Notes on harmonic measure

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1 Introduction

In these notes we provide a straightforward introduction to the topic of harmonic measure. This is an area where many advances have been obtained in the last years and we think that this book can be useful for people interested in this topic.

In the first Chapters 2-6 we have followed classical references such as [Fol95], [Car98], [GM05], [Lan72], [AG01], and [Ran95], as well as some private notes of Jonas Azzam. A large part of the content of Chapter 7 is based on Kenig's book [Ken94], and on papers by Aikawa, Hofmann, Martell, and many others. Chapter 8 is based on a paper by Jerison and Kenig [JK82]. In Chapter 9, the proof of Jones-Wolff theorem about the dimension of harmonic measure in the plane follows the presentation of [CVT18][1]. In some parts of Chapter 10 we follow the book of Caffarelli and Salsa [CS05] and some work by Mourgoglou and the second named author of these notes. Most of the last chapter follows [AHM$^+$16].

We apologize in advance for possible inaccuracies or lack of citation. Anyway, we remark that this work is still under construction and we plan to add more content as well as more accurate citations in future versions of these notes.

[1] We thank J. Cufí and J. Verdera for allowing us to reproduce a large part of the content from [CVT18].

2 Harmonic functions

2.1 Definition and basic properties

Given an open set $\Omega \subset \mathbb{R}^d$ we say that a real-valued function u is harmonic in Ω if $u \in C^2(\Omega)$ and

$$\Delta u(x) = \sum_{j=1}^{d} \partial_j^2 u(x) = 0$$

for every $x \in \Omega$ (later on we will see that the C^2 hypothesis can be replaced by just locally integrable if we consider the distributional Laplacian).

Let κ_d denote the area of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, that is,

$$\kappa_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)}$$

see [Fol95, Proposition 0.7] for instance, and $d\sigma$ denote the surface measure. Recall that the volume of the unit ball is then $|B_1(0)| = \frac{\kappa_d}{d}$ (see [Fol95, Corollary 0.8]). Below, we denote $B_r(x)$ the open ball centered at x with radius r, and $S_r(x) = \partial B_r(x)$.

Throughout the notes, $\fint_U f\, d\mu$ stands for the average integral with respect to the measure μ, i.e., $\frac{1}{\mu(U)} \int_U f\, d\mu$.

Lemma 2.1 (Mean value theorem). *Let $\Omega \subset \mathbb{R}^d$ be open. If $u \in C^2(\Omega)$ is harmonic, then*

$$u(x_0) = \fint_{B_r(x_0)} u(y)dy = \fint_{B_1(0)} u(x_0 + ry)dy \quad \textit{for every } \overline{B_r(x_0)} \subset \Omega \subset \mathbb{R}^d. \tag{2.1}$$

Moreover

$$u(x_0) = \fint_{S_r(x_0)} u(y)d\sigma(y) = \fint_{S_1(0)} u(x_0 + ry)d\sigma(y) \quad \textit{for every } \overline{B_r(x_0)} \subset \Omega \subset \mathbb{R}^d. \tag{2.2}$$

Proof. Changing variables, we have that

$$A(\rho) := \frac{1}{\rho^d} \int_{B_\rho(x_0)} u(x)dx = \int_{B_1} u(\rho x + x_0)dx.$$

On the other hand, set

$$\begin{aligned}\widetilde{A}(\rho) &:= \int_{B_1} \nabla u(\rho x + x_0) \cdot x\, dx \\ &= \int_{B_\rho(x_0)} \frac{\nabla u(x) \cdot (x - x_0)}{\rho} \frac{dx}{\rho^d} = \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \nabla u(x) \cdot \nabla |x - x_0|^2\, dx.\end{aligned}$$

Since u satisfies that $\Delta u = 0$ in Ω, we can apply Green's formula twice to obtain

$$\begin{aligned}\widetilde{A}(\rho) &= \frac{1}{2\rho^{d+1}} \int_{S_\rho(x_0)} |x - x_0|^2 \, \nabla u(x) \cdot \nu \, dx - \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, |x - x_0|^2 \, dx \\ &= \frac{1}{2\rho^{d-1}} \int_{S_\rho(x_0)} \nabla u(x) \cdot \nu \, dx = 0, \end{aligned} \tag{2.3}$$

where ν stands for the normal vector to the sphere pointing outward.

Since $u \in C^2(\Omega)$, for every x we have $\int_\rho^r \nabla u(tx + x_0) \cdot x \, dt = u(rx + x_0) - u(\rho x + x_0)$ by the fundamental theorem of calculus. Applying Fubini's Theorem we get

$$\begin{aligned} 0 &\overset{(2.3)}{=} \int_\rho^r \widetilde{A}(t) \, dt = \int_{B_1} \int_\rho^r \nabla u(tx + x_0) \cdot x \, dt \, dx = \int_{B_1} (u(rx + x_0) - u(\rho x + x_0)) \, dx \\ &= A(r) - A(\rho). \end{aligned} \tag{2.4}$$

So $A(r) = A(\rho)$ for all $\rho < r$.

On the other hand, taking the mean and using the continuity of u we obtain

$$\left| u(x_0) - \frac{d}{\kappa_d} \lim_{\rho \to 0} A(\rho) \right| = \lim_{\rho \to 0} \frac{1}{|B_\rho(x_0)|} \left| \int_{B_\rho(x_0)} (u(x_0) - u(x)) \, dx \right| \leqslant \lim_{\rho \to 0} o_{\rho \to 0}(1) = 0.$$

To see the coincidence with the average on spheres, note that in polar coordinates we have

$$A(\rho) = \frac{1}{\rho^d} \int_{S_1(0)} \int_0^\rho u(t\theta) t^{d-1} dt \, d\theta.$$

From this formula one can easily show that (2.2) implies (2.1), but we need to prove the converse. Let us differentiate this expression. We get that

$$\begin{aligned} 0 = A'(\rho) &= \frac{-d}{\rho^{d+1}} \int_{S_1(0)} \int_0^\rho u(t\theta) t^{d-1} dt \, d\theta + \frac{1}{\rho^d} \int_{S_1(0)} u(\rho\theta) \rho^{d-1} d\theta \\ &= \frac{-d}{\rho} A(\rho) + \frac{1}{\rho^d} \int_{S_\rho(x_0)} u(\rho\theta) d\theta. \end{aligned} \tag{2.5}$$

Since $u(x_0) = \frac{d}{\kappa_d} A(\rho)$ by (2.1), we readily get (2.2) multiplying the last expression times $\frac{\rho}{\kappa_d}$. □

Remark 2.2. Arguing as above, it follows that if $u \in C^2(\Omega)$ satisfies $\Delta u \geqslant 0$ in Ω, then

$$u(x_0) \leqslant \fint_{B_r(x_0)} u(y) dy \leqslant \fint_{S_r(x_0)} u(y) d\sigma(y) \tag{2.6}$$

whenever $\overline{B_r(x_0)} \subset \Omega \subset \mathbb{R}^d$. Indeed, instead of (2.3), we have

$$\begin{aligned} \widetilde{A}(\rho) &= \frac{1}{2\rho^{d-1}} \int_{S_\rho(x_0)} \nabla u(x) \cdot \nu \, dx - \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, |x - x_0|^2 \, dx \\ &= \frac{1}{2\rho^{d-1}} \int_{S_\rho(x_0)} \Delta u(x) \, dx - \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, |x - x_0|^2 \, dx \\ &= \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, (\rho^2 - |x - x_0|^2) \, dx \geqslant 0. \end{aligned}$$

Then, as in (2.4), we deduce that

$$A(r) - A(\rho) \geqslant 0 \quad \text{if } \rho < r.$$

Then, letting $\rho \to 0$, the first inequality in (2.6) follows.

Further, notice that the preceding discussion shows that $A'(\rho) \geqslant 0$, and then by (2.5) it follows that

$$0 \leqslant \frac{-d}{\rho} A(\rho) + \frac{1}{\rho^d} \int_{S_\rho(x_0)} u(\rho\theta) d\theta,$$

which is equivalent to the last inequality in (2.6).

And the converse is true:

Theorem 2.3 (Converse of the mean value Theorem). *If $u \in C(\Omega)$ satisfies (2.1) or (2.2), then $u \in C^\infty$ and it is harmonic.*

Proof. Note that we have seen that (2.1) and (2.2) are in fact equivalent. Thus, it suffices to assume that u satisfies (2.2).

Let $\psi \in C^\infty([0,1])$ be a non-negative function with $\int_0^\infty \psi(t) t^{d-1} dt = 1$. Define $\phi_\varepsilon(x) := \frac{1}{\kappa_d \varepsilon^d} \psi\left(\frac{|x|}{\varepsilon}\right)$. Then $\int \phi_\varepsilon = 1$ for every ε. Next consider the subset $\Omega_\varepsilon := \{x \in \Omega : \overline{B_\varepsilon(x)} \subset \Omega\}$. If $x \in \Omega_\varepsilon$ then we claim that

$$u(x) = \int u(y) \phi_\varepsilon(x-y)\, dy.$$

Indeed,

$$\begin{aligned} u(x) - \int u(y)\phi_\varepsilon(x-y)\, dy &= \int (u(x) - u(y))\phi_\varepsilon(x-y)\, dy \\ &= \int_0^\varepsilon \frac{\psi(\frac{\rho}{\varepsilon})}{\kappa_d \varepsilon^d} \int_{S_1(0)} (u(x) - u(x+\rho\theta))\, d\theta\, d\rho \overset{(2.2)}{=} 0. \end{aligned}$$

We can conclude that u is C^∞ in Ω_ε and, therefore, in the whole of Ω.

To get the harmonicity, note that the derivative with respect to r of $\int_{S_1(0)} u(x+ry) d\sigma(y)$ is zero by assumption. That is

$$\begin{aligned} 0 &= \frac{d}{dr} \fint_{S_r(x)} u(y) d\sigma(y) = c \frac{d}{dr} \int_{S_1(0)} u(x+ry) d\sigma(y) = c \int_{S_1(0)} \partial_\nu u(x+ry)\, d\sigma \\ &= c \fint_{S_r(x)} \partial_\nu u\, d\sigma \overset{\text{Green Thm}}{=} \frac{c}{r^{d-1}} \int_{B_r(x)} \Delta u\, dx. \end{aligned}$$

Since the Laplacian vanishes on every ball, we deduce that it is actually zero everywhere. ∎

In particular, every harmonic C^2 function is C^∞. Therefore we can restate the definition of harmonic function:

Definition 2.4. We say that a function $u : \Omega \to \mathbb{R}$ is harmonic if $u \in C(\Omega)$ and it satisfies the mean value property (2.1).

As we have seen, every harmonic function satisfies also the mean value property in spheres, it is $C^\infty(\Omega)$ and $\Delta u = 0$. This self-improvement property is also true for harmonic distributions, we will see that later on.

Theorem 2.5 (The maximum principle)**.** *Let Ω be a domain (i.e. open and connected set). If u is harmonic and real-valued and $A := \sup_\Omega u < \infty$, then either $u(x) < A$ for every $x \in \Omega$ or $u(x) = A$ for every $x \in \Omega$.*

Proof. $\{x \in \Omega : u(x) = A\}$ is relatively closed by continuity and open by the mean value theorem. □

Corollary 2.6. *Let Ω be a bounded open set. If $u \in C(\Omega)$ is harmonic and real-valued, then the supremum and the infimum are attained at the boundary.*

Proof. Assume that the supremum is not attained at the boundary. Then, by compactness it must be attained in the interior. This implies that u is constant in some component of Ω, which in turn implies that the supremum is also attained at the boundary of that component, a contradiction. Also the infimum is attained at the boundary since $\inf_\Omega u = -\sup_\Omega(-u)$. □

Theorem 2.7 (Uniqueness theorem)**.** *Let Ω be a bounded open set. If $u_1, u_2 \in C(\overline{\Omega})$ are harmonic in Ω, and $u_1|_{\partial\Omega} \equiv u_2|_{\partial\Omega}$, then $u_1|_\Omega \equiv u_2|_\Omega$.*

Proof. Apply the corollary to $u_1 - u_2$. □

Theorem 2.8 (Liouville's theorem)**.** *Let u be a bounded harmonic function in $\mathbb{R}^n$. Then u is constant.*

Proof. Note that for $r > 2|x|$

$$
\begin{aligned}
|u(x) - u(0)| &= \left| \fint_{B_r(x)} u(y)dy - \fint_{B_r(0)} u(y)dy \right| \leqslant \frac{d}{\kappa_d r^d} \int_{B_{r+|x|}(0)\setminus B_{r-|x|}(0)} |u(y)|dy \\
&\leqslant \frac{d\|u\|_\infty}{\kappa_d} \frac{|B_{r+|x|}(0)\setminus B_{r-|x|}(0)|}{r^d} \lesssim_d \frac{|x|\|u\|_\infty}{r} \xrightarrow{r\to\infty} 0.
\end{aligned}
$$

□

2.2 The Caccioppoli inequality

We have shown that every harmonic function $u \in C(\Omega)$ is $C^\infty(\Omega)$. Next we turn our attention to weakly harmonic functions.

Definition 2.9. Given an open set $\Omega \subset \mathbb{R}^d$, we say that $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is *weakly harmonic* if every test function $\varphi \in C_c^\infty(\Omega)$ satisfies that

$$\langle \Delta u, \varphi \rangle := -\langle \nabla u, \nabla \varphi \rangle = 0. \tag{2.7}$$

We say that $u \in D'(\Omega)$ is *distributionally harmonic* if, instead, test functions satisfy

$$\langle \Delta u, \varphi \rangle := \langle u, \Delta \varphi \rangle = 0. \tag{2.8}$$

Arguing by density, if u is weakly harmonic then equation (2.7) is verified also for every $\varphi \in W_c^{1,2}(\Omega)$. Note that every harmonic function is weakly harmonic, and every weakly harmonic function is distributionally harmonic, but the converse has not been established yet (see Proposition 2.19 below).

Lemma 2.10 (Caccioppoli Inequality). *Let $\Omega \subset \mathbb{R}^d$ be an open set, and let u be weakly harmonic in Ω. Then for every ball $B \subset \Omega$ of radius r we have*

$$\int_B |\nabla u|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2,$$

where $rt \leqslant \operatorname{dist}(B, \partial\Omega)$

Proof. Let η be a Lipschitz function such that $\chi_B \leqslant \eta \leqslant \chi_{(t+1)B}$ and with $|\nabla \eta| \leqslant \frac{1}{rt}$. Since u is weakly harmonic and η is compactly supported in Ω, we have that

$$0 = \int_{(t+1)B} \nabla u \cdot \nabla(u\eta^2).$$

By the Leibniz rule, the former identity can be written as

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 = -\int_{(t+1)B} 2u\eta \nabla u \cdot \nabla \eta,$$

and using Hölder's inequality we get

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \left(\int_{(t+1)B} 4u^2 |\nabla \eta|^2 \right)^{\frac{1}{2}} \left(\int_{(t+1)B} \eta^2 |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Thus,

$$\int_B |\nabla u|^2 \leqslant \int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \int_{(t+1)B} 4u^2 |\nabla \eta|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2.$$

□

The Caccioppoli inequality is also valid for subharmonic functions, see Section 5.1. This inequality implies the universal control for the gradient in terms of the distance to the boundary and the L^∞ norm of u:

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^d$ be an open set, and let u be harmonic in Ω. Then*

$$|\nabla u(x)| \lesssim \frac{\|u\|_{L^\infty(\Omega)}}{d_\Omega(x)}, \tag{2.9}$$

where $d_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$.

Proof. Since the derivatives of u are harmonic, by the mean value theorem and the Caccioppoli inequality

$$\begin{aligned} |\nabla u(x)| &= \left| \fint_{B_{\frac{1}{2}d_\Omega(x)}(x)} \nabla u \, dm \right| \leqslant \left(\fint_{B_{\frac{1}{2}d_\Omega(x)}(x)} |\nabla u|^2 \, dm \right)^{\frac{1}{2}} \\ &\leqslant \left(\frac{4}{(\frac{1}{2}d_\Omega(x))^2} \fint_{B_{d_\Omega(x)}(x)} |u|^2 \, dm \right)^{\frac{1}{2}} \lesssim \frac{1}{d_\Omega(x)} \|u\|_{L^\infty(\Omega)}, \end{aligned}$$

as claimed. □

By iterating the estimate in Lemma 2.10, we immediately obtain the following.

Lemma 2.12. *Let u be a harmonic function in $B_1(0)$. Then, for all $k \geqslant 1$,*

$$\|u\|_{C^k(B_{1/2}(0))} \leqslant C(k) \, \|u\|_{L^\infty(B_1(0))}.$$

Then we deduce the following generalization of Liouville's theorem.

Proposition 2.13. *Let $\gamma > 0$ and let u be harmonic in $\mathbb{R}^d$ such that $|u(x)| \leqslant C(1+|x|)^\gamma$ for all $x \in \mathbb{R}^d$. Then u is a polynomial of degree at most $\lfloor \gamma \rfloor$.*

Proof. For $r > 0$, consider the function $u_r(x) = u(r\,x)$. Since u_r is harmonic, for any $k > 1$, by Lemma 2.12 we have

$$\begin{aligned} \|D^k u\|_{L^\infty(B_{r/2}(0))} &= \frac{1}{r^k} \|D^k u_r\|_{L^\infty(B_{1/2}(0))} \leqslant \frac{C(k)}{r^k} \|u_r\|_{L^\infty(B_1(0))} \\ &= \frac{C(k)}{r^k} \|u\|_{L^\infty(B_r(0))} \leqslant \frac{C'(k)(1+r)^\gamma}{r^k}. \end{aligned}$$

For $k = \lfloor \gamma \rfloor + 1$, the term on the right hand side tends to 0 as $r \to \infty$, and thus $D^k u$ vanishes identically in $\mathbb{R}^d$. Consequently, u is a polynomial of degree at most $k - 1 = \lfloor \gamma \rfloor$. □

Lemma 2.14. *Every sequence of uniformly bounded harmonic functions in an open set Ω is locally equicontinuous, it has a converging partial subsequence, and the limit is harmonic as well.*

Proof. Let $\{u_n\}_n$ with $\Delta u_n = 0$ in Ω and $\|u_n\|_{L^\infty(\Omega)} \leqslant C < \infty$.

By assumption u_n is a sequence of uniformly bounded and, by Lemma 2.11, uniformly locally equicontinuous functions. By the Ascoli-Arzelá theorem, u_n has a partial converging uniformly in every compact subset of Ω.

To see that the limit is also harmonic just apply the converse to the mean value theorem (see Theorem 2.3) to the limiting function. □

2.3 Harnack's inequality

Lemma 2.15 (Harnack's inequality)**.** *Let B be a ball and let $u \geqslant 0$ be a harmonic function in $2B$. Then*

$$\sup_B u \leqslant C \inf_B u.$$

Proof. Set $B = B(x_0, r)$. To prove the lemma it suffices to show that, for all $y, z \in B$, $u(y) \lesssim u(z)$, with the implicit constant depending only on d. Suppose first that $|y - z| \leqslant r/4$. Then we have $B(y, r/4) \subset B(z, r/2) \subset\subset 2B$, and so we have, by the mean value property,

$$u(y) = \fint_{B(y,r/4)} u\, dx \lesssim \fint_{B(z,r/2)} u\, dx = u(z).$$

In the case when $|y - z| > r/4$, we partition the segment $[y, z]$ into eight segments I_j with equal length and disjoint interiors. So we write

$$[y, z] = \bigcup_{0 \leqslant j \leqslant 7} [y_j, y_{j+1}],$$

and we assume that $y = y_0$, $z = y_8$. Since the length of $[y, z]$ is at most $\operatorname{diam}(B) = 2r$, it holds $|y_j - y_{j+1}| \leqslant r/4$ for each j. By the previous estimate, then we have $u(y_j) \lesssim u(y_{j+1})$ for each j. Thus,

$$u(y) = u(y_0) \lesssim u(y_1) \lesssim \cdots \lesssim u(y_8) = u(z).$$

□

Note that by modifying the argument above we can get that for every $t \geqslant 0$ there exists an optimal constant $\epsilon(t)$ so that every harmonic function $u \geqslant 0$ in $(1 + t)B$ satisfies

$$\sup_B u \leqslant (1 + \varepsilon(t)) \inf_B u.$$

The reader can prove that ε is non-increasing and $\varepsilon(t) \xrightarrow{t \to 0} \infty$. But the interesting asymptotic behavior is for $t \to \infty$:

Lemma 2.16 (Asymptotic Harnack inequality)**.** *There exists a nonnegative function $\epsilon(t) \xrightarrow{t \to \infty} 0$ so that every harmonic function $u \geqslant 0$ in $(1 + t)B$ satisfies that*

$$\sup_B u \leqslant (1 + \varepsilon(t)) \inf_B u.$$

Proof. The proof follows by an argument very similar to the one in the preceding lemma. Indeed, assume $t \geqslant 8$, say, and consider arbitrary points $x, z \in B$. Furthermore, assume without loss of generality that $r(B) = 1$. Then we have $B(x, t/2) \subset B(z, 2+t/2) \subset (1+t)B$ and so

$$\begin{aligned} u(x) &= \frac{1}{|B(x,t/2)|} \int_{B(x,t/2)} u\, dy \leqslant \frac{1}{|B(x,t/2)|} \int_{B(z,2+t/2)} u\, dy \\ &= \frac{|B(z, 2 + t/2)|}{|B(x, t/2)|} u(z) = \left(\frac{4+t}{t}\right)^d u(z). \end{aligned}$$

So we may choose $\epsilon(t) = \left(\frac{4+t}{t}\right)^d - 1$. □

Lemma 2.17. *Let $\Omega \subset \mathbb{R}^d$ be a domain and let $x, y \in \Omega$. Then there is a constant $C_{x,y} > 0$ depending just on x, y, and Ω such that for any positive harmonic function u in Ω, it holds*

$$C_{x,y}^{-1}\, u(x) \leqslant u(y) \leqslant C_{x,y}\, u(y).$$

Remark that the important fact about the estimate above is that the constant $C_{x,y}$ does not depend on the particular function u.

Proof. Let $\gamma \subset \Omega$ be a compact curve contained in Ω whose end points are x and y, and let $\delta = \operatorname{dist}(\gamma, \partial\Omega)$. By the compactness of γ, there is a finite covering of γ by open balls B_i, $i = 1, \ldots, m$, centered in γ with $\operatorname{rad}(B_i) = \delta/2$ (with m depending on Ω and γ).

We reorder the balls B_i as follows. Suppose that $x \in B_1$ without loss of generality. If $m \geqslant 2$, because of the connectivity of γ, there exists another ball B_i, call it B_2, such that $B_1 \cap B_2 \neq \varnothing$. Next, if $m \geqslant 3$, by the connectivity of γ again, there exists another ball, call it B_3, such that $(B_1 \cup B_2) \cap B_3 \neq \varnothing$, and so on. Denote $U_k = \bigcup_{1 \leqslant i \leqslant k} B_i$, so that $U_k = U_{k-1} \cup B_k$, $U_{k-1} \cap B_k \neq \varnothing$, and $\gamma \subset U_m$.

Given u harmonic and positive in Ω, by Harnack's inequality $u(z) \approx u(z')$ for all $z, z' \in B_i$ (since $2B_i \subset \Omega$). Then, by induction it follows easily that $u(z) \approx u(z')$ for all $z, z' \in U_k$ (with the implicit constant depending on k), for $k = 1, \ldots, m$. In particular, $u(x) \approx_m u(y)$. □

2.4 The fundamental solution

To conclude this chapter, we will see that every harmonic distribution (see Definition 2.9) is in fact a C^∞ function. This is a quite general fact for elliptic partial differential equations with C^∞ fundamental solutions, see [Fol95, Theorem 1.58] for the details.

Let us define

$$\mathcal{E}(x) = \begin{cases} \dfrac{|x|^{2-d}}{(d-2)\kappa_d} & \text{if } d > 2, \\[2ex] \dfrac{-\log|x|}{2\pi} & \text{if } d = 2, \end{cases} \tag{2.10}$$

Note that, since $\kappa_2 = 2\pi$, for every $n \geqslant 1$ its gradient is

$$\nabla\mathcal{E}(x) = \frac{-x}{\kappa_d |x|^d}. \tag{2.11}$$

Proposition 2.18. *The fundamental solution of $(-\Delta)$ in $\mathbb{R}^d$ is precisely $\mathcal{E}$, i.e. $-\Delta\mathcal{E}$ is the Dirac delta distribution δ_0.*

The preceding proposition must be understood in the sense that for every test function $\varphi \in D(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$, we have

$$\varphi(0) =: \langle \delta_0, \varphi \rangle = -\langle \Delta\mathcal{E}, \varphi \rangle = -\langle \mathcal{E}, \Delta\varphi \rangle.$$

Proof of Proposition 2.18. Consider $\epsilon > 0$ and let ν be the normal vector to S_ϵ pointing towards the origin. For $\varphi \in C_c^\infty$ we have

$$-\langle \mathcal{E}, \Delta\varphi \rangle = \int \nabla\mathcal{E} \cdot \nabla\varphi. \tag{2.12}$$

Indeed,

$$\begin{aligned}
\left| -\langle \mathcal{E}, \Delta\varphi \rangle - \int \nabla\mathcal{E} \cdot \nabla\varphi \right| &= \left| \int_{B_\epsilon} \mathcal{E}\Delta\varphi + \int_{B_\epsilon^c} \mathcal{E}\Delta\varphi + \int \nabla\mathcal{E} \cdot \nabla\varphi \right| \\
&\overset{\text{Green}}{\leqslant} \left| \int_{B_\epsilon} \mathcal{E}\Delta\varphi \right| + \left| \int_{B_\epsilon^c} \nabla\mathcal{E} \cdot \nabla\varphi - \int \nabla\mathcal{E} \cdot \nabla\varphi \right| + \left| \int_{S_\epsilon} \mathcal{E}\nabla\varphi \cdot \nu \right| \\
&\lesssim \|\Delta\varphi\|_\infty \|\mathcal{E}\|_{L^1(B_\epsilon)} + \left| \int_{B_\epsilon} \nabla\mathcal{E} \cdot \nabla\varphi \right| + \|\mathcal{E}\|_{L^\infty(S_\epsilon)} \|\nabla\varphi\|_\infty \epsilon^{d-1}.
\end{aligned}$$

For $d = 2$, using (2.10) we have $\|\mathcal{E}\|_{L^1(B_\epsilon)} \approx \int_0^\epsilon r|\log r| dr \xrightarrow{\epsilon \to 0} 0$ and $\|\mathcal{E}\|_{L^\infty(S_\epsilon)} = c|\log\epsilon|$. In case $d > 2$, then using (2.10) we have $\|\mathcal{E}\|_{L^1(B_\epsilon)} \approx \int_0^\epsilon r dr \xrightarrow{\epsilon \to 0} 0$ and $\|\mathcal{E}\|_{L^\infty(S_\epsilon)} = c\epsilon^{2-d}$. All in all, letting $\epsilon \to 0$ we get (2.12).

Moreover,

$$\begin{aligned}
| -\langle \mathcal{E}, \Delta\varphi \rangle - \varphi(0)| &\overset{(2.12)}{=} \left| \int \nabla\mathcal{E} \cdot \nabla\varphi - \varphi(0) \right| = \left| \int_{B_\epsilon} \nabla\mathcal{E} \cdot \nabla\varphi + \int_{B_\epsilon^c} \nabla\mathcal{E} \cdot \nabla\varphi - \varphi(0) \right| \\
&\overset{\text{Green}}{\lesssim} \|\nabla\varphi\|_\infty \int_{B_\epsilon} |x|^{1-d} + \left| \int_{S_\epsilon^c} \nabla\mathcal{E} \cdot \nu\varphi - \varphi(0) \right| + \left| \int_{B_\epsilon^c} \Delta\mathcal{E}\varphi \right|
\end{aligned}$$

Now, $\int_{B_\epsilon} |x|^{1-d} \approx \epsilon \xrightarrow{\epsilon \to 0} 0$, and $\Delta\mathcal{E} \equiv 0$ in B_ϵ^c. Moreover, for $y \in S_\epsilon$ we get

$$\nabla\mathcal{E}(y) \cdot \nu(y) = \frac{-y}{\kappa_d |y|^d} \cdot \frac{-y}{|y|} = \frac{1}{\kappa_d \epsilon^{d-1}} = \frac{1}{\sigma(S_\epsilon)}.$$

Thus,

$$| -\langle \mathcal{E}, \Delta\varphi \rangle - \varphi(0)| \lesssim \left| \fint_{S_\epsilon^c} \varphi - \varphi(0) \right| \xrightarrow{\epsilon \to 0} 0,$$

as claimed by the continuity of φ at the origin.

□

The preceding proposition implies that for every test function $\varphi \in D(\Omega)$, we have

$$-\Delta(\mathcal{E} * \varphi)(x) = \varphi(x). \tag{2.13}$$

Note that $\mathcal{E} * \varphi \in C^\infty$ because $\mathcal{E} \in L^1_{\text{loc}}$.

In fact we obtain the following:

Proposition 2.19. *Let u be a harmonic distribution in an open set Ω. Then $u \in C^\infty(\Omega)$.*

Remark that a distribution is called harmonic if it is distributionally harmonic.

Proof. Given a distibution T with compact support contained in a bounded open set V, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ we can define

$$\langle \mathcal{E} * T, \varphi \rangle := \langle T, \psi(\mathcal{E} * (\varphi_-))_- \rangle,$$

where ψ is any cuttof function $\psi \in C_c^\infty$ with $\chi_{\mathrm{supp}T} \leqslant \psi \leqslant \chi_V$, and $f_-(x) := f(-x)$. This definition does not depend on the particular choice of ψ, because the test function in the right-hand side will not vary in the support of T. Moreover, we claim that this distribution is in fact C^∞ out of the support of T. Indeed, for any test function φ with $\mathrm{supp}\varphi \cap \mathrm{supp}T = \varnothing$, one can consider $\varepsilon := \mathrm{dist}(\mathrm{supp}\varphi, \mathrm{supp}T)$, and given a C^∞ function ϕ such that $\chi_{B_{\varepsilon/4}} \leqslant \phi \leqslant \chi_{B_{\varepsilon/2}}$, one can infer that $\langle \mathcal{E} * T, \varphi \rangle = \langle ((1-\phi)\mathcal{E}) * T, \varphi \rangle$. The latter can be shown to be a C^∞ distribution arguing as in the proof of [Gra08, Theorem 2.3.20].

When u is a distribution in an open set Ω such that $\Delta u = 0$, given a ball $B \subset \Omega$ we can define a cut-off function $\psi_B \in C^\infty$ such that $\chi_{\frac{1}{2}B} \leqslant \psi_B \leqslant \chi_B$. Then $\Delta(\psi_B u)$ is a distribution supported in $\overline{B} \backslash \frac{1}{2}B$ and therefore $\mathcal{E} * (\Delta(\psi_B u))$ is a well-defined distribution. Given $\varphi \in D(\Omega) := C_c^\infty(\Omega)$, assuming if necessary that $\psi_B \nabla \psi \equiv 0$, we have

$$\langle \mathcal{E} * (-\Delta(\psi_B u)), \varphi \rangle = \langle (-\Delta(\psi_B u)), \psi(\mathcal{E} * (\varphi_-))_- \rangle = \langle \psi_B u, -\Delta(\mathcal{E} * (\varphi_-))_- \rangle \overset{(2.13)}{=} \langle \psi_B u, \varphi \rangle,$$

i.e. $\mathcal{E} * (-\Delta(\psi_B u)) = \psi_B u$ in the distributional sense. Since the former is in fact C^∞ out of the support of $\Delta(\psi_B u)$, we conclude in particular that in $\frac{1}{2}B$, the function $u = \psi_B u$ is C^∞. □

The approach above can be slightly modified in order to obtain the hypoellipticity of the laplacian:

Theorem 2.20 ([Fol95, Theorem 1.58])**.** *The laplacian Δ is hypoelliptic, i.e., if u is a distribution on a bounded open set Ω such that $\Delta u \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.*

Remark 2.21. Note that $\mathcal{E} \in L^p_{\mathrm{loc}}$ for every $p < \frac{d}{d-2}$, and $\nabla \mathcal{E} \in L^p_{\mathrm{loc}}$ for every $p < \frac{d}{d-1}$. The integrability at infinity is obtained for $p > \frac{d}{d-2}$, and $p > \frac{d}{d-1}$ respectively.

3 The Dirichlet Problem

3.1 The weak formulation

Consider the problem of finding a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ in an open set $\Omega \subset \mathbb{R}^d$ to the Dirichlet problem with boundary data $f \in C(\partial\Omega)$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

To obtain a general theory of existence and uniqueness, we can work in Sobolev spaces with only one derivative, and this requires a weak formulation of the Dirichlet problem. Assume that $u \in C^1(\overline{\Omega})$, and let $\varphi \in C_c^\infty(\Omega)$. Then Green's theorem implies that

$$0 = \int_\Omega \varphi \Delta u = -\int_\Omega \nabla u \cdot \nabla \varphi + \int_{\partial\Omega} \varphi \nabla u \cdot \nu \, d\sigma = -\int_\Omega \nabla u \cdot \nabla \varphi. \tag{3.2}$$

Equation (3.2) provides us with a weak formulation of $\Delta u = 0$. But how can we encode the boundary behavior? Set

$$H^1(\Omega) := W^{1,2}(\Omega) := \{f \in L^2(\Omega) : \partial_i f \in L^2(\Omega) \text{ for } 1 \leqslant i \leqslant n+1\},$$

and we define

$$H_0^1(\Omega) := \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$$

and the quotient space

$$H^{1/2}(\partial\Omega) := H^1(\Omega)/H_0^1(\Omega)$$

(see [Sch02, Theorem 3.13], for instance). Given $f \in H^1(\Omega)$, its class in $H^{1/2}(\partial\Omega)$ is often called "the trace of f". Now, in a bounded open set Ω, if $u = f$ in $\partial\Omega$ and $u, f \in C^2(\overline{\Omega})$, then one can show that $u - f \in H_0^1(\Omega)$. Moreover, the identity (3.2) can be extended by density to $\varphi \in H_0^1(\Omega)$.

All in all, in an open set Ω, we say that $u \in H^1(\Omega)$ is a (weak) solution to the Dirichlet problem (3.1) if

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi = 0 & \text{for every } \varphi \in H_0^1(\Omega), \text{ and} \\ f - u \in H_0^1(\Omega). \end{cases} \tag{3.3}$$

Note that if $u \in C^2(\overline{\Omega}) \cap H^1(\Omega)$ is a weak solution (3.3), then it is also a solution to (3.1) for f regular enough.

Let us write $v := u - f$. Solving (3.3) is equivalent to finding $v \in H^1_0(\Omega)$ solving

$$\int_\Omega \nabla v \cdot \nabla \varphi = -\int_\Omega \nabla f \cdot \nabla \varphi \quad \text{for every } \varphi \in H^1_0(\Omega), \tag{3.4}$$

which in the strong formulation reads as

$$\begin{cases} \Delta v = \Delta f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in H^1_0(\Omega)$ be a harmonic function. Then it is the null function.*

Proof. There exist C^∞_c functions ψ_i such that $\psi_i \to u$ in H^1. Note that

$$\int \nabla \psi_i \cdot \nabla \psi_i = \int \nabla \psi_i \cdot \nabla (u - \psi_i) + \int \nabla \psi_i \cdot \nabla u.$$

But the last integral is null because u is harmonic. Thus, using the Cauchy-Schwartz inequality we get

$$\|\nabla \psi_i\|^2_{L^2} \leqslant \|\nabla \psi_i\|_{L^2} \|\nabla (u - \psi_i)\|_{L^2},$$

i.e.

$$\|\nabla \psi_i\|_{L^2} \leqslant \|\nabla (u - \psi_i)\|_{L^2}.$$

Taking limits,

$$\|\nabla u\|_{L^2} = \lim_{i\to\infty} \|\nabla \psi_i\|_{L^2} \leqslant \lim_{i\to\infty} \|\nabla (u - \psi_i)\|_{L^2} = 0.$$

Thus, u is constant and has trace 0, so it is the null function. □

Remark 3.2. Note that the preceding result does not apply to $\log|x|$ in the complement of B_1, since it does not have trace 0 according to the definitions, neither to x_d in $\mathbb{R}^d_+$. Indeed, C^∞_c functions cannot approach in L^2 norm a function which does not belong to L^2. The condition $u \in H^1(\Omega)$ is not satisfied in this case.

Theorem 3.3 (Riesz representation theorem for Hilbert spaces, see [Sch02, Theorem 2.1])**.** *Let H be a Hilbert space with inner product $(\cdot,\cdot)$, and let H^* be its dual. Then for each $u^* \in H^*$ there exists a unique $u \in H$ such that*

$$\langle u^*, v\rangle = (u, v).$$

Corollary 3.4. *Let Ω be open and let $f \in H^{\frac{1}{2}}(\partial\Omega)$. If the Dirichlet problem (3.1) has a solution $u \in H^1(\Omega)$, then this is unique and moreover $u \in C^\infty(\Omega)$. If Ω is bounded, then the solution exists.*

Proof. The uniqueness of the solution comes from Proposition 3.1 and the smoothness from hypoellipticity (see Section 2.4).

Suppose now that Ω is bounded. Then $\|\nabla v\|_{L^2(\Omega)}$ is a norm for the functions $v \in H_0^1(\Omega)$ (because of the Poincaré inequality) and the associated scalar product equals

$$(v, \varphi) = \int \nabla v \cdot \nabla \varphi \quad \text{for all } v, \varphi \in H_0^1(\Omega).$$

Let F denote a representative of f in H^1. Consider the linear functional

$$-\int_\Omega \nabla F \cdot \nabla \varphi \quad \text{for every } \varphi \in H_0^1(\Omega).$$

By the Riesz representation theorem, there exists a unique $v \in H_0^1(\Omega)$ solving (3.4). Note that v does not depend on the particular choice of F. Indeed, let $F_1, F_2 \in H^1(\Omega)$ with $F_1 - F_2 \in H_0^1(\Omega)$, and let v_1, v_2 be the solutions to (3.4) with functions F_1, F_2 respectively. Then

$$\int \nabla(v_1 - v_2) \cdot \nabla \varphi = -\int_\Omega \nabla f \cdot \nabla \varphi + \int_\Omega \nabla f \cdot \nabla \varphi = 0.$$

Thus, $v_1 - v_2$ is weakly harmonic. Moreover, $v_1 - v_2$ has trace 0. By Proposition 3.1 it is the null function.

Let $u := v + F$. Then u solves (3.3). $\square$

3.2 The Green function

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, let $x \in \Omega$, and define the fundamental solution (to $-\Delta$) with pole at x as

$$\mathcal{E}^x(y) := \mathcal{E}(x - y),$$

see (2.10). Note that $\mathcal{E}^0 = \mathcal{E}$. The equation

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = -\mathcal{E}^x(\cdot) & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

has a unique weak solution $v^x \in H^1(\Omega)$ by Corollary 3.4. Then we define the *Green function* with pole at x as

$$G^x(y) := v^x(y) + \mathcal{E}^x(y). \tag{3.6}$$

The thoughtful reader may notice that $\mathcal{E}^x$ is not an H^1 function, but this can be fixed by multiplying $\mathcal{E}$ times $\psi_{\partial\Omega}^x$, which is defined to be a C^∞ function vanishing in a neighborhood of x such that $\psi_{\partial\Omega}^x \equiv 1$ in a neighborhood of $\partial\Omega$, i.e., v^x is the weak solution to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = -\psi_{\partial\Omega}^x \mathcal{E}^x & \text{on } \partial\Omega. \end{cases}$$

Definition 3.5. Given $x \in \Omega$, define $\mathrm{d}_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$ and call $U_x := B_{\frac{1}{2}\mathrm{d}_\Omega(x)}(x)$. Then, since $\overline{U_x} \cap \partial\Omega = \varnothing$, we can find a compact set K_x and open sets V_x, $\widetilde{V}_x$ such that $\partial\Omega \subset V_x \subset \widetilde{V}_x \subset K_x \subset \overline{U_x}^c$ and a bump function $\psi^x_{\partial\Omega} \in C^\infty(\mathbb{R}^d)$ satisfying

$$\chi_{V_x} \leqslant \psi^x_{\partial\Omega} \leqslant \chi_{\widetilde{V}_x}. \tag{3.7}$$

Note that for every $\varphi \in C^\infty_c(\Omega)$ one has

$$\begin{aligned}\int \nabla G^x(y) \cdot \nabla\varphi(y)\, dy &= \int \nabla v^x(y) \cdot \nabla\varphi(y)\, dy + \int \nabla \mathcal{E}^x(y) \cdot \nabla\varphi(y)\, dy \\ &= 0 + \int \nabla\mathcal{E}(z) \cdot \nabla_z \varphi(x+z)\, dz \overset{\text{P.2.18}}{=} \varphi(x).\end{aligned} \tag{3.8}$$

That is $\Delta G^x = -\delta_x$ as a distribution in $D'(\Omega)$, with "vanishing" boundary values, i.e., with $\psi^x_{\partial\Omega} G^x \in H^1_0(\Omega)$ (see (3.7) above and Remark 2.21), so we say that G^x is the weak solution to

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega, \\ G^x = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

For any given $\varphi \in C^\infty_c(\Omega)$, we can write

$$\varphi(x) = \int_\Omega \nabla\varphi(z) \cdot \nabla G^x(z)$$

by (3.8). We want to apply this identity to $G^x(y)$, but it is not a test function.

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Then*

$$G^x(y) = \int_\Omega \nabla G^x \cdot \nabla G^y \, dm,$$

whenever $x, y \in \Omega$ are different points. In particular,

$$G^x(y) = G^y(x).$$

In other words, the Green function is symmetric and, therefore, it is harmonic also with respect to x. As a consequence, $v^x(y) = v^y(x)$ and it is harmonic with respect to $x \in \Omega$ as well. Note that for the lemma to make sense, we need that $\nabla G^x \cdot \nabla G^y \in L^1(\Omega)$. A priori one may think that $\mathcal{E}^x \in W^{1,\frac{d}{d-1}}_{\mathrm{loc}}(\mathbb{R}^d)$ implies $G^x \in W^{1,\frac{d}{d-1}}_{\mathrm{loc}}(\mathbb{R}^d)$, and this fact is not enough to grant integrability of $\nabla G^x \cdot \nabla G^y$. However, both terms are C^∞ away from the pole, and since $x \neq y$, then integrability comes from the local boundedness of the Green function away from the pole together with the integrability of the singularity.

Proof of Lemma 3.6. In order to apply (3.8), we need to substitute the Green function by a suitable test function approximating it. Let $\psi := \psi^x_{\partial\Omega}\psi^y_{\partial\Omega}$, and consider

$$G^x = (1-\psi)G^x + \psi G^x. \tag{3.10}$$

Let $U := (\widetilde{V}_x \cup \widetilde{V}_y) \backslash \Omega^c$ (see Definition 3.5) so that $\operatorname{supp}(\psi) \cap \Omega \subset U$. Since $\psi G^x \in H_0^1(U)$, there exists $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(U)$ so that

$$\varphi_k \xrightarrow[H^1(\Omega)]{k\to\infty} \psi G^x, \tag{3.11}$$

which allows us to approximate the last term in (3.10). On the other hand, let $\eta \in C^\infty(\mathbb{R})$ such that $\chi_{(0,1/2)} \leqslant \eta \leqslant \chi_{(0,1)}$ and write $\eta_k(z) := \eta(k|x-z|)$, which allows us to approximate the Green function around the pole $(1-\psi)G^x$ in (3.10) by $(1-\eta_k-\psi)G^x$.

Next, we define

$$f_k(z) := (1-\eta_k(z)-\psi(z))G^x(z) + \varphi_k(z),$$

which is in $C_c^\infty(\Omega)$ for k large enough. Note that subtracting η_k skips the pole x where the Green function is not C^∞, and subtracting ψ skips the boundary, while the values of ψG^x are substituted by the approximation φ_k. Since $\psi(y) = \varphi_k(y) = \eta_k(y) = 0$, for k large enough

$$\begin{aligned} G^x(y) = f_k(y) &\overset{(3.8)}{=} \int_\Omega \nabla f_k \cdot \nabla G^y \, dm \\ &= \int_\Omega \nabla G^x \cdot \nabla G^y \, dm + \int_\Omega \nabla(f_k - G^x) \cdot \nabla G^y \, dm. \end{aligned} \tag{3.12}$$

The lemma follows if we prove that

$$\left| \int_\Omega \nabla(f_k - G^x) \cdot \nabla G^y \, dm \right| \xrightarrow{k\to\infty} 0 \tag{3.13}$$

Indeed,

$$G^x - f_k = (\eta_k + \psi)G^x - \varphi_k,$$

and

$$\nabla(G^x - f_k) = \nabla\eta_k G^x + \eta_k \nabla G^x + \nabla(\psi G^x - \varphi_k).$$

Since $y \notin \operatorname{supp}\nabla(G^x - f_k)$, ∇G^y stays bounded in the integral (3.13). For $z \in U \subset\subset \mathbb{R}^d \backslash \{x\}$ also G^x and ∇G^x stay bounded. Therefore we only need to show that

$$\boxed{1} := \int_U |\nabla(\psi G^x - \varphi_k)| \xrightarrow{k\to\infty} 0,$$

and

$$\boxed{2} := \int_{B_{1/k}(x)} |\nabla\eta_k(z) G^x(z) + \eta_k(z)\nabla G^x(z)| \xrightarrow{k\to\infty} 0.$$

By the Cauchy-Schwartz inequality, since $|U| < \infty$, using (3.11) we get the integrability of the first term:

$$\boxed{1} \leqslant |U|^{\frac{1}{2}} \|\nabla(\psi G^x - \varphi_k)\|_2 \xrightarrow{k\to\infty} 0.$$

Finally, for $d \geqslant 3$ and k large enough, we can neglect the v^x term and bound the last term by

$$\boxed{2} \lesssim \int_{B_{1/k}(x)} k|x-z|^{2-d} + |x-z|^{1-d} \leqslant k\frac{1}{k^2} + \frac{1}{k} \xrightarrow{k\to\infty} 0,$$

proving (3.13). When $d = 2$ the limit is also 0:

$$\int_{B_{1/k}(x)} k|\log(|x-z|)| + |x-z|^{-1} \lesssim k\frac{1}{k^2}\left(-\log(k) + \frac{1}{2}\right) + \frac{1}{k} \xrightarrow{k\to\infty} 0.$$

$\square$

Consider $f \in C_c^\infty(\Omega)$. Then define

$$v(x) := -\int_\Omega G^x(y)f(y)\,dy = -f * \mathcal{E}(x) - \int_\Omega v^x(y)f(y)\,dy.$$

Since v^x is harmonic, $\Delta v = f$ in Ω. Moreover, if G^x is continuous up to the boundary, then $G^x(y)$ vanishes for $x \in \partial\Omega$. So v is the natural candidate to be the solution to the Dirichlet problem

$$\begin{cases} \Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Assuming regularity on $\partial\Omega$, we can define the Poisson kernel

$$P^x(\xi) := -\partial_\nu G^x(\xi) \qquad \text{for every } x \in \Omega,\ \xi \in \partial\Omega.$$

If $u \in C(\overline{\Omega})$ is harmonic in Ω, then we can write formally

$$\begin{aligned} u(x) = \int u(z)\delta_x(z) &= \int_\Omega (u(z)(-\Delta G^x(z)) + \Delta u(z) G^x(z)) \\ &\overset{\text{Green}}{=} \int_{\partial\Omega} (-u(\zeta)\partial_\nu G^x(\zeta) + \partial_\nu u(\zeta) G^x(\zeta))d\zeta. \end{aligned}$$

If G^x vanishes continuously in the boundary, we get that

$$u(x) = \int_{\partial\Omega} u(\zeta)P^x(\zeta)d\zeta.$$

Therefore, we expect that the Dirichlet problem (3.1) may be solved by integrating the boundary values times the Poisson kernel for regular enough domains. Harmonic measure will be a generalization of the Poisson kernel to more rough domains.

3.3 Limitations of the weak formulation

The weak solution to the Dirichlet problem exposed above is only half-satisfactory. We get existence and uniqueness for every domain, but it is not quite clear what does it mean to have 0 trace. In practical applications of (3.1) we would like to prescribe boundary values f only in the boundary of the domain, and not in a neighborhood of it. Moreover, one should expect that in case f is continuous, then the solution u is continuous up to the boundary, with $u|_{\partial\Omega} \equiv f$. However, the weak solutions above may not be continuous up to the boundary.

Example 3.7. Let $\Omega = B_1\backslash\{0\} \subset \mathbb{R}^d$ with $d \geqslant 3$, and take $f = 0$ in $\partial B_1(0)$ and $f(0) = 1$. A natural candidate to "represent" f in $H^1(\Omega)$ is the function $F(x) = 1 - |x|\chi_{B_1}$ is in $H^1(\Omega)$. Let us see that its class in $H_0^1(\Omega)$ coincides with the class of $G(x) \equiv 0$, i.e., let's show that $F - G = F \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$.

Let $\eta \in C^\infty(\mathbb{R})$ such that $\chi_{(-\infty,1/2)} \leqslant \eta \leqslant \chi_{(-\infty,1)}$. Then let $\varphi_\varepsilon(x) = \eta(\varepsilon^{-1}|x|)$ and let $\psi_\varepsilon(x) = \eta(\varepsilon^{-1}(|x| - 1 + \varepsilon))$, and consider $h_\varepsilon := \psi_\varepsilon(1 - \varphi_\varepsilon)F \in C_c^\infty(\Omega)$. Then we have that $F = h_\varepsilon$ in $B_1^c \cup (B_{1-\varepsilon}\backslash B_\varepsilon)$

$$\|F - h_\varepsilon\|_2 = \|(1 - \psi_\varepsilon(1 - \varphi_\varepsilon))(1 - |x|\chi_{B_1})\|_2 \leqslant (|B_1\backslash(B_{1-\varepsilon} \cup B_\varepsilon|)^{\frac{1}{2}} \xrightarrow{\varepsilon\to 0} 0.$$

On the other hand, since

$$\|\nabla\varphi_\varepsilon\|_\infty + \|\nabla\psi_\varepsilon\|_\infty \leqslant \varepsilon^{-1}\|\eta'\|_\infty,$$

and using that the support of $F - h_\varepsilon$ is contained in $\overline{B_1}\backslash B_{1-\varepsilon} \cup \overline{B_\varepsilon}$, using the product rule we deduce that

$$\begin{aligned}\|\nabla(F - h_\varepsilon)\|_2 &= \|\nabla[(1 - \psi_\varepsilon(1 - \varphi_\varepsilon))(1 - |x|\chi_{B_1})]\|_{L^2(B_1\backslash B_{1-\varepsilon}\cup B_\varepsilon)} \\ &\leqslant \left(\|\varepsilon\nabla\psi_\varepsilon\|^2_{L^2(B_1\backslash B_{1-\varepsilon})} + \|\nabla\varphi_\varepsilon\|^2_{L^2(B_\varepsilon)}\right)^{\frac{1}{2}} + \|\nabla(|x|\chi_{B_1})\|_{L^2(B_1\backslash B_{1-\varepsilon}\cup B_\varepsilon))} \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

We have seen that $F \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$ and therefore $F \equiv 0$ in $H_0^1(\Omega)$. Thus, the weak solution to the Dirichlet problem

$$\begin{cases}\Delta u = 0 & \text{in } \Omega, \\ u = F & \text{on } \partial\Omega\end{cases} \tag{3.15}$$

is $u = 0$.

The example above is related to the fact that a point has capacity zero in $\mathbb{R}^d$ for every $d \geqslant 2$, see Chapter 6. We will see in further chapters that, in fact, there exists no harmonic function u in $\Omega = B_1\backslash\{0\} \subset \mathbb{R}^d$ such that $\lim_{z\to 0} u(z) = 1$ for $d \geqslant 2$.

Further, is there a one-to-one relation between $H^{\frac{1}{2}}(\partial\Omega)$ and some class of functions defined in $\partial\Omega$? If the boundary of the domain is regular enough (existence of local bi-lipschitz, C^1 parameterizations should suffice, for instance), then the traces $H^{\frac{1}{2}+\varepsilon}(\partial\Omega)$ of $W^{1+\varepsilon,2}$ coincide with the Besov space $B^{1/2+\varepsilon}_{2,2}(\partial\Omega)$, with an appropriate definition using partitions of the unity and local parameterizations, see [Tri83, Section 3.3.3], for instance.

3.4 Solvability of the Dirichlet problem for continuous functions: the case of the unit ball

Definition 3.8. We say that the Dirichlet problem (3.1) in an open set Ω is *solvable for continuous functions* if there exists a function $u_f \in C(\overline{\Omega})$ for every $f \in C(\partial\Omega)$ such that $\Delta u = 0$ in Ω and $u(y) = f(y)$ for $y \in \partial\Omega$.

Note that such a solution would be unique by the Uniqueness Theorem 2.7.

Next we will study the sovability of the Dirichlet problem for continuous functions in the case Ω is the unit ball. First we will need to introduce the Green function in the unit ball, which has a nice algebraic expression.

Lemma 3.9. *Let $x, y \in \mathbb{R}^d \backslash \{0\}$. Then*

$$\left| \frac{x}{|x|} - |x|y \right| = \left| |y|x - \frac{y}{|y|} \right|.$$

Proof. Let $t \in \mathbb{R}$, $t > 0$. Then

$$\left| \frac{x}{t} - ty \right|^2 = \frac{|x|^2}{t^2} - 2x \cdot y + t^2 |y|^2.$$

Evaluating for $t = |x|$ and for $t = |y|^{-1}$ we reach the same expression. □

Define

$$v^x(y) := \begin{cases} -\mathcal{E}(\frac{x}{|x|} - |x|y) & \text{if } x \neq 0, \\ -\mathcal{E}(e_1) & \text{if } x = 0. \end{cases}$$

Note that for $|\xi| = 1$, $x \neq 0$ we get that $\left| \frac{x}{|x|} - |x|\xi \right| = |x - \xi|$ from the previous lemma, so $v^x(\xi) = -\mathcal{E}(x - \xi)$. The same happens when $x = 0$ because the fundamental solution depends only on the modulus. Moreover, for fixed $x \in B_1$, v^x has no singularity in B_1, given that

$$\frac{x}{|x|} - |x|y = 0 \implies y = \frac{x}{|x|^2} \implies y \notin B_1.$$

Therefore $v^x \in C^1(\overline{B_1}) \subset H^1(\Omega)$ and $\Delta v^x = 0$ in B_1. So the Green function (3.6) in the unit ball is

$$G^x(y) := \begin{cases} \mathcal{E}(x - y) - \mathcal{E}(\frac{x}{|x|} - |x|y) & \text{if } x \neq 0, \\ \mathcal{E}(-y) - \mathcal{E}(e_1) & \text{if } x = 0. \end{cases}$$

Note that $G^x(y) = G^y(x)$ by Lemma 3.9.

Now we can compute the Poisson kernel: for $x = 0$, $|\xi| = 1$, it is

$$\partial_\nu G^0(\xi) = \xi \cdot \nabla \mathcal{E}(\xi) \overset{(2.11)}{=} \xi \cdot \frac{-\xi}{\kappa_d |\xi|^d} = -\frac{1}{\kappa_d},$$

and for $x \neq 0$, $|\xi| = 1$ we get

$$\begin{aligned} \partial_\nu G^x(\xi) &= \xi \cdot \nabla_y \left(\mathcal{E}(x - y) - \mathcal{E}\left(\frac{x}{|x|} - |x|y \right) \right) |_{y=\xi} \\ &\overset{(2.11)}{=} \xi \cdot \left(\frac{x - \xi}{\kappa_d |x - \xi|^d} - \frac{\frac{x}{|x|} - |x|\xi}{\kappa_d \left| \frac{x}{|x|} - |x|\xi \right|^d} |x| \right) \\ &\overset{\text{L. 3.9}}{=} \xi \cdot \left(\frac{x - \xi - (x - |x|^2 \xi)}{\kappa_d \, |x - \xi|^d} \right) = |\xi|^2 \frac{|x|^2 - 1}{\kappa_d \, |x - \xi|^d} = \frac{|x|^2 - 1}{\kappa_d \, |x - \xi|^d}. \end{aligned}$$

Summing up, for $x \in B_1$ and $|\xi| = 1$ we get

$$P^x(\xi) = \frac{1-|x|^2}{\kappa_d\,|x-\xi|^d}. \tag{3.16}$$

Theorem 3.10. *Let $f \in L^1(\partial B_1)$ and define*

$$u_f(x) := \int_{\partial B_1} P^x(\zeta) f(\zeta)\, d\sigma(\zeta) \qquad \text{for } x \in B_1.$$

Then u is harmonic on B_1. If f is continuous, then $u_f \in C(\overline{B_1})$, with $u_f|_{\partial B_1} = f$. If $f \in L^p(\partial B_1)$, then $u_f(r\cdot) \to f$ in $L^p(\partial B_1)$ as $r \to 1$.

Proof. The function u_f is well defined because the Poisson kernel is bounded for x fixed. Since G is harmonic on x, P is also harmonic on x and so is u_f.

We claim that for every $x \in \partial B_1$, $P^x\, d\sigma$ is a probability measure, i.e.,

$$\int_{\partial B_1} P^x\, d\sigma = 1. \tag{3.17}$$

Indeed, for $x = 0$ it is trivial. By (3.16), the mean value theorem and Lemma 3.9 we get

$$\frac{1}{\kappa_d} \overset{(3.16)}{=} P^0\left(\frac{x}{|x|}\right) \overset{(2.2)}{=} \fint P^{|x|\xi}\left(\frac{x}{|x|}\right) d\sigma(\xi) \overset{\text{L. }3.9}{=} \fint P^x\,(\xi)\, d\sigma(\xi),$$

as claimed.

If f is continuous and $\xi \in \partial B_1$, then

$$\begin{aligned}
|f(\xi) - u_f(r\xi)| &\overset{(3.17)}{=} \left| \int_{\partial B_1} P^{r\xi}(\zeta)(f(\xi) - f(\zeta))\, d\sigma(\zeta) \right| \\
&\leqslant \int_{|\zeta-\xi|\leqslant\delta} \left|P^{r\xi}(\zeta)\right| |f(\xi) - f(\zeta)|\, d\sigma(\zeta) + \int_{|\zeta-\xi|>\delta} \left|P^{r\xi}(\zeta)\right| |f(\xi) - f(\zeta)|\, d\sigma(\zeta) \\
&\overset{(3.17)}{\leqslant} \sup_{|\zeta-\xi|\leqslant\delta} |f(\xi) - f(\zeta)| + 2\|f\|_\infty \sup_{|\zeta-\xi|>\delta} \left|P^{r\xi}(\zeta)\right|.
\end{aligned}$$

The first term in the right-hand side of the last estimate can be made arbitrarily small by fixing δ small enough, and then the second term can also be made small by choosing r close enough to 1. Choices can be made independently of ξ. This shows that $u_f(r\cdot)$ converges uniformly to u_f, and this implies global continuity.

If $f \in L^p(\partial B_1)$, then we can use the density of C^∞ on L^p to find a function $f_\varepsilon \in C^\infty(\partial B_1)$ with $\|f - f_\varepsilon\|_{L^p(\partial B_1)} \leqslant \varepsilon$. Now,

$$\|f - u_f(r\cdot)\|_{L^p(\partial B_1)} \leqslant \|f - f_\varepsilon\|_{L^p(\partial B_1)} + \|f_\varepsilon - u_{f_\varepsilon}(r\cdot)\|_{L^p(\partial B_1)} + \|u_{f_\varepsilon}(r\cdot) - u_f(r\cdot)\|_{L^p(\partial B_1)}.$$

Choosing ε small enough and r close enough to 1, the two first terms can be made arbitrarily small.

Regarding the last one, we claim that $\|u_{f_\varepsilon}(r\cdot) - u_f(r\cdot)\|_{L^p(\partial B_1)} \leqslant \|f_\varepsilon - f\|_{L^p(\partial B_1)}$. Indeed, for $p = 1$ we have

$$\|u_g(r\cdot)\|_{L^1(\partial B_1)} \leqslant \int_{\partial B_1}\int_{\partial B_1} P^{r\xi}(\zeta)\,|g(\zeta)|\,d\sigma(\zeta)\,d\sigma(\xi) \leqslant \|g\|_{L^1(\partial B_1)}\int_{\partial B_1} P^{r\xi}(\zeta)\,d\sigma(\xi).$$

Note that the mean value theorem

$$\int_{\partial B_1} P^{r\xi}(\zeta)\,d\sigma(\xi) = \kappa_d P^0(\zeta) = 1,$$

so $g \mapsto u_g$ is bounded in $L^1(\partial B_1)$ with norm 1. On the other hand,

$$\|u_g(r\cdot)\|_{L^\infty(\partial B_1)} \leqslant \sup_{\xi\in\partial B_1}\int_{\partial B_1} P^{r\xi}(\zeta)\,|g(\zeta)|\,d\sigma(\zeta) \leqslant \|g\|_\infty \sup_{\xi\in\partial B_1}\int_{\partial B_1} P^{r\xi}(\zeta)\,d\sigma(\zeta) \overset{(3.17)}{=} \|g\|_\infty.$$

By interpolation we get that $f \mapsto u_f(r\cdot)$ is a bounded operator in $L^p(\partial B_1)$ with norm 1. This fact proves the claim and, therefore, the L^p convergence follows. □

Remark 3.11. For the ball $B_r(0)$, with $r > 0$, we have a similar result. In this case the Poisson kernel for $B_r(0)$ equals

$$P^x_{B_r(0)}(\xi) = \frac{r^2 - |x|^2}{\kappa_d\, r\, |x-\xi|^d}.$$

Then the same result as in Theorem 3.10 holds for $f \in L^1(\partial B_r(0))$, with $P^x(\zeta)$ replaced by $P^x_{B_r(0)}(\zeta)$. That is, the function

$$u_f(x) := \int_{\partial B_r(0)} P^x_{B_r(0)}(\zeta)\,d\sigma(\zeta) \qquad \text{for } x \in B_r(0),$$

solves the Dirichlet problem with boundary data f in $B_r(0)$ when f is continuous. Also, for $f \in L^p(\partial B_r(0))$, we have that $u_f(r\cdot) \to f$ in $L^p(\partial B_r(0))$ as $r \to 1$.

3.5 Double layer potential: exploiting the jump formulas

When a domain Ω has bounded and smooth boundary, say $\partial\Omega \in C^{1+\epsilon}$, then a usual way to solve the Dirichlet problem (3.1) for continuous functions is via the double layer potential. We will not prove here the results, but we will sketch the main ideas, which can be found for instance in [Fol95, Chapter 3].

Consider the gradient of the fundamental solution

$$\nabla\mathcal{E}^x(y) = \frac{(x-y)}{\kappa_d|x-y|^d},$$

which is the kernel of the so-called Riesz transform of homogeneity $1-d$. In particular, the normal derivative of $\mathcal{E}$ in the boundary of Ω,

$$K^x(\zeta) := \partial_\nu \mathcal{E}^x(\zeta) = \nu(\zeta)\cdot\nabla\mathcal{E}^x(\zeta) = \frac{(x-\zeta)\cdot\nu(\zeta)}{\kappa_d|x-\zeta|^d}$$

for $\zeta \in \partial\Omega$ and $x \in \mathbb{R}^d\backslash\{\zeta\}$ is well defined whenever $\partial\Omega$ has C^1 parameterizations. Then for every $g \in C(\partial\Omega)$ and every $x \in \mathbb{R}^d\backslash\partial\Omega$, we can consider the double layer potential

$$\mathcal{D}g(x) := \int_{\partial\Omega} K^x(\zeta)g(\zeta)d\sigma(\zeta),$$

which is harmonic in $(\partial\Omega)^c$.

The double layer potential is not well defined a priori in the boundary of the domain, but it makes sense to define its principal value for $\xi \in \partial\Omega$ as

$$T_K(g)(\xi) := \text{p.v.}\mathcal{D}g(\xi) = \lim_{\varepsilon\to 0}\int_{\partial\Omega\backslash B_\varepsilon(\xi)} K^x(\zeta)g(\zeta)d\sigma(\zeta). \tag{3.18}$$

This pointwise definition does not coincide with the (non-tangential) limit of the double layer potential,

$$\mathcal{D}g(\xi) := \text{n.t.}\lim_{x\to\xi}\mathcal{D}g(x) = \lim_{x\to\xi:2\mathrm{d}_\Omega(x)\geq|x-\xi|}\mathcal{D}g(x),$$

where $\mathrm{d}_\Omega(x) = \mathrm{dist}(x,\partial\Omega)$. However, they are related by the so-called jump formula:

$$\mathcal{D}g(\xi) = \frac{1}{2}g(\xi) + T_K(g)(\xi),$$

which is a consequence of the identities

$$\int K^x(\zeta)\,d\sigma(\zeta) = \begin{cases} 1 & \text{if } x\in\Omega, \\ 1/2 & \text{if } x\in\partial\Omega, \text{ understood as a principal value,} \\ 0 & \text{if } x \in \overline{\Omega}^c. \end{cases}$$

When the boundary has parameterizations in $C^{1+\varepsilon}$, the normal vector becomes Hölder continuous and the singularity of K^x is of homogeneity below $d-1$, and it is therefore integrable with respect to the surface measure, so we can omit the principal value in (3.18). Then the kernel K^x becomes somewhat *smoothing* in this case, in the sense that T_K maps $L^\infty(\partial\Omega)$ to $C(\partial\Omega)$ for instance, and it is compact in $L^2(\partial\Omega)$, and the operator $\frac{1}{2}\mathrm{I}+T_K$ is Fredholm in $L^2(\partial\Omega)$. Moreover, if $(\frac{1}{2}\mathrm{I}+T_K)(g)\in C(\Omega)$ with $g\in L^2(\partial\Omega)$, then $g \in C(\partial\Omega)$.

In fact, if Ω is simply connected and $C^{1+\varepsilon}$, then $\frac{1}{2}\mathrm{I}+T_K$ happens to be invertible in $L^2(\partial\Omega)$. Thus, given $f\in C(\Omega)$, one can find a unique solution to the Dirichlet problem by finding the unique solution to the equation $f = (\frac{1}{2}\mathrm{I}+T_K)(g)$. Then $u := \mathcal{D}(g)$, i.e. $u = \mathcal{D}(\frac{1}{2}\mathrm{I}+T_K)^{-1}(f)$ satisfies (3.1) in the sense that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \text{n.t.}\lim_{x\to\xi} u(x) = f(\xi) & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

If Ω is multiply connected, some modifications related to the connectivity of the complement need to be done in order to find an inverse operator in a suitable function space.

The Dirichlet problem in the unbounded component can also be solved in this way, and assuming a priori that the solution u_f satisfies that $u_f(x) = O_{x\to\infty}(|x|^{3-d})$ one can get also uniqueness.

4 Basic results from measure theory

4.1 Measures

Following [Mat95], we will define a measure on a set X as a function on the parts of X, regardless of the σ-algebra of measurable sets. This is often called exterior measure in some references, but it is quite elementary to define the σ-algebra of measurable sets once the (exterior) measure is given. Conversely, every countably additive non-negative set function on a σ-algebra of subsets of X can be extended to every set, see [Mat95]. Let us assume that X is a metric space.

Definition 4.1. We say that $\mu : \{A : A \subset X\} \to \mathbb{R}$ is a measure if

1. $\mu(\varnothing) = 0$,
2. $\mu(A) \leqslant \mu(B)$ whenever $A \subset B \subset X$ and
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$, whenever $A_i \subset X$ for every $1 \leqslant i < \infty$.

We say that $A \subset X$ is μ-measurable if

$$\mu(E) = \mu(E \cap A) + \mu(E \backslash A) \qquad \text{for every } E \subset X.$$

Definition 4.2. Given a set X, we say that a collection Σ of subsets of X is a *σ-algebra* whenever Σ is closed under complement, countable unions, and countable intersections. When X is a topological space, we define the collection of *Borel sets of X* as the minimal σ-algebra containing all the open sets in the topology.

Lemma 4.3. *The measurable sets form a σ-algebra. If $\{A_i\}_{i=1}^{\infty}$ is a collection of μ-measurable and pairwise disjoint sets, then*

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i). \tag{4.1}$$

If $B_i \nearrow B$, i.e., if $B_1 \subset B_2 \subset \cdots$ and $B = \bigcup_i B_i$, then $\mu(B) = \lim_i \mu(B_i)$.

If $C_i \searrow C$, i.e., if $C_1 \supset C_2 \supset \cdots$ and $C = \bigcap_i C_i$, and moreover $\mu(C_1) < +\infty$, then $\mu(C) = \lim_i \mu(C_i)$.

Definition 4.4. Let μ be a measure on a metric space X.

1. μ is a Borel measure if all Borel sets are μ-measurable.
2. μ is a Radon measure if it is Borel,

a) $\mu(K) < \infty$ for every compact set $K \subset X$,

b) $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact}\}$ for every open set $V \subset X$,

c) $\mu(A) = \inf\{\mu(V) : V \supset A \text{ is open}\}$ for every set $A \subset X$.

