For I_2, using also [To1, Lemma 3.4] in the first inequality, we get

$$
\begin{aligned}
I_2 &\lesssim \sum_{k\geq 0}\frac{\ell(Q)^\varepsilon}{\ell(Q^k)^{n+1+\varepsilon}}\left(\sum_{0\leq j\leq k+2}\alpha(Q^j)\right)\ell(Q^{k+2})^{n+1} \\
&\lesssim \sum_{k\geq 0}\left(\frac{\ell(Q)}{\ell(Q^k)}\right)^\varepsilon\left(\sum_{0\leq j\leq k+2}\alpha(Q^j)\right) \\
&\lesssim \sum_{R\in\mathcal{D}:R\supset Q}\;\sum_{P\in\mathcal{D}:Q\subset P\subset R}\alpha(P)\left(\frac{\ell(Q)}{\ell(R)}\right)^\varepsilon \\
&= \sum_{P\in\mathcal{D}:P\supset Q}\alpha(P)\sum_{R\in\mathcal{D}:R\supset P}\left(\frac{\ell(Q)}{\ell(R)}\right)^\varepsilon \\
&\approx \sum_{P\in\mathcal{D}:P\supset Q}\alpha(P)\left(\frac{\ell(Q)}{\ell(P)}\right)^\varepsilon.
\end{aligned}
\tag{4.8}
$$

Therefore by (4.5), (4.7) and (4.8), for $x \in \frac{1}{2}B_Q \cap L_Q$ and $\ell(Q) \leq r \leq 2\ell(Q)$,

$$\left|\int h_r(y-x)\,d\mu(y)\right| \lesssim \sum_{P\in\mathcal{D}:P\supset Q} \alpha(P)\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}. \tag{4.9}$$

On the other hand, given an arbitrary $x \in Q$, let x' be its orthogonal projection on L_Q (notice that $x' \in \frac{1}{2}B_Q$). We have

$$\begin{aligned}\left|\int h_r(y-x)\,d\mu(y)\right| &\leq \left|\int h_r(y-x')\,d\mu(y)\right| + \int_{B_Q} |h_r(y-x)-h_r(y-x')|\,d\mu(y)\\ &\quad + \int_{\mathbb{R}^d\setminus B_Q} |h_r(y-x)-h_r(y-x')|\,d\mu(y)\\ &:= I_3 + I_4 + I_5.\end{aligned} \tag{4.10}$$

For $\ell(Q) \leq r \leq 2\ell(Q)$, by (4.9),

$$I_3 \lesssim \sum_{P\in\mathcal{D}:P\supset Q} \alpha(P)\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}. \tag{4.11}$$

We can now estimate I_4 and I_5 using (4.1). First

$$I_4 \lesssim \int_{B_Q} \frac{|x-x'|}{\ell(Q)^{n+1}}d\mu(y) \lesssim \frac{\operatorname{dist}(x,L_Q)}{\ell(Q)^{n+1}}\ell(Q)^n = \frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}. \tag{4.12}$$

Moreover, noticing that if $y \notin B_Q$ and $\xi \in [y-x, y-x']$ we have that $|y-x| \approx |\xi|$,

$$\begin{aligned}I_5 &\lesssim \int \frac{|x-x'|}{\ell(Q)^{n+1}} \sup_{\xi\in[y-x,y-x']} |\nabla(h_r)(\xi)|d\mu(y)\\ &\lesssim \frac{|x-x'|}{\ell(Q)^{n+1}} \int \frac{\ell(Q)^{n+1+\varepsilon}}{(\ell(Q)+|y-x|)^{n+1+\varepsilon}}d\mu(y)\\ &\lesssim \operatorname{dist}(x,L_Q)\ell(Q)^{\varepsilon}\ell(Q)^{-1-\varepsilon} = \frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}.\end{aligned} \tag{4.13}$$

Hence by (4.10), (4.11), (4.12) and (4.13), we get the following pointwise estimate for $x \in Q$ and $\ell(Q) \leq r \leq 2\ell(Q)$:

$$|h_r * \mu(x)| \lesssim \frac{\operatorname{dist}(x,L_Q)}{\ell(Q)} + \sum_{P\in\mathcal{D}:P\supset Q} \alpha(P)\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}. \tag{4.14}$$

Therefore,

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}:Q\subset R_0}\int_Q\int_{\ell(Q)}^{2\ell(Q)}|h_r*\mu(x)|^2\,\frac{dr}{\ell(Q)}d\mu(x)\\
&\quad\lesssim \sum_{Q\in\mathcal{D}:Q\subset R_0}\int_Q\int_{\ell(Q)}^{2\ell(Q)}\left(\frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}\right)^2\frac{dr}{\ell(Q)}d\mu(x)\\
&\qquad+\sum_{Q\in\mathcal{D}:Q\subset R_0}\int_Q\int_{\ell(Q)}^{2\ell(Q)}\left(\sum_{P\in\mathcal{D}:P\supset Q}\alpha(P)\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\right)^2\frac{dr}{\ell(Q)}d\mu(x)\\
&\quad\lesssim \sum_{Q\in\mathcal{D}:Q\subset R_0}\int_Q\left(\frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}\right)^2d\mu(x)+\\
&\qquad+\sum_{Q\in\mathcal{D}:Q\subset R_0}\left(\sum_{P\in\mathcal{D}:P\supset Q}\alpha(P)^2\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\right)\left(\sum_{P\in\mathcal{D}:P\supset Q}\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\right)\mu(Q),
\end{aligned}$$

where we used Cauchy-Schwarz for the last inequality. By [To1, Lemmas 5.2 and 5.4],

$$\sum_{Q\in\mathcal{D}:Q\subset R_0}\int_Q\frac{\operatorname{dist}(x,L_Q)^2}{\ell(Q)^2}\,d\mu(x)\lesssim\mu(R_0).$$

Finally,

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}:Q\subset R_0}\left(\sum_{P\in\mathcal{D}:P\supset Q}\alpha(P)^2\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\right)\left(\sum_{P\in\mathcal{D}:P\supset Q}\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\right)\mu(Q)\\
&\qquad\lesssim\sum_{Q\in\mathcal{D}:Q\subset R_0}\ \sum_{P\in\mathcal{D}:Q\subset P\subset R_0}\alpha(P)^2\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\mu(Q)\\
&\qquad\quad+\sum_{Q\in\mathcal{D}:Q\subset R_0}\ \sum_{P\in\mathcal{D}:P\supset R_0}\alpha(P)^2\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\mu(Q)\\
&\qquad\lesssim\sum_{P\in\mathcal{D}:P\subset R_0}\alpha(P)^2\sum_{Q\in\mathcal{D}:Q\subset P}\left(\frac{\ell(Q)}{\ell(P)}\right)^{\varepsilon}\mu(Q)+\sum_{Q\in\mathcal{D}:Q\subset R_0}\left(\frac{\ell(Q)}{\ell(R_0)}\right)^{\varepsilon}\mu(Q)\\
&\qquad\lesssim\sum_{P\in\mathcal{D}:P\subset R_0}\alpha(P)^2\mu(P)+\mu(R_0)\lesssim\mu(R_0),
\end{aligned}$$

where the last inequality follows from Theorem 2.1. □

Theorem 1.2 now follows from Theorem 3.9, Corollary 3.11 and Theorem 4.1.

5. Uniform rectifiabilty implies boundedness of square functions: the non-smooth case

By Corollary 3.10 we already know that condition (1.3) implies the uniform n-rectifiability of μ, assuming μ to be n-AD-regular. So to complete the proof of Theorem 1.1 it remains to show that (1.3) holds for any ball $B(x_0,R)$ centered at $\operatorname{supp}(\mu)$ if μ is uniformly n-rectifiable. To this end, we would like to argue as in the preceding section, setting

$$\phi_r = \frac{1}{r^n}\,\chi_{B(0,r)}(x), \qquad x\in\mathbb{R}^d,$$

and

$$h_r = \phi_r - \phi_{2r}.$$

The main obstacle is the lack of smoothness of h_r. To solve this problem we will decompose h_r using wavelets as follows.

Consider a family of C^1 compactly supported orthonormal wavelets in $\mathbb{R}^n$. Tensor products of Daubechies compactly supported wavelets with 3 vanishing moments will suffice for our purposes, see e.g. [Mal, Section 7.2.3]. We denote this family of functions by $\{\psi_I^\epsilon\}_{I\in\mathcal{D}(\mathcal{R}^n),1\leq\epsilon\leq 2^n-1}$, where $\mathcal{D}(\mathbb{R}^n)$ is the standard grid of dyadic cubes in $\mathbb{R}^n$. Each ψ_I^ϵ is a C^1 function supported on $5I$, which satisfies $\|\psi_I^\epsilon\|_2=1$, and moreover

$$\|\psi_I^\epsilon\|_\infty \lesssim \frac{1}{\ell(I)^{n/2}}, \qquad \|\nabla\psi_I^\epsilon\|_\infty \lesssim \frac{1}{\ell(I)^{1+n/2}} \qquad \text{for all } I\in\mathcal{D}(\mathbb{R}^n) \text{ and } 1\leq\epsilon\leq 2^n-1,$$

where $\ell(I)$ is the sidelength of the cube I. Recall that any function $f\in L^2(\mathbb{R}^n)$ can be written as

$$f=\sum_{I\in\mathcal{D}(\mathbb{R}^n)} \langle f,\psi_I^\epsilon\rangle\,\psi_I^\epsilon.$$

To simplify notation and avoid using the ϵ index, we consider 2^n-1 copies of $\mathcal{D}(\mathbb{R}^n)$ and we denote by $\widetilde{\mathcal{D}}(\mathbb{R}^n)$ their union. Then we can write

$$f=\sum_{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n)} \langle f,\psi_I\rangle\,\psi_I,$$

with the sum converging in $L^2(\mathbb{R}^n)$.

In particular, we have

$$\widetilde{h} := \chi_{B_n(0,1)} - \frac{1}{2^n}\,\chi_{B_n(0,2)} = \sum_{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n)} a_I\,\psi_I, \tag{5.1}$$

where $B_n(0,r)$ stands for the ball centered at 0 with radius r in $\mathbb{R}^n$ and

$$a_I=\Big\langle \chi_{B_n(0,1)}-\frac{1}{2^n}\,\chi_{B_n(0,2)},\,\psi_I\Big\rangle.$$

So we have

$$\tilde{h}_r := \frac{1}{r^n}\,\chi_{B_n(0,r)}(x) - \frac{1}{(2r)^n}\,\chi_{B_n(0,2r)}(x) = \sum_{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n)} a_I\,\frac{1}{r^n}\psi_I\left(\frac{x}{r}\right).$$

Notice that we have been talking about wavelets in $\mathbb{R}^n$ although the ambient space of the measure μ and the function h_r is $\mathbb{R}^d$, with $d\geq n$. We remark that we chose to work with wavelets in $\mathbb{R}^n$ mainly because the functions $\tilde{h}_r$ have zero mean in $\mathbb{R}^n$, while this is not the case for the functions h_r in $\mathbb{R}^d$.