3. In those cases, if the metric space is separable we say that $\operatorname{supp}\mu := \bigcap_{F=\overline{F}:\, \mu(F^c)=0} F$.

4.2 Integration

Let μ be a measure in $\mathbb{R}^d$. We say that $\phi : \mathbb{R}^d \to \mathbb{R}$ is a simple function whenever there exist a finite number of μ-measurable sets $\{A_j\}_{j=1}^N$ and coefficients $\{\alpha_j\}_{j=1}^N \subset \mathbb{R}$ such that

$$\phi = \sum_{j=1}^N \alpha_j \chi_{A_j}.$$

We can define its integral by

$$\int \phi \, d\mu := \sum_{j=1}^N \alpha_j \mu(A_j).$$

The set of simple functions is denoted by $\mathcal{S}_\mu$. Note that for $\phi \in \mathcal{S}_\mu$, the decomposition described above is not unique, but its choice does not change the value of the integral. Given a non-negative measurable function $f : \mathbb{R}^d \to \mathbb{R}$ (i.e., a function such that $f^{-1}(r, +\infty)$ is measurable for every $r \in \mathbb{R}$), we define its integral

$$\int f \, d\mu := \sup\left\{ \int \phi \, d\mu : \phi \in \mathcal{S}_\mu \text{ with } 0 \leqslant \phi \leqslant f \right\}.$$

Integration in measurable subsets is defined as

$$\int_A f \, d\mu := \int f \chi_A \, d\mu.$$

Theorem 4.5 (Fubini's theorem)**.** *Suppose that μ, ν are locally finite Borel measures on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. If f is a non-negative Borel function on $\mathbb{R}^{d_1+d_2}$, then*

$$\int\int f(x,y) \, d\mu(x) \, d\nu(y) = \int\int f(x,y) \, d\nu(y) \, d\mu(x).$$

Corollary 4.6. *Suppose that μ is a locally finite Borel measure on $\mathbb{R}^d$. If f is a non-negative Borel function on $\mathbb{R}^d$, then*

$$\int f(x) \, d\mu(x) = \int_0^\infty \mu(\{x \in \mathbb{R}^d : f(x) \geqslant t\}) \, dt.$$

Given a μ-measurable function $f : \mathbb{R}^d \to \mathbb{R}$, and $0 < p < \infty$, we say that $f \in L^p(\mu)$ whenever $\int |f|^p < +\infty$. In case $f \in L^1(\mu)$, we can define

$$\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu,$$

where

$$f_+ := \max\{f, 0\}, \qquad \text{and} \qquad f_- := \max\{-f, 0\}.$$

Note that $f = f_+ - f_-$, with $f_+, f_- \geqslant 0$.

4.3 The Riesz representation theorem

Theorem 4.7 (Riesz representation Theorem)**.** *Let X be a locally compact metric space and $L : C_c(X) \to \mathbb{R}$ a positive linear functional. Then there is a unique Radon measure μ such that*

$$L_f = \int f \, d\mu \qquad \text{for } f \in C_c(X).$$

The approach presented below is based on the proof of [Rud87, Chapter 2], where the reader may find all the details and the proofs of every single lemma used here.

Proof. Given an open set $V \subset X$ we write $f \prec V$ whenever $f \in C_c(V)$, and $0 \leqslant f \leqslant \chi_V$. We define

$$\mu(V) := \sup\{L_f : f \prec V\}.$$

Note that for open sets $U \subset V$ it follows immediately that $\mu(U) \leqslant \mu(V)$. Therefore it makes sense to define for every $E \subset X$

$$\mu(E) := \inf\{\mu(V) : V \supset E \text{ and } V \text{ is open}\}.$$

We will use often the following immediate consequence of the positivity of L_f:

$$\text{If } f, g \in C_c(X) \text{ are such that } 0 \leqslant f \leqslant g, \text{ then } L_f \leqslant L_g \tag{4.2}$$

First we claim that μ is a measure.

1. Since $\varnothing$ is open, $\mu(\varnothing) = \sup\{L_f : f \prec \varnothing\} = L_0 = 0$.

2. Given sets $A \subset B \subset X$,

$$\{V : V \supset A \text{ and } V \text{ is open}\} \supset \{V : V \supset B \text{ and } V \text{ is open}\}$$

trivially, and taking infimum in a subset always increases the result, so

$$\mu(A) \leqslant \mu(B). \tag{4.3}$$

3. Let $A_i \subset X$ for $1 \leqslant i < \infty$, and let $\varepsilon > 0$. Consider open sets $V_i \supset A_i$ such that $\mu(V_i) \leqslant \mu(A_i) + \frac{\epsilon}{2^i}$, and let $f \prec V := \bigcup_i V_i$ so that $\mu(V) \leqslant L_f + \varepsilon$.

   Since $K := \operatorname{supp} f$ is compactly contained in V we infer that there exist $n \in \mathbb{N}$ and a finite subcovering, i.e., a subset $\{i_j\}_{j=1}^n \subset \mathbb{N}$ so that $K \subset \bigcup_{j=1}^n V_{i_j}$.

   There exists a partition of the unity in K for the covering V_{i_j}, i.e., there exist functions $h_j \prec V_{i_j}$ with $\chi_K \leqslant \sum_j h_j \leqslant 1$. Then

$$\begin{aligned} \mu\left(\bigcup_i A_i\right) &\leqslant \mu(V) \leqslant L_f + \varepsilon = L_{f\sum_j h_j} + \varepsilon = \sum_j L_{fh_j} + \varepsilon \\ &\leqslant \sum_j \mu\left(V_{i_j}\right) + \varepsilon \leqslant \sum_i \left(\mu(A_i) + \frac{\varepsilon}{2^i}\right) + \varepsilon \leqslant \sum_i \mu(A_i) + 2\varepsilon, \end{aligned} \tag{4.4}$$

   concluding the proof that μ is a mesaure.

Next we show that μ is in fact a Radon measure. To show that we begin by $a) - c)$ in Definition 4.4:

a) Let $K \subset X$ be a compact set. Then K is contained in a ball B. Consider a continuous function $\chi_K \leqslant f \leqslant \chi_B$, which exists by Urysohn's lemma. Then call $V := \{x : f(x) > 1/2\}$. Every function $g \prec V$ satisfies that $g \leqslant 2f$. Therefore

$$\mu(K) \leqslant \mu(V) = \sup\{L_g : g \prec V\} \overset{(4.2)}{\leqslant} 2L_f < \infty.$$

b) Let V be an open set. We will prove that its measure coincides with the supremum of the measures of its compact subsets. Let $\varepsilon > 0$ and $f \prec V$ such that $\mu(V) \leqslant L_f + \varepsilon$. Then write $K := \operatorname{supp} f$ and consider an open set $U \supset K$. It is clear that $f \prec U$ and thus $\mu(U) > L_f$. Since this holds for every such U, passing to the infimum we can infer that $\mu(K) \geqslant L_f$. All in all,

$$\mu(V) \leqslant L_f + \varepsilon \leqslant \mu(K) + \varepsilon.$$

   Since such a compact set can be obtained for every ε, we conclude that

$$\mu(V) \leqslant \sup\{\mu(K) : K \subset V\}.$$

   The converse inequality follows from (4.3).

c) $\mu(E) := \inf\{\mu(V) : V \supset E \text{ and } V \text{ is open}\}$ follows by definition.

To complete the proof that μ is Radon, we will check that it is Borel regular. First of all, let K_1, K_2 be compact, disjoint subsets of X. We claim that

$$\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2). \tag{4.5}$$

Indeed, it is well known that there exist open sets $V_i \supset K_i$, such that $V_1 \cap V_2 = \varnothing$ (see [Rud87, Theorem 2.7], for instance), and also there exists an open set $W \supset K_1 \cup K_2$

such that $\mu(W) < \mu(K_1 \cup K_2) + \varepsilon$. Moreover, there exist functions $f_i \prec V_i \cap W$ so that $\mu(V_i \cap W) \leqslant L_{f_i} + \varepsilon$. Then, since the supports of f_i are disjoint, $f_1 + f_2 \prec W$ and we get

$$\begin{aligned} \mu(K_1) + \mu(K_2) &\overset{(4.3)}{\leqslant} \mu(V_1 \cap W) + \mu(V_2 \cap W) \leqslant L_{f_1} + L_{f_2} + 2\varepsilon \\ &= L_{f_1+f_2} + 2\varepsilon \leqslant \mu(W) + 2\varepsilon < \mu(K_1 \cup K_2) + 3\varepsilon, \end{aligned}$$

proving the claim.

Since the μ-measurable sets form a σ-algebra, to show that μ is a Borel measure we only need to check that every open set V is μ-measurable, i.e., every $E \subset X$ satisfies that

$$\mu(E) = \mu(E \cap V) + \mu(E \cap V^c).$$

By the subadditivity shown in (4.4), it suffices to prove that

$$\mu(E) \geqslant \mu(E \cap V) + \mu(E \cap V^c) \tag{4.6}$$

and for this we may assume that $\mu(E) < \infty$.

First let us assume that E is an open set with finite measure. Then write $\widetilde{V} = V \cap E$, so $E \cap V^c = E \cap (V^c \cup E^c) = E \cap (V \cap E)^c = E \cap \widetilde{V}^c$, i.e. we have to show that

$$\mu(E) \geqslant \mu(\widetilde{V}) + \mu(E \cap \widetilde{V}^c).$$

Let $K_1 \subset \widetilde{V}$ be a compact set such that

$$\mu(\widetilde{V}) \leqslant \mu(K_1) + \varepsilon.$$

Then consider an open set $U \supset E \cap \widetilde{V}^c$ so that $\mu(U) \leqslant \mu(E \cap \widetilde{V}^c) + \varepsilon$. Define $\widetilde{U} := U \cap E \cap K_1^c$ which is again an open set. Then

$$\mu(\widetilde{U}) \overset{(4.3)}{\leqslant} \mu(U) \leqslant \mu(E \cap \widetilde{V})^c + \varepsilon,$$

and

$$E \cap \widetilde{V}^c = U \cap E \cap \widetilde{V}^c \subset U \cap E \cap K_1^c = \widetilde{U} \subset K_1^c \cap E. \tag{4.7}$$

To end consider a compact set $K_2 \subset \widetilde{U}$ such that $\mu(\widetilde{U}) \leqslant \mu(K_2) + \varepsilon$. All in all,

$$\begin{aligned} \mu(\widetilde{V}) + \mu(E \cap \widetilde{V}^c) &\overset{(4.7)}{\leqslant} \mu(K_1) + \varepsilon + \mu(\widetilde{U}) \leqslant \mu(K_1) + \mu(K_2) + 2\varepsilon \\ &\overset{(4.5)}{=} \mu(K_1 \cup K_2) + 2\varepsilon \overset{(4.3)}{\leqslant} \mu(E) + 2\varepsilon, \end{aligned}$$

and (4.6) follows for open sets.

Consider a set $E \subset X$ (without the openness assumption). Then there exists an open set $V_E \supset E$ such that $\mu(V_E) \leqslant \mu(E) + \varepsilon$. Then

$$\mu(E \cap V) + \mu(E \cap V^c) \overset{(4.3)}{\leqslant} \mu(V_E \cap V) + \mu(V_E \cap V^c) = \mu(V_E) \leqslant \mu(E) + \varepsilon,$$

proving (4.6) for general sets.

To end we have to check that $L_f = \int f\,d\mu$ for every $f \in C_c(X)$. For simplicity we may assume that f is real valued. Moreover, it suffices to show

$$L_f \leqslant \int f\,d\mu, \tag{4.8}$$

since we can apply the same inequality to $-f$ to obtain the converse estimate.

Let $[a,b] \cup \{0\}$ be the range of f. For every n consider $\{y_i\}_{i=0}^{n+1}$ with $y_0 < a$, $y_{n+1} = b$ and $0 < y_{i+1} - y_i \leqslant (b-a)/n =: \varepsilon$ for every $i \leqslant n$. Let $E_i := f^{-1}((y_{i-1}, y_i]) \cap \operatorname{supp} f$, which are Borel sets and, thus, measurable. Consider open sets $V_i \supset E_i$ with $\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{n+1}$ and such that $f(x) < y_i + \varepsilon$ for every $x \in V_i$; and let h_i be a partition of the unity of $\operatorname{supp} f$ with respect to the covering $\{V_i\}$, that is $h_i \prec V_i$ with $\chi_{\{\operatorname{supp} f\}} \leqslant \sum_i h_i \leqslant 1$. Then

$$\begin{aligned} L_f &= \sum_i L_{h_i f} \overset{(4.2)}{\leqslant} \sum_i (y_i + \varepsilon) L_{h_i} \leqslant \sum_i (y_i + \varepsilon)\mu(V_i) \leqslant \sum_i (y_i - \varepsilon + 2\varepsilon)\left(\mu(E_i) + \frac{\varepsilon}{n+1}\right) \\ &= \sum_i \mu(E_i)(y_i - \varepsilon) + 2\varepsilon \sum_i \mu(E_i) + \frac{\varepsilon}{n+1}\sum_i y_i + \varepsilon^2 \overset{(4.1)}{\leqslant} \int f\,d\mu + \varepsilon(2\mu(\operatorname{supp} f) + b + \varepsilon) \end{aligned}$$

and (4.8) follows choosing ε arbitrarily small.

As for uniqueness, assume that μ_1, μ_2 are Radon measures satisfying the hypotheses of the Theorem. Since Radon measures are determined by their values on compact sets, we only need to check that $\mu_1(K) = \mu_2(K)$ for every compact set $K \subset X$. Consider such a compact set, and let $V \supset K$ be an open set such that $\mu_2(V) \leqslant \mu_2(K) + \varepsilon$. By Urysohn's lemma, there exists $f \prec V$ such that $\chi_K \leqslant f$. Then

$$\mu_1(K) = \int \chi_K\,d\mu_1 \leqslant \int f\,d\mu_1 = L_f = \int f\,d\mu_2 \leqslant \int \chi_V\,d\mu_2 = \mu_2(V) \leqslant \mu_2(K) + \varepsilon.$$

□

4.3.1 Image measure

Definition 4.8. The image of a measure μ under a mapping $f : X \to Y$ (also known as *push-forward measure*) is defined by $f_\# \mu(A) = \mu(f^{-1}(A))$ for $A \subset Y$.

Theorem 4.9. *If X, Y are separable metric spaces, f is continuous and μ is a compactly supported Radon measure, then $f_\#\mu$ is a Radon measure, with $\operatorname{supp} f_\#\mu = f(\operatorname{supp}\mu)$.*

Theorem 4.10. *If X, Y are metric spaces, f is a Borel mapping, μ is a Borel measure and g is a nonnegative Borel function, then*

$$\int g\,df_\#\mu = \int (g \circ f)\,d\mu.$$

4.3.2 Weak convergence

Let $\{\mu_i\}_{i=0}^{\infty}$ be a collection of Radon measures in a metric space X. We say that μ_i converge weakly to μ, and write

$$\mu_i \rightharpoonup \mu_0,$$

if

$$\lim_{i\to\infty} \int \varphi \, d\mu_i = \int \varphi \, d\mu \quad \text{for every } \varphi \in C_c(X).$$

As a consequence of the Riesz representation theorem, one can prove that a uniformly locally finite collection of measures has a weakly convergent subsequence:

Theorem 4.11. *If $\{\mu_i\}_{i=1}^{\infty}$ is a collection of Radon measures in $\mathbb{R}^d$, with*

$$\sup_i \mu_i(K) < +\infty,$$

for every compact set $K \subset \mathbb{R}^d$, then there is a weakly convergent subsequence $\{\mu_{i_k}\}_{k=1}^{\infty}$, and a Radon measure μ with

$$\mu_{i_k} \rightharpoonup \mu.$$

Consider the Dirac delta measure δ_i in $i \in \mathbb{N}$. Note that the sequence $\delta_i \rightharpoonup 0$. This example shows that the weak convergence of measures does not imply the convergence of the measure of a particular set. However, the following semicontinuity properties hold:

Theorem 4.12. *Let $\{\mu_i\}_{i=0}^{\infty}$ be a collection of Radon measures in a locally compact metric space X. If $\mu_i \rightharpoonup \mu_0$, $K \subset X$ is compact and $G \subset X$ is open, then*

$$\mu(K) \geqslant \limsup_{i\to\infty} \mu_i(K),$$

and

$$\mu(G) \leqslant \liminf_{i\to\infty} \mu_i(G).$$

4.4 Hausdorff measure and dimension

For every subset $A \subset \mathbb{R}^d$, $0 \leqslant s < +\infty$ and $0 < \delta \leqslant +\infty$, define

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_i \operatorname{diam}(E_i)^s : A \subset \bigcup_i E_i \text{ with } \operatorname{diam}(E_i) \leqslant \delta \right\},$$

and let

$$\mathcal{H}^s(A) := \lim_{\delta \searrow 0} \mathcal{H}^s(A)$$

be the *s-dimensional Hausdorff measure of A.* The quantity $\mathcal{H}^s_\infty(A)$ also plays an important role and is called *s-dimensional Hausdorff content of A.* The Hausdorff measure happens to be a Radon measure. The 0-dimensional Hausdorff measure is the counting measure, the 1-dimensional measure is a generalization of the length measure in $\mathbb{R}^d$, and the d-dimensional measure is a multiple of the Lebesgue measure.

If A is a set with $\mathcal{H}^s(A) < +\infty$, then $\mathcal{H}^s|_A$ is locally finite and, in fact, it happens to be a Radon measure (see [Mat95, chapter 4]).

Another interesting fact is that although

$$\mathcal{H}^s_\infty(A) \leqslant \mathcal{H}^s_\delta(A) \nearrow \mathcal{H}^s(A),$$

having null Hausdorff content is equivalent to having zero Hausdorff measure:

$$\mathcal{H}^s_\infty(A) = 0 \iff \mathcal{H}^s(A) = 0.$$

Theorem 4.13. *For $0 \leqslant s < t < \infty$ and $A \subset \mathbb{R}^d$,*

1. *$\mathcal{H}^s(A) < +\infty$ implies $\mathcal{H}^t(A) = 0$, and*
2. *$\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s(A) = +\infty$.*

This leads to the concept of Hausdorff dimension:

Definition 4.14. The *Hausdorff dimension* of a set $A \subset \mathbb{R}^d$ is

$$\dim_{\mathcal{H}} A = \sup\{s : \mathcal{H}^s(A) > 0\}.$$

From the previous theorem, one can infer that

$$\dim_{\mathcal{H}} A = \sup\{s : \mathcal{H}^s(A) = +\infty\} = \inf\{s : \mathcal{H}^s(A) < +\infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

4.5 Frostman's lemma

The following result is Frostman's Lemma, which is a fundamental tool in geometric measure theory and in potential theory.

Theorem 4.15. *Let E be a Borel set in $\mathbb{R}^d$. Then $\mathcal{H}^s(E) > 0$ if and only if there exists a finite Radon measure μ compactly supported in E such that*

$$\mu(B_r(x)) \leqslant r^s \quad \text{for every } x \in \mathbb{R}^d \text{ and } r > 0.$$

Further,

$$\mathcal{H}^s_\infty(E) \approx \sup\left\{\mu(E) : \operatorname{supp}\mu \subset E,\, \mu(B_r(x)) \leqslant r^s \text{ for every } x \in \mathbb{R}^d \text{ and } r > 0\right\},$$

with the implicit constant depending only on d.

Below we provide a proof for the case when E is a compact set. The case when E is σ-compact is easily deduced from this. These two cases suffice for the purposes of these notes.

Proof. Suppose first that such a measure μ exists, and let us see that $\mathcal{H}^s_\infty(E) \gtrsim \mu(E)$. Indeed, consider a covering $\bigcup_i A_i \supset E$, and take for each i a point $x_i \in A_i$. Since the union of the balls $B_{\mathrm{diam}(A_i)}(x_i)$ covers E, we get

$$\sum_i \mathrm{diam}(A_i)^s \geqslant c^{-1} \sum_i \mu\big(B_{\mathrm{diam}(A_i)}(x_i)\big) \geqslant c^{-1}\mu(E).$$

Taking the infimum over all possible coverings of E, we obtain $\mathcal{H}^s_\infty(E) \geqslant c^{-1}\,\mu(E)$.

For the converse implication of the theorem, assume that E is contained in a dyadic cube Q_0. The measure μ will be constructed as a weak limit of measures μ_n, $n \geqslant 0$. The first measure is

$$\mu_0 = \mathcal{H}^s_\infty(E)\,\frac{\mathcal{L}^d|_{Q_0}}{\mathcal{L}^d(Q_0)}.$$

For $n \geqslant 1$, each measure μ_n vanishes in $\mathbb{R}^d \backslash Q_0$, it is absolutely continuous with respect to Lebesgue measure, and in each cube from $\mathcal{D}_n(Q_0)$ (this is the family of dyadic n-descendants of Q_0), it has constant density. It is defined from μ_{n-1} as follows. If $P \in \mathcal{D}_n(Q_0)$ and P is a dyadic child of $Q \in \mathcal{D}_{n-1}(Q_0)$ (then we write $P \in \mathcal{C}h(Q)$), we set

$$\mu_n(P) = \frac{\mathcal{H}^s_\infty(P \cap E)}{\sum_{R \in \mathcal{C}h(Q)} \mathcal{H}^s_\infty(R \cap E)}\,\mu_{n-1}(Q). \tag{4.9}$$

Observe that

$$\sum_{P \in \mathcal{C}h(Q)} \mu_n(P) = \mu_{n-1}(Q) \qquad \text{for all } Q \in \mathcal{D}_{n-1}(Q_0),$$

and thus $\mu_n(\mathbb{R}^d) = \mu_{n-1}(\mathbb{R}^d)$.

As said above, μ is just a weak limit of the measures μ_n. The fact that μ is supported on E is easy to check: from the definition of μ_n in (4.9), $\mu_n(P) = 0$ if $P \in \mathcal{D}_n(Q_0)$ does not intersect E. As a consequence, $\mu_k(P) = 0$ for all $k \geqslant n$ too, and thus,

$$\mathrm{supp}(\mu_k) \subset \mathcal{U}_{2^{-n+1}\mathrm{diam}(Q_0)}(E) \qquad \text{for all } k \geqslant n.$$

From this condition, one gets that $\mathrm{supp}(\mu) \subset \mathcal{U}_{2^{-n+1}\mathrm{diam}(Q_0)}(E)$, for all $n \geqslant 0$, which proves the claim.

Next we will show that

$$\mu_n(P) \leqslant \mathcal{H}^s_\infty(P \cap E) \qquad \text{for all } P \in \mathcal{D}_n(Q_0).$$

This follows easily by induction: it is clear for $n = 0$, and if it holds for $n-1$ and Q is the dyadic parent of P, then

$$\mu_{n-1}(Q) \leqslant \mathcal{H}^s_\infty(Q \cap E) \leqslant \sum_{R \in \mathcal{C}h(Q)} \mathcal{H}^s_\infty(R \cap E).$$

Thus, from (4.9), we infer that $\mu_n(P) \leqslant \mathcal{H}^s_\infty(P \cap E)$, as claimed. As a consequence, for all $j \geqslant n$,

$$\mu_j(P) \leqslant \mathcal{H}^s_\infty(P \cap E) \qquad \text{for all } P \in \mathcal{D}_n(Q_0).$$

Moreover, by construction, all the dyadic cubes which do not intersect Q_0 have zero measure μ_j.

Since every open ball B_r of radius r with $2^{-n-1}\ell(Q_0) \leqslant r < 2^{-n}\ell(Q_0)$ is contained in a union of at most 2^d dyadic cubes P_k with side length $2^{-n}\ell(Q_0)$, we get

$$\mu_j(B_r) \leqslant \sum_{k=1}^{2^d} \mu_j(P_k) \leqslant \sum_{k=1}^{2^d} \mathcal{H}^s_\infty(P_k \cap E) \leqslant 2^d \operatorname{diam}(P_k)^s \leqslant c\, r^s,$$

for all $j \geqslant n$. Letting $j \to \infty$, we infer that $\mu(B_r) \leqslant c\, r^s$.

So we have constructed a measure μ supported on E such that $\mu(E) = \mathcal{H}^s_\infty(E)$ with $\mu(B_r(x)) \leqslant r^s$ for all $x \in \mathbb{R}^d$ and all $r > 0$, which implies

$$\mathcal{H}^s_\infty(E) \lesssim \sup\left\{\mu(E) : \operatorname{supp}\mu \subset E,\, \mu(B_r(x)) \leqslant r^s \;\forall x \in \mathbb{R}^d, r > 0\right\}.$$

□

5 Harmonic measure via Perron's method

To solve the Dirichlet problem for a very general class of open sets, it is convenient to use harmonic measure. Before introducing this notion, we will introduce subharmonic functions and we will show the solution of the Dirichlet problem via Perron's method.

5.1 Subharmonic functions

Definition 5.1. For $\Omega \subset \mathbb{R}^d$ open, we say that $u : \Omega \to [-\infty, \infty)$ is *subharmonic* if it is upper semicontinuous in Ω and $u(x) \leqslant ⨍_{B_r(x)} u$ whenever $B_r(x) \subset\subset \Omega$.

On the other hand, $u : \Omega \to (-\infty, +\infty]$ is *superharmonic* if it lower semicontinuous and $u(x) \geqslant ⨍_{B_r(x)} u$ whenever $B_r(x) \subset\subset \Omega$.

Recall that u is called upper semicontinuous at $x \in \Omega$ if $\limsup_{y\to x} u(y) \leqslant u(x)$, and it is lower semicontinuous if $\liminf_{y\to x} u(y) \geqslant u(x)$. It is easily checked that, if K is compact and $u : K \to [-\infty, \infty)$ is upper semicontinuous, then u attains the maximum on K. Analogously, if $u : K \to (-\infty, \infty]$ is lower semicontinuous, then u attains the minimum on K. Note that upper semicontinuity does not imply local Lebesgue integrability. However, the function is locally bounded above and therefore, the average $⨍_{B_r(x)} u$ in the previous definition is in $[-\infty, +\infty)$.

Of course, any function that is harmonic in Ω is both subharmonic and superharmonic. Further, u is subharmonic if and only if $-u$ is superharmonic. Other immediate properties are stated below.

Lemma 5.2. *If u, v are subharmonic in Ω, then $u+v$ and $\max(u, v)$ are both subharmonic in Ω. On the other hand, if u, v are superharmonic in Ω, then $u + v$ and $\min(u, v)$ are both superharmonic in Ω.*

Proof. This is immediate. □

Subharmonic functions satisfy the maximum principle (and superharmonic functions satisfy the minimum principle):

Lemma 5.3 (Maximum principle). *If u is a subharmonic function in a bounded open set Ω such that*

$$\limsup_{x\to\xi} u(x) \leqslant 0 \quad \text{for every } \xi \in \partial\Omega,$$

then $u \leqslant 0$ in Ω. If moreover Ω is connected, then either $u \equiv 0$ or $u < 0$ in Ω.

Proof. By considering each component of Ω separately, we can assume that Ω is connected and it is enough to prove the second statement of the lemma. Suppose first that u does not achieve a supremum in Ω. If $x_j \in \Omega$ is such that $\lim_j u(x_j) = \sup_\Omega u$, then $\lim_j \operatorname{dist}(x_j, \partial\Omega) = 0$, for otherwise we could extract a subsequence converging to a point inside Ω and obtain a contradiction. Using that Ω is bounded, by passing to a subsequence we may assume that $x_j \to \xi \in \partial\Omega$. By assumption, this implies that every $x \in \Omega$ satisfies

$$u(x) < \sup_\Omega u = \lim_j u(x_j) \leqslant \limsup_{y\to\xi} u(y) \leqslant 0.$$

If u achieves the supremum at some $x \in \Omega$, then there exists r such that $B_r(x) \subset \Omega$. Assume that there exists $y \in B_r(x)$ such that $u(y) < u(x) = \sup_\Omega u$. Then, by upper semicontinuity we would get

$$\sup_\Omega u = u(x) \leqslant \fint_{B_r(x)} u < \sup_\Omega u,$$

reaching a contradiction. Therefore, the function is constant in the ball $B_r(x)$. This implies that the set where the supremum is achieved is open. But it is also relatively closed in Ω by semicontinuity and so u is constant in Ω. □

Next we give a couple of characterizations of subharmonicity under a certain priori regularity conditions. First, we check the behavior of the Laplacian when a subharmonic function has two derivatives, and then we use it to show that the fundamental solution to $-\Delta$, see (2.10), is an example of superharmonic function.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in C^2(\Omega)$. The function u is subharmonic in Ω if and only if $\Delta u \geqslant 0$ in Ω.*

Proof. The fact that $\Delta u \geqslant 0$ in Ω implies the subharmonicity of u is a direct consequence of Remark 2.2. To prove the converse implication, we have to show that $\Delta u(x) \geqslant 0$ for every $x \in \Omega$. To this end, consider the function

$$v(y) = u(y) - u(x) - \nabla u(x)\,(y - x).$$

Since u is subharmonic and any affine function is harmonic, it follows that v is also subharmonic. The Taylor expansion of v in x equals

$$v(y) = \frac{1}{2}\,(y - x)^T\, D^2u(x)\,(y - x) + o(|y - x|^2),$$

where $D^2u(x)$ is the Hessian matrix of u. For any ball $\overline{B_r(x)} \subset \Omega$, we have

$$\begin{aligned} 0 = v(x) \leqslant \fint_{B_r(x)} v\,dy &= \frac{1}{2}\fint_{B_r(x)} (y - x)^T\, D^2u(x)\,(y - x)\,dy + o(r^2) \\ &= \frac{1}{2}\sum_{i,j} \partial_{i,j}u(x) \fint_{B_r(x)} (y_i - x_i)\,(y_j - x_j)\,dy + o(r^2) \\ &= c\,\Delta u(x)\,r^2 + o(r^2), \end{aligned}$$

where we took into account that $\int_{B_r(x)}(y_i - x_i)\,(y_j - x_j)\,dy$ vanishes if $i \neq j$ and is positive otherwise. Dividing by cr^2, we deduce

$$\Delta u(x) + o(1) \geqslant 0,$$

with $o(1) \to 0$ as $r \to 0$. This implies that $\Delta u(x) \geqslant 0$, and the proof of the lemma is concluded. □

Lemma 5.5. *The fundamental solution of $-\Delta$ is harmonic in $\mathbb{R}^d \backslash \{0\}$ and superharmonic in $\mathbb{R}^d$.*

Proof. Harmonicity can be easily checked. To prove superharmonicity, notice first that $\mathcal{E}$ is lower semicontinuous. Next, for every $\varepsilon > 0$ let φ_ε be a C^∞, positive, radially decreasing, function supported on $B_\varepsilon(0)$ with $\int \varphi_\varepsilon = 1$. Then $\mathcal{E} * \varphi_\varepsilon \in C^\infty(\mathbb{R}^d)$. Further,

$$\Delta(\mathcal{E} * \varphi_\varepsilon) = -\varphi_\varepsilon \leqslant 0.$$

Thus, by Lemma 5.4, $\mathcal{E} * \varphi_\varepsilon$ is superharmonic in $\mathbb{R}^d$. Consequently, for any ball B centered in $x_0 \neq 0$ and any $\varepsilon > 0$,

$$⨍_B \mathcal{E} * \varphi_\varepsilon \leqslant \mathcal{E} * \varphi_\varepsilon(x_0).$$

Letting $\varepsilon \to 0$, we deduce

$$⨍_B \mathcal{E} \leqslant \mathcal{E}(x_0).$$

In case $x_0 = 0$, we have $\mathcal{E}(x_0) = +\infty$ and the last inequality is satisfied trivially. □

Next we characterize continuous subharmonic functions as those functions whose interior values in balls lie below the solution to the Dirichlet problem with the same boundary values.

Lemma 5.6. *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in C(\Omega)$. Then u is subharmonic if and only if for every ball $B \subset\subset \Omega$ and every harmonic function v such that $u(x) \leqslant v(x)$ for every $x \in \partial B$, it holds either $v > u$ or $v \equiv u$ in B.*

Proof. The only if implication follows by the maximum principle to the subharmonic function $u - v$. To see the converse, let $B_r(x) \subset\subset \Omega$ and let v be the harmonic function in B_r continuous up to the boundary that agrees with u on ∂B_r (see Theorem 3.10). Then

$$⨍_{\partial B_r} u\,d\sigma = ⨍_{\partial B_r} v\,d\sigma = v(x) \geqslant u(x).$$

Thus,

$$⨍_{B_r} u\,dm = \frac{d}{\kappa_d r^d} \int_0^r \int_{\partial B_t} u\,d\sigma\,dt = \frac{d}{r^d} \int_0^r ⨍_{\partial B_t} u\,d\sigma\, t^{d-1} dt \geqslant \frac{du(x)}{r^d} \int_0^r t^{d-1}\,dt = u(x).$$

□

Let $u \in C(\Omega)$ be subharmonic in a ball B. Let $\tilde{u}$ be the harmonic function in B that agrees with u on ∂B and set $U := \chi_{\Omega \setminus B} u + \chi_B \tilde{u}$. Note that $U \geqslant u$ by Lemma 5.6. This is called the *harmonic lift* of u in B.

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^d$ be open. If $u \in C(\Omega)$ is subharmonic in Ω, $x \in \Omega$ and $B = B_r(x) \subset\subset \Omega$, then the harmonic lift of u in B is also subharmonic in Ω.*

Proof. Let U be the harmonic lift of u in B. Consider v harmonic in a ball $B' \subset \Omega$ with $B' \cap B \neq \varnothing$ and $v \geqslant U$ in the boundary of B'. We want to prove that either $v > U$ or $v \equiv U$ in B'.

Case 1: $\partial B \cap B' = \varnothing$, that is $B' \subset B$ and U is harmonic in B'. Then the claim follows by Lemma 5.6 applied to U.

Case 2: $\partial B \cap B' \neq \varnothing$ and $v(y) > U(y)$ in $\partial B \cap B'$. Using the continuity of U and the maximum principle applied to $U - v$ in $B' \setminus B$ and $B' \cap B$ separately, we get that $v > U$ in B'.

Case 3: $\partial B \cap B' \neq \varnothing$ and there exists $y \in \partial B \cap B'$ such that $v(y) \leqslant U(y) = u(y)$. In this case, since $v \geqslant u$ in $\partial B'$, Lemma 5.6 implies that $v \equiv u$ in B'. If $\partial B' \cap B \neq \varnothing$, the identity $v \equiv u$ in B' implies the existence of a point in $\partial B' \cap B \neq \varnothing$ where $u(y) \leqslant U(y) \leqslant v(y) = u(y)$ and therefore $U \equiv u$ by Lemma 5.6. If, instead, $\partial B' \cap B = \varnothing$, that is if $B \subset B'$, then u is harmonic in B and, therefore, $U \equiv u$ as well and the claim follows. □

Next we provide a couple of properties of subharmonic functions, again under certain a priori conditions. First we see that subharmonicity is preserved by an approximation of the identity. Then we use this fact to show that subharmonic Sobolev functions are weakly subharmonic, see Remark 5.10 below. This properties will be used to show the Caccioppoli inequality for subharmonic functions.

Lemma 5.8. *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in L^1_{\mathrm{loc}}(\Omega)$ be subharmonic. For $\rho > 0$, denote $\Omega_\rho = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > \rho\}$. Then following holds:*

(a) If μ is a (non-negative) Radon measure supported in $B_\rho(0)$ and $u * \mu$ is upper semicontinuous in Ω_ρ, then $u * \mu$ is subharmonic in Ω_ρ.

(b) If φ be a continuous non-negative function supported in $B_\rho(0)$, then $u * \varphi$ is subharmonic in Ω_ρ.

Proof. Clearly, the statement (b) is a consequence of (a), since $u * \varphi$ is continuous because φ is continuous and compactly supported. To prove (a), we have to check that for any $x \in \Omega_\rho$ and $r > 0$ such that $B_r(x) \subset \Omega_\rho$, we have $u * \mu(x) \leqslant \fint_{B_r(x)} u * \mu \, dm$. Without loss of generality, assume $x = 0$ and that $B_r(0) \subset \Omega_\rho$. Denoting $\widetilde{u}(y) = u(-y)$ and $\widehat{\chi}_{B_r(0)} = m(B_r(0))^{-1} \chi_{B_r(0)}$, we have

$$\fint_{B_r(0)} u * \mu \, dm = \left\langle u * \mu, \widehat{\chi}_{B_r(0)} \right\rangle = \left\langle \mu, \widetilde{u} * \widehat{\chi}_{B_r(0)} \right\rangle.$$

Notice now that for any $y \in \operatorname{supp}\mu$, $B_\rho(y) \subset \Omega$ (because $B_r(0) \subset \Omega_\rho$ and $\operatorname{supp}\mu \subset B_\rho(0)$) and so

$$\widetilde{u} * \widehat{\chi}_{B_r(0)}(y) = \fint_{B_r(y)} \widetilde{u} \, dm \geqslant \widetilde{u}(y).$$

Consequently,

$$\fint_{B_r(0)} u * \mu \, dm \geqslant \langle \mu, \widetilde{u} \rangle = u * \mu(0).$$

□

Lemma 5.9. *Let $\Omega \subset \mathbb{R}^d$ be open, let $u \in L^1_{\mathrm{loc}}(\Omega)$ be subharmonic in Ω, and $\varphi \in C_c^\infty(\Omega)$, with $\varphi \geqslant 0$. Then, its distributional derivatives satisfy*

$$\langle \nabla u, \nabla \varphi \rangle \leqslant 0.$$

Consequently, if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ with $1 < p \leqslant \infty$ and $\varphi \in W^{1,p'}_c(\Omega)$ with $\varphi \geqslant 0$, we have

$$\int \nabla u \cdot \nabla \varphi \leqslant 0. \tag{5.1}$$

Proof. For every $\varepsilon > 0$, let ψ_ε be a C^∞, positive, radially decreasing, function supported on $B_\varepsilon(0)$ with $\int \psi_\varepsilon = 1$. Let $\Omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \Omega_c) > \varepsilon\}$ and take ε small enough such that $\mathrm{supp}\varphi \subset \Omega_\varepsilon$. Then we have

$$\langle \nabla u, \nabla \varphi \rangle = -\int u \, \Delta\varphi \, dx = -\lim_{\varepsilon \to 0} \int (u * \psi_\varepsilon) \, \Delta\varphi \, dx = -\lim_{\varepsilon \to 0} \int \Delta(u * \psi_\varepsilon) \, \varphi \, dx.$$

Since $u * \psi_\varepsilon$ is C^∞ and subharmonic in Ω_ε, it follows that $\Delta(u * \psi_\varepsilon) \geqslant 0$ in Ω_ε, see Lemmas 5.4 and 5.8. Thus,

$$\int \Delta(u * \psi_\varepsilon) \, \varphi \, dx \geqslant 0$$

for any $\varepsilon > 0$ small enough, and so $\langle \nabla u, \nabla \varphi \rangle \leqslant 0$.

The second statement in the lemma follows easily by a density argument. □

Remark 5.10. A function $f \in W^{1,2}(\Omega)$ satisfying (5.1) is called weakly subharmonic. Note that we don't ask for semicontinuity in this definition. What we call weakly subharmonic is sometimes called a *subsolution* to $\Delta u = 0$, see [Ken94, Section 1.1], for instance.

Lemma 5.11 (Caccioppoli Inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be weakly subharmonic in Ω and non-negative. Then for every ball $B \subset \Omega$ of radius r we have*

$$\int_B |\nabla u|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2,$$

where $t = \mathrm{dist}(B, \partial\Omega)$

Proof. The arguments are very similar to the ones in Lemma 2.10. Let η be a Lipschitz function such that $\chi_B \leqslant \eta \leqslant \chi_{(t+1)B}$ and with $|\nabla \eta| \leqslant \frac{1}{rt}$. Since u is weakly subharmonic, η is compactly supported, and $u\eta^2 \geqslant 0$, by Leibniz' rule and Lemma 5.9 we have

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 = \int_{(t+1)B} \nabla u \cdot \nabla(u\eta^2) - \int_{(t+1)B} 2u\eta \nabla u \cdot \nabla \eta \leqslant -\int_{(t+1)B} 2u\eta \nabla u \cdot \nabla \eta.$$

By Hölder's inequality we get

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \left(\int_{(t+1)B} 4u^2 |\nabla \eta|^2 \right)^{\frac{1}{2}} \left(\int_{(t+1)B} \eta^2 |\nabla u|^2 \right)^{\frac{1}{2}},$$

and so

$$\int_B |\nabla u|^2 \leqslant \int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \int_{(t+1)B} 4u^2 |\nabla \eta|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2.$$

□

5.2 Perron classes and resolutive functions

Throughout this section we assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set (not necessarily connected).

For $f \in C(\partial\Omega)$, the Perron method, that we will describe below, associates a harmonic function $u_f : \Omega \to \mathbb{R}$ to f. Even if f is continuous, the function u_f may not extend continuously to the boundary. However, We will see that if Ω is regular enough in some sense, then u_f extends continuously to $\partial\Omega$ and its boundary values coincide with f.

Definition 5.12. Given a bounded function $f : \partial\Omega \to \mathbb{R}$, define the lower Perron class as

$$\mathcal{L}_f = \left\{ u \in C(\Omega) : \text{ is subharmonic and } \limsup_{x \to \xi} u(x) \leqslant f(\xi) \text{ for all } \xi \in \partial\Omega \right\},$$

and the upper Perron class as

$$\mathcal{U}_f = \left\{ u \in C(\Omega) : u \text{ is superharmonic and } \liminf_{x \to \xi} u(x) \geqslant f(\xi) \text{ for all } \xi \in \partial\Omega \right\}.$$

Note that the constant function $x \mapsto \sup_{\partial\Omega} f$ is an element of $\mathcal{U}_f$ (and $x \mapsto \inf_{\partial\Omega} f$ is an element of $\mathcal{L}_f$). Therefore, $\mathcal{U}_f$ and $\mathcal{L}_f$ are non-empty and we can define the real-valued functions

$$\underline{H}_f(x) = \sup_{u \in \mathcal{L}_f} u(x), \qquad \overline{H}_f(x) = \inf_{u \in \mathcal{U}_f} u(x)$$

for $x \in \Omega$, which we call *lower Perron solution* and *upper Perron solution* respectively.

Remark 5.13. If $f \in C(\overline{\Omega})$ is harmonic in Ω, for every $u \in \mathcal{L}_f$ we can apply the maximum principle (see Lemma 5.3) to $u - f$ to infer that $u \leqslant f$ in Ω. In particular, we deduce that $f = \underline{H}_f = \overline{H}_f$. So if the solution of the Dirichlet problem with continuous boundary data exists, then it coincides with the lower and upper Perron solutions.

Lemma 5.14. *For every bounded function $f : \partial\Omega \to \mathbb{R}$, the functions $\underline{H}_f$ and $\overline{H}_f$ are harmonic.*

Proof. We will show only the case $\underline{H}_f$. The other follows by noting that $\overline{H}_f = -\underline{H}_{-f}$.

Fix $x \in \Omega$ and $B = B_r(x) \subset\subset \Omega$. Let $\{u_j\}_{j=1}^{\infty} \subset \mathcal{L}_f$ be a sequence of subharmonic functions so that $u_j(x) \xrightarrow{j \to \infty} \underline{H}_f(x)$. By replacing u_j by $\max(u_j, \inf_{\partial\Omega} f)$ if necessary (see

Lemma 5.2), we may assume that the sequence of functions u_j is uniformly bounded from below.

Let U_j be the harmonic lift of u_j in B, which is subharmonic by Claim 5.7 and therefore $U_j \leqslant \underline{H}_f$. This sequence is uniformly bounded above by $\sup_{\partial\Omega} f$ by the maximum principle and it is also bounded below since the u_j's are uniformly bounded from below. Thus, passing to a subsequence if necessary, we may assume that U_j converges pointwise in B to a harmonic function U (see Lemma 2.14). As we have seen, $u_j \leqslant U_j \leqslant \underline{H}_f$ and, therefore, $U(x) = \underline{H}_f(x)$.

We claim that $U \equiv \underline{H}_f$ in B. Assume not. Then there is $y \in B$ so that $U(y) < \underline{H}_f(y)$, and by definition of $\underline{H}_f$, there must be $v \in \mathcal{L}_f$ so that $U(y) < v(y) \leqslant \underline{H}_f(y)$. Set $v_j = \max\{U_j, v\}$ (which is again subharmonic by Lemma 5.1) and let V_j be the harmonic lift of v_j in B, so now V_j is harmonic in B. Passing to a subsequence, we may assume V_j converges pointwise to a harmonic function V in B. Since $U_j \leqslant V_j$, we have that $U \leqslant V \leqslant \underline{H}_f$ in B, and so $U(x) = V(x) = \underline{H}_f(x)$, which implies $U = V$ in B by the maximum principle. However, $U(y) < v(y) \leqslant V_j(y)$ which implies $U(y) < V(y)$, a contradiction. □

Lemma 5.15. *Every bounded function $f : \partial\Omega \to \mathbb{R}$ satisfies $\underline{H}_f \leqslant \overline{H}_f$.*

Proof. Let $u \in \mathcal{U}_f$ and $v \in \mathcal{L}_f$. Then $v - u$ is subharmonic with $\limsup_{x\to\xi}(v - u) \leqslant f(\xi) - f(\xi) = 0$ for all $\xi \in \partial\Omega$, and so by the maximum principle, $v \leqslant u$. Taking infimum and supremum over $\mathcal{U}_f$ and $\mathcal{L}_f$ respectively, we get $\underline{H}_f \leqslant \overline{H}_f$. □

Definition 5.16. We say that a bounded function $f : \partial\Omega \to \mathbb{R}$ is resolutive if $\underline{H}_f = \overline{H}_f$.

Lemma 5.17. *If f, g are resolutive so are $-f$ and $f + g$.*

Proof. Note that if $u \in \mathcal{U}_f$ and $v \in \mathcal{U}_g$, then $u + v \in \mathcal{U}_{f+g}$, and so $\overline{H}_{f+g} \leqslant u + v$. Therefore, $\overline{H}_{f+g} \leqslant \overline{H}_f + \overline{H}_g$. Similarly, $\underline{H}_{f+g} \geqslant \underline{H}_f + \underline{H}_g = \overline{H}_f + \overline{H}_g$. Therefore $\overline{H}_{f+g} \leqslant \underline{H}_{f+g}$ and the converse inequality follows from Lemma 5.15.

Also being f resolutive implies that $\underline{H}_{-f} = -\overline{H}_f = -\underline{H}_f = \overline{H}_{-f}$. □

Lemma 5.18. *If $f \in C(\overline{\Omega})$ is subharmonic in Ω, then $f|_{\partial\Omega}$ is resolutive.*

Proof. Since f is subharmonic and continuous up to the boundary, we have $f \in \mathcal{L}_f$, and so $f \leqslant \underline{H}_f$. Note that $\underline{H}_f$ is harmonic (hence superharmonic) and $\liminf_{x\to\xi} \underline{H}_f(x) \geqslant \liminf_{x\to\xi} f(x) = f(\xi)$, so $\underline{H}_f \in \mathcal{U}_f$, hence $\underline{H}_f \geqslant \overline{H}_f$. □

Lemma 5.19. *Polynomials are resolutive in every bounded open set.*

Proof. Let u be a polynomial. Note that the function $v(x) = |x|^2$ satisfies $\Delta v = 2d > 0$. In particular v is subharmonic in $\mathbb{R}^d$ by Lemma 5.4. Since Δu is a polynomial, it is bounded in any bounded open set Ω. Thus, for $k > 0$ large enough, $\Delta(u + kv) > 0$ in Ω. So both v and $u + kv$ are subharmonic in Ω and continuous in $\overline{\Omega}$. Hence they are resolutive, and therefore $u = (u + kv) - kv$ is resolutive too. □

Theorem 5.20 (Wiener)**.** *$C(\partial\Omega)$ functions are resolutive.*

Proof. Let $f \in C(\partial\Omega)$ and $\varepsilon > 0$. By the Stone-Weierstrass theorem [Sto48], we may find a polynomial u such that $|f - u| < \varepsilon$ on $\partial\Omega$. Thus,

$$\overline{H}_f \leqslant \overline{H}_{u+\varepsilon} = \overline{H}_u + \varepsilon = \underline{H}_u + \varepsilon \leqslant \underline{H}_f + 2\varepsilon,$$

and letting $\varepsilon \to 0$ gives that f is resolutive. □

In this way, we can associate to a continuous function f a harmonic function $H_f := \underline{H}_f = \overline{H}_f$. The fact that f is resolutive is not the reason we can define an association. For example, we could just associate to any bounded function f on the boundary the harmonic function $\overline{H}_f$. The property of being resolutive is not significant for us because it allows us define a harmonic extension of f. Instead, this property will be useful in using maximum principle arguments when trying to prove continuity at the boundary of the Perron solution.

As mentioned earlier, H_f may not coincide with f at the boundary, even if f is continuous. To give an example, consider $\Omega = B_1(0)\backslash\{0\} \subset \mathbb{R}^d$, and let $f(\xi) = 0$ for $\xi \in \partial B_1(0)$, $f(0) = 1$. Define

$$u_\varepsilon(x) := \frac{\varepsilon}{|x|^{d-2}}$$

for $d \geqslant 3$ (for $d = 2$ use the logarithm). Since $u_\varepsilon > 0$ is harmonic and goes to $+\infty$ at the origin, we immediately get $u_\varepsilon \in \mathcal{U}_f$, so

$$H_f(x) \leqslant \frac{\varepsilon}{|x|^{d-2}} \xrightarrow{\varepsilon\to 0} 0.$$

Since $0 \in \mathcal{L}_f$ trivially, we get that $H_f(x) \geqslant 0$ and Lemma 5.15 implies that $H_f(x) = 0$. That is, H_f is the same for $\Omega = B_1(0)$ and for $\Omega = B_1(0)\backslash\{0\}$.

5.3 Harmonic measure via Perron's method

Throughout this section we assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set, unless otherwise stated. Next we provide the definition of harmonic measure via the so-called *Perron's method*.

Definition 5.21. Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $x \in \Omega$. The harmonic measure for Ω based at x (or with pole in x) is the unique Radon measure ω^x on $\partial\Omega$ such that

$$H_f(x) = \int_{\partial\Omega} f(\xi)d\omega^x(\xi) \qquad \text{for all } f \in C(\partial\Omega).$$

The existence and uniqueness of ω^x is ensured by the Riesz representation theorem, i.e. Theorem 4.7. Abusing notation we extend ω^x by 0 to the whole $\mathbb{R}^d$, that is $\omega^x(\mathbb{R}^d\backslash\partial\Omega) := 0$.

Remark 5.22. Note that $1 \in \mathcal{L}_1 \cap \mathcal{U}_1$, so $H_1(x) = 1$ regardless of any consideration on the geometry of Ω by Lemma 5.15. Therefore

$$\omega^x(\partial\Omega) = \int 1 d\omega^x = H_1(x) = 1.$$

So ω^x is a probability measure.

Example 5.23. Consider the case of the unit ball B_1. We showed in Theorem 3.10 that the Dirichlet problem is solvable in B_1 and that, for any $f \in C(\partial B_1)$, its harmonic extension equals

$$u_f(x) = \int_{\partial B_1} P^x(\zeta) f(\zeta)\, d\sigma(\zeta) \qquad \text{for } x \in B_1,$$

where $P^x(\xi)$ is the Poisson kernel:

$$P^x(\xi) = \frac{1 - |x|^2}{\kappa_d\, |x - \xi|^d}.$$

Since $u_f = H_f$ for all $f \in C(\partial B_1)$, by the uniqueness of ω^x it follows that

$$d\omega^x(\xi) = P^x(\xi)\, d\sigma(\xi).$$

In the case $x = 0$, we have

$$d\omega^0(\xi) = \frac{1}{\kappa_d}\, d\sigma(\xi).$$

That is, ω^0 is the normalized surface measure on the unit sphere.

In many geometric and qualitative analytic properties of harmonic measure, the choice of the pole plays no role. This is due to the fact that harmonic measures with different poles are mutually absolutely continuous in (connected) domains. To prove this fact, we start by checking the harmonicity with respect to the pole of the harmonic measure of a given compact set.

Lemma 5.24. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let ω^x be the harmonic measure for Ω. Let $K \subset \partial\Omega$ be compact. Then the function $u(x) := \omega^x(K)$ is harmonic in Ω.*

Proof. For each $n \geqslant 1$, let U_n be the $(1/n)$-neighborhood of K, i.e. $U_n = \{x : \operatorname{dist}(x, K) < 1/n\}$. Consider a sequence of functions $f_n \in C(\partial\Omega)$ such that $\chi_K \leqslant f_n \leqslant \chi_{U_n \cap \partial\Omega}$, so that $f_n \to \chi_k$ pointwise in $\partial\Omega$.

By dominated convergence theorem, it follows that, for any fixed $x \in \Omega$,

$$u(x) = \omega^x(K) = \lim_{n\to\infty} \int f_n\, d\omega^x \leqslant \omega^x(U_1) \leqslant 1.$$

Since $u_n(x) := \int f_n d\omega^x$, with $n \geqslant 1$, is a uniformly bounded sequence of harmonic functions, the limit is also harmonic (see Lemma 2.14). $\square$

Lemma 5.25. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let ω^x be the harmonic measure for Ω. For all $x, y \in \Omega$, the measures ω^x and ω^y are mutually absolutely continuous.*

Proof. By the inner regularity of Radon measures, it suffices to show that $\omega^x(K) \approx \omega^y(K)$ for any compact set K, with the implicit constant depending only on Ω, x, y, but not on K. This is an immediate consequence of Lemma 2.17, as $u(x) := \omega^x(K)$ is a positive harmonic function in Ω, □

As a matter of fact, the harmonicity with respect to the pole is also satisfied when the set is Borel regular. The proof in this case is a bit more technical, since the approximating open sets given by Borel regularity in Definition 4.4 depend on the particular pole.

Lemma 5.26. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, let ω^x be the harmonic measure for Ω, and let $A \subset \partial\Omega$ be a Borel set. Then the function $u(x) := \omega^x(A)$ is harmonic in Ω.*

Proof. If A is compact, this has already been shown in Lemma 5.24. If A is open, then $\omega^x(A^c)$ is harmonic and we write $u(x) = \omega^x(A) = 1 - \omega^x(A^c)$. So u is harmonic in Ω.

Let $A \subset \Omega$ be now an arbitrary Borel set A and fix $x \in \Omega$. By the regularity of ω^x, there exists a sequence of open sets $U_n \supset A$ such that $\omega^x(U_n \backslash A) \leqslant 1/n$. Moreover, we can take $U_{n+1} \subset U_n$ by redefining the sequence suitably. Then, letting $G = \bigcap_{n \geqslant 1} U_n$, we have $\omega^x(G \backslash A) = 0$. By the mutual absolute continuity of all the harmonic measures ω^y, with $y \in \Omega$, it follows that $\omega^y(G \backslash A) = 0$ for all $y \in \Omega$. Thus, since A is Borel (and therefore, it is measurable), we get

$$\omega^y(G) = \omega^y(G \backslash A) + \omega^y(G \cap A) = \omega^y(A) = u(y)$$

for all $y \in \Omega$.

Now it just remains to notice that $\omega^y(G)$ is a harmonic function, since it equals a pointwise limit of uniformly bounded harmonic functions, because Lemma 4.3 implies

$$\omega^y(G) = \lim_{n \to \infty} \omega^y(U_n).$$

□

Remark 5.27. In the preceding lemma we have considered Borel sets because they are measurable for every pole. There may be sets which are not Borel, but which are measurable for certain ω^x, however mesurability for other poles should be discussed in this setting. However, the preceding lemma and its proof can be extended to any set A using the exterior measure of possibly non-measurable sets $\omega^y(A) := \inf\{\omega^y(E) : A \subset E \text{ with } E \text{ measurable}\}$, see [Mat95] for instance.

The next result will be useful in other chapters when studying the properties of harmonic measure.

Lemma 5.28. *Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}^d$ be bounded open sets such that $\widetilde{\Omega} \subset \Omega$ and $\partial\Omega \cap \partial\widetilde{\Omega} \neq \varnothing$. Denote by ω_Ω and $\omega_{\widetilde{\Omega}}$ the respective harmonic measures for Ω and $\widetilde{\Omega}$. For any $x \in \widetilde{\Omega}$ and any Borel set $A \subset \partial\Omega \cap \partial\widetilde{\Omega}$, it holds*

$$\omega^x_{\widetilde{\Omega}}(A) \leqslant \omega^x_\Omega(A).$$

Proof. To simplify notation we write $\omega = \omega_\Omega$ and $\widetilde{\omega} = \omega_{\widetilde{\Omega}}$. By the regularity properties of harmonic measure, it suffices to prove that $\widetilde{\omega}^x(A) \leqslant \omega^x(A)$ for any compact subset $A \subset \partial\Omega \cap \partial\widetilde{\Omega}$. Consider an arbitrary function $\varphi \in C(\partial\Omega)$ such that $\varphi = 1$ on A. To illustrate the main idea of the proof, suppose first that Dirichlet problem is solvable in Ω for any continuous boundary data, so that the Perron solution $v = H_\varphi$ in Ω of the Dirichlet problem with boundary data φ extends continuously to $\partial\Omega$ and $v|_{\partial\Omega} = \varphi$. Then,

$$\widetilde{\omega}^x(A) \leqslant \int_{\partial\widetilde{\Omega}} v\, d\widetilde{\omega}^x = v(x) = \int_{\partial\Omega} \varphi\, d\omega^x.$$

Then taking the infimum over all the functions $\varphi \in C(\partial\Omega)$ as above, we deduce that $\widetilde{\omega}^x(A) \leqslant \omega^x(A)$.

In the general case, we need a more careful argument. For φ as above and any $\varepsilon > 0$, let $u \in \mathcal{U}_\varphi^\Omega$ (the upper Perron class for φ in Ω) be such

$$\int_{\partial\Omega} \varphi\, d\omega^x \geqslant u(x) - \varepsilon.$$

By the definition of $\mathcal{U}_\varphi^\Omega$, we have

$$\liminf_{y\to\xi} u(y) \geqslant \varphi(\xi) = 1 \quad \text{ for all } \xi \in A.$$

Then, by the compactness of A, there exists δ-neighborhood $U_\delta(A)$ such that $u(y) \geqslant 1-\varepsilon$ for all $y \in U_\delta(A) \cap \Omega$. Consider now a function $\widetilde{\varphi} \in C(\partial\widetilde{\Omega})$ supported on $U_\delta(A) \cap \partial\widetilde{\Omega}$ which equals 1 on A and is bounded above uniformly by 1. Then we claim that $u|_{\widetilde{\Omega}} \in \mathcal{U}^{\widetilde{\Omega}}_{(1-\varepsilon)\widetilde{\varphi}}$ (the upper Perron class for $(1-\varepsilon)\widetilde{\varphi}$ in $\widetilde{\Omega}$). Indeed, u is superharmonic in $\widetilde{\Omega}$ and

$$\liminf_{y\to\xi} u(y) \geqslant 0 = \widetilde{\varphi}(\xi) \quad \text{for all } \xi \in \partial\widetilde{\Omega}\backslash U_\delta(A),$$

and

$$\liminf_{y\to\xi} u(y) \geqslant 1-\varepsilon \geqslant (1-\varepsilon)\widetilde{\varphi}(\xi) \quad \text{for all } \xi \in \partial\widetilde{\Omega} \cap U_\delta(A).$$

Therefore,

$$(1-\varepsilon)\,\widetilde{\omega}^x(A) \leqslant \int_{\partial\widetilde{\Omega}} (1-\varepsilon)\widetilde{\varphi}\, d\widetilde{\omega}^x \leqslant u(x) \leqslant \int_{\partial\Omega} \varphi\, d\omega^x + \varepsilon.$$

Since ε is arbitrarily small, we have $\widetilde{\omega}^x(A) \leqslant \int_{\partial\Omega} \varphi\, d\omega^x$. Taking the infimum over all the functions $\varphi \in C(\partial\Omega)$ such that $\varphi = 1$ on A, we derive $\widetilde{\omega}^x(A) \leqslant \omega^x(A)$. □

5.4 Wiener regularity

In this section we continue to assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set, unless stated otherwise. In view of Lemma 5.26 it is tempting to refer to the harmonic measure of any set $A \subset \partial\Omega$ as the harmonic function in Ω having boundary values χ_A. Unfortunately, χ_A

is not a continuous function, and it is not clear what does it mean to have a discontinuous function as trace, for instance, when A is a dense subset with null harmonic measure. If the boundary is regular enough, this limit may be understood in the L^p sense, for instance, see Theorem 3.10, but the limit would be defined almost everywhere in some sense. We could expect, however, that $\lim_{x\to\xi}\omega^x(A) = 1$ if $\operatorname{dist}(\xi, \partial\Omega \cap A^c) > 0$, and $\lim_{x\to\xi}\omega^x(A) = 0$ if $\operatorname{dist}(\xi, A) > 0$. Unfortunately, we cannot grant yet that $H_f|_{\partial\Omega} \equiv f$ for continuous functions. We need to describe when this happens, that is, we need to study regular points.

Definition 5.29. We say that $\xi \in \partial\Omega$ is a *regular point* if whenever $f \in C(\partial\Omega)$, $H_f(x) \to f(\xi)$ as $\Omega \ni x \to \xi$, i.e.

$$\int_{\partial\Omega} f(\zeta)d\omega^x(\zeta) \xrightarrow{\Omega\ni x\to\xi} f(\xi). \tag{5.2}$$

We say that Ω is *Wiener regular* if every point in the boundary is regular.

From the definition above, it follows easily that if a domain Ω is Wiener regular, then the support of harmonic measure is the whole boundary of Ω.

A method for proving regularity at a point $\xi \in \partial\Omega$ consists in showing the existence of a barrier function for ξ, that is, a function $v : \Omega \to \mathbb{R}$ such that

1. v is superharmonic in Ω.
2. $\liminf_{y\to\zeta} v(y) > 0$ for all $\zeta \in \partial\Omega\backslash\{\xi\}$.
3. $\lim_{y\to\xi} v(y) = 0$.

Notice that, by the minimum principle applied to each component of Ω, $v > 0$ in Ω.

Theorem 5.30. *If $\xi \in \partial\Omega$ has a barrier function, then for any bounded function f on $\partial\Omega$ which is continuous at ξ, we have*

$$\lim_{x\to\xi} \underline{H}_f(x) = \lim_{x\to\xi} \overline{H}_f(x) = f(\xi).$$

In particular, ξ is a regular point.

Proof. Let v be a barrier for ξ and let $\varepsilon > 0$. Since f is continuous in ξ, there is $\delta > 0$ so that $|\zeta - \xi| \leqslant \delta$ implies $|f(\zeta) - f(\xi)| < \varepsilon$. Since v is superharmonic, the infimum of v in $\Omega_\delta := \Omega\backslash\bar{B}_\delta(\xi)$ is attained in $\partial\Omega_\delta$, see Lemma 5.3. That is, there exists some $y \in \partial\Omega_\delta$ such that

$$\inf_{\Omega_\delta} v = \liminf_{z\to y} v(z).$$

If $y \in \partial\Omega$, then $\liminf_{z\to y} v(z) > 0$ by the definition of barrier, and if $y \in \Omega \cap \partial B_\delta(\xi)$, then $\liminf_{z\to y} v(z) \geqslant v(y) > 0$ too, by the lower semicontinuity of v and the fact that $v > 0$ in Ω. Thus $\inf_{\Omega_\delta} v > 0$. So we can pick $k > 0$ such that

$$k \liminf_{z\to\zeta} v(z) > 2\sup|f|$$

on $\partial\Omega\backslash\bar{B}_\delta(\xi)$ (we can do this because f is bounded).

Now, since $f(\zeta) < f(\xi) + \varepsilon$ on $\bar{B}_\delta(\xi) \cap \partial\Omega$ and $f(\zeta) \leqslant 2\sup|f| + f(\xi)$ on $\partial\Omega\backslash\bar{B}_\delta(\xi)$, we have

$$f(\zeta) \leqslant k \liminf_{z\to\zeta} v(z) + f(\xi) + \varepsilon \quad \text{ for all } \zeta \in \partial\Omega.$$

Thus, $k\,v + f(\xi) + \varepsilon \in \mathcal{U}_f$ and therefore $\overline{H}_f(x) \leqslant kv(x) + f(\xi) + \varepsilon$ in Ω and so

$$\limsup_{x\to\xi} \overline{H}_f(x) \leqslant \limsup_{x\to\xi} k\,v(x) + f(\xi) + \varepsilon \leqslant 0 + f(\xi) + \varepsilon.$$

Letting $\varepsilon \to 0$ we get $\limsup_{x\to\xi} \overline{H}_f(x) \leqslant f(\xi)$, and arguing analogously we can also prove that $\liminf_{x\to\xi} \underline{H}_f(x) \geqslant f(\xi)$. The theorem is an immediate consequence of this fact, by Lemma 5.15. □

The preceding theorem asserts that the existence of a barrier for $\xi \in \partial\Omega$ implies that ξ is a regular point. The converse result is also true:

Theorem 5.31. *Let Ω be a bounded open set and let $\xi \in \partial\Omega$ be a regular point. Then there exists a barrier for ξ. This barrier can be chosen to be harmonic in Ω.*

Proof. Let $u(x) = |x-\xi|^2$. Obviously, $f := u|_{\partial\Omega} \in C(\partial\Omega)$. We claim that $v = H_f$ is a barrier for ξ. Indeed, this is harmonic in Ω and $\lim_{y\to\xi} H_f(y) = f(\xi)$ by the regularity of ξ. Also, u is subharmonic (because $\Delta u > 0$) and so $u \in \mathcal{L}_f$ and then $u \leqslant \underline{H}_f = H_f = v$ in Ω. Therefore, for all $\zeta \in \partial\Omega\backslash\{\xi\}$,

$$\liminf_{y\to\zeta} v(y) \geqslant \liminf_{y\to\zeta} u(y) = u(\zeta) > 0.$$

□

As a consequence, the harmonic measure of any open set with pole approaching to a boundary point interior to this set tends to 1.

Corollary 5.32. *Let Ω be a bounded open set and let $\xi \in \partial\Omega$ be a regular point. For every open set $A \subset \mathbb{R}^d$ containing ξ,*

$$\lim_{\Omega\ni x\to\xi} \omega^x(A) = 1.$$

Also

$$\lim_{\Omega\ni x\to\xi} \omega^x(\overline{A}^c) = 0.$$

Proof. By Urysohn's lemma, there exists a continuous function $f : \partial\Omega \to \mathbb{R}$ such that $f(\xi) = 1$ and $f|_{A^c\cap\partial\Omega} \equiv 0$. Then we have

$$H_f(x) = \int f\, d\omega^x \leqslant \int \chi_A\, d\omega^x = \omega^x(A)$$

by the monotonicity of integration. Since ξ is a regular point we have

$$1 \geqslant \limsup_{\Omega\ni x\to\xi} \omega^x(A) \geqslant \liminf_{\Omega\ni x\to\xi} \omega^x(A) \geqslant \lim_{\Omega\ni x\to\xi} H_f(x) = f(\xi) = 1.$$

The other estimate follows by an analogous reasoning assuming $f(\xi) = 0$ and $f|_{\overline{A}^c\cap\partial\Omega} \equiv 1$. □

Remark 5.33. There is a *thickness* property described in terms of capacity which characterizes regularity as well, see Chapter 6 for more details.

Remark 5.34. One easy criterion for ξ to have a barrier is the existence of an exterior tangent ball, that is, the existence of $B = B_r(y) \subset \Omega^c$ so that $\partial\Omega \cap \partial B = \{\xi\}$. In this way, the function $w(x) = |\xi - y|^{2-d} - |x - y|^{2-d}$ is a barrier function at ξ.

Note that harmonic measure associates a function $H_f(x)$ to each continuous function f on the boundary, although we don't necessarily know if it is a "true" extension in the sense that it is continuous up to the boundary and coincides with f there; all we know is that it is a harmonic function. If it happens that Ω is Wiener regular, then $\int f d\omega^x = H_f(x)$ is a harmonic function continuous up to the boundary with boundary values f.

5.5 The Dirichlet problem in unbounded domains with compact boundary

In order to study the properties of harmonic measure it is convenient to extend the study of the Dirichlet problem to unbounded open sets with compact boundary and to define the harmonic measure for this type of domains too. This the objective of this section.

Let $\Omega \subsetneq \mathbb{R}^d$ be un unbounded open set with compact boundary. Solving the Dirichlet problem in Ω for a function $f \in C(\partial\Omega)$ consists in finding a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the following:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega, \\ u = f \quad \text{on } \partial\Omega, \\ \|u\|_{\infty,\Omega} < \infty, \\ \text{when } d \geqslant 3, \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{5.3}$$

Proposition 5.35. *Let $\Omega \subsetneq \mathbb{R}^d$ be un unbounded open set with compact boundary and let $f \in C(\partial\Omega)$. If there exists a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying (5.6), then it is unique.*

Proof. Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ be two solutions of (5.6) and let us check that they are equal. Suppose first that $d \geqslant 3$. For $R > 0$, denote $\Omega_r = \Omega \cap B_r(0)$. Let r be large enough so that $\partial\Omega \subset B_r(0)$. For $0 < r_0 < r$, by the maximum principle, taking into account that $u = v$ on $\partial\Omega$,

$$\|u - v\|_{\infty,\Omega_{r_0}} \leqslant \|u - v\|_{\infty,\Omega_r} = \|u - v\|_{\infty,\partial\Omega_r} = \|u - v\|_{\infty,S_r(0)} \leqslant \|u\|_{\infty,S_r(0)} + \|v\|_{\infty,S_r(0)}.$$

By the last condition in (5.6), $\|u\|_{\infty,S_r(0)} + \|v\|_{\infty,S_r(0)} \to 0$ as $r \to \infty$, and so $u = v$ in Ω_{r_0}, with r_0 arbitrarily large.

Next we consider the case $d = 2$. Without loss of generality, we assume that $\partial\Omega \subset B_{1/4}(0)$. Let $\xi \in \partial\Omega$, and for a given $\delta > 0$, consider the function

$$h_\delta(x) = u(x) - v(x) - \delta \log|x - \xi|.$$

By the continuity of u and v at ξ, for any $\varepsilon > 0$ there exists some $\rho \in (0, 1/4)$ such that

$$|u(x) - v(x)| \leqslant \varepsilon \quad \text{ for all } x \in \Omega \text{ such that } |x - \xi| \leqslant r.$$

For $r \gg \rho$, consider the domain $\Omega_{\rho,r} = \Omega \cap B_r(\xi)\backslash\overline{B}_\rho(\xi)$. We assume r large enough so that $\partial\Omega \subset B_r(\xi)$. Notice that

$$\partial\Omega_{\rho,r} \subset \partial\Omega \cup (\Omega \cap S_\rho(\xi)) \cup S_r(\xi).$$

Notice that $|u - v| \leqslant \varepsilon$ and $|\log|\cdot - \xi|| \leqslant \delta|\log\rho|$ in $\partial\Omega \cup (\Omega \cap S_\rho(\xi)) \subset B_{1/2}(0)$. Thus,

$$|h_\delta| \leqslant \varepsilon + \delta|\log\rho| \quad \text{ in } \partial\Omega \cup (\Omega \cap S_\rho(\xi)).$$

On the other hand, for $x \in S_r(\xi)$, $\log|x - \xi| = \log r$. So for a given $\delta > 0$, if r is large enough taking into account also that u and v are bounded, we have

$$h_\delta \leqslant 0 \quad \text{ in } S_r(\xi).$$

From the last estimates and the maximum principle, we deduce that

$$h_\delta \leqslant \varepsilon + \delta|\log\rho| \quad \text{ in } \Omega_{\rho,r},$$

Letting $r \to \infty$, we get infer that the same estimate is valid in $\Omega\backslash\overline{B}_\rho(\xi)$. That is,

$$u(x) - v(x) - \delta\,\log|x - \xi| \leqslant \varepsilon + \delta|\log\rho(\varepsilon)| \quad \text{ for all } x \in \Omega_{\rho(\varepsilon)},$$

where we wrote $\rho(\varepsilon)$ to emphasize the dependence of ρ on ε. Since this inequality holds for all $\delta > 0$, we derive that $u \leqslant v + \varepsilon$ in $\Omega_{\rho(\varepsilon)}$. Finally, letting $\varepsilon \to 0$ and $\rho(\varepsilon) \to 0$, it follows that $u \leqslant v$ in Ω. Interchanging the roles of u and v in the arguments above, we deduce $v \leqslant u$ in Ω, and so we are done. □

Definition 5.36. Let Ω be an unbounded open set with bounded boundary. We say that Ω is Wiener regular if for $r > 0$ such that $\partial\Omega \subset B_r(0)$, the set $\Omega_r := \Omega \cap B_r(0)$ is Wiener regular. Also, we say that $\xi \in \partial\Omega$ is a regular point for Ω if it is regular for Ω_r.

Let us check that the definition does not depend on the precise $r > 0$ such that $\partial\Omega \subset B_r(0)$. Notice first that $\partial\Omega_r = \partial\Omega \cup \partial B_r(0)$. By the exterior tangent ball criterion in Remark 5.34 it follows all the points $\xi \in \partial B_r(0)$ are Wiener regular (for the open set Ω_r). To deal with the points from $\partial\Omega$, let $0 < r_1 < r_2$ be such that $\partial\Omega \subset B_{r_1}(0)$. If v_2 is barrier for $\xi \in \partial\Omega$ in Ω_{r_2}, then it is also a barrier in Ω_{r_1}, and so the Wiener regularity of ξ in Ω_{r_2} implies the Wiener regularity in Ω_{r_1}. Conversely, let v_1 be a barrier for ξ in Ω_{r_1} and consider $r_0 < r_1$ such that we still have $\partial\Omega \subset B_{r_0}(0)$. Then

$$m_{r_0} := \inf_{\partial B_{r_0}(0)} v_1(x) > 0$$

because of the superharmonicity of v_1, the other properties in the definition of a barrier, and the minimum principle. Then we define

$$v_2(x) = \begin{cases} \min(v_1(x), m_r) & \text{in } \Omega \cap B_{r_0}(0), \\ m_r & \text{in } B_{r_2}(0) \backslash B_{r_0}(0). \end{cases}$$

It is easy to check that v_2 is superharmonic in Ω_{r_2} and moreover it is a barrier for this set at ξ. Thus the Wiener regularity of ξ in Ω_{r_1} implies the Wiener regularity in Ω_{r_2}.

We will show below that if $\Omega \subsetneq \mathbb{R}^d$ is an unbounded open set with compact boundary which is Wiener regular, then the Dirichlet problem in (5.6) is solvable for all $f \in C(\partial\Omega)$. The main step is contained in the following theorem.

Theorem 5.37. *Let $\Omega \subsetneq \mathbb{R}^d$ be an unbounded open set with compact boundary and let $f \in C(\partial\Omega)$. For $r > 0$ such that $\partial\Omega \subset B_r(0)$, denote $\Omega_r = \Omega \cap B_r(0)$ and let H_f^r be the Perron solution of the Dirichlet problem in Ω_r with boundary data equal to f in $\partial\Omega$ and equal to 0 in $S_r(0)$. Then the following holds:*

(a) The functions H_f^r converge uniformly in bounded subsets of Ω to a function harmonic and bounded in Ω as $r \to \infty$.

(b) In the case $d \geqslant 3$, the limiting function H_f satisfies $\lim_{x\to\infty} H_f(x) = 0$.

(c) If $\xi \in \partial\Omega$ is a regular point, then $\lim_{\Omega \ni x \to \xi} H_f(x) = f(\xi)$.

Remark that (a) asserts that the convergence of the functions H_f^r to H_f is uniform in $\Omega \cap B_{r_1}(0)$ for any $r_1 > 0$. This a stronger statement than just asking for the local uniform convergence in compact subsets of Ω.

By the theorem above, it is clear that if $\Omega \subsetneq \mathbb{R}^d$ is a Wiener regular unbounded open set with compact boundary, then H_f is the solution of the Dirichlet problem stated in (5.6).

Proof of Theorem 5.37. We claim that it suffices to prove the theorem for $f \geqslant 0$. Indeed, for an arbitrary function $f \in C(\partial\Omega)$, we can write $f = f^+ - f^-$, so that the functions $f^\pm$ are non-negative and continuous. Then we have

$$H_f^r = H_{f^+}^r - H_{f^-}^r,$$

and it is enough to prove the statements (a), (b), (c) for $f^\pm$.

(a) Let $r_0 > 0$ be such that $\partial\Omega \subset B_{r_0/2}(0)$. The fact that $0 \leqslant f \leqslant \sup_{\partial\Omega} f$, ensures that

$$0 \leqslant H_f^r \leqslant \sup_{\partial\Omega} f \quad \text{in } \Omega_r, \text{ for all } r \geqslant r_0. \tag{5.4}$$

Next we will show that, for $r_0 < r < R$,

$$H_f^r \leqslant H_f^R \quad \text{in } \Omega_r. \tag{5.5}$$

This is an easy consequence of the maximum principle. Indeed, for $s > r_0$ denote by $\mathcal{L}_f^s$ and $\mathcal{U}_f^s$ the respective lower and upper Perron classes in Ω_s for the function f_s which equals f on $\partial\Omega$ and vanishes in $S_s(0)$. Given $u \in \mathcal{L}_f^r$, let $\widetilde{u} : \Omega_R \to \mathbb{R}$ be defined by

$$\widetilde{u} = \begin{cases} \max(u,0) & \text{in } \Omega_r, \\ \\ 0 & \text{in } B_R(0)\backslash B_r(0). \end{cases} \tag{5.6}$$

It is immediate to check that $\widetilde{u}$ is subharmonic in Ω_R and so that $\widetilde{u} \in \mathcal{L}_f^R$. So for all $x \in \Omega_r$ we have

$$u(x) \leqslant \widetilde{u}(x) \leqslant \underline{H}_f^R(x) = H_f^R(x).$$

Taking the supremum over all $u \in \mathcal{L}_f^r$, we deduce $H_f^r(x) \leqslant H_f^R(x)$, so that (5.5) holds.

From the monotonicity of the family of function $\{H_f^r\}_{r>0}$ ensured by (5.5) and the bound in (5.4), we infer that the limit $\lim_{r\to\infty} H_f^r(x)$ exists for all $x \in \Omega$ and that the limit function H_f is bounded. Since the functions H_f^r, for $r > 0$, are harmonic and uniformly bounded, it follows that the preceding limit is uniform on compact subsets of Ω.

Next we will show that for any $r_1 > r_0$, the functions H_f^r converge uniformly on Ω_{r_1}. Observe first that they converge uniformly in $S_{r_1}(0)$ since this is a compact subset of Ω. So given $\varepsilon > 0$, there exists $r_2 > r_1$ such that

$$\|H_f^s - H_f\|_{\infty,S_{r_1}(0)} < \varepsilon \quad \text{ for all } s > r_2.$$

For $R > r > r_2$, consider now two arbitrary functions $u_r \in \mathcal{U}_f^r$ and $u_R \in \mathcal{L}_f^R$. Notice that

$$\limsup_{\Omega\ni x\to\xi} u_R(x) \leqslant f(\xi) \leqslant \liminf_{\Omega\ni x\to\xi} u_r(x) \quad \text{ on } \partial\Omega.$$

Since $\|H_f^r - H_f^R\|_{\infty,S_{r_1}(0)} < 2\varepsilon$, we also have

$$u_R \leqslant H_f^R \leqslant H_f^r + 2\varepsilon \leqslant u_r + 2\varepsilon \quad \text{ in } S_{r_1}(0).$$

Using that $u_R - u_r$ is subharmonic in Ω_{r_1} and the maximum principle, it follows that

$$u_R \leqslant u_r + 2\varepsilon \quad \text{ in } \Omega_{r_1}.$$

Taking the supremum over all $u_R \in \mathcal{L}_f^R$ and the infimum over all $u_r \in \mathcal{U}_f^r$ and using that continuous functions are resolutive, we deduce that

$$H_f^R \leqslant H_f^r + 2\varepsilon \quad \text{ in } \Omega_{r_1}.$$

Together with (5.5), this implies $\|H_f^r - H_f^R\|_{\infty,\Omega_{r_1}} \leqslant 2\varepsilon$. Letting $R \to \infty$, it follows that

$$\|H_f^r - H_f\|_{\infty,\Omega_{r_1}} \leqslant 2\varepsilon \quad \text{ for all } r > r_2,$$

which proves (a).

(b) Suppose $d \geqslant 3$. Let $M > 0$ be large enough so that

$$u(\xi) \leqslant M\,\mathcal{E}(\xi) \qquad \text{for all } \xi \in \partial\Omega.$$

By the maximum principle, we easily infer that $u \leqslant M\,\mathcal{E}$ in Ω_r for all $u \in \mathcal{L}_f^r$, for $r > r_0$. This implies that $H_f^r \leqslant M\,\mathcal{E}$ in Ω_r. Letting $r \to \infty$, it follows that $H_f \leqslant M\,\mathcal{E}$ in Ω, and so

$$\limsup_{x\to\infty} H_f(x) \leqslant \limsup_{x\to\infty} \mathcal{E}(x) = 0.$$

Since H_f is non-negative, this implies that H_f vanishes at infinity.

(c) For all $r > r_0$, since $\xi \in \partial\Omega$ is regular point for Ω_r, then $\lim_{\Omega\ni x\to\xi} H_f^r(x) = f(\xi)$. Together with the uniform convergence of H_f^r to H_f in Ω_{r_1} for any given $r_1 > r_0$, this easily yields $\lim_{\Omega\ni x\to\xi} H_f(x) = f(\xi)$. □

Under the assumptions and notation of Theorem 5.37, it is immediate to check that, for any $x \in \Omega$, the functional $C(\partial\Omega) \ni f \mapsto H_f(x)$ is linear and bounded. Indeed, the linearity is due to the linearity of $C(\partial\Omega) \ni f \mapsto H_f^r(x)$ and the boundedness follows from the fact that $\inf_{\partial\Omega} f \leqslant H_f^r \leqslant \sup_{\partial\Omega} f$ for all $r \geqslant r_0$, which yields

$$\|H_f\|_{\infty,\Omega} \leqslant \|f\|_{\infty,\partial\Omega} \tag{5.7}$$

letting $r \to 0$.

Definition 5.38. Let $\Omega \subset \mathbb{R}^d$ be an unbounded open set with compact boundary and let $x \in \Omega$. The harmonic measure for Ω with pole at x is the unique Radon measure ω^x on $\partial\Omega$ such that

$$H_f(x) = \int_{\partial\Omega} f(\xi)\,d\omega^x(\xi) \qquad \text{for all } f \in C(\partial\Omega),$$

where is H_f defined as in Theorem 5.37. The existence and uniqueness of ω^x is ensured by the Riesz representation theorem, i.e. Theorem 4.7. Abusing notation we extend ω^x by 0 to the whole $\mathbb{R}^d$, that is $\omega^x(\mathbb{R}^d \backslash \partial\Omega) := 0$.

Remark 5.39. By the definition, for any unbounded open set with compact boundary $\Omega \subset \mathbb{R}^d$, for any $f \in C(\partial\Omega)$, and any $x \in \Omega$, we have

$$\int_{\partial\Omega} f(\xi)\,d\omega^x(\xi) = \lim_{r\to\infty} \int_{\partial\Omega} f(\xi)\,d\omega^x_{\Omega_r}(\xi).$$

By Theorem 5.37, the convergence is uniform in bounded subsets of Ω.

Observe that, by (5.7) it follows that

$$0 \leqslant \omega^x(\partial\Omega) \leqslant 1 \qquad \text{for all } x \in \Omega. \tag{5.8}$$

The following proposition provides additional information.

Proposition 5.40. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary and let $x \in \Omega$. In the case $d = 2$, $\omega^x(\partial\Omega) = 1$, that is, ω^x is a probability measure. In the case $d = 3$, if x belongs to the unbounded component of Ω, then $0 < \omega^x(\partial\Omega) < 1$.*

In particular, the proposition implies that the statement (b) in Theorem 5.37 may fail in the case $d = 2$. Without the Wiener regular assumption on Ω, further information will be obtained later in Proposition 6.35.

Proof. Since Ω is Wiener regular, in the case $d = 2$ the function identically 1 in Ω solves the Dirichlet problem (5.6) for $f = 1$ in $\partial\Omega$. By the uniqueness of the solution, $H_f = 1$ indentically in Ω and thus $\omega^x(\partial\Omega) = 1$.

In the case $d \geqslant 3$, again we have $\omega^x(\partial\Omega) = H_1(x)$ by Theorem 5.37. On the other hand, the statement (b) in the same theorem asserts that $H_1(x) \to 0$ as $x \to \infty$. So H_1 is a non constant non negative harmonic function in the unbounded component of Ω which is bounded above by 1, by (5.7). By the strong maximum principle (applied to $\Omega \cap B_r(0)$ and r large enough) it follows that $0 < \omega^x(\partial\Omega) = H_1(x) < 1$. □

Example 5.41. Let $\Omega = \mathbb{R}^d \backslash \bar{B}_1(0)$ for $d \geqslant 3$. The solution of the Dirichlet problem for $f \equiv 1$ in $\partial\Omega$ is the function $u(x) = |x|^{2-d}$. Thus,

$$\omega^x(\partial\Omega) = \frac{1}{|x|^{d-2}} \qquad \text{for all } x \in \Omega.$$

Next we wish to show that, in the case $d = 2$, we can easily define the notion of harmonic measure with pole at ∞. First we need the following auxiliary result, which has its own interest.

Proposition 5.42. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $x_0 \in \Omega$. Let $u : \Omega\backslash\{x_0\} \to \mathbb{R}$ be a harmonic function such that $u(x) = o(\mathcal{E}(x - x_0))$ as $x \to x_0$. Then u extends as a harmonic function to the whole Ω.*

Of course, the proposition applies to the particular case where u is bounded and harmonic in $\Omega\backslash\{x_0\}$. See also Theorem 6.34 for a related result.

Proof. Let $\bar{B}_r(x_0)$ be a closed ball contained in Ω, with $r < 1$, and let v be the solution of the Dirichlet problem in $B_r(x_0)$ with boundary data $u|_{S_r(x_0)}$. For any $\varepsilon > 0$, consider the function

$$h_\varepsilon(x) = u(x) - v(x) - \varepsilon\,\mathcal{E}(x - x_0), \qquad \text{for } x \in B_r(x_0)\backslash\{x_0\}.$$

This is harmonic in $B_r(x_0)\backslash\{x_0\}$ and $\lim_{x\to x_0} h_\varepsilon(x) = -\infty$. By the maximum principle applied to any annulus $A_{s,r}(x_0)$ with s sufficiently small, we deduce that $h_\varepsilon \leqslant 0$ in $B_r(x_0)\backslash\{x_0\}$. Since this holds for any $\varepsilon > 0$, we get $u \leqslant v$ in $B_r(x_0)\backslash\{x_0\}$. Reversing the roles of u and v, we obtain the opposite inequality. Thus $u = v$ in $B_r(x_0)\backslash\{x_0\}$ and so the proposition follows just letting $u = v$ in the whole $B_r(x_0)$. □

Corollary 5.43. *For some $r > 0$, let $u : \mathbb{C}\backslash\bar{B}_r(0) \to \mathbb{R}$ be a harmonic and bounded function. Then $\lim_{z\to\infty} u(z)$ exists and the function defined by $v(z) := u(1/z)$ can be extended to a harmonic function in $B_{1/r}(0)$.*

Proof. The function $v(z) := u(1/z)$ is harmonic and bounded in $B_{1/r}(0)\backslash\{0\}$. So it extends to a harmonic function in $B_{1/r}(0)$ by the preceding proposition. Thus,

$$\lim_{z\to\infty} u(z) = \lim_{z\to 0} v(z)$$

exists. □

Now we can define harmonic measure with pole at ∞ for unbounded open set with compact boundary in the plane as in Definition 5.38, just putting $x = \infty$ there:

Definition 5.44. Let $\Omega \subset \mathbb{R}^2$ be an unbounded open set with compact boundary. The harmonic measure for Ω with pole at ∞ is the unique Radon measure ω^∞ on $\partial\Omega$ such that

$$\lim_{x\to\infty} H_f(x) = \int_{\partial\Omega} f(\xi)d\omega^\infty(\xi) \qquad \text{for all } f \in C(\partial\Omega),$$

where H_f is defined as in Theorem 5.37. The existence and uniqueness of ω^∞ is ensured by the Riesz representation theorem.

Obviously, for any function $f \in C(\partial\Omega)$ (and Ω as in the definition),

$$\int_{\partial\Omega} f(\xi)d\omega^\infty(\xi) = \lim_{z\to\infty} \int_{\partial\Omega} f(\xi)d\omega^z(\xi).$$

Observe that for any z belonging to the unbounded component of Ω, the measures ω^z and ω^∞ are mutually absolutely continuous. Indeed, for any Borel set $E \subset \partial\Omega$, it follows easily from the strong maximum principle applied to the function $v(z) = \omega^{1/z}(E)$ in a neighborhood of the origin that $v(0) = 0$ if and only if v vanishes identically.

In the case $d \geqslant 3$, one can also the define the notion of harmonic measure with pole at ∞ for unbounded open set with compact boundary in $\mathbb{R}^d$, at least under the assumption of Wiener regularity, following a different approach. We postpone this task to Chapter 7.

6 Potential theory

6.1 Potentials

Recall that the fundamental solution of the minus Laplacian in $\mathbb{R}^d$ equals

$$\mathcal{E}(x) = \begin{cases} \dfrac{|x|^{2-d}}{(d-2)\kappa_d} & \text{if } d \geqslant 3, \\ \\ \dfrac{-\log|x|}{2\pi} & \text{if } d = 2, \end{cases}$$

For a Radon measure μ in $\mathbb{R}^d$, we consider the potential U_μ defined by

$$U_\mu(x) = \mathcal{E} * \mu(x) = \int \mathcal{E}(x-y)\, d\mu(y), \tag{6.1}$$

and the energy integral

$$I(\mu) := \iint \mathcal{E}(x-y) d\mu(y) d\mu(x). \tag{6.2}$$

For $d \geqslant 3$, U_μ is called the Newtonian potential of μ, and for $d = 2$, the logarithmic or Wiener potential of μ.

Lemma 6.1 (Semicontinuity properties)**.** *For non-negative Radon measures $\mu_n \rightharpoonup \mu$ with compact support we have:*

(a) $\liminf_{y \to x} U_\mu(y) \geqslant U_\mu(x)$ *for all $x \in \mathbb{R}^d$. So the potential U_μ is lower semicontinuous in $\mathbb{R}^d$.*

(b) $\liminf_{n\to\infty} U_{\mu_n}(x) \geqslant U_\mu(x)$ *for all $x \in \mathbb{R}^d$.*

(c) $\liminf_{n\to\infty} I(\mu_n) \geqslant I(\mu)$.

(d) The potential U_μ is superharmonic.

The proof of this lemma is an easy exercise that we leave for the reader. The superharmonicity of U_μ is a consequence of the lower semicontinuity of U_μ, the superharmonicity of $\mathcal{E}$, and Lemma 5.8 (a). For more details, alternatively, the reader may have a look at [Lan72] or [Ran95].

Theorem 6.2 (Continuity principle for potentials)**.** *Given a compactly supported Radon measure μ in $\mathbb{R}^d$, if $U_\mu \in C(\text{supp}\mu)$, then $U_\mu \in C(\mathbb{R}^d)$.*

Proof. In the case $d = 2$, by a suitable contraction we can assume that $\operatorname{diam}(\operatorname{supp}\mu) \leqslant 1/2$, so that $\mathcal{E}(x-y) > 0$ for all $x, y \in \operatorname{supp}\mu$.

Since U_μ is continuous in $\mathbb{R}^d \backslash \operatorname{supp}\mu$ we only have to check the continuity in $\operatorname{supp}\mu$. For each $n \geqslant 1$, let

$$f_\delta(x) = \int_{|x-y| \geqslant \delta} \mathcal{E}(x-y)\, d\mu(y).$$

Since the family of functions $\{f_\delta\}$ is monotone in δ and $U_\mu|_{\operatorname{supp}\mu}$ is continuous, the convergence of f_δ to U_μ is uniform in $\operatorname{supp}\mu$, by Dini's theorem. Equivalently, $U_{\chi_{B_\delta(x)}\mu}(x) \to 0$ uniformly on $x \in \operatorname{supp}\mu$ as $\delta \to 0$.

To prove the continuity of U_μ at a given $x \in \operatorname{supp}\mu$, fix $\varepsilon > 0$, and take $\delta \in (0, 1/4)$ such that $U_{\chi_{B_\delta(z)}\mu}(z) < \varepsilon$ for all $z \in \operatorname{supp}\mu$ and such that $\mu(B_\delta(x)) < \varepsilon$ (that the latter condition holds for δ small enough is due to the fact that μ has no point masses, because $U_\mu(z) < \infty$ for all $z \in \operatorname{supp}\mu$). For $y \in B_{\delta/4}(x)$, we write

$$\begin{aligned}|U_\mu(x) - U_\mu(y)| \leqslant &\int_{|x-z|<\delta/2} \mathcal{E}(x-z)\, d\mu(z) + \int_{|x-z|<\delta/2} \mathcal{E}(y-z)\, d\mu(z) \\ &+ \left| \int_{|x-z| \geqslant \delta/2} \left(\mathcal{E}(x-z) - \mathcal{E}(y-z)\right)\, d\mu(z) \right|.\end{aligned}$$

The first integral on the right hand side is bounded above by ε. The third one tends to 0 as $y \to x$, because for a fixed $\delta > 0$, the function $g(y) = \int_{|x-z| \geqslant \delta/2} \mathcal{E}(y-z)\, d\mu(z)$ is continuous in $B_{\delta/4}(x)$. To estimate the second integral on the right hand side, let y' be the closest point to y from $\operatorname{supp}\mu$. Notice that $|y'-y| \leqslant |x-y| \leqslant \delta/4$, and thus $y' \in B_{\delta/2}(x)$. It is immediate to check that then

$$|z-y'| \lesssim |z-y| \quad \text{ for all } z \in \operatorname{supp}\mu.$$

Thus, in the case $d \geqslant 3$, $\mathcal{E}(y-z) \lesssim \mathcal{E}(y'-z)$, and so, using that $y' \in \operatorname{supp}\mu$,

$$\int_{|x-z|<\delta/2} \mathcal{E}(y-z)\, d\mu(z) \lesssim \int_{B_{\delta/2}(x)} \mathcal{E}(y'-z)\, d\mu(z) \lesssim \int_{B_\delta(y')} \mathcal{E}(y'-z)\, d\mu(z) \lesssim \varepsilon.$$

In the case $d = 2$, we have $|y-z| \geqslant |y'-z|$ for $z \in B_{|y-y'|}(y')$ and so $\mathcal{E}(y-z) \leqslant \mathcal{E}(y'-z)$ for such z. On the other hand, for $z \in \operatorname{supp}\mu \backslash B_{|y-y'|}(y')$, we have $|y-z| \approx |y'-z|$ and thus

$$\mathcal{E}(y-z) = \mathcal{E}(y'-z) + \frac{1}{2\pi} \log \frac{|y'-z|}{|y-z|} \leqslant \mathcal{E}(y'-z) + C.$$

Therefore,

$$\begin{aligned}\int_{|x-z|<\delta/2} \mathcal{E}(y-z)\, d\mu(z) &\leqslant \int_{B_{\delta/2}(x)} \mathcal{E}(y'-z)\, d\mu(z) + C\, \mu(B_{\delta/2}(x)) \\ &\leqslant \int_{B_\delta(y')} \mathcal{E}(y'-z)\, d\mu(z) + C\, \mu(B_{\delta/2}(x)) \lesssim \varepsilon.\end{aligned}$$

So for any dimension, we have

$$\limsup_{y\to x}|U_\mu(x)-U_\mu(y)|\lesssim\varepsilon+\limsup_{y\to x}\left|\int_{|x-z|\geqslant\delta/2}\left(\mathcal{E}(x-z)-\mathcal{E}(y-z)\right)\,d\mu(z)\right|\approx\varepsilon.$$

Since ε is arbitrary, we have that $U_\mu(y)\to U_\mu(x)$ as $y\to x$. $\square$

Theorem 6.3 (Maximum principle for potentials)**.** *Given a compactly supported Radon measure μ in $\mathbb{R}^d$, if $U_\mu(x)\leqslant 1$ μ-a.e., then $U_\mu(x)\leqslant 1$ everywhere in $\mathbb{R}^d$.*

Proof. Again, by contracting suitably $\mathrm{supp}\mu$, we can assume that $\mathrm{diam}(\mathrm{supp}\mu)\leqslant 1/2$ in the case $d=2$.

Let $E=\mathrm{supp}\mu$. For any $\tau>0$, by Egorov's theorem, there is a compact subset $F=F_\tau\subset E$ such that $\mu(E\backslash F)<\tau$ and so that $U_{\chi_{B_\varepsilon(x)}\mu}(x)$ converges uniformly to 0 in F as $\varepsilon\to 0$.

We claim that $U_{\chi_F\mu}$ is continuous in $\mathbb{R}^d$. Indeed, by the preceding theorem, if suffices to show that $U_{\chi_F\mu}\in C(F)$. To prove this, for any $\varepsilon\in(0,1/2)$ and $x,x'\in F$ such that $|x-x'|\leqslant\varepsilon^d$, we write

$$\begin{aligned}|U_{\chi_F\mu}(x)-U_{\chi_F\mu}(x')|\leqslant&\int_{|x-y|\leqslant\varepsilon}\mathcal{E}(x-y)d\mu|_F(y)+\int_{|x-y|\leqslant\varepsilon}\mathcal{E}(x'-y)d\mu|_F(y)\\&+\int_{|x-y|>\varepsilon}\left|\mathcal{E}(x-y)-\mathcal{E}(x'-y)\right|d\mu|_F(y)\end{aligned}$$

The first integral on the right hand side tends to 0 as $\varepsilon\to 0$ (uniformly on $x\in F$), and the same happens with the second one, taking into account that $\{y:|x-y|\leqslant\varepsilon\}\subset\{y:|x'-y|\leqslant 2\varepsilon\}$. For the last one, in the case $d\geqslant 3$, for $y,x,x'\in F$ such that $|x-y|>\varepsilon$ and $|x-x'|\leqslant\varepsilon^d$, we have

$$\left|\mathcal{E}(x-y)-\mathcal{E}(x'-y)\right|=\left|\frac{c}{|x-y|^{d-2}}-\frac{c}{|x'-y|^{d-2}}\right|\lesssim\frac{|x-x'|}{|x-y|^{d-1}}\lesssim\varepsilon.$$

In the case $d=2$, observe that

$$\left|\frac{|x'-y|}{|x-y|}-1\right|\leqslant\frac{|x'-x|}{|x-y|}\leqslant\varepsilon,\quad\text{for } y,x,x' \text{ such that } |x-y|>\varepsilon \text{ and } |x-x'|\leqslant\varepsilon^2,$$

and thus, for some constant $C>0$,

$$\left|\mathcal{E}(x-y)-\mathcal{E}(x'-y)\right|\approx\left|\log\frac{|x'-y|}{|x-y|}\right|\lesssim\varepsilon.$$

Then, for any dimension d,

$$\int_{|x-y|>\varepsilon}\left|\mathcal{E}(x-y)-\mathcal{E}(x'-y)\right|d\mu|_F(y)\lesssim\varepsilon\mu(F).$$

Therefore,

$$\lim_{\varepsilon \to 0} \sup_{x,x' \in F : |x-x'| \leqslant \varepsilon^2} |U_{\chi_F \mu}(x) - U_{\chi_F \mu}(x')| = 0,$$

and thus the claim holds.

Notice that $U_{\chi_F \mu}(x) \leqslant U_\mu(x) \leqslant 1$ for all $x \in F$. Further, in the case $d \geqslant 3$, $U_{\chi_F \mu}(x) \to 0$ when $x \to \infty$, while in the case $d = 2$ we get $U_{\chi_F \mu}(x) \to -\infty$. Since $U_{\chi_F \mu}$ is harmonic in $\mathbb{R}^d \backslash F$ and continuous in $\mathbb{R}^d$, by the maximum principle (applied to $\Omega_R = B_R(0) \backslash F$ and letting $R \to \infty$), we deduce that $U_{\chi_F \mu}(x) \leqslant 1$ for all $x \in \mathbb{R}^d \backslash E \subset \mathbb{R}^d \backslash F$. Now we just have to write

$$U_\mu(x) = U_{\chi_F \mu}(x) + U_{\chi_{E \backslash F} \mu}(x) \leqslant 1 + U_{\chi_{E \backslash F} \mu}(x),$$

and note that $U_{\chi_{E \backslash F} \mu}(x) \to 0$ for any $x \in \mathbb{R}^d \backslash E$, as $\tau \to 0$ (recall that $\mu(E \backslash F) \leqslant \tau$). □

6.2 Capacity

Definition 6.4. Given a bounded set $E \subset \mathbb{R}^d$, we define its capacity $\mathrm{Cap}(E)$ by

$$\mathrm{Cap}(E) = \frac{1}{\inf_{\mu \in M_1(E)} I(\mu)}, \tag{6.3}$$

where the infimum is taken over all *probability* measures μ supported on E. When $d \geqslant 3$, $\mathrm{Cap}(E)$ is also called the Newtonian capacity of E, and for $d = 2$, the Wiener capacity of E.

In the case $d = 2$, quite often we will write $\mathrm{Cap}_W(E)$ instead of $\mathrm{Cap}(E)$. Remark that $\mathrm{Cap}_W(E)$ may be negative, and we allow this to be infinite too. On the other hand, if $\mathrm{diam}(E) < 1$, then $\mathcal{E}(x-y) \geqslant (2\pi)^{-1} \log \frac{1}{\mathrm{diam}(E)} > 0$ for all $x, y \in E$, and it follows that $\inf_{\mu \in M_1(E)} I(\mu) > 0$, and so $0 \leqslant \mathrm{Cap}_W(E) < \infty$.[1]

Definition 6.5. Given a set $E \subset \mathbb{R}^2$, we define its logarithmic capacity by

$$\mathrm{Cap}_L(E) = e^{-2\pi \inf_{\mu \in M_1(E)} I(\mu)} = e^{-\frac{2\pi}{\mathrm{Cap}_W(E)}}.$$

It is immediate to check that if $E \subset F$, then $\mathrm{Cap}(E) \leqslant \mathrm{Cap}(F)$ for $d \geqslant 3$ and $\mathrm{Cap}_L(E) \leqslant \mathrm{Cap}_L(F)$ for $d = 2$.[2] Another trivial property is that the capacities Cap, Cap_W, and Cap_L are invariant by translations. Further, the Newtonian capacity is homogeneous of degree $d-2$ when $d \geqslant 3$. That is, for a given $\lambda > 0$ and $E \subset \mathbb{R}^d$, we have

$$\mathrm{Cap}(\lambda E) = \lambda^{d-2} \, \mathrm{Cap}(E).$$

This follows easily from the fact that the fundamental solution $\mathcal{E}$ is homogeneous of degree $2-d$ in $\mathbb{R}^d$, $d \geqslant 3$. In the case $d = 2$, $\mathcal{E}$ is not homogeneous, and the behavior of Cap_W

[1] We will see below that this also holds if $\overline{E}$ is contained in $B_1(0)$.

[2] In the case $d = 2$, the inequality $\mathrm{Cap}_W(E) \leqslant \mathrm{Cap}_W(F)$ fails if $\mathrm{Cap}_W(F) < 0$, and it holds if $\mathrm{Cap}_W(F) > 0$, and in particular if $\mathrm{diam}(F) < 1$.

under dilations is more complicated. To study this, denote $T_\lambda(x) = \lambda\, x$, so that if μ is a probability measure supported on E, then the image measure $T_{\lambda\#}\mu$ (see definition 4.8) is another probability measure supported on λE. Then, by Theorem 4.10 we have

$$\begin{aligned} I(T_{\lambda\#}\mu) &= \frac{1}{2\pi}\iint \log\frac{1}{|x-y|}\, dT_{\lambda\#}\mu(x)\, dT_{\lambda\#}\mu(y) \\ &= \frac{1}{2\pi}\iint \log\frac{1}{|\lambda x-\lambda y|}\, d\mu(x)\, d\mu(y) = I(\mu) - \frac{1}{2\pi}\log\lambda. \end{aligned}$$

Taking the infimum, we derive

$$\inf_{\eta\in M_1(\lambda E)} I(\eta) = \inf_{\mu\in M_1(E)} I(\mu) - \frac{1}{2\pi}\log\lambda,$$

So we get

$$\mathrm{Cap}_W(\lambda E) = \frac{1}{\dfrac{1}{\mathrm{Cap}_W(E)} - \frac{1}{2\pi}\log\lambda}.$$

In particular, notice that for λ big enough we have $\mathrm{Cap}_W(\lambda E) < 0$ [3]. On the contrary, in the case $d \geqslant 3$, Newtonian capacity is always non-negative. The rather strange behavior of the Wiener capacity under dilations and other related technical issues is one of the motivations for the introduction of logarithmic capacity. Clearly, $\mathrm{Cap}_L(E) \geqslant 0$ for any compact set E, and moreover for any $\lambda > 0$,

$$\mathrm{Cap}_L(\lambda E) = e^{-\frac{2\pi}{\mathrm{Cap}_W(E)} + \log\lambda} = \lambda\,\mathrm{Cap}_L(E).$$

So the logarithmic capacity is homogeneous of degree 1.

Remark 6.6. Note that given a bounded set E, the potential of the Lebesgue measure restricted to E is bounded. In particular, if E has positive Lebesgue measure then its capacity is not zero. One can also check that if U_μ is a bounded potential, then μ must vanish for sets of capacity zero.

Lemma 6.7 (Outer regularity of capacity)**.** *For any compact set $E \subset \mathbb{R}^d$ and let V_n, $n \geqslant 1$, a decreasing sequence (i.e., $V_n \supset V_{n+1}$) of open sets such that and $E = \bigcap_n V_n$. Then*

$$\lim_{n\to\infty} \mathrm{Cap}(V_n) = \mathrm{Cap}(E) \quad \textit{for } d \geqslant 3$$

and

$$\lim_{n\to\infty} \mathrm{Cap}_L(V_n) = \mathrm{Cap}_L(E) \quad \textit{for } d = 2.$$

Proof. This is a straightforward consequence of the semicontinuity property of the energies $I(\mu_n)$ in Lemma 6.1 and Theorems 4.11 and 4.12. We leave the details for the reader. □

[3] Also, formally, $\mathrm{Cap}_W(\lambda E) = \infty$ in case that $\dfrac{1}{\mathrm{Cap}_W(E)} = \frac{1}{2\pi}\log\lambda$.

We say that a property holds *q.e.* (quasi everywhere) if it holds except on a set of capacity zero.

Theorem 6.8 (Existence of equilibrium measure)**.** *Let $E \subset \mathbb{R}^d$ be a compact set with $\mathrm{Cap}(E) > 0$. There exists a Radon probability measure μ supported on E such that*

$$\mathrm{Cap}(E) = \frac{1}{I(\mu)}.$$

Further, any such measure satisfies $U_\mu(x) = (\mathrm{Cap}\, E)^{-1}$ q.e. $x \in E$ and $U_\mu(x) \leqslant (\mathrm{Cap}\, E)^{-1}$ for all $x \in E$.

Proof. Remark first that, for the case $d = 2$, by contracting E suitably, we can assume that $\mathrm{diam}(E) \leqslant 1/2$, so that $\mathcal{E}(x - y) > 0$ for all $x, y \in E$.

Let

$$\gamma := \inf\{I(\mu) : \mathrm{supp}\mu \subset E \text{ and } \mu(E) = 1\}. \tag{6.4}$$

By the lower semicontinuity of I, see Lemma 6.1 c), there exists a measure μ realizing this infimum. Since all the measures in the infimum are supported in the compact set E, so is the minimizer μ, which is also a probability measure, see Theorems 4.11 and 4.12.

Next we claim that

$$U_\mu(x) \geqslant \gamma \text{ q.e. } x \in E. \tag{6.5}$$

We prove this claim by contradiction. Let

$$T_\varepsilon := \{x \in E : U_\mu(x) < \gamma - \varepsilon\}$$

and assume that $\mathrm{Cap}(T_\varepsilon) > 0$. Then there exists a probability measure τ supported on T_ε with $I(\tau) < \infty$. By Chebyshev and reducing and rescaling τ if necessary, we may assume that $U_\tau(x) \leqslant K < \infty$ for a suitable $K > 0$. For $\delta \in (0, 1)$, let

$$\mu_\delta := (1 - \delta)\mu + \delta\tau,$$

which is also a probability measure. Note that

$$\begin{aligned} I(\mu_\delta) &= \iint \mathcal{E}(x - y)\,((1 - \delta)d\mu(y) + \delta d\tau(y))\,((1 - \delta)\,d\mu(x) + \delta\,d\tau(x)) \\ &= (1 - \delta)^2 I(\mu) + 2\delta(1 - \delta) \iint \mathcal{E}(x - y)\,d\mu\,d\tau + \delta^2 I(\tau) \\ &= \gamma - 2\delta\gamma + 2\delta \int U_\mu d\tau + o(\delta^2) \leqslant \gamma - 2\delta\gamma + 2\delta(\gamma - \varepsilon) + o(\delta^2) < \gamma \end{aligned}$$

for δ small enough. This contradicts the fact that μ minimizes (6.4). Therefore, $\mathrm{Cap}(T_\varepsilon) = 0$ for every $\varepsilon > 0$, that is, the claim (6.5) holds.

We also claim that

$$U_\mu(x) \leqslant \gamma \text{ for every } x \in E. \tag{6.6}$$

Let $\nu := \mu|_{T_\varepsilon}$. Then $U_\nu(x) \leqslant U_\mu(x) < \gamma - \varepsilon$ for $x \in T_\varepsilon$. By the maximum principle U_ν is bounded and therefore $\nu(T_\varepsilon) = 0$ (see Remark 6.6), i.e., $\mu(T_\varepsilon) = 0$. Since $T_\varepsilon \nearrow T_0$, by Lemma 4.3 we get that $\mu(T_0) = 0$. We have that

$$\gamma = I(\mu) = \int_{\{U_\mu > \gamma\}} U_\mu \, d\mu + \int_{\{U_\mu = \gamma\}} U_\mu \, d\mu + \int_{\{U_\mu < \gamma\}} U_\mu \, d\mu.$$

The third integral is zero and therefore, since μ is a probability measure, we infer that the first integral must be zero as well, so $\mu(\{U_\mu > \gamma\}) = 0$ and therefore (6.6) holds μ-almost everywhere. The lower semicontinuity property of U_μ (see Lemma 6.1 a)) implies that (6.6) holds everywhere in the support of μ and by the maximum principle it holds everywhere. □

We will show soon that, for a compact set E with positive capacity, the probability measure μ supported on E such that $\mathrm{Cap}(E) = \frac{1}{I(\mu)}$ is unique. This probability measure μ is called the equilibrium measure of E, and its potential U_μ, the equilibrium potential of E.

Corollary 6.9. *Let E be compact with $\mathrm{Cap}(E) > 0$ and let μ be an equilibrium measure of E. Let ν be another Radon measure and let $A = \{x \in E : U_\nu(x) < \infty\}$. Then U_μ equals $(\mathrm{Cap}\, E)^{-1}$ ν-a.e. in A.*

Proof. In the case $d = 2$, we assume that $E \subset B_{1/2}(0)$. For $k > 1$, let $A_k = \{x \in E : U_\nu(x) \leqslant k$ and $U_\mu(x) < (\mathrm{Cap}(E))^{-1}\}$. If $\nu(A_k) > 0$, then the (non-zero) measure $\tau = \nu|_{A_k}$ satisfies

$$U_\tau(x) \leqslant U_\nu(x) \leqslant k \quad \text{for all } x \in A_k.$$

So we deduce that $I(\tau) < +\infty$ and so $\mathrm{Cap}(A_k) > 0$. This contradicts the fact that $U_\mu(x) = (\mathrm{Cap}(E))^{-1}$ q.e. in E. □

Before proving the uniqueness of the equilibrium measure, we need to prove the following positivity result for the energy of signed measures. Remark that for a signed measure, its potential and its energy are defined in the same way as in (6.1) and (6.2), as soon as the corresponding integrals make sense.

Theorem 6.10. *Let ν be a compactly supported Radon signed measure in $\mathbb{R}^d$ such that $I(|\nu|) < \infty$. Assume also that $\nu(\mathbb{R}^d) = 0$ in the case $d = 2$. Then*

$$I(\nu) \geqslant 0.$$

Further, $I(\nu) > 0$ unless $\nu = 0$.

The fact that $I(\nu)$ is always non-negative (under the assumptions above) is quite remarkable. Observe that in the case $d = 2$ the assumption that $\nu(\mathbb{R}^d) = 0$ cannot be eliminated. Indeed, if E is a compact set with $\mathrm{Cap}_L(E) > 1$, then its equilibrium measure μ satisfies $I(\mu) < 0$.

Proof. Assume first that, besides satisfying the assumptions in the theorem, ν is of the form $\nu = g\,\mathcal{L}^d$, where $\mathcal{L}^d$ is the Lebesgue measure and $g \in C_c^\infty(\mathbb{R}^d)$. Then $\mathcal{E} * g$ is a C^∞ function and we have

$$g = -\Delta(\mathcal{E} * g).$$

In the case $d \geqslant 3$, since $0 \leqslant \mathcal{E}(x) \lesssim |x|^{2-d}$, we have

$$|\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d-2}} \quad \text{and} \quad |\nabla\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d-1}} \tag{6.7}$$

as $x \to \infty$. Then, by integrating by parts, it easily follows that

$$I(g\mathcal{L}^d) = \int (\mathcal{E} * g)\, g\, d\mathcal{L}^d = -\int (\mathcal{E} * g)\, \Delta(\mathcal{E} * g)\, d\mathcal{L}^d \overset{(6.7)}{=} \int |\nabla(\mathcal{E} * g)|^2\, d\mathcal{L}^d \tag{6.8}$$

(notice that all the integrals above make sense because of (6.7). In the case $d = 2$, since $\nu(\mathbb{R}^d) = 0$, it is immediate to check that we have the improved decay

$$|\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d-1}} \quad \text{and} \quad |\nabla\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d}} \tag{6.9}$$

as $x \to \infty$. Then we can integrate by parts again to deduce that (6.8) also holds. In any case, in particular, the identity (6.8) shows that $I(g\mathcal{L}^d) \geqslant 0$.

Consider now an arbitrary signed measure satisfying the assumptions of the theorem. Consider a radial non-increasing C^∞ bump function φ such that $0 \leqslant \varphi \leqslant \chi_{B_2(0)}$ with $\int \varphi = 1$ and, for $\varepsilon > 0$, set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(\varepsilon^{-1}x)$. Then the measure $\nu_\varepsilon = \varphi_\varepsilon * \nu$ is of the form $\nu_\varepsilon = g_\varepsilon$, with $g_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, and has zero mean in the case $d = 2$. So by (6.8) it holds

$$I(\nu_\varepsilon) = \int |\nabla\mathcal{E} * \nu_\varepsilon|^2\, d\mathcal{L}^d \geqslant 0. \tag{6.10}$$

So to prove that $I(\nu) \geqslant 0$ it suffices to show that $I(\nu_\varepsilon) \to I(\nu)$ as $\varepsilon \to 0$. To this end, applying Fubini we write

$$I(\nu_\varepsilon) = \int (\varphi_\varepsilon * \mathcal{E} * \nu)\, \varphi_\varepsilon * \nu\, d\mathcal{L}^d = \int (\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E} * \nu)\, d\nu.$$

Observe now that, for any $x \in \mathbb{R}^d$, since $\varphi_\varepsilon * \varphi_\varepsilon$ is C^∞ with unitary mass, radial non-increasing, and compactly supported, then it is a convex combination of functions of the form $\frac{1}{m(B_r(0))}\chi_{B_r(0)}$ (see the proof of Lemma 5.8). Since $\mathcal{E}$ is superharmonic, by Lemma 5.8,

$$\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E}(x) \leqslant \mathcal{E}(x) \quad \text{for all } x \in \mathbb{R}^d \tag{6.11}$$

(this could also be checked by a direct computation), and also $\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E}(x) \to \mathcal{E}(x)$ as $\varepsilon \to 0$ for all $x \neq 0$.

We claim that in the case $d = 2$ we can assume that $\operatorname{supp}\nu \subset B_{1/4}(0)$. Indeed, for any $\lambda > 0$, consider the dilation $T_\lambda x = \lambda x$. Then, for a suitable $\lambda > 0$, it turns out that the image measure $(T_\lambda)_\# \nu$ is supported on $B_{1/4}(0)$ and it satisfies

$$\begin{aligned} I((T_\lambda)_\#\nu) &= \frac{1}{2\pi}\iint \log\frac{1}{|x-y|}\,d(T_\lambda)_\#\nu(x)\,d(T_\lambda)_\#\nu(y) \\ &= \frac{1}{2\pi}\iint \log\frac{1}{|\lambda x-\lambda y|}\,d\nu(x)\,d\nu(y) = I(\nu) - \frac{1}{2\pi}\,\nu(\mathbb{R}^d)^2\log\lambda = I(\nu), \end{aligned}$$

which yields the claim.

So for any $d \geqslant 2$ and ε small enough we can assume that $\mathcal{E}(x-y) > 0$ for all $x, y \in \mathrm{supp}\nu \cup \mathrm{supp}\nu_\varepsilon$. Then, by the dominated convergence theorem, for all $x \in \mathrm{supp}\nu$ such that $\mathcal{E} * |\nu|(x) < \infty$, taking into account (6.11) and the fact that $\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E}(x) \to \mathcal{E}(x)$ for all $x \neq 0$, it follows that

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E} * \nu(x) = \mathcal{E} * \nu(x),$$

and moreover $\mathcal{E} * \nu(x) \leqslant \mathcal{E} * |\nu|(x)$. By another application of dominated convergence, since $I(|\nu|) < \infty$, we infer that

$$\lim_{\varepsilon\to 0} I(\nu_\varepsilon) = \lim_{\varepsilon\to 0}\int (\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E} * \nu)\,d\nu = I(\nu), \tag{6.12}$$

which concludes the proof of the fact that $I(\nu) \geqslant 0$.

Next suppose that $I(\nu) = 0$. From the identity in (6.10) and (6.12) , we deduce that

$$\lim_{\varepsilon\to 0}\int |\nabla\mathcal{E} * \nu_\varepsilon|^2\,d\mathcal{L}^d = 0.$$

By an easy application of Fubini's theorem, it follows that $\mathcal{E} * \nu \in L^1_{loc}(\mathbb{R}^d)$. Now, we can compute the distributional Laplacian of the induced distribution, which happens to be precisely $\Delta(\mathcal{E} * \nu) = -\nu$. On the other hand, it is well know that $\mathcal{E} * \nu_\varepsilon = \varphi_\varepsilon * \mathcal{E} * \nu$ tends to $\mathcal{E} * \nu$ in $L^1_{loc}(\mathbb{R}^d)$, that is in $L^1(B_r(0))$ for any $r > 0$. Together with the Poincaré inequality, denoting by $m_{B_r(0)}(\mathcal{E} * \nu)$ the mean of $\mathcal{E} * \nu$ in $B_r(0)$, this implies

$$\begin{aligned} \fint_{B_r(0)} |\mathcal{E} * \nu - m_{B_r(0)}(\mathcal{E} * \nu)|\,d\mathcal{L}^d &= \lim_{\varepsilon\to 0}\fint_{B_r(0)} |\mathcal{E} * \nu_\varepsilon - m_{B_r(0)}(\mathcal{E} * \nu_\varepsilon)|\,d\mathcal{L}^d \\ &\lesssim \lim_{\varepsilon\to 0}\left(\fint_{B_r(0)} |\nabla(\mathcal{E} * \nu_\varepsilon)|^2\,d\mathcal{L}^d\right)^{1/2} r(B) = 0. \end{aligned}$$

So we deduce that $\mathcal{E} * \nu$ is constant a.e. with respect to Lebesgue measure. Since this happens for any ball $B_r(0)$ and $\mathcal{E} * \nu$ tends to 0 at ∞, it turns out that $\mathcal{E} * \nu$ vanishes a.e. Then, from the fact that $\nu = -\Delta(\mathcal{E} * \nu)$ in the sense of distributions, we infer that $\nu = 0$. □

Theorem 6.11. *Let $E \subset \mathbb{R}^d$ be a compact set with* $\mathrm{Cap}(E) > 0$. *Then the equilibrium measure for E is unique.*

Proof. Aiming for a contradiction, suppose that there are two equilibrium measures μ and ν for E. For $t \in (0,1)$, consider the measure

$$\sigma_t = t\,\mu + (1-t)\,\nu.$$

Obviously, σ_t is a probability measure. Let us see that $I(\sigma_t) < I(\mu)$ for t small enough. Indeed, we have

$$\begin{aligned} I(\sigma_t) = \int \mathcal{E} * \sigma_t\, d\sigma_t &= t^2\, I(\mu) + t(1-t)\int \mathcal{E} * \mu\, d\nu + t(1-t)\int \mathcal{E} * \nu\, d\mu + (1-t)^2\, I(\nu) \\ &= (1-2t)\, I(\nu) + t\int \mathcal{E} * \mu\, d\nu + t\int \mathcal{E} * \nu\, d\mu + O(t^2). \end{aligned}$$

The sum of the two integrals on the right hand side can be rewritten as

$$\begin{aligned} \int \mathcal{E} * \mu\, d\nu + \int \mathcal{E} * \nu\, d\mu &= \int \mathcal{E} * (\mu-\nu)\, d\nu + I(\nu) + \int \mathcal{E} * (\nu-\mu)\, d\mu + I(\mu) \\ &= 2I(\nu) - \int \mathcal{E} * (\mu-\nu)\, d(\mu-\nu) = 2\, I(\nu) - I(\mu-\nu) \end{aligned}$$

From the identities above, we deduce

$$I(\sigma_t) = (1-2t)\, I(\nu) + 2t\, I(\nu) - tI(\mu-\nu) + O(t^2) = I(\nu) - tI(\mu-\nu) + O(t^2).$$

By Theorem 6.10, if $\mu \neq \nu$, then $I(\mu-\nu) > 0$, and so $I(\sigma_t) < I(\nu) = I(\mu)$ for t small enough, which yields the desired contradiction. □

Theorem 6.12. *Let $E \subset \mathbb{R}^d$ be compact, and suppose also that $\operatorname{diam}(E) < 1$ in the case $d = 2$. Then we have*

$$\operatorname{Cap}(E) = \sup\Big\{\mu(E) : \mu \in M_+(\mathbb{R}^d),\ \operatorname{supp}\mu \subset E,\ \sup_{\mathbb{R}^d} U_\mu \leqslant 1\Big\}. \tag{6.13}$$

Here $M_+(E)$ stands for the set of (non-negative) Radon measure supported on E.

Proof. The fact that $\operatorname{diam}(E) < 1$ in the case $d = 2$ implies that $\mathcal{E}(x-y) \geqslant \frac{1}{2\pi}\log\frac{1}{\operatorname{diam}(E)} > 0$ for all $x, y \in E$, which in turn implies that $I(\mu)$ is positive and bounded away from 0 for any measure μ supported on E, and so $\operatorname{Cap}_W(E) = \operatorname{Cap}(E) \geqslant 0$.

Denote by S_E the supremum in (6.13). In case $\operatorname{Cap}(E) = 0$, then every $\mu \in M_+(E)$ satisfies $I(\mu) = +\infty$. In particular, we infer that the potential U_μ is not bounded above in the support of μ. Thus, the only measure in the left-hand side of (6.13) is the null measure and $S_E = 0 = \operatorname{Cap}(E)$.

Let us assume $\operatorname{Cap}(E) > 0$. The fact that $\operatorname{Cap}(E) \geqslant S_E$ is immediate: for $\varepsilon > 0$, let μ be supported on E such that $\sup_{\mathbb{R}^d} U_\mu \leqslant 1$ and such that $\mu(E) + \varepsilon \geqslant S_E$. Consider the probability measure $\nu = \mu(E)^{-1}\mu$. Then

$$I(\nu) = \mu(E)^{-2}\, I(\mu) = \mu(E)^{-2}\int U_\mu(x)\, d\mu(x) \leqslant \mu(E)^{-1}.$$

Therefore,

$$\operatorname{Cap}(E) \geqslant I(\nu)^{-1} \geqslant \mu(E) \geqslant S_E - \varepsilon.$$

For the converse inequality, consider the equilibrium measure ν of E, so that $U_\nu(x) \leqslant \operatorname{Cap}(E)^{-1}$ for all $x \in \mathbb{R}^d$, by Theorem 6.8 and Theorem 6.3. Then the measure $\mu = \operatorname{Cap}(E)\,\nu$ satisfies $\sup_{\mathbb{R}^d} U_\mu \leqslant 1$ in $\mathbb{R}^d$ and thus $S_E \geqslant \mu(E) = \operatorname{Cap}(E)$. □

Remark that the supremum in (6.13) is attained for E uniquely by the measure $\operatorname{Cap}(E)\,\nu$, where ν stands for the equilibrium measure of E. This can be shown arguing as in Theorem 6.12.

Corollary 6.13 (Subadditivity of capacity)**.** *For Borel sets $E_n \subset \mathbb{R}^d$, with $\operatorname{diam}(\bigcup_n E_n) < 1$ in the case $d = 2$, we have*

$$\operatorname{Cap}\left(\bigcup_n E_n\right) \leqslant \sum_n \operatorname{Cap}(E_n).$$

Proof. Let $F \subset \bigcup_n E_n$ be compact and let μ be supported on $\bigcup_n E_n$ be such that $\|U_\mu\|_\infty \leqslant 1$ in $\mathbb{R}^d$ and $\mu(F) = \operatorname{Cap}(F)$. Then $\|U_{\chi_{E_n \cap F}\mu}\|_\infty \leqslant \|U_\mu\|_\infty \leqslant 1$ for any n, and thus $\mu(E_n \cap F) \leqslant \operatorname{Cap}(E_n \cap F) \leqslant \operatorname{Cap}(E_n)$. Therefore,

$$\operatorname{Cap}(F) = \mu(F) \leqslant \sum_n \mu(E_n \cap F) \leqslant \sum_n \operatorname{Cap}(E_n).$$

Since this holds for any compact set $F \subset \bigcup_n E_n$, we are done since, by the definition of capacity,

$$\operatorname{Cap}(E) = \sup_{F \subset E : F \text{ is compact}} \operatorname{Cap} F.$$

□

Lemma 6.14. *For any Radon measure μ in $\mathbb{R}^d$ with compact support and let $\lambda > 0$. In the case $d \geqslant 3$ we have*

$$\operatorname{Cap}\left(\{x \in \mathbb{R}^d : U_\mu(x) \geqslant \lambda\}\right) \leqslant \frac{\|\mu\|}{\lambda}.$$

In the case $d = 2$,

$$\operatorname{Cap}\left(\{x \in B_{1/2}(0) : U_\mu(x) \geqslant \lambda\}\right) \leqslant \frac{\|\mu\|}{\lambda}.$$

Proof. Consider a compact set $E \subset \{x \in \mathbb{R}^d : U_\mu(x) \geqslant \lambda\}$ (in the case $d = 2$, $E \subset \{x \in B_{1/2}(0) : U_\mu(x) \geqslant \lambda\}$) and let ν be supported on E be such that $\sup_{\mathbb{R}^d} U_\nu \leqslant 1$ and $\operatorname{Cap}(E) = \nu(E)$. Then we have

$$\operatorname{Cap}(E) = \nu(E) \leqslant \frac{1}{\lambda} \int U_\mu \, d\nu = \frac{1}{\lambda} \int U_\nu \, d\mu \leqslant \frac{\|\mu\|}{\lambda}.$$

Taking the supremum on such sets E, the lemma follows. □

Proposition 6.15. *For a ball $\bar{B} \subset \mathbb{R}^d$, we have*

$$\mathrm{Cap}(\bar{B}) = (d-2)\kappa_d\, r(\bar{B})^{d-2} \qquad \text{if } d \geqslant 3,$$

and

$$\mathrm{Cap}_L(\bar{B}) = r(\bar{B}) \qquad \text{if } d = 2.$$

Proof. Without loss of generality, assume that $\bar{B}$ is centered in the origin and that it is closed. In the case $d = 2$, by homogeneity we can assume $r(\bar{B}) < 1/2$. Let $x \in \bar{B}^c$ and notice that $\mathcal{E}^x(y) := \mathcal{E}(x-y)$ is harmonic in the interior of $\bar{B}$. Let σ be the surface measure on $\partial\bar{B}$. Then by the mean value theorem,

$$U_\sigma(x) = \int_{\partial\bar{B}} \mathcal{E}(x-y)\, d\sigma(y) = \sigma(\partial\bar{B})\, \mathcal{E}(x-0) = \sigma(\partial\bar{B})\, \mathcal{E}(x).$$

Note that U_σ is constant in $\partial\bar{B}$ by symmetry, and therefore it is continuous in $\mathbb{R}^d$ by the continuity principle. Thus, the same identity holds on $\partial\bar{B}$. Therefore, using also the maximum principle, in the case $d \geqslant 3$, we get

$$\sup_{\mathbb{R}^d} U_\sigma = \sup_{\partial\bar{B}} U_\sigma = \sigma(\partial\bar{B})\, \mathcal{E}(r(\bar{B})) = \frac{\kappa_d\, r(\bar{B})^{d-1}}{(d-2)\kappa_d\, r(\bar{B})^{d-2}} = \frac{r(\bar{B})}{d-2}.$$

Therefore, the measure $\mu = (d-2)r(\bar{B})^{-1}\sigma$ satisfies $\sup_{\mathbb{R}^d} U_\mu = 1$ and so

$$\mathrm{Cap}(\bar{B}) \geqslant \mu(\bar{B}) = (d-2)r(\bar{B})^{-1}\sigma(\bar{B}) = (d-2)\kappa_d\, r(\bar{B})^{d-2}.$$

For the converse estimate, remark that in fact the measure μ satisfies $U_\mu \equiv 1$ in $\partial\bar{B}$. Since μ is supported on $\partial\bar{B}$ and U_μ is harmonic in the interior of $\bar{B}$ and continuous in its closure, by the maximum principle it is identically 1 in the whole $\bar{B}$. Then, from Lemma 6.14 we deduce that $\mathrm{Cap}(\bar{B}) \leqslant \mu(\bar{B}) = (d-2)\kappa_d\, r(\bar{B})^{d-2}$, which proves the lemma in the case $d \geqslant 3$.

In the case $d = 2$ we argue analogously. Indeed, it is straightforward to check that, for all $x \in \partial\bar{B}$ we have we have $U_\sigma(x) = r(\bar{B}) \log \frac{1}{r(\bar{B})}$. Then, by the same arguments as before, it follows that

$$\mathrm{Cap}_W(\bar{B}) = \frac{2\pi}{\log \frac{1}{r(\bar{B})}},$$

and so $\mathrm{Cap}_L(\bar{B}) = r(\bar{B})$. ☐

As a corollary of the preceding estimate for the logarithmic capacity, we obtain:

Corollary 6.16. *Let μ be Radon measure supported on the (open) ball $B_1(0) \subset \mathbb{R}^2$. Then $I(\mu) > 0$.*

Proof. Let $E = \mathrm{supp}\mu$. Since $E \subset B_1(0)$, there exists some $\rho \in (0,1)$ such that $E \subset B_\rho(0)$. Consequently, $\mathrm{Cap}_L(E) \leqslant \mathrm{Cap}_L(\bar{B}_\rho(0)) = \rho < 1$. Thus, $e^{-2\pi I(\mu)} < 1$, which implies that $I(\mu) > 0$. ☐

A quick inspection of the arguments above shows that $\mathrm{Cap}(\bar{B}) = \mathrm{Cap}(\partial \bar{B})$ for any ball. This also holds for any arbitrary compact set. In fact, we show below that the capacity of a compact set equals the capacity of its outer boundary. For $E \subset \mathbb{R}^d$ compact, its outer boundary, denoted by $\partial_o E$, is the boundary of the unbounded component of $\mathbb{R}^d \backslash E$.

Theorem 6.17. *For any compact set $E \subset \mathbb{R}^d$, we have $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$ (and so $\mathrm{Cap}_L(E) = \mathrm{Cap}_L(\partial_o E)$ in the case $d = 2$).*

Proof. First we show that $\mathrm{Cap}(E) = \mathrm{Cap}(\partial E)$. To this end, it suffices to show that the equilibrium measure μ of E is supported on ∂E (in the case $d = 2$, if necessary, we can assume that $E \subset B_{1/2}(0)$). To prove this, recall that by Theorem 6.8 $U_\mu(x) = (\mathrm{Cap}\, E)^{-1}$ q.e. $x \in E$. In particular, this holds a.e. in the interior of E with respect to Lebesgue measure, see Remark 6.6. Since $-\Delta U_\mu = \mu$ in the sense of distributions, for any C^∞ function φ supported on the interior of E, it holds

$$\int \varphi \, d\mu = -\langle U_\mu, \Delta \varphi \rangle = -(\mathrm{Cap}\, E)^{-1} \int_{\mathrm{supp}\varphi} \Delta \varphi = 0.$$

Thus μ vanishes identically on the interior of E, which shows that $\mathrm{supp}\mu \subset \partial E$.

To show that $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$, let Ω be the unbounded component of $\mathbb{R}^d \backslash E$ and let $\widehat{E} = \mathbb{R}^d \backslash \Omega$ (so that $\widehat{E}$ coincides with the union of E and the bounded components of $\mathbb{R}^d \backslash E$). Then we have $\partial_o E = \partial \widehat{E}$ and

$$\partial_o E \subset \partial E \subset E \subset \widehat{E}.$$

Since $\mathrm{Cap}(\widehat{E}) = \mathrm{Cap}(\partial_o E)$, we also have $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$. $\square$

Remark 6.18. From the uniqueness of the equilibrium measure and the fact that $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$, it follows that the equilibrium measure of E is supported on $\partial_o E$.

6.3 Relationship between Hausdorff content and capacity

Lemma 6.19. *Let $E \subset \mathbb{R}^d$ be compact and $d - 2 < s \leqslant d$. In the case $d \geqslant 3$, we have*

$$\mathcal{H}^s_\infty(E)^{\frac{d-2}{s}} \lesssim_{s,d} \mathrm{Cap}(E) \lesssim_d \mathcal{H}^{d-2}_\infty(E).$$

In the case $d = 2$, we have

$$\mathrm{Cap}_L(E) \gtrsim_s \mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

Proof. First we consider the case $d \geqslant 3$. To check that $\mathrm{Cap}(E) \lesssim \mathcal{H}^{d-2}_\infty(E)$, for any $\varepsilon > 0$ we consider a covering of E by a family of open balls B_i, $i \geqslant 1$, such that

$$\sum_i r(B_i)^{d-2} \lesssim_d \mathcal{H}^{d-2}_\infty(E) + \varepsilon.$$

Since E is compact, we may assume that the family of balls B_i is finite. Then, using the subadditivity of the Newtonian capacity (see Corollary 6.13) and Proposition 6.15, we get

$$\mathrm{Cap}(E) \leqslant \sum_i \mathrm{Cap}(\bar{B}_i) \approx \sum_i r(B_i)^{d-2} \lesssim_d \mathcal{H}^{d-2}_\infty(E) + \varepsilon,$$

which shows that $\mathrm{Cap}(E) \lesssim_d \mathcal{H}^{d-2}_\infty(E)$.