We identify $\mathbb{R}^n$ with the "horizontal" subspace of $\mathbb{R}^d$ given by $\mathbb{R}^n \times \{0\} \times \ldots \times \{0\}$ and we consider the following circular projection $\Pi : \mathbb{R}^d \to \mathbb{R}^n$. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we denote $x^H := (x_1, \ldots, x_n)$ and $x^V = (x_{n+1}, \ldots, x_d)$. If $x^H \neq 0$ we set

$$\Pi(x) = \frac{|x|}{|x^H|}\, x^H.$$

If $x^H = 0$, we set $\Pi(x) = (|x|, 0, \ldots, 0)$, say. Observe that in any case $|x| = |\Pi(x)|$.

Notice also that

$$h_r(x) = \frac{1}{r^n}\, \chi_{B_n(0,r)}(\Pi(x)) - \frac{1}{(2r)^n}\, \chi_{B_n(0,2r)}(\Pi(x)) = \sum_{I \in \widetilde{\mathcal{D}}(\mathbb{R}^n)} a_I\, \frac{1}{r^n} \psi_I\left(\frac{\Pi(x)}{r}\right).$$

Thus,

$$h_r * \mu(x) = \sum_{I \in \widetilde{\mathcal{D}}(\mathbb{R}^n)} a_I\, \frac{1}{r^n} \psi_I\left(\frac{\Pi(\cdot)}{r}\right) * \mu(x). \tag{5.2}$$

Observe that the functions ψ_I are smooth, and so one can guess that the α coefficients of [To1] will be useful to estimate $\psi_I\left(\frac{\Pi(\cdot)}{r}\right) * \mu(x)$. Concerning the coefficients a_I we have:

Lemma 5.1. *For $I \in \widetilde{\mathcal{D}}(\mathbb{R}^n)$, we have:*

(a) *If $5I \cap \big(\partial B_n(0,1) \cup \partial B_n(0,2)\big) = \varnothing$, then $a_I = 0$.*

(b) *If $\ell(I) \gtrsim 1$, then $|a_I| \lesssim \ell(I)^{-1-n/2}$.*

(c) *If $\ell(I) \lesssim 1$, then $|a_I| \lesssim \ell(I)^{n/2}$.*

Proof. The first statement follows from the fact that the wavelets ψ_I have zero mean in $\mathbb{R}^n$ and that $\widetilde{h} = \chi_{B_n(0,1)} - \frac{1}{2^n}\, \chi_{B_n(0,2)}$ is constant on $\operatorname{supp} \psi_I$ if $5I \cap \big(\partial B_n(0,1) \cup \partial B_n(0,2)\big) = \varnothing$.

The statement (c) is immediate:

$$|a_I| = \left| \int \widetilde{h}\, \psi_I\, dx \right| \leq \|\psi_I\|_1 \lesssim \ell(I)^{n/2}\, \|\psi_I\|_2 = \ell(I)^{n/2}.$$

Finally (b) follows from the smoothness of ψ_I and the fact that $\widetilde{h}$ has zero mean. Indeed,

$$|a_I| = \left| \int_{B_n(0,2)} \widetilde{h}(x)\, (\psi_I(x) - \psi_I(0))\, dx \right| \leq 2 \|\nabla \psi_I\|_\infty \int |\widetilde{h}|\, dx \lesssim \frac{1}{\ell(I)^{1+n/2}}.$$

$\square$

By estimating $\psi_I\left(\frac{\Pi(\cdot)}{r}\right) * \mu(x)$ in terms of the $\alpha(Q)$'s, using some arguments in the spirit of the ones in [MT], below we will prove the following.

Theorem 5.2. *Let μ be an n-AD-regular measure in $\mathbb{R}^d$. If μ is uniformly n-rectifiable, then there exists a constant c such that*

$$\int_0^R \int_{x \in B(x_0,R)} |h_r * \mu(x)|^2\, d\mu(x)\, \frac{dr}{r} \leq c\, R^n, \tag{5.3}$$

for all $x_0 \in \operatorname{supp}(\mu)$, $R > 0$.

5.1. **Preliminaries for the proof of Theorem 5.2.** It is immediate to check that the estimate (5.3) holds if and only if for all $R \in \mathcal{D}$

$$\sum_{Q\in\mathcal{D}:Q\subset R} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |h_r * \mu(x)|^2 \frac{dr}{\ell(Q)}\, d\mu(x) \leq c\, \mu(R). \tag{5.4}$$

Let $\delta > 0$ be some small constant to be fixed below. To estimate the preceding integral we can assume that $\alpha(1000Q) \leq \delta^2$. Otherwise we have

$$|h_r * \mu(x)| \lesssim 1 \leq \frac{\alpha(1000Q)}{\delta^2}$$

and, by Theorem 2.1,

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{D}(R)\\ \alpha(1000Q)\geq\delta^2}} &\frac{1}{\ell(Q)} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |h_r * \mu(x)|^2 dr d\mu(x)\\ &\lesssim \frac{1}{\delta^4} \sum_{Q\in\mathcal{D}(R)} \alpha(1000Q)^2 \mu(Q) \lesssim \frac{1}{\delta^4}\mu(R).\end{aligned} \tag{5.5}$$

Since the functions h_r are even, we have

$$h_r * \mu(x) = \int h_r(y-x) d\mu(y).$$

Recalling (5.2), we get

$$h_r * \mu(x) = \frac{1}{r^n} \sum_{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n)} a_I \int \psi_I\left(\frac{\Pi(y-x)}{r}\right) d\mu(y).$$

By Lemma 5.1, $a_I = 0$ whenever $5I \cap \big(\partial B_n(0,1) \cup \partial B_n(0,2)\big) = \varnothing$. Therefore it will be enough to sum over those I such that $5I \cap \big(\partial B_n(0,1) \cup \partial B_n(0,2)\big) \neq \varnothing$ and the domain of integration of each $\psi_I\left(\frac{\Pi(\cdot)}{r}\right)$ is $\Pi^{-1}(r \cdot 5I)$.

Notice that $5I$ stands for the cube from $\mathbb{R}^n$ concentric with I with side length equal to $5\ell(I)$. On the other hand, given a set $A \subset \mathbb{R}^n$, we write

$$r \cdot A = \{r \cdot x \in \mathbb{R}^n : x \in A\}.$$

So $r \cdot 5I = r \cdot (5I)$ is a cube in $\mathbb{R}^n$ with side length $5r\ell(I)$ which is not concentric with I unless I is centered at the origin.

We set

$$\begin{aligned}h_r * \mu(x) = \frac{1}{r^n} &\sum_{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n):\ell(I)\geq 1/100} a_I \int \psi_I\left(\frac{\Pi(y-x)}{r}\right) d\mu(y)\\ &+ \frac{1}{r^n} \sum_{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n):\ell(I)< 1/100} a_I \int \psi_I\left(\frac{\Pi(y-x)}{r}\right) d\mu(y)\\ &=: F_r(x) + G_r(x),\end{aligned} \tag{5.6}$$

so that

$$
\begin{aligned}
&\sum_{Q\in\mathcal{D}:Q\subset R}\int_Q\int_{\ell(Q)}^{2\ell(Q)}|h_r*\mu(x)|^2\,\frac{dr}{\ell(Q)}\,d\mu(x)\\
&\qquad\lesssim \sum_{\substack{Q\in\mathcal{D}:Q\subset R,\\ \alpha(1000Q)\geq\delta^2}}\int_Q\int_{\ell(Q)}^{2\ell(Q)}|h_r*\mu(x)|^2\,\frac{dr}{\ell(Q)}\,d\mu(x)\\
&\qquad\quad+\sum_{\substack{Q\in\mathcal{D}:Q\subset R,\\ \alpha(1000Q)\leq\delta^2}}\int_Q\int_{\ell(Q)}^{2\ell(Q)}|F_r(x)|^2\,\frac{dr}{\ell(Q)}\,d\mu(x)\\
&\qquad\quad+\sum_{\substack{Q\in\mathcal{D}:Q\subset R,\\ \alpha(1000Q)\leq\delta^2}}\int_Q\int_{\ell(Q)}^{2\ell(Q)}|G_r(x)|^2\,\frac{dr}{\ell(Q)}\,d\mu(x)\\
&\qquad=: I_0+I_1+I_2.
\end{aligned}
\tag{5.7}
$$

As shown in (5.5), we have

$$I_0\lesssim\frac{1}{\delta^4}\,\mu(R).$$

Thus to prove Theorem 5.2 it is enough to show that $I_1+I_2\leq c(\delta)\,\mu(R)$.

5.2. **Estimate of the term I_1 in** (5.7)**.** We first need to estimate $F_r(x)$. To this end, we take $Q\in\mathcal{D}$ and $r>0$ such that $x\in Q$ and $\ell(Q)\leq r<2\ell(Q)$. We also assume that L_Q (the best approximating plane for $\alpha(Q)$) is parallel to $\mathbb{R}^n$.

Let $I\in\widetilde{\mathcal{D}}(\mathbb{R}^n)$ be such that $\ell(I)\geq 1/100$ and $5I\cap\big(\partial B_n(0,1)\cup\partial B_n(0,2)\big)\neq\varnothing$. Let $P:=P(I)\in\mathcal{D}$ be some cube containing Q such that $\ell(P)\approx r\ell(I)\approx\ell(Q)\ell(I)$. Let also ϕ_P be a smooth bump function such that $\chi_{3P}\leq\phi_P\leq\chi_{B_P}$, $\|\nabla\phi_P\|_\infty\leq 1$, and $\phi_P=1$ on $x+\Pi^{-1}(r\cdot 5I)$. Then

$$\int\psi_I\left(\frac{\Pi(y-x)}{r}\right)d\mu(y)=\int_{3P}\psi_I\left(\frac{\Pi(y-x)}{r}\right)d\mu(y)=\int\phi_P(y)\,\psi_I\left(\frac{\Pi(y-x)}{r}\right)d\mu(y).$$

Lemma 5.3. *Let $I\in\widetilde{\mathcal{D}}(\mathbb{R}^n)$ be such that $\ell(I)\geq 1/100$ and $5I\cap\big(\partial B_n(0,1)\cup\partial B_n(0,2)\big)\neq\varnothing$ and let $P=P(I)$ as above. We have*

$$\left|\int\psi_I\left(\frac{\Pi(y-x)}{r}\right)d\mu(y)\right|\lesssim\left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2}\left(\frac{\operatorname{dist}(x,L_Q)}{\ell(P)}+\sum_{S\in\mathcal{D}:Q\subset S\subset P}\alpha(2S)\right)\ell(P)^n.\tag{5.8}$$