To see that $\mathrm{Cap}(E) \gtrsim_{s,d} \mathcal{H}^s_\infty(E)^{\frac{d-2}{s}}$, we apply Frostman's Lemma 4.15. This tells us that there exists some Borel measure μ supported on E such that

$$\mathcal{H}^s_\infty(E) \approx_d \mu(E) \tag{6.14}$$

and

$$\mu(B_r(x)) \leqslant r^s \quad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0. \tag{6.15}$$

Then, for all $x \in \mathbb{R}^d$ we have

$$\begin{aligned} c\, U_\mu(x) &= \int \frac{1}{|x-y|^{d-2}}\, d\mu(y) = \int_0^\infty \mu\big(\{y : |x-y|^{2-d} > t\}\big)\, dt \\ &= \int_0^\infty \mu\big(B\big(x, t^{\frac{1}{2-d}}\big)\big)\, dt \overset{(6.15)}{\leqslant} \int_0^{\mu(E)^{\frac{2-d}{s}}} \mu(E)\, dt + \int_{\mu(E)^{\frac{2-d}{s}}}^\infty t^{\frac{s}{2-d}}\, dt \approx_{s,d} \mu(E)^{1-\frac{d-2}{s}}. \end{aligned}$$

Therefore,

$$\mathrm{Cap}(E) \overset{(6.13)}{\geqslant} \frac{\mu(E)}{\|U_\mu\|_\infty} \gtrsim_{s,d} \frac{\mu(E)}{\mu(E)^{1-\frac{d-2}{s}}} = \mu(E)^{\frac{d-2}{s}} \overset{(6.14)}{\approx}_d \mathcal{H}^s_\infty(E)^{\frac{d-2}{s}}.$$

In the case $d = 2$, we may and will assume that $\mathrm{diam}(E) < 1$ since, for any $\lambda > 0$.

$$\mathrm{Cap}_L(\lambda E) = \lambda\, \mathrm{Cap}_L(E) \quad \text{ and } \quad \mathcal{H}^s_\infty(\lambda E)^{\frac{1}{s}} = \lambda\, \mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

We apply again Frostman's Lemma to get a measure μ supported on E satisfying (6.14) and (6.15). Then, for any $\tau \geqslant 0$ for $x \in \mathrm{supp}\mu$ we have

$$\begin{aligned} 2\pi\, U_\mu(x) &= \int \log \frac{1}{|x-y|}\, d\mu(y) = \int_0^\infty \mu\Big(\Big\{y : \log\frac{1}{|x-y|} > t\Big\}\Big)\, dt \\ &= \int_0^\infty \mu\big(B\big(x, e^{-t}\big)\big)\, dt \overset{(6.15)}{\leqslant} \int_0^\tau \mu(E)\, dt + \int_\tau^\infty e^{-ts}\, dt = \tau\, \mu(E) + \frac{1}{s} e^{-\tau s}. \end{aligned}$$

We choose $\tau = -\frac{1}{s}\log\mu(E)$ (notice that $\tau \geqslant 0$ because $\mu(E) \overset{(6.15)}{<} 1$, since $\mathrm{diam}(E) < 1$), and then we obtain

$$2\pi\, U_\mu(x) \leqslant \frac{\mu(E)}{s}\left(\log\frac{1}{\mu(E)} + 1\right).$$

Hence, for the probability measure $\sigma = \mu(E)^{-1}\mu$, we have

$$2\pi\, I(\sigma) \leqslant \frac{1}{s}\left(\log\frac{1}{\mu(E)} + 1\right).$$

Therefore,

$$\mathrm{Cap}_W(E) \geqslant \frac{1}{I(\sigma)} \geqslant \frac{2\pi s}{\log \frac{1}{\mu(E)} + 1},$$

or equivalently,

$$\mathrm{Cap}_L(E) \geqslant e^{\frac{\log \mu(E) - 1}{s}} = C(s)\, \mu(E)^{\frac{1}{s}} \overset{(6.14)}{\approx}_s \mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

□

Remark 6.20. It can be shown that if $\mathcal{H}^{d-2}(E) < \infty$, then $\mathrm{Cap}(E) = 0$. See [Mat95, Theorem 8.7], for example.

6.4 Wiener's criterion

Given a bounded open set $\Omega \subset \mathbb{R}^d$, by Theorem 5.30 and Theorem 5.31, a point $\xi \in \partial\Omega$ is regular (for the Dirichlet problem) if and only if there is a barrier function for ξ in Ω. In this section we show a characterization of more metric-geometric type. This is the so called Wiener's criterion.

Theorem 6.21 (Wiener's criterion)**.** *For $d \geqslant 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $\xi \in \partial\Omega$. The following are equivalent:*

(a) ξ is a regular point.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k}) \backslash \Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty.$

Here $\bar{A}(\xi, r_1, r_2)$ denotes the closed annulus centered at ξ with inner radius r_1 and outer radius r_2. Recall also that in the case $d \geqslant 3$, $\mathrm{Cap}(\bar{B}(\xi, 2^{-k})) \approx 2^{-k(d-2)}$, and in the case $d = 2$, $\mathrm{Cap}(\bar{B}(\xi, 2^{-k})) = \mathrm{Cap}_W(\bar{B}(\xi, 2^{-k})) \approx 1/k$. Thus, in the latter case, the condition (b) is equivalent to

(b') $\displaystyle\sum_{k=1}^{\infty} k\; \mathrm{Cap}_W(\bar{A}(\xi, 2^{-k-1}, 2^{-k}) \backslash \Omega) = \infty.$

Remark 6.22. In the case $d \geqslant 3$, the condition (b) is equivalent to

(b") $\displaystyle\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}) \backslash \Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty.$

Indeed, it is trivial that (b) $\Rightarrow$ (b"). To see that (b") $\Rightarrow$ (b) we use the subadditivity of Newtonian capacity to write

$$\sum_{k\geqslant 1}\frac{\mathrm{Cap}(\bar{B}(\xi,2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi,2^{-k}))}\lesssim\sum_{k\geqslant 1}\sum_{j\geqslant k}\frac{\mathrm{Cap}(\bar{A}(\xi,2^{-j-1},2^{-j})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi,2^{-j}))}$$
$$=\sum_{j\geqslant 1}\mathrm{Cap}(\bar{A}(\xi,2^{-j-1},2^{-j})\backslash\Omega)\sum_{k\leqslant j}\frac{1}{\mathrm{Cap}(\bar{B}(\xi,2^{-j}))}.$$

Now observe that the last sum on the right hand side is comparable to $\sum_{k\leqslant j}2^{k(d-2)}\approx 2^{j(d-2)}\approx\mathrm{Cap}(\bar{B}(\xi,2^{-j}))^{-1}$. Thus,

$$\sum_{k\geqslant 1}\frac{\mathrm{Cap}(\bar{B}(\xi,2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi,2^{-k}))}\lesssim\sum_{j\geqslant 1}\frac{\mathrm{Cap}(\bar{A}(\xi,2^{-j-1},2^{-j})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi,2^{-j}))},$$

which yields the desired implication.

6.4.1 Sufficiency of the criterion for Wiener regularity

Proof of (b) $\Rightarrow$ (a) in Theorem 6.21 in the case $d\geqslant 3$. We will construct a barrier $\widetilde{w}:\Omega\to\mathbb{R}$ for the point ξ. We will show that there exists a harmonic function $w:\Omega\to\mathbb{R}$ satisfying:

(i) $\lim_{\Omega\ni x\to\xi}w(x)=1$.

(ii) $\limsup_{x\to\zeta}w(x)<1$ for all $\zeta\in\partial\Omega\backslash\{\xi\}$.

Then we just have to take $\widetilde{w}=1-w$ to get the desired barrier.

To shorten notation, write $\bar{A}_k=\bar{A}(\xi,2^{-k-1},2^{-k})$, $B_k=B(\xi,2^{-k})$, and $\bar{B}_k=\overline{B_k}$. For a fixed large constant $\Lambda\geqslant 10$ to be chosen below and for any $n_0>1$, the condition (b) ensures the existence of natural numbers N,M, with $n_0\leqslant N<M$ such that

$$\Lambda\leqslant\sum_{N\leqslant k\leqslant M}\frac{\mathrm{Cap}(\bar{A}_k\backslash\Omega)}{\mathrm{Cap}(\bar{B}_k)}\leqslant\Lambda+1$$

(notice that each summand in the sum above is at most 1). For each $k\geqslant n_0$, if $\mathrm{Cap}(\bar{A}_k\backslash\Omega)=0$, define $\mu_k\equiv 0$ and if $\mathrm{Cap}(\bar{A}_k\backslash\Omega)>0$ let μ_k be the equilibrium measure for $\bar{A}_k\backslash\Omega$. Consider the function

$$u_k(x)=\mathrm{Cap}(\bar{A}_k\backslash\Omega)\,U_{\mu_k}(x);$$

and set

$$v(x)=\sum_{N\leqslant k\leqslant M}u_k(x).$$

Claim 6.23. *Let $d\geqslant 3$. For any $\varepsilon>0$, if $\Lambda=\Lambda(\varepsilon)$ is chosen large enough, the function v satisfies*

$$v(\xi)\approx\Lambda,\tag{6.16}$$

$$v(x) \leqslant (1+\varepsilon)\, v(\xi) \quad \textit{for all } x \in \Omega, \tag{6.17}$$

$$|v(x) - v(\xi)| \leqslant C \frac{|x-\xi|}{r(\bar{B}_M)}\, v(\xi) \quad \textit{for all } x \in \Omega \cap \bar{B}_M, \tag{6.18}$$

and

$$v(x) \leqslant \frac{1}{10}\, v(\xi) \quad \textit{for all } x \in \Omega \backslash \bar{B}_{N-k_0} \textit{ if } k_0 \geqslant 2 \textit{ is large enough.} \tag{6.19}$$

Remark that the constant k_0 in the last estimate does not depend on ε. In the case $N - k_0 \leqslant 0$, we understand that $\bar{B}_{N-k_0} = 2^{k_0} \bar{B}_N$.

Proof of the Claim. The estimate (6.16) is easy: for each $k \in [N, M]$ we have

$$u_k(\xi) = \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, U_{\mu_k}(\xi) \approx \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \mathcal{E}(r(B_k)) \approx \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)}.$$

Thus,

$$v(\xi) \approx \sum_{N \leqslant k \leqslant M} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} \approx \Lambda. \tag{6.20}$$

Next we turn our attention to (6.17), which is the most delicate part of the claim. Notice first that, by the maximum principle, it suffices to prove this for $x \in \bar{B}_N \backslash B_M = \bigcup_{N \leqslant i \leqslant M} \bar{A}_i$. So fix $x \in A_i$, with $N \leqslant i \leqslant M$. For some $h \geqslant 1$ to be chosen soon, we write

$$v(x) = \sum_{k=N}^{i-h-1} u_k(x) + \sum_{k=N \vee i-h}^{M \wedge i+h} u_k(x) + \sum_{k=i+h+1}^{M} u_k(x) =: v_a(x) + v_b(x) + v_c(x).$$

To estimate $v_b(x)$ we just take into account that

$$u_k(y) \leqslant \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, U_{\mu_k}(y) \leqslant 1 \quad \text{for all } y \in \mathbb{R}^d,$$

by Theorem 6.8. So we deduce

$$v_b(x) \leqslant 2h + 1.$$

To deal with $v_a(x)$, we will use the fact that, $|x - \xi| \leqslant r(\bar{B}_i) < 2^{-k} 2^{-h}$ for $k < i - h$, implying

$$\begin{aligned} u_k(x) &= u_k(\xi) + (u_k(x) - u_k(\xi)) = u_k(\xi) + \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, (U_{\mu_k}(x) - U_{\mu_k}(\xi)) \\ &\leqslant u_k(\xi) + C\, \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \frac{|x-\xi|}{\mathrm{dist}(\xi, \bar{A}_k)^{d-1}} \\ &\leqslant u_k(\xi) + C\, \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \frac{r(\bar{B}_i)}{r(\bar{B}_k)^{d-1}} \\ &\leqslant u_k(\xi) + C\, 2^{-h} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)}. \end{aligned} \tag{6.21}$$

For $v_c(x)$, we take into account that for $k > i + h$ we get $r(\bar{B}_i) > 2^h r(\bar{B}_k)$, so

$$u_k(x) \leqslant C \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{dist}(x, \bar{A}_k)^{d-2}} \leqslant C \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{r(\bar{B}_i)^{d-2}} \leqslant C\, 2^{-h(d-2)} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)}.$$

Consequently, gathering the estimates obtained for $k < i - h$ and for $k > i + h$ and using also (6.20), we get

$$v_a(x) + v_c(x) \leqslant \sum_{N \leqslant k \leqslant M} u_k(\xi) + C\,2^{-h} \sum_{N \leqslant k \leqslant M} \frac{\operatorname{Cap}(\bar{A}_k \backslash \Omega)}{\operatorname{Cap}(\bar{B}_k)} \leqslant v(\xi) + C\,2^{-h}\,v(\xi).$$

Thus,

$$v(x) = v_a(x) + v_b(x) + v_c(x) \leqslant v(\xi) + (2h+1) + C\,2^{-h}\,v(\xi) \leqslant v(\xi)\Big(1 + \frac{C\,h}{\Lambda} + C\,2^{-h}\Big).$$

So choosing h large enough and then Λ large enough as well, (6.17) follows.

To prove (6.18), we can assume $x \in \frac{1}{2}\bar{B}_M$ because of (6.17). Arguing as in (6.21), we obtain

$$|u_k(x) - u_k(\xi)| \leqslant C\ \operatorname{Cap}(\bar{A}_k \backslash \Omega)\,\frac{|x-\xi|}{\operatorname{dist}(\xi, \bar{A}_k)^{d-1}} \lesssim C\,\frac{\operatorname{Cap}(\bar{A}_k \backslash \Omega)}{\operatorname{Cap}(\bar{B}_k)}\,\frac{|x-\xi|}{r(\bar{B}_M)}.$$

Summing over $k \in [N, M]$ and using (6.20), we deduce (6.18).

Finally we deal with (6.19). So we take $x \in \Omega \backslash \bar{B}_{N-k_0}$, for $k_0 \geqslant 2$. Then we have

$$u_k(x) \approx \frac{\operatorname{Cap}(\bar{A}_k \backslash \Omega)}{\operatorname{dist}(x, \bar{B}_k)^{d-2}} \leqslant \frac{\operatorname{Cap}(\bar{A}_k \backslash \Omega)}{2^{(d-2)k_0}\,r(\bar{B}_k)^{d-2}} \approx 2^{(2-d)k_0}\,u_k(\xi).$$

Hence, summing on $k \in [N, M]$, we obtain

$$v(x) \lesssim 2^{(2-d)k_0} \sum_{N \leqslant k \leqslant M} u_k(\xi) = 2^{(2-d)k_0}\,v(\xi).$$

□

Applying the preceding claim, we construct sequences of natural numbers N_j, M_j, and functions v_j, for $j \geqslant 1$, as follows. We choose $N_0 = 1$, $M_0 = 2$. Assuming that $N_{j-1} < M_{j-1}$ have already been chosen, by applying Claim 6.23 with some $\varepsilon \in (0, 1/2)$ to be fixed below and $n_0 = M_{j-1} + k_0$, for some $k_0 \geqslant 2$ to be fixed below too, we find $M_j > N_j \geqslant n_0$ so that the function

$$v_j(x) = \sum_{N_j \leqslant k \leqslant M_j} u_k(x)$$

satisfies (6.16), (6.17), (6.18), and (6.19) (with v_j in place of v). Now we define

$$w(x) = \sum_{j \geqslant 1} 2^{-j}\,\frac{v_j(x)}{v_j(\xi)}. \tag{6.22}$$

Obviously, $w(\xi) = 1$ and it is easy to check that w is superharmonic in $\mathbb{R}^d$ (since each function v_j is superharmonic by Lemma 6.1). Consequently,

$$\liminf_{y \to \xi} w(y) \geqslant w(\xi) = 1. \tag{6.23}$$

Our next objective is to show that

$$\limsup_{y\to\zeta} w(y) < 1 \text{ for all } \zeta \in \partial\Omega\backslash\{\xi\} \quad\text{and}\quad w(y) < 1 \text{ for all } y \in \Omega. \tag{6.24}$$

Observe that the latter condition together with (6.23) implies the condition (i) above, i.e., $\lim_{\Omega\ni y\to\xi} w(y) = 1$. To prove (6.24) it suffices to show that for any $h \geqslant 1$ there exists $\delta_h > 0$ such that

$$w(x) \leqslant 1 - \delta_h \qquad \text{for all } x \in \bar{B}_{M_h}\backslash\bar{B}_{M_{h+1}}. \tag{6.25}$$

To prove this, for a given $x \in \bar{B}_{M_h}\backslash\bar{B}_{M_{h+1}}$, we split

$$w(x) = \sum_{j=1}^{h-1} 2^{-j}\,\frac{v_j(x)}{v_j(\xi)} + 2^{-h}\frac{v_h(x)}{v_h(\xi)} + 2^{-h-1}\frac{v_{h+1}(x)}{v_{h+1}(\xi)} + \sum_{j\geqslant h+2} 2^{-j}\,\frac{v_j(x)}{v_j(\xi)} =: S_1 + S_2 + S_3 + S_4. \tag{6.26}$$

By (6.18), the first sum satisfies

$$\begin{aligned} S_1 &= \sum_{j=1}^{h-1} 2^{-j}\,\frac{v_j(x)}{v_j(\xi)} \leqslant \sum_{j=1}^{h-1} 2^{-j} + \sum_{j=1}^{h-1} 2^{-j}\,\frac{|v_j(x) - v_j(\xi)|}{v_j(\xi)} \\ &\leqslant (1 - 2^{-h+1}) + C\sum_{j=1}^{h-1} 2^{-j}\frac{r(\bar{B}_{M_h})}{r(\bar{B}_{M_j})} \leqslant (1 - 2^{-h+1}) + C\sum_{j=1}^{h-1} 2^{-j}\,2^{k_0(j-h)}, \end{aligned}$$

where we took into account that $r(\bar{B}_{M_{j+1}}) \leqslant 2^{-k_0} r(\bar{B}_{M_j})$ for each j, by the construction of the sequence M_j. For $k_0 \geqslant 3$, we have

$$\sum_{j=1}^{h-1} 2^{-j}2^{k_0(j-h)} = \frac{2^{-h}}{2^{k_0-1}-1} \leqslant \frac{2^{-h}}{2^{k_0-2}} = 2^{-h-k_0+2}.$$

Thus,

$$S_1 \leqslant (1 - 2^{-h+1}) + C2^{-h-k_0}.$$

For S_2 and S_3 we apply (6.17):

$$S_2 + S_3 \leqslant (1+\varepsilon)(2^{-h} + 2^{-h-1}).$$

Finally we estimate S_4. For this term we use the fact that if $x \notin \bar{B}_{M_{h+1}}$ and $j \geqslant h+2$, then by (6.19) we have $v_j(x) \leqslant \frac{1}{10}\,v_j(\xi)$, assuming k_0 large enough. Therefore,

$$S_4 \leqslant \frac{1}{10}\sum_{j\geqslant h+2} 2^{-j} = \frac{1}{10}\,2^{-h-1}. \tag{6.27}$$

Gathering the estimates for $S_1,\dots,S_4$, we obtain

$$\begin{aligned} w(x) &\leqslant (1 - 2^{-h+1}) + C2^{-h-k_0} + (1+\varepsilon)(2^{-h} + 2^{-h-1}) + \frac{1}{10}\,2^{-h-1} \\ &= 1 - 2^{-h}\left(\frac{9}{20} - C2^{-k_0} - \frac{3\varepsilon}{2}\right). \end{aligned}$$

Then, choosing ε small enough and k_0 large, we derive $w(x) \leqslant 1 - 2^{-h-2}$, which proves (6.25) and completes the proof of (b) $\Rightarrow$ (a). $\square$

Proof of (b) ⇒ (a) in Theorem 6.21 in the case $d = 2$. The proof is very similar to the one above for $d \geqslant 3$ and so we only point out the differences in the argument. Given $1 < n_0 \leqslant N < M$, we define the functions u_k and v as above. Then the estimates (6.16), (6.17), and (6.18) in Claim 6.23 also hold if Λ is chosen large enough, while for (6.19) we require now that $k_0 \geqslant 10N/11$ and N large enough.

The proof of this variant of Claim 6.23 for the case $d = 2$ is very similar to the one for $d = 3$. Indeed, (6.16) has the same proof. Regarding (6.17), we split $v(x) = v_a(x)+v_b(x)+v_c(x)$ as in the case $d \geqslant 3$. We have $v_b(x) \leqslant 2h+1$ by the same arguments as for $d \geqslant 3$. To deal with $v_a(x)$ we estimate the functions u_k for $k < i - h$ by arguments quite similar to the ones in (6.21). Indeed, notice that

$$|U_{\mu_k}(x) - U_{\mu_k}(\xi)| \lesssim \int \left| \log \frac{|x-y|}{|\xi - y|} \right| d\mu_k(y)$$

Writing

$$\left| \log \frac{|x-y|}{|\xi - y|} \right| = \left| \log \left(1 + \frac{|x-y| - |x-\xi|}{|\xi - y|} \right) \right| \leqslant \frac{|x-\xi|}{|x-y|},$$

we deduce

$$|U_{\mu_k}(x) - U_{\mu_k}(\xi)| \lesssim \frac{|x-\xi|}{\operatorname{dist}(\xi, \bar{A}_k)}.$$

Thus,

$$\begin{aligned} u_k(x) &= u_k(\xi) + \operatorname{Cap}(\bar{A}_k \backslash \Omega)\,(U_{\mu_k}(x) - U_{\mu_k}(\xi)) \\ &\leqslant u_k(\xi) + C\ \operatorname{Cap}(\bar{A}_k \backslash \Omega)\,\frac{|x-\xi|}{\operatorname{dist}(\xi, \bar{A}_k)} \\ &\leqslant u_k(\xi) + C\ \operatorname{Cap}(\bar{A}_k \backslash \Omega)\,\frac{r(\bar{B}_i)}{r(\bar{B}_k)} \\ &\leqslant u_k(\xi) + C\,2^{-h}\ \operatorname{Cap}(\bar{A}_k \backslash \Omega)\,U_{\mu_k}(\xi), \end{aligned} \tag{6.28}$$

where we used the trivial bound $U_{\mu_k}(\xi) \geqslant 1$ in the last inequality for N large enough. For $v_c(x)$, we take into account that for $k > i + h$ we have

$$\begin{aligned} u_k(x) &\leqslant \operatorname{Cap}(\bar{A}_k \backslash \Omega)\,\mathcal{E}(\operatorname{dist}(x, \bar{A}_k)) \leqslant \operatorname{Cap}(\bar{A}_k \backslash \Omega)\,\mathcal{E}(c\,r(\bar{B}_i)) \\ &\leqslant \operatorname{Cap}(\bar{A}_k \backslash \Omega) \int \mathcal{E}(\xi - y)\,d\mu_k(y)\ \frac{\mathcal{E}(c\,r(\bar{B}_i))}{\inf_{y \in \bar{A}_k} \mathcal{E}(\xi - y)} \leqslant u_k(\xi), \end{aligned}$$

since $\mathcal{E}(c\,r(\bar{B}_i)) \leqslant \inf_{y \in \bar{A}_k} \mathcal{E}(\xi - y)$ for $k > i + h$ with h large enough.

Consequently, gathering the estimates obtained for $k < i - h$ and for $k > i + h$ and using also (6.16) and (6.20), we get

$$v_a(x) + v_c(x) \leqslant (1 + C2^{-h}) \sum_{N \leqslant k \leqslant M} u_k(\xi) = (1 + C2^{-h})\,v(\xi).$$

Thus,

$$v(x) = v_a(x) + v_b(x) + v_c(x) \leqslant v(\xi) + (2h+1) + C\,2^{-h}\,v(\xi) \leqslant v(\xi)\Big(1 + \frac{C\,h}{\Lambda} + C\,2^{-h}\Big).$$

So choosing h large enough and then Λ large enough, we get (6.17).

The proof of (6.18) also follows by arguments very similar to the ones for the case $d = 2$ and so we skip them.

Finally we deal with (6.19). So we take $x \in \Omega \backslash \bar{B}_{N-k_0}$, for $k_0 \geqslant 10N/11$ and N large enough. For $x \in B_{1/2}(\xi)$, then we have

$$U_{\mu_k}(x) = \int \mathcal{E}(x-y)\, d\mu_k(y) \leqslant \int \mathcal{E}(\xi - y)\, d\mu_k(y)\, \frac{\sup_{y \in \bar{A}_k} \mathcal{E}(x-y)}{\inf_{y \in \bar{A}_k} \mathcal{E}(\xi - y)}$$
$$\leqslant U_{\mu_k}(\xi)\, \frac{\log(c\, 2^{k_0}\, r(\bar{B}_N))}{\log(c'\, r(\bar{B}_N))} \leqslant U_{\mu_k}(\xi)\, \frac{C + N - k_0}{C' + N}.$$

From the condition that $k_0 \geqslant 10N/11$ we deduce that $N - k_0 \leqslant N/11$, and thus for N large enough it holds $\frac{C+N-k_0}{C'+N} \leqslant \frac{1}{10}$. Hence, multiplying by $\operatorname{Cap}(\bar{A}_k \backslash \Omega)$ and summing on $k \in [N, M]$, we obtain

$$v(x) \leqslant \frac{1}{10} \sum_{N \leqslant k \leqslant M} u_k(\xi) = \frac{1}{10}\, v(\xi) \quad \text{ for all } x \in \Omega \backslash \bar{B}_{N-k_0}.$$

To complete the proof of (b) $\Rightarrow$ (a) we choose sequences N_j and M_j as in the case $d \geqslant 3$, but with the additional requirement that $N_j \geqslant 20 M_{j-1}$ for each j, say. This condition ensures that we will be able to apply (6.19) to estimate the term S_4 in (6.26) arguing as in (6.27). Then almost the same arguments as the ones for the case $d \geqslant 3$ show that the function w defined in (6.22) is barrier for ξ. We leave the details for the reader. $\square$

6.4.2 Necessity of the criterion for Wiener regularity

Recall that in Definition 5.36 we introduced the notion of Wiener regularity for unbounded open sets with compact boundary. Before proving the necessity part in Theorem 6.21, i.e., the implication (a) $\Rightarrow$ (b), we need the following auxiliary result.

Lemma 6.24. *Let $E \subset \mathbb{R}^d$ be compact with $\operatorname{Cap}(E) > 0$ and let Ω_E be the unbounded component of $\mathbb{R}^d \backslash E$. Suppose that Ω_E is Wiener regular and let μ be the equilibrium measure for E. Then the equilibrium potential U_μ is continuous in $\mathbb{R}^d$ and $U_\mu = (\operatorname{Cap}(E))^{-1}$ identically on E.*

Proof. Without loss of generality, we assume that $E \subset B_{1/2}(0)$. For $r > 2$ we denote $\Omega_{E,r} = \Omega_E \cap B_r(0)$ and we let u_r be the solution of the Dirichlet problem in $\Omega_{E,r}$ with boundary data:

$$u_r = \begin{cases} (\operatorname{Cap}(E))^{-1} & \text{in } \partial\Omega_E, \\ U_\mu & \text{in } \partial B_r(0). \end{cases}$$

We extend u_r to $\widehat{E} = \mathbb{R}^d \backslash \Omega_E$ by setting $u_r(x) = (\operatorname{Cap}(E))^{-1}$ for $x \in \widehat{E}$, so that u_r is continuous in $B_r(0)$, by the Wiener regularity of $\Omega_{E,r}$.

Observe that, for all $\xi \in \partial\Omega_E$,

$$0 \leqslant \limsup_{x \to \xi} (u_r(x) - U_\mu(x)) \leqslant (\operatorname{Cap}(E))^{-1}.$$

Therefore, since $u_r = U_\mu$ in $\partial B_r(0)$, by the maximum principle we get

$$\|u_r - U_\mu\|_{\infty,\Omega_{E,r}} \leqslant (\mathrm{Cap}(E))^{-1}.$$

As this estimate is uniform in r, we deduce that there exists a sequence $r_k \to \infty$ such that u_{r_k} converges locally uniformly on compact subsets of Ω_E to some function u harmonic in Ω_E. In particular, it converges uniformly on $\partial B_1(0)$. Since u_{r_k} equals $(\mathrm{Cap}(E))^{-1}$ in $\partial\Omega_E$ for all k, by the maximum principle it follows that the convergence is also uniform in $\overline{\Omega_E} \cap \bar{B}_1(0)$. Then we deduce that u is continuous in $\overline{\Omega_E}$ and so it extends continuously to the whole $\mathbb{R}^d$. Further, u equals $(\mathrm{Cap}(E))^{-1}$ in $\widehat{E}$, $u \leqslant (\mathrm{Cap}(E))^{-1}$ in Ω_E, and together with the fact that u is continuous in $\mathbb{R}^d$ and harmonic in Ω_E, this implies that u is superharmonic in $\mathbb{R}^d$. Notice also that

$$\|u - U_\mu\|_{\infty,\mathbb{R}^d} \leqslant (\mathrm{Cap}(E))^{-1}.$$

The preceding estimate implies that u is non-constant in the case $d = 2$, since $U_\mu(x) \to -\infty$ as $|x| \to \infty$. In the case $d \geqslant 3$, it is also easy to check that u is non-constant. Indeed, let $\widetilde{u}_r : \bar{A}_{1,r}(0) \to \mathbb{R}$ be defined by

$$\widetilde{u}_r(x) = \mathrm{Cap}(E)^{-1}\,\mathcal{E}(1)^{-1}\,\mathcal{E}(x) + \max_{\partial B_r(0)} U_\mu,$$

where, abusing notation, we wrote $\mathcal{E}(1) = \mathcal{E}(y)$ for $|y| = 1$. It is immediate to check that $u_r \leqslant \widetilde{u}_r$ in $\partial\bar{A}_{1,r}(0)$, and thus also in $A_{1,r}(0)$ by the maximum principle. Then, letting $r \to \infty$, it follows that $u(x) \leqslant \mathrm{Cap}(E)^{-1}\,\mathcal{E}(1)^{-1}\,\mathcal{E}(x)$ for $|x| > 1$, which implies that u is non-constant.

The superharmonicity of u in $\mathbb{R}^d$ implies that $-\Delta u$ is a non-negative measure in the sense of distributions. This is an immediate consequence of Lemma 5.9 and the Riesz representation theorem. The fact that u is non-constant and the maximum principle ensures that Δu is not the zero measure.

Now we claim that there exists some constant $c_0 \in \mathbb{R}$ such that

$$u = -\mathcal{E} * \Delta u + c_0 \tag{6.29}$$

in the $L^1_{loc}(\mathbb{R}^d)$ sense. To prove this, observe first that the function $v := u + \mathcal{E} * \Delta u$ is harmonic in $\mathbb{R}^d$, and for $|x| \gg 1$ it satisfies

$$|v(x)| \leqslant |u(x)| + |\mathcal{E} * \Delta u(x)| \leqslant (\mathrm{Cap}(E))^{-1} + U_\mu(x) + |\mathcal{E} * \Delta u(x)| \leqslant C_0 + C_1\,|\mathcal{E}(|x|)|,$$

where C_0 and C_1 depend on u. In the case $d \geqslant 3$, this implies that v is bounded and so it is constant, by Liouville's theorem. In the case $d = 2$, we also deduce that v is constant. This follows easily from Lemma 2.11 applied to v in $B_R(0)$, letting $R \to \infty$:

$$\|\nabla v\|_{\infty,B_{R/2}(0)} \lesssim \frac{\|v\|_{\infty,B_R(0)}}{R} \lesssim \frac{C_0 + C_1 \log R}{R} \to 0.$$

So in any case (6.29) holds.

Let us see now that the pointwise identity

$$u(x) = -\mathcal{E} * \Delta u(x) + c_0 \tag{6.30}$$

holds for *all* $x \in \mathbb{R}^d$. Indeed, this holds in Ω_E by the continuity of $\mathcal{E} * \Delta u$ and u in Ω_E. So it remains to show that

$$(\mathrm{Cap}(E))^{-1} = -\mathcal{E} * \Delta u(x) + c_0 \quad \text{ for all } x \in \widehat{E}.$$

To this end, notice that for each $t > 0$, by the identity (6.29) in the L^1_{loc} sense and the continuity of u,

$$c_0 + \fint_{B_t(x)} \mathcal{E} * (-\Delta u)\, dm = \fint_{B_t(x)} u\, dm \xrightarrow{t\to 0} u(x).$$

On the other hand, by the superharmonicity of $\mathcal{E} * (-\Delta u)$ (recall that $-\Delta u$ is a positive measure), $\fint_{B_t(x)} \mathcal{E} * (-\Delta u)\, dm \leqslant \mathcal{E} * (-\Delta u)(x)$, and so

$$\mathrm{Cap}(E)^{-1} = u(x) = c_0 + \limsup_{t\to 0} \fint_{B_t(x)} \mathcal{E} * (-\Delta u)\, dm \leqslant c_0 + \mathcal{E} * (-\Delta u)(x).$$

For the converse inequality, we take into account that $c_0 + \mathcal{E} * (-\Delta u) \leqslant \mathrm{Cap}(E)^{-1}$ a.e. in $\mathbb{R}^d$, and thus the same estimate happens everywhere in $\mathbb{R}^d$ by the lower semicontinuity of $\mathcal{E} * (-\Delta u)$ (see Lemma 6.1(a)). So (6.30) holds for all $x \in \mathbb{R}^d$.

From (6.30) we deduce that

$$\mathcal{E} * (-\Delta u)(x) = (\mathrm{Cap}(E))^{-1} - c_0 =: c_1 \quad \text{ for all } x \in \widehat{E}.$$

Since $-\Delta u$ is a non-zero positive measure supported on $\widehat{E} \subset B_{1/2}(0)$, it follows that $c_1 > 0$. So letting $k = (c_1 \mathrm{Cap}(E))^{-1}$, it turns out that $\mathcal{E} * (-k\Delta u)(x) = (\mathrm{Cap}(E))^{-1}$ for all $x \in E$. Next we will show that this implies that $-k\Delta u = \mu$. To this end, by Theorem 6.10 it suffices to prove that $-k\Delta u$ is a probability measure and that $I(\mu + k\Delta u) = 0$.

To prove that $-k\Delta u$ is a probability measure we first apply Theorem 6.12, taking into account that $\|\mathcal{E} * (-k\,\mathrm{Cap}(E)\,\Delta u)\|_\infty = 1$, and then we derive $\mathrm{Cap}(E) \geqslant \| - k\,\mathrm{Cap}(E)\Delta u\|$, or equivalently, $\| - k\Delta u\| \leqslant 1$. For the converse inequality we apply Lemma 6.14 and we obtain $\mathrm{Cap}(E) \leqslant \| - k\,\mathrm{Cap}(E)\,\Delta u\|$, so that $\| - k\Delta u\| = 1$.

Next we will show that $I(\mu + k\Delta u) = 0$. Notice first that $I(|\mu + k\Delta u|) < +\infty$ because both $\mathcal{E} * \mu$ and $\mathcal{E} * (-k\Delta u)$ are uniformly bounded in E. We write

$$I(\mu + k\Delta u) = \int U_{(\mu+k\Delta u)}\, d(\mu + k\Delta u) = \int \big(U_\mu - U_{(-k\Delta u)}\big)\, d\mu + k \int \big(U_\mu - U_{(-k\Delta u)}\big)\, d(\Delta u).$$

Both integrals on the right hand side vanish because $U_{(-k\Delta u)}$ equals identically $(\mathrm{Cap}\, E)^{-1}$ in $E \supset \mathrm{supp}\mu$, while U_μ equals $(\mathrm{Cap}\, E)^{-1}$ μ-a.e. and $(-k\Delta u)$-a.e. by Corollary 6.9. Hence, $I(\mu + k\Delta u) = 0$ and thus $\mu = -k\Delta u$. In turn, this implies that $U_\mu = -k\,\mathcal{E} * \Delta u$, and so U_μ is continuous in $\mathbb{R}^d$ and identically equal to $(\mathrm{Cap}\, E)^{-1}$ in $\widehat{E}$. $\square$

Proof of (a) ⇒ (b) in Theorem 6.21. As above, we write $\bar{A}_k = \bar{A}(\xi, 2^{-k-1}, 2^{-k})$, $B_k = B_{2^{-k}}(\xi)$, and $\bar{B}_k = \overline{B_k}$. To get a contradiction, suppose that $\xi \in \partial\Omega$ is a regular point such that

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} < \infty.$$

Without loss of generality, assume also that $\Omega \subset B_{1/2}(0)$.

We will replace Ω by an auxiliary Wiener regular open subset $\widetilde{\Omega} \subset \Omega$ so that $\xi \in \partial\Omega \cap \partial\widetilde{\Omega}$. We define $\widetilde{\Omega}$ as follows. For each $k \geqslant 1$ such that $\bar{A}_k \backslash \Omega \neq \varnothing$, let $\rho_k \in (0, 2^{-k-3})$ be such that

$$\mathrm{Cap}(\mathcal{U}_{\rho_k}(\bar{A}_k \backslash \Omega)) \leqslant \mathrm{Cap}(\bar{A}_k \backslash \Omega) + 2^{-k}\, \mathrm{Cap}(\bar{B}_k),$$

where $\mathcal{U}_\ell(G)$ stands for the ℓ-neighborhood of G. We cover $\bar{A}_k \backslash \Omega$ by a finite number of closed balls $B_{k,j}$ centered in $\bar{A}_k \backslash \Omega$ with the same radius ρ_k, and we let $E_k = \bigcup_j B_{j,k}$. In case that $\bar{A}_k \backslash \Omega = \varnothing$, then we let $E_k = \varnothing$ be a closed ball $B_{k,1}$ contained in $\bar{A}_k$ such that $\mathrm{Cap}(B_{k,1}) = 2^{-k}\, \mathrm{Cap}(\bar{B}_k)$. Finally, we let

$$\widetilde{\Omega} = \Omega \backslash \bigcup_{k \geqslant 1} E_k.$$

It is easy to check that $\widetilde{\Omega}$ is open. Further,

$$\sum_{k \geqslant 1} \frac{\mathrm{Cap}(\bar{A}_k \backslash \widetilde{\Omega})}{\mathrm{Cap}(\bar{B}_k)} \leqslant \sum_{k \geqslant 1} \frac{\mathrm{Cap}(E_{k-1} \cup E_k \cup E_{k+1})}{\mathrm{Cap}(\bar{B}_k)}.$$

Using that $\mathrm{Cap}(E_{k-1} \cup E_k \cup E_{k+1}) \leqslant \mathrm{Cap}(E_{k-1}) + \mathrm{Cap}(E_k) + \mathrm{Cap}(E_{k+1})$ and that $\mathrm{Cap}(\bar{B}_{k-1}) \approx \mathrm{Cap}(\bar{B}_k) \approx \mathrm{Cap}(\bar{B}_{k+1})$, it follows that

$$\sum_{k \geqslant 1} \frac{\mathrm{Cap}(\bar{A}_k \backslash \widetilde{\Omega})}{\mathrm{Cap}(\bar{B}_k)} \lesssim \sum_{k \geqslant 1} \frac{\mathrm{Cap}(E_k)}{\mathrm{Cap}(\bar{B}_k)} \leqslant \sum_{k \geqslant 1} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} + \sum_{k \geqslant 1} 2^{-k} < \infty. \tag{6.31}$$

Also $\xi \in \partial\widetilde{\Omega}$ because the preceding estimate implies that, for k large enough, $\mathrm{Cap}(\bar{A}_k \backslash \widetilde{\Omega}) \ll \mathrm{Cap}(\bar{B}_k) \approx \mathrm{Cap}(\bar{A}_k)$, so that $\bar{A}_k \cap \widetilde{\Omega} \neq \varnothing$.

To check that $\widetilde{\Omega}$ is Wiener regular, notice first that ξ is a Wiener regular point for $\widetilde{\Omega}$, because if $v : \Omega \to \mathbb{R}$ is a barrier for ξ in Ω, then $v|_{\widetilde{\Omega}}$ is a barrier of ξ in $\widetilde{\Omega}$. Further, it is immediate to check that any other point $\zeta \in \partial\widetilde{\Omega}$ with $\zeta \neq \xi$ belongs to the boundary of some ball $B_{k.j}$, and so ζ is Wiener regular because of the existence of an outer tangent ball in ζ (namely, $B_{k.j}$). So $\widetilde{\Omega}$ satisfies the required properties.

For $k \geqslant 1$ we denote

$$F_k = \{\xi\} \cup \bigcup_{j \geqslant k} E_j.$$

Notice that F_k is a compact set such that $F_k \subset \bar{B}_{k-1}$, and by the same arguments as above, it follows easily that $\mathbb{R}^d \backslash F_k$ is Wiener regular and that $\xi \in \partial F_k$.

Next we will derive a contradiction from the fact that ξ is a regular point for $\widetilde{\Omega}$ and the condition (6.31). For $0 < \varepsilon < 1/4$, let $N \geqslant 2$ be such that

$$\sum_{k \geqslant N} \frac{\operatorname{Cap}(E_k)}{\operatorname{Cap}(\bar{B}_k)} < \varepsilon. \tag{6.32}$$

Because of the Wiener regularity of $\widetilde{\Omega}$, there exists a function $f \in C(\overline{\widetilde{\Omega}})$, harmonic in $\widetilde{\Omega}$, with $0 \leqslant f \leqslant 1$, with $f(\xi) = 0$ and $f = 0$ in $\partial\widetilde{\Omega} \backslash \bar{B}_{N+1}$. By the continuity of f, there exists $s < 2^{-N-1}$ such that $f(x) > 1 - \varepsilon$ in $\overline{\widetilde{\Omega}} \cap \bar{B}_s(\xi)$.

Let us see that there exists $M \geqslant 1$ large enough such that $2^{-M} < s/4$ and such that the equilibrium potential U_{F_M} for F_M satisfies

$$\operatorname{Cap}(F_M)\, U_{F_M}(x) \leqslant \varepsilon \quad \text{ for all } x \in \mathbb{R}^d \backslash \bar{B}_s(\xi).$$

Indeed, we have

$$\operatorname{Cap}(F_M)\, U_{F_M}(x) \leqslant \operatorname{Cap}(\bar{B}_{M-1})\, \mathcal{E}(\operatorname{dist}(F_M, \partial B_s(\xi)) \lesssim \frac{\mathcal{E}(s)}{\mathcal{E}(2^{-M+1})},$$

which tends to 0 as $M \to \infty$. We denote $V_{F_M} = \operatorname{Cap}(F_M)\, U_{F_M}$.

Let $A_{N,M} = \bigcup_{N \leqslant k \leqslant M} E_k$. Again, $\mathbb{R}^d \backslash A_{N,M}$ is Wiener regular because because $A_{N,M}$ is the union of a finite number of balls, and we can apply the criterion of the outer tangent ball. Let $U_{A_{N,M}}$ be the equilibrium potential of $A_{N,M}$ and denote $V_{A_{N,M}} = \operatorname{Cap}(A_{N,M})\, U_{A_{N,M}}$. By Lemma 6.24, it turns out that V_{F_M} and $V_{A_{N,M}}$ are continuous and $V_{F_M} + V_{A_{N,M}} \geqslant 1$ on $F_M \cup A_{N,M}$. Then, by the definition of f and the maximum principle it follows that $V_{F_M} + V_{A_{N,M}} \geqslant f$ in $\widetilde{\Omega}$. Therefore,

$$V_{A_{N,M}} \geqslant f - V_{F_M} \geqslant 1 - 2\varepsilon \quad \text{ in } \partial B_s(\xi) \cap \widetilde{\Omega}.$$

We also have $V_{A_{N,M}} = 1 > 1 - 2\varepsilon$ in $A_{N,M}$, and so by the maximum principle applied to the set $B_s(\xi) \backslash A_{N,M}$ (recall that $2^{-M+2} < s < 2^{-N-1}$), it follows that

$$V_{A_{N,M}}(\xi) \geqslant 1 - 2\varepsilon. \tag{6.33}$$

Now we intend to contradict this estimate. To this end, notice that for $x \in \partial B_{1/2}(\xi)$,

$$\begin{aligned} V_{A_{N,M}}(x) &= \operatorname{Cap}(A_{N,M})\, U_{A_{N,M}}(x) \\ &\leqslant \operatorname{Cap}(B_{N-1})\, \mathcal{E}(\operatorname{dist}(x, A_{N,M})) \lesssim \operatorname{Cap}(B_{N-1}) \approx \mathcal{E}(2^{-N})^{-1}. \end{aligned}$$

In $A_{N,M}$ we also have

$$V_{A_{N,M}}(x) = 1 \leqslant \sum_{N \leqslant k \leqslant M} V_{E_k}(x) = \sum_{N \leqslant k \leqslant M} \operatorname{Cap}(E_k)\, U_{E_k}(x).$$

Then, by the maximum principle and by (6.32),

$$\begin{aligned} V_{A_{N,M}}(\xi) &\leqslant \sum_{N \leqslant k \leqslant M} \operatorname{Cap}(E_k)\, U_{E_k}(\xi) + C\, \mathcal{E}(2^{-N})^{-1} \\ &\approx \sum_{N \leqslant k \leqslant M} \frac{\operatorname{Cap}(E_k)}{\operatorname{Cap}(\bar{B}_k)} + \mathcal{E}(2^{-N})^{-1} \lesssim \varepsilon + \mathcal{E}(2^{-N})^{-1}, \end{aligned}$$

which contradicts (6.33). $\square$

6.5 Kellogg's theorem

A set $E \subset \mathbb{R}^d$ is called polar if $\mathrm{Cap}(E) = 0$. Of course, in the case $d = 2$, this is equivalent to saying that $\mathrm{Cap}_L(E) = 0$. Kellogg's theorem asserts that, for any bounded open set $\Omega \subset \mathbb{R}^d$, the set of (Wiener) irregular points is polar. In order to prove this, we will need some auxiliary results, which have their own interest.

Recall that in Section 5.4 we introduced the notion of barrier functions, whose existence characterizes the regularity of boundary points. Next we introduce the weaker notion of generalized barrier, which also can be used to characterize regular points, as we will see below. Given an open set $\Omega \subset \mathbb{R}^d$, function $v : \Omega \to \mathbb{R}$ is called a generalized barrier for Ω at $\xi \in \partial\Omega$ if

1. v is superharmonic in $V \cap \Omega$,
2. $v > 0$ in Ω, and
3. $\lim_{x\to\xi} v(x) = 0$.

It is immediate to check that a barrier for ξ is also a generalized barrier. The converse statement is not true. However, we have the following key result.

Theorem 6.25. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. A point $\xi \in \partial\Omega$ is regular for Ω if and only there exists a generalized barrier for Ω at ξ.*

To prove this theorem, we will use the following simple result:

Lemma 6.26. *For $r > 0$, let $V \subset S_r(0)$ be relatively open in $S_r(0)$, and for any $x \in B_r(0)$ let*

$$g(x) = \int_{S_r(0)} P^x_{B_r(0)}(\zeta)\, \chi_{V_{r,\varepsilon}}(\zeta)\, d\sigma(\zeta),$$

where σ is the surface measure on $S_r(0)$. Then,

$$\lim_{B_r(0)\ni x\to\xi} g(x) = 1 \qquad \textit{for all } \xi \in V.$$

Recall that $P^x_{B_r(0)}$ is the Poisson kernel for the ball $B_r(0)$, which was introduced in Remark 3.11.

Proof. For $\xi \in V$, let $\varphi \in C(S_r(0))$ be such that $\varphi(\xi) = 1$, $0 \leqslant \varphi \leqslant 1$, and $\mathrm{supp}\varphi \subset V$, so that

$$\varphi \leqslant \chi_V \leqslant 1.$$

Since the Poisson kernel is a positive function, for all $x \in B_r(0)$ we have

$$\int_{S_r(0)} P^x_{B_r(0)}(\zeta)\, \varphi(\zeta)\, d\sigma(\zeta) \leqslant \int_{S_r(0)} P^x_{B_r(0)}(\zeta)\, \chi_V(\zeta)\, d\sigma(\zeta) \leqslant \int_{S_r(0)} P^x_{B_r(0)}(\zeta)\, d\sigma(\zeta). \quad (6.34)$$

The integral on the left hand side equals the harmonic extension of φ to $B_r(0)$ evaluated at x, and this tends to $\varphi(\xi) = 1$ as $x \to \xi$, by Theorem 3.10 and Remark 3.11. On the other hand, the last integral is identically 1 for all $x \in B_r(0)$. Thus, letting $x \to \xi$ in (6.34), the lemma follows. $\square$

Proof of Theorem 6.25. The statement in the theorem is equivalent to saying that there exists a barrier at $\xi \in \partial\Omega$ for Ω if and only if there exists a generalized barrier. Since any barrier is also a generalized barrier, we are left wit showing that the existence of a generalized barrier at $\xi \in \partial\Omega$ for Ω implies the existence of a "usual" barrier. To this end, consider the function $\varphi : \overline{\Omega} \to \mathbb{R}$ defined by $\varphi(x) = |x - \xi|^2$. The fact that $\Delta\varphi \geqslant 0$ away from ξ ensures that φ is subharmonic in Ω. The function $f := \varphi|_{\partial\Omega}$ is continuous in $\partial\Omega$, and thus it is also resolutive. Further, since $\varphi \in \mathcal{L}_f$ (recall that this is the lower Perron class for Ω, introduced in Definition 5.12), we have $v := H_f = \underline{H}_f \geqslant \varphi$ in Ω. Thus, v is a positive harmonic function in Ω such that for all $\zeta \in \partial\Omega\backslash\{\xi\}$,

$$\liminf_{\Omega \ni x \to \zeta} v(x) \geqslant f(\zeta) > 0.$$

Hence to show that v is a "usual" barrier for ξ, it suffices to prove that

$$\lim_{\Omega \ni x \to \xi} v(x) = 0. \tag{6.35}$$

To prove (6.35), without loss of generality, assume that $\xi = 0$. Let u be a generalized barrier at 0 for Ω and let $r > 0$ be such that $S_r(0) \cap \Omega \neq \varnothing$. For a given $\varepsilon > 0$, consider a compact subset $E_{r,\varepsilon} \subset S_r(0) \cap \Omega$ such that $\sigma((S_r(0) \cap \Omega)\backslash E_{r,\varepsilon}) \leqslant \varepsilon\, \sigma(S_r(0))$, where σ is the surface measure on $S_r(0)$. Notice that $\gamma_{r,\varepsilon} = \inf_{E_{r,\varepsilon}} u > 0$ (recall that u is lower semicontinuous in Ω and so the infimum on any compact subset of Ω is attained in that compact subset). Consider the set $V_{r,\varepsilon} = (S_r(0) \cap \Omega)\backslash E_{r,\varepsilon}$, which is relatively open in $S_r(0)$. Let $g : S_r(0) \to \mathbb{R}$ be defined by the "harmonic extension" of $\chi_{V_{r,\varepsilon}}$ to $B_r(0)$, that is,

$$g(x) = \int_{S_r(0)} P^x_{B_r(0)}(\zeta)\, \chi_{V_{r,\varepsilon}}(\zeta)\, d\sigma(\zeta).$$

Let $h : \Omega \cap B_r(0)$ be the function defined by

$$h = r^2 + \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2\, u + \operatorname{diam}(\Omega)^2\, g,$$

where $P^x_{B_r(0)}$ is the Poisson kernel for $B_r(0)$ with pole at x. Notice that h is superharmonic in Ω. We claim that for any function $s \in \mathcal{L}_f$ (recall that this means that $s \in C(\Omega)$ is a subharmonic function such that $\limsup_{x\to\eta} s(x) \leqslant f(\eta)$ for all $\eta \in \partial\Omega$), it holds that

$$\liminf_{x\to\eta} h(x) \geqslant \limsup_{x\to\eta} s(x) \qquad \text{for all } \eta \in \partial(\Omega \cap B_r(0)). \tag{6.36}$$

Indeed, if $\eta \in V_{r,\varepsilon} = \overline{B}_r(0) \cap \partial\Omega$, then

$$\liminf_{x\to\eta} h(x) \geqslant r^2 \geqslant f(\eta) \geqslant \limsup_{x\to\eta} s(x).$$

On the other hand, if $\eta \in E_{r,\varepsilon}$, since u is lower semicontinuous in Ω,

$$\liminf_{x\to\eta} h(x) \geqslant \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2 \liminf_{x\to\eta} u \geqslant \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2\, u(\eta) \geqslant \operatorname{diam}(\Omega)^2 \geqslant f(\eta).$$

Finally, for $\eta \in S_r(0) \cap \Omega \backslash E_{r,\varepsilon}$, by Lemma 6.26,

$$\liminf_{x\to\eta} h(x) \geqslant \operatorname{diam}(\Omega)^2 \liminf_{x\to\eta} g(x) = \operatorname{diam}(\Omega)^2 \geqslant f(\eta).$$

So our claim holds.

From the superharmonicity of $h - s$ and the maximum principle in Lemma 5.3 (applied to $s - h$) and (6.36), we deduce that

$$s(x) \leqslant h(x) \qquad \text{for all } x \in B_r(0) \cap \Omega.$$

Since this estimate holds for all $s \in \mathcal{L}_f$, we deduce that $H_f(x) \leqslant h(x)$ for all $x \in B_r(0) \cap \Omega$. Thus,

$$\begin{aligned}\limsup_{x\to 0} H_f(x) &\leqslant r^2 + \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2 \limsup_{x\to 0} u + \operatorname{diam}(\Omega)^2 \limsup_{x\to 0} g \\ &= r^2 + 0 + g(0) = r^2 + \operatorname{diam}(\Omega)^2 \, \frac{\sigma(V_{r,\varepsilon})}{\sigma(S_r(0))} \leqslant r^2 + \operatorname{diam}(\Omega)^2 \, \varepsilon.\end{aligned}$$

Choosing $\varepsilon = r^2 \operatorname{diam}(\Omega)^{-2}$, we get $\limsup_{x\to 0} H_f(x) \leqslant 2\, r^2$. Since r can be taken arbitrarily small and H_f is positive, we deduce that

$$\lim_{x\to 0} v(x) = \lim_{x\to 0} H_f(x) = 0,$$

as wished. □

Theorem 6.27. *Let $E \subset \mathbb{R}^d$ be compact with $\operatorname{Cap}(E) > 0$ and let Ω_E be the unbounded component of $\mathbb{R}^d \backslash E$. Let μ be the equilibrium measure for E. If a point $\xi \in \partial\Omega_E$ is irregular for Ω_E, then $U_\mu(\xi) < \operatorname{Cap}(E)^{-1}$. In particular, the set of irregular points for Ω_E is polar, and moreover it is contained in an F_σ polar set.*

Proof. Let us see that if $U_\mu(\xi) \geqslant \operatorname{Cap}(E)^{-1}$, then ξ is regular. Remark that the inequality $U_\mu(\xi) \geqslant \operatorname{Cap}(E)^{-1}$ is equivalent to $U_\mu(\xi) = \operatorname{Cap}(E)^{-1}$ because $\|U_\mu\|_{\infty,\mathbb{R}^d} \leqslant \operatorname{Cap}(E)^{-1}$. We claim that the function $v = \operatorname{Cap}(E)^{-1} - U_\mu$ is a generalized barrier at ξ for Ω_E (i.e., for $\Omega_E \cap B_r(0)$ for any $r > 0$ such that $E \subset B_r(0)$). To check this, notice first that v is harmonic and that $v > 0$ in Ω_E. The latter assertion follows from the fact that v is non-constant and non-negative in Ω and Ω is connected. By the semicontinuity property (a) in Lemma 6.1, we know that $\liminf_{y\to\xi} U_\mu(y) \geqslant U_\mu(\xi)$. Consequently, $\liminf_{y\to\xi} v(y) \leqslant v(\xi) = 0$. So v is a generalized barrier at ξ for Ω_E, and by Theorem 6.25 ξ is a regular point for Ω_E.

To prove the second statement of the theorem observe that, by what we have just proved, the set of irregular points for Ω_E is contained in the set

$$S = \{x \in E : U_\mu(x) < \operatorname{Cap}(E)^{-1}\},$$

which is a polar set, by Theorem 6.8. Therefore, the set of irregular points for Ω_E is also polar. Further, writing $S = \bigcup_{j\geqslant 1} S_j$, with

$$S_j = \left\{x \in E : U_\mu(x) \leqslant \operatorname{Cap}(E)^{-1} - \tfrac{1}{j}\right\},$$

by the lower semicontinuity of U_μ it is clear that S is an F_σ set, since each S_j is closed. □

Remark 6.28. In fact, the converse of the first statement in Theorem 6.27 also holds. That is, for Ω_E and μ as in Theorem 6.27, a point $\xi \in \partial\Omega_E$ is irregular if and only if $U_\mu(\xi) < \mathrm{Cap}(E)^{-1}$. However, we will not need this result and so we skip the proof.

Theorem 6.29. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. A point $\xi \in \partial\Omega$ is irregular for Ω if and only if there exists some component Ω_0 of Ω such that $\xi \in \partial\Omega_0$ and x is irregular for Ω_0. In particular, if x is not in the boundary of any component of Ω, then it is regular for Ω.*

Proof. Denote by $\{\Omega_j\}_{j\in J}$ the family of components of Ω. If $\xi \in \partial\Omega_j$ and ξ is irregular for Ω_j, then there is not any barrier at ξ for Ω_j, which it readily implies that there is not any barrier at ξ for Ω. Thus, ξ is irregular for Ω.

In the converse direction, suppose that there is not any Ω_j such that ξ is irregular for Ω_j. To prove that ξ is regular for Ω, we intend to define a generalized barrier v at ξ for Ω. For any Ω_j such that $\xi \in \partial\Omega_j$, since ξ is regular for Ω_j, there exists a barrier v_j at ξ for Ω. For such Ω_j, we define $v = \min(v_j, 1/j)$. For the components Ω_j such that $\xi \notin \partial\Omega_j$, we let $v = 1/j$ on Ω_j.

To check that v is a generalized barrier at ξ for Ω, notice first that v is superharmonic and positive in Ω. To see that $\lim_{x\to\xi} v(x) = 0$, let $\varepsilon > 0$ and consider the finite set $J_\varepsilon = \{j \in J : j \leqslant \varepsilon^{-1}\}$. If $J_\varepsilon = \varnothing$, then $u \leqslant \varepsilon$ on Ω. Otherwise, for each $j \in J_\varepsilon$ there exists an open neighborhood V_j of ξ such that either $V_j \cap \Omega_j = \varnothing$ or $v \leqslant \varepsilon$ in $V_j \cap \Omega_j$. So letting $V = \bigcup_{j\in J_\varepsilon} V_j$ it turns out that V is an open neighborhood of y where $v \leqslant \varepsilon$ on V. So $\lim_{x\to\xi} v(x) = 0$ as wished, and thus v is the desired generalized barrier. □

Theorem 6.30 (Kellogg's theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then the set of irregular points for Ω is polar. Further, this is contained in an F_σ polar set.*

Proof. By Theorem 6.29, it suffices to show that the set of irregular points for any component of Ω is irregular, taking into account that the number of components is at most countable and that a finite or countable union of polar sets is polar. So to prove the theorem we can assume that Ω is connected.

Given a bounded connected set Ω, for any $\xi \in \partial\Omega$ let B_ξ be an open ball centered in ξ such that $\Omega \cap \partial B_\xi \neq \varnothing$. Consider the domain $\Omega_\xi = \Omega \cup (\mathbb{R}^d \backslash \overline{B_\xi})$. Notice that Ω_ξ is an unbounded connected set with bounded boundary, and then by Theorem 6.27 the set of irregular points for Ω_ξ is polar (we can assume that $\mathrm{Cap}(\partial\Omega_\xi) > 0$ because otherwise any subset of $\partial\Omega_\xi$ is polar) and it is contained in an F_σ polar set. Now remark that $B_\xi \cap \partial\Omega \subset \partial\Omega_\xi$ and that any point $\xi \in B_\xi \cap \partial\Omega$ which is irregular for Ω is also irregular for Ω_ξ. This follows immediately from Wiener's criterion for regularity (although it could be also easily deduced from the characterization of regularity in terms of existence of barriers). Therefore, the subset of irregular points for Ω that belong to $B_\xi \cap \partial\Omega$ is polar and it is contained in an F_σ polar set.

Finally, since $\partial\Omega$ is compact, there exists a finite covering of $\partial\Omega$ with balls B_{ξ_i}, for a finite subset of points $\xi_i \in \partial\Omega$. By the preceding discussion, the set of irregular points for

Ω that belong to $B_{\xi_i} \cap \partial\Omega$ is polar. Since a finite union of polar sets is also polar and a finite unions of F_σ sets is an F_σ set, the theorem follows. □

Remark 6.31. In fact, the set of irregular points for an open set $\Omega \subset \mathbb{R}^d$ with compact boundary is itself an F_σ set. This follows easily from Wiener's criterion. Indeed, it is immediate to check that an equivalent form of the criterion is the following: a point $\xi \in \partial\Omega$ is regular for the Dirichlet problem in Ω if and only if

$$F(\xi) := \int_0^1 \frac{\operatorname{Cap}(A(\xi, r, 2r) \cap \Omega^c)}{\operatorname{Cap}(B(\xi, r))} \frac{dr}{r} = \infty.$$

so that x is regular if and only if $S(x) = \infty$. Since F is lower semicontinuous, for all $\lambda > 0$ the set $\{x \in \mathbb{R}^{n+1} : F(x) > \lambda\}$ is open and thus the set of Wiener regular point is a G_δ set (relative to $\partial\Omega$). Thus the set of the irregular points from $\partial\Omega$ is an F_σ set.

6.6 Removability of polar sets

Theorem 6.32. *Let $\Omega \subset \mathbb{R}^d$ be bounded and open, and let $Z \subset \partial\Omega$ be a Borel polar set. Then, for any $x \in \Omega$,*

$$\omega^x(Z) = 0.$$

Proof. In the case $d = 2$, we will assume that $\Omega \subset B_{1/2}(0)$. The measure ω^x is Radon and thus it is inner regular. Then it is enough to prove the theorem for Z being a compact (polar) set. Under this assumption, by the outer regularity of capacity (see Lemma 6.7), for any $\varepsilon > 0$ there is an open set $V \supset Z$ such that $\operatorname{Cap}(V) < \varepsilon$. By the compactness of Z, we can find finitely many open balls B_i, $i = 1, \ldots, m$, centered on Z such that $2B_i \subset V \cap B_{1/2}(0)$ and

$$Z \subset \bigcup_{1 \leqslant i \leqslant m} B_i.$$

Consider the compact set $E = \bigcup_{1 \leqslant i \leqslant m} \overline{B_i}$ and let $\Omega_E = \mathbb{R}^d \backslash E$. Since E consists of a union of finitely many balls, it follows either by Wiener's criterion or by the exterior ball criterion in Remark 5.34 that Ω_E is Wiener regular. Then, by Lemma 6.24, if μ stands for the equilibrium measure for E, the potential U_μ is continuous in $\mathbb{R}^d$ and $U_\mu = (\operatorname{Cap}(E))^{-1}$ identically on E.

Consider now the function $f(x) = \operatorname{Cap}(E)\, U_\mu(x)$, and notice that it is superharmonic and continuous in $\mathbb{R}^d$, and it equals 1 on E. Also, it is positive in $\overline{\Omega}$ since $\Omega \subset B_{1/2}(0)$ in the planar case. So we have

$$\omega^x(Z) \leqslant \omega^x(E) \leqslant \int f\, d\omega^x. \tag{6.37}$$

By definition, letting $g = f|_{\partial\Omega}$, the last integral above equals $H_g(x)$. Since f belongs to the upper Perron class for g, we have $H_g(x) \leqslant f(x)$. Thus,

$$\omega^x(Z) \leqslant f(x) = \operatorname{Cap}(E)\, U_\mu(x) \leqslant \operatorname{Cap}(V)\, U_\mu(x) \leqslant \varepsilon\, U_\mu(x). \tag{6.38}$$

As μ is a probability measure supported on E,

$$U_\mu(x) = \int \mathcal{E}(x-y)\, d\mu(y) \leqslant \sup_{y\in E} \mathcal{E}(x-y) \to \sup_{y\in Z} \mathcal{E}(x-y) \qquad \text{as } \varepsilon \to 0.$$

Since $\sup_{y\in Z} \mathcal{E}(x-y) < \infty$, letting $\varepsilon \to 0$ in (6.43), we deduce that $\omega^x(Z) = 0$. $\square$

Definition 6.33. Let Ω be a bounded open set and let $E \subset \Omega$ be a compact set. We say that E is removable for bounded harmonic functions in Ω if every function $f : \Omega \backslash E \to \mathbb{R}$ which is harmonic and bounded can be extended to the whole Ω as a harmonic function.

Theorem 6.34. *Let Ω be a bounded open set and let $E \subset \Omega$ be a compact polar set. Then E is removable for bounded harmonic functions in Ω if and only if E is polar.*

Notice that, in particular, the removability of a compact set E for bounded harmonic functions does not depend on the bounded open set Ω containing E.

Proof. First we show that if $\mathrm{Cap}(E) > 0$ then E is not removable. To this end, let μ be the equilibrium measure of E and U_μ the corresponding equilibrium potential. Then U_μ is a bounded harmonic function in $\Omega\backslash E$. Further, it is easy to check that U_μ cannot be extended harmonically to a function f harmonic in the whole Ω. Otherwise, f would be a function continuous in $\overline{\Omega}$ and harmonic in Ω such that $\max_{\overline{\Omega}} f$ is not attained in $\partial\Omega$, because $\sup_E f = \mathrm{Cap}(E)^{-1} > \max_{\partial\Omega} f$. So we get a contradiction.

To prove the converse implication, let $\Omega \subset \mathbb{R}^d$ be bounded and open and let $E \subset \Omega$ be a compact polar set. Without loss of generality we can assume that $\overline{\Omega} \subset B_{1/2}(0)$ in the case $d = 2$. We claim that there exists a Wiener regular open set $\widetilde{\Omega}$ which contains E and such that $\overline{\widetilde{\Omega}} \subset \Omega$. For example $\widetilde{\Omega}$ can be constructing as the interior of the union of finitely many dyadic cubes of the same size in a suitable way. We leave the details for the reader.

Given $\varepsilon > 0$, let V_ε be an open set such that $E \subset V_\varepsilon$ and $\mathrm{Cap}(V_\varepsilon) < \varepsilon$. By the compactness of E, we can find finitely many open balls B_i, $i = 1, \ldots, m$, centered on Z such that $3B_i \subset V \cap B_{1/2}(0)$ and

$$E \subset \bigcup_{1\leqslant i\leqslant m} B_i.$$

Consider the compact set $F_\varepsilon = \bigcup_{1\leqslant i\leqslant m} 2\overline{B_i}$ and let $\widetilde{\Omega}_\varepsilon = \widetilde{\Omega}\backslash F_\varepsilon$. Notice that

$$\partial\widetilde{\Omega}_\varepsilon = \partial\widetilde{\Omega} \cup \partial F_\varepsilon.$$

For $x \in \widetilde{\Omega}_\varepsilon$, we bound $\omega^x_{\widetilde{\Omega}_\varepsilon}(\partial F_\varepsilon)$ as in Theorem 6.32: by considering the equilibrium measure μ of F_ε, as in (6.44) we deduce that

$$\omega^x_{\widetilde{\Omega}_\varepsilon}(\partial F_\varepsilon) \leqslant \mathrm{Cap}(F_\varepsilon)\, U_\mu(x) \leqslant \varepsilon\, U_\mu(x) \leqslant C(x)\, \varepsilon,$$

with $C(x)$ independent of ε (assuming ε small enough).

Next we will show that if $f : \Omega\backslash E \to \mathbb{R}$ is harmonic and bounded, then f extends to the whole Ω as a harmonic function. To this end, let g be the harmonic extension of $f|_{\partial\widetilde{\Omega}}$ to $\widetilde{\Omega}$ and fix $x \in \widetilde{\Omega}$. Take $\varepsilon > 0$ small enough such that $x \in \widetilde{\Omega}_\varepsilon$. Observe that both f and g are harmonic in $\widetilde{\Omega}_\varepsilon$ and continuous in $\overline{\widetilde{\Omega}_\varepsilon}$ and their boundary values coincide in $\partial\widetilde{\Omega}$. So we have

$$f(x) - g(x) = \int_{\partial\widetilde{\Omega}_\varepsilon} (f-g)\, d\omega^x_{\widetilde{\Omega}_\varepsilon} = \int_{\partial F_\varepsilon} (f-g)\, d\omega^x_{\widetilde{\Omega}_\varepsilon} \leqslant \|f-g\|_{\infty,\overline{\widetilde{\Omega}}}\, \omega^x_{\widetilde{\Omega}_\varepsilon}(F_\varepsilon) \lesssim \|f\|_{\infty,\Omega}\, C(x)\, \varepsilon.$$

Since ε is a positive constant which can be taken arbitrarily small, we infer that $f(x) = g(x)$. So we deduce that $f = g$ in $\widetilde{\Omega}$. That is, f extends harmonically to the whole $\widetilde{\Omega}$, just defining $f = g$ in E. □

Next we will apply some of the results obtained in this chapter to prove an enhanced version of Proposition 5.40 about the harmonic measure for unbounded open set with compact boundary.

Proposition 6.35. *Let $\Omega \subset \mathbb{R}^d$ be an unbounded open set with compact boundary and let $x \in \Omega$. Then the following holds:*

(a) If $\mathrm{Cap}(\partial\Omega) = 0$, then $\omega^x(\partial\Omega) = 0$.

(b) If $\mathrm{Cap}(\partial\Omega) > 0$ and $d = 2$, then $\omega^x(\partial\Omega) = 1$, that is, ω^x is a probability measure.

(c) If $\mathrm{Cap}(\partial\Omega) > 0$ and $d \geqslant 3$, then $0 < \omega^x(\partial\Omega) < 1$ whenever x belongs to the unbounded component of Ω.

Proof. **(a)** Suppose that $\mathrm{Cap}(\partial\Omega) = 0$. Recall that

$$\omega^x(\partial\Omega) = \lim_{r\to\infty} H^r_f(x) =: H_f(x),$$

where H^r_f is the Perron solution of the Dirichlet problem in $\Omega_r := \Omega \cap B_r(0)$ with boundary data equal to 1 in $\partial\Omega$ and to 0 in $S_r(0)$. So $H^r_f(x) = \omega^x_{\Omega_r}(\partial\Omega)$. For r large enough so that $\partial\Omega \subset B_r(0)$, we have $\omega^x_{\Omega_r}(\partial\Omega) = 0$, by Theorem 6.32. Thus, $H^r_f(x) = 0$ for any r large enough and so $\omega^x(\partial\Omega) = 0$.

(b) Suppose now that $\mathrm{Cap}(\partial\Omega) > 0$ and $d = 2$. By (5.8), $\omega^x(\partial\Omega) \leqslant 1$, so we only have to show the converse inequality. Consider the function

$$u_\varepsilon = 1 + \varepsilon\, U_\mu,$$

where μ is the equilibrium measure for $\partial\Omega$. Since $U_\mu(x) \to -\infty$ as $x \to \infty$, for any r large enough we have $\partial\Omega \subset B_r(0)$ and moreover $u_\varepsilon < 0$ on $S_r(0)$. Notice also that $u_\varepsilon \leqslant 1 + \varepsilon\, \mathrm{Cap}(\partial\Omega)^{-1}$ on $\mathbb{R}^2$. So the function

$$v_\varepsilon = \frac{1}{1 + \varepsilon\, \mathrm{Cap}(\partial\Omega)^{-1}}\, u_\varepsilon$$

belongs to the class $\mathcal{L}_f^r$, the lower Perron class in Ω_r for the function f_r which equals f on $\partial\Omega$ and vanishes on $S_0(0)$. Thus, for any $x \in \Omega_r$,

$$H_f^r(x) \geqslant v_\varepsilon(x) = \frac{1}{1+\varepsilon\ \mathrm{Cap}(\partial\Omega)^{-1}}\,(1+\varepsilon\, U_\mu(x)).$$

Recalling that this holds for any r large enough, we can take the limit as $r \to \infty$ to deduce that the same estimate holds for $H_f(x)$. That is,

$$\omega^x(\partial\Omega) \geqslant \frac{1}{1+\varepsilon\ \mathrm{Cap}(\partial\Omega)^{-1}}\,(1+\varepsilon\, U_\mu(x)).$$

Letting $\varepsilon \to 0$, we infer that $\omega^x(\partial\Omega) \geqslant 1$, which completes the proof of (b).

(c) In this case $\mathrm{Cap}(\partial\Omega) > 0$ and $d \geqslant 3$. Denote by Ω_o the unbounded component of Ω. The same arguments as in Proposition 5.40 show that $\omega^x(\partial\Omega) < 1$ for $x \in \Omega_o$. So we only have to check that $\omega^x(\partial\Omega) > 0$. By Theorem 5.37 (c), if $\xi \in \partial\Omega$ is a regular point, then

$$\lim_{\Omega\ni x\to\xi} \omega^x(\partial\Omega) = \lim_{\Omega\ni x\to\xi} H_f(x) = 1. \tag{6.39}$$

By Theorem 6.17,

$$\mathrm{Cap}(\partial\Omega_o) = \mathrm{Cap}(\mathbb{R}^2\backslash\Omega_o) \geqslant \mathrm{Cap}(\partial\Omega) > 0.$$

By Kellogg's theorem, the set of irregular points is polar, and thus there exists some regular point $\xi \in \partial\Omega_o$. Therefore, (6.39) holds for this point ξ, and thus $\omega^x(\Omega)$ does not vanish identically in Ω_o. Since $\omega^x(\partial\Omega) \geqslant 0$ for all $x \in \Omega$, by the strong maximum principle it follows that $\omega^x(\partial\Omega) > 0$ in the whole Ω_o. □

6.7 Reduction to Wiener regular open sets

In this section we show some results which will be used later in these notes to reduce the proof of some properties for harmonic measure in general open sets to the case when these sets are Wiener regular. More precisely, the results in this section will be used to prove the Jones-Wolff theorem about the dimension of harmonic in the plane and to show the rectifiability of harmonic measure when it is absolutely continuous with Hausdorff measure of codimension 1 in $\mathbb{R}^d$.

Proposition 6.36. *Let $\Omega \subset \mathbb{R}^d$ be open with compact boundary and let $p \in \Omega$. Let $Z \subset \partial\Omega$ be the family of irregular points of Ω. For any $\varepsilon > 0$, then there exists a covering of Z by a countable or finite family of closed balls $\{B_i\}_{i\in I}$ satisfying the following properties:*

(i) The balls B_i are centered in $\partial\Omega$ and they have bounded overlap.

(ii) $\mathrm{Cap}(\bigcup_{i\in I} 2B_i) \leqslant \varepsilon$.

(iii) $\widetilde{\Omega} := \Omega\backslash\bigcup_{i\in I} B_i$ *is open.*

(iv) $\partial\widetilde{\Omega} \subset \left(\partial\Omega \backslash \bigcup_{i\in I} B_i\right) \cup \bigcup_{i\in I} \partial B_i$.

(v) $\widetilde{\Omega}$ *is Wiener regular.*

(vi) *For any* $x \in \widetilde{\Omega}$*, if either* $d = 2$ *with* $\Omega \subset B_{1/2}(0)$*, or* $d \geqslant 3$*, we have*

$$\omega^x_{\widetilde{\Omega}}\Big(\bigcup_{i\in I} 2B_i\Big) \leqslant \varepsilon \sup_{y\in\partial\widetilde{\Omega}} \mathcal{E}(x-y). \tag{6.40}$$

In the case when $d = 2$ *and* Ω *is unbounded, suppose that* $\mathrm{Cap}(\partial\Omega) > 0$*, that* x *belongs to the unbounded component of* Ω*, and that* ε *is small enough. Then,*

$$\omega^x_{\widetilde{\Omega}}\Big(\bigcup_{i\in I} 2B_i\Big) \leqslant C\varepsilon. \tag{6.41}$$

Proof. Let $Z \subset \partial\Omega$ be the subset of irregular points of $\partial\Omega$. By Kellogg's theorem $\mathrm{Cap}(Z) = 0$, and moreover Z is contained in an F_σ set Z_0 such that $\mathrm{Cap}(Z_0) = 0$. By the outer regularity of capacity for compact sets and the fact that Z_0 is an F_σ set, we deduce that there exists an open set U containing Z_0 with $\mathrm{Cap}(U) \leqslant \varepsilon$. Now, for each $x \in Z_0$ we consider a closed ball B_x contained in U, and by Besicovitch covering theorem we find a subamily $\{B_i\}_{i\in I} \subset \{B_x\}_{x\in Z_0}$ with bounded overlap which covers Z_0, so that the properties (i) and (ii) in the lemma hold.

Next we will show that the set $\widetilde{\Omega} = \Omega \backslash \bigcup_{i\in I} B_i$ is open. Indeed, we claim that

$$\overline{\bigcup_{i\in I} B_i} \backslash \bigcup_{i\in I} B_i \subset \partial\Omega. \tag{6.42}$$

This inclusion implies that

$$\Omega \backslash \overline{\bigcup_{i\in I} B_i} = \Omega \backslash \left[\left(\overline{\bigcup_{i\in I} B_i} \backslash \bigcup_{i\in I} B_i\right) \cup \bigcup_{i\in I} B_i\right] = \Omega \backslash \bigcup_{i\in I} B_i = \widetilde{\Omega},$$

and thus ensures that $\widetilde{\Omega}$ is open.

To show the claim (6.42) consider $x \in \overline{\bigcup_{i\in I} B_i} \backslash \bigcup_{i\in I} B_i$ and recall that, by construction each ball B_i is closed. Then x must be the limit of a sequence of points belonging to infinitely many different balls B_{i_k}, $i_k \in I$. It turns out that then we have $r(B_{i_k}) \to 0$. This is a straightforward consequence of the fact that any family of balls B_j, $j \in J \subset I$, such that $\mathrm{dist}(B_j, x) \leqslant 1$ and $0 < \varepsilon \leqslant r(B_j) \leqslant 1$ must be finite, by the finite overlap of the family $\{B_i\}_{i\in I}$. The fact that $r(B_{i_k}) \to 0$ implies that $x \in \partial\Omega$, since the balls $B_{i,k}$ are centered in $\partial\Omega$.

To prove (iv), write

$$\begin{aligned}\partial\widetilde{\Omega} = \partial\Big(\Omega \backslash \bigcup_{i\in I} B_i\Big) &\subset \partial\Omega \cup \overline{\bigcup_{i\in I} B_i} = \partial\Omega \cup \left(\overline{\bigcup_{i\in I} B_i} \backslash \bigcup_{i\in I} B_i\right) \cup \bigcup_{i\in I} B_i \\ &= \partial\Omega \cup \bigcup_{i\in I} B_i = \left(\partial\Omega \backslash \bigcup_{i\in I} B_i\right) \cup \bigcup_{i\in I} B_i.\end{aligned}$$

On the other hand, by construction the interior of each ball B_i lies in the exterior of $\widetilde{\Omega}$, and thus

$$\partial\widetilde{\Omega} = \partial\widetilde{\Omega}\backslash\mathrm{ext}(\widetilde{\Omega}) \subset \Big[\Big(\partial\Omega\backslash\bigcup_{i\in I} B_i\Big) \cup \bigcup_{i\in I} B_i\Big]\backslash\mathrm{ext}(\widetilde{\Omega}) \subset \Big(\partial\Omega\backslash\bigcup_{i\in I} B_i\Big) \cup \bigcup_{i\in I}\partial B_i,$$

which proves (iv).

Next we check that $\widetilde{\Omega}$ is Wiener regular. That is, all the points $x \in \partial\widetilde{\Omega}$ are Wiener regular for $\widetilde{\Omega}$. We have to show that

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash\widetilde{\Omega})}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty$$

for all $x \in \partial\widetilde{\Omega}$. By (iv) we know that either $x \in (\partial\Omega\backslash\bigcup_{i\in I} B_i)$ or $x \in \partial B_i$ for some $i \in I$. In the latter case we have

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash\widetilde{\Omega})}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} \geqslant \sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash B_i\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty,$$

since the complement of any ball B_i is Wiener regular. If $x \in \partial\Omega\backslash\bigcup_{i\in B_i} B_i$, then we know that x is Wiener regular for Ω, because $Z \subset \bigcup_{i\in I} B_i$. Thus, using just that $\widetilde{\Omega}^c \supset \Omega^c$, we obtain

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash\widetilde{\Omega})}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} \geqslant \sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty.$$

So the proof that $\widetilde{\Omega}$ is Wiener regular is concluded.

The arguments to prove (vi) are quite similar to the ones for Theorem (6.32). For any $d \geqslant 2$ we consider any finite subfamily $J \subset I$ of the closed balls B_i, and we let $E = \bigcup_{i\in J} B_i$, so that E is compact and $\mathrm{Cap}(E) \leqslant \varepsilon$, by (ii). Since E consists of a union of finitely many closed balls, it follows either by Wiener's criterion or by the exterior ball criterion in Remark 5.34 that Ω_E is Wiener regular. Then, by Lemma 6.24, if μ_E stands for the equilibrium measure for E, the potential U_{μ_E} is continuous in $\mathbb{R}^d$ and $U_{\mu_E} = (\mathrm{Cap}(E))^{-1} \geqslant \varepsilon^{-1}$ in E.

Suppose first that $d \geqslant 3$ or $d = 2$ with $\Omega \subset B_{1/2}(0)$. Consider the function $f(x) = \mathrm{Cap}(E)\, U_{\mu_E}(x)$, and notice that it is superharmonic and continuous in $\mathbb{R}^d$, and it equals 1 on E. Also, it is positive in $\overline{\Omega}$ since $\Omega \subset B_{1/2}(0)$ in the planar case. So we have

$$\omega^x_{\widetilde{\Omega}}(E) \leqslant \int f\, d\omega^x_{\widetilde{\Omega}}. \tag{6.43}$$

By definition, letting $g = f|_{\partial\widetilde{\Omega}}$, the last integral above equals $H_g(x)$. Since f belongs to the upper Perron class for g in $\widetilde{\Omega}$, we have $H_g(x) \leqslant f(x)$. Thus,

$$\omega^x_{\widetilde{\Omega}}(E) \leqslant f(x) = \mathrm{Cap}(E)\, U_{\mu_E}(x) \leqslant \varepsilon\, U_{\mu_E}(x) \leqslant \varepsilon\, U_{\mu_E}(x) \leqslant \varepsilon \sup_{y\in E} \mathcal{E}(x-y), \tag{6.44}$$

using that μ is a probability measure supported on E for the last inequality. Since the estimate above holds for any finite subfamily $J \subset I$, (6.40) holds.

In the case when $d = 2$ and Ω is unbounded, we can assume that $\mathrm{Cap}(\partial\Omega) > 0$. Then consider the function

$$g(x) = U_{\mu_E}(x) - U_{\mu_{\partial\Omega}}(x),$$

where $\mu_{\partial\Omega}$ is the equilibrium measure for $\partial\Omega$. Notice that g is superharmonic in Ω and

$$g(x) \geqslant \frac{1}{\mathrm{Cap}(E)} - \frac{1}{\mathrm{Cap}(\partial\Omega)} \geqslant \frac{1}{\varepsilon} - \frac{1}{\mathrm{Cap}\,\partial\Omega} \qquad \text{for } x \in E.$$

Then for ε small enough, $g(x) \geqslant \frac{1}{2\varepsilon} > 0$ on E, and since g vanishes at ∞, by the maximum principle g is positive in the unbounded component of Ω. Thus, for x in this component,

$$\begin{aligned}
\omega^x_{\widetilde{\Omega}}(E) &\leqslant 2\varepsilon\, g(x) = 2\varepsilon(U_{\mu_E}(x) - U_{\mu_{\partial\Omega}}(x)) \\
&= \frac{\varepsilon}{\pi}\int \log\frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x,\partial\Omega)}{|x-y|}\,d\mu_E(y) - \frac{\varepsilon}{\pi}\int \log\frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x,\partial\Omega)}{|x-y|}\,d\mu_\Omega(y) \\
&\leqslant \frac{\varepsilon}{\pi}\int \log\frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x,\partial\Omega)}{|x-y|}\,d\mu_E(y) \leqslant \frac{\varepsilon}{\pi}\log\frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x,\partial\Omega)}{\mathrm{dist}(x,E)},
\end{aligned}$$

where in the before to last inequality we took into account that $\log\frac{\mathrm{diam}\partial\Omega+\mathrm{dist}(x,\partial\Omega)}{|x-y|}$ is positive in $\partial\Omega$. For ε small enough, $\mathrm{dist}(x,E) \geqslant \frac{1}{2}\mathrm{dist}(x,\partial\Omega)$, and then (6.41) follows. $\square$

Lemma 6.37. *Let $\Omega \subset \mathbb{R}^d$ be open with compact boundary and let $p \in \Omega$. For any $\varepsilon > 0$, denote by $\widetilde{\Omega}_\varepsilon$ the Wiener regular set $\widetilde{\Omega}$ constructed in Proposition 6.36. Suppose either that Ω is bounded with $d \geqslant 3$, or that $d = 2$ and $\mathrm{Cap}_L(\partial\Omega) > 0$. Then, for any Borel set $A \subset \partial\Omega$,*

$$\lim_{\varepsilon\to 0}\omega^p_{\widetilde{\Omega}_\varepsilon}(A) = \omega^p_\Omega(A). \tag{6.45}$$

In fact, the lemma also holds in the case $d \geqslant 3$ and Ω unbounded with compact boundary. However, we will not need this result and so we will not show this.

Proof. In the case $d = 2$ we can assume that $\partial\Omega \subset B_{1/2}(0)$ by a suitable dilation. Let $A \subset \partial\Omega$ be a Borel set. Then, by Lemma 5.28,

$$\omega^p_{\widetilde{\Omega}_\varepsilon}(A) = \omega^p_{\widetilde{\Omega}_\varepsilon}(A \cap \partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon) \leqslant \omega^p_\Omega(A \cap \partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon) \leqslant \omega^p_\Omega(A).$$

To estimate $\omega^p_\Omega(A)$ in terms of $\omega^p_{\widetilde{\Omega}_\varepsilon}(A)$, we take into account that $\omega_\Omega(\partial\Omega) = \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon) = 1$ and we apply the previous estimate to $\partial\Omega\backslash A$: write

$$\begin{aligned}
\omega^p_\Omega(A) = 1 - \omega^p_\Omega(\partial\Omega\backslash A) &\leqslant 1 - \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon\backslash A) \\
&= \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega) + \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \cap \partial\Omega) - \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \cap \partial\Omega\backslash A) \\
&= \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega) + \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \cap \partial\Omega \cap A) = \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega) + \omega^p_{\widetilde{\Omega}_\varepsilon}(A).
\end{aligned}$$

Hence,

$$|\omega^p_{\widetilde{\Omega}_\varepsilon}(A) - \omega^p_\Omega(A)| \leqslant \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \backslash \partial\Omega). \tag{6.46}$$

Since $\partial\widetilde{\Omega}_\varepsilon$ is contained in the union of the balls B_i, $i \in I$, in Proposition 6.36, by the proeprty (vi) in the proposition $\omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \backslash \partial\Omega)$ tends to 0 aas $\varepsilon \to 0$. □

Notice that, by (6.46), the convergence in (6.45) is uniform on the set $A \subset \partial\Omega$.

7 Harmonic measure and Green function in Wiener regular open sets

In this section we will assume that Ω is an open Wiener regular set.

7.1 The Green function in terms of harmonic measure in bounded open sets

For a bounded open Wiener regular set $\Omega \subset \mathbb{R}^d$, we may write the Green function in terms of harmonic measure. Let us see how.

Given $x \in \Omega$, define the harmonic extension

$$v^x(y) := -\int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) \quad \text{for } y \in \Omega, \tag{7.1}$$

where $\mathcal{E}^x$ is the fundamental solution of the minus Laplacian with pole at z. Note that $\mathcal{E}^x$ is continuous in $z \in \partial\Omega$ and Ω is Wiener regular, then $v^x \in C(\overline{\Omega})$ and its boundary values are opposite to those of the fundamental solution. Thus,

$$G^x(y) = \begin{cases} \mathcal{E}^x(y) + v^x(y) & \text{for } y \in \Omega \backslash \{x\}, \\ 0 & \text{otherwise,} \end{cases} \tag{7.2}$$

is continuous away from the pole, and harmonic in $\mathbb{R}^d \backslash \partial\Omega$.

Thus, in a sense G is the continuous solution to the Dirichlet problem

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega, \\ G^x = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular bounded open set. The Green function for Ω is non-negative in Ω, and positive in the component of Ω that contains x. Further, it is subharmonic in $\mathbb{R}^d \backslash \{x\}$.*

Proof. To prove the first statement, notice that $G^x \equiv 0$ in any component V of Ω which does not contain x, by the maximum principle, since G^x is harmonic in V and vanishes continuously in ∂V. If V_x is the component of Ω that contains x, we consider any $\varepsilon > 0$ small enough such that $\bar{B}_{2\varepsilon}(x) \subset V_x$, and we set $V_{x,\varepsilon} = V_x \backslash \bar{B}_\varepsilon(x)$. For ε small enough, $G^x > 0$ in $\partial\bar{B}_\varepsilon(x)$, and then by the maximum principle, it follows that $G^x > 0$ in $V_{x,\varepsilon}$. So $G^x > 0$ in V_x.

Regarding the second statement, using the maximum principle for harmonic functions, one can check that the Green function satisfies the condition in Lemma 5.6, implying the subharmonicity of the Green function (7.2) away from the pole. $\square$

Here there is a small trouble. We have defined the Green function in two different ways, solving the Dirichlet problem in the Sobolev sense and in the continuous sense. Fortunately, both definitions coincide in Wiener regular open sets:

Lemma 7.2. *Let v_x and G^x be defined as in (7.1) and (7.2), and let ψ^x be a bump function satisfying $\chi_{B_{2t}(x)^c} \leqslant \psi^x \leqslant \chi_{B_t(x)^c}$ for $t < \frac{1}{2}\mathrm{dist}(x, \partial\Omega)$. Then $v_x \in H^1(\Omega)$, and $\psi^x G^x \in H^1_0(\Omega)$. So G^x coincides with the other Green function defined in Section 3.2.*

Proof. First we will check that $G^x \in H^1(\Omega \backslash B_{2t}(x))$. Since Ω is bounded, it is enough to check that $\|G^x\|_{L^2(B\cap\Omega)} < +\infty$ for every ball B such that $2B \cap B_{2t}(x) = \varnothing$. To show this fact we will use Caccioppoli inequality, but in order to apply it, we need to know *a priori* the finiteness of the L^2 norm of the gradient. To avoid a circular argument, we need to define $u_\varepsilon(y) := \max\{G^x(y) - \varepsilon, 0\}$ for $y \in B_{2t}(x)^c$.

Let us check the properties of u_ε. First, since $G^x \in C^\infty(\Omega \backslash B_{2t}(x))$, we can infer that $u_\varepsilon \in H^1(2B)$ (see [EG15, Theorem 4.4]). On the other hand, since G^x is subharmonic away from the pole, also u_ε is subharmonic. Moreover, it is non-negative. Finally, we can apply the Caccioppoli inequality and the maximum principle to get

$$\int_B |\nabla u_\varepsilon|^2 \lesssim \int_{2B} |u_\varepsilon|^2 \leqslant \int_{2B} (G^x)^2 \leqslant |2B| \max_{\partial B_{2t}(x)} (G^x)^2,$$

which is independent of ε.

By the monotone convergence theorem, we get

$$\int_{B\cap\Omega} |\nabla G^x|^2 = \lim_{\varepsilon\to 0} \int_B |\nabla u_\varepsilon|^2 \lesssim |2B| \max_{\partial B_{2t}(x)} (G^x)^2 < +\infty,$$

i.e.,

$$G^x \in H^1_{\mathrm{loc}}(\Omega \backslash B_\varepsilon(x)),$$

and thus $v^x = G^x - \mathcal{E}^x \in H^1_{\mathrm{loc}}(\Omega \backslash B_\varepsilon(x))$ as well. Since it is C^∞ in a neighborhood of the pole, we get $v^x \in H^1_{\mathrm{loc}}(\Omega)$.

It remains to check $\psi^x G^x \in H^1_0(\Omega)$. For every $y \in \Omega$ define $u_\varepsilon(y) := \max\{\psi^x(y)G^x(y) - \varepsilon, 0\}$. Then

$$\lim_{\varepsilon\to 0} u_\varepsilon(y) = \psi^x(y)G^x(y), \quad \text{and} \quad \lim_{\varepsilon\to 0} \nabla u_\varepsilon(y) = \nabla(\psi^x G^x)(y).$$

Moreover, by the triangle inequality

$$\|u_\varepsilon - \psi^x G^x\|_{H^1(\Omega)} \leqslant \|u_\varepsilon\|_{H^1(\Omega)} + \|\psi^x G^x\|_{H^1(\Omega)} \leqslant 2\|\psi^x G^x\|_{H^1(\Omega)}.$$

Thus, by the dominated convergence theorem, we get

$$\|u_\varepsilon - \psi^x G^x\|_{H^1} \xrightarrow{\varepsilon\to 0} 0.$$

Note that u_ε is compactly supported in $\Omega \backslash B_t(x)$, and it is Lipschitz. Thus, we have shown the existence of Lipschitz functions (not C^∞ in general) with compact support converging to $\psi^x G^x$ in the Sobolev norm. Proving that this implies that $\psi^x G^x \in H^1_0(\Omega)$ is an exercise left for the reader. $\square$

Remark 7.3. In fact, when a Sobolev function vanishes continuously in the boundary, its gradient can be extended by zero in the complement of the open set, the proof is similar to [EG15, Theorem 4.4]. Thus, we have shown that $G^x \in H^1_{\text{loc}}(\mathbb{R}^d \backslash B_\varepsilon(x))$, with $\nabla G^x(y) \equiv 0$ for $y \in \Omega^c$.

For $x \in \mathbb{R}^d \backslash \Omega$ and $y \in \Omega$, we will also set

$$G^x(y) = 0. \tag{7.3}$$

This choice, together with Lemmas 3.6 and 7.2 implies that

$$G^x(y) = G^y(x) \qquad \text{for all } (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \backslash \Omega^c \times \Omega^c \text{ with } x \neq y. \tag{7.4}$$

Note that the equation (7.2) is still valid for $x \in \mathbb{R}^d \backslash \overline{\Omega}$ and $y \in \Omega$. The case when $x \in \partial\Omega$ and $y \in \Omega$ is more delicate and the identity (7.2) may fail. However, we have the following partial result:

Lemma 7.4. *Let $\Omega \subset \mathbb{R}^d$ be bounded and Wiener regular and let $y \in \Omega$. For m-almost all $x \in \Omega^c$ we have*

$$\mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) = 0. \tag{7.5}$$

Clearly, in the particular case where $m(\partial\Omega) = 0$, this result is a consequence of the aforementioned fact that (7.2) also holds for all $x \in \mathbb{R}^d \backslash \overline{\Omega}$, $y \in \Omega$, with $G^x(y) = 0$.

Proof. Let $A \subset \Omega^c$ be a compact set with $m(A) > 0$. Observe that the potential $U_A := U_{\chi_A m} = \mathcal{E} * \chi_A$ is continuous, bounded in $\mathbb{R}^d$, and harmonic in A^c, see Remark 6.6. Then, by Fubini we have for all $y \in \Omega$,

$$\begin{aligned} \int_A \left(\mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) \right) dm(x) &= U_A(y) - \int_{\partial\Omega} \int_A \mathcal{E}^x(z)\, dm(x)\, d\omega^y(z) \\ &= U_A(y) - \int_{\partial\Omega} U_A(z)\, d\omega^y(z) = 0, \end{aligned}$$

using that U_A is harmonic in $\Omega \subset A^c$ and bounded on $\partial\Omega$ for the last identity. Since the compact set $A \subset \Omega^c$ is arbitrary, the lemma follows. $\square$

Remark 7.5. As a corollary of the preceding lemma we deduce that

$$G^x(y) = \mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) \qquad \text{for } m\text{-a.e. } x \in \mathbb{R}^d.$$

Lemma 7.6. *For all $x \in \Omega$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\int \varphi\, d\omega^x(y) - \varphi(x) = \int_\Omega \Delta\varphi(y)\, G^x(y)\, dy = -\int_\Omega \nabla\varphi(y) \cdot \nabla G^x(y)\, dy.$$

Proof. The first identity follows from Lemma 3.6 and (7.3), the preceding remark, and Fubini. Indeed,

$$\begin{aligned}\int_{\Omega} \Delta\varphi(y)\, G^x(y)\, dy &= \int_{\mathbb{R}^d} \Delta\varphi(y)\, G^y(x)\, dy = \int \Delta\varphi(y) \left(\mathcal{E}^y(x) - \int_{\partial\Omega} \mathcal{E}^y(z)\, d\omega^x(z)\right) dy \\ &= (\Delta\varphi * \mathcal{E})(x) - \int_{\partial\Omega} (\Delta\varphi * \mathcal{E})(z)\, d\omega^x(z) \\ &= -\varphi(x) + \int_{\partial\Omega} \varphi(z)\, d\omega^x(z).\end{aligned}$$

The last identity in the lemma follows integrating by parts and a density argument. □

Notice that, by the preceding lemma, in the sense of distributions, that is in the dual space $\mathcal{D}'(\mathbb{R}^d)$ (here, as in the literature in functional analysis, $\mathcal{D}$ stands for C^∞ functions with compact support, equipped with a certain topology, see [Rud91, Chapter 6]), we have

$$\Delta G^x = \omega^x - \delta_x \quad \text{for all } x \in \Omega.$$

For smooth domains with smooth Green function, we have the following:

Proposition 7.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1 domain, $x \in \Omega$ and suppose that $G^x \in C^1(\overline{\Omega})$. Then*

$$\omega^x = -(\partial_\nu G^x)\, \sigma,$$

where ν is the unit outer normal to $\partial\Omega$ and σ is the surface measure on $\partial\Omega$.

Proof. It suffices to show that for any $\varphi \in \mathcal{D} = C_c^\infty(\mathbb{R}^d)$ it holds

$$\int_{\partial\Omega} \varphi\, d\omega^x(y) = -\int_{\partial\Omega} \varphi(y)\, \partial_\nu G^x(y)\, d\sigma(y).$$

We may assume that φ vanishes in a neighborhood of x by modifying suitably φ far away from $\partial\Omega$, since the domain of integration in both integrals above is $\partial\Omega$. So consider $r > 0$ such that $B_{2r}(x) \subset \Omega$ and $\operatorname{supp}\varphi \subset \mathbb{R}^d \backslash B_{2r}(x)$. Denote $\Omega^r = \Omega \backslash \bar{B}_r(x)$. Using that G^x is harmonic in Ω^r and that φ vanishes in $B_{2r}(x)$, by Lemma 7.6 and Green's formula we have

$$\begin{aligned}\int \varphi\, d\omega^x(y) &= \int_{\Omega} \Delta\varphi(y)\, G^x(y)\, dy = \int_{\Omega^r} \Delta\varphi(y)\, G^x(y)\, dy \\ &= -\int_{\partial\Omega^r} \varphi(y)\, \partial_\nu G^x(y)\, d\sigma(y) = -\int_{\partial\Omega} \varphi(y)\, \partial_\nu G^x(y)\, d\sigma(y).\end{aligned}$$

□

Lemma 7.8. *Let B be a ball centered in $\partial\Omega$ and let $x \in \Omega \backslash 2B$. Then,*

$$\omega^x(B) \lesssim r(B)^{d-2} \fint_{2B} G^x(y)\, dy.$$

Proof. Let φ be a bump function such that $\chi_B \leqslant \varphi \leqslant \chi_{2B}$ with $\|D^2\varphi\| \lesssim \frac{1}{r(B)^2}$. By Lemma 7.6, we have

$$\omega^x(B) \leqslant \int \varphi\, d\omega^x = \int \Delta\varphi(y)\, G^x(y)\, dy \lesssim \frac{1}{r(B)^2} \int_{2B} G^x(y)\, dy = r(B)^{d-2} \fint_{2B} G^x(y)\, dy.$$

□

As we shall see in further chapters, when Ω is an NTA or CDC uniform domain, for x and B as in the preceding lemma, we have

$$\omega^x(B) \approx r(B)^{d-2}\, G^x(X_B^{\text{in}}),$$

where X_B^{in} is an interior corkscrew point for B. One can view the result in the preceding lemma as a weak version of the estimate $\omega^x(B) \lesssim r(B)^{d-2}\, G^x(X_B^{\text{in}})$. In the next sections we will obtain some estimates in the converse direction.

7.2 The Green function in unbounded open sets with compact boundary

Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. In the case $d \geqslant 3$, we defined the Green function for Ω in the same we did for bounded open sets. That is, given $x \in \Omega$, we consider the harmonic extension

$$v^x(y) := -\int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) \quad \text{for } y \in \Omega, \tag{7.6}$$

Then we define the Green function with pole at x as follows:

$$G^x(y) = \begin{cases} \mathcal{E}^x(y) + v^x(y) & \text{for } y \in \Omega\backslash\{x\}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.7}$$

Notice that G^x is continuous away from the pole, harmonic in $\mathbb{R}^d\backslash\partial\Omega$, and $G^x(y) \to 0$ as $y \to \infty$.

In the case $d = 2$ we cannot define G^x as above because otherwise this will have a pole at ∞, which is not convenient. Instead we want G^x to be bounded at ∞. If Ω is not dense in $\mathbb{R}^d$, we can take a point $\xi \in \mathbb{R}^2\backslash\overline{\Omega}$ and we can define G^x as above, replacing $\mathcal{E}^x$ in (7.6) and (7.7) by $\mathcal{E}^x - \mathcal{E}^\xi$. Notice that $\mathcal{E}^x - \mathcal{E}^\xi$ has a logarithmic singularity (i.e., a pole) at x, it is continuous in $\partial\Omega$, and it is bounded at ∞. Then it easily follows that the Green function G^x defined in this way has a pole at x, it is bounded at ∞, and vanishes continuously on $\partial\Omega$.

For an arbitrary a Wiener regular unbounded open set with compact boundary in the plane, we define G^x as in (7.6) and (7.7), replacing $\mathcal{E}^x$ by $\mathcal{E}^x - U_\mu$, where μ is the equilibrium measure for $\partial\Omega$. Again it turns out that the Green function G^x defined in this way has a pole at x, it is bounded at ∞, and vanishes continuously on $\partial\Omega$. Indeed, recall that the equilibrium potential is continuous in $\mathbb{R}^d$ when Ω is Wiener regular. Further, this can be written as follows, for $y \in \Omega$,

$$G^x(y) = \frac{1}{2\pi} \int_{\partial\Omega} \log \frac{|y-\xi|}{|y-x|}\, d\mu(\xi) - \frac{1}{2\pi} \int_{\partial\Omega} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega^y(z). \tag{7.8}$$

The analog of Lemma 7.1 holds for unbounded domains with compact boundary:

Lemma 7.9. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. The Green function for Ω is non-negative in Ω, and positive in the component of Ω that contains x. Further, it is subharmonic in $\mathbb{R}^d \backslash \{x\}$. In the case $d = 3$, G^x vanishes at ∞, and in the case $d = 2$, it is bounded at ∞*

The proof is similar to the one of Lemma 7.1 and we leave this for the reader.

Next we show that the Green function G^x is "locally" in the Sobolev space $H_0^1(\Omega)$. More precisely:

Lemma 7.10. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary and let $x \in \Omega$. Let G^x be defined as in (7.7) in the case $d \geqslant 3$ and as in (9.19) in the case $d = 2$. For $0 < t < \frac{1}{2}\mathrm{dist}(x, \partial\Omega)$, let ψ^x be a bump function satisfying $\chi_{B_{2t}(x)^c} \leqslant \psi^x \leqslant \chi_{B_t(x)^c}$. For any $r > 0$ such that $\partial\Omega \subset B_r(0)$, let ψ_r be a bump function such that $\chi_{B_r(x)} \leqslant \psi_r \leqslant \chi_{B_{2r}(x)}$. Then $\psi^x\, \psi_r\, G^x \in H_0^1(\Omega)$.*

The arguments for this lemma are similar to the ones for Lemma 7.2 and so we omit them again.

Lemma 7.11. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. For $r > 0$ such that $\partial\Omega \subset B_r(0)$, let $\Omega_r = \Omega \cap B_r(0)$. For $x \in \Omega$ and $r > |x|$, let G^x and G_r^x the respective harmonic functions for Ω and Ω_r with pole at x. Then $G_r^x \to G^x$ as $r \to \infty$ uniformly on bounded sets.*

Proof. In the the case $d \geqslant 3$, for $x, y \in \Omega$ with $x \neq y$, we have

$$G^x(y) = \mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_\Omega^y(z).$$

The same identity holds for G_r^x, replacing $\partial\Omega$ and ω_Ω by $\partial\Omega_r$ and ω_{Ω_r}, respectively. Thus,

$$\begin{aligned} G_r^x(y) - G^x(y) &= \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_\Omega^y(z) - \int_{\partial\Omega_r} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z) \\ &= \left(\int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_\Omega^y(z) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z) \right) - \int_{\partial B_r(0)} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z). \end{aligned}$$

By Remark 5.39, the term in parentheses on the right hand side tends to 0 as $r \to \infty$. On the other hand, the second term can be bounded as follows:

$$\left| \int_{\partial B_r(0)} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z) \right| \lesssim \frac{1}{\mathrm{dist}(x, \partial B_r(0))^{d-2}}\, \omega_{\Omega_r}^y(\partial B_r(0)) \leqslant \frac{1}{\mathrm{dist}(x, \partial B_r(0))^{d-2}},$$

which also tends to 0 uniformly on bounded subsets of Ω.

In the case $d = 2$, the Green function G^x for Ω can be written as in (9.19). The Green function G_r^x for Ω_r can be written in a similar fashion, for $y \in \Omega_r$:

$$G_r^x(y) = \frac{1}{2\pi}\int_{\partial\Omega} \log\frac{|y-\xi|}{|y-x|}\,d\mu(\xi) - \frac{1}{2\pi}\int_{\partial\Omega_r}\int_{\partial\Omega} \log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)\,d\omega_{\Omega_r}^y(z). \tag{7.9}$$

Here μ is the equilibrium measure for $\partial\Omega$. To check the preceding identity, notice that μ is a probability measure and we have

$$\frac{1}{2\pi}\int_{\partial\Omega} \log|y-\xi|\,d\mu(\xi) - \frac{1}{2\pi}\int_{\partial\Omega_r}\int_{\partial\Omega} \log|z-\xi|\,d\mu(\xi)\,d\omega_{\Omega_r}^y(z) = 0,$$

because the function $g(y) := \frac{1}{2\pi}\int_{\partial\Omega}\log|y-\xi|\,d\mu(\xi)$ is harmonic and continuous in $\overline{\Omega_r}$. Then, by (9.19) and (7.9), we get

$$\begin{aligned}
G_r^x(y) - G^x(y) &= \int_{\partial\Omega}\int_{\partial\Omega} \log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)\,d\omega_{\Omega}^y(z) - \int_{\partial\Omega_r}\int_{\partial\Omega} \log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)\,d\omega_{\Omega_r}^y(z)\\
&= \left(\int_{\partial\Omega}\int_{\partial\Omega} \log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)\,d\omega_{\Omega}^y(z) - \int_{\partial\Omega}\int_{\partial\Omega} \log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)\,d\omega_{\Omega_r}^y(z)\right)\\
&\quad - \int_{\partial B_r(0)}\int_{\partial\Omega} \log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)\,d\omega_{\Omega_r}^y(z).
\end{aligned}$$

By Remark 5.39 (applied with $f(z) := \int_{\partial\Omega}\log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)$), it follows that the first term in parentheses tends to 0 uniformly in bounded subsets of Ω. Using the fact that $f(z) \to 0$ as $z \to \infty$, we also get easily that that the last term tends to 0 uniformly in bounded subsets of Ω. $\square$

Thanks to the preceding lemma, many of the results obtained in the previous section for the Green function in Wiener regular bounded open sets can be extended to the case of unbounded open sets with compact boundaries. First, we easily get that the Green function is symmetric:

Lemma 7.12. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. For all $x, y \in \Omega$, with $x \neq y$, the Green function for Ω satisfies $G^x(y) = G^y(x)$.*

Proof. Let $\Omega_r = \Omega \cap B_r(0)$, with $r > 0$ big enough so that $\partial\Omega \subset B_r(0)$ and $x, y \in \Omega_r$. Let G_r denote the Green function for Ω_r. Then we have

$$G^x(y) = \lim_{r\to\infty} G_r^x(y) = \lim_{r\to\infty} G_r^y(x) = G^y(x).$$

$\square$

From now on, quite often we will write

$$G(x,y) = G^x(y) = G^y(x).$$

Lemma 7.13. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. For all $x \in \Omega$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\int \varphi \, d\omega^x(y) - \varphi(x) = \int_\Omega \Delta\varphi(y) \, G^x(y) \, dy = -\int_\Omega \nabla\varphi(y) \cdot \nabla G^x(y) \, dy.$$

Proof. The first identity follows from the one derived for bounded open sets in Lemma 7.6 and from the uniform convergence of G_r^x to G^x in bounded subsets of Ω. The second one follows from the first one by integration by parts. $\square$

Proposition 7.14. *Let $\Omega \subset \mathbb{R}^d$ be an unbounded C^1 domain with compact boundary, $x \in \Omega$ and suppose that $G^x \in C^1(\overline{\Omega})$. Then*

$$\omega^x = -(\partial_\nu G^x) \, \sigma,$$

where ν is the unit outer normal to $\partial\Omega$ and σ is the surface measure on $\partial\Omega$.

Proof. This follows from the preceding lemma, arguing as in Proposition 7.7. $\square$

Lemma 7.15. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. Let B be a ball centered in $\partial\Omega$ and let $x \in \Omega \backslash 2B$. Then,*

$$\omega^x(B) \lesssim r(B)^{d-2} \fint_{2B} G^x(y) \, dy.$$

Proof. This is proven in the same way as Lemma 7.8 for the case of bounded open sets. $\square$

7.3 Newtonian capacity, harmonic measure, and Green's function in the case $d \geqslant 3$

In this whole section we assume either that Ω is a Wiener regular open set in $\mathbb{R}^d$, with $d \geqslant 3$, and that either it is bounded or it is unbounded with compact boundary.

Lemma 7.16. *Let $d \geqslant 3$ and $\Omega \subset \mathbb{R}^d$ be an open Wiener regular set with compact boundary. Let B be a closed ball centered at $\partial\Omega$. Then*

$$\omega^x(B) \geqslant c(d) \frac{\operatorname{Cap}(\frac{1}{4}B \backslash \Omega)}{r(B)^{d-2}} \qquad \text{for all } x \in \tfrac{1}{4}B \cap \Omega,$$

with $c(d) > 0$.

Proof. We can assume that Ω is bounded. Otherwise, the estimate above follows from the analogous estimate applied to $\Omega_r = \Omega \cap B_r(0)$ letting $r \to \infty$.

Let $\mu_{\frac{1}{4}B \cap \partial\Omega}$ be the equilibrium measure for $\frac{1}{4}B \backslash \Omega$, and let $\mu = \operatorname{Cap}(\frac{1}{4}B \backslash \Omega) \, \mu_{\frac{1}{4}B \backslash \Omega}$, so that $\|U_\mu\|_\infty \leqslant 1$ and $\|\mu\| = \operatorname{Cap}(\frac{1}{4}B \backslash \Omega)$. Notice that, for all $x \in B^c$,

$$U_\mu(x) = \int \frac{c_d}{|x-y|^{d-2}} \, d\mu(y) \leqslant \frac{c_d \|\mu\|}{(\frac{3}{4} r(B))^{d-2}}.$$

Consider the function $f(x) = U_\mu(x) - \frac{c_d\|\mu\|}{(\frac{3}{4}(B))^{d-2}}$. Using that $f(x) \leqslant 0$ in B^c, $\|f\|_\infty \leqslant 1$, and that f is harmonic in Ω, by Corollary 5.32 and the maximum principle we deduce that, for all $x \in \Omega$,

$$\omega^x(B) \geqslant f(x).$$

In particular, for $x \in \frac{1}{4}B$ we have

$$\begin{aligned}\omega^x(B) &\geqslant \int \frac{c_d}{|x-y|^{d-2}}\,d\mu(y) - \frac{c_d\|\mu\|}{(\frac{3}{4}r(B))^{d-2}} \\ &\geqslant \frac{c_d\|\mu\|}{(\frac{1}{2}r(B))^{d-2}} - \frac{c_d\|\mu\|}{(\frac{3}{4}r(B))^{d-2}} = c_d\big(2^{d-2} - (\tfrac{4}{3})^{d-2}\big)\,\frac{\operatorname{Cap}(\frac{1}{4}B\backslash\Omega)}{r(B)^{d-2}},\end{aligned}$$

which proves the lemma. $\square$

Lemma 7.17. *Let $d \geqslant 3$ and $\Omega \subset \mathbb{R}^d$ be an open Wiener regular set with compact boundary. Let B be a closed ball centered at $\partial\Omega$. Then, for all $a > 2$,*

$$\omega^x(aB) \gtrsim \inf_{z\in 2B\cap\Omega} \omega^z(aB)\, r(B)^{d-2}\, G^x(y) \quad \text{for all } x \in \Omega\backslash 2B \text{ and } y \in B\cap\Omega, \tag{7.10}$$

with the implicit constant independent of a.

Proof. We can assume that Ω is bounded. Otherwise, the estimate above follows from the one applied to $\Omega_r = \Omega \cap B_r(0)$ letting $r \to \infty$.

Fix $y \in B\cap\Omega$ and note that for every $x \in \partial(2B)\cap\Omega$ we have $\inf_{z\in 2B\cap\Omega}\omega^z(aB) \leqslant \omega^x(aB)$ and, therefore

$$G^x(y) \leqslant \mathcal{E}^x(y) \approx \frac{1}{|x-y|^{d-2}} \leqslant \frac{c}{r(B)^{d-2}} \leqslant \frac{c\,\omega^x(aB)}{r(B)^{d-2}\,\inf_{z\in 2B\cap\Omega}\omega^z(aB)}. \tag{7.11}$$

Let us observe that the two non-negative functions

$$u(x) = c^{-1}\,G^x(y)\,r(B)^{d-2} \inf_{z\in 2B\cap\Omega}\omega^z(aB) \qquad \text{and} \qquad v(x) = \omega^x(aB)$$

are harmonic, hence continuous, in $\Omega\backslash\overline{B}$. Note that (7.11) says that $u \leqslant v$ in $\partial(2B)\cap\Omega$ and hence $\lim_{\Omega\backslash\overline{2B}\ni z\to x}(v-u)(z) = (v-u)(x) \geqslant 0$ for every $x \in \partial(2B)\cap\Omega$. On the other hand, for a fixed $y \in B\cap\Omega$, one has that $\lim_{\Omega\ni z\to x} G^z(y) = 0$ for every $x \in \partial\Omega$. Gathering all these we conclude that

$$\liminf_{\Omega\backslash\overline{2B}\ni z\to x} (v-u)(z) \geqslant 0$$

for every $x \in \partial(\Omega\backslash\overline{2B})$. The lemma follows by the maximum principle. $\square$

Combining the two preceding lemmas, choosing $a = 8$, we obtain:

Lemma 7.18. *Let $d \geqslant 3$ and $\Omega \subset \mathbb{R}^d$ be an open Wiener regular set with compact boundary. Let B be a closed ball centered at $\partial\Omega$. Then,*

$$\omega^x(8B) \gtrsim_n \operatorname{Cap}(2B\backslash\Omega)\, G^x(y) \quad \text{for all } x \in \Omega\backslash 2B \text{ and } y \in B\cap\Omega. \tag{7.12}$$

Observe that, in the case when Ω is an NTA domain, we have $\omega^x(8B) \approx \omega^x(B)$ and $\mathrm{Cap}(2B\backslash\Omega) \approx \mathrm{Cap}(B) = r(B)^{d-2}$, so that we recover the estimate

$$\omega^x(B) \gtrsim r(B)^{d-2}\, G^x(y),$$

for $y \in \frac{1}{4}B$. Thus, Lemma 7.18 is a weak version of the converse inequality to the one in 7.8. Lemma 7.17 can be shown without the assumption on Wiener regularity adapting the same proof above, but at certain inequalities are to be shown modulo *polar sets*. The appropriate maximum principle can be found in [Hel14, Lemma 5.2.21], and requires as an extra step to check the boundedness above of $u - v$.

7.4 Logarithmic capacity, harmonic measure, and Green's function in the plane

Lemma 7.19. *Let $\Omega \subset \mathbb{R}^2$ be a Wiener regular open set with compact boundary and let B be a closed ball centered at $\partial\Omega$. Then*

$$\omega^x(B) \gtrsim \frac{1}{\log \dfrac{\mathrm{Cap}_L(B)}{\mathrm{Cap}_L(\frac{1}{4}B\backslash\Omega)}} = \frac{1}{\log \dfrac{r(B)}{\mathrm{Cap}_L(\frac{1}{4}B\backslash\Omega)}} \qquad \textit{for all } x \in \tfrac{1}{4}B \cap \Omega.$$

Remark the estimate in the lemma is equivalent to saying that

$$\omega^x(B) \gtrsim \frac{1}{\dfrac{1}{\mathrm{Cap}_W(\frac{1}{4}B\backslash\Omega)} - \dfrac{1}{\mathrm{Cap}_W(B)}} \qquad \text{for all } x \in \tfrac{1}{4}B \cap \Omega.$$

Proof. We can assume that Ω is bounded by proving first the estimate above for $\Omega_t = \Omega \cap B_t(0)$ and then letting $t \to \infty$. We denote $r = r(B)$. Replacing Ω by $\frac{1}{4r}\,\Omega$ if necessary, we can assume that $\mathrm{diam}(B) < 1$. Then, denoting $E = \frac{1}{4}B\backslash\Omega$, the identity (6.13) holds.

Let μ be the optimal measure for the supremum in (6.13), so that $\mathrm{supp}\mu \subset E$, $\mu(E) = \mathrm{Cap}_W(E)$, and the function $u := \mathcal{E} * \mu$ is harmonic out of E and it satisfies $\|u\|_\infty \leqslant 1$. For all $z \in \frac{1}{4}B$ and all $y \in E$ we have $|z-y| \leqslant \frac{1}{2}r$. Therefore,

$$u(z) = \frac{1}{2\pi}\int \log\frac{1}{|z-y|}\,d\mu(y) \geqslant \frac{1}{2\pi}\int \log\frac{2}{r}\,d\mu(y) = \frac{\mu(E)}{2\pi}\,\log\frac{2}{r} \qquad \text{for all } z \in \tfrac{1}{4}B.$$

Also, for $z \in B^c$, we have $\mathrm{dist}(z, \mathrm{supp}E) \geqslant \frac{3}{4}r(B)$, and thus

$$u(z) \leqslant \frac{1}{2\pi}\int \log\frac{4}{3r}\,d\mu(y) = \frac{\mu(E)}{2\pi}\,\log\frac{4}{3r} \qquad \text{for all } z \in B^c.$$

Consider now the function

$$v = u - \frac{\mu(E)}{2\pi}\,\log\frac{4}{3r}.$$

Observe that

$$v(z) \geqslant \frac{\mu(E)}{2\pi} \log \frac{2}{r} - \frac{\mu(E)}{2\pi} \log \frac{4}{3r} = \frac{\mu(E)}{2\pi} \log \frac{3}{2} \qquad \text{for all } z \in \tfrac{1}{4}B$$

and

$$v(z) \leqslant 0 \qquad \text{for all } z \in B^c.$$

Combining the maximum principle with Corollary 5.32, and using the fact that $x \in \frac{1}{4}B$ we deduce that

$$\omega^x(B) \geqslant \frac{v(x)}{\sup v} \geqslant \frac{\mu(E)}{2\pi \sup v} \log \frac{3}{2} = c\, \frac{\mathrm{Cap}_W(E)}{\sup v}.$$

Regarding $\sup v$, taking into account that $\|u\|_\infty \leqslant 1$, it is clear that

$$\sup v \leqslant 1 - \frac{1}{2\pi} \log \frac{4}{3r}\, \mu(E) = 1 - \frac{1}{2\pi} \log \frac{4}{3r}\, \mathrm{Cap}_W(E) \leqslant 1 - \frac{1}{2\pi} \log \frac{1}{r}\, \mathrm{Cap}_W(E).$$

Therefore,

$$\omega^x(B) \geqslant c\, \frac{\mathrm{Cap}_W(E)}{1 - \frac{1}{2\pi} \log \frac{1}{r}\, \mathrm{Cap}_W(E)} = c' \frac{1}{\log \dfrac{1}{\mathrm{Cap}_L(E)} - \log \dfrac{1}{r}} = c' \frac{1}{\log \dfrac{r}{\mathrm{Cap}_L(E)}}.$$

$\square$

Remark 7.20. It is easy to check that the constant $1/4$ in the preceding lemma can be replaced by any constant $\alpha \in (1/4, 1/3)$, with the implicit constant depending on α.

Lemma 7.21. *Let $\Omega \subset \mathbb{R}^2$ be an open Wiener regular set with compact boundary and let B be a closed ball centered at $\partial\Omega$. Then, for all $a > 2$,*

$$\omega^x(aB) \gtrsim \inf_{z \in 2B \cap \Omega} \omega^z(aB) \fint_B |G^x(y) - m_B(G^x)|\, dy \qquad \textit{for all } x \in \Omega \backslash 2B. \tag{7.13}$$

Proof. We can assume that Ω is bounded by proving first the estimate above for $\Omega_t = \Omega \cap B_t(0)$ and then letting $t \to \infty$.

Let $f(x) = \frac{\omega^x(aB)}{\inf_{z \in 2B \cap \Omega} \omega^z(aB)}$. Then (7.13) can be written as

$$\fint_B |G^x(y) - m_B(G^x)|\, dy \leqslant f(x).$$

Consider a continuous function φ_B such that $\chi_{\frac{3}{2}B} \leqslant \varphi_B \leqslant \chi_{\frac{7}{4}B}$. For $x \in \Omega \backslash 2B$, we write using (7.4)

$$2\pi\, G^x(y) = 2\pi\, G^y(x) = \log \frac{1}{|x-y|} - \int \log \frac{1}{|\xi - y|}\, d\omega^x(\xi) = g_1(y) + g_2(y),$$

with
$$g_1(y) = \log \frac{1}{|x-y|} - \int (1-\varphi_B(\xi)) \log \frac{1}{|\xi - y|} \, d\omega^x(\xi)$$
and
$$g_2(y) = -\int \varphi_B(\xi) \log \frac{1}{|\xi - y|} \, d\omega^x(\xi),$$
for every fixed x. We will treat separately the local and the non-local parts:
$$\fint_B |G^x(y) - m_B(G^x)| \, dy \leqslant \fint_B |g_1 - m_B g_1| \, dy + \fint_B |g_2 - m_B g_2| \, dy =: I_1 + I_2.$$

First we will estimate the *local* term I_2. To this end, let r denote the radius of B and let
$$\widetilde{g}_2(y) = -\int \varphi_B(\xi) \log \frac{4r}{|\xi - y|} \, d\omega^x(\xi),$$
so that $\widetilde{g}_2 = g_2 - C(B,r)$, for a suitable constant $C(B,r)$. Then we have
$$\begin{aligned} I_2 = \fint_B |\widetilde{g}_2 - m_B \widetilde{g}_2| \, dy \leqslant 2\, m_B |\widetilde{g}_2| &= 2 \fint_B \int \varphi_B(\xi) \log \frac{4r}{|\xi - y|} \, d\omega^x(\xi) \, dy \\ \leqslant 2 \int_{2B} \fint_B \log \frac{4r}{|\xi - y|} \, dy \, d\omega^x(\xi) &\lesssim \int_{2B} \fint_{B(\xi,3r)} \log \frac{4r}{|\xi - y|} \, dy \, d\omega^x(\xi), \end{aligned}$$
By a change of variable, we have
$$\fint_{B(\xi,3r)} \log \frac{4r}{|\xi - y|} \, dy = \fint_{B(0,3)} \log \frac{4}{|y|} \, dy = C,$$
and thus
$$I_2 \lesssim \omega^x(2B) \leqslant \omega^x(aB) \leqslant \frac{\omega^x(aB)}{\inf_{z \in 2B \cap \Omega} \omega^z(aB)} = f(x)$$
for any $a \geqslant 2$.

To deal with the *non-local* term I_1, we write
$$\begin{aligned} I_1 &\leqslant \fint_B \fint_B |g_1(y) - g_1(z)| \, dy \, dz \\ &\leqslant \fint_B \fint_B \left| \log \frac{|x-z|}{|x-y|} - \int (1-\varphi_B(\xi)) \log \frac{|\xi - z|}{|\xi - y|} \, d\omega^x(\xi) \right| dy \, dz. \end{aligned}$$
Denote
$$A_{y,z}(x) = \log \frac{|x-z|}{|x-y|} - \int (1-\varphi_B(\xi)) \log \frac{|\xi - z|}{|\xi - y|} \, d\omega^x(\xi),$$
so that
$$I_1 \leqslant \sup_{y,z \in B} |A_{y,z}(x)|.$$

To estimate $A_{y,z}(x)$ (for $y, z \in B$) notice that both $A_{y,z}$ and f are harmonic in $\Omega \backslash 2B$. Further, since
$$\frac{|x-z|}{|x-y|} \approx \frac{|\xi - z|}{|\xi - y|} \approx 1 \qquad \text{for all } x \in \Omega \backslash 2B,\ \xi \in \partial\Omega \backslash \tfrac{3}{2}B, \text{ and } y, z \in B,$$

we infer that

$$|A_{y,z}(x)| \lesssim 1 \quad \text{for all } x \in \Omega\backslash 2B \text{ and } y, z \in B.$$

Further, using (5.2) it is immediate to check that

$$\lim_{\Omega \ni x \to \zeta} A_{y,z}(x) = 0 \qquad \text{for all } \zeta \in \partial\Omega\backslash 2B \text{ and } y, z \in B.$$

On the other hand,

$$f(x) \geqslant 1 \qquad \text{for all } x \in \Omega \cap aB$$

and

$$f(x) \geqslant 0 \qquad \text{for all } x \in \Omega.$$

Then, by the maximum principle, it follows that

$$A_{y,z}(x) \leqslant C\, f(x) \qquad \text{for all } x \in \Omega\backslash 2B \text{ and all } y, z \in B.$$

Consequently,

$$I_1 = I_1(x) \leqslant \sup_{y,z \in B} |A_{y,z}(x)| \lesssim f(x).$$

Together with the estimate we obtained for I_2, this proves the lemma. □

Lemma 7.22. *Let $\Omega \subset \mathbb{R}^2$ be an open Wiener regular set with compact boundary. Let $\bar{B}$ be a closed ball centered at $\partial\Omega$. Then,*

$$G^x(y) \lesssim \omega^x(8\bar{B}) \left(\log \frac{\mathrm{Cap}_L(\bar{B})}{\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega)} \right)^2 \qquad \text{for all } x \in \Omega\backslash 2\bar{B} \text{ and } y \in \tfrac{1}{5}\bar{B} \cap \Omega. \tag{7.14}$$

Proof. We can assume that Ω is bounded by proving first the estimate above for $\Omega_t = \Omega \cap B_t(0)$ and then letting $t \to \infty$.

To prove the lemma we will estimate $\fint_{\frac{1}{4}B} G^x(z)\, dm(z)$ in terms of $\fint_B |G^x(z) - m_B G^x|\, dm(z)$ and then we will apply Lemmas 7.21 and 7.19.

Let $\bar{B} = \bar{B}_r(\xi)$, with $\xi \in \partial\Omega$. For $\frac{9}{10}r < s \leqslant r$, consider the open set $\Omega_s = B_s(\xi) \cap \Omega$. Then, for all $x \in \Omega\backslash 2\bar{B}$ and $y \in \frac{1}{4}B \cap \Omega$, we have

$$G^x(y) = \int_{\partial\Omega_s} G^x(z)\, d\omega^y_{\Omega_s}(z) = \int_{\partial B_s(\xi)} G^x(z)\, d\omega^y_{\Omega_s}(z),$$

where ω_{Ω_s} is the harmonic measure for Ω_s and we took into account that $G^x(z)$ vanishes when $z \in \partial\Omega$. Notice that Ω_s may not be connected, in this case the harmonic measure is defined to be zero outside the boundary of the component containing the pole.

Remark that, for all $y \in \frac{1}{4}B \cap \Omega$ there exists some function $\rho^y_s : \partial B_s(\xi) \to [0, \infty)$ such that

$$\omega^y_{\Omega_s}|_{\partial B_s(\xi)} = \rho^y_s \frac{\mathcal{H}^1|_{\partial B_s(\xi)}}{2\pi s},$$

with $\|\rho_s^y\|_\infty \lesssim 1$. This follows easily from the fact that, by the maximum principle,

$$\omega_{\Omega_s}^y(E) \leqslant \omega_{B_s(\xi)}^y(E) \qquad \text{for all } E \subset \partial B_s(\xi)$$

and the explicit formula for $\omega_{B_s(\xi)}^y$, see Example 5.23. Writing

$$\rho^y(z) = \rho^y_{|z-\xi|}(z),$$

by Fubini we have

$$\begin{aligned} G^x(y) &= \frac{1}{0.1r} \int_{0.9r}^{r} \int_{\partial B_s(\xi)} G^x(z)\, d\omega_{\Omega_s}^y(z)\, ds \\ &= \frac{10}{r} \int_{0.9r}^{r} \int_{\partial B_s(\xi)} G^x(z)\, \rho^y(z) d\mathcal{H}^1(z)\, \frac{ds}{2\pi s} = \int_{A(\xi,0.9r,r)} G^x(z)\, d\mu^y(z), \end{aligned} \tag{7.15}$$

where μ^y is the measure

$$d\mu^y(z) = \frac{10}{2\pi\, r\, |z-\xi|}\, \rho^y(z)\, dm|_{A(\xi,0.9r,r)}(z).$$

Averaging (7.15) over $y \in \frac{1}{4}B$ and applying Fubini, we get

$$m_{\frac14 B} G^x = \fint_{\frac14 B} \int_{A(\xi,0.9r,r)} G^x(z)\, d\mu^y(z)\, dy = \int_{A(\xi,0.9r,r)} G^x(z)\, d\mu(z), \tag{7.16}$$

where

$$d\mu(z) = \rho(z)\, dm|_{A(\xi,0.9r,r)}(z), \qquad \rho(z) = \frac{10}{2\pi\, r\, |z-\xi|} \fint_{\frac14 B} \rho^y(z)\, dy$$

understanding that $\rho^y(z) \equiv 0$ when $y \notin \Omega$. Notice that $\|\rho\|_\infty \lesssim r^{-2}$, since $\|\rho^y\|_\infty \lesssim 1$ for all $y \in \frac14 B$.

Observe now that, by Lemma 7.19 and the subsequent remark, we have

$$\omega_{\Omega_s}^y(B_{0.9s}(\xi)) \gtrsim \frac{1}{\log \dfrac{s}{\mathrm{Cap}_L(B_{0.29s}(\xi)\backslash\Omega)}} \qquad \text{for all } y \in B_{0.29s}(\xi) \cap \Omega_s.$$

Since $\frac14 B \subset B(\xi, 0.29s)$ for $\frac{9}{10}r < s \leqslant r$, we infer that

$$\omega_{\Omega_s}^y(B(\xi,0.9s)) \gtrsim \frac{1}{\log \dfrac{s}{\mathrm{Cap}_L(\frac14 B\backslash\Omega)}} \approx \frac{1}{\log \dfrac{r}{\mathrm{Cap}_L(\frac14 B\backslash\Omega)}} \qquad \text{for all } y \in \tfrac14 B \cap \Omega_s.$$

Thus,

$$\omega_{\Omega_s}^y(\partial B_s(\xi)) \leqslant 1 - \varepsilon_0,$$

where

$$\varepsilon_0 = \frac{c}{\log \dfrac{r}{\mathrm{Cap}_L(\frac14 B\backslash\Omega)}},$$

for some $c > 0$. Thus,

$$\|\mu\| = \mu(A(\xi, 0.9r, r)) = \fint_{\frac{1}{4}B} \frac{1}{0.1r} \int_{0.9r}^{r} \omega_{\Omega_s}^{y}(\partial B_s(\xi))\, ds\, dy \leqslant 1 - \varepsilon_0.$$

Next we consider the measure

$$\nu = \frac{1}{2}\Big(\mu + \frac{m|_{\frac{1}{4}B}}{m(\frac{1}{4}B)}\Big),$$

so that

$$\frac{1}{2} \leqslant \nu(B) = \frac{1}{2}\big(\mu(B) + 1\big) \leqslant 1 - \frac{\varepsilon_0}{2}.$$

From (7.16) and this estimate we infer that

$$\begin{aligned} m_{\frac{1}{4}B} G^x &= \frac{1}{2} \int_{A(\xi, 0.9r, r)} G^x(z)\, d\mu(z) + \frac{1}{2}\, m_{\frac{1}{4}B} G^x \\ &= \nu(B) \fint_B G^x(z)\, d\nu(z) \leqslant \Big(1 - \frac{\varepsilon_0}{2}\Big) \fint_B G^x(z)\, d\nu(z). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\varepsilon_0}{2} \fint_B G^x(z)\, d\nu(z) &\leqslant \fint_B G^x(z)\, d\nu(z) - m_{\frac{1}{4}B} G^x \\ &\leqslant \Big| \fint_B G^x(z)\, d\nu(z) - m_B G^x \Big| + \Big| m_B G^x - m_{\frac{1}{4}B} G^x \Big| \\ &\leqslant \fint_B \big|G^x(z) - m_B G^x\big|\, d\nu(z) + \fint_{\frac{1}{4}B} \big|G^x(z) - m_B G^x\big|\, dm(z). \end{aligned} \tag{7.17}$$

Recall now that $\nu(B) \approx 1$ and that

$$\nu = \frac{1}{2}\Big(\rho\, \chi_{A(\xi, 0.9r, r)} + \frac{1}{m(\frac{1}{4}B)}\, \chi_{\frac{1}{4}B}\Big)\, m|_B =: \widetilde{\rho}\, m|_B,$$

it is clear that $\|\widetilde{\rho}\|_{L^\infty(B)} \lesssim r^{-2}$. Hence,

$$\begin{aligned} \fint_B \big|G^x(z) - m_B G^x\big|\, d\nu(z) &\lesssim \frac{1}{r^2} \int_B \big|G^x(z) - m_B G^x\big|\, dm(z) \\ &\lesssim \fint_B \big|G^x(z) - m_B G^x\big|\, dm(z). \end{aligned}$$

By the definition of ν, (7.17), and the preceding estimate, we obtain

$$\frac{\varepsilon_0}{4} \fint_{\frac{1}{4}B} G^x(z)\, dm(z) \leqslant \frac{\varepsilon_0}{2} \fint_B G^x(z)\, d\nu(z) \lesssim \fint_B \big|G^x(z) - m_B G^x\big|\, dm(z),$$

From the preceding estimate, taking into account that G^x is subharmonic in $\mathbb{R}^2\backslash\{x\}$ and using Lemmas 7.21 and 7.19, for all $y \in \frac{1}{5}B$ we get

$$\begin{aligned} G^x(y) &\lesssim \fint_{\frac{1}{4}B} G^x(z)\,dm(z) \lesssim \varepsilon_0^{-1} \fint_B \left|G^x(z) - m_B G^x\right| dm(z) \\ &\lesssim \frac{\omega^x(8B)}{\inf_{z\in 2B\cap\Omega}\omega^z(8B)}\,\log\frac{r}{\mathrm{Cap}_L(\frac{1}{4}B\backslash\Omega)} \lesssim \omega^x(8B)\,\log\frac{8r}{\mathrm{Cap}_L(2B\backslash\Omega)}\,\log\frac{r}{\mathrm{Cap}_L(\frac{1}{4}B\backslash\Omega)} \\ &\lesssim \omega^x(8B)\left(\log\frac{r}{\mathrm{Cap}_L(\frac{1}{4}B\backslash\Omega)}\right)^2. \end{aligned}$$

□

Notice that, in the case when Ω is an NTA domain, we have $\omega^x(8B) \approx \omega^x(B)$ and $\mathrm{Cap}_L(\frac{1}{4}B\backslash\Omega) \approx \mathrm{Cap}_L(B) = r(B)$, so that we recover the estimate

$$\omega^x(B) \gtrsim G^x(y),$$

for $y \in \frac{1}{5}B$, as in the case $d \geqslant 3$.

7.5 Capacity density condition

7.5.1 The CDC and Wiener regularity

Let $\Omega \subsetneq \mathbb{R}^d$ be an open set in $\mathbb{R}^d$ and let $\xi \in \partial\Omega$ and $r_0 > 0$. We say that Ω satisfies the (ξ, r_0)-local capacity density condition if there exists some constant $c > 0$ such that, for any $r \in (0, r_0)$,

$$\mathrm{Cap}(B_r(\xi)\backslash\Omega) \geqslant c\,r^{d-2} \qquad \text{in the case } d \geqslant 3,$$

and

$$\mathrm{Cap}_L(B_r(\xi)\backslash\Omega) \geqslant c\,r \qquad \text{in the case } d = 2.$$

We say that Ω satisfies the capacity density condition (CDC) if it satisfies the (ξ, r_0)-local capacity density condition for all $\xi \in \partial\Omega$ and all $r_0 > 0$. For example, a Jordan domain in the plane satisfies the CDC, or more generally, any planar bounded domain whose boundary consists of finitely many curves (we do not allow degenerate curves consisting of a single point).

The CDC can be understood as a strong form of Wiener regularity. In fact, we have:

Proposition 7.23. *Let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary and let $\xi \in \partial\Omega$ and $r_0 > 0$. If the (ξ, r_0)-local capacity density holds for Ω, then ξ is a regular point for the Dirichlet problem.*

As a corollary, if Ω satisfies the CDC, then it is Wiener regular.

Proof. This is an easy consequence of the Wiener criterion, more precisely of the implication (b) $\Rightarrow$ (a) in Theorem 6.21. Indeed, we just have to check that the (ξ, r_0)-local capacity density condition implies that

$$\sum_{k=1}^{\infty} \frac{\operatorname{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k}) \backslash \Omega)}{\operatorname{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty.$$

As shown in Remark 6.22, in the case $d \geqslant 3$ this is equivalent to the fact that

$$\sum_{k=1}^{\infty} \frac{\operatorname{Cap}(\bar{B}_{2^{-k}}(\xi) \backslash \Omega)}{\operatorname{Cap}(\bar{B}_{2^{-k}}(\xi))} = \infty.$$

Now we just have to observe that (ξ, r_0)-local capacity density condition is equivalent to the fact that $\operatorname{Cap}(B_r(\xi) \backslash \Omega) \geqslant c \operatorname{Cap}(B_r(\xi))$ for $0 < r < r_0$, which clearly implies the above estimate.