Proof. Without loss of generality we assume that $x=0$. Let L_0 be the plane parallel to L_Q passing through 0 (that is, $L_0=\mathbb{R}^n$) and denote by $\Pi^\perp$ the orthogonal projection onto

L_0. Then

$$\begin{aligned}\int \psi_I\left(\frac{\Pi(y)}{r}\right) d\mu(y) &= \int \phi_P(y)\psi_I\left(\frac{\Pi(y)}{r}\right) d\mu(y)\\ &= \int \phi_P(y)\ \left(\psi_I\left(\frac{\Pi(y)}{r}\right) - \psi_I\left(\frac{\Pi^\perp(y)}{r}\right)\right) d\mu(y)\\ &\quad + \int \phi_P(y)\ \psi_I\left(\frac{\Pi^\perp(y)}{r}\right) d(\mu - c_P\mathcal{H}^n_{|L_0})(y)\\ &\quad + c_P\int_{L_0} \phi_P(y)\ \psi_I\left(\frac{\Pi^\perp(y)}{r}\right) d\mathcal{H}^n(y)\\ &=: A_1 + A_2 + A_3.\end{aligned} \tag{5.9}$$

Since $\psi_I\left(\frac{\Pi^\perp(y)}{r}\right) = \psi_I\left(\frac{y}{r}\right)$ for $y \in L_0$, and $\phi_P = 1$ on $r\cdot 5I$, we get

$$A_3 = c_P\int_{5I} \psi_I\left(\frac{y}{r}\right) dy = 0. \tag{5.10}$$

We now proceed to estimate A_2:

$$\begin{aligned}|A_2| &\leq \left|\int \phi_P(y)\ \psi_I\left(\frac{\Pi^\perp(y)}{r}\right) d(\mu - c_P\mathcal{H}^n_{|L_P})(y)\right|\\ &\quad + \left|c_P\int \phi_P(y)\ \psi_I\left(\frac{\Pi^\perp(y)}{r}\right) d(\mathcal{H}^n_{|L_P} - \mathcal{H}^n_{|L_0})(y)\right|\\ &\lesssim \left\|\nabla\left(\phi_P\ \psi_I\left(\frac{\Pi^\perp(\cdot)}{r}\right)\right)\right\|_\infty \big(\alpha(P)\,\ell(P)^{n+1} + \operatorname{dist}_H(L_P\cap B_P, L_0\cap B_P)\ell(P)^n\big),\end{aligned} \tag{5.11}$$

from the definition of the α numbers and the fact that $c_P \approx 1$. Using the gradient bounds for the functions ϕ_P and ψ_I, and the fact that $\ell(P) \approx r\ell(I) \approx \ell(Q)\ell(I)$, we get

$$\begin{aligned}\left\|\nabla\left(\phi_P\ \psi_I\left(\frac{\Pi^\perp(\cdot)}{r}\right)\right)\right\|_\infty &\lesssim \|\nabla\phi_P\|_\infty\|\psi_I\|_\infty + \left\|\nabla\left(\psi_I\left(\frac{\Pi^\perp(\cdot)}{r}\right)\right)\right\|_\infty\\ &\lesssim \frac{1}{\ell(P)}\frac{1}{\ell(I)^{n/2}} + \frac{1}{\ell(I)^{n/2+1}}\frac{1}{r} \approx \frac{1}{\ell(Q)}\frac{1}{\ell(I)^{n/2+1}} \approx \frac{1}{\ell(Q)}\left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2+1}.\end{aligned} \tag{5.12}$$

We also remark that in the previous estimate we used the fact that $\|\Pi^\perp\|_\infty \leq 1$, which does not hold for the spherical projection Π.

Furthermore, by [To1, Lemma 5.2 and Remark 5.3],

$$\begin{aligned}\operatorname{dist}_H(L_P\cap B_P, L_0\cap B_P) &\leq \operatorname{dist}_H(L_P\cap B_P, L_Q\cap B_P) + \operatorname{dist}(0, L_Q)\\ &\leq \sum_{S\in\mathcal{D}:Q\subset S\subset P} \alpha(S)\ell(P) + \operatorname{dist}(0, L_Q).\end{aligned} \tag{5.13}$$

Therefore, by (5.11), (5.12), and (5.13),

$$|A_2| \lesssim \left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2} \ell(P)^n \left(\sum_{S\in\mathcal{D}:Q\subset S\subset P} \alpha(S) + \frac{\operatorname{dist}(0, L_Q)}{\ell(P)}\right). \tag{5.14}$$

We now estimate the term A_1:

$$\begin{aligned} |A_1| &= \left| \int \phi_P(y) \, \left(\psi_I \left(\frac{\Pi(y)}{r} \right) - \psi_I \left(\frac{\Pi^\perp(y)}{r} \right) \right) d\mu(y) \right| \\ &\lesssim \frac{\|\nabla \psi_I\|_\infty}{r} \int_{B_P} |\Pi(y) - \Pi^\perp(y)| \, d\mu(y) \\ &\lesssim \frac{1}{\ell(Q)} \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2+1} \int_{B_P} |\Pi(y) - \Pi^\perp(y)| \, d\mu(y). \end{aligned} \tag{5.15}$$

It is easy to check that

$$|\Pi(y) - \Pi^\perp(y)| \lesssim \operatorname{dist}(y, L_0). \tag{5.16}$$

Furthermore, as in (5.13), for $y \in B_P$,

$$\begin{aligned} \operatorname{dist}(y, L_0) &\le \operatorname{dist}(0, L_Q) + \operatorname{dist}(y, L_Q) \\ &\le \operatorname{dist}(0, L_Q) + \operatorname{dist}(y, L_P) + \operatorname{dist}_H(L_P \cap 3B_P, L_Q \cap 3B_P) \\ &\lesssim \operatorname{dist}(0, L_Q) + \operatorname{dist}(y, L_P) + \sum_{S \in \mathcal{D}: Q \subset S \subset P} \alpha(S) \ell(P). \end{aligned} \tag{5.17}$$

Therefore, by (5.15), (5.16), and (5.17),

$$\begin{aligned} |A_1| \lesssim & \frac{1}{\ell(Q)} \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2+1} \Big(\operatorname{dist}(0, L_Q) \, \ell(P)^n \\ & + \int_{B_P} \operatorname{dist}(y, L_P) \, d\mu(y) + \ell(P)^{n+1} \sum_{S \in \mathcal{D}: Q \subset S \subset P} \alpha(S) \Big) \\ \lesssim & \left(\frac{\ell(Q)}{\ell(P)} \right)^{n/2} \left(\frac{\operatorname{dist}(0, L_Q)}{\ell(P)} + \sum_{S \in \mathcal{D}: Q \subset S \subset P} \alpha(2S) \right) \ell(P)^n, \end{aligned} \tag{5.18}$$

where we used that, by [To1, Remark 3.3],

$$\int_{B_P} \operatorname{dist}(y, L_P) d\mu(y) \lesssim \alpha(2P) \ell(P)^{n+1}.$$

The lemma follows from the estimates (5.9), (5.10), (5.14), and (5.18). $\square$

Lemma 5.4. *We have*

$$|F_r(x)| \lesssim \frac{\operatorname{dist}(x, L_Q)}{\ell(Q)} + \sum_{S \in \mathcal{D}: S \supset Q} \alpha(2S) \frac{\ell(Q)}{\ell(S)}. \tag{5.19}$$

Proof. Recalling that $P = P(I) \supset Q$, by (5.8),

$$|F_r(x)| \lesssim \frac{1}{\ell(Q)^n} \sum_{\substack{I \in \widetilde{\mathcal{D}}(\mathbb{R}^n): \\ \ell(I) \ge 1/100}} |a_I| \left(\frac{\ell(Q)}{\ell(P(I))} \right)^{n/2} \left(\frac{\operatorname{dist}(x, L_Q)}{\ell(P(I))} + \sum_{\substack{S \in \mathcal{D}: \\ Q \subset S \subset P(I)}} \alpha(2S) \right) \ell(P(I))^n.$$

Using (b) from Lemma 5.1,

$$
\begin{aligned}
|F_r(x)| &\lesssim \sum_{\substack{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n):\ell(I)\geq 1/100,\\ P(I)\supset Q}} \frac{\ell(Q)}{\ell(P(I))}\left(\frac{\operatorname{dist}(x,L_Q)}{\ell(P(I))}+\sum_{S\in\mathcal{D}:Q\subset S\subset P(I)}\alpha(2S)\right)\\
&\lesssim \sum_{P\in\mathcal{D}:\,P\supset Q}\frac{\ell(Q)}{\ell(P)}\left(\frac{\operatorname{dist}(x,L_Q)}{\ell(P)}+\sum_{S\in\mathcal{D}:Q\subset S\subset P}\alpha(2S)\right)\\
&= \sum_{P\in\mathcal{D}:\,P\supset Q}\frac{\operatorname{dist}(x,L_Q)\,\ell(Q)}{\ell(P)^2}+\sum_{S\in\mathcal{D}:S\supset Q}\alpha(2S)\sum_{P\in\mathcal{D}:\,P\supset S}\frac{\ell(Q)}{\ell(P)}\\
&\lesssim \frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}+\sum_{S\in\mathcal{D}:S\supset Q}\alpha(2S)\frac{\ell(Q)}{\ell(S)}.
\end{aligned}
$$

□

Lemma 5.5. *The term I_1 in* (5.7) *satisfies*

$$I_1 \lesssim \mu(R).$$

Proof. By (5.19),

$$I_1 \lesssim \sum_{Q\in\mathcal{D}(R)}\int_Q\left(\frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}+\sum_{S\in\mathcal{D}:S\supset Q}\alpha(2S)\frac{\ell(Q)}{\ell(S)}\right)^2 d\mu(x). \tag{5.20}$$

By Cauchy-Schwartz,

$$\left(\sum_{S\in\mathcal{D}:S\supset Q}\alpha(2S)\frac{\ell(Q)}{\ell(S)}\right)^2 \leq \sum_{S\in\mathcal{D}:S\supset Q}\alpha(2S)^2\frac{\ell(Q)}{\ell(S)}\cdot\sum_{S\in\mathcal{D}:S\supset Q}\frac{\ell(Q)}{\ell(S)}. \tag{5.21}$$

Since $\sum_{S\in\mathcal{D}:S\supset Q}\frac{\ell(Q)}{\ell(S)}\lesssim 1$,

$$
\begin{aligned}
I_1 &\lesssim \sum_{Q\in\mathcal{D}(R)}\int_Q\left(\frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}\right)^2 d\mu(x)+\sum_{Q\in\mathcal{D}(R)}\sum_{S\in\mathcal{D}:S\supset Q}\alpha(2S)^2\frac{\ell(Q)}{\ell(S)}\mu(Q)\\
&=: S_1+S_2.
\end{aligned}
\tag{5.22}
$$