The case $d = 2$ a little trickier. Notice first that, for $r \in (0, 1)$ the estimate $\operatorname{Cap}_L(\bar{B}_r(\xi) \backslash \Omega) \geqslant c\, r$ implies that

$$\frac{\operatorname{Cap}_W(\bar{B}_r(\xi) \backslash \Omega)}{\operatorname{Cap}_W(\bar{B}_r(\xi))} = \frac{\log \frac{1}{\operatorname{Cap}_L(\bar{B}_r(\xi))}}{\log \frac{1}{\operatorname{Cap}_L(\bar{B}_r(\xi) \backslash \Omega)}} \geqslant \frac{\log \frac{1}{r}}{\log \frac{1}{c\, r}} = \frac{\log \frac{1}{r}}{\log \frac{1}{r} - C} \geqslant \frac{1}{2},$$

assuming r small enough in the last inequality. Observe now that $\operatorname{Cap}_W(\bar{B}_{r^4}(\xi)) = \frac{1}{4} \operatorname{Cap}_W(\bar{B}_{r^4}(\xi))$. Then, by the subadditivity of Cap_W we deduce

$$\frac{1}{2} \leqslant \frac{\operatorname{Cap}_W((\bar{B}_r(\xi) \backslash \Omega) \backslash B_{r^4}(\xi)) + \operatorname{Cap}_W(B_{r^4}(\xi))}{\operatorname{Cap}_W(\bar{B}_r(\xi))} = \frac{\operatorname{Cap}_W(\bar{A}_{r^4,r}(\xi) \backslash \Omega)}{\operatorname{Cap}_W(\bar{B}_r(\xi))} + \frac{1}{4}.$$

Hence

$$\frac{\operatorname{Cap}_W(\bar{A}_{r^4,r}(\xi) \backslash \Omega)}{\operatorname{Cap}_W(\bar{B}_r(\xi))} \geqslant \frac{1}{4}. \tag{7.18}$$

Now we can estimate the Wiener's series from below as follows, considering j_0 large enough,

$$\sum_{j \geqslant j_0} \sum_{4^j \leqslant k \leqslant 4^{j+1}-1} \frac{\operatorname{Cap}_W(\bar{A}(\xi, 2^{-k-1}, 2^{-k}) \backslash \Omega)}{\operatorname{Cap}_W(\bar{B}(\xi, 2^{-k}))}$$
$$\geqslant \sum_{j \geqslant j_0} \sum_{4^j \leqslant k \leqslant 4^{j+1}-1} \frac{\operatorname{Cap}_W(\bar{A}(\xi, 2^{-k-1}, 2^{-k}) \backslash \Omega)}{\operatorname{Cap}_W(\bar{B}(\xi, 2^{-4^j}))} \geqslant \sum_{j \geqslant j_0} \frac{\operatorname{Cap}_W(\bar{A}(\xi, 2^{-4^{j+1}}, 2^{-4^j}) \backslash \Omega)}{\operatorname{Cap}_W(\bar{B}(\xi, 2^{-4^j}))}.$$

By (7.18), each of the summands on the right hand side is at least $1/4$ and so the sum is infinite. $\square$

Remark that, by Lemmas 7.16, 7.19, 7.17, and 7.21, if Ω satisfies the CDC, then it holds

$$\omega^x(B) \gtrsim 1 \quad \text{ for all } x \in \tfrac{1}{4} B \cap \Omega$$

and

$$G^x(y) \lesssim \frac{\omega^x(8B)}{r(B)^{d-2}} \quad \text{ for all } x \in \Omega \backslash 2B \text{ and } y \in \tfrac{1}{5} B \cap \Omega.$$

7.5.2 Hölder continuity at the boundary

Lemma 7.24. *Let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary, let $\xi \in \partial\Omega$, and let $r > 0$. Suppose that Ω satisfies the (ξ, r_0)-local capacity density condition. Let u be a nonnegative function which is continuous in $\overline{B_r(\xi) \cap \Omega}$ and harmonic in $B_r(\xi) \cap \Omega$, and vanishes on $B_r(\xi) \cap \partial\Omega$. Then there is $\alpha > 0$ such that for all $r \in (0, r_0)$,*

$$u(x) \lesssim \left(\frac{|x-\xi|}{r}\right)^{\alpha} \sup_{B_r(\xi)\cap\Omega} u \quad \text{for all } x \in \Omega \cap B_r(\xi). \tag{7.19}$$

Proof. For very $k \geqslant 0$, let $B_k = B_{6^{-k}r}(\xi)$ and $\Omega_k = \Omega \cap B_k$. Since u vanishes identically on $\partial\Omega \cap B_k$, for all $x \in \partial B_{k+1} \cap \Omega$ we have

$$u(x) = \int_{\partial\Omega_k} u(y)\, d\omega^x_{\Omega_k}(y) = \int_{\partial B_k \cap \Omega} u(y)\, d\omega^x_{\Omega_k}(y) \leqslant \omega^x_{\Omega_k}(\partial B_k \cap \Omega) \sup_{\partial B_k \cap \Omega} u.$$

By the (ξ, r_0)-local capacity density condition and Lemmas 7.16 and 7.19,

$$\omega^x_{\Omega_k}(\partial B_k \cap \Omega) = 1 - \omega^x_{\Omega_k}(\partial\Omega \cap B_k) \leqslant 1 - c_0$$

for some $c_0 \in (0, 1)$. Thus,

$$\sup_{\partial B_{k+1}\cap\Omega} u \leqslant (1-c_0) \sup_{\partial B_k \cap \Omega} u.$$

By the maximum principle and iterating, we deduce that

$$\sup_{B_k\cap\Omega} u = \sup_{\partial B_k\cap\Omega} u \leqslant (1-c_0)^k \sup_{\partial B_0 \cap \Omega} u.$$

This readily proves the lemma. □

As an easy corollary we get a result about Hölder regularity:

Lemma 7.25. *Let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary and let B be a ball with radius r_0 centered in $\partial\Omega$. Suppose that Ω satisfies the (ξ, r_0)-local CDC for every $\xi \in \partial\Omega \cap 2B$. Let u be a nonnegative function which is continuous in $\overline{2B \cap \Omega}$ and harmonic in $2B \cap \Omega$, and vanishes continuously on $2B \cap \partial\Omega$. Then there is $\alpha > 0$ such that*

$$|u(x) - u(y)| \lesssim \left(\frac{|x-y|}{r_0}\right)^{\alpha} \sup_{2B\cap\Omega} u \quad \text{for all } x, y \in B \cap \Omega. \tag{7.20}$$

Proof. By replacing Ω by $\Omega \cap 2B$ if necessary, we can assume that the (ξ, r_0)-local CDC holds for all $\xi \in \partial\Omega$, so that in particular Ω is Wiener regular.

To prove the lemma, clearly we may assume that $|x - y| \leqslant r/4$. Denote as usual $\mathrm{d}_\Omega(z) := \mathrm{dist}(z, \partial\Omega)$, and suppose first that

$$|x-y| \leqslant \frac{1}{2}\max(\mathrm{d}_\Omega(x), \mathrm{d}_\Omega(y)) =: \frac{1}{2}\mathrm{d}_\Omega(x,y).$$

Assume that $\mathrm{d}_\Omega(y) \leqslant \mathrm{d}_\Omega(x) = \mathrm{d}_\Omega(x,y)$, say, and consider the ball $B' = B(x, \mathrm{d}_\Omega(x,y))$. Notice that $B' \subset \Omega \cap 2B$ and $x, y \in \frac{1}{2}B'$. So by standard arguments it follows that

$$\begin{aligned} |u(x) - u(y)| &\leqslant \|\nabla u\|_{\infty,\frac{1}{2}B'} |x-y| \lesssim \|u\|_{\infty,B'} \frac{|x-y|}{r(B')} \leqslant \|u\|_{\infty,2B} \frac{|x-y|}{\mathrm{d}_\Omega(x,y)} \\ &\leqslant \|u\|_{\infty,2B} \left(\frac{|x-y|}{\mathrm{d}_\Omega(x,y)}\right)^\alpha. \end{aligned} \tag{7.21}$$

Notice also that the same estimate holds trivially in case that $|x-y| > \frac{1}{2}\mathrm{d}_\Omega(x,y)$.

On the other hand, by Lemma 7.24,

$$u(x) \lesssim \left(\frac{\mathrm{d}_\Omega(x)}{r_0}\right)^\alpha \|u\|_{\infty,2B},$$

and the same estimate holds replacing x by y. Thus,

$$\begin{aligned} |u(x) - u(y)| \leqslant u(x) + u(y) &\lesssim \left(\frac{\mathrm{d}_\Omega(x)}{r_0}\right)^\alpha \|u\|_{\infty,2B} + \left(\frac{\mathrm{d}_\Omega(y)}{r_0}\right)^\alpha \|u\|_{\infty,2B} \\ &\lesssim \left(\frac{\mathrm{d}_\Omega(x,y)}{r_0}\right)^\alpha \|u\|_{\infty,2B}. \end{aligned} \tag{7.22}$$

Taking the geometric mean of (7.21) and (7.22), the lemma follows (with $\alpha/2$ instead of α). □

As another immediate corollary of Lemma 7.24 we get the following:

Lemma 7.26. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular open set with compact boundary, let $\xi \in \partial\Omega$, and let $r_0 > 0$. Suppose that Ω satisfies the (ξ, r_0)-local capacity density condition. Then there is $\alpha > 0$ such that, for all $r \in (0, r_0)$,*

$$\omega^x(B(\xi,r)^c) \lesssim \left(\frac{|x-\xi|}{r}\right)^\alpha \qquad \text{for } x \in \Omega \cap B_r(\xi). \tag{7.23}$$

7.5.3 Improving property of the CDC

As shown in Lemma 6.19, if a set $E \subset \mathbb{R}^d$ satisfies $\mathrm{Cap}(E) > 0$, then $\mathcal{H}^{d-2}_\infty(E) > 0$. Further, this estimate is sharp in the sense that one cannot infer that $\mathcal{H}^s_\infty(E) > 0$ for any $s > d-2$. In fact, it is not difficult to construct a compact set $E \subset \mathbb{R}^d$ such that $\mathrm{Cap}(E) > 0$ with $\dim_H(E) = d-2$. On the other hand, if $\Omega \subset \mathbb{R}^d$ satisfies the CDC, then it easily follows that

$$\mathcal{H}^{d-2}_\infty(\Omega^c \cap B_r(\xi)) \gtrsim r^{d-2} \qquad \text{for all } \xi \in \partial\Omega,\ r > 0.$$

From the previous discussion, it would appear that the exponent $d-2$ in this estimate might be sharp. Surprisingly, this can be improved, as the following theorem shows.

Theorem 7.27. *Let $r_0 > 0$ and let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary satisfying the (ξ, r_0)-local capacity density condition for every $\xi \in \partial\Omega$. Then there exists some $s > d-2$ and some $c > 0$ such that*

$$\mathcal{H}^s_\infty(\Omega^c \cap B_r(\xi)) \geqslant c\, r^s \quad \text{for all } \xi \in \partial\Omega,\ 0 < r \leqslant r_0.$$

The constant $c > 0$ and the precise $s > d-2$ depend only on d and on the constant involved in the local CDC.

Proof. We consider first the case $d \geqslant 3$. Denote $E = \Omega^c$. Observe first that the fact that Ω satisfies the (ξ, r_0)-local CDC for every $\xi \in \partial\Omega$ is equivalent to saying that

$$\mathrm{Cap}(E \cap B_r(x)) \gtrsim r^{d-2} \quad \text{for all } x \in E,\ 0 < r \leqslant r_0.$$

Fix now a point $\xi \in \partial\Omega$ and $0 < R \leqslant r_0$, and let us see that $\mathcal{H}^s_\infty(E \cap B_R(\xi)) \gtrsim R^s$ for some $s > d-2$, with both s and the implicit constant depending only on the local CDC. To this end, define $E_1 = B_{R/4}(\xi)$ and, inductively, for $m \geqslant 2$,

$$E_m = E \cap \bigcup_{x \in E_{m-1}} B_{2^{-m}R}(x).$$

It is immediate to check that the closure F of $\bigcup_{m \geqslant 1} E_m$ is contained in $B_R(\xi) \cap E$ and satisfies

$$\mathrm{Cap}(F \cap B_r(x)) \gtrsim r^{d-2} \quad \text{for all } x \in F,\ 0 < r \leqslant R.$$

Equivalently, the open set $\mathbb{R}^d \backslash F$ satisfies the CDC.

Let μ_F be the equilibrium measure of F, and denote $\eta_s = R^s \mu_F$. We intend to show that there exists some $s > d-2$ such that

$$\eta_s(B_r(x)) \lesssim r^s \quad \text{for all } x \in F,\ 0 < r \leqslant R. \tag{7.24}$$

By Frostman's lemma, clearly this implies that

$$\mathcal{H}^s_\infty(E \cap B_R(\xi)) \geqslant \mathcal{H}^s_\infty(F) \gtrsim R^s,$$

as wished. To prove (7.24), let $\eta = \eta_{d-2} = R^{d-2}\, \mu_F$, and notice that the CDC satisfied by F^c ensures that F^c is Wiener regular, so that by Lemma 6.24,

$$U_\eta(x) = R^{d-2}\, \frac{1}{\mathrm{Cap}(F)} \quad \text{for all } x \in F.$$

So the function

$$f(x) = R^{d-2}\, \frac{1}{\mathrm{Cap}(F)} - U_\eta(x)$$

is continuous in $\mathbb{R}^d$, harmonic in F^c, it vanishes in F, and it is non-negative in F^c, by the properties of the equilibrium potential. Further $\|f\|_\infty \leqslant R^{d-2}\, \frac{1}{\mathrm{Cap}(F)} \lesssim 1$. So by Lemma 7.25, f is Hölder continuous and, for some $\alpha > 0$ depending on the CDC it holds

$$|U_\eta(x) - U_\eta(y)| = |f(x) - f(y)| \lesssim \left(\frac{|x-y|}{R}\right)^\alpha \quad \text{for all } x, y \in B_{2R}(\xi). \tag{7.25}$$

To prove (7.24), fix $x \in F$ and $0 < r \leqslant R$, and let φ be a bump function such that $\chi_{B_r(x)} \leqslant \varphi_r \leqslant \chi_{B_{2r}(x)}$, with $\|\nabla\varphi_r\|_\infty \lesssim 1/r$. Since $-\Delta U_\eta = \eta$ in the sense of distributions, we have

$$\eta(B_r(x)) \leqslant \int \varphi_r \, d\eta = -\langle \Delta U_\eta, \varphi_r\rangle = -\int U_\eta \, \Delta\varphi_r \, dy = -\int (U_\eta(y) - U_\eta(x)) \, \Delta\varphi_r \, dy,$$

where, in the last identity, we used the fact that $\int \Delta\varphi_r \, dy = 0$. Plugging the estimate (7.25), we deduce

$$\eta(B_r(x)) \lesssim \frac{1}{r^2} \int_{B(x,2r)} |U_\eta(y) - U_\eta(x)| \, dy \lesssim r^{d-2} \left(\frac{r}{R}\right)^\alpha,$$

or equivalently,

$$\eta_{d-2+\alpha}(B_r(x)) \lesssim r^{d-2+\alpha}.$$

So (7.24) holds with $s = d - 2 + \alpha$.

In the case $d = 2$, by a suitable dilation, we may assume that $R = 1/4$, say. Then the arguments above work in a similar fashion, so that at the end we deduce that $\eta_\alpha(B_r(x)) \lesssim r^\alpha$. □

7.6 Harmonic measure and Green's function with pole at infinity

In this section we will study the connection between harmonic measure with pole at infinity and Green's function with pole at infinity for unbounded open sets with compact boundary. We will study first the case of the plane, which is simpler, and later the higher dimensional case.

7.6.1 The case of the plane

Recall that for an unbounded open set with compact boundary the notion of harmonic measure with pole at ∞ was introduced in Definition 5.44. From that definition, it follows that for any function $f \in C(\partial\Omega)$,

$$\int_{\partial\Omega} f(\xi) d\omega^\infty(\xi) = \lim_{z\to\infty} \int_{\partial\Omega} f(\xi) d\omega^x(\xi). \tag{7.26}$$

Analogously, for any Borel set $E \subset \partial\Omega$, we have $\omega^z(E) \to \omega^\infty(E)$ as $z \to \infty$.

In the context above, denote by $G : \Omega \times \Omega \to \mathbb{R}$ the Green function for Ω. For any fixed point $y \in \Omega$, the function $G(y, \cdot)$ is harmonic at ∞ (i.e., it has a removable singularity at ∞), by Corollary 5.43. Thus we can define

$$G^\infty(y) = G(y, \infty) = \lim_{z\to\infty} G(y, z). \tag{7.27}$$

Theorem 7.28. *Let $\Omega \subset \mathbb{R}^2$ be a Wiener regular unbounded open set with compact boundary. Let $\{p_k\}_k \subset \Omega$ be a sequence of points such that $p_k \to \infty$. Then the functions G^{p_k} converge uniformly in bounded subsets of Ω to G^∞, the measures $\omega^{p_k}|_{\partial\Omega}$ converge weakly to ω^∞, and the following holds:*

(a) ω^∞ is a probability measure which is mutually absolutely continuous with ω^p, for every p belonging to the unbounded component of Ω.

(b) For every $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\int_\Omega G^\infty(z)\, \Delta\varphi(z)\, dm(z) = \int \varphi\, d\omega^\infty.$$

(c) ω^∞ coincides with the equilibrium measure of $\partial\Omega$ and moreover, for every $z \in \Omega$,

$$G^\infty(z) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} - \frac{1}{2\pi}\int_{\partial\Omega} \log\frac{1}{|\xi - z|}\, d\omega^\infty(\xi).$$

Proof. The weak convergence of ω^{p_k} to ω^∞ is equivalent to (7.26). It is clear that this implies that ω^∞ is a probability measure (this can also be derived directly from the definition of ω^∞ and the Riesz representation theorem). Further, we already discussed the mutual absolute continuity of ω^∞ and ω^p after Definition 5.44.

From the pointwise convergence given by (7.27) and an easy application of the Arzela-Ascoli theorem, it follows that the functions G^{p_k} converge uniformly in compact subsets of Ω to G^∞ as $p_k \to \infty$. To prove the uniform convergence in bounded subsets of Ω, let $r > 0$ be an arbitrary radius such that $\partial\Omega \subset S_r(0)$. Since the functions G^{p_k} vanish continuously on $\partial\Omega$, by the maximum principle the sequence $\{G^{p_k}\}_{k\geqslant 1}$ is a uniform Cauchy sequence in $\Omega \cap B_r(0)$, and so the convergence in uniform in $\Omega \cap B_r(0)$. So the convergence in uniform in bounded subsets of Ω.

The statement (b) of the theorem is a consequence of the fact that, for $\varphi \in C_c^\infty(\mathbb{R}^2)$ and ξ away from the support of φ,

$$\int_\Omega G^\xi(z)\, \Delta\varphi(z)\, dm(z) = \int \varphi\, d\omega^\xi.$$

Then we let $\xi \to \infty$ and use the uniform convergence of G^ξ to G^∞ in bounded sets and the weak convergence of ω^ξ to ∞, and (b) follows.

To prove (c), recall that

$$G(z,\xi) = \frac{1}{2\pi}\int_{\partial\Omega} \log\frac{|\xi - x|}{|\xi - z|}\, d\mu(x) - \frac{1}{2\pi}\int_{\partial\Omega}\int_{\partial\Omega} \log\frac{|y - x|}{|y - z|}\, d\mu(x)\, d\omega^\xi(y),$$

where μ is the equilibrium measure of $\partial\Omega$. Letting $\xi \to \infty$, we obtain

$$\begin{aligned} 2\pi\, G^\infty(z) &= 0 - \int_{\partial\Omega}\int_{\partial\Omega} \log\frac{|y - x|}{|y - z|}\, d\mu(x)\, d\omega^\infty(y) \\ &= \iint \log\frac{1}{|y - x|}\, d\mu(x)\, d\omega^\infty(y) - \iint \log\frac{1}{|y - z|}\, d\mu(x)\, d\omega^\infty(y). \end{aligned}$$

Since μ is a probability measure, the double integral term on the right hand side equals $\int \log \frac{1}{|y-z|}\, d\omega^\infty(y)$. For the first summand we take into account that

$$\mathcal{E} * \mu(w) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} \qquad \text{for all } w \in \partial\Omega,$$

since Ω is Wiener regular, and so

$$2\pi\, G^\infty(z) = \frac{2\pi}{\mathrm{Cap}_W(\partial\Omega)} - \int \log \frac{1}{|y-z|}\, d\omega^\infty(y).$$

By continuity, this identity also holds for all $z \in \partial\Omega$, and so integrating with respect to ω^∞ we get

$$0 = \int G^\infty(z)\, d\omega^\infty(z) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} - \frac{1}{2\pi} \iint \log \frac{1}{|y-z|}\, d\omega^\infty(y)\, d\omega^\infty(z).$$

So the energy associated with the measure ω^∞ coincides with the equilibrium energy $\frac{1}{\mathrm{Cap}_W(\partial\Omega)}$. Since any measure σ supported on $\partial\Omega$ minimizing the energy $\int \mathcal{E} * \sigma\, d\sigma$ coincides with the equilibrium measure μ, we infer that $\omega^\infty = \mu$. □

7.6.2 The higher dimensional case

For $d \geqslant 3$, let $\Omega \subset \mathbb{R}^d$ be an unbounded Wiener regular open set with compact boundary. In this case we cannot define the harmonic measure with pole at infinity directly as the weak limit of the measures ω^p with $p \to \infty$ because this limit is always zero. Instead we can define harmonic measure and the Green function with pole at infinity by a limiting process involving renormalization The construction is summarized in the following theorem:

Theorem 7.29. *For $d \geqslant 3$, let $\Omega \subset \mathbb{R}^d$ be an unbounded Wiener regular open set with compact boundary. Let $\{p_k\}_k \subset \Omega$ be a sequence of points such that $p_k \to \infty$. Then the functions $\mathcal{E}(p_k)^{-1} G^{p_k}$ converge uniformly in bounded subsets of Ω to some function $G^\infty : \Omega \to \mathbb{R}$, the measures $\mathcal{E}(p_k)^{-1}\omega^{p_k}$ converge weakly to some measure ω^∞ supported in $\partial\Omega$, and the following holds:*

- *(a) The limiting function G^∞ and the limiting measure ω^∞ do not depend on the chosen sequence $\{p_k\}_k$.*
- *(b) G^∞ is harmonic and positive in Ω.*
- *(c) ω^∞ is mutually absolutely continuous with ω^p, for every $p \in \Omega$.*
- *(d) For every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,*

$$\int_\Omega G^\infty(x)\, \Delta\varphi(x)\, dx = \int \varphi\, d\omega^\infty.$$

(e) *ω^∞ is the equilibrium measure of $\partial\Omega$ times* $\mathrm{Cap}(\partial\Omega)$ *(and, so $\|\omega^\infty\| = \mathrm{Cap}(\partial\Omega)$) and moreover, for every $x \in \Omega$,*

$$G^\infty(x) = 1 - \mathcal{E} * \omega^\infty(x) = 1 - \omega^x(\partial\Omega).$$

Proof. Let μ be the equilibrium measure of $\partial\Omega$. Observe first that, for all $p \in \Omega$,

$$\omega^p(\partial\Omega) = \mathrm{Cap}(\partial\Omega)\, U_\mu(p), \tag{7.28}$$

since the right hand side is a function that is harmonic in Ω and continuous in $\overline{\Omega}$, it equals 1 in $\partial\Omega$, and vanishes at ∞.

Consider now an arbitrary sequence $\{p_k\}_k \subset \Omega$ such that $p_k \to \infty$. We write

$$\mathcal{E}(p_k)^{-1}\omega^{p_k} = \mathrm{Cap}(\partial\Omega)\,\frac{U_\mu(p_k)}{\mathcal{E}(p_k)}\,\frac{1}{\omega^{p_k}(\partial\Omega)}\,\omega^{p_k}. \tag{7.29}$$

It is immediate to check that

$$\lim_{p_k\to\infty} \frac{U_\mu(p_k)}{\mathcal{E}(p_k)} = 1.$$

Thus there exists a subsequence $\{p_{k_j}\}_j$ such that $\mathcal{E}(p_{k_j})^{-1}\omega^{p_{k_j}}$ converges weakly $*$ to some measure $\widetilde{\omega}$ supported on $\partial\Omega$, with total mass $\mathrm{Cap}(\partial\Omega)$.

Notice also that the Green function satisfies

$$\mathcal{E}(p_k)^{-1}G(x,p_k) \leqslant \mathcal{E}(p_k)^{-1}\mathcal{E}(x-p_k) \to 1 \quad \text{as } k\to\infty, \quad \text{for all } x\in\Omega.$$

Thus there exists another subsequence $\{p_{k_h}\}_h$ such that the functions $\mathcal{E}(p_{k_h})^{-1}G^{p_{k_h}}$ converge locally uniformly in compact subsets of Ω to some harmonic function $\widetilde{g} : \Omega \to \mathbb{R}$ such that $\|\widetilde{g}\|_\infty \leqslant 1$. Without loss of generality, we may assume that the subsequences $\{p_{k_j}\}_j$ and $\{p_{k_h}\}_h$ coincide. Using that the functions $\mathcal{E}(p_{k_h})^{-1}G^{p_{k_h}}$ vanish continuously in $\partial\Omega$, and using the maximum principle, as in the proof of Theorem, it follows that they converge uniformly on bounded subsets of Ω.

Given $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, we have

$$\mathcal{E}(p_{k_j})^{-1}\int_\Omega g(x,p_{k_j})\,\Delta\varphi(x)\,dx = -\mathcal{E}(p_{k_j})^{-1}\varphi(p_{k_j}) + \mathcal{E}(p_{k_j})^{-1}\int \varphi\, d\omega^{p_{k_j}}.$$

By the uniform convergence of $\mathcal{E}(p_{k_j})^{-1}g(\cdot,p_{k_j})$ to $\widetilde{g}$ in bounded subsets of Ω, the left hand side converges to $\int_\Omega \widetilde{g}\,\Delta\varphi\,dx$ as $j\to\infty$, and by the weak $*$ convergence of $\mathcal{E}(p_{k_j})^{-1}\omega^{p_{k_j}}$ and the fact that $\varphi(p_{k_j}) = 0$ for j big enough, it is clear that the right hand side converges to $\int \varphi\,d\widetilde{\omega}$. So we deduce that

$$\int_\Omega \widetilde{g}\,\Delta\varphi\,dx = \int \varphi\,d\widetilde{\omega}.$$

From this fact, it is clear that $\widetilde{g}$ does not vanish identically on Ω. Taking into account that $\widetilde{g}$ is non-negative by construction and harmonic in Ω, it follows that $\widetilde{g}$ is (strictly) positive in Ω.

Next we will show that $\widetilde{\omega}$ coincides with the measure $\mathrm{Cap}(\partial\Omega)\,\mu$. To this end, recall that for any $x \in \Omega$,

$$G^{p_{k_j}}(x) = \mathcal{E}(x - p_{k_j}) - \int \mathcal{E}(x - z)\, d\omega^{p_{k_j}}(z).$$

Hence,

$$\mathcal{E}(p_{k_j})^{-1} G^{p_{k_j}}(x) = \mathcal{E}(p_{k_j})^{-1}\mathcal{E}(x - p_{k_j}) - \mathcal{E}(p_{k_j})^{-1} \int \mathcal{E}(x - z)\, d\omega^{p_{k_j}}(z).$$

The left side converges to $\widetilde{g}(x)$ as $j \to \infty$, while the first term on the right hand side tends to 1 and the last one to $\int \mathcal{E}(x - z)\, d\widetilde{\omega}(z)$. So we deduce that

$$\widetilde{g}(x) = 1 - \int \mathcal{E}(x - z)\, d\widetilde{\omega}(z) = 1 - U_{\widetilde{\omega}}(x). \tag{7.30}$$

Since $\widetilde{g}(x)$ is positive in Ω, we deduce that $U_{\widetilde{\omega}}(x) < 1$ for all $x \in \Omega$, and thus $U_{\widetilde{\omega}}(x) \leqslant 1$ for all $x \in \partial\Omega$. Since $\|\widetilde{\omega}\| = \mathrm{Cap}(\partial\Omega)$, by the uniqueness of the equilibrium measure μ of $\partial\Omega$, it follows that $\widetilde{\omega} = \mathrm{Cap}(\partial\Omega)\,\mu$, as claimed.

In particular, the identity $\widetilde{\omega} = \mathrm{Cap}(\partial\Omega)\,\mu$ ensures that the measure $\widetilde{\omega}$ does not depend on the chosen subsequence $\{p_{k_j}\}_j$, which in turn implies that the initial sequence of measures $\mathcal{E}(p_k)^{-1}\omega^{p_k}$ converges to $\widetilde{\omega}$. From the relationship between $\widetilde{g}$ and $\widetilde{\omega}$ in (7.30), we deduce that $\widetilde{g}$ does not depend on the subsequence $\{p_{k_j}\}_j$ either, and analogously this implies the local uniform convergence in bounded subsets of Ω of the functions $\mathcal{E}(p_k)^{-1}G^{p_k}$.

The preceding arguments show that setting $\omega^\infty = \widetilde{\omega}$ and $G^\infty = \widetilde{g}$, the properties (a), (b), (d) and (e) hold. In particular, notice that the identities stated in (e) follow from (7.30) and (7.28). So it just remains to prove (c).

Consider a ball $B \subset \mathbb{R}^{n+1}$ centered at the origin such that $\partial\Omega \subset \frac{1}{2}B$. It suffices to show that ω^∞ is absolutely continuous with respect to $p \in \partial B$. To this end, observe first that, by a Harnack chain argument,

$$\omega^p(E) \approx \omega^{p'}(E) \qquad \text{for all } p, p' \in \partial B \text{ and all } E \subset \partial\Omega,$$

with the implicit constant independent of $p, p' \in E$. Consider the function

$$f_E(x) = \frac{r(B)^{n-1}}{|x|^{n-1}}\,\omega^p(E).$$

Observe that $f_E(p) = \omega^p(E) \approx \omega^q(E)$ for all $q \in \partial B$. Also, $\lim_{q\to\infty} f_E(q) = \lim_{q\to\infty}\omega^q(E) = 0$. So by the maximum principle we deduce that $f_E(x) \approx \omega^x(E)$ uniformly for all $x \in B^c$ and $E \subset \partial\Omega$. So we get

$$\frac{\omega^x(E)}{\omega^p(E)} \approx \frac{f_E(x)}{f_E(p)} = \frac{r(B)^{n-1}}{|x|^{n-1}} = \frac{f_{\partial\Omega}(x)}{f_{\partial\Omega}(p)} \approx \frac{\omega^x(\partial\Omega)}{\omega^p(\partial\Omega)}.$$

Thus,

$$\frac{\omega^p(E)}{\omega^p(\partial\Omega)} \approx \frac{\omega^x(E)}{\omega^x(\partial\Omega)} \qquad \text{for all } x \in B^c,$$

and then

$$\frac{\omega^p(E)}{\omega^p(\partial\Omega)} \approx \limsup_{y\to\infty} \frac{\omega^y(E)}{\omega^y(\partial\Omega)} \approx \liminf_{y\to\infty} \frac{\omega^y(E)}{\omega^y(\partial\Omega)}.$$

By the identity (7.29) and for k large enough, it follows that for $p \in \partial B$,

$$\frac{\mathcal{E}(p_k)^{-1}\omega^{p_k}(E)}{\mathrm{Cap}(\partial\Omega)} = \frac{U_\mu(p_k)}{\mathcal{E}(p_k)} \frac{\omega^{p_k}(E)}{\omega^{p_k}(\partial\Omega)} \approx \frac{U_\mu(p_k)}{\mathcal{E}(p_k)} \frac{\omega^p(E)}{\omega^p(\partial\Omega)}.$$

Letting $k \to \infty$, we derive

$$\mathrm{Cap}(\partial\Omega)^{-1}\omega^\infty(E) \approx \frac{\omega^p(E)}{\omega^p(\partial\Omega)}$$

for any measurable set $E \subset \partial\Omega$, which proves (c). □

Remark 7.30. Notice that the estimate in Lemma 7.17 also holds for the harmonic measure and the Green function with pole at ∞. To check this, just multiply the inequality (7.10) by $\mathcal{E}(x)^{-1}$ and take the limit as $x \to \infty$.

8 Harmonic measure in CDC uniform and NTA domains

This chapter deals with properties of harmonic measure on CDC uniform and NTA domains. Most of the material is based on [JK82]. For simplicity, in this chapter we assume that the domain Ω is bounded. We will use the following notation.

Definition 8.1. Let $\Omega \subset \mathbb{R}^d$. For every $\xi \in \partial\Omega$ and $r > 0$ we write the boundary ball

$$\Delta_{r,\xi} := \Delta_r(\xi) := B_r(\xi) \cap \partial\Omega.$$

We also use the classical notation for rescaled balls to the boundary balls:

$$t\Delta_{r,\xi} := \Delta_{tr,\xi}$$

8.1 CDC, uniform, and NTA domains

Definition 8.2. A CDC domain is a domain satisfying the CDC condition.

Recall that CDC domains are Wiener regular.

Definition 8.3. A domain $\Omega \subset \mathbb{R}^d$ satisfies the exterior corkscrew condition if for every $\xi \in \partial\Omega$ and $r < r_0$ there exists a point $X_r^{\text{ex}}(\xi) = X_{r,\xi}^{\text{ex}} = X_{\Delta_{r,\xi}}^{\text{ex}} \in \overline{\Omega}^c$ such that $|X_r^{\text{ex}}(\xi) - \xi| < r$ and $\mathrm{d}_\Omega(X_r^{\text{ex}}(\xi)) := \mathrm{dist}(X_r^{\text{ex}}(\xi), \partial\Omega) > A^{-1}r$. We call $X_r^{\text{ex}}(\xi)$ an *exterior corkscrew point* of ξ at scale r, and $B_{\Delta_{r,\xi}}^{\text{ex}} := B_{r,\xi}^{\text{ex}} := B_{\frac{r}{2A}}(X_{r,\xi}^{\text{ex}})$ is called exterior corkscrew ball. Note that $B_{r,\xi}^{\text{ex}} \subset 2B_{r,\xi}^{\text{ex}} \subset \overline{\Omega}^c$.

It is immediate to check that, for any bounded domain, the exterior corkscrew condition implies the CDC condition, and thus the Wiener regularity of Ω.

Next we recall one of the Hölder regularity properties already shown for CDC domains.

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^d$ be a CDC domain, let $u \in C^0(B_r(\xi) \cap \Omega)$ be non-negative harmonic, vanishing continuously on $\Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $r < r_0$. Then there are constants C_0 and α depending on d and the CDC character so that*

$$u(x) \leqslant C_0 \left(\frac{|x - \xi|}{r} \right)^\alpha \sup_{B_r(\xi) \cap \Omega} u \quad \textit{for every } x \in B_r(\xi) \cap \Omega.$$

Definition 8.5. A uniform domain $\Omega \subset \mathbb{R}^d$ is a domain satisfying

- Interior corkscrew condition: For every $\xi \in \partial\Omega$ and $r < r_0$ there exists a point $X_r^{\text{in}}(\xi) = X_{r,\xi}^{\text{in}} = X_{\Delta_{r,\xi}}^{\text{in}} \in \Omega$ such that $|X_r^{\text{in}}(\xi) - \xi| < r$ and $\mathrm{d}_\Omega(X_r^{\text{in}}(\xi)) > A^{-1}r$. We call $X_r^{\text{in}}(\xi)$ a *(interior) corkscrew point* of ξ at scale r, and $B_{\Delta_{r,\xi}}^{\text{in}} := B_{r,\xi}^{\text{in}} := B_{\frac{r}{2A}}(\xi)$ is called interior corkscrew ball. Note that $B_{r,\xi}^{\text{in}} \subset 2B_{r,\xi}^{\text{in}} \subset \overline{\Omega}^c$.
- Harnack chain condition: for $\varepsilon > 0$ and $x_1, x_2 \in \Omega$ with $\mathrm{d}_\Omega(x_j) > \varepsilon$ and $|x_1 - x_2| = r < r_0$, there exists N depending only on $\frac{r}{\varepsilon}$ and a collection of balls $\{B_j\}_{j=1}^N$ with $x_1 \in B_1$, $x_2 \in B_N$ such that $2B_j \subset \Omega$ for every $0 \leqslant j \leqslant N$ and $B_j \cap B_{j-1} \neq \varnothing$ for every $1 \leqslant j \leqslant N$. This collection of balls is called a *Harnack chain* joining x_1 and x_2.

From now on, for short we will say that a domain is CDC uniform it is both CDC and uniform.

Lemma 8.6. *A domain $\Omega \subset \mathbb{R}^d$ is uniform if and only if for every $x_0, x_1 \in \Omega$ with $|x_0 - x_1| < r_0$ there exists a path $\gamma : [0,1] \to \Omega$ such that*

1. *$\gamma(j) = x_j$ for $j \in \{0,1\}$,*
2. *the length of the curve $\ell(\gamma) \leqslant \widetilde{A}|x - y|$ and*
3. *for $t \in (0,1)$ we have $d_\Omega(\gamma(t)) \geqslant \operatorname{dist}(\gamma(t), \{x_0, x_1\})/\widetilde{A}$.*

Proof. We can show first the 'if' part. Let $\xi \in \partial\Omega$, $r < \min\{r_0, \operatorname{diam}\Omega\}$. Consider $x_0 \in B_{\frac{r}{4}}(\xi) \cap \Omega$ and $x_1 \in \partial B_r(\xi) \cap \Omega$ (which exists by connectedness) and consider the path γ connecting x_0 and x_1. Then the point $X_r^{\text{in}}(\xi) := y \in \gamma(0,1) \cap \partial B_{\frac{r}{2}}(\xi)$ is a corkscrew point, so Ω satisfies de corkscrew condition.

Let us prove that the Harnack chain condition is also satisfied. To this end just consider $\varepsilon > 0$ and $x_1, x_2 \in \Omega$ with $\operatorname{dist}(x_j, \partial\Omega) > \varepsilon$ and $|x_1 - x_2| = r < r_0$. Take the collection of balls $\{B_{\frac{1}{10}d_\Omega(y)}(y)\}_{y \in \gamma([0,1])}$. By the $5r$-covering theorem there exists a subcollection of disjoint balls B_j such that $5B_j$ cover $\gamma([0,1])$. The radii of the balls are bounded below by a constant times $\operatorname{dist}(\gamma([0,1]), \partial\Omega) > C_{\widetilde{A}}^{-1}\varepsilon$ by the third condition.

We claim that for every $k > 0$ the number of balls with $2^k C_{\widetilde{A}}^{-1}\varepsilon \leqslant r(B_j) < 2^{k+1} C_{\widetilde{A}}^{-1}\varepsilon$ is bounded by a constant C_1 depending on d and perhaps on $\widetilde{A}$. It is enough to consider the balls whose center is closer to the endpoint x_0 and $\varepsilon = d_\Omega(x_0)$.

First consider k so that $2^k C_{\widetilde{A}}^{-1} \leqslant 1$. Writing $x(B)$ for the center of the ball and $r(B)$ for its radius, in this case,

$$\operatorname{dist}(x(B), x_0) \leqslant d_\Omega(x(B)) \approx r(B) \leqslant 2\varepsilon$$

and, therefore, since the balls are disjoint, the number of such balls is bounded by a dimensional constant times $C_{\widetilde{A}}^{-d}$.

So let us consider the balls such that $d_\Omega(x_0) \leqslant 2^k C_{\widetilde{A}}^{-1}\varepsilon \leqslant r(B) < 2^{k+1} C_{\widetilde{A}}^{-1}\varepsilon$. Since $10B \subset \Omega$, we can infer that $d_\Omega(x_0) \leqslant 9|x(B) - x_0|$. By the third property and the triangle inequality, it follows that

$$|x(B) - x_0| \lesssim_{\widetilde{A}} d_\Omega(x(B)) \leqslant |x(B) - x_0| + d_\Omega(x_0) \approx |x(B) - x_0|.$$

The number of disjoint balls whose size is comparable to their distance from a point is bounded by a dimensional constant, and the claim follows.

Also the maximum size of the balls is bounded by $r(B) = \frac{1}{10} d_\Omega(x(B)) \leqslant \ell(\gamma) + \varepsilon \lesssim r + \varepsilon$ by the second condition. Thus, the number of balls is bounded by

$$N \leqslant C_1(\log_2(r+\varepsilon) - \log_2(C_{\widetilde{A}}^{-1}\varepsilon)) = C_1 \log_2\left(\frac{r+\varepsilon}{C_{\widetilde{A}}^{-1}\varepsilon}\right) = C_1 \log_2\left(\frac{\frac{r}{\varepsilon}+1}{C_{\widetilde{A}}^{-1}}\right).$$

To show the converse, assume that Ω is uniform and let $x_0, x_1 \in \Omega$ with $\varepsilon \leqslant |x_0 - x_1| < r_0$. Let $\xi_j \in \partial\Omega$ be points minimizing $\operatorname{dist}(x_j, \xi)$, and for every $0 \leqslant k \leqslant k_0 := \lfloor \log_2(\frac{|x_0-x_1|}{\varepsilon}) \rfloor$ consider the corkscrew point $y_k^j := X_{2^k\varepsilon}^{\text{in}}(\xi_j)$. The number of balls in a Harnack chain between two consecutive points y_k^j and y_{k+1}^j is uniformly bounded. The same can be said about the Harnack chain joining $y_{k_0}^0$ and $y_{k_0}^1$. Joining the centers of the balls in these Harnack chains between consecutive points we find a path satisfying the three conditions above. Indeed 1 holds trivially, 2 is a consequence of the fact that the number of balls of each scale is uniformly bounded and, therefore, the length of the curve can be controlled by a geometric sum whose bigger term is comparable to $|x_0 - x_1|$. The third condition follows from the fact that for every ball B from the Harnack chains d_Ω is comparable with $r(B)$ in $\gamma([0,1]) \cap B$ and the distance from the ball to the closest end-point is bounded again by a geometric series whose bigger term is comparable to $r(B)$. □

Put in plain words, the definition we give here of uniform domains in terms of corkscrew points and Harnack chains coincides with the definition in terms of "cigar paths" from the Sobolev extension domains in [Jon81]. Also from the previous proof we can infer that the definition coincides with the one in [GO79], where the distance $\operatorname{dist}(\gamma(t), \{x_0, x_1\})$ in the third condition is replaced by the arc-length distance to the endpoints.

Roughly speaking, the domain cannot have outer cusps, thin tubes or slits. In two dimensions inner cusps are also banned.

The Harnack chain condition gives us that, whenever u is a positive harmonic function on Ω,

$$C^{-N(\Lambda)} u(y) \leqslant u(x) \leqslant C^{N(\Lambda)} u(y) \text{ whenever } \frac{|x-y|}{d_\Omega(\{x,y\})} \leqslant \Lambda.$$

By the previous proof, uniformity tells us that for $k \geqslant 1$ we have $N(2^k) \leqslant C_1 \log_2\left(C_A 2^k\right) \leqslant C_1(k + \log_2(C_A))$, that is whenever $|x - y| \leqslant \min\{2^k d_\Omega(\{x,y\}), r_0\}$ with $k \geqslant 2$ we have

$$C_A^{-k} u(y) \leqslant u(x) \leqslant C_A^k u(y). \tag{8.1}$$

Note that the value of C_A may have increased in our reasoning, but depends only on the constant A and the dimension d.

In particular, for CDC uniform domains, by the results in Chapter 7 and the Harnack chain property we have:

Lemma 8.7. *Let $\Omega \subset \mathbb{R}^d$ be a CDC uniform domain and let $\xi \in \partial\Omega$ and $r < r_0$. Then*

$$\omega^{X_{r,\xi}^{\text{in}}}(\Delta_{r,\xi}) \geqslant c_A.$$

Definition 8.8. A non-tangentially accessible domain (NTA domain for short) is a uniform domain satisfying also the exterior corkscrew condition.

It is clear that any NTA domain is CDC uniform. The notion of NTA domain was introduced by Jerison and Kenig in [JK82]. In this work they studied the behavior of harmonic measure in this type of domains.

Roughly speaking, NTA domains cannot have outer cusps, inner cusps, thin tubes, slits or isolated points in the boundary. In fact, for every $E \neq \varnothing$ contained in a simply connected NTA domain $\Omega \subset \mathbb{R}^d$ with $|E| = 0$, $\Omega \backslash E$ is not an NTA domain. In other words, if Ω is bounded, consider $\widetilde{\Omega}$ to be the complement of the unbounded component of the complement of Ω, which is a simply connected containing Ω. Then $\widetilde{\Omega} \backslash \Omega$ consists of a (perhaps empty) collection of connected closed sets with positive Lebesgue measure. This is in contrast to uniform domains, since the complement of the planar 1/4-Cantor set is a uniform domain in $\mathbb{R}^2$.

If the domain is bounded, we may assume without loss of generality that $r_0 = \operatorname{diam}(\Omega)$. Indeed, just by taking worse constants depending on the ratio $\frac{r_0}{\operatorname{diam}(\Omega)}$ we can check that both corkscrew conditions and the Harnack chain condition are satisfied as well for $r_0 < r < \operatorname{diam}(\Omega)$.

8.2 Green's function for CDC uniform domains

Next we show that the supremum of a nonnegative harmonic function in a ball coincides modulo constant with the value at the corkscrew point:

Lemma 8.9. *Let Ω be a CDC uniform domain. Let $u \geqslant 0$ harmonic in Ω, vanishing continuously on $\Delta_{2r,\xi}$ with $\xi \in \partial\Omega$ and $2r < r_0$, then we have*

$$\sup_{\Omega \cap B_r(\xi)} u \leqslant C_A u(X^{\text{in}}_{r,\xi}).$$

Proof. To simplify notation, let us assume that $4r < r_0$, let us assume that u vanishes on 4Δ with $\Delta := \Delta_{r,\xi}$, and let us assume that $u(X^{\text{in}}_{2\Delta}) = 1$. We will prove that

$$\sup_{\Omega \cap B_{2r}(\xi)} u \lesssim 1.$$

Theorem 8.4 implies the existence of a constant $A_1 > 1$ s.t. for every $\zeta \in 3\Delta$ and every $s < r$

$$\sup_{B(\zeta, A_1^{-1}s)\cap\Omega} u \leqslant \frac{1}{2} \sup_{B(\zeta,s)\cap\Omega} u. \tag{8.2}$$

The second observation is about the quantitative behavior of Harnack chains described in (8.1): if $x \in B_r(\zeta) \cap \Omega$ with $\zeta \in 3\Delta$, $n \in \mathbb{N}$, and $d_\Omega(x) \geqslant A_1^{-n} r$, then

$$|X^{\text{in}}_{2r,\xi} - x| < 6r \leqslant 6A_1^n d_\Omega(x) \implies C_A^{-k} u(x) \leqslant u(X^{\text{in}}_{2r,\xi}) = 1,$$

where $k = 1 + \lfloor \log_2(6A_1^n) \rfloor \approx n$. Thus, we can pick $A_2 := C_A^{k/n} > 1$ above, and we deduce that whenever $x \in B_{2r}(\zeta) \cap \Omega$, we have

$$u(x) > A_2^n \implies \mathrm{d}_\Omega(x) < A_1^{-n} r. \tag{8.3}$$

Now we argue by contradiction: consider N so that $2^N > A_2$ and let $n = N+3$. Assume that there exists $y_0 \in \Omega \cap B_{2r}(\xi)$ with $u(y_0) > A_2^n$. Then, by (8.3) we can find $\xi_0 \in \partial\Omega$ satisfying that

$$|y_0 - \xi_0| < A_1^{-n} r.$$

Note also that

$$|\xi - \xi_0| \leqslant |\xi - y_0| + |y_0 - \xi_0| \leqslant 2r + A_1^{-n} r < 3r.$$

and by (8.2) we have

$$\sup_{B(\xi_0, A_1^{-n+N} r)} u > 2^N \sup_{B(\xi_0, A_1^{-n} r)} u > A_2 \cdot A_2^n = A_2^{n+1}.$$

We have proven the existence of $y_1 \in B(\xi_0, A_1^{-n+N} r)$ with $u(y_1) > A_2^{n+1}$. Since $N - n < 0$, we can apply (8.3) to find $\xi_1 \in \partial\Omega$ so that

$$|y_1 - \xi_1| < A_1^{-n-1} r.$$

Note also that

$$|\xi - \xi_1| \leqslant |\xi - \xi_0| + |\xi_0 - y_1| + |y_1 - \xi_1| \leqslant (2 + A_1^{-n} + A_1^{-n+N} + A_1^{-n-1}) r < 3r,$$

and by (8.2) we have

$$\sup_{B(\xi_1, A_1^{-n-1+N} r)} u > 2^N \sup_{B(\xi_1, A_1^{-n-1} r)} u > A_2 \cdot u(y_1) > A_2^{n+2}.$$

Iterating the construction, we find $y_k \in B(\xi_{k-1}, A_1^{-n+N-k+1} r)$ with $u(y_k) > A_2^{n+k}$. We can apply (8.3) to find $\xi_k \in \partial\Omega$ so that

$$|y_k - \xi_k| < A_1^{-n-k} r.$$

Note also that

$$|\xi - \xi_k| \leqslant |\xi - \xi_{k-1}| + |\xi_{k-1} - y_k| + |y_k - \xi_k| \leqslant \left(2 + A_1^{-n} + \sum_{j=1}^{k} \left(A_1^{-n+N-j+1} + A_1^{-n-j}\right)\right) r < 3r,$$

form A_1 large enough, and by (8.2) we have

$$\sup_{B(\xi_1, A_1^{-n-k+N} r)} u > 2^N \sup_{B(\xi_1, A_1^{-n-k} r)} u > A_2 \cdot u(y_k) > A_2^{n+k+1},$$

so the induction can be carried on.

Note that y_k is a Cauchy sequence converging to a point in 3Δ. Therefore, we reach a contradiction with the continuity of u. $\square$

Recall that for a bounded Wiener regular domain (and so for a CDC domain) the Green function equals, for $x \in \Omega$ and $y \in \mathbb{R}^d \backslash \{x\}$:

$$G_\Omega^x(y) = \begin{cases} \mathcal{E}^y(x) - \int \mathcal{E}^y(\xi) d\omega^x(\xi) \geqslant 0 & \text{if } y \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the following holds, as shown in the preceding chapter.

Lemma 8.10. *Let Ω be a CDC uniform domain and let $G := G_\Omega$ be its Green function. For every $x \in \Omega$ and a.e. $y \in \mathbb{R}^d$ we have*

$$G^x(y) = \mathcal{E}^y(x) - \int \mathcal{E}^y(\xi) d\omega^x(\xi).$$

Moreover, G^x vanishes continuously in $\partial\Omega$, and therefore it is continuous in $\mathbb{R}^d \backslash \{x\}$.

Lemma 8.11. *Let Ω be a CDC uniform domain, let $G := G_\Omega$ be its Green function and let $x \in \Omega \backslash B(\xi, 8r)$, with $\xi \in \partial\Omega$ and $4r < r_0$. Then the boundary ball $\Delta := \Delta_{r,\xi}$ satisfies*

$$\omega^x(\Delta) \leqslant C_A r^{d-2} G^x(X_\Delta^{\text{in}})$$

Proof. Let $\phi \in C^\infty$ bump function so that $\chi_{B_r(\xi)} \leqslant \phi \leqslant \chi_{B_{3r/2}(\xi)}$ (so $\phi(x) = 0$) and $|D^2\phi| \lesssim r^{-2}$. Then

$$\begin{aligned} \int_\Omega G^x(y) \Delta\phi(y)\, dm(y) &= \int_{\mathbb{R}^d} \left(\mathcal{E}^y(x) - \int \mathcal{E}^y(\xi) d\omega^x(\xi) \right) \Delta\phi(y)\, dm(y) \\ &= -\phi(x) - \int \int_{\mathbb{R}^d} \Delta\phi(y)\, \mathcal{E}^\xi(y)\, dm(y)\, d\omega^x(\xi) \\ &= 0 + \int \phi(\xi)\, d\omega^x(\xi) \geqslant \omega^x(\Delta). \end{aligned}$$

Now, since G^x is harmonic in the CDC uniform domain $\Omega \backslash B_{\frac{1}{2}\mathrm{d}_\Omega(x)}(x)$ (with perhaps worse constants than the original one) and vanishes on the boundary, we can use Lemma 8.9 to conclude that $G^x(y) \lesssim_A G^x(X_{2\Delta}^{\text{in}})$ on $B_{2r}(\xi) \cap \Omega$. Thus,

$$\omega^x(\Delta) \leqslant \int_\Omega G^x(y) \Delta\phi(y)\, dm(y) \leqslant \int_{B_{2r}(\xi)} G^x(y) |\Delta\phi(y)|\, dm(y) \overset{\text{L } 8.9}{\lesssim}_{A,d} r^{d-2} G^x(X_{2\Delta}^{\text{in}}).$$

The lemma follows by the Harnack chain condition. □

Next we want to show that the two terms in the conclusion of Lemma 8.11 are in fact comparable.

By the results in Chapter 7 and the Harnack chain condition, the following also holds.

Lemma 8.12. *Let Ω be a CDC uniform domain, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $r < r_0$. If $x \in \Omega \backslash B_{\frac{1}{2}\Delta}^{\text{in}}$, then*

$$r^{d-2} G^x(X_{\frac{1}{2}\Delta}^{\text{in}}) \lesssim_A \omega^x(\Delta)$$

Combining Lemmas 8.11 and 8.12 we get the following remarkable fact.

Theorem 8.13. *Let Ω be a uniform CDC domain, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $4r < r_0$. For $x \in \Omega \backslash B(\xi, 8r)$*

$$\frac{\omega^x(\Delta)}{r^{d-2}G^x(X_\Delta^{\text{in}})} \approx 1$$

with constants depending on A.

8.3 The doubling condition

Lemma 8.14 (Doubling condition)**.** *Let Ω be a CDC uniform domain. If $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $x \in \Omega$, then*

$$\omega^x(2\Delta) \leqslant C\omega^x(\Delta),$$

with C depending on $\mathrm{d}_\Omega(x)$, d, r_0, A and $\mathrm{diam}(\Omega)$.

Proof. Without loss of generality, we may assume that $r_0 < \mathrm{diam}\Omega$. Then take $r_1 = A^{-1}r_0$ Let us assume that $2\mathrm{d}_\Omega(x) > r_1$.

The case $16r > r_1$ follows by Lemma 8.7 and the Harnack inequality. Indeed, we can find a finite family of points ξ_j so that $\Delta(\xi_j, r_0/8)$ cover the boundary, so

$$\omega^x(\Delta_{r_0/8,\xi}) \overset{\text{Harnack}}{\gtrsim} \omega^{X^{\text{in}}_{r_0/8,\xi}}(\Delta_{r_0/8,\xi}) \overset{\text{L. 8.7}}{\geqslant} c_A,$$

the constants of the first estimate depending only on d, r_1 and $\mathrm{diam}(\Omega)$. Now, there is a ξ_{j_0} so that $\xi \in \Delta(\xi_{j_0}, r_0/8)$ and thus $\Delta(\xi_{j_0}, r_0/8) \subset \Delta$. Therefore

$$\omega^x(\Delta) \geqslant \omega^x(\Delta_{r_0/8,\xi}) \gtrsim c_A \geqslant c_A\omega^x(2\Delta).$$

If $16r < r_1$, then we can use Theorem 8.13 twice and the Harnack chain:

$$\omega^x(2\Delta) \overset{\text{T 8.13}}{\approx} cr^{n-2}G^x(X_{2\Delta}^{\text{in}}) \overset{\text{Harnack}}{\approx} cr^{n-2}G^x(X_\Delta^{\text{in}}) \overset{\text{T 8.13}}{\approx} \omega^x(\Delta).$$

For the cases not included in the previous ones, consider x_0 so that $\mathrm{d}_\Omega(x_0) \approx r_1$. Then, since $\omega^x(\Delta)$ and $\omega^x(2\Delta)$ are harmonic functions, we get that

$$\omega^x(\Delta) \approx_x \omega^{x_0}(\Delta) \leqslant c\omega^{x_0}(2\Delta) \approx_x \omega^x(2\Delta).$$

$\square$

Lemma 8.15. *Let Ω be a CDC uniform domain. There exists a constant C_A such that for every $\xi \in \partial\Omega$ and $r \leqslant C_A r_0$, there exists a CDC uniform domain $\Omega_{r,\xi}$ such that*

$$\Omega \cap B_{C_A^{-1}r}(\xi) \subset \Omega_{r,\xi} \subset \Omega \cap B_{C_A r}(\xi).$$

The constants of the CDC uniform domain are independent of ξ and r. Moreover, for $\zeta \in \partial\Omega_{r,\xi} \backslash B_{\frac{r}{2}}(\xi)$, we have that $\mathrm{d}_\Omega(\zeta) \gtrsim c_A r$.

Proof. Consider a Whitney covering of Ω. That is, denote by $\mathcal{W} := \mathcal{W}(\Omega)$ the set of maximal dyadic cubes $Q \subset \Omega$ such that $4Q \cap \Omega^c = \varnothing$. These cubes have disjoint interiors and can be easily shown to satisfy the following properties:

(a) $\operatorname{dist}(Q, \Omega^c) \lesssim \ell(Q) \lesssim \operatorname{dist}(Q, \Omega^c)$, where $\ell(Q)$ denotes the side length of the cube.

(b) If $Q, R \in \mathcal{W}$ and $4Q \cap 4R \neq \varnothing$, then $\ell(Q) \approx_d \ell(R)$.

(c) $\sum_{Q \in \mathcal{W}} \chi_{2Q} \lesssim_d \chi_\Omega$.

Now, let $\Delta := \Delta_{A^{-1}r,\xi_0}$. For every $\zeta \in \Delta$ and $\rho < r$, there exists $Q^{\text{in}}_{r,\zeta} \in \mathcal{W}$ so that $\ell(Q) \approx A^{-1}r$ and $Q^{\text{in}}_{r,\zeta} \cap B^{\text{in}}_{r,\zeta} \neq \varnothing$. Denote

$$\mathcal{F}_1 := \{Q \in \mathcal{W} : Q = Q^{\text{in}}_{r,\zeta} \text{ for some } \zeta \in \Delta \text{ and } \rho < r\}.$$

We can identify $Q \in \mathcal{F}_1$ with a pair (r_Q, ζ_Q) so that $Q = Q^{\text{in}}_{r_Q,\zeta_Q}$. Then, for $Q, R \in \mathcal{F}_1$ there exists a Harnack chain of balls $\{B^{Q,R}_j\}_{j=1}^{N_{Q,R}}$ joining $B^{\text{in}}_{r_Q,\zeta_Q}$ with $B^{\text{in}}_{r_R,\zeta_R}$ as in Definition 8.5, that is, $N_{Q,R} \lesssim \frac{\mathrm{D}(Q,R)}{\min\{\ell(Q),\ell(R)\}}$, $B^{Q,R}_j \cap B^{Q,R}_{j+1} \neq \varnothing$ and $r(B^{Q,R}_j) = \operatorname{dist}(B^{Q,R}_j, \partial\Omega)$. Note that

$$B^{Q,R}_j \subset \{x \in \Omega : \operatorname{dist}(x, \Delta) \leqslant 2r\}$$

and by Lemma 8.6 we get

$$\operatorname{dist}(B^{Q,R}_j, \Delta) \leqslant \min\{\operatorname{dist}(B^{Q,R}_j, Q) + C\ell(Q), \operatorname{dist}(B^{Q,R}_j, R) + C\ell(R)\} \leqslant C_A r(B^{Q,R}_j).$$

Next we define

$$\mathcal{F}_2 := \{Q \in \mathcal{W} : Q \cap B^{R,S}_j \neq \varnothing \text{ for some } R, S \in \mathcal{F}_1 \text{ and } j \leqslant N_{R,S}\}.$$

At this point the reader may note that every pair of cubes in $\mathcal{F}_1$ can be connected by a chain of cubes in $\mathcal{F}_2$, whatever that means. However, we still need to show the existence of Harnack chains joining cubes in $\mathcal{F}_2 \backslash \mathcal{F}_1$.

Given $Q \in \mathcal{F}_2$, we claim that there exists $\Psi(Q) \in \mathcal{F}_1$ so that

$$\ell(Q) \approx \ell(\Psi(Q)) \approx \mathrm{D}(Q, \Psi(Q)). \tag{8.4}$$

Indeed, note that there exists a couple of cubes $R_Q, S_Q \in \mathcal{F}_1$ so that $Q \cap B^{R_Q,S_Q}_j \neq \varnothing$ for some $j \leqslant N_{R_Q,S_Q}$. In particular,

$$\operatorname{dist}(Q, \Delta) \leqslant \operatorname{dist}(B^{R_Q,S_Q}_j, \Delta) + 2r(B^{R_Q,S_Q}_j) \leqslant \min\{4r, C_A\ell(Q)\}.$$

Let $\zeta_Q \in C_A Q \cap \Delta$. Then $\Psi(Q) := Q^{\text{in}}_{\zeta_Q, A\ell(Q)}$ satisfies (8.4).

Next we define

$$\mathcal{F}_3 := \{Q \in \mathcal{W} : \operatorname{dist}(Q, \Delta) \leqslant \min\{4r, C_A\ell(Q)\}\}.$$

We get that $\mathcal{F}_2 \subset \mathcal{F}_3$ as discussed above. Moreover, for $Q \in \mathcal{F}_3$ we can reason as above to define $\zeta_Q \in C_A Q \cap \Delta$, so that $\Psi(Q) := Q^{\text{in}}_{\zeta_Q, A\ell(Q)}$ satisfies (8.4) as well.

Estimate (8.4) means in particular that all the balls in the chain $\{B_j^{Q,\Psi(Q)}\}$ joining Q and $\Psi(Q)$ are roughly of the same size and their number is bounded by universal constants depending only on A and d. Therefore, we define

$$\mathcal{F}_4 := \{R \in \mathcal{W} : B_j^{Q,\Psi(Q)} \cap R \neq \varnothing \text{ for some } Q \in \mathcal{F}_2,\, j \leqslant N_{Q,\Psi(Q)}\},$$

and let

$$\widetilde{\Omega} := \bigcup_{Q \in \mathcal{F}_4} (1 + c_d)Q.$$

The Harnack chain condition is satisfied by construction: Ψ can easily be extended to $\mathcal{F}_4$ so that (8.4) is satisfied. Now, for points in neighboring Whitney cubes the chain can be constructed thanks to the dilation $(1 + c_d)$. For points in Whitney cubes Q_1, Q_2 further away, connect each cube Q_j to $\Psi(Q_j)$ and then connect $\Psi(Q_1)$ and $\Psi(Q_2)$ by a Harnack chain of balls $B_j^{\Psi(Q_1),\Psi(Q_2)}$. Then the number of balls depends only on $\frac{\mathrm{D}(Q_1,Q_2)}{\min\{\ell(Q_1),\ell(Q_2)\}}$.

To see that $\widetilde{\Omega}$ satisfies the interior corkscrew condition, just notice that if $\zeta \in \partial\widetilde{\Omega}$ and $\operatorname{dist}(\zeta, \Delta) \leqslant \rho/2$, then there are interior corkscrew balls contained in $B_\rho(\zeta)$ which are also interior corkscrew balls (with perhaps worse constants) for the new domain. If, instead, $\operatorname{dist}(\zeta, \Delta) > \rho/2$, then we have that $\rho \lesssim \ell(Q)$ for any $Q \in \mathcal{W}$ such that $\zeta \in \overline{Q}$. Since $\zeta \in \partial\widetilde{\Omega}$, then there is an interior cube $Q_1 \subset \widetilde{\Omega}$ and a cube Q_2 with $(1 - c_d)Q_2 \subset \overline{\widetilde{\Omega}}^c$ (perhaps decreasing the constant c_2), so that $\zeta \in Q_j$. Finding corkscrew balls of size comparable to ρ is possible because cubes are also uniform CDC domains. The fact that the CDC condition holds for $\widetilde{\Omega}$ is proved by similar arguments. □

Theorem 8.16 (Uniform boundary Harnack principle)**.** *Let Ω be a CDC uniform domain, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $r < C_A^{-1} r_0$. Let $u, v \geqslant 0$ harmonic in Ω vanishing continuously on $C_A\Delta$ and $u(X_\Delta^{\text{in}}) = v(X_\Delta^{\text{in}})$. Then $\frac{u}{v} \approx 1$ on $C_A^{-1}B_r(\xi) \cap \Omega$.*

Proof. Consider the intermediate domain $\widetilde{\Omega} := \Omega_{2r,\xi}$ from Lemma 8.15. We write $\widetilde{\Delta}_{r,\xi} := \widetilde{\Omega} \cap B_r(\xi)$, $\widetilde{\omega}$ for the harmonic measure in $\widetilde{\Omega}$ and so on.

Denote

$$L_1 := \{\zeta \in \partial\widetilde{\Omega} \backslash \partial\Omega : \operatorname{dist}(\zeta, \partial\Omega) < (C_A)^{-1} r\}$$

and

$$L_2 := \partial\widetilde{\Omega} \backslash (L_1 \cup \partial\Omega).$$

Take a minimal covering of L_1 with surface balls $\widetilde{\Delta}_j = \widetilde{\Delta}_j(\zeta_j, (10C_A)^{-1}r) \subset \widetilde{\Omega}$ with $j \in \{1, N\}$. Since the covering is minimal, N only depends on d and A.

On the other hand, there is a point $\zeta_0 \in \partial\widetilde{\Omega} \backslash B_{2C_A^{-1}r}(\xi)$. Then the surface ball in $\partial\widetilde{\Omega}$ defined as $\widetilde{\Delta}_0 = \widetilde{\Delta}_{(C_A)^{-1}r,\zeta_0} \subset L_2$.

Now, by Theorem 8.13 and the Harnack chain condition, we get

$$\widetilde{\omega}^x(\widetilde{\Delta}_j) \approx \left(\frac{r}{10C_A}\right)^{d-2} G^x(X_{\widetilde{\Delta}_j}^{\text{in}}) \approx \left(\frac{r}{10C_A}\right)^{d-2} G^x(X_{\widetilde{\Delta}_0}^{\text{in}}) \approx \widetilde{\omega}^x(\widetilde{\Delta}_0),$$

and therefore

$$\widetilde{\omega}^x(L_1) \leqslant \sum_{j=1}^{N} \widetilde{\omega}^x(\widetilde{\Delta}_j) \approx N\,\widetilde{\omega}^x(\widetilde{\Delta}_0) \lesssim \widetilde{\omega}^x(L_2), \tag{8.5}$$

the constants not depending on $x \in C_A^{-1}B_r(\xi) \cap \Omega$.

Applying Lemma 8.9 an the Harnack chain condition applied in Ω, assuming C_A large enough, we obtain

$$\sup_{\widetilde{\Omega}} u \lesssim_A u(X_\Delta^{\text{in}}). \tag{8.6}$$

On the other hand, by Harnack inequality again $\inf_{L_2} v \gtrsim_A v(X_\Delta^{\text{in}}) = u(X_\Delta^{\text{in}})$. All in all we get, for $x \in C_A^{-1}B_r(\xi) \cap \Omega$,

$$u(x) \overset{\text{Max.P.}}{\leqslant} \omega^x_{\widetilde{\Omega}}((\partial\Omega)^c) \sup_{\widetilde{\Omega}} u \overset{(8.6)}{\lesssim} \omega^x_{\widetilde{\Omega}}((\partial\Omega)^c) u(X_\Delta^{\text{in}}) \overset{(8.5)}{\lesssim} \omega^x_{\widetilde{\Omega}}(L_2) \inf_{L_2} v \overset{\text{Max.P.}}{\leqslant} v(x).$$

□

Lemma 8.17 (Universal doubling constant)**.** *In Lemma 8.14, if $x \in B_r(\xi) \cup B_{8r}(\xi)^c$ then C does not depend on x.*

Proof. The case $16r > r_1 = A^{-1}r_0$ and $2\mathrm{d}_\Omega(x) > r_1$ is already settled in the proof of Lemma 8.14.

If $x \in B(\xi, r)$ and $2\mathrm{d}_\Omega(x) < r_0$, then we have

$$\omega^x(\Delta) \geqslant \omega^x(B_{2\mathrm{d}_\Omega(x)}(x) \cap \Delta) \overset{\text{L 8.7}}{\geqslant} c_A \geqslant c_A \omega^x(2\Delta),$$

and the lemma follows.

Note also that the case $x \in B(\xi, r)$ and $2\mathrm{d}_\Omega(x) \geqslant r_1$, $16r \leqslant r_1$ cannot happen.