By [To1, Lemmas 5.2 and Lemma 5.4] and Theorem 2.1, we obtain $S_1\lesssim\mu(R)$. We now deal with the term S_2:

$$
\begin{aligned}
S_2 &= \sum_{Q\in\mathcal{D}(R)}\sum_{S\in\mathcal{D}:Q\subset S\subset R}\alpha(2S)^2\frac{\ell(Q)}{\ell(S)}\mu(Q)+\sum_{Q\in\mathcal{D}(R)}\sum_{S\in\mathcal{D}:S\supsetneq R}\alpha(2S)^2\frac{\ell(Q)}{\ell(S)}\mu(Q)\\
&=: S_{21}+S_{22}.
\end{aligned}
\tag{5.23}
$$

Using just that $\alpha(2S)\lesssim 1$,

$$S_{22}\lesssim\sum_{Q\in\mathcal{D}(R)}\sum_{S\in\mathcal{D}:S\supset R}\frac{\ell(Q)}{\ell(S)}\mu(Q)\lesssim\sum_{Q\in\mathcal{D}(R)}\mu(Q)\frac{\ell(Q)}{\ell(R)}\lesssim\mu(R). \tag{5.24}$$

Finally, using Fubini and Theorem 2.1,

$$S_{21} \leq \sum_{S \in \mathcal{D}(R)} \alpha(2S)^2 \sum_{Q \in \mathcal{D}: Q \subset S} \frac{\ell(Q)}{\ell(S)} \mu(Q) \lesssim \sum_{S \in \mathcal{D}(R)} \alpha(2S)^2 \mu(S) \lesssim \mu(R). \tag{5.25}$$

By (5.22), (5.23), (5.24), and (5.25) we obtain $I_1 \lesssim \mu(R)$. $\square$

5.3. **Estimate of the term I_2 in** (5.7)**.** It remains to show that $I_2 \lesssim \mu(R)$. Recall that the cubes in the sum corresponding to I_2 in (5.7) satisfy $\alpha(1000\,Q) \leq \delta^2$.

We need now to estimate $G_r(x)$ (see (5.6)) for $x \in Q$ and $\ell(Q) \leq r < 2\ell(Q)$. Recall that

$$G_r(x) = \frac{1}{r^n} \sum_{I \in \widetilde{\mathcal{D}}(\mathbb{R}^n): \ell(I) < 1/100} a_I \int \psi_I \left(\frac{\Pi(y-x)}{r} \right) d\mu(y). \tag{5.26}$$

The arguments will be more involved than the ones we used for $F_r(x)$.

To estimate $G_r(x)$ we now introduce a *stopping time condition* for $P \in \mathcal{D}$: P belongs to $\mathcal{G}_0$ if

(1) $P \subset 1000\,Q$, and
(2) $\sum_{S \in \mathcal{D}: P \subset S \subset 1000Q} \alpha(100S) \leq \delta$.

The maximal cubes in $\mathcal{D} \setminus \mathcal{G}_0$ may vary significantly in size, even if they are neighbors, and this would cause problems. For this reason we use a quite standard *smoothing* procedure. We define

$$\ell(y) := \inf_{P \in \mathcal{G}_0} (\ell(P) + \operatorname{dist}(y, P)), \quad y \in \mathbb{R}^d, \tag{5.27}$$

and

$$d(z) := \inf_{y \in \Pi^{-1}(z)} \ell(y), \quad z \in \mathbb{R}^n. \tag{5.28}$$

Lemma 5.6. *The function $\ell(\cdot)$ is 1-Lipschitz, and the function $d(\cdot)$ is 3-Lipschitz.*

Proof. For simplicity we assume that $x = 0$. The function $\ell(\cdot)$ is 1-Lipschitz, as the infimum of the family of 1-Lipschitz functions $\{\ell(P) + \operatorname{dist}(\cdot, P)\}_{P \in \mathcal{G}_0}$.

Let us turn our attention to $d(\cdot)$. Let $z, z' \in \mathbb{R}^n$ and $\varepsilon > 0$. Let $y \in \Pi^{-1}(z)$ such that $\ell(y) \leq d(z) + \varepsilon$. Consider the points

$$y_0 = \frac{|z'|}{|y|}\, y \quad \text{and} \quad z_0 = \frac{|z'|}{|z|}\, z.$$

Notice that $\Pi(y_0) = z_0$. Let L_{y_0} be the n-plane parallel to $\mathbb{R}^n$ which contains y_0 and consider the point $\{y'\} = \Pi^{-1}(z') \cap L_{y_0}$. That is, y' is the point which fulfils the following properties:

$$|y'| = |z'|, \quad y'^H = \frac{|y'^H|}{|z'|}\, z', \quad y'^V = y_0^V.$$

Observe that

$$|y_0| = |z_0| = |z'| = |y'|.$$

Since $y_0^V = y'^V$, this implies that $|y_0^H| = |y'^H|$. Furthermore,

$$|y_0 - y'| = |y_0^H - y'^H| = \left| \frac{|y_0^H|}{|y_0|} z_0 - \frac{|y'^H|}{|y'|} z' \right| = \frac{|y'^H|}{|y'|} |z_0 - z'| \le |z_0 - z'|.$$

Moreover,

$$|y - y_0| = |z - z_0| = \left| z - \frac{|z'|}{|z|}\, z \right| = \big| |z| - |z'| \big| \le |z - z'|.$$

Hence,

$$\begin{aligned} |y - y'| &\le |y - y_0| + |y_0 - y'| \le |z - z'| + |z_0 - z'| \\ &\le |z - z'| + |z_0 - z| + |z - z'| \le 3|z - z'|. \end{aligned}$$

Then, using that ℓ is 1-Lipschitz and (5.3),

$$d(z') \le \ell(y') \le |y - y'| + \ell(y) \le 3|z - z'| + d(z) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we deduce that $d(z') \le 3|z - z'| + d(z)$. In the same way one gets that $d(z) \le 3|z - z'| + d(z')$. □

For δ small enough, the condition $\alpha(1000\, Q) \le \delta^2$ guarantees that any cube $P \subset 1000\, Q$ such that $\ell(P) = \ell(Q)$ belongs to $\mathcal{G}_0$, in particular $\ell(y) \le \ell(Q)$ for all $y \in 1000\, Q$. Furthermore since $\mathcal{G}_0 \neq \varnothing$, we deduce that $\ell(y)$, $d(z) < \infty$ for all $y \in \mathbb{R}^d$, $z \in \mathbb{R}^n$.

Now we consider the family $\mathcal{F}$ of cubes $I \in \mathcal{D}(\mathbb{R}^n)$ such that

$$r \operatorname{diam}(I) \le \frac{1}{5000} \inf_{z \in r \cdot I} d(z). \tag{5.29}$$

Let $\mathcal{F}_0 \subset \mathcal{F}$ be the subfamily of $\mathcal{F}$ consisting of cubes with maximal length. In particular the cubes in $\mathcal{F}_0$ are pairwise disjoint. Moreover it is easy to check that if $I, J \in \mathcal{F}_0$ and

$$20\, I \cap 20\, J \neq \varnothing, \tag{5.30}$$

then $\ell(I) \approx \ell(J)$.

We denote by $\mathcal{G}(x, r)$ the family of cubes $I \in \widetilde{\mathcal{D}}(\mathbb{R}^n)$ which satisfy

- $\ell(I) \le \frac{1}{100}$,
- $5I \cap (\partial B_n(0,1) \cup \partial B_n(0,2)) \neq \varnothing$,
- $\big(x + \Pi^{-1}(r \cdot 5I)\big) \cap \operatorname{supp}(\mu) \neq \varnothing$, and
- I is not contained in any cube from $\mathcal{F}_0$.

We denote by $\mathcal{T}(x, r)$ the family of cubes $I \in \widetilde{\mathcal{D}}(\mathbb{R}^n)$ which satisfy

- $\ell(I) \le \frac{1}{100}$,
- $5I \cap (\partial B_n(0,1) \cup \partial B_n(0,2)) \neq \varnothing$,
- $\big(x + \Pi^{-1}(r \cdot 5I)\big) \cap \operatorname{supp}(\mu) \neq \varnothing$, and
- $I \in \mathcal{F}_0$.

Now we write

$$
\begin{aligned}
G_r(x) &= \frac{1}{r^n}\sum_{I\in\mathcal{G}(x,r)} a_I \int \psi_I\left(\frac{\Pi(y-x)}{r}\right) d\mu(y) \\
&\quad + \frac{1}{r^n}\sum_{I\in\mathcal{T}(x,r)}\;\sum_{J\in\widetilde{\mathcal{D}}(\mathbb{R}^n):\,J\subset I} a_J \int \psi_J\left(\frac{\Pi(y-x)}{r}\right) d\mu(y) \\
&=: G_{r,1}(x) + G_{r,2}(x),
\end{aligned}
\tag{5.31}
$$

so that

$$
\begin{aligned}
I_2 &\leq \sum_{\substack{Q\in\mathcal{D}:Q\subset R,\\ \alpha(1000Q)\leq\delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |G_{r,1}(x)|^2\,\frac{dr}{\ell(Q)}\,d\mu(x) \\
&\quad + \sum_{\substack{Q\in\mathcal{D}:Q\subset R,\\ \alpha(1000Q)\leq\delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |G_{r,2}(x)|^2\,\frac{dr}{\ell(Q)}\,d\mu(x) \\
&=: I_{21} + I_{22}.
\end{aligned}
\tag{5.32}
$$

First we will deal with the term $G_{r,1}(x)$. To this end we need several auxiliary lemmas.

Lemma 5.7. *If $I \in \mathcal{G}(x,r)$, then there exists $P := P(I) \in \mathcal{D}$ with $\ell(P) \approx r\ell(I)$ such that*

$$\operatorname{supp}(\mu) \cap \big(x + \Pi^{-1}(r\cdot 5I)\big) \subset 3P.$$

Proof. Notice that, by definition, $\operatorname{supp}(\mu) \cap (x + \Pi^{-1}(r\cdot 5I)) \neq \varnothing$. Observe also that the conclusion of the lemma holds if $\ell(r\cdot 5I) \approx \ell(Q)$ because $\alpha(1000Q) \leq \delta^2$.

So assume that $\ell(r\cdot 5I) \ll \ell(Q)$ and consider $z \in r\cdot 5I$. Since $I \in \mathcal{G}(x,r)$, $I \notin \mathcal{F}_0$, and d is 3-Lipschitz, we have

$$d(z) \leq c_2\, r\ell(I),$$

for some absolute constant c_2. Take $y \in x + \Pi^{-1}(r\cdot 5I)$ such that

$$\ell(y) \leq 2c_2\, r\ell(I).$$

Let $\varepsilon = c_2\, r\ell(I)$. By definition, there exists some cube $P' \in \mathcal{G}_0$ such that

$$\ell(P') + \operatorname{dist}(y, P') \leq \ell(y) + \varepsilon \leq 3c_2\, r\,\ell(I).$$

Let $A > 10$ be some big constant to be fixed below. Suppose that there are two cubes $P_0, P_1 \in \mathcal{D}$ which satisfy the following properties

(i) $r\ell(I) \leq \ell(P_0) = \ell(P_1) \leq 10\, r\ell(I)$,
(ii) $\operatorname{dist}(P_0, P_1) \geq A\ell(P_0)$,
(iii) $P_i \cap \big(x + \Pi^{-1}(r\cdot 5I)\big) \neq \varnothing$ for $i = 1, 2$.

Suppose that $\operatorname{dist}(P_0, P') \geq \operatorname{dist}(P_1, P')$. Then from (ii) we infer that

$$\operatorname{dist}(P_0, P') \gtrsim A\,\ell(P_0).$$

Let $P'' \in \mathcal{D}$ such that $P_0 \cup P' \subset 3P''$ with minimal side length, so that $\ell(P'') \approx \ell(P_0) + \ell(P') + \operatorname{dist}(P_0, P')$. Since $\alpha(1000Q) \leq \delta^2$ and $\ell(P_0), \ell(P_1), \ell(P') \ll \ell(Q)$, it follows easily that we must also have $\ell(P'') \ll \ell(Q)$. It is not difficult to check that either

$$\beta_1(P'') \gg \delta, \quad \text{or} \quad \measuredangle(L_{P''}, L_Q) \gg \delta.$$

In either case one has

$$\sum_{S\in\mathcal{D}:\,P''\subset S\subset Q}\alpha(S)\gg\delta.$$

We deduce that

$$\sum_{S\in\mathcal{D}:\,P'\subset S\subset Q}\alpha(100S)\gg\delta,$$

because $P'\subset 3P''$. This contradicts the fact that $P'\in\mathcal{G}_0$.

We have shown that a pair of cubes P_0, P_1 such as the ones above does not exist. Thus, if $P_0\in\mathcal{D}$ satisfies

$$r\ell(I)\le\ell(P_0)\le 10\,r\ell(I),$$

and

$$P_0\cap\big(x+\Pi^{-1}(r\cdot 5I)\big)\neq\varnothing,$$

then any other cube P_1 for which these properties also hold must be contained in the ball $B(x_{P_0}, c_3\,A\,\ell(P_0))$, where x_{P_0} stands for the center of P_0 and c_3 is some absolute constant. Hence letting $P=P(I)$ be some suitable ancestor of P_0, the lemma follows. □

Lemma 5.8. *Let $I\in\mathcal{G}(x,r)$ and let $P=P(I)\in\mathcal{D}$ be the cube from Lemma 5.7, so that* $\operatorname{supp}(\mu)\cap(x+\Pi^{-1}(r\cdot 5I))\subset 3P$. *We have*

(5.33)
$$\left|\int\psi_I\left(\frac{\Pi(y-x)}{r}\right)d\mu(y)\right|\lesssim\left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2}\left(\sum_{S\in\mathcal{D}:P\subset S\subset Q}\alpha(S)+\frac{\operatorname{dist}(x,L_Q)}{\ell(Q)}\right)\ell(P)^n.$$

Proof. Without loss of generality we assume that $x=0$ and as before we let $L_0=\mathbb{R}^n$ be the n-plane parallel to L_Q containing 0. Let also $y_P\in B_P\cap\operatorname{supp}(\mu)$ be such that $\operatorname{dist}(y_P,L_P)\lesssim\alpha(P)\ell(P)$. The existence of such point follows from [To1, Remark 3.3] and Chebychev's inequality. We also denote by $\widetilde{L}_P$ the n-plane parallel to L_0 which contains y_P. We set $\sigma_P=c_P\mathcal{H}^n_{|L_P}$ and $\widetilde{\sigma}_P=c_P\mathcal{H}^n_{|\widetilde{L}_P}$. Let ϕ_P be a smooth function such that $\chi_{B_P}\le\phi_P\le\chi_{3B_P}$ and $\|\nabla\phi_P\|_\infty\lesssim\ell(P)^{-1}$. Since $\alpha(P)$ is assumed to be very small, we have $\Pi^{-1}(r\cdot 5I)\cap\widetilde{L}_P\subset B_P$. Then we write

$$\begin{aligned}\int\psi_I\left(\frac{\Pi(y)}{r}\right)d\mu(y)&=\int\phi_P(y)\,\psi_I\left(\frac{\Pi(y)}{r}\right)d\mu(y)\\&=\int\phi_P(y)\psi_I\left(\frac{\Pi(y)}{r}\right)(d\mu(y)-d\sigma_P(y))\\&\quad+\int\phi_P(y)\psi_I\left(\frac{\Pi(y)}{r}\right)(d\sigma_P(y)-d\widetilde{\sigma}_P(y))+\int\psi_I\left(\frac{\Pi(y)}{r}\right)d\widetilde{\sigma}_P(y)\\&=:A_1+A_2+A_3.\end{aligned}\tag{5.34}$$

Now we turn our attention to A_1:

$$\begin{aligned}|A_1| &= \left|\int \phi_P(y)\psi_I\left(\frac{\Pi(y)}{r}\right)(d\mu(y) - d\sigma_P(y))\right| \\ &\le \left\|\nabla\left(\phi_P\psi_I\left(\frac{\Pi(\cdot)}{r}\right)\right)\right\|_\infty \alpha(P)\ell(P)^{n+1} \\ &\lesssim \left(\frac{1}{\ell(P)}\frac{1}{\ell(I)^{n/2}} + \frac{1}{\ell(I)^{n/2+1}}\frac{1}{r}\right)\alpha(P)\ell(P)^{n+1} \\ &\approx \left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2}\alpha(P)\ell(P)^n,\end{aligned} \tag{5.35}$$

where we used that $\ell(P) \approx \ell(I)\ell(Q)$ and that $\|\nabla\Pi\|_\infty \lesssim 1$ on B_P since B_P lies far from the subspace $\Pi^{\perp^{-1}}(\{0\})$.

We will now estimate the term A_2. We have

$$|A_2| = \left|\int \phi_P(y)\psi_I\left(\frac{\Pi(y)}{r}\right)(d\sigma_P(y) - d\widetilde{\sigma}_P(y))\right|.$$

As in [To1, Lemma 5.2],

$$\measuredangle(L_P, \widetilde{L}_P) = \measuredangle(L_P, L_Q) \lesssim \sum_{S\in\mathcal{D}:P\subset S\subset Q}\alpha(S).$$

Therefore,

$$\operatorname{dist}_H(\widetilde{L}_P\cap B_P, L_P\cap B_P) \lesssim \sum_{S\in\mathcal{D}:P\subset S\subset Q}\alpha(S)\ell(P),$$

and, as in (5.35),

$$\begin{aligned}|A_2| &\lesssim \left\|\nabla\left(\phi_P\psi_I\left(\frac{\Pi(\cdot)}{r}\right)\right)\right\|_\infty \ell(P)^n \operatorname{dist}_H(\widetilde{L}_P\cap B_P, L_P\cap B_P) \\ &\lesssim \left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2}\sum_{S\in\mathcal{D}:P\subset S\subset Q}\alpha(S)\ell(P)^n.\end{aligned} \tag{5.36}$$

We now consider A_3. Let B be a ball centered in L_0 such that supp $\psi_I\left(\frac{\cdot}{r}\right) \subset B$ and $\operatorname{diam}(B) \lesssim \ell(P)$. For some constant c_*, with $0 \le c_* \lesssim 1$, to be fixed below, we write

(5.37)

$$\begin{aligned}\left|\int \psi_I\left(\frac{\Pi(y)}{r}\right)d\widetilde{\sigma}_P(y)\right| &= \left|c_P\int \psi_I\left(\frac{y}{r}\right)d(\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P})(y)\right| \\ &\le \left|c_P\int \psi_I\left(\frac{y}{r}\right)d(\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P})(y) - c_*\,c_P\int \psi_I\left(\frac{y}{r}\right)d\mathcal{H}^n_{|L_0}(y)\right| \\ &\quad + \left|c_*\,c_P\int \psi_I\left(\frac{y}{r}\right)d\mathcal{H}^n_{|L_0}(y)\right| \\ &\lesssim \frac{\|\nabla\psi_I\|_\infty}{\ell(Q)}\operatorname{dist}_B(\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P}, c_*\mathcal{H}^n_{|L_0}),\end{aligned}$$

where in the last inequality we took into account that $c_*\,c_P \lesssim 1$ and that $\int_{\mathbb{R}^n}\psi_I\left(\frac{y}{r}\right)dy = 0$.

Notice that the map $\Pi_{|\widetilde{L}_P\to L_0}$ need not be affine and so the term $\operatorname{dist}_B(\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P}, c_*\mathcal{H}^n_{|L_0})$ requires some careful analysis. Anyway, we claim that, for some appropriate constant $c_*\lesssim 1$,

$$\operatorname{dist}_B(\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P}, c_*\mathcal{H}^n_{|L_0}) \lesssim \left(\sum_{S\in\mathcal{D}:P\subset S\subset Q}\alpha(S)+\frac{\operatorname{dist}(0,L_Q)}{\ell(Q)}\right)\ell(P)^{n+1}, \tag{5.38}$$

which implies that

$$|A_3|\lesssim\left(\frac{\ell(Q)}{\ell(P)}\right)^{n/2}\left(\sum_{S\in\mathcal{D}:P\subset S\subset Q}\alpha(S)+\frac{\operatorname{dist}(0,L_Q)}{\ell(Q)}\right)\ell(P)^n.$$

Notice that the lemma is an immediate consequence of the estimates we have for A_1, A_2 and A_3.