If $16r < r_1$ and $x \in B(\xi, 8r)^c$, then we can use Theorem 8.13 twice and the Harnack chain:

$$\omega^x(2\Delta) \overset{\text{T 8.13}}{\approx} cr^{n-2}G^x(X_{2\Delta}^{\text{in}}) \overset{\text{Harnack}}{\approx} cr^{n-2}G^x(X_\Delta^{\text{in}}) \overset{\text{T 8.13}}{\approx} \omega^x(\Delta).$$

The case $16r > r_1$, $\mathrm{d}_\Omega(x) < r_1$, $x \in B(\xi, 8r)^c$ can be obtained using the boundary Harnack principle. Indeed, let x_0 be such that $\mathrm{d}_\Omega(x_0) \geqslant r_1$

$$\sup_{\substack{\mathrm{d}_\Omega(x) < r_1,\, x \in B(\xi, 8r)^c \\ \mathrm{d}_\Omega(x) < A^{-2}r}} \frac{\omega^x(2\Delta)}{\omega^x(\Delta)} \overset{\text{L 8.16}}{\lesssim} \sup_{\substack{\mathrm{d}_\Omega(x) < r_1,\, x \in B(\xi, 8r)^c \\ \mathrm{d}_\Omega(x) \geqslant A^{-2}r_1}} \frac{\omega^x(2\Delta)}{\omega^x(\Delta)} \approx \frac{\omega^{x_0}(2\Delta)}{\omega^{x_0}(\Delta)}.$$

□

Note that one cannot expect to avoid the dependence on x: if $x \to 2\Delta \backslash \overline{\Delta}$, then $\omega^x(\Delta) \to 0$ and $\omega^x(2\Delta) \to 1$.

Theorem 8.18. *Let Ω be a CDC uniform domain, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $9r < r_0$. Assume that $\frac{\operatorname{dist}(x,\Delta)}{\operatorname{dist}(x,2\Delta)} > 2$. Then there exits constants α and β depending on the dimension and A so that*

$$\left(\frac{\operatorname{dist}(x, \Delta)}{\operatorname{dist}(x, 2\Delta)}\right)^{\alpha} \lesssim_{d,A} \frac{\omega^x(2\Delta)}{\omega^x(\Delta)} \lesssim_{d,A} \left(\frac{\operatorname{dist}(x, \Delta)}{\operatorname{dist}(x, 2\Delta)}\right)^{\beta}.$$

Proof. Let $\lambda_x := \frac{\mathrm{dist}(x,\Delta)}{\mathrm{dist}(x,2\Delta)}$, and let $\Omega_k := \{x \in \Omega : 2^{k-1} < \lambda_x \leqslant 2^k\}$. We divide Ω_k in two subregions.

$$\Omega_k^1 := \{x \in \Omega_k : \mathrm{d}_\Omega(x) \geqslant \mathrm{dist}(x, 2\Delta)\},$$

$$\Omega_k^2 := \left\{x \in \Omega_k : \mathrm{d}_\Omega(x) < \frac{\mathrm{dist}(x, 2\Delta)}{2C_A^2}\right\},$$

and

$$\Omega_k^3 = \Omega_k \backslash (\Omega_k^1 \cup \Omega_k^2).$$

By Lemma 8.17, we may assume that $x \in B_{8r}(\xi)^c \backslash B_r(\xi)$. First let us consider $x \in \Omega_k^2$. Note that $\mathrm{dist}(x,\Delta)-r \leqslant \mathrm{dist}(x,2\Delta) < 8r$ implies in particular that $\mathrm{dist}(x,\Delta) < 9r < r_0$. Applying the boundary Harnack principle from Theorem 8.16, it is enough to show the result for Ω_k^3.

But in this case the result can be compared to Ω_λ^1 using a Harnack chain.

It remains to study the case $x \in \Omega_k^1 \backslash \Omega_k^2$. If $0 \leqslant k \leqslant 1$, then we can compare to the case $x \in B_r(\xi)$ by a Harnack chain. Therefore we may assume that $k \geqslant 2$. For $x \in \Omega_\lambda^1$, let $\xi_x \in 2\Delta$ such that $|x - \xi_x| = \mathrm{d}_\Omega(x)$. Then

$$\omega^x(2\Delta) \overset{\text{Harnack}}{\approx} \omega^{X^{\text{in}}_{\mathrm{d}_\Omega(x),\xi_x}}(2\Delta) \overset{\text{L 8.7}}{\approx} 1.$$

To estimate $\omega^x(\Delta)$ from above we use the exterior corkscrew: by Theorem 8.4 we obtain

$$\omega^x(\Delta) \leqslant C\left(\frac{\mathrm{d}_\Omega(x)}{\mathrm{dist}(\xi,\Delta)}\right)^\alpha \leqslant C\left(\frac{\mathrm{dist}(x,\Delta)}{\mathrm{dist}(x,\Delta) - \mathrm{dist}(x,\Delta)}\right)^\alpha \leqslant C\left(\frac{\mathrm{dist}(x,\Delta)}{2\mathrm{dist}(x,\Delta)}\right)^\alpha,$$

implying the first estimate.

To estimate $\omega^x(\Delta)$ from below, let $\rho := \mathrm{dist}(\xi,\Delta) \leqslant 2\mathrm{dist}(x,\Delta)$. Then we use the uniform character (8.1). To do so, note that

$$|x - X^{\text{in}}_{\rho,\xi_x}| \leqslant |x - \xi_x| + |\xi_x - X^{\text{in}}_{\rho,\xi_x}| \leqslant \mathrm{d}_\Omega(x) + 2\mathrm{dist}(x,\Delta) \approx 2^k \mathrm{d}_\Omega(x).$$

Therefore, we get

$$\omega^x(\Delta) \overset{(8.1)}{\geqslant} C_A^{-k}\omega^{X^{\text{in}}_{\rho,\xi_x}}(\Delta) \overset{\text{L 8.7}}{\geqslant} c\left(\frac{\mathrm{dist}(x,\Delta)}{2\mathrm{dist}(x,\Delta)}\right)^\beta.$$

$\square$

Lemma 8.19 (Change of pole formula)**.** *Let $2r < r_0$, $\Delta_{s,\xi} \subset \Delta_{r/2,\xi_0}$ and $x \in \Omega \backslash B_{8r}(\xi_0)$. Then*

$$\omega^{X^{\text{in}}_{r,\xi_0}}(\Delta_{s,\xi}) \approx \frac{\omega^x(\Delta_{s,\xi})}{\omega^x(\Delta_{r,\xi_0})}$$

Proof. By Theorem 8.13, the lemma is equivalent to showing that

$$G^{X^{\text{in}}_{r,\xi_0}}(X^{\text{in}}_{s,\xi}) \approx r^{2-d}\frac{G^x(X^{\text{in}}_{s,\xi})}{G^x(X^{\text{in}}_{r,\xi_0})}.$$

Now this estimate can be obtained using Theorem 8.16 with $u = G^{X^{\text{in}}_{r,\xi_0}}$ and $v = G^x$ after normalizing with the value of an appropriate point of the boundary of the corkscrew ball: let $Y \in \partial B^{\text{in}}_{r,\xi_0}$. Then

$$G^{X^{\text{in}}_{r,\xi_0}}(Y) \overset{\text{T } 8.13 \ \& \text{ L } 8.7}{\approx} r^{2-d}.$$

Thus,

$$\frac{G^{X^{\text{in}}_{r,\xi_0}}(X^{\text{in}}_{s,\xi})}{G^x(X^{\text{in}}_{s,\xi})} \overset{\text{L } 8.16}{\approx} \frac{G^{X^{\text{in}}_{r,\xi_0}}(Y)}{G^x(Y)} \overset{\text{Harnack}}{\approx} \frac{r^{2-d}}{G^x(X^{\text{in}}_{r,\xi_0})}, \tag{8.7}$$

and the lemma follows. Note that the Green function is not harmonic in the domain, but in the domain minus a ball, which is CDC uniform with worse constants. Thus, to establish the first estimate in (8.7) one needs to apply the Harnack inequality to localize to a region where the conditions for Theorem 8.16 to apply hold, the details are left to the reader. □

8.4 Estimates for the Radon-Nikodym derivative

Remark 8.20. Fix a pole x_0 and $\omega := \omega^{x_0}$. Then the Radon-Nykodim derivative $K(x,\xi) = \frac{d\omega^x}{d\omega}(\xi)$ equals $\lim_{r\to 0} \frac{\omega^x(\Delta_{r,\xi})}{\omega(\Delta_{r,\xi})}$ for ω-a.e. ξ.

Proof. To see that $\omega^x \ll \omega^{x_0}$, note that given a Borel set $E \subset \partial\Omega$ with $\omega^{x_0}(E) = 0$, there exists an open set $U_n \supset E$ such that $\omega^{x_0}(U_n) < \frac{1}{n}$. Moreover, there exists a compact set $K_n \subset U_n$ so that $\omega^x(U_n) < \omega^x(K_n) + \frac{1}{n}$. Consider u_n to be a harmonic function with value 1 in $K_n \cap \partial\Omega$ and value 0 in $U_n^c \cap \partial\Omega$. Then

$$\omega^x(E) \leqslant \omega^x(U_n) < \omega^x(K_n) + \frac{1}{n} \leqslant u_n(x) + \frac{1}{n} \overset{\text{Harnack ineq.}}{\approx_{x_0,x}} u_n(x_0) + \frac{1}{n} \leqslant \omega^{x_0}(U_n) + \frac{1}{n} \leqslant \frac{2}{n}.$$

Once this is settled, by [Rud87, Theorem 6.9] we obtain that $\int_{\Delta_{r,\xi}} \frac{d\omega^x}{d\omega}(\zeta)d\omega(\zeta) = \omega^x(\Delta_{r,\xi})$ and, therefore, using the Lebesgue differentiation theorem (see [Mat95, Corollary 2.14]), we get

$$\frac{\omega^x(\Delta_{r,\xi})}{\omega(\Delta_{r,\xi})} = \frac{1}{\omega(\Delta_{r,\xi})} \int_{\Delta_{r,\xi}} \frac{d\omega^x}{d\omega}(\zeta)d\omega(\zeta) \xrightarrow{r\to 0} \frac{d\omega^x}{d\omega}(\xi) \quad \text{for } \omega - a.e.\, \xi \in \mathbb{R}^n.$$

□

Lemma 8.21. *Let $x = X^{\text{in}}_{r,\xi_0}$, $\Delta_j = \Delta_{2^j r,\xi_0}$ and $R_j = \Delta_j \backslash \Delta_{j-1}$. Then*

$$\sup_{\xi\in R_j} K(x,\xi) \leqslant \frac{C_{x_0} C 2^{-\gamma j}}{\omega(\Delta_j)},$$

with $\gamma, C > 0$ depending only on Ω.

Proof. Note that the Harnack chain condition implies that for $\widetilde{x} \in \Omega$, we have

$$\omega^{\widetilde{x}}(\Delta) \approx_{\widetilde{x},x_0} \omega^{x_0}(\Delta).$$

In particular, open sets have comparable measures and, therefore, the measures are comparable. Thus, without loss of generality, we may assume that $\operatorname{dist}(x, \partial\Omega) \geqslant r_1 = C_A^{-1} r_0$.

For $2^j r < r_0$ and $\Delta' \subset R_j$, the idea is to combine Theorem 8.4, Lemma 8.9, and Harnack's inequality to get

$$\omega^x(\Delta') \leqslant C_A \omega^{X^{\text{in}}_{\Delta_j}}(\Delta') \left(\frac{|y - \xi_0|}{2^j r}\right)^\gamma.$$

After that, use the change of pole formula (see Lemma 8.19). For $2^j r > r_0$, just use that the number of R_j is finite and apply Lemma 8.9 and Harnack's inequality. □

Lemma 8.22. *Let $r < r_0$. Then*

$$\sup_{\xi \in \partial\Omega \setminus \Delta_{r,\xi_0}} K(x, \xi) \xrightarrow{x \to \xi_0} 0.$$

Proof. Apply Lemma 8.9 and Harnack's inequality to get

$$\omega^x(\Delta_{\epsilon,\xi}) \leqslant C_A \omega^{x_0}(\Delta_{\epsilon,\xi})$$

Using that $\omega^x(\Delta_{\epsilon,\xi})$ is a harmonic function vanishing at $\Delta_{r/2,\xi_0}$, one can use Theorem 8.4 to get the quantitative estimates of Hölder type. □

Let ξ be a boundary point, $r < r_1$ (so that $x_0 \notin B_r(\xi)$). Consider the intermediate domain $\widetilde{\Omega} = \Omega_{r,\xi}$ as in Lemma 8.15, $x = X^{\text{in}}_{r,\xi}$ with respect to Ω, $y \in B(x, A^{-3/2} r) \setminus B(x, A^{-2} r)$, $\Delta = \Delta(\xi, A^{-2} R)$. Then

$$G_{\widetilde{\Omega}}(y, x) \approx r^{2-n}$$

and, by Theorem 8.13 and Harnack,

$$G_\Omega(y, x_0) \approx r^{2-n} \omega(\Delta).$$

Compare both functions on y using Theorem 8.16 to get

Claim 8.23. *For $z \in B_{A^{-2} r}(\xi) \cap \Omega$*

$$G_{\widetilde{\Omega}}(z, x) \approx \frac{G_\Omega(z, x_0)}{\omega(\Delta)}.$$

By Claim 8.23 and Theorem 8.13 we get

Claim 8.24. *For every surface ball $\Delta' \subset \Delta$, we have*

$$\omega^x_{\widetilde{\Omega}}(\Delta') \approx \frac{\omega(\Delta')}{\omega(\Delta)}$$

Finally, from Claim 8.24 and Lemma 8.19 (maybe it is enough to use $5r$-covering) we obtain

Claim 8.25. *For every Borel set $E \subset \Delta$, we have*

$$\omega^x_{\widetilde{\Omega}}(E) \approx \frac{\omega(E)}{\omega(\Delta)}.$$

8.5 Global boundary behavior of harmonic functions in CDC uniform domains

An immediate consequence of Theorem 8.16 is the following global boundary Harnack principle.

Theorem 8.26 (Global boundary Harnack principle). *Let Ω be a CDC uniform domain, and let V be an open set. For any compact set $K \subset V$, there exists a constant C such that for all positive harmonic functions u, v in Ω that vanish continuously on $\partial\Omega \cap V$, then for every $x, y \in \Omega \cap K$*

$$C^{-1}\frac{u(x)}{v(x)} \leqslant \frac{u(y)}{v(y)} \leqslant C\frac{u(x)}{v(x)}.$$

Lemma 8.27. *Let Ω be a CDC uniform domain. Let u be harmonic and positive in Ω, with $\xi \in \partial\Omega$. If u vanishes continuously on $\partial\Omega\backslash\Delta$ where $\Delta := \Delta_{r,\xi}$ with $r < r_0$, then for all $x \in \Omega\backslash B_{2r}$,*

$$u(x) \approx_A u(X_\Delta^{\text{in}})\omega^x(\Delta).$$

Proof. Cover $\partial B_{2r}(\xi) \cap \partial\Omega$ with balls $B_\rho(\xi_j)$ of radius $\rho := C_A^{-2}\min\{r_0, r\}$, where C_A is the constant from Theorem 8.16, so that every $x \in B_\rho(\xi_j)$ satisfies that

$$\frac{u(x)}{\omega^x(\Delta)} \overset{\text{T } 8.16}{\approx} \frac{u(X_{\rho,\xi_j}^{\text{in}})}{\omega^{X_{\rho,\xi_j}^{\text{in}}}(\Delta)} \overset{\text{Harnack}}{\approx} \frac{u(X_\Delta^{\text{in}})}{\omega^{X_\Delta^{\text{in}}}(\Delta)} \overset{\text{L } 8.7}{\approx} u(X_\Delta^{\text{in}}).$$

The estimates extend to $x \in \partial B_{2r}(\xi) \cap \overline{\Omega}$ by the Harnack inequality, and the lemma follows by the maximum principle. $\square$

A *kernel function* in Ω at $\xi \in \partial\Omega$ is a positive harmonic function u in Ω that vanishes continuously on $\partial\Omega\backslash\{\xi\}$ and such that $u(x_0) = 1$. Note that $\limsup_{x\to\xi} u(x) = \infty$. Otherwise $\{\xi\}$ would have positive harmonic measure, and this cannot happen (one can check that sets with zero capacity have always zero harmonic measure).

Lemma 8.28. *Let Ω be a CDC uniform domain. There exists a kernel function u at every boundary point.*

Proof. Let $\xi \in \partial\Omega$, and denote

$$u_m(x) = \frac{\omega^x(\Delta_{2^{-m},\xi})}{\omega(\Delta_{2^{-m},\xi})},$$

so that $u_m(x_0) = 1$.

By Harnack's inequality and Lemma 2.14 there is a partial $u_{m_j} \xrightarrow{j\to\infty} u$ uniformly on compact subsets of Ω, with u positive and harmonic in Ω.

Let $r < r_0$ and let $\Delta := \Delta_{r,\xi}$. For j big enough, we get

$$u_{m_j}(x) \overset{\text{L } 8.27}{\approx}_A u_{m_j}(X_\Delta^{\text{in}})\omega^x(\Delta) \overset{\text{Harnack}}{\approx}_{r,x_0,A} u_{m_j}(x_0)\omega^x(\Delta) = \omega^x(\Delta)$$

for every $x \in \Omega \backslash B_{2r}$. Therefore,

$$u(x) \approx \omega^x(\Delta) \quad \text{for every } x \in \Omega \backslash B_{2r}$$

and therefore u vanishes in $\partial\Omega \backslash 2\Delta$. The lemma follows letting $r \to 0$. $\square$

Lemma 8.29. *Let Ω be a CDC uniform domain. Assume that u_1 and u_2 are kernel functions for Ω at ξ. Then*

$$u_1(x) \approx_A u_2(x) \quad \textit{for every } x \in \Omega.$$

Proof. Let $r > 0$ be small enough and $\Delta := \Delta_{r,\xi}$. By Lemma 8.27

$$1 = u_j(x_0) \approx_A u_j(X_\Delta^{\text{in}})\omega(\Delta).$$

and

$$u_j(x) \approx_A u_j(X_\Delta^{\text{in}})\omega^x(\Delta).$$

Therefore

$$u_1(x) \approx_A \frac{\omega^x(\Delta)}{\omega(\Delta)} \approx_A u_2(x)$$

for all $x \in \Omega \backslash B_{2r}(\xi)$ for r small enough. $\square$

Theorem 8.30. *Let Ω be a CDC uniform domain. For every boundary point the kernel function is unique.*

Proof. We follow the approach of [CFMS81, Theorem 3.1]. Assume that u_1, u_2 are kernel functions for Ω at $\xi \in \partial\Omega$. Then, for $x \in \Omega$ we have $\frac{u_1(x)}{u_2(x)} \leqslant C_0 \frac{u_1(x_0)}{u_2(x_0)}$ by Lemma 8.29. Therefore

$$u_1 \leqslant C_0 u_2. \tag{8.8}$$

holds for every pair of kernel functions u_1, u_2.

If $C_0 = 1$ the lemma follows, so we may assume that $C_0 > 1$. In that case,

$$\frac{C_0}{C_0 - 1} u_2 - \frac{1}{C_0 - 1} u_1 = u_2 + \frac{1}{C_0 - 1}(u_2 - u_1)$$

is a kernel function as well. Therefore (8.8) holds for this function, namely

$$u_1 \leqslant C_0 \left(u_2 + \frac{1}{C_0 - 1}(u_2 - u_1) \right)$$

so

$$\frac{C_0}{C_0 - 1} \left(u_2 + \frac{1}{C_0 - 1}(u_2 - u_1) \right) - \frac{1}{C_0 - 1} u_1 = u_2 + \frac{2}{C_0 - 1}(u_2 - u_1) + \frac{1}{(C_0 - 1)^2}(u_2 - u_1)$$

is also a kernel function.

In general, if

$$u_2 + \left(\frac{k}{C_0 - 1} + t_k \right)(u_2 - u_1) \tag{8.9}$$

is a kernel function, then (8.8) holds for this function as well, namely

$$u_1 \leqslant C_0 \left(u_2 + \left(\frac{k}{C_0 - 1} + t_k \right) (u_2 - u_1) \right),$$

so

$$\frac{C_0}{C_0 - 1} \left(u_2 + \left(\frac{k}{C_0 - 1} + t_k \right) (u_2 - u_1) \right) - \frac{1}{C_0 - 1} u_1$$
$$= u_2 + \frac{k+1}{C_0 - 1}(u_2 - u_1) + \frac{k + t_k(C_0 - 1)}{(C_0 - 1)^2}(u_2 - u_1)$$

is also a kernel function. By induction, a kernel function as in (8.9) can be obtained for every $k \in \mathbb{N}$ with $t_k > 0$.

Now, applying (8.8) again, we get that for every k

$$u_2 + \frac{k}{C_0 - 1}(u_2 - u_1) \leqslant u_2 + \left(\frac{k}{C_0 - 1} + t_k \right) (u_2 - u_1) \leqslant C_0 u_2.$$

This implies that $u_2 \leqslant u_1$. But interchanging the roles of u_1 and u_2 we obtain the converse inequality and the lemma follows.

□

Definition 8.31. A *non-tangential region* at $\xi \in \partial\Omega$ is denoted by

$$\Gamma_\alpha(\xi) := \{ x \in \Omega : \, |x - \xi| < (1 + \alpha) \mathrm{d}_\Omega(x) \}.$$

The *non-tangential maximal function* is denoted

$$\mathcal{N}_\alpha u(\xi) := \sup_{\Gamma_\alpha(\xi)} |u|$$

for u defined in Ω.

Usually the value of α is of little importance when dealing with harmonic functions because typically the boundedness of the operator $\mathcal{N}_\alpha$ does not depend on α. Therefore we usually denote Nu for some value of α.

The centered Hardy-Littlewood maximal function with respect to ω is defined as

$$M_\omega f(\xi) := \sup_r \fint_{\Delta_{r,\xi}} |f| \, d\omega$$

for every $f \in L^1_{\mathrm{loc}}(\omega)$, and, more generally,

$$M_\omega \mu(\xi) := \sup_r \frac{\mu(\Delta_{r,\xi})}{\omega \Delta_{r,\xi}}$$

for every $\mu \in \mathcal{M}(\partial\Omega) := \{\text{Finite Radon measures supported in } \partial\Omega\}$.

We say that u converges to f *non-tangentially at* ξ if for any α,

$$\lim_{\Gamma_\alpha(\xi) \ni x \to \xi} u(x) = f(\xi).$$

The maximal function satisfies a weak-$(1,1)$ estimate, i.e.

$$\omega\{M_\omega f > \lambda\} \leqslant \frac{C}{\lambda}\|f\|_{L^1(\omega)}, \tag{8.10}$$

and for every $1 < p \leqslant \infty$

$$\|M_\omega f\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}, \tag{8.11}$$

see [Mat95, Theorem 2.19], for instance. In fact the weak estimate also holds for Radon measures, by the same covering arguments used to prove the weak $(1,1)$ bounds:

Lemma 8.32. *For $\mu \in \mathcal{M}(\partial\Omega)$ we have*

$$\omega\{M_\omega \mu > \lambda\} \leqslant \frac{C}{\lambda}|\mu(\partial\Omega)|. \tag{8.12}$$

Theorem 8.33. *Let Ω be a CDC uniform domain. If μ is a finite Borel measure on $\partial\Omega$ with Radon-Nykodim decomposition (see [Mat95, Theorem 2.17]) $d\mu = f\,d\omega + d\nu$, where ν is mutually singular with ω, and $u_\mu(x) := \int K(x,\zeta)\,d\mu(\zeta)$, then $\mathcal{N}_\alpha u_\mu \leqslant C_\alpha M_\omega \nu$, and u converges to f non-tangentially at ω-a.e. boundary point.*

Proof. Consider the operator $\widetilde{N}$ defined on $\mathcal{M}(\partial\Omega)$ by

$$\widetilde{N}\mu := \mathcal{N}_\alpha u_\mu,$$

where α is fixed (and the constants may depend on its value). First we claim that

$$\widetilde{N}\mu \leqslant C M_\omega \mu. \tag{8.13}$$

Indeed, let us assume that $y \in \Gamma_\alpha(\xi)$, with $\operatorname{dist}(y,\xi) \leqslant r \ll r_0$, and let $\Delta := \Delta_{r,\xi}$. By the Harnack inequality we have that

$$u_\mu(y) \overset{\text{Harnack}}{\approx_{\alpha,A}} u_\mu(X_\Delta^{\text{in}}) = \int K(X_\Delta^{\text{in}},\zeta)\,d\mu(\zeta).$$

Decomposing as in Lemma 8.21 we get

$$u_\mu(y) \lesssim \sum_j \int_{R_j} K(X_\Delta^{\text{in}},\zeta)\,d\mu(\zeta) \overset{\text{L } 8.21}{\lesssim_A} \sum_j \frac{2^{-\gamma_A j}}{\omega(\Delta_j)} \int_{R_j} d\mu(\zeta) \leqslant M_\omega\mu(\xi) \sum_j 2^{-\gamma_A j} \lesssim_A M_\omega \mu(\xi).$$

Since $\widetilde{N}\mu(\xi) = \sup_{y\in\Gamma_\alpha(\xi)} |u_\mu(y)|$, estimate (8.13) follows.

Note that combining (8.12) with (8.13) we obtain the weak type estimate

$$\omega\{\widetilde{N}\mu > \lambda\} \leqslant \frac{C}{\lambda}|\mu(\partial\Omega)|. \tag{8.14}$$

It remains to compute the nontangential limit of u_μ, proving that it coincides with f at ω-a.e. boundary point. Let us write $\text{n.t.}\limsup_{y\to\xi} := \limsup_{\Gamma_\alpha(\xi)\ni y\to\xi}$. Given $\varepsilon, \lambda > 0$, we want to prove that

$$\boxed{0_{\mu,\lambda}} := \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_\mu(y) - f(\xi)| > \lambda\right\} < \varepsilon. \tag{8.15}$$

First we will compute the case $\nu = 0$. Whenever $f \in C(\partial\Omega)$, we have that

$$u_f(x) = \int f(\zeta) K(x,\zeta)\, d\omega(\zeta) = \int f(\zeta)\, d\omega^x(\zeta) = Hf(x),$$

so

$$u_f(x) \to f(\xi) \text{ as } x \to \xi \in \partial\Omega \tag{8.16}$$

by Wiener regularity.

For $f \in L^1(\partial\Omega)$, consider simple functions $\{f_n\}_n$ converging in $L^1(\omega)$ to f. Since ω is a Radon measure, we can find continuous functions $\{f_{n,j}\}_j$ converging to f in $L^1(\omega)$. By a diagonal argument, we find a sequence of continuous functions $\{g_n\}_n$ converging in $L^1(\omega)$ to f.

Using the triangle inequality, we can decompose the left-hand side of (8.15) as

$$\begin{aligned}\boxed{\boxed{0_{f\omega,\lambda}}} &\leqslant \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_f(y) - u_{g_n}(y)| > \frac{\lambda}{3}\right\} \\ &\quad + \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_{g_n}(y) - g_n(\xi)| > \frac{\lambda}{3}\right\} \\ &\quad + \omega\left\{|g_n(\xi) - f(\xi)| > \frac{\lambda}{3}\right\} = \boxed{1} + \boxed{2} + \boxed{3}.\end{aligned}$$

By (8.10),

$$\boxed{3} \leqslant \frac{C}{\lambda}\|f - g_n\|_{L^1(\omega)}.$$

The continuity of g_n implies that $u_{g_n} = H_{g_n}$. By (8.16) Since Ω is Wiener regular, we get that

$$\boxed{2} = 0.$$

Finally,

$$\boxed{1} \leqslant \omega\left\{\widetilde{N}(f - g_n)(\xi) > \frac{\lambda}{3}\right\} \overset{(8.14)}{\leqslant} \frac{C}{\lambda}\|f - g_n\|_{L^1(\omega)}.$$

Combining the three estimates, we obtain

$$\omega\left\{\left|\text{n.t.}\limsup_{y\to\xi} u_f(y) - f(\xi)\right| > \lambda\right\} \leqslant \frac{C}{\lambda}\|f - g_n\|_{L^1(\omega)} < \varepsilon$$

for n big enough (depending on λ and f), so (8.15) is settled whenever $\nu = 0$.

If $\nu \neq 0$, we write

$$\begin{aligned}\boxed{\boxed{0_{\mu,\lambda}}} &\leqslant \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_\mu(y) - f(\xi)| > \lambda/2\right\} + \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_\nu(y) - 0| > \lambda/2\right\} \\ &= \boxed{\boxed{0_{f\omega,\lambda/2}}} + \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_\nu(y) - 0| > \lambda\right\}.\end{aligned}$$

Let $E \subset \partial\Omega$ be a measurable given by the Radon-Nykodim decomposition, i.e. so that $\omega(E) = 0 = \nu(\partial\Omega\backslash E)$. Since ν, ω are Radon measures, we can find a compact set $K \subset E$ and an open set $U \supset E$ so that $\nu(E\backslash K) < \delta$ and $\omega(U) < \delta$.

Now,

$$\boxed{\boxed{0_{\mu,\lambda}}} \leqslant \boxed{\boxed{0_{f\omega,\lambda/2}}} + \omega\left\{\text{n.t.}\limsup_{y\to\xi}\left|u_{\nu|_{E\backslash K}}(y)\right| > \lambda/4\right\} + \omega\left\{\text{n.t.}\limsup_{y\to\xi}\left|u_{\nu|_K}(y)\right| > \lambda/4\right\}$$
$$= \boxed{\boxed{0_{f\omega,\lambda/2}}} + \boxed{\boxed{4}} + \boxed{\boxed{5}}.$$

We have already shown that $\boxed{\boxed{0_{f\omega,\lambda/2}}} \leqslant \varepsilon/3$ for every $\varepsilon > 0$. The weak estimate (8.14) implies that

$$\boxed{\boxed{4}} \leqslant \omega\left\{\text{n.t.}\limsup_{y\to\xi}\left|\tilde{N}\nu_{E\backslash K}(y)\right| > \lambda/4\right\} \leqslant \frac{C}{\lambda}\nu(E\backslash K) \leqslant \frac{C}{\lambda}\delta.$$

Note also that

$$\boxed{\boxed{5}} \leqslant \omega(U) + \omega\left\{\xi \in U^c : \text{n.t.}\limsup_{y\to\xi}\left|u_{\nu|_K}(y)\right| > \lambda/4\right\}.$$

We claim that $r := \operatorname{dist}(K, U^c) > 0$. Indeed, for every $x \in K$ there exists a ball B_x so that $2B_x \subset U$ and by compactness, there is a finite collection of balls $\{B_j\}$ so that $K \subset \bigcup B_j$ with $2B_j \subset U$. Since the collection is finite, it has a minimal radius, which is a lower bound for the distance, implying the claim.

Now, for every $\xi \in U^c$, $y \in \Gamma_\alpha(\xi)$ we have that

$$u_{\nu|_K}(y) := \int_K K(x,\zeta)\,d\nu(\zeta) \leqslant \nu(K) \sup_{\zeta\in\partial\Omega\backslash\Delta_{r,\xi}} K(y,\zeta) \xrightarrow[y\to\xi]{\text{L } 8.22} 0,$$

so

$$\omega\left\{\xi \in U^c : \limsup_{y\to\xi}\left|u_{\nu|_K}(y)\right| > \lambda/4\right\} = 0.$$

Combining all the estimates, we get

$$\boxed{\boxed{0_{\mu,\lambda}}} \leqslant \varepsilon/3 + \frac{C}{\lambda}\delta + \delta < \varepsilon$$

as long as we take δ small enough. □

Remark 8.34. Note that we can say that $u_f = u_{f\omega}$ is the harmonic extension of f.

9 Harmonic measure in the complex plane

9.1 Harmonic measure and conformal mappings

One of the basic facts that makes the study of harmonic measure in the plane different from higher dimensions is the availability of many formal mappings in the plane and the good behavior of harmonic measure under conformal mappings.

Proposition 9.1. *Let $\Omega, \Omega' \subset \mathbb{C}$ be bounded Wiener regular domains, and let $\varphi : \overline{\Omega} \to \overline{\Omega'}$ be a continuous surjective map such that $\varphi(\partial\Omega) = \partial\Omega'$. Suppose also that φ is holomorphic in Ω, and let $x \in \Omega$ and $x' = \varphi(x)$. Denote by ω_Ω and $\omega_{\Omega'}$ the respective harmonic measures for Ω and Ω'. Then,*

$$\omega_{\Omega'}^{x'} = \varphi_\# \omega_\Omega^x.$$

In particular, for any Borel set $A \subset \partial\Omega'$, we have $\omega_{\Omega'}^{x'}(A) = \omega_\Omega^x(\varphi^{-1}(A))$.

Recall that give a continuous map $\varphi : G \to G'$ and a Borel measure μ on G, then the image measure $\varphi_\#\mu$ is a measure on G' defined by

$$\varphi_\#\mu(A) = \mu(\varphi^{-1}(A))$$

for any Borel set $A \subset G'$. Then, for any Borel function $f : G' \to \mathbb{R}$, it holds

$$\int f \circ \varphi \, d\mu = \int f \, d\varphi_\#\mu.$$

See Chapter 1 from [Mat95], for more details.

Proof. Let $f : \partial\Omega' \to \mathbb{R}$ be an arbitrary continuous function and let $u_{\Omega',f}$ be its harmonic extension to Ω'. Then $u_{\Omega',f} \circ \varphi$ is continuous in $\overline{\Omega}$, harmonic in Ω, and it it coincides with the harmonic extension of $f \circ \varphi : \Omega \to \mathbb{R}$, i.e., $u_{\Omega',f} \circ \varphi = u_{\Omega, f\circ\varphi}$. Therefore,

$$\int f \, d\omega_{\Omega'}^{x'} = u_{\Omega',f}(x') = u_{\Omega',f}(\varphi(x)) = u_{\Omega,f\circ\varphi}(x) = \int f \circ \varphi \, d\omega_\Omega^x = \int f \, d\varphi_\#\omega_\Omega^x.$$

Since this holds for any continuous function f on $\partial\Omega'$, the proposition follows. □

Corollary 9.2. *Let $\Omega \subset \mathbb{R}^d$ be simply connected. Let $\varphi : B_1(0) \to \Omega$ be a conformal mapping which extends to a continuous map $\bar{B}_1(0) \to \overline{\Omega}$. Then*

$$\omega_\Omega^{\varphi(0)} = \frac{1}{2\pi} \varphi_\# \mathcal{H}^1|_{\partial B_1(0)}.$$

Proof. By topological arguments, $\varphi(\partial B_1(0)) = \partial\Omega$. By Proposition 9.1, we deduce that

$$\omega_\Omega^{\varphi(0)} = \varphi_\# \omega^0_{B_1(0)} = \frac{1}{2\pi}\, \varphi_\# \mathcal{H}^1|_{\partial B_1(0)}.$$

□

Remark that, by Cathédory's theorem, if Ω is a Jordan domain, then the conformal mapping $\varphi : B_1(0) \to \Omega$ extends continuously to $\partial B_1(0)$, and thus the preceding corollary applies. Notice also that whenever we know how to find the conformal map $\varphi : B_1(0) \to \Omega$, we know how to find the harmonic measure ω_Ω.

9.2 The Riesz brothers theorem

In this section and the following one in this chapter we state some important theorems about harmonic measure for domains in the complex plane. For the moment, we skip the proofs.

Theorem 9.3 (F. and M. Riesz Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a simply connected domain such that $\partial\Omega$ has finite length, and let $\varphi : \mathbb{D} \to \Omega$ be conformal. Then, for any Borel set $A \subset \partial\Omega$,*

$$\omega(A) = 0 \iff \mathcal{H}^1(A) = 0.$$

Notice that the preceding result. does not depend on the precise pole for harmonic measure, since harmonic measures for different poles (and the same domain) are mutually absolutely continuous. We also have the following version result in terms of the Hardy space $H^1(\mathbb{D})$.

Theorem 9.4 (F. and M. Riesz Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a Jordan domain and let $\varphi : \mathbb{D} \to \Omega$ be conformal. Then $\partial\Omega$ has finite length if and only if $\varphi' \in H^1(\mathbb{D})$. If $\varphi \in H^1(\mathbb{D})$, then*

$$\|\varphi'\|_{H^1(\mathbb{D})} = \mathcal{H}^1(\partial\Omega)$$

and for any Borel set $A \subset \partial\mathbb{D}$,

$$\mathcal{H}^1(\varphi(A)) = \frac{1}{2\pi}\int_A |\varphi'|\, d\mathcal{H}^1.$$

In these notes we do not include the proofs of these important theorems (for the moment). See Chapter VI from [GM05], for example.

9.3 The dimension of harmonic measure in the plane

The dimension of a Borel measure μ in $\mathbb{R}^d$ is defined as follows.

$$\dim(\mu) = \inf\{\dim(G) : G \subset \mathbb{R}^d \text{ Borel }, \mu(G^c) = 0\}.$$

This does not have to be confused with the dimension of suppμ. For example, let $\mathbb{Q} = \{q_k\}_{k\geqslant 1}$ be the set of all rational numbers, ordered in some way. Then consider the following measure in $\mathbb{R}$:

$$\mu = \sum_{k\geqslant 1} 2^{-k}\,\delta_{q_k},$$

where δ_{q_k} is the Dirac delta on q_k. It is immediate to check that $\dim \mu = 0$, while supp$\mu = \mathbb{R}$ and so $\dim(\text{supp}\mu) = 1$.

For simply connected domains Makarov [Mak85] proved in 1985 the following:

Theorem 9.5. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then $\dim \omega = 1$. Further, $\omega(E) = 0$ for any set $E \subset \partial\Omega$ with Hausdorff dimension $\dim(E) < 1$.*

Remark that the dimension of harmonic measure is independent of the chosen pole in the domain. For arbitrary planar domains, Jones and Wolff proved the following result in 1988 [JW88]:

Theorem 9.6. *For any open set $\Omega \subset \mathbb{C}$, the associated harmonic measure satisfies*

$$\dim(\omega) \leqslant 1.$$

Observe that the boundary of a planar domain may have Hausdorff dimension larger than 1. This is the case, for example, of the Jordan domain enclosed by the von Koch snowflake. It is well known that this curve has dimension $\log 4/\log 3$. Further, it is easy to check that, because of connectedness, the (closed) support of harmonic measure coincides with the full boundary for any domain Ω. In spite of this fact, the dimension of harmonic measure is always at most 1. So there is a set $G \subset \partial\Omega$ with $\dim G \leqslant 1$ with full harmonic measure. Clearly, such set G must be dense in $\partial\Omega$.

The Jones-Woff theorem was sharpened by Wolff [Wol93] a few years later:

Theorem 9.7. *For any open set $\Omega \subset \mathbb{C}$, there exists a set $E \subset \partial\Omega$ with σ-finite length and full harmonic measure.*

The rest of this chapter is devoted to the proof of the Jones-Wolff Theorem 9.6. We will not prove the other theorems by Makarov and Wolff mentioned above.

9.4 Preliminary reductions for the proof of the Jones–Wolff Theorem

We will prove Theorem 9.6 assuming $\partial\Omega$ to be bounded, since we have defined harmonic measure in this case. The case where $\partial\Omega$ is unbounded easily follows from the bounded case (once harmonic measure is properly defined). We will show first below that we may assume that Ω is Wiener regular.

Lemma 9.8. *To prove Theorem 9.6, it suffices to prove it when Ω is Wiener regular.*

Proof. For each $\varepsilon = 1/k$, let $\widetilde{\Omega}_k$ be the Wiener regular open set constructed in Proposition 6.36 (denoted by $\widetilde{\Omega}$ there). Also, denote by F_k the union of the balls B_i, $i \in I$, in the construction of $\widetilde{\Omega}_k$. Suppose $k \geqslant k_0$ small enough so that $p \in \widetilde{\Omega}_k$. Denote by ω and ω_k the respective harmonic measures for Ω and $\widetilde{\Omega}_k$. By Theorem 9.6 applied to $\widetilde{\Omega}_k$, there exists a subset $G_k \subset \partial\widetilde{\Omega}_k$ with full harmonic measure ω_k^p and with Hausdorff dimension at most 1. Since $\widetilde{\Omega}_k \subset \Omega$, by the maximum principle (see Lemma 5.28),

$$\omega^p(\partial\Omega \cap \partial\widetilde{\Omega}_k \backslash G_k) \leqslant \omega_k^p(\partial\Omega \cap \partial\widetilde{\Omega}_k \backslash G_k) = 0.$$

Since $\partial\Omega = (F_k \cap \partial\Omega) \cup (\partial\widetilde{\Omega}_k \cap \partial\Omega)$, the set $(F_k \cap \partial\Omega) \cup G_k$ has full harmonic measure ω^p for each $k \geqslant k_0$. So $\bigcap_{k\geqslant k_0}((F_k \cap \partial\Omega) \cup G_k)$ has also full measure ω^p. Now notice that

$$\bigcap_{k\geqslant k_0}((F_k \cap \partial\Omega) \cup G_k) \subset \bigcap_{k\geqslant k_0}(F_k \cap \partial\Omega) \cup \bigcup_{k\geqslant k_0} G_k.$$

The set $G := \bigcup_{k\geqslant k_0} G_k$ has Hausdorff dimension at most 1, and $F := \bigcap_{k\geqslant k_0}((F_k \cap \partial\Omega) \cup G_k)$ has zero capacity, because the $\operatorname{Cap}(F_k) \leqslant 1/k$ for all k. In particular, $\mathcal{H}^1(F) = 0$. So $G \cup F$ has full harmonic measure ω^p and has Hausdorff dimension at most 1. Thus, $\dim \omega^p \leqslant 1$. □

The next reduction is the following.

Lemma 9.9. *To prove Theorem 9.6, we may assume that Ω is an unbounded domain with compact boundary and that the pole for harmonic measure is ∞.*

Proof. We may assume that Ω is connected because the harmonic measure for Ω with pole at $p \in \Omega$ coincides with the harmonic measure for the component of Ω containing p, with pole at p.

Suppose now that $p \neq \infty$. Consider the map $\varphi(z) = 1/(z-p)$. This is a conformal mapping of the Riemann sphere, and by Proposition 9.1 (which also holds for unbounded domains with compact boundary), denoting $\Omega' = \varphi(\Omega)$, we have

$$\omega_{\Omega'}^{\infty} = \varphi_{\#}\omega_{\Omega}^{p}.$$

Hence, assuming that Theorem 9.6 holds for $\omega_{\Omega'}^{\infty}$, we infer that there exists some subset $G \subset \partial\Omega'$ with $\dim_{\mathcal{H}} G \leqslant 1$ and full measure $\omega_{\Omega'}^{\infty}$. Then $\varphi^{-1}(G)$ has full measure ω_{Ω}^{p} and, since $\varphi|_{\partial\Omega} : \partial\Omega \to \partial\Omega'$ is bilipschitz, we also have $\dim_{\mathcal{H}} \varphi^{-1}(G) \leqslant 1$. □

Recall that in Theorem 7.28 we showed the following properties for the harmonic measure and for the Green function with pole at ∞, for any unbounded Wiener regular domain Ω with compact boundary:

(i) For every $\varphi \in C_c^{\infty}(\mathbb{R}^2)$,

$$\int_{\Omega} G^{\infty}(z)\, \Delta\varphi(z)\, dm(z) = \int \varphi\, d\omega^{\infty}.$$

(ii) ω^∞ coincides with the equilibrium measure of $\partial\Omega$ and moreover, for every $z \in \Omega$,

$$G^\infty(z) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} - \frac{1}{2\pi}\int_{\partial\Omega} \log\frac{1}{|\xi - z|}\, d\omega^\infty(\xi). \tag{9.1}$$

Recall also that, for any compact set $E \subset \mathbb{C}$,

$$\frac{1}{\mathrm{Cap}_W(E)} = \inf_\mu I(\mu) = \inf_\mu \int \mathcal{E} * \mu \, d\mu,$$

where the infimum is taken over all probability measure supported on E. The number

$$\gamma_E = \frac{1}{\mathrm{Cap}_W(E)}$$

is called the Robin constant of E. So we have $\mathrm{Cap}_L(E) = e^{-2\pi\gamma_E^{-1}}$.

Lemma 9.10. *To prove Theorem 9.6, it is enough to prove that for any $\varepsilon > 0$ the following holds:*

$$\text{For each } \eta > 0 \text{ there is a set } A \subset K \text{ with } \mathcal{H}^{1+\varepsilon}_\infty(A) < \eta \text{ and } \omega(K \setminus A) < \eta. \tag{9.2}$$

Proof. The statement (9.2) implies that for $\eta > 0$ there is a set $A \subset \partial\Omega$ with $\mathcal{H}^{1+\varepsilon}_\infty(A) < \eta$ and $\omega(\partial\Omega\setminus A) = 0$, which in turn implies that there is $A \subset \partial\Omega$ with $\mathcal{H}^{1+\varepsilon}_\infty(A) = 0$ and $\omega(\partial\Omega\setminus A) = 0$. Now taking $\varepsilon_n \to 0$, one gets sets $A_n \subset \partial\Omega$ with $\mathcal{H}^{1+\varepsilon_n}_\infty(A_n) = 0$ and $\omega(\partial\Omega\setminus A_n) = 0$. Letting $G = \bigcap_n A_n$ we have $\mathcal{H}^{1+\varepsilon_n}_\infty(G) = 0$, for each n, which gives that the Hausdorff dimension of G is less than or equal to one, and $\omega(\partial\Omega\setminus G) = 0$. □

Sketch of the proof of Theorem 9.6

One makes a reduction to the case in which $K := \partial\Omega$ is a finite union of pieces of small diameter and rather well separated. Then one constructs an auxiliary compact K^*, which is a finite union of closed discs, using two special modification methods, which one calls "the disc construction" and the "annulus construction". It is crucial to compare the harmonic measure associated with Ω and that associated with the new domain $\Omega^* = \mathbb{C}^*\setminus K^*$. This is simple for the annulus construction, but much more delicate for the disc construction; Lemma 9.11 below takes care of this issue. The gradient of the Green function g of Ω^* with pole at ∞ can be estimated on some special curves surrounding K^* and contained in level sets of g. All these ingredients allow to estimate the harmonic measure of Ω in terms of the integral of the gradient of g on these curves. Lemma 9.14 is the main tool to end the proof estimating this integral in the appropriate way. An ingredient in the proof of Lemma 9.14 yields in the limiting case, assuming $\partial\Omega$ smooth, the formula

$$\int_{\partial\Omega} |\partial_\nu g| \log|\partial_\nu g| \, ds > -c_0,$$

where g is now the Green function of Ω with pole at ∞, ν is the outer unit normal to $\partial\Omega$ and $c_0 > 0$. By Proposition 7.14, the harmonic measure is (in the smooth case)

$$d\omega^\infty(z) = -\partial_\nu g(z)\, ds.$$

Assume that at the point z the "dimension" of ω^∞ at z is $d(z)$, which means that $\omega(B(z,r)) \sim r^{d(z)}$. Since

$$|\partial_\nu g(z)| = \lim_{r\to 0} \frac{\omega^\infty(B(z,r))}{2r},$$

we have

$$\lim_{r\to 0} \int_{\partial\Omega} (d(z)-1)\log(2r)\, d\omega^\infty(z) \geqslant -c_0.$$

From this fact, we deduce that the integrand in the left hand side of the preceding identity does not tend ot $-\infty$ in a set of positive measure as $r \to 0$, that is $d(z) \leqslant 1$ for ω^∞-a.e. $z \in \partial\Omega$, and so, ω^∞ lives in a set of dimension not greater than 1.

From now on, in the rest of this chapter, unless otherwise stated, we assume that Ω is a Wiener regular unbounded domain with compact boundary, and we denote by ω its harmonic measure with pole at ∞. We will also write $K = \partial\Omega$.

9.5 The disc and the annulus construction

Let us start with the disc construction.

Disc construction

Fix $\varepsilon > 0$. Let Q be a square with sides parallel to the axes and side length $\ell = \ell(Q)$ and set $E = Q \cap K$. Replace E by a closed disc B with the same center as Q and radius $r(B)$ defined by

$$r(B) = \frac{1}{2}\frac{\mathrm{Cap}_L(E)^{1+\varepsilon}}{\ell^\varepsilon} = \frac{1}{2}\frac{e^{-\gamma_E(1+\varepsilon)}}{\ell^\varepsilon}. \tag{9.3}$$

So we get a new compact set $\widetilde{K} = (K\backslash E) \cup B$, a new domain $\widetilde{\Omega} = \mathbb{C}^*\backslash\widetilde{K} = (\Omega \cup E)\backslash B$ and a new harmonic measure $\widetilde{\omega} = \widetilde{\omega}^\infty_{\widetilde{\Omega}}$.

Note that $B \subset Q$. In fact, since the logarithmic capacity of a disc is the radius

$$\mathrm{Cap}_L(E) \leqslant \frac{\sqrt{2}}{2}\ell,$$

so that

$$r(B) \leqslant \frac{1}{2}\frac{\left(\sqrt{2}/2\right)^{1+\varepsilon}\cdot \ell^{1+\varepsilon}}{\ell^\varepsilon} = \frac{\ell}{2}\left(\sqrt{2}/2\right)^{1+\varepsilon} \leqslant \ell/2.$$

Annulus construction

Let Q be a square with sides parallel to the axis and take the square RQ, where R is a number larger than 1 that will be chosen later. One has to think that R is very large. Delete $K \cap (RQ\backslash Q)^0$ from K to obtain a new domain $\widetilde{\Omega} = \Omega \cup (RQ\backslash Q)^0$ and a new harmonic measure $\widetilde{\omega} = \omega_{\widetilde{\Omega}}$.

It is important to have some control on the harmonic measure of the new domain obtained after performing the disc or the annulus construction. For the annulus this is easy: any part of K which has not been removed has larger or equal harmonic measure. In other words, if A satisfies $A \cap (RQ\backslash Q) = \varnothing$, then $\widetilde{\omega}(A) \geqslant \omega(A)$. This is a consequence of the fact that $A \subset \partial\Omega \cap \partial\widetilde{\Omega}$ and $\Omega \subset \widetilde{\Omega}$ (the domain increases and the set lies in the common boundary).

Estimating the harmonic measure after the disc construction is a difficult task. The result is the following.

Lemma 9.11. *Let Q be a square with sides parallel to the axis. Fix $\varepsilon > 0$ and perform the disc construction for this ε. Assume that $RQ\backslash Q \subset \Omega$. Then there exists a number $R_0(\varepsilon)$ such that for $R \geqslant R_0(\varepsilon)$ one has*

(a) $\widetilde{\omega}(B) \geqslant C(\varepsilon)\,\omega(Q \cap K)$, where $C(\varepsilon)$ is a positive constant depending only on ε.

(b) $\widetilde{\omega}(A) \geqslant \omega(A)$, if $A \subset \partial\Omega\backslash RQ$.

Above $\widetilde{\omega}$ and ω are harmonic measures with pole at ∞.

The proof of Lemma 9.11 will be presented in Section 9.10 and we will use it as a black box in the arguments below.

9.6 The Main Lemma and the domain modification

Let $\Omega = \mathbb{C}^*\backslash K$, $\mathrm{Cap}_L K > 0$ and assume that $K \subset \{|z| < 1/2\}$ (this assumption will be convenient later on, but it is not essential). Fix $\varepsilon > 0$ and let $R > 2 + R_0(\varepsilon)$, R integer, where $R_0(\varepsilon)$ is the constant given by Lemma 9.11. We let M stand for a large constant that will be chosen later and we let ρ be a small constant so that $M \leqslant \log 1/\rho$, and $\rho = \frac{1}{2^N}$, N a positive integer. Consider the grid $\mathcal{G}$ of dyadic squares of side length ρ and lower left corner at the points of the form $\{(m + ni)\rho;\, m, n \in \mathbb{Z}\}$. For each $1 \leqslant p, q \leqslant R$, let $\mathcal{G}_{pq}$ be the family of (closed) squares $Q \in \mathcal{G}$ with $(m, n) \equiv (p, q) \pmod{R \times R}$. Then $\mathcal{G} = \bigcup_{p,q=1}^{R} \mathcal{G}_{pq}$.

Write $K_{pq} = \bigcup_{Q \in \mathcal{G}_{pq}} K \cap Q$, $\Omega_{pq} = \mathbb{C}^*\backslash K_{pq}$, $\omega_{pq}(A) = \omega^{\infty}_{\Omega_{pq}}$. We will show the following:

Main Lemma 9.12. *For any $\varepsilon > 0$ and for any $\eta > 0$, one can choose $R(\varepsilon) > 0$ large enough and $\rho(\eta, \varepsilon)$ small enough so that for all $1 \leqslant p, q \leqslant R$ there is a Borel set $A_{pq} \subset K_{pq}$ satisfying*

$$\mathcal{H}^{1+\varepsilon}_{\infty}(A_{pq}) < \eta \quad \text{and} \quad \omega_{pq}(K_{pq}\backslash A_{pq}) < \eta. \tag{9.4}$$

An important fact about the previous statement is that the constant $R = R(\varepsilon)$ does not depend on η, so that η can be chosen later depending on $R(\varepsilon)$.

Let us see how Lemma 9.10, and so the Jones-Wolff theorem, is derived from Main Lemma 9.12. Write $A = \bigcup_{1 \leqslant p,q \leqslant R} A_{pq}$. Then, we have

$$\mathcal{H}^{1+\varepsilon}_{\infty}(A_{pq}) \leqslant \sum_{1 \leqslant p,q \leqslant R} \mathcal{H}^{1+\varepsilon}_{\infty}(A_{pq}) \leqslant R^2\,\eta,$$

and, by Lemma 5.28,

$$\omega(K\backslash A) \leqslant \sum_{1\leqslant p,q\leqslant R} \omega(K_{pq}\backslash A) \leqslant \sum_{1\leqslant p,q\leqslant R} \omega(K_{pq}\backslash A_{pq}) \leqslant \sum_{1\leqslant p,q\leqslant R} \omega_{pq}(K_{pq}\backslash A_{pq}) \leqslant R^2\eta.$$

Recalling that η can be taken arbitrarily small, for any given R, (9.2) follows.

Our next objective is to prove the Main Lemma 9.12. To this end, we need to perform a domain modification which we proceed to describe.

Domain modification.
From now on we fix p, q and let $\Omega = \Omega_{pq}$, $K = K_{pq}$, $\omega = \omega_{pq}$. We let $\{Q_j\}_j$ be the family of squares in $\mathcal{G}_{pq}$. We remark that, by the construction, for each square Q_j one has $RQ_j\backslash Q \subset \Omega$, so that we will be able to apply Lemma 9.11.

Fix $\varepsilon > 0$ and perform the disc construction for ε in every square Q_j, so that we get a finite family of closed discs $\{B_j\}$, whose union is a compact set K_1, a new domain $\Omega_1 = \mathbb{C}^*\backslash K_1$ and a new harmonic measure $\omega_1 = \omega^{\infty}_{\Omega_1}$.

Next choose a dyadic square Q^1 of largest side $\ell(Q^1)$, not necessarily from $\mathcal{G}_{pq}$, such that

$$\ell(Q^1) \geqslant \rho \quad \text{and} \quad \omega_1(Q^1) \geqslant M\ell(Q^1).$$

If such Q^1 does not exist we stop the domain modification. If Q^1 exists we perform the annulus construction on Q^1 (with constant R) and after this we perform the disc construction on the square Q^1, replacing $K_1 \cap Q^1$ by a disc B^1. So we obtain a new compact K_2, a new domain $\Omega_2 = \mathbb{C}^*\backslash K_2$ and a new harmonic measure $\omega_2 = \omega^{\infty}_{\Omega_2}$.

Now we continue and take Q^2 dyadic with largest side such that $Q^2 \not\subset Q^1$, $\ell(Q^2) \geqslant \rho$ and $\omega_2(Q^2) \geqslant M\ell(Q^2)$. If such Q^2 does not exist we stop. Otherwise we perform the annulus construction on Q^2 but with a special rule: If $B^1 \cap (\partial(RQ^2\backslash Q^2)) \neq \varnothing$, then we do not remove the set $B^1 \cap (RQ^2\backslash Q^2)$ from K_2. The reason for this rule is to get full balls in all cases.

After that we perform the disc construction on Q^2, replacing $K_2 \cap Q^2$ by the corresponding disc B^2, getting a new compact K_3, a new domain Ω_3 and a new harmonic measure ω_3.

We continue this process so that if $K_1 \cap Q^1$, $K_2 \cap Q^2, \dots, K_{n-1} \cap Q^{n-1}$ have been substituted by $B^1, \dots, B^{n-1}$ we choose now (if there exists) a dyadic cube Q^n with largest side so that

$$Q^n \not\subset Q^j, \quad j = 1, \dots, n-1, \quad \ell(Q^n) \geqslant \rho, \quad \omega_n(Q^n) \geqslant M\ell(Q^n).$$

Then (if we do not stop) we perform the annulus construction with respect to Q^n but without removing $B^j \cap (RQ^n\backslash Q^n)$, $j = 1, \dots, n-1$ in case that $B^j \cap (\partial(RQ^n\backslash Q^n)) \neq \varnothing$ (this is the special rule). Finally we perform the disc construction on Q^n, getting B^n, K_{n+1}, Ω_{n+1} and ω_{n+1}.

At each step there are only finitely many candidate dyadic squares, because $\rho \leqslant \ell(Q) \leqslant 1/M$. Since no Q^j can be repeated (because $Q^j \not\subset Q^\ell$, $\ell = 1, \dots, j-1$) the modification process stops after finitely many steps. Let $K^*, \Omega^* = \mathbb{C}\backslash K^*$, $\omega^* = \omega^{\infty}_{\Omega^*}$ be the final outcome so that K^* is the disjoint union of the non removed discs; more precisely,

$$K^* = \bigcup_{k\in S} B^k \cup \bigcup_{j\in T} B_j \quad \text{(some finite sets of indices } S \text{ and } T\text{)},$$

where the B_j are the original discs and the B^k are the new discs produced after performing the annulus and the disc constructions.

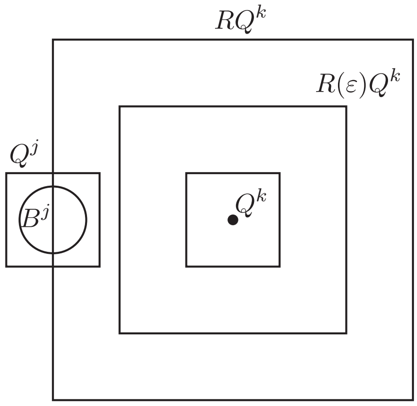
Now we want to prove by means of Lemma 9.11 the following estimates:

$$\omega^*(B_j) \geqslant C(\varepsilon)\,\omega(Q_j), \qquad j \in T, \tag{9.5}$$

$$\omega^*(Q^j) \geqslant C(\varepsilon)\,M\ell(Q^j), \quad j \in S. \tag{9.6}$$

For (9.5) note first that we always have $RQ_j\backslash Q_j \subset \Omega$. Since Q_j has survived all steps we cannot have $RQ^k \supset Q_j$ at some step k. Since RQ^k is a union of dyadic squares, the other possibility is $RQ^k \cap Q_j = \varnothing$ for all k and we can apply both inequalities in Lemma 9.11.

For (9.6), when we select Q^j we have $\omega_j(Q^j) \geqslant M\ell(Q^j)$ and after performing the annulus and the disc constructions, we get $\omega_{j+1}(B^j) \geqslant C(\varepsilon)\,\omega_j(Q^j) \geqslant C(\varepsilon)\,M\ell(Q^j)$. If $k > j$ there are three possibilities: i) $B^j \subset RQ^k\backslash Q^k$, in which case B^j has disappeared and j would not be in S; ii) $B^j \cap (RQ^k\backslash Q^k) = \varnothing$ in which case $\omega_{k+1}(B^j) \geqslant \omega_{j+1}(B^j)$ and iii) $B^j \cap \partial(RQ^k\backslash Q^k) \neq \varnothing$.



In this last case we have $\ell(Q^k) \geqslant \ell(Q^j)$ since otherwise Q^k would had disappeared. But now since $R = 2 + R_0(\varepsilon)$ we get that $B^j \cap (R_0(\varepsilon)Q^k\backslash Q^k) = \varnothing$ and so $\omega_{k+1}(B^j) \geqslant \omega_{j+1}(B^j)$ by Lemma 9.11 part b). At the end we obtain

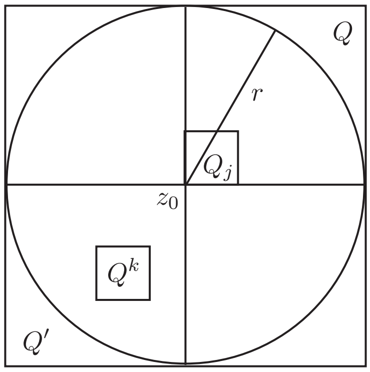
$$\omega^*(Q^j) \geqslant \omega^*(B^j) \geqslant \omega_{k+1}(B^j) \geqslant \omega_{j+1}(B^j) \geqslant C(\varepsilon)\,\omega_j(Q^j) \geqslant C(\varepsilon)\,M\ell(Q^j).$$

We will also need the following estimate.

If $z_0 \in Q_j$, $j \in T$ (or $z_0 \in Q^k$, $k \in S$) and $r \geqslant \ell(Q_j)$ ($r \geqslant \ell(Q^k)$), then

$$\omega^*\{|z - z_0| < r\} \leqslant CMr. \tag{9.7}$$

Let us discuss the case of Q_j, $z_0 \in Q_j$. We remark that if Q is a dyadic square with $Q \supset Q_j$, then one has $\omega^*(Q) \leqslant M\ell(Q)$ because otherwise the process would not have been stopped.



Take now a dyadic square $Q \supset Q_j$ with side length $2^m\ell(Q_j)$ such that $r \leqslant 2^m\ell(Q_j) \leqslant 2r$. We just said that $\omega^*(Q) \leqslant Mr$. Now the disc $\{|z-z_0| < r\}$ is contained in 4 dyadic squares of the same side length as Q. Take one of these squares Q' different from Q. If Q' does not contain any $Q_{j'}$ or Q^k then $\omega^*(Q') = 0$. Otherwise $\omega^*(Q') \leqslant Mr$.
The case $z_0 \in Q^k$ is dealt with similarly.

The next lemma shows that the union of the family of squares $\{Q_j\}_{j\in T}$ and a dilation of the family $\{Q_k\}_{k\in S}$ contains K.

Lemma 9.13. $K \subset \bigcup_{k\in S} 2RQ^k \cup \bigcup_{j\in T} Q_j$.

Proof. Recall that now $K = K_{pq} = \bigcup_{Q\in\mathcal{G}_{pq}} K \cap Q$. So let $Q \in \mathcal{G}_{pq}$ and $E = K \cap Q$. If $Q = Q_j$ for some $j \in T$ then $E \subset Q_j$ and so $E \subset \bigcup_k 2RQ^k \cup \bigcup_{j\in T} Q_j$.

If $Q \neq Q_j$ for every $j \in T$ then there is a first index j_1 such that $Q \subset RQ^{j_1}\backslash Q^{j_1}$; if $j_1 \in S$ then $Q \subset RQ^{j_1}$, $j_1 \in S$, and we are done. If $j_1 \notin S$ there is a first index j_2 such that $Q^{j_1} \subset RQ^{j_2}\backslash Q^{j_2}$. In this case $\ell(Q^{j_2}) \geqslant 2\ell(Q^{j_1})$ because if we had $\ell(Q^{j_1}) \geqslant \ell(Q^{j_2})$ then $Q^{j_2} \subset RQ^{j_1}$ and $Q^{j_2} \subset RQ^{j_1}\backslash Q^{j_1}$, so that Q^{j_2} would have disappeared. If $j_2 \in S$ we have $Q \subset RQ^{j_2}$ and we are done. If $j_2 \notin S$ there is a first j_3 such that

$$Q^{j_2} \subset RQ^{j_3}\backslash Q^{j_3}$$

and so on.

We get a sequence $j_1 < j_2 < \cdots < j_n$ with $j_1, \ldots, j_{n-1} \notin S$, $j_n \in S$ so that $Q^{j_k} \subset RQ^{j_{k+1}}\backslash Q^{j_{k+1}}$ and $\ell(Q^{j_{i+1}}) \geqslant 2\ell(Q^{j_i})$, which implies $Q \subset 2RQ^{j_n}$. The double radius appears because we need to argue on two steps: in the first we use that $Q^{j_{n-1}} \subset RQ^{j_n}$ and in the second that $Q \subset RQ^{j_{n-1}}$. □

9.7 Surrounding K^* by level curves of the Green function

To continue the proof of the Theorem, let Q be a square $Q = Q_j$, $j \in T$ or $Q = Q^k$, $k \in S$ and let B be the corresponding disc. Let $g(z) = g_{\Omega^*}(z, \infty)$ be the Green function of the domain Ω^* with pole at ∞. The goal of this section is to find a closed curve σ surrounding B, contained in a level set of g, and such that

$$|\nabla g(z)| \leqslant CM^2 \log 1/\ell(Q), \quad z \in \sigma, \tag{9.8}$$

for a positive constant C.

The Green function g is the logarithmic potential of the equilibrium measure plus the Robin constant, that is,

$$\begin{aligned} g(z) &= \frac{1}{2\pi}\int_{K^*} \log|z-w|\,d\omega^*(w) + \gamma_{K^*} \\ &= \frac{1}{2\pi}\int_B \log|z-w|\,d\omega^*(w) + \frac{1}{2\pi}\int_{K^*\backslash B} \log|z-w|\,d\omega^*(w) + \gamma_{K^*} =: u(z)+v(z)+\gamma_{K^*}. \end{aligned}$$

We have the estimate

$$|\nabla v(z)| \leqslant C\int_{K^*\backslash B} \frac{d\omega^*(w)}{|z-w|} \leqslant CM\log 1/\ell(Q), \quad z\in Q\backslash B. \tag{9.9}$$

To show this inequality, fix $z\in Q\backslash B$ and set $\omega^*(t)=\omega^*(B(z,t))$. We have

$$\begin{aligned} \int_{K^*\backslash B} \frac{d\omega^*(w)}{|z-w|} &\leqslant \int_{\ell(Q)}^1 \frac{d\omega^*(t)}{t} \leqslant \omega^*(B(z,1)) + \int_{\ell(Q)}^1 \frac{\omega^*(t)}{t^2}\,dt \\ &\leqslant 1 + CM\int_{\ell(Q)}^1 \frac{dt}{t} \leqslant 1 + CM\log 1/\ell(Q) \leqslant CM\log 1/\ell(Q), \end{aligned}$$

where we have used (9.7).

We would like to estimate the derivative $\frac{\partial u}{\partial r}(z)$ from below. Assume for simplicity that the center of the square Q, and so of the disc B, is the origin, and write $z=re^{i\theta}$.

Since

$$2\pi\, u(re^{i\theta}) = \frac{1}{2}\int_B \log|re^{i\theta}-w|^2\,d\omega^*(w),$$

we have

$$\begin{aligned} 2\pi\,\frac{\partial u}{\partial r}(z) &= \frac{1}{2}\int_B \frac{1}{|re^{i\theta}-w|^2}\,\frac{\partial}{\partial r}\Big((re^{i\theta}-w)(re^{-i\theta}-\bar w)\Big)\,d\omega^*(w) \\ &= \int_B \operatorname{Re}\left(\frac{(z-w)\,\bar z}{|z-w|^2\,|z|}\right)\,d\omega^*(w), \end{aligned}$$

which in particular tells us that $\frac{\partial u}{\partial r}(z)\geqslant 0$.

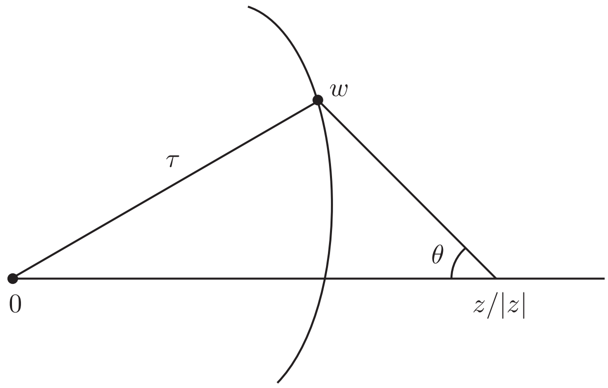
Now we write

$$\operatorname{Re}\left(\frac{(z-w)\,\bar z}{|z-w|^2|z|}\right) = \frac{1}{|z-w|}\left\langle \frac{z-w}{|z-w|}, \frac{z}{|z|}\right\rangle$$

and we look for the minimum value of $\left\langle \frac{z-w}{|z-w|}, \frac{z}{|z|}\right\rangle$ when $|w|=\tau$, τ being the radius $r(B)$ of B.

Assuming that $\frac{z}{|z|}=1$, set $\left\langle \frac{z-w}{|z-w|}, 1\right\rangle = \cos\theta$ (see the figure). The cosine Theorem yields

$$\cos\theta = \frac{1}{2|z|}\left(|z-w| + \frac{|z|^2-\tau^2}{|z-w|}\right)$$



so that the minimum is attained for

$$|z-w| = \sqrt{|z|^2 - \tau^2},$$

that is, when $z-w$ is orthogonal to w.