To conclude, it remains to prove the claim (5.38). This task requires some preliminary calculations and we defer it to Lemma 5.9. □

Our next objective consists in comparing the measures $\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P}$ and $\mathcal{H}^n_{|L_0}$ from the preceding lemma. To this end, we consider the map $\widetilde{\Pi}:=\Pi_{|\widetilde{L}_P\to L_0}$. Abusing notation, identifying both $\widetilde{L}_P$ and L_0 with $\mathbb{R}^n$, we also denote by $\widetilde{\Pi}$ the corresponding mapping in $\mathbb{R}^n$, that is $\widetilde{\Pi}:\mathbb{R}^n\to\mathbb{R}^n$. Then, writing $h=y_P^V$, for $y=(y_1,\dots,y_n,h)$ we have

$$\widetilde{\Pi}_i(y)=y_i\sqrt{\frac{y_1^2+\cdots+y_n^2+|h|^2}{y_1^2+\cdots+y_n^2}}=y_i\sqrt{1+\frac{|h|^2}{y_1^2+\cdots+y_n^2}},$$

for $i=1,\dots,n$. Hence, for $i,j=1,\dots,n$,

$$\partial_j\widetilde{\Pi}_i=\delta_{ij}\frac{|y|}{|y^H|}-\frac{|h|^2\,y_i\,y_j}{|y||y_H|^3}=\frac{|y|}{|y_H|}\left(\delta_{ij}-|h|^2\frac{y_iy_j}{|y|^2|y_H|^2}\right),$$

where δ_{ij} denotes Kronecker's delta. For $y\in P$,

$$|\partial_j\widetilde{\Pi}_i(y)-\partial_j\widetilde{\Pi}_i(y_P)|\leq\left|\frac{|y|}{|y^H|}-\frac{|y_P|}{|y_P^H|}\right|+|h|^2\left|\frac{y_i\,y_j}{|y|\,|y^H|}-\frac{y_{P\,i}\,y_{P\,j}}{|y_P|\,|y_P{}^H|}\right|.$$

Moreover,

$$\begin{aligned}\left|\frac{|y|}{|y^H|}-\frac{|y_P|}{|y_P^H|}\right| &\approx\left|\frac{|y|^2}{|y^H|^2}-\frac{|y_P|^2}{|y_P^H|^2}\right|=\left|\frac{|y|^2|y_P^H|^2-|y_P|^2|y^H|^2}{|y^H|^2\,|y_P^H|^2}\right|\\ &=\left|\frac{(|y^H|^2+|h|^2)|y_P^H|^2-(|y_P^H|^2+|h|^2)|y^H|^2}{|y^H|^2\,|y_P^H|^2}\right|\\ &=\left|\frac{|h|^2(|y_P^H|^2-|y^H|^2)}{|y^H|^2\,|y_P^H|^2}\right|\approx\frac{|h|^2\,|y|\,||y_P^H|-|y^H||}{|y|^4}\\ &\lesssim\frac{|h|^2\,\ell(P)}{r^3},\end{aligned}$$

and in a similar manner we get

$$|h|^2\left|\frac{y_i\,y_j}{|y|\,|y^H|}-\frac{y_{P\,i}\,y_{P\,j}}{|y_P|\,|y_P{}^H|}\right|\lesssim\frac{|h|^2\,\ell(P)}{r^3}.$$

Hence

$$|\partial_j \widetilde{\Pi}_i(y) - \partial_j \widetilde{\Pi}_i(y_P)| \lesssim \frac{|h|^2\,\ell(P)}{r^3}. \tag{5.39}$$

Now we write

(5.40)

$$\begin{aligned}|J\widetilde{\Pi}(y) - J\widetilde{\Pi}(y_P)| &= \left|\sum_\sigma \mathrm{sgn}(\sigma)\prod_{j=1}^n \partial_j\widetilde{\Pi}_{\sigma(j)}(y) - \sum_\sigma \mathrm{sgn}(\sigma)\prod_{j=1}^n \partial_j\widetilde{\Pi}_{\sigma(j)}(y_P)\right| \\ &\le c(n)\sup_{i,j}\big|\partial_j\widetilde{\Pi}_i(y) - \partial_j\widetilde{\Pi}_i(y_P)\big| \,\big(\sup_{i,j}|\partial_j\widetilde{\Pi}_i(y)|^{n-1} + \sup_{i,j}|\partial_j\widetilde{\Pi}_i(y_P)|^{n-1}\big) \\ &\lesssim \sup_{i,j}\big|\partial_j\widetilde{\Pi}_i(y) - \partial_j\widetilde{\Pi}_i(y_P)\big|,\end{aligned}$$

where the sum is computed over all permutations of $\{1,\dots,n\}$ and $\mathrm{sgn}(\sigma)$ denotes the signature of the permutation σ. Moreover, in the last inequality we used again that $\|\nabla\Pi\|_\infty \lesssim 1$ on B_P since B_P lies far from the subspace ${\Pi^\perp}^{-1}(\{0\})$.

Therefore, by (5.40) and (5.39),

$$|J\widetilde{\Pi}(y) - J\widetilde{\Pi}(y_P)| \lesssim \frac{|h|^2\,\ell(P)}{r^3} \qquad \text{for } y\in P. \tag{5.41}$$

Lemma 5.9. *Let B be a ball centered in $\Pi(P)$ with $\mathrm{diam}(B)\lesssim \ell(P)$. Then*

$$\mathrm{dist}_B(\Pi_\sharp \mathcal{H}^n_{|\widetilde{L}_P}, c_*\mathcal{H}^n_{|L_0}) \lesssim \left(\sum_{S\in\mathcal{D}:P\subset S\subset Q}\alpha(S) + \frac{\mathrm{dist}(0,L_Q)}{\ell(Q)}\right)\ell(P)^{n+1},$$

where $c_* = (J\widetilde{\Pi}(y_P))^{-1}$.

Proof. Let f be 1-Lipschitz with $\mathrm{supp}\, f\subset B$. Then, recalling that $\widetilde{\sigma}_P = c_P\mathcal{H}^n_{|\widetilde{L}_P}$,

$$\begin{aligned}\left|\int f\,d(\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P}) - c_*\int f\,d\mathcal{H}^n_{|L_0}\right| &\approx \left|\frac{1}{c_*}\int f(\Pi(y))d\mathcal{H}^n_{|\widetilde{L}_P} - \int f(y)d\mathcal{H}^n_{|L_0}\right| \\ &= \left|\frac{1}{c_*}\int_{\mathbb{R}^n} f(\widetilde{\Pi}(y))dy - \int_{\mathbb{R}^n} f(y)dy\right| \\ &= \left|\int_{\mathbb{R}^n} f(\widetilde{\Pi}(y))\,J\widetilde{\Pi}(y_P)dy - \int_{\mathbb{R}^n} f(\widetilde{\Pi}(y))\,J\widetilde{\Pi}(y)dy\right|,\end{aligned}$$

where we changed variables in the last line. Now notice that $\mathrm{supp}\, f\circ\widetilde{\Pi}\subset B'$, where B' is a ball concentric with B such that $\mathrm{diam}(B)\lesssim \ell(P)$. In addition, since $\mathrm{supp}\, f\subset B$ and $\|\nabla f\|_\infty \le 1$ we also get $\|f\|_\infty \lesssim \ell(P)$. Hence, by (5.41),

$$\begin{aligned}\left|\int f\,d(\Pi_\sharp\mathcal{H}^n_{|\widetilde{L}_P}) - c_*\int f\,d\mathcal{H}^n_{|L_0}\right| &\lesssim \int_{\mathbb{R}^n}|f(\widetilde{\Pi}(y))||J\widetilde{\Pi}(y_P) - J\widetilde{\Pi}(y)|dy \\ &\lesssim \frac{|h|^2\,\ell(P)}{r^3}\int_{B'}\ell(P)dy \lesssim \frac{|h|\,\ell(P)^{n+1}}{r}.\end{aligned}$$

Moreover, by [To1, Remark 5.3] and the choice of y_P,

$$|h| = \operatorname{dist}(y_P, L_0) \leq \operatorname{dist}(y_P, L_Q) + \operatorname{dist}(L_0, L_Q) \lesssim \sum_{S\in\mathcal{D}:P\subset S\subset Q} \alpha(S)\ell(S) + \operatorname{dist}(0, L_Q).$$

Hence

$$\left|\int f\, d(\Pi_\sharp \mathcal{H}^n_{|\widetilde{L}_P}) - c_* \int f\, d\mathcal{H}^n_{|L_0}\right| \lesssim \frac{\sum_{S\in\mathcal{D}:P\subset S\subset Q} \alpha(S)\ell(Q) + \operatorname{dist}(0, L_Q)}{\ell(Q)}\, \ell(P)^{n+1},$$

and the lemma follows. □

We denote

$$\widetilde{\mathcal{G}}(x, r) := \{P(I)\}_{I\in\mathcal{G}(x,r)}.$$

We need the following auxiliary result.

Lemma 5.10. *For every $a \geq 1$ and every $S \in \mathcal{D}$,*

$$\sum_{P\in\widetilde{\mathcal{G}}(x,r):\, P\subset a\, S} \mu(P) \lesssim \mu(S),$$

with the implicit constant depending on a.

Proof. We assume $x = 0$ for simplicity. Notice that for every $P \in \widetilde{\mathcal{G}}(0, r)$ such that $P \subset a\, S$ there exists some $I \in \mathcal{G}(0, r)$ such that $r\ell(I) \approx \ell(P)$ and $r \cdot I \subset a'\, \Pi(B_S)$ where a' only depends on a. Therefore

$$\begin{aligned}
&\sum_{P\in\widetilde{\mathcal{G}}(0,r);\, P\subset a\, S} \mu(P) \lesssim \sum\{\ell(r\cdot I)^n : I \in \mathcal{G}(0, r);\, r \cdot I \subset a'\Pi(B_S)\}\\
&\lesssim \sum\{\ell(r\cdot I)^n : I \in \widetilde{\mathcal{D}}(\mathbb{R}^n);\, r\cdot I \subset a'\Pi(B_S); r\cdot 5I \cap (\partial B_n(0, r) \cup \partial B_n(0, 2r)) \neq \varnothing\}\\
&\leq c(a)\, \ell(S)^n \approx c(a)\, \mu(S).
\end{aligned}$$

□

We can now estimate the term $G_{r,1}(x)$ in (5.31).

Lemma 5.11. *We have*

$$|G_{r,1}(x)| \lesssim \sum_{P\in\widetilde{\mathcal{G}}(x,r)} \left(\alpha(aP) + \frac{d(x, L_Q)}{\ell(Q)}\right) \frac{\mu(P)}{\mu(Q)}, \tag{5.42}$$

for some absolute constant $a \geq 1$.

Proof. Using (5.33) and (c) from Lemma 5.1,

$$\begin{aligned}
|G_{r,1}(0)| &= \left|\frac{1}{r^n} \sum_{I\in\mathcal{G}(0,r)} a_I \int \psi_I\left(\frac{\Pi(y)}{r}\right) d\mu(y)\right|\\
&\lesssim \frac{1}{\ell(Q)^n} \sum_{I\in\mathcal{G}(0,r)} \left(\sum_{S\in\mathcal{D}:P(I)\subset S\subset Q} \alpha(S) + \frac{\operatorname{dist}(0, L_Q)}{\ell(Q)}\right) \ell(P(I))^n.
\end{aligned} \tag{5.43}$$

Notice that by the definition of $\mathcal{G}(0,r)$, for every $I \in \mathcal{G}(0,r)$

$$\#\{P \in \mathcal{D} : P = P(I)\} \lesssim 1.$$

Then

$$|G_{r,1}(0)| \lesssim \sum_{P \in \widetilde{\mathcal{G}}(0,r)} \sum_{S \in \mathcal{D}: P \subset S \subset Q} \alpha(S) \frac{\ell(P)^n}{\ell(Q)^n} + \sum_{P \in \widetilde{\mathcal{G}}(0,r)} \frac{\operatorname{dist}(0, L_Q)}{\ell(Q)} \frac{\ell(P)^n}{\ell(Q)^n}. \tag{5.44}$$

If $S \in \mathcal{D}$ is such that $P \subset S \subset Q$, then there exists $\widetilde{S} \in \widetilde{\mathcal{G}}(0,r)$, with $\ell(\widetilde{S}) \approx \ell(S)$, such that $S \subset a\widetilde{S}$ for some $a \geq 1$. In fact, since $P \in \widetilde{\mathcal{G}}(0,r)$ we can find $I' \in \mathcal{G}(0,r)$ with $\ell(r \cdot I') \approx \ell(S)$ such that $\Pi(S) \cap r \cdot I' \neq \varnothing$. Therefore we can take $\widetilde{S} := P(I')$.

Hence for $P \in \widetilde{\mathcal{G}}(0,r)$,

$$\sum_{S \in \mathcal{D}: P \subset S \subset Q} \alpha(S) \lesssim \sum_{\substack{S \in \widetilde{\mathcal{G}}(0,r): \\ P \subset a\, S \subset a\, Q}} \alpha(a\, S). \tag{5.45}$$

Thus, using also Lemma 5.10,

$$\begin{aligned} \sum_{P \in \widetilde{\mathcal{G}}(0,r)} \sum_{S \in \mathcal{D}: P \subset S \subset Q} \alpha(S) \frac{\ell(P)^n}{\ell(Q)^n} &\lesssim \sum_{P \in \widetilde{\mathcal{G}}(0,r)} \sum_{\substack{S \in \widetilde{\mathcal{G}}(0,r): \\ P \subset a\, S \subset a\, Q}} \alpha(a\, S) \frac{\ell(P)^n}{\ell(Q)^n} \\ &\approx \sum_{S \in \widetilde{\mathcal{G}}(0,r):\, S \subset a\, Q} \alpha(a\, S) \sum_{P \in \widetilde{\mathcal{G}}(0,r):\, P \subset a\, S} \frac{\mu(P)}{\mu(Q)} \\ &\lesssim \sum_{S \in \widetilde{\mathcal{G}}(0,r):\, S \subset a\, Q} \alpha(a\, S) \frac{\mu(S)}{\mu(Q)}. \end{aligned} \tag{5.46}$$

Together with (5.44), this yields (5.42). □

Now we will deal with the term I_{21} in (5.32).

Lemma 5.12. *We have*

$$I_{21} \lesssim \mu(R).$$

Proof. 0 By Lemmas 5.10 and 5.11, and Cauchy-Schwarz,

$$\begin{aligned} |G_{r,1}(x)|^2 &\lesssim \left(\sum_{P \in \widetilde{\mathcal{G}}(x,r)} \left(\alpha(aP) + \frac{d(x, L_Q)}{\ell(Q)} \right)^2 \frac{\mu(P)}{\mu(Q)} \right) \left(\sum_{P \in \widetilde{\mathcal{G}}(x,r)} \frac{\mu(P)}{\mu(Q)} \right) \\ &\lesssim \sum_{P \in \widetilde{\mathcal{G}}(x,r)} \left(\alpha(aP) + \frac{d(x, L_Q)}{\ell(Q)} \right)^2 \frac{\mu(P)}{\mu(Q)}. \end{aligned}$$

Then

$$
\begin{aligned}
I_{21} &\lesssim \sum_{Q\in\mathcal{D}(R)} \frac{1}{\ell(Q)^{n+1}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \sum_{P\in\widetilde{\mathcal{G}}(x,r)} \left(\alpha(aP)^2 + \frac{d(x,L_Q)^2}{\ell(Q)^2}\right) \mu(P)\, dr d\mu(x) \\
&\lesssim \sum_{Q\in\mathcal{D}(R)} \frac{1}{\ell(Q)^{n+1}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} \sum_{\substack{P\subset a''Q:\\ cB_P\cap(\partial B(x,r)\cup\partial B(x,2r))\neq\varnothing}} \left(\alpha(aP)^2 + \frac{d(x,L_Q)^2}{\ell(Q)^2}\right) \mu(P)\, dr d\mu(x).
\end{aligned}
$$

By Fubini,

$$
\begin{aligned}
&\int_{\ell(Q)}^{2\ell(Q)} \sum_{\substack{P\subset a''Q:\\ cB_P\cap\partial B(x,r)\neq\varnothing}} \left(\alpha(aP)^2 + \frac{d(x,L_Q)^2}{\ell(Q)^2}\right) \mu(P)\, dr \\
&\qquad = \sum_{P\subset a''Q} \left(\alpha(aP)^2 + \frac{d(x,L_Q)^2}{\ell(Q)^2}\right) \mu(P) \int_{\{r:\, cB_P\cap\partial B(x,r)\neq\varnothing\}} dr \\
&\qquad \lesssim \sum_{P\subset a''Q} \left(\alpha(aP)^2 + \frac{d(x,L_Q)^2}{\ell(Q)^2}\right) \mu(P)\ell(P),
\end{aligned}
$$

where we used the fact that if $r>0$ is such that $cB_P\cap\partial B(x,r)\neq\varnothing$ then

$$
|x_P| - c\,\ell(P) \le r \le |x_P| + c\,\ell(P),
$$

where x_P is the center of B_P. Therefore,

$$
\begin{aligned}
I_{21} &\lesssim \sum_{Q\in\mathcal{D}(R)} \frac{1}{\ell(Q)^n} \int_Q \sum_{P\subset a''Q} \left(\alpha(aP)^2 + \frac{d(x,L_Q)^2}{\ell(Q)^2}\right) \frac{\ell(P)}{\ell(Q)} \mu(P) d\mu(x) \\
&\lesssim \sum_{Q\in\mathcal{D}(R)} \sum_{P\subset a''Q} \alpha(aP)^2 \frac{\ell(P)}{\ell(Q)}\, \mu(P) \\
&\qquad + \sum_{Q\in\mathcal{D}(R)} \frac{1}{\ell(Q)^n} \int_Q \frac{d(x,L_Q)^2}{\ell(Q)^2} d\mu(x) \sum_{P\in\mathcal{D}:P\subset a''Q} \frac{\ell(P)}{\ell(Q)}\, \mu(P) \qquad (5.47) \\
&\lesssim \sum_{P\in\mathcal{D}:P\subset a''R} \alpha(aP)^2\mu(P) \sum_{Q\in\mathcal{D}:a''Q\supset P} \frac{\ell(P)}{\ell(Q)} + \sum_{Q\in\mathcal{D}(R)} \int_Q \frac{d(x,L_Q)^2}{\ell(Q)^2} d\mu(x) \\
&\lesssim \mu(R).
\end{aligned}
$$

□

Finally we turn our attention to I_{22}. Recall that

$$
I_{22} = \sum_{\substack{Q\in\mathcal{D}:Q\subset R,\\ \alpha(1000Q)\le\delta^2}} \int_Q \int_{\ell(Q)}^{2\ell(Q)} |G_{r,2}(x)|^2 \frac{dr}{\ell(Q)}\, d\mu(x).
$$

For $x\in\operatorname{supp}(\mu)$ and $r>0$ set

$$
f_{x,r}(y) = \sum_{I\in\mathcal{T}(x,r)} \sum_{J\in\widetilde{\mathcal{D}}(\mathbb{R}^n):\, J\subset I} a_J\, \psi_J\left(\frac{\Pi(y-x)}{r}\right),
$$

so that

$$G_{r,2}(x) = \frac{1}{r^n} \int f_{x,r}(y)\, d\mu(y).$$

Lemma 5.13. *The functions $f_{x,r}$ satisfy*

- $\operatorname{supp} f_{x,r} \subset \bigcup_{I\in\mathcal{T}(x,r)} 3P(\hat{I})$*, where $\hat{I}$ is the father of I,*
- $\|f_{x,r}\|_\infty \lesssim 1$.

Proof. We assume again that $x = 0$. Notice that $\operatorname{supp} f_{x,r} \subset \Pi^{-1}(r\cdot 5I) \cap \operatorname{supp}(\mu)$ and since $I \in \mathcal{F}_0$, we have $\hat{I} \in \mathcal{G}(x,r)$. Therefore by Lemma 5.7, $\Pi^{-1}(r\cdot 5I) \cap \operatorname{supp}(\mu) \subset 3P(\hat{I})$.

We will now show that $\|f_{x,r}\|_\infty \lesssim 1$. Recalling (5.30) if $I, J \in \mathcal{F}_0$ and $20I \cap 20J \neq \varnothing$, then $\ell(I) \approx \ell(J)$. If $I \in \mathcal{F}_0 \setminus \mathcal{T}(x,r)$ or $I \subset J$ for some $J \in \mathcal{F}_0 \setminus \mathcal{T}(x,r)$, then by Lemma 5.1 $a_I = 0$. Therefore,

$$f_{x,r}(y) = \sum_{I\in\mathcal{F}_0} \sum_{J\in\widetilde{\mathcal{D}}(\mathbb{R}^n):\, J\subset I} a_J \psi_J\left(\frac{\Pi(y)}{r}\right).$$

We now consider the function

$$\widetilde{f}(z) = \sum_{I\in\mathcal{F}_0} \sum_{J\subset I} a_J \psi_J(z).$$

The second assertion in the lemma follows after checking that $\|\widetilde{f}\|_\infty \lesssim 1$. To this end, recall that by (5.1), for any $k \in \mathbb{Z}$, we have $\widetilde{h} = \sum_{I\in\widetilde{\mathcal{D}}(\mathbb{R}^n)} a_I \psi_I$. We can also write

$$\widetilde{h}(z) = \sum_{I\in\widetilde{\mathcal{D}}_k(\mathbb{R}^n)} \sum_{J\subset I} a_J \psi_J(z) + \sum_{I\in\mathcal{D}_k(\mathbb{R}^n)} \beta_I \phi_I(z), \tag{5.48}$$

where $\beta_I = \langle \widetilde{h}, \phi_I \rangle$ and the functions ϕ_I satisfy

- $\operatorname{supp} \phi_I \subset 7I$,
- $\|\phi_I\|_\infty \lesssim \frac{1}{\ell(I)^{n/2}}$,
- $\|\nabla\phi_I\|_\infty \lesssim \frac{1}{\ell(I)^{n/2+1}}$,
- $\|\phi_I\|_2 = 1$.

See [Mal, Theorem 7.9]. We note that $\operatorname{supp} \psi_I \subset 5I$ and $\operatorname{supp} \phi_I \subset 7I$ since we are taking Daubechies wavelets with 3 vanishing moments, see [Mal, p. 250].

Now let $z \in I_0$ for some $I_0 \in \mathcal{F}_0$ with $\ell(I_0) = 2^{-k}$. Notice that

$$\begin{aligned}\|\beta_I\phi_I\|_\infty &\lesssim |\beta_I|\ell(I)^{-n/2} \le \int |\widetilde{h}(y)\, \phi_I(y)| dy\ \ell(I)^{-n/2} \\ &\lesssim \|\phi_I\|_1\, \ell(I)^{-n/2} \lesssim \ell(I)^{n/2}\|\phi_I\|_2 \ell(I)^{-n/2} = 1.\end{aligned} \tag{5.49}$$

By the finite superposition of $\operatorname{supp}\phi_I$ for $I \in \mathcal{D}_k$, (5.49) implies that

$$\left|\sum_{I\in\mathcal{D}_k(\mathbb{R}^n)} \beta_I \phi_I(z)\right| \lesssim 1.$$

Therefore by (5.48) we deduce that

$$\left|\sum_{I\in\widetilde{\mathcal{D}}_k(\mathbb{R}^n)}\sum_{J\subset I} a_J\psi_J(z)\right| \lesssim 1. \tag{5.50}$$

We will now prove that

$$\left|\widetilde{f}(z) - \sum_{I\in\widetilde{\mathcal{D}}_k(\mathbb{R}^n)}\sum_{J\subset I} a_J\psi_J(z)\right| \lesssim 1.$$

Together with (5.50), this shows that $|\widetilde{f}(z)| \lesssim 1$ and proves the lemma. We have

$$\begin{aligned}\left|\widetilde{f}(z) - \sum_{I\in\widetilde{\mathcal{D}}_k(\mathbb{R}^n)}\sum_{J\subset I} a_J\psi_J(z)\right| &= \left|\sum_{\substack{I\in\mathcal{F}_0:\\ 5I\cap I_0\neq\varnothing}}\sum_{J\subset I} a_J\psi_J(z) - \sum_{\substack{I\in\mathcal{D}_k(\mathbb{R}^n):\\ 5I\cap I_0\neq\varnothing}}\sum_{J\subset I} a_J\psi_J(z)\right| \\ &\lesssim \sum_{J\in A_1\triangle A_2} |a_J\psi_J(z)|,\end{aligned} \tag{5.51}$$

where

$$A_1 = \{J\in\widetilde{\mathcal{D}}(\mathbb{R}^n) : J\subset I, \text{ for some } I\in\mathcal{F}_0 \text{ such that } 5I\cap I_0\neq\varnothing\}$$

and

$$A_2 = \{J\in\widetilde{\mathcal{D}}(\mathbb{R}^n) : J\subset I, \text{ for some } I\in\mathcal{D}_k(\mathbb{R}^n) \text{ such that } 5I\cap I_0\neq\varnothing\}.$$

It follows as in (5.49) that $\|a_J\psi_J\|_\infty \lesssim 1$. Therefore,

$$\left|\widetilde{f}(z) - \sum_{I\in\widetilde{\mathcal{D}}_k(\mathbb{R}^n)}\sum_{J\subset I} a_J\psi_J(z)\right| \lesssim \#A \lesssim 1.$$

This follows from the fact that if $I\in\mathcal{F}_0$ such that $5I\cap I_0\neq\varnothing$ then $\ell(I)\approx\ell(I_0)$. □

Lemma 5.14. *We have*

$$I_{22}\lesssim\mu(R).$$

Proof. Lemma 5.13 implies that

$$|G_{r,2}(x)| \lesssim \frac{1}{\ell(Q)^n}\int |f_{x,r}(y)|\,d\mu(y) \lesssim \frac{1}{\ell(Q)^n}\sum_{I\in\mathcal{T}(x,r)}\mu(P(\hat{I})).$$

As noted earlier, for $I\in\mathcal{T}(x,r)$, the parent of I, denoted by $\hat{I}$, belongs to $\mathcal{G}(x,r)$. Observe also that

$$r\operatorname{diam}(I)\leq\frac{1}{5000}\inf_{z\in r\cdot I} d(z),$$

because $\mathcal{T}(x,r)\in\mathcal{F}_0$. So every $z'\in r\cdot I\subset r\cdot\hat{I}$ satisfies $d(z')\geq 5000\,r\operatorname{diam}(I) = 2500\,r\operatorname{diam}(\hat{I})$. This implies that $d(z)\gtrsim r\,\ell(I)$ for all $z\in r\cdot\hat{I}$, because $d(\cdot)$ is 3-Lipschitz. As a consequence, by the definition of $d(\cdot)$, there exists some $y\in P(\hat{I})$ such that $\ell(y)\gtrsim$

$r\,\ell(I) \approx \ell(P(\hat{I}))$. Then it follows easily that there exists some descendant U of $P(\hat{I})$ with $\ell(U) \approx \ell(P(\hat{I}))$ such that

$$\sum_{S\in\mathcal{D}:U\subset S\subset 1000Q} \alpha(100S) \geq \delta.$$

This clearly implies that either

$$\sum_{S\in\mathcal{D}:P(\hat{I})\subset S\subset 1000Q} \alpha(100S) \geq \frac{\delta}{2},$$

or

$$\sum_{S\in\mathcal{D}:U\subset S\subset P(\hat{I})} \alpha(100S) \geq \frac{\delta}{2}.$$

Since $\ell(U) \approx \ell(P(\hat{I}))$, from the second condition one infers that $\alpha(100P(\hat{I})) \geq c\delta$. Hence in either case, for some small constant $c > 0$,

$$\sum_{S\in\mathcal{D}:P\subset S\subset 1000Q} \alpha(100S) \geq c\delta. \tag{5.52}$$

Therefore,

$$\begin{aligned}|G_{r,2}(x)| &\lesssim_\delta \frac{1}{\ell(Q)^n} \sum_{I\in\mathcal{T}(x,r)} \mu(P(\hat{I})) \sum_{S\in\mathcal{D}:P\subset S\subset 1000Q} \alpha(100S)\\ &\lesssim \frac{1}{\ell(Q)^n} \sum_{P\in\widetilde{\mathcal{G}}(x,r)} \mu(P) \sum_{S\in\mathcal{D}:P\subset S\subset 1000Q} \alpha(100S).\end{aligned}$$

Notice that

$$\sum_{P\in\widetilde{\mathcal{G}}(x,r)} \frac{\ell(P)^n}{\ell(Q)^n} \sum_{S\in\mathcal{D}:P\subset S\subset 1000Q} \alpha(100S)$$

is smaller, modulo the constants 1000 and 100, than the right side in (5.44). Therefore by the same arguments we used for I_{21} we get $I_{22} \lesssim \mu(R)$. □

From Lemmas 5.12 and 5.14 we deduce that $I_2 \lesssim \mu(R)$. Together with Lemma 5.5 this completes the proof of Theorem 5.2.

6. Proof of Proposition 1.3

We will only prove the equivalence (a)⇔(b), as (a)⇔(c) is very similar.

By Theorem 4.1, it is clear that uniform n-rectifiability implies the boundedness of the square function in (b) for any positive integer k. As for the converse, next we show that Lemma 3.1 holds with $\Delta_{\mu,\varphi}$ replaced by $\Delta^k_{\mu,\varphi}$.

Lemma 6.1. *Let k be a positive integer. For all $\varepsilon > 0$ there exists $\delta > 0$ such that all n-AD-regular measures μ with constant c_0 and $0 \in \operatorname{supp}(\mu)$ such that*

$$\int_\delta^{\delta^{-1}} \int_{x\in\bar{B}(0,\delta^{-1})} |\Delta^k_{\mu,\varphi}(x,r)|\, d\mu(x)\, dr \leq \delta,$$

satisfy

$$\operatorname{dist}_{B(0,1)}(\mu, \mathcal{U}(\varphi, c_0)) < \varepsilon.$$

Proof. Suppose that there exists an $\varepsilon > 0$, and for each $m \geq 1$ there exists an n-AD-regular measure μ_m such that $0 \in \operatorname{supp}(\mu_m)$,

$$\int_{1/m}^{m} \int_{x \in \bar{B}(0,m)} |\Delta^k_{\mu_m,\varphi}(x,r)|\, d\mu_m(x)\, dr \leq \frac{1}{m}, \tag{6.1}$$

and

$$\operatorname{dist}_{B(0,1)}(\mu_m, \mathcal{U}(\varphi, c_0)) \geq \varepsilon. \tag{6.2}$$

By (1.1) we can replace $\{\mu_m\}$ by a subsequence converging weak * (i.e. when tested against compactly supported continuous functions) to a measure μ and it is easy to check that $0 \in \operatorname{supp}(\mu)$ and that μ is also n-dimensional AD-regular with constant c_0. We claim that

$$\int_0^\infty \int_{x \in \mathbb{R}^d} |\Delta^k_{\mu,\varphi}(x,r)|\, d\mu(x)\, dr = 0.$$

The proof of this statement is elementary and is almost the same as the analogous one in Lemma 3.1. We leave the details for the reader.

Our next objective consists in showing that $\mu \in \mathcal{U}(\varphi, c_0)$. To this end, denote by G the subset of those points $x \in \operatorname{supp}(\mu)$ such that

$$\int_0^\infty |\Delta^k_{\mu,\varphi}(x,r)|\, dr = 0. \tag{6.3}$$

It is clear that G has full μ-measure. For $x \in G$ and $r > 0$, consider the function $f_x(r) = \varphi_r * \mu(x)$. Then $f_x : (0, +\infty) \to \mathbb{R}$ is bounded and C^∞, and it follows from (6.3) that f_x is a polynomial in r of degree at most $k-1$, whose coefficients may depend on x. However, since μ is n-AD-regular, it follows easily that there exists some constant c such that

$$|f_x(r)| = |\varphi_r * \mu(x)| \leq c \quad \text{for all } r > 0.$$

Thus f_x must be constant on r. So for all $x \in G$ and $0 < R_1 \leq R_2$,

$$\varphi_{R_1} * \mu(x) = \varphi_{R_2} * \mu(x).$$

This is the same estimate we obtained in (3.11) in Lemma 3.1. So proceeding exactly in the same way as there we deduce then that

$$\varphi_{R_1} * \mu(x) = \varphi_{R_2} * \mu(y) \quad \text{for all } x, y \in \operatorname{supp} \mu \text{ and all } 0 < R_1 \leq R_2.$$

That is, $\mu \in \mathcal{U}(\varphi, c_0)$. However, by condition (6.2), letting $m \to \infty$, we have

$$\operatorname{dist}_{B(0,1)}(\mu, \mathcal{U}(\varphi, c_0)) \geq \varepsilon,$$

because $\operatorname{dist}_{B(0,1)}(\cdot, \mathcal{U}(\varphi, c_0))$ is continuous under the weak * topology. So $\mu \notin \mathcal{U}(\varphi, c_0)$, which is a contradiction. $\square$

Applying the previous lemma and arguing in the same way as in Section 3 one proves the implication (b)$\Rightarrow$(a) of Proposition 1.3.

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