We then have

$$\operatorname{Re}\left(\frac{(z-w)\,\bar z}{|z-w|^2|z|}\right) \geqslant \frac{\sqrt{|z|-\tau}\,\sqrt{|z|+\tau}}{|z-w||z|} \geqslant \frac{\sqrt{|z|-\tau}}{|z|\,\sqrt{|z|+\tau}},$$

and also

$$\frac{\sqrt{|z|-\tau}}{|z|\,\sqrt{|z|+\tau}} \geqslant \frac{1}{|z|}\left(1-\frac{\tau}{|z|}\right).$$

Returning to the case of a square Q centered at the point z_0 with $\tau = r(B)$ we get the estimate of $\frac{\partial u}{\partial r}(z)$ we are looking for, namely,

$$2\pi\,\frac{\partial u}{\partial r}(z) \geqslant \frac{\sqrt{|z-z_0|-r(B)}}{\sqrt{|z-z_0|+r(B)}}\,\frac{\omega^*(B)}{|z-z_0|} \geqslant \frac{\omega^*(B)}{|z-z_0|} - \frac{r(B)\,\omega^*(B)}{|z-z_0|^2}, \quad |z-z_0| > r(B). \tag{9.10}$$

We are now ready to estimate the gradient of the Green function g. Define

$$\alpha = \alpha(B) = \max\left(\frac{\omega^*(B)}{M^2\log 1/\ell(Q)}, 2r(B)\right)$$

and distinguish two cases:

Case 1: $\alpha = 2r(B)$, that is, $\dfrac{\omega^*(B)}{M^2\log 1/\ell(Q)} \leqslant 2r(B)$.

We let σ to be the circle ∂B so that we need to prove the estimate

$$|\nabla g(z)| \leqslant CM^2 \log 1/\ell(Q), \quad z \in \partial B.$$

This is a consequence of the inequality

$$\sup_{\partial B} |\nabla g| \leqslant C \inf_{\partial B} |\nabla g| \tag{9.11}$$

for some constant C.

In fact, using (9.11) one gets

$$\omega^*(B) = -\int_{\partial B} \partial_\nu g\, ds \geqslant \inf_{\partial B} |\nabla g|\, r(B)$$

and for $z \in \partial B$

$$|\nabla g(z)| \leqslant \sup_{\partial B} |\nabla g(z)| \leqslant C \inf_{\partial B} |\nabla g(z)| \leqslant C\, \frac{\omega^*(B)}{r(B)} \leqslant CM^2 \log 1/\ell(Q). \tag{9.12}$$

In order to prove (9.11) assume that $z_0 = 0$ and take two points z and z' with $|z| = |z'| = 2r(B)$. Then we have

$$m^{-1} g(z') \leqslant g(z) \leqslant m g(z')$$

for some constant m; this follows by applying Harnack's inequality to discs of radius $\delta < r(B)$ centered at points on the circle $\{|z| = 2r(B)\}$, chosen so that the discs of radius $\delta/2$ cover this circle.

Take now z and z' with $r(B) < |z| = |z'| < 2r(B)$. We also have

$$m^{-1} g(z') \leqslant g(z) \leqslant m g(z').$$

Indeed, for $\theta \in [0, 2\pi]$, write $g_\theta(z) = g(e^{i\theta} z)$, then

$$m^{-1} g_\theta(z) \leqslant g(z) \leqslant m g_\theta(z)$$

holds for $|z| = 2r(B)$, and trivially also holds for $|z| = r(B)$, $\theta \in [0, 2\pi]$. By the maximum principle we get

$$m^{-1} g_\theta(z) \leqslant g(z) \leqslant m g_\theta(z), \quad r(B) \leqslant |z| \leqslant 2r(B), \quad \theta \in [0, 2\pi].$$

As a consequence, for $|z| = |z'| = r(B)$ and n, n' the unit exterior normal vectors to ∂B at z and z', we have

$$\frac{m^{-1} g(z' + tn')}{t} \leqslant \frac{g(z + tn)}{t} \leqslant \frac{m g(t' + tn')}{t}$$

and so

$$m^{-1} |\partial_\nu g|(z') \leqslant |\partial_\nu g|(z) \leqslant m |\partial_\nu g|(z'), \quad |z| = |z'| = r(B)$$

and finally $\sup\limits_{|z| = r(B)} |\nabla g| \leqslant C \inf\limits_{|z| = r(B)} |\nabla g|$, as required.

Case 2: $\alpha > 2r(B)$, that is, $\alpha = \dfrac{\omega^*(B)}{M^2 \log 1/\ell(Q)}$.

We note that

$$\alpha \leqslant \frac{\omega^*(Q)}{M^2 \log 2} \leqslant \frac{2M\ell(Q)}{M^2 \log 2} \leqslant \frac{4}{M}\, \ell(Q). \tag{9.13}$$

The inequality $\omega^*(Q) \leqslant M\ell(Q)$, for $Q = Q_j$, comes from the fact that Q_j has survived the process to get to ω^*. If $Q = Q^k$, take the dyadic square $\widetilde{Q}$ with side length $2\,\ell(Q^k)$ and containing Q^k. Since the process has stopped, $\omega^*(Q^k) \leqslant \omega^*(\widetilde{Q}) \leqslant M\ell(\widetilde{Q}) = 2M\ell(Q)$.

Taking in (9.13) $M > 8$, we obtain $\alpha \leqslant \ell(Q)/2$ and so $\{|z - z_0| = \alpha\} \subset Q$.

Now we want to prove that

$$|\nabla g(z)| \leqslant 4\,M^2 \log 1/\ell(Q), \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha, \tag{9.14}$$

where μ is such that $\mu > e^{20\pi}$, a condition that will be used later. Choosing $M > 8\mu$ we obtain $\alpha\mu < \ell(Q)/2$, by (9.13). Hence the annulus $\alpha \leqslant |z - z_0| \leqslant \mu\alpha$ is contained in $Q\backslash B$, a fact that will be used in the sequel without further mention.

Let us show

$$\frac{\partial u}{\partial r}(z) \geqslant |\nabla v(z)|, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha. \tag{9.15}$$

By (9.10) we get

$$2\pi\,\frac{\partial u}{\partial r}(z) \geqslant \frac{\sqrt{|z - z_0| - r(B)}}{\sqrt{|z - z_0| + r(B)}}\,\frac{\omega^*(B)}{|z - z_0|} \geqslant \frac{\sqrt{\alpha - r(B)}}{\sqrt{\alpha + r(B)}}\,\frac{\omega^*(B)}{\mu\alpha}, \quad \alpha < |z - z_0| \leqslant \mu\alpha,$$

where we have used that the function $x \to \frac{\sqrt{x - r(B)}}{\sqrt{x + r(B)}}$ is increasing.

Since $\alpha > 2r(B)$, taking the quotient M/μ big enough, we have

$$\begin{aligned} 2\pi\,\frac{\partial u}{\partial r}(z) &\geqslant \frac{1}{\sqrt{3}}\,\frac{\omega^*(B)}{\mu\alpha} \\ &= \frac{1}{\sqrt{3}\,\mu}\,M^2 \log 1/\ell(Q) \geqslant C\,M \log 1/\ell(Q) \geqslant |\nabla v(z)|, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha, \end{aligned}$$

by (9.9).

Therefore

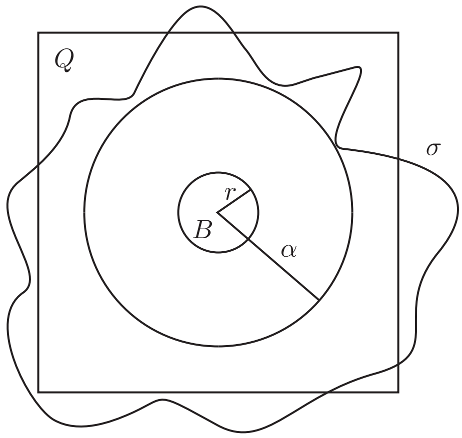
$$|\nabla g(z)| \leqslant |\nabla u(z)| + |\nabla v(z)| \leqslant 2|\nabla u(z)| \leqslant C \int_{\partial B} \frac{d\omega^*(w)}{|z - w|}, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha,$$

and $|z - w| \geqslant |z - z_0| - |w - z_0| \geqslant \alpha - r(B) \geqslant \frac{\alpha}{2}$, which gives

$$|\nabla g(z)| \leqslant C\,\frac{\omega^*(B)}{\alpha} = C\,M^2 \log 1/\ell(Q), \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha,$$

as required.

Assume $z_0 = 0$, let $c = \sup\{g(z) : |z| = \alpha\}$ and take as σ the connected component of $\{g = c\}$ that contains a point on $|z| = \alpha$. The curve σ encloses a domain that contains the disc $\{|z| < \alpha\}$.



We claim that σ remains inside $\{|z| \leqslant \mu\alpha\}$, which, in view of (9.14), yields the required estimate (9.8).

We have

$$2\pi\,|\nabla u(z)| \leqslant \int_B \frac{d\omega^*(w)}{|z-w|} \leqslant 2\frac{\omega^*(B)}{|z|}, \quad |z| > \alpha,$$

because

$$|z-w| \geqslant |z| - |w| \geqslant \frac{|z|}{2} + \frac{\alpha}{2} - r(B) > \frac{|z|}{2}.$$

By (9.10)

$$2\pi\,\frac{\partial u}{\partial r}(z) \geqslant \frac{\omega^*(B)}{|z|} - \frac{r(B)\omega^*(B)}{|z|^2}, \quad |z| > r(B).$$

Note that

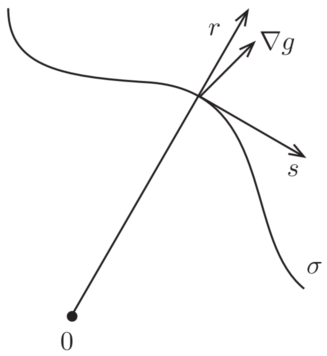
$$\frac{r(B)\omega^*(B)}{|z|^2} \leqslant \frac{1}{2}\frac{\omega^*(B)}{|z|}$$

because $|z| \geqslant \alpha \geqslant 2\,r(B)$. Then, for $|z| > \alpha$,

$$2\pi\,\frac{\partial u}{\partial r}(z) \geqslant \frac{1}{2}\frac{\omega^*(B)}{|z|} \quad \text{and} \quad |\nabla u(z)| \leqslant 4\frac{\partial u}{\partial r}(z).$$

Therefore, by (9.15),

$$|\nabla g(z)| \leqslant |\nabla u(z)| + |\nabla v(z)| \leqslant 5\frac{\partial u}{\partial r}(z), \quad \alpha \leqslant |z| \leqslant \mu\alpha. \tag{9.16}$$



Note that since the quotient M/μ can be taken as large as we want, we can improve (9.15) to

$$\frac{1}{2}\frac{\partial u}{\partial r}(z) \geqslant |\nabla v(z)|, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha.$$

Then

$$\begin{aligned}\frac{\partial g}{\partial r}(z) &= \frac{\partial u}{\partial r}(z) + \frac{\partial v}{\partial r}(z) \\ &\geqslant \frac{\partial u}{\partial r}(z) - |\nabla g(z)| > 0, \quad \alpha \leqslant |z| \leqslant \mu\alpha.\end{aligned} \tag{9.17}$$

The curve σ contains at least a point a on the circle $\{|z| = \alpha\}$. Consider the maximal subarc τ of σ containing a and contained in the disc $\{|z| \leqslant \mu\alpha\}$. By (9.17), each ray emanating from the origin intersects τ only once, and so τ can be parametrized by the polar angle θ in the form $r(\theta)e^{i\theta}$ with $\theta_1 \leqslant \theta \leqslant \theta_2$. Without loss of generality assume $\theta_1 < 0 < \theta_2$ and $r(0) = a$.

If $\tau = \sigma$ we are done. If not, $r(\theta_2) = \mu\alpha$ and we will reach a contradiction. If r is the radial direction and s is the orthogonal direction to r, then (9.16) yields

$$\left|\frac{\partial g}{\partial s}(z)\right| \leqslant |\nabla g(z)| \leqslant 5\frac{\partial u}{\partial r}(z) \leqslant 10\frac{\partial g}{\partial r}(z).$$

Since $g(r(\theta)e^{i\theta}) = c$, taking the derivative with respect to θ one gets

$$0 = \langle \nabla g(r(\theta)e^{i\theta}), r'(\theta)e^{i\theta} + ir(\theta)e^{i\theta}\rangle = r'(\theta)\frac{\partial g}{\partial r} + r(\theta)\frac{\partial g}{\partial s}$$

that gives

$$|r'(\theta)|\,\frac{\partial g}{\partial r} = r(\theta)\left|\frac{\partial g}{\partial s}\right|$$

and so

$$\frac{|r'(\theta)|}{r(\theta)} \leqslant 10.$$

Therefore

$$\log\frac{r(\theta_2)}{r(0)} = \int_0^{\theta_2}\frac{r'(\theta)}{r(\theta)}\,d\theta \leqslant \int_0^{\theta_2}\frac{|r'(\theta)|}{r(\theta)}\,d\theta \leqslant 20\pi$$

and, recalling the way μ has been chosen,

$$r(\theta_2) \leqslant e^{20\pi}r(0) = e^{20\pi}\alpha < \mu\alpha,$$

which is a contradiction. By (9.14) we obtain the desired inequality (9.8).

9.8 The estimate of the gradient of Green's function on the level curves

In the previous section we have exhibited for each disc $B = B_j$, $j \in T$ or $B = B^k$, $k \in S$, a simple curve σ contained in a level curve of g and surrounding B, on which the estimate (9.8) holds. Let now Γ be the curve formed by the set of σ's corresponding to each disc B_j or B^k. Then Γ separates K^* from infinity.

In this section we prove the estimate

$$\int_\Gamma |\log|\nabla g|\,\partial_\nu g|\,ds \leqslant C\,\log\log(1/\rho). \tag{9.18}$$

Since we are assuming that $M \leqslant \log(1/\rho)$, we have, by (9.8),

$$\log^+|\nabla g(z)| \leqslant \log(CM^2 \log 1/\ell(Q)) \leqslant C\,\log\log(1/\rho), \quad z \in \Gamma.$$

Note that

$$-\int_\Gamma \partial_\nu g\,ds = \sum_\sigma \int_\sigma \partial_\nu g\,ds = \sum_B \omega^*(B)$$

which is clear for those terms for which $\sigma = \partial B$ and follows from the divergence theorem for the others, because σ surrounds ∂B.

Hence

$$\int_\Gamma |\partial_\nu g|\log^+|\nabla g|\,ds \leqslant C\,\log\log(1/\rho)\int_\Gamma |\partial_\nu g|\,ds$$
$$= C\,\log\log(1/\rho)\sum_B \omega^*(B) \leqslant C\,\log\log(1/\rho).$$

In order to estimate the integral on Γ of $\partial_\nu g \log^-|\nabla g|$ we need the following lemma.

Lemma 9.14. *Let $g(z) = g_\Omega(z,\infty)$ be the Green function of the domain Ω with pole at infinity and let $\Gamma = \bigcup_{j=1}^N \Gamma_j$ be the union of finitely many closed Jordan curves Γ_j so that $\Gamma \subset \{|z| < 1\}$, Γ separates $K = \mathbb{C}^* \backslash \Omega$ from infinity and there are constants c_j, $j = 1,\ldots,N$ such that $\Gamma_j \subset \{g(z) = c_j\}$, $j = 1,\ldots,N$. Then*

$$\int_\Gamma |\partial_\nu g|\,\log|\nabla g|\,ds > -\log 4\pi,$$

where n is the outward unit normal to Γ.

The proof of this lemma will be discussed in Section 9.10.

By Lemma 9.14 we have

$$\int_\Gamma |\partial_\nu g|\,\log^-|\nabla g|\,ds \leqslant \int_\Gamma |\partial_\nu g|\,\log^+|\nabla g|\,ds + \log 4\pi,$$

which completes the proof of (9.18).

9.9 End of the proof of the Main Lemma 9.12 and of the Jones-Wolff Theorem

Recall from (9.2) that for a fixed $\varepsilon > 0$ and for each $\eta > 0$ we have to find a set $A \subset K$ with $\mathcal{H}^{1+\varepsilon}_\infty(A) < \eta$ and $\omega(K\backslash A) < \eta$.

Decompose the set of indices T as $T = T_1 \cup T_2$ with

$$T_1 = \{j \in T : \omega^*(B_j) \geqslant \rho^{\varepsilon/2}\, r_j\},$$
$$T_2 = \{j \in T : \omega^*(B_j) \leqslant \rho^{\varepsilon/2}\, r_j\},$$

where $r_j = r(B_j)$.

Set

$$A = \left[K \cap \left(\bigcup_{k\in S} 2RQ^k\right)\right] \cup \left[K \cap \left(\bigcup_{j\in T_1} Q_j\right)\right].$$

We know, by Lemma 9.13, that

$$K\backslash A = \bigcup_{j\in T_2} (K \cap Q_j).$$

The inequality (9.6) yields, using that $\sum_{k\in S} \omega^*(Q^k) \leqslant 1$,

$$\mathcal{H}^{1+\varepsilon}_\infty\left(K \cap \left(\bigcup_{k\in S} 2RQ^k\right)\right) \lesssim (2R)^{1+\varepsilon} \sum_{k\in S} \ell(Q^k)^{1+\varepsilon}$$
$$\leqslant \frac{R^{1+\varepsilon}}{(M\,C(\varepsilon))^{1+\varepsilon}} \sum_{k\in S} \omega^*(Q^k)^{1+\varepsilon} \leqslant \left(\frac{R}{M\,C(\varepsilon)}\right)^{1+\varepsilon} \leqslant \eta$$

for M big enough. By Lemma 6.19 with $s = 1+\varepsilon$ and the definition of the radius of B_j in the disc construction (9.3) we obtain

$$\left(\bigcup_{j\in T_1} (K \cap Q_j)\right) \leqslant \sum_{j\in T_1} \mathcal{H}^{1+\varepsilon}_\infty(K \cap Q_j) \leqslant C \sum_{j\in T_1} \mathrm{Cap}_L(K \cap Q_j)^{1+\varepsilon}$$
$$= C \sum_{j\in T_1} r_j\, \rho^{\varepsilon} = C \sum_{j\in T_1} r_j\, \rho^{\varepsilon/2}\, \rho^{\varepsilon/2}$$
$$\leqslant C \sum_{j\in T_1} \rho^{\varepsilon/2}\, \omega^*(B_j) \leqslant C\rho^{\varepsilon/2} \leqslant \eta$$

provided ρ is small enough.

We have got $\mathcal{H}^{1+\varepsilon}_\infty(A) < \eta$ and it remains to estimate $\omega(K\backslash A)$.

By inequality (9.5)

$$\omega(K\backslash A) = \omega\left(\bigcup_{j\in T_2} (K \cap Q_j)\right) \leqslant \frac{1}{C(\varepsilon)} \sum_{j\in T_2} \omega^*(B_j).$$

Now we remark that for $j \in T_2$ we are in the Case 1 of the Section 9.7, that is

$$\frac{\omega^*(B_j)}{M^2 \log(1/\rho)} \leqslant 2r_j.$$

Indeed, since $\omega^*(B_j) \leqslant \rho^{\varepsilon/2} r_j$ it is enough to see that

$$\rho^{\varepsilon/2} \leqslant 2M^2 \log(1/\rho),$$

which clearly holds for ρ sufficiently small.

For $z \in \partial B_j$, $j \in T_2$, we know by (9.12) that

$$|\nabla g(z)| \leqslant C \, \frac{\omega^*(B_j)}{r_j} \leqslant C \, \rho^{\varepsilon/2},$$

so that

$$\log |\nabla g(z)| \leqslant \log C + \frac{\varepsilon}{2} \log \rho \ \leqslant \frac{\varepsilon}{4} \log \rho,$$

for small enough ρ. Hence, for such small ρ,

$$|\log|\nabla g(z)|| \geqslant \frac{\varepsilon}{4} \log(1/\rho).$$

We then get

$$\begin{aligned}
\omega(K \backslash A) &\leqslant \frac{1}{C(\varepsilon)} \sum_{j \in T_2} \omega^*(B_j) \leqslant \frac{1}{C(\varepsilon)} \sum_{j \in T_2} \int_{\partial B_j} |\partial_\nu g| \, ds \\
&\leqslant \frac{C}{C(\varepsilon) \, \varepsilon \log(1/\rho)} \sum_{j \in T_2} \int_{\partial B_j} |\partial_\nu g| \, |\log|\nabla g|| \, ds \\
&\leqslant \frac{C}{C(\varepsilon) \, \varepsilon \log(1/\rho)} \int_\Gamma |\partial_\nu g| \, |\log|\nabla g|| \, ds \\
&\leqslant \frac{C}{\varepsilon \, C(\varepsilon)} \, \frac{\log \log(1/\rho)}{\log(1/\rho)},
\end{aligned}$$

due to (9.18). Thus $\omega(K \backslash A) < \eta$ if ρ is small enough. Therefore for fixed $\varepsilon > 0$ and given $\eta > 0$, we can choose M and ρ such that the set A satisfies the desired conclusion.

9.10 Proof of the lemmas

9.10.1 Proof of Lemma 9.11

Changing scale we may assume that $\ell(Q) = 1$. Let ξ_0 stand for the center of Q. Further, by applying Proposition 6.36 and Lemma 6.37 and using an approximation argument, we can assume that Ω is Wiener regular.

Proof of a). Denote by μ the equilibrium measure for $(\Omega \cup E)^c$ By (9.19) $\Omega \cup E$ an unbounded domain, the Green function $G(z, \xi)$ of the domain $\Omega \cup E$ with pole at ξ can be written in the form

$$G(z,\xi) = \frac{1}{2\pi}\int \log\frac{|z-a|}{|z-\xi|}\,d\mu(a) + \frac{1}{2\pi}\iint \log\frac{|w-\xi|}{|w-a|}\,d\mu(a)\,d\omega^z_{\Omega\cup E}(w), \quad z \in \Omega\cup E. \tag{9.19}$$

Note that both measures μ and $\omega^z_{\Omega\cup E}$ are supported in $\partial\Omega\backslash RQ$. From (9.19) it is clear that the Green function can also be written in the form

$$G(z,\xi) = \frac{1}{2\pi}\log\frac{1}{|z-\xi|} + h(z,\xi), \quad z \in \Omega \cup E, \quad \xi \in \Omega \cup E, \tag{9.20}$$

with

$$h(z,\xi) = \frac{1}{2\pi}\iint \log\frac{|w-\xi|\,|z-a|}{|w-a|}\,d\omega^z_{\Omega\cup E}(w)\,d\mu(a), \quad z \in \Omega \cup E, \quad \xi \in \Omega \cup E \tag{9.21}$$

Clearly

$$|\nabla_\xi h(z,\xi)| \leqslant \frac{1}{2\pi}\left|\int_{\partial\Omega\backslash RQ} \frac{1}{\overline{w}-\overline{\xi}}\,d\omega^z_{\Omega\cup E}(w)\right| \leqslant O\left(\frac{1}{R}\right), \quad \xi \in Q, \quad z \in \Omega \cup E. \tag{9.22}$$

Next, for a given $z_0 \in \partial Q$, we wish to estimate $h(\xi_0, z_0)$ from below. To this end, note that, for all $a \in \operatorname{supp}\mu \subset \partial\Omega\backslash RQ$, $|z_0 - a| \geqslant \frac{1}{2}(R-1) \geqslant R/4 \geqslant \frac{1}{2}|\xi_0 - z_0|$ (because we assume $R \geqslant 2$), and thus, for all $w \in \partial\Omega\backslash RQ$,

$$|w-a| \leqslant |w-\xi_0| + |\xi_0 - z_0| + |z_0 - a| \leqslant |w-\xi_0| + 3|z_0 - a|.$$

Thus, using the two estimates $|z_0 - a| \geqslant R/4$ and $|w - \xi_0| \geqslant \frac{1}{2}R$, we derive

$$|w-a| \leqslant |w-\xi_0|\,\frac{|z_0-a|}{R/4} + 3|z_0-a|\,\frac{|w-\xi_0|}{R/2} = \frac{10|w-\xi_0|\,|z_0-a|}{R}.$$

Hence,

$$\log\frac{|w-\xi_0|\,|z_0-a|}{|w-a|} \geqslant \log\frac{R}{10}, \quad w \in \partial\Omega\backslash RQ, \quad a \in \partial\Omega\backslash RQ.$$

Plugging this into (9.21), we obtain

$$h(z_0,\xi_0) \geqslant \frac{1}{2\pi}\log\frac{R}{10}. \tag{9.23}$$

Let now μ_E and μ_B be the equilibrium measures of E and B respectively and set

$$u(z) := \int_B G(z,\xi)\,d\mu_B(\xi), \quad v(z) := \int_E G(z,\xi)\,d\mu_E(\xi).$$

For every $z_0 \in \partial Q$ one has

$$u(\eta) = \gamma_B + h(z_0,\xi_0) + O(1/R), \quad \eta \in B,$$
$$v(\eta) = \gamma_E + h(z_0,\xi_0) + O(1/R), \quad \eta \in E,$$

where the constant in $O(1/R)$ is independent of z_0. To see this just write

$$h(\eta,\xi) = (h(\eta,\xi) - h(\eta,\xi_0)) + (h(\xi_0,\eta) - h(\xi_0,z_0)) + h(z_0,\xi_0),$$

use (9.22), the symmetry of the Green's function and the fact that the equilibrium potential of a compact set is equal to the Robin constant on the set (except for an exceptional set of zero capacity).

Now since $u = v = 0$ on $\partial\Omega\backslash RQ$ one gets

$$\begin{aligned} u(z) &= \int_{\partial\tilde{\Omega}} u(\xi)\, d\omega^z_{\tilde{\Omega}}(\xi) = \int_{\partial B} u(\xi)\, d\omega^z_{\tilde{\Omega}}(\xi), \\ v(z) &= \int_{\partial\Omega} v(\xi)\, d\omega^z_{\Omega}(\xi) = \int_{\partial E} v(\xi)\, d\omega^z_{\Omega}(\xi). \end{aligned}$$

Hence, for $z \notin K \cup Q$,

$$\begin{aligned} u(z) &= (\gamma_B + h(z_0,\xi_0) + O(1/R))\, \omega^z_{\tilde{\Omega}}(B), \\ v(z) &= (\gamma_E + h(z_0,\xi_0) + O(1/R))\, \omega^z_{\Omega}(E). \end{aligned}$$

Assume for the sake of simplicity that $\xi_0 = 0$. Then by plugging the identity (9.20) into the above definitions of u and v we obtain

$$\begin{aligned} u(z) &= \log\frac{1}{|z|} + \int_B h(z,\xi)\, d\mu_B(\xi), && z \notin B, \\ v(z) &= \int_E \log\frac{1}{|z-\xi|}\, d\mu_E(\xi) + \int_E h(z,\xi)\, d\mu_E(\xi), && z \notin E. \end{aligned}$$

Set

$$\begin{aligned} \varphi(z) := u(z) - v(z) &= \int_E \left(\log\frac{1}{|z|} - \log\frac{1}{|z-\xi|}\right) d\mu_E(\xi) \\ &\quad + \int_B h(z,\xi)\, d\mu_B(\xi) - \int_E h(z,\xi)\, d\mu_E(\xi) \\ &= \int_E \left(\log\frac{1}{|z|} - \log\frac{1}{|z-\xi|}\right) d\mu_E(\xi) + \int_B (h(z,\xi) - h(z,0))\, d\mu_B(\xi) \\ &\quad - \int_E (h(z,\xi) - h(z,0))\, d\mu_E(\xi). \end{aligned}$$

Thus, for $z \in \Omega\backslash RQ$,

$$\begin{aligned} |\varphi(z)| &\leqslant \left|\int_E \log\frac{|z-\xi|}{|z|}\, d\mu_E(\xi)\right| + \left|\int_B (h(z,\xi) - h(z,0))\, d\mu_B(\xi)\right| \\ &\quad + \left|\int_E (h(z,\xi) - h(z,0))\right| d\mu_E(\xi) = O(1/R). \end{aligned}$$

We have used that for $\xi \in E$

$$\log \frac{|z-\xi|}{|z|} \leqslant \log \frac{|z|+|\xi|}{|z|} \leqslant \log\left(1+\frac{2}{|z|}\right) = O\left(\frac{1}{|z|}\right)$$

and

$$\log \frac{|z|}{|z-\xi|} \leqslant \log\left(1-\frac{|z-\xi|-|z|}{|z-\xi|}\right) \leqslant \log\left(1+\frac{|\xi|}{|z-\xi|}\right) = O\left(\frac{1}{|z|}\right).$$

Therefore

$$u(z) = v(z) + O(1/|z|), \quad z \in \Omega\backslash RQ.$$

Recalling that $\operatorname{Cap}_L(B) = \frac{1}{2}\operatorname{Cap}_L(E)^{1+\varepsilon}$ one gets

$$\begin{aligned}\omega_\Omega^z(B) &= \frac{u(z)}{\gamma_B + h(z_0,0) + O(1/R)} = \frac{v(z)+O(1/|z|)}{\gamma_E(1+\varepsilon)+\log 2 + h(z_0,0)+O(1/R)} \\ &= \frac{(\gamma_E + h(z_0,0)+O(1/R))\,\omega(\Omega,E,z)+O(1/|z|)}{\gamma_E(1+\varepsilon)+\log 2+h(z_0,0)+O(1/R)}.\end{aligned}$$

Clearly there exists $R_0(\varepsilon)$ such that for $R > R_0(\varepsilon)$ we have

$$\omega_\Omega^z(B) \geqslant \frac{1}{2}\frac{\gamma_E + h(z_0,0)}{\gamma_E(1+\varepsilon)+\log 2 + h(z_0,0)}\,\omega_\Omega^z(E) + O\left(\frac{1}{|z|}\right),$$

since the denominator $\gamma_E(1+\varepsilon)+\log 2+h(z_0,0)$ is bounded below away from 0 by (9.23). Appealing again to (9.23) we obtain that, for $R > R_0(\varepsilon)$,

$$\frac{\gamma_E + h(z_0,0)}{\gamma_E(1+\varepsilon)+\log 2 + h(z_0,0)} \geqslant \frac{1}{2}$$

and so

$$\omega_\Omega^z(B) \geqslant \frac{1}{4}\omega_\Omega^z(E) + O\left(\frac{1}{|z|}\right).$$

Letting $z \to \infty$ completes the proof of a) in the lemma. □

Proof of b). Assume that $\xi_0 = 0$ and let $U = \{|z| < R\}$. The Green function $g = g_U$ of U is

$$G(w,\xi) = \log\left|\frac{1-\frac{w}{R}\frac{\overline{\xi}}{R}}{\frac{w}{R}-\frac{\xi}{R}}\right|.$$

Let G_B be the Green function of $U\backslash B$ and G_E the Green function of $U\backslash E$. We claim that

$$G_B(z,\xi) = G(z,\xi) - \int_{\partial B} G(w,\xi)\,d\omega_{U\backslash B}^z(w), \quad z,\xi \in U\backslash B. \tag{9.24}$$

On one hand, the right hand side $\phi(z,\xi)$ is a harmonic function of z except for $z = \xi$ where it has a logarithmic pole. On the other hand, if z tends to a point in $\partial(U\backslash B)$ then

$\phi(z,\xi)$ tends to 0, owing to the fact that $\int_{\partial B} G(w,\xi)\,d\omega(U\backslash B,w,z)$ is the solution of the Dirichlet problem in $U\backslash B$ with boundary values $G(z,\xi)$ with ξ fixed.

Analogously one obtains

$$G_E(z,\xi) = G(z,\xi) - \int_{\partial E} G(w,\xi)\,d\omega^z_{U\backslash B}(w), \quad z,\xi \in U\backslash E. \tag{9.25}$$

The goal is to prove the inequality

$$\frac{\partial G_B}{\partial n}(z,\xi) \geqslant \frac{\partial G_E}{\partial n}(z,\xi), \quad |z| = \frac{R}{2}, \quad \xi \in \partial U, \tag{9.26}$$

which follows from

$$G_B(z,\xi) \geqslant G_E(z,\xi), \quad |z| = \frac{R}{2}, \quad \frac{3}{4}R \leqslant |\xi| < R. \tag{9.27}$$

Since $G_B(z,\xi) = G_E(z,\xi),\ |\xi| = R,$ then, by the maximum principle, it is enough to show (9.27) for $|\xi| = \frac{3}{4}R$.

We start by proving

$$\log\left(\frac{4}{3}\right) - \frac{C}{R} \leqslant G(w,\xi) \leqslant \log\left(\frac{4}{3}\right) + \frac{C}{R}, \quad |w| \leqslant 1, \quad |\xi| = \frac{3}{4}R, \tag{9.28}$$

where C is a positive constant and R is sufficiently large. We have

$$G(w,\xi) = \log\left(\frac{4}{3}\right) + G(w,\xi) - G(0,\xi) = \log\left(\frac{4}{3}\right) + \log\left|1 - \frac{w\bar{\xi}}{R^2}\right| - \log\left|1 - \frac{w}{\xi}\right|.$$

The absolute value of each of the last two terms is less than or equal to C/R for some constant C and (9.28) follows.

Inserting (9.28) into (9.24) and (9.25) we get

$$G_B(z,\xi) \geqslant G(z,\xi) - \left(\log\left(\frac{4}{3}\right) + \frac{C}{R}\right)\omega^z_{U\backslash B}(B), \quad |z| = \frac{R}{2}, \quad |\xi| = \frac{3}{4}R,$$

$$G_E(z,\xi) \leqslant G(z,\xi) - \left(\log\left(\frac{4}{3}\right) - \frac{C}{R}\right)\omega^z_{U\backslash E}(E), \quad |z| = \frac{R}{2}, \quad |\xi| = \frac{3}{4}R.$$

Clearly (9.27) is a consequence of the two preceding inequalities and the following claim.

Claim 9.15. *For R large enough one has*

$$\left(\log\left(\frac{4}{3}\right) + \frac{C}{R}\right)\omega^z_{U\backslash B}(B) \leqslant \left(\log\left(\frac{4}{3}\right) - \frac{C}{R}\right)\omega^z_{U\backslash E}(E), \quad |z| = \frac{R}{2}.$$

We postpone the proof of the Claim and we proceed to complete the argument for Lemma 9.11.

Consider a subset A of $\partial\Omega\backslash RQ$. We want to prove

$$\omega^z(A) \leqslant \tilde{\omega}^z(A), \quad |z| = \frac{R}{2}, \tag{9.29}$$

where $\omega^z(A) = \omega^z_\Omega(A)$ and $\widetilde{\omega}^z(A) = \omega^z_{\widetilde{\Omega}}(A)$. Take a point z_0 with $|z_0| = \frac{R}{2}$ such that

$$\sup_{|z|=R/2} \frac{\omega^z(A)}{\widetilde{\omega}^z(A)} = \frac{\omega^{z_0}(A)}{\widetilde{\omega}^{z_0}(A)}.$$

Assume, to get a contradiction, that $\frac{\omega^{z_0}(A)}{\widetilde{\omega}^{z_0}(A)} = \lambda > 1$. Then

$$\lambda\widetilde{\omega}^z(A) - \omega^z(A) = \lambda - 1 > 0, \quad z \in A,$$

and

$$\lambda\widetilde{\omega}^z(A) - \omega^z(A) \geqslant 0, \quad |z| = \frac{R}{2}.$$

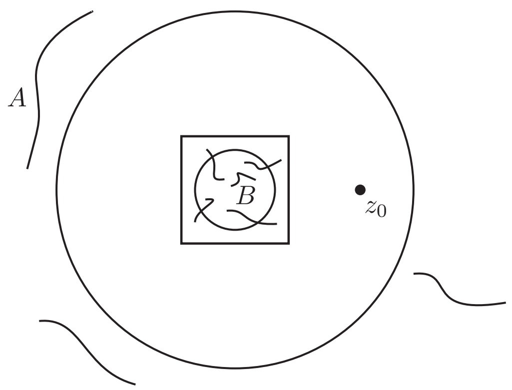
The maximum principle yields

$$\lambda\widetilde{\omega}^z(A) - \omega^z(A) > 0, \quad z \in \partial U.$$

Since $\omega^\xi(A)$ is a harmonic function on $U\backslash E$ vanishing on ∂E and, similarly, $\widetilde{\omega}^\xi(A)$ is a harmonic function on $U\backslash B$ vanishing on ∂B, we get, by (9.26),

$$\begin{aligned} 0 &= \lambda\widetilde{\omega}^{z_0}(A) - \omega^{z_0}(A) \\ &= \frac{1}{2\pi}\int_{\partial U} \frac{\partial G_B}{\partial n}(z_0,\xi)\lambda\,\widetilde{\omega}^\xi(A)\,ds(\xi) - \frac{1}{2\pi}\int_{\partial U} \frac{\partial G_E}{\partial n}(z_0,\xi)\,\omega^\xi(A)\,ds(\xi) \\ &\geqslant \frac{1}{2\pi}\int_{\partial U} \frac{\partial G_B}{\partial n}(z_0,\xi)\left(\lambda\widetilde{\omega}^\xi(A) - \omega^\xi(A)\right)\,ds(\xi) > 0, \end{aligned}$$

which is a contradiction. Then (9.29) holds.



By (9.29) and the maximum principle, $\omega^z(A) \leqslant \widetilde{\omega}^z(A)$ for $z \in \Omega$ and $|z| \geqslant \frac{R}{2}$, and letting $|z| \to \infty$, item b) of Lemma 9.11 follows.

Proof of the Claim. Recall that we are assuming $\ell(Q) = 1$, so that for all compact sets K, $\mathrm{Cap}_L(E) = \mathrm{Cap}_L(K \cap Q) \leqslant 1/\sqrt{2}$ and hence $\gamma_E \geqslant \log\sqrt{2} > 0$.

Moreover
$$\gamma_B = \gamma_E(1+\varepsilon) + \log 2 > \gamma_E.$$
Let $r = r(B)$ be the radius of B. The function
$$\log\left(\frac{R}{|z|}\right)\frac{1}{\log(R/r)}, \quad z \in U\backslash B,$$
is harmonic on $U\backslash B$, vanishes on $|z| = R$ and is 1 on $|z| = r$. Thus it is precisely $\omega^z_{U\backslash B}(B)$. Since $-\log r(B) = \gamma_B$ we have
$$\omega^z_{U\backslash B}(B) = \log\left(\frac{R}{|z|}\right)\frac{1}{\log R + \gamma_B}, \quad z \in U\backslash B. \tag{9.30}$$

We turn now our attention on $\omega^z_{U\backslash E}(E)$. Consider the function
$$f(z) = \int_E \log\frac{R}{|z-w|}\,d\mu_E(w)\,\frac{1}{\log R + \gamma_E} \quad z \in U\backslash E.$$
Since $\int_E \log\frac{1}{|z-w|}\,d\mu_E(w) = \gamma_E$ for $z \in E$, except for a set of zero logarithmic capacity, $f(z) = 1$, $z \in E$, except for a set of zero logarithmic capacity.

If $w \in E$, $z \in \partial U$ one has $|z-w| = R + O(1)$ and so
$$\log\frac{R}{|z-w|} = -\log\left(1 - \frac{R - |z-w|}{R}\right) = -\log(1 + O(1/R)) = O(1/R)$$
and
$$\log\frac{|z-w|}{R} = -\log\left(1 - \frac{|z-w| - R}{|z-w|}\right) = -\log\left(1 + O(1/R)\right) = O(1/R).$$
Since $f(z) = 1$, $z \in E$, we conclude that
$$|f(z)| \leqslant \frac{O(1/R)}{\log R + \gamma_E}, \quad z \in \partial U,$$
so that the function
$$\tilde{f}(z) = f(z) - \frac{C/R}{\log R + \gamma_E}$$
satisfies $\tilde{f}(z) \leqslant 1$, $z \in E$, and $\tilde{f}(z) \leqslant 0$, $z \in \partial U$, for an appropriate large constant C. It follows that
$$\tilde{f}(z) \leqslant \omega^z_{U\backslash E}(E), \quad z \in U\backslash E.$$
To estimate this harmonic measure we write
$$\begin{aligned}\omega^z_{U\backslash E}(E) \geqslant \frac{-C}{R(\log R + \gamma_E)} + \frac{1}{\log R + \gamma_E}\int_E\left(\log\frac{R}{|z-w|} - \log\frac{R}{|z|}\right)d\mu_E(w)\\ + \frac{1}{\log R + \gamma_E}\log\frac{R}{|z|} = T_1 + T_2 + T_3.\end{aligned}$$

By (9.30)

$$T_3 = \frac{1}{\log R + \gamma_B} \log \frac{R}{|z|} + \left(\frac{1}{\log R + \gamma_E} - \frac{1}{\log R + \gamma_B} \right) \log \frac{R}{|z|} = \omega^z_{U \setminus B}(B) + T_4.$$

For the term T_4 we have

$$T_4 = \frac{\gamma_B - \gamma_E}{(\log R + \gamma_E)(\log R + \gamma_B)} \log \frac{R}{|z|} \geqslant \frac{\varepsilon \gamma_E + \log 2}{(\log R + 2\gamma_E + \log 2)^2},$$

provided $\varepsilon < 1$, because $\gamma_B \leqslant 2\gamma_E + \log 2$.
For the term T_2 we have

$$|T_2| \leqslant \frac{1}{\log R + \gamma_E} \int_E \left| \log \frac{|z-w|}{|z|} \right| \, d\mu_E(\omega)$$

with

$$\log \frac{|z-w|}{|z|} = \log \left(1 + \frac{|z-w| - |z|}{|z|} \right) = \log(1 + O(1/R)) = O(1/R)$$

and the same estimate also holds for $\log \frac{|z|}{|z-w|}$. Hence

$$|T_2| \leqslant \frac{C}{R(\log R + \gamma_E)}.$$

Since $|T_1|$ obviously satisfies the same estimate, we conclude that

$$\omega^z_{U \setminus E}(E) \geqslant \omega^z_{U \setminus B}(B) + \frac{\varepsilon \gamma_E + \log 2}{(\log R + 2\gamma_E + \log 2)^2} - \frac{C}{R(\log R + \gamma_E)}, \tag{9.31}$$

for some positive constant C.

Recall that the claim is

$$\left(\log \left(\frac{4}{3} \right) + \frac{C}{R} \right) \omega^z_{U \setminus B}(B) \leqslant \left(\log \left(\frac{4}{3} \right) - \frac{C}{R} \right) \omega^z_{U \setminus E}(E), \quad |z| = \frac{R}{2}.$$

From now to the end of the proof of the claim z denotes a point satisfying $|z| = \frac{R}{2}$. By (9.31) we get, for $R \geqslant R_0(\varepsilon)$,

$$\omega^z_{U \setminus E}(E) \geqslant \omega^z_{U \setminus B}(B) + C \frac{\varepsilon \gamma_E}{(\log R + \gamma_E)^2}.$$

It is sufficient to show

$$\left(\log \left(\frac{4}{3} \right) + \frac{C}{R} \right) \omega^z_{U \setminus B}(B) \leqslant \left(\log \left(\frac{4}{3} \right) - \frac{C}{R} \right) \left(\omega^z_{U \setminus B}(B) + C \frac{\varepsilon \gamma_E}{(\log R + \gamma_E)^2} \right)$$

or

$$\frac{C \omega^z_{U \setminus B}(B)}{R} \leqslant -\frac{C}{R} \omega^z_{U \setminus B}(B) + \left(\log \left(\frac{4}{3} \right) - \frac{C}{R} \right) C \frac{\varepsilon \gamma_E}{(\log R + \gamma_E)^2},$$

which amounts to, for $R \geqslant R_0(\varepsilon)$,

$$\frac{\omega^z_{U\setminus B}(B)}{R} \leqslant C\,\frac{\varepsilon\gamma_E}{(\log R+\gamma_E)^2}.$$

By (9.30), for $|z| = R/2$, we have

$$\omega^z_{U\setminus B}(B) = \frac{2}{\log R+\gamma_B} = \frac{2}{\log R+(1+\varepsilon)\gamma_E+\log 2} \leqslant \frac{2}{\log R+\gamma_E}.$$

Then, for $R \geqslant R_0(\varepsilon)$, we get

$$\frac{\omega^z_{U\setminus B}(B)}{R} \leqslant \frac{2}{R(\log R+\gamma_E)} \leqslant C\,\frac{\varepsilon\gamma_E}{(\log R+\gamma_E)^2},$$

where the last inequality is equivalent to

$$2(\log R+\gamma_E) \leqslant C\,R\varepsilon\gamma_E,$$

which is clearly true for R large enough, because $\gamma_E \geqslant \log\sqrt{2}$. □

9.10.2 Proof of Lemma 9.14

We note that in the statement of Lemma 9.14 one has to understand that no curve Γ_j lies inside another curve Γ_k; in other words, the bounded connected components of $\mathbb{C}\setminus\Gamma_j$, $1 \leqslant j \leqslant N$, are disjoint. Also, replacing K by $\{g \leqslant \varepsilon\}$ for small $\varepsilon > 0$, we can assume Ω is a finitely connected domain with smooth boundary.

Recall that we can write the Green function g as

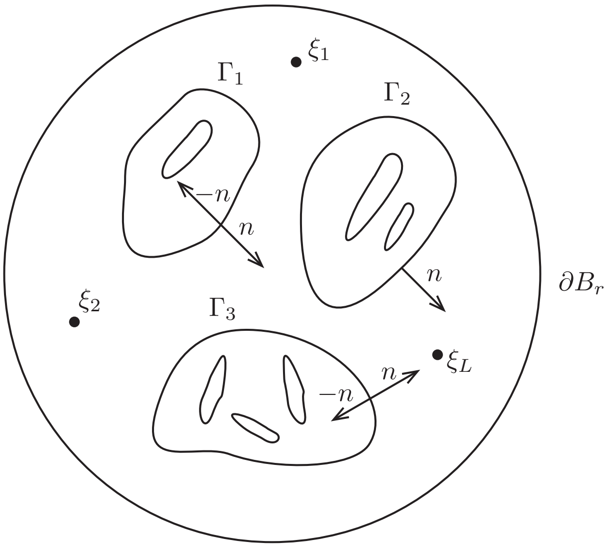
$$g(z) = \frac{1}{2\pi}\log|z| + \gamma_K + h_0(z), \tag{9.32}$$

where

$$h_0(z) = \frac{a_1}{z} + \frac{\bar a_1}{\bar z} + \frac{a_2}{z^2} + \frac{\bar a_2}{\bar z^2} + \cdots$$

is harmonic and satisfies $h_0(\infty) = 0$.

Let $\{\xi_k\}$ be the set of critical points of g that lie outside Γ. First of all we note that there is only a finite number of these critical points. Indeed, the ξ_k's are the zeros of ∂g, which is a holomorphic function on $\Omega = \mathbb{C}^*\setminus K$ vanishing at infinity. Hence the critical points can accumulate only on K and so outside Γ there are only finitely many, say $\xi_1,\dots,\xi_L$.



To simplify notation, we denote by n the inner normal at $\partial\Omega$, and by ν the outer normal. We want to show the equality

$$\int_\Gamma \frac{\partial g}{\partial n} \log|\nabla g|\, ds = 2\pi \sum_{k=1}^{L} g(\xi_k) + \sum_{j=1}^{N} c_j \int_{\Gamma_j} \frac{\partial}{\partial n} \log|\nabla g|\, ds + 2\pi\gamma_K - \log 2\pi.$$

Let B_r be the disc centered at the origin of radius r big enough to contain the unit disc and all the critical points of g. Green's formula gives

$$\begin{aligned} -\int_\Gamma \frac{\partial g}{\partial n} \log|\nabla g|\, ds &+ \int_{\partial B_r} \frac{\partial g}{\partial n} \log|\nabla g|\, ds \\ &= -\int_\Gamma g \frac{\partial}{\partial n} \log|\nabla g|\, ds + \int_{\partial B_r} g \frac{\partial}{\partial n} \log|\nabla g|\, ds - 2\pi \sum_{k=1}^{L} g(\xi_k), \end{aligned}$$

where we used that $\Delta \log|\partial g| = 2\pi \sum_{k=1}^{L} \delta_{\xi_k}$. Equivalently

$$\begin{aligned} \int_\Gamma \frac{\partial g}{\partial n} \log|\nabla g|\, ds = 2\pi \sum_{k=1}^{L} g(\xi_k) &+ \sum_{j=1}^{N} c_j \int_{\Gamma_j} \frac{\partial}{\partial n} \log|\nabla g|\, ds \\ &+ \int_{\partial B_r} \left(\frac{\partial g}{\partial n} \log|\nabla g| - g \frac{\partial}{\partial n} \log|\nabla g| \right) ds \end{aligned}$$

and we need to prove

$$\lim_{r\to\infty} \int_{\partial B_r} \left(\frac{\partial g}{\partial n} \log|\nabla g| - g \frac{\partial}{\partial n} \log|\nabla g| \right) ds = 2\pi\gamma_K - \log 2\pi. \tag{9.33}$$

On ∂B_r the normal derivative $\frac{\partial}{\partial n}$ is the partial derivative $\frac{\partial}{\partial r}$. By (9.32)

$$\frac{\partial g}{\partial r}(z) = \frac{1}{2\pi r} + \frac{\partial}{\partial r} h_0(z) = \frac{1}{2\pi r} + O\left(\frac{1}{r^2}\right), \quad |z| = r,$$

and similarly

$$\nabla g(z) = \frac{1}{2\pi} \nabla \log |z| + \nabla h_0(z) = \frac{1}{2\pi \bar{z}} + O\left(\frac{1}{|z|^2}\right), \quad |z| = r.$$

Thus

$$\log |\nabla g(z)| = \log \frac{1}{2\pi r} + O\left(\frac{1}{r}\right)$$

and

$$\frac{\partial}{\partial r} \log |\nabla g(z)| = -\frac{1}{r} + O\left(\frac{1}{r^2}\right).$$

The integral in (9.33) becomes

$$\int_{\partial B_r} \left(\frac{1}{2\pi r} \log \frac{1}{2\pi r} + \left(\frac{1}{2\pi} \log r + h_0 + \gamma_K\right) \frac{1}{r}\right) ds + O\left(\frac{1}{r}\right)$$
$$= \left(2\pi h_0 + 2\pi\gamma_K + \log \frac{1}{2\pi}\right) + O\left(\frac{1}{r}\right),$$

which tends to $2\pi\gamma_K - \log 2\pi$ as $r \to \infty$, because $h_0(r) \to 0$.

The next step is to prove the identities

$$\int_{\Gamma_j} \frac{\partial}{\partial n} \log |\nabla g| \, ds = -2\pi, \quad j = 1, 2, \ldots, N.$$

Since $\nabla g = 2\bar{\partial} g$,

$$\begin{aligned}
\int_{\Gamma_j} \frac{\partial}{\partial n} \log |\nabla g| \, ds &= \int_{\Gamma_j} \frac{\partial}{\partial n} \log |\bar{\partial} g| \, ds \\
&= \int_{\Gamma_j} \langle 2\bar{\partial} \log |\bar{\partial} g|, n \rangle \, ds = \int_{\Gamma_j} \left\langle \frac{\bar{\partial}^2 g}{\bar{\partial} g}, n \right\rangle ds \\
&= \mathrm{Re}\left(\int_{\Gamma_j} \frac{\partial^2 g}{\partial g} n \, ds\right) = \mathrm{Re}\left(\frac{1}{i} \int_{\Gamma_j} \frac{\partial^2 g}{\partial g} \, dz\right) \\
&= \mathrm{Var}\arg_{\Gamma_j}(\partial g) = \mathrm{Var}\arg_{\Gamma_j}(\overline{\nabla g}) = -2\pi.
\end{aligned}$$

Therefore

$$\int_{\Gamma} \frac{\partial g}{\partial n} \log |\nabla g| \, ds = 2\pi \sum_{k=1}^{L} g(\xi_k) - 2\pi \sum_{j=1}^{N} c_j + 2\pi\gamma_K - \log 2\pi$$

and the proof of the lemma is reduced to

$$2\pi \sum_{j=1}^{N} c_j \leqslant 2\pi \sum_{k=1}^{L} g(\xi_k) + 2\pi\gamma_K - \log 2\pi + \log 4\pi$$

or, equivalently,

$$\sum_{j=1}^{N} c_j \leqslant \sum_{k=1}^{L} g(\xi_k) + \gamma_K + \frac{\log 2}{2\pi} \tag{9.34}$$

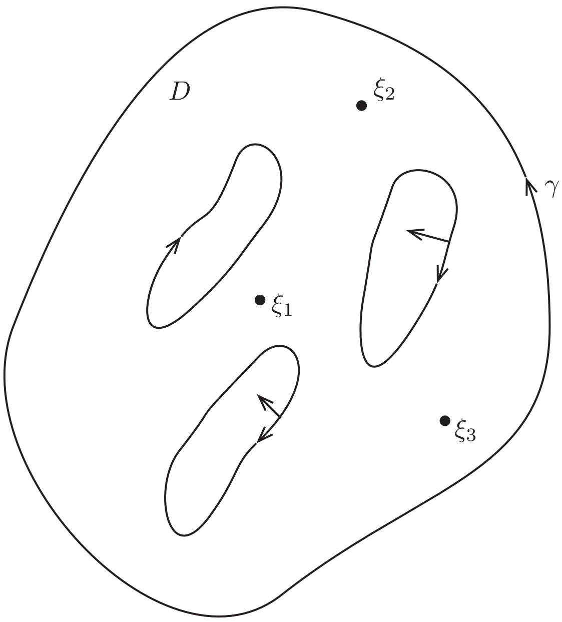
Let μ_K be equilibrium measure of K. Then

$$g(z) = \gamma_K + \frac{1}{2\pi}\int_K \log|z-w|\, d\mu_K(w),$$

and so, recalling that $\Gamma \subset \{|z|<1\}$,

$$g(z) \leqslant \gamma_K + \frac{\log 2}{2\pi}, \quad |z|<1. \tag{9.35}$$

Now we make a remark. Let γ be a Jordan curve which is contained in a level set of g and that surrounds a number β of connected components of K. Then the number of critical points of g inside γ is $\beta-1$.



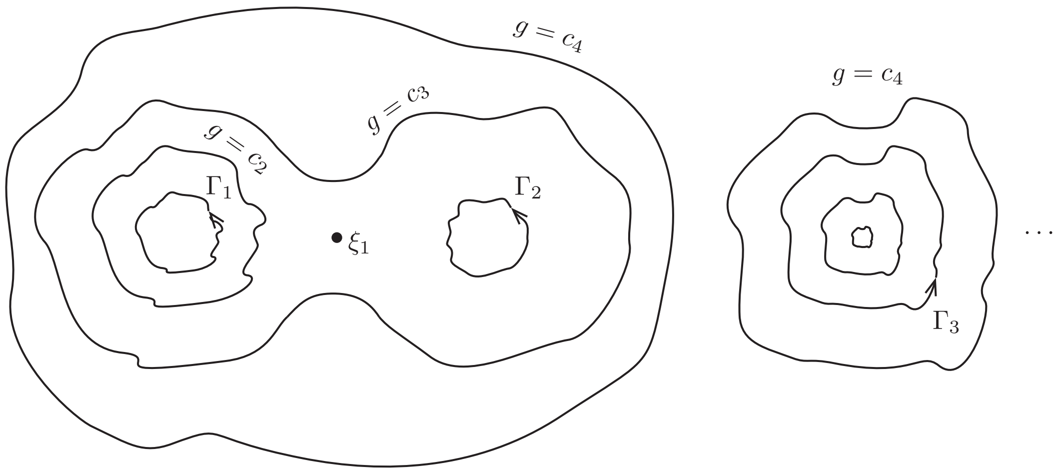
To see this, let D stand for the domain bounded by γ and K; then ∇g is orthogonal to the boundary of D and when we travel along ∂D the argument of ∇g increases by 2π over γ and decreases by 2π over the boundary of each component of K. So the total variation of $\arg(\partial g)$ on ∂D is $(\beta-1)\,2\pi$ and, by the argument principle, ∂g has $\beta-1$ zeros in D.

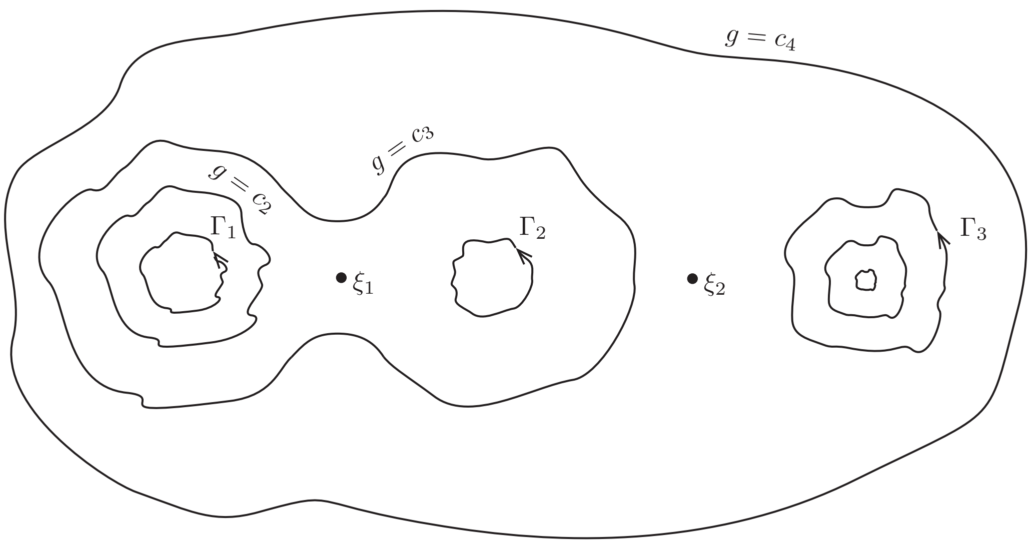
Take now γ containing all critical points of g and K. Then the total number of critical points of g is the number of components of K minus 1. Assuming that Γ_j contains β_j components of K, $j=1,\ldots,N$, we know that the number of critical points inside γ_j is

$\beta_j - 1$ and so the number of critical points outside Γ is $N-1$. Replacing in (9.34) the number L of critical points outside Γ by $N-1$, the inequality to be proven is

$$\sum_{j=1}^{N} c_j \leqslant \sum_{k=1}^{N-1} g(\xi_k) + \gamma_K + \frac{\log 2}{2\pi}. \tag{9.36}$$

To show (9.36) let us assume that the constants c_j are different and ordered so that $c_1 < c_2 < \cdots < c_N$. We would like to understand how the $N-1$ critical points outside Γ appear.





The critical points of g appear when two components of a level set of g touch. The critical point may have a multiplicity if more than two components coincide at a point; in this case, the multiplicity is, by the argument principle, the number of components that are joining minus one. Assume, for instance, that two components of $\{z : g(z) = c\}$

intersect at ξ_1 and c is the least number with this property. On one hand, $\nabla g(\xi_1) = 0$, since otherwise $\{z : g(z) = c\}$ would be a smooth curve around ξ_1, which is not the case. On the other hand, the domain bounded by $\{z : g(z) = c\}$ contains two Γ_j, which must be Γ_1 and Γ_2. Thus $g(\xi_1) \geqslant c_2$. If there were three components of $\{z : g(z) = c\}$ which join at ξ_1, then Γ_1, Γ_2 and Γ_3 would be inside the domain bounded by $\{z : g(z) = c\}$. Hence $g(\xi_1) \geqslant c_3$. Arguing inductively in this way we finally obtain that the $N-1$ critical points of g outside Γ satisfy

$$\sum_{k=1}^{N-1} g(\xi_k) \geqslant \sum_{j=2}^{N} c_j.$$

Since $c_1 = g(\tau)$ for some τ, (9.35) gives $c_1 \leqslant \gamma_K + \log 2$ and (9.36) follows. $\square$

10 Ahlfors regular domains

10.1 Dahlberg's theorem

10.1.1 Introduction

We need to introduce the notion of Lipschitz domain. We say that $Z \subset \mathbb{R}^{n+1}$ is a (d, ℓ)-cylinder if there is a coordinate system $x = (\bar{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$Z = \{(\bar{x}, x_{n+1}) : |\bar{x}| \leqslant d, -10\ell d \leqslant |x_{n+1}| \leqslant 10\ell d\}.$$

Also, for all $s > 0$, we denote

$$sZ = \{(\bar{x}, x_{n+1}) : |\bar{x}| \leqslant sd, -10\ell d \leqslant |x_{n+1}| \leqslant 10\ell d\}.$$

We say that Ω is a Lipschitz domain with Lipschitz character (ℓ, N, C_0) is there is $r_0 > 0$ and at most N (d, ℓ)-cylinders Z_j, $j = 1, \ldots, N$, with $C_0^{-1} r_0 \leqslant d \leqslant C_0 r_0$ such that

- $8Z_j \cap \partial\Omega$ is the graph of a Lipschitz function A_j with $\|\nabla A_j\|_\infty \leqslant \ell$, $A_j(0) = 0$,
- $\partial\Omega = \bigcup_j (Z_j \cap \partial\Omega)$,
- We have that
$$8Z_j \cap \Omega = \{(\bar{x}, x_{n+1}) \in 8Z_j : x_{n+1} > A_j(\bar{x})\}, \tag{10.1}$$
in the coordinate system associated with Z_j.

We also say that Ω is a *Lipschitz domain* with Lipschitz constant ℓ.

On the other hand we say that $\Omega \subset \mathbb{R}^{n+1}$ ia a special Lipschitz domain if there is a coordinate system $x = (\bar{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ and a Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Omega = \{(\bar{x}, x_{n+1}) : x_{n+1} > A(\bar{x})\}.$$

Our objective in this section is to prove the following fundamental theorem of Dahlberg [Dah77]:

Theorem 10.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be either a bounded Lipschitz domain or a special Lipschitz domain and denote by σ the surface measure in Ω. Let B be a ball centered in $\partial\Omega$ and $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, 2B \cap \partial\Omega) \geqslant C_1^{-1} r(B)$. Then the following holds:*

(a) The harmonic measure ω^{x_0} and σ are mutually absolutely continuous.

(b) *We have*

$$\left(\fint_{B\cap\partial\Omega} \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \right)^{1/2} \leqslant C \fint_{B\cap\partial\Omega} \frac{d\omega^{x_0}}{d\sigma}\, d\sigma = C\,\frac{\omega^{x_0}(B)}{\sigma(B)}, \tag{10.2}$$

where C depends only on n, the Lipschitz character of Ω, and C_1.

(c) $\omega^{x_0} \in A_\infty(\sigma)$*, with the A_∞ constants depending only on on n, the Lipschitz character of Ω, C_1, and* $\mathrm{dist}(x_0, \partial\Omega)$*.*

10.1.2 Strategy for the proof of Dahlberg's theorem

Notice first that a Lipschitz domain is NTA, and thus its associated harmonic measure is doubling. Using this doubling property it is immediate to check that it suffices to prove the theorem for a ball B small enough such that $x_0 \notin 4B$ and $4B$ is contained in $2Z_j$, where Z_j is one of the cylinders defined above.

Suppose that the boundary of Ω is smooth and that the Green function belongs to $C^2(\overline{\Omega})$, so that Green's formula can be applied to $g := G(x_0, \cdot)$ and to its partial derivatives (away from x_0). In this case ω^{x_0} and σ are mutually absolutely continuous and

$$\frac{d\omega^{x_0}}{d\sigma} = -\partial_\nu g,$$

where $\partial_\nu g$ is the normal derivative of g in $\partial\Omega$ (we assume that ν is the outer unit normal for Ω). Since g is constantly equal to 0 in $\partial\Omega$, the tangential derivative of g vanishes in $\partial\Omega$, and moreover

$$-\partial_\nu g = |\partial_\nu g| \approx \partial_{n+1} g \quad \text{in } 8Z_j \cap \partial\Omega,$$

in the coordinate system for Z_j. Therefore,

$$\int_{B\cap\partial\Omega} \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \approx -\int_{B\cap\partial\Omega} \partial_\nu g\, \partial_{n+1} g\, d\sigma.$$

Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a bump function which equals 1 in B and vanishes away from $2B$. Since both g and $\partial_{n+1} g$ are harmonic in $2B$, by Green's formula

$$\begin{aligned}
\int_{B\cap\partial\Omega} \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma &\lesssim -\int_{\partial\Omega} \varphi\, \partial_\nu g\, \partial_{n+1} g\, d\sigma = -\int_{\partial\Omega} \partial_\nu(\varphi\, g)\, \partial_{n+1} g\, d\sigma \\
&= \int_\Omega \big(-\Delta(\varphi\, g)\, \partial_{n+1} g + \varphi\, g\, \Delta(\partial_{n+1} g) \big)\, dm = -\int_\Omega \Delta(\varphi\, g)\, \partial_{n+1} g\, dm \\
&= -\int_\Omega \big(\Delta\varphi\, g\, \partial_{n+1} g - 2\partial_{n+1} g\, \nabla\varphi \cdot \nabla g \big)\, dm.
\end{aligned}$$

By the definition of φ, Theorem 8.13, and Caccioppoli's inequality, we obtain

$$\begin{aligned}
&\int_\Omega \big|\Delta\varphi\, g\, \partial_{n+1} g - 2\partial_{n+1} g\, \nabla\varphi\cdot\nabla g\big|\, dm \qquad (10.3)\\
&\quad\lesssim \frac{1}{r(B)^2}\left(\int_{\Omega\cap 2B} g^2\, dm\right)^{1/2}\left(\int_{\Omega\cap 2B} |\partial_{n+1} g|^2\, dm\right)^{1/2} + \frac{1}{r(B)}\int_{\Omega\cap 2B} |\nabla g|^2\, dm\\
&\quad\lesssim \frac{1}{r(B)^3}\int_{\Omega\cap 3B} g^2\, dm \lesssim \frac{1}{r(B)^3}\left(\frac{\omega^{x_0}(B)}{r(B)^{n-1}}\right)^2 m(B) \approx \frac{\omega^{x_0}(B)^2}{\sigma(B)},
\end{aligned}$$

which yields (10.2). The fact that ω^{x_0} is an $A_\infty(\sigma)$ weight follows then easily from the this reverse Hölder property.

For arbitrary Lipschitz domains the argument above does not work because we cannot assume a priori that $\partial_\nu g$ and $\partial_{n+1} g$ are defined in $\partial\Omega$ and that the Green formula applied above holds. To prove Dahlberg's theorem with full rigor, first we will consider the case when the boundary $\partial\Omega$ is of class C^1 and we will prove a discrete version of (10.2) following an approach based on the arguments above. Later we will deduce the full result by an approximation argument

10.1.3 Two auxiliary lemmas

Lemma 10.2. *Let u be a positive harmonic function in the upper half space $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ and continuous in $\overline{H}$ which vanishes in ∂H. Then there exists some constant $\lambda > 0$ such that*

$$u(x) = \lambda\, x_{n+1} \quad \text{for all } y \in H.$$

Proof. Let $x_0 = e_{n+1}$. We choose $\lambda = u(x_0)$ and we let $v(x) = \lambda\, x_{n+1}$ for $x \in \overline{H}$. Since both u and v are positive and harmonic in H and vanish continuously in ∂H, by the boundary Harnack principle (see Theorem 8.16) we have that $u(x) \approx v(x)$ for all $x \in H$. Thus, u grows at most linearly at ∞.

Since u vanishes in ∂H, it can be extended by reflection to lower half space. Next we use the fact that that any harmonic function in $\mathbb{R}^{n+1}$ satisfying $|u(x)| \leqslant C(1 + |x|)$ in $\mathbb{R}^{n+1}$ is a polynomial of degree at most 1, by Proposition 2.13. From this fact one easily gets that $u = \lambda\, x_{n+1}$.

□

We need now to introduce the Jones' β coefficients used to measure the flatness of sets. Given a set $E \subset \mathbb{R}^{n+1}$, a ball $B := B_r(x) \subset \mathbb{R}^{n+1}$, and an n-plane $L \subset \mathbb{R}^{n+1}$, we let

$$\beta_{\infty,E}(B, L) = \beta_{\infty,E}(x, r, L) = \sup_{y\in E\cap B_r(x)} \frac{\operatorname{dist}(y, E)}{r}.$$

Lemma 10.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain, let B a ball centered in $\partial\Omega$, and let $H = \{y : y_{n+1} > 0\}$ and $L = \partial H$. For any $\varepsilon > 0$ there exists some $\delta > 0$ (depending on ε*

and the NTA character of Ω) such that the following holds. Suppose that $\Omega \cap \delta^{-1}B \subset \overline{H}$ and that $\beta_{\infty,\partial\Omega}(\delta^{-1}B, L) \leqslant \delta$. Let $u : \overline{\Omega \cap \delta^{-1}B} \to \mathbb{R}$ be a continuous function vanishing identically in $\partial\Omega \cap \delta^{-1}B \to \mathbb{R}$ and positive and harmonic in $\Omega \cap \delta^{-1}B$. Then there exists some constant $\lambda > 0$, depending on u, such that

$$|u(y) - \lambda\, y_{n+1}| \leqslant \varepsilon\, \|u\|_{\infty,B} \quad \text{for all } y \in \overline{\Omega \cap B}, \tag{10.4}$$

Further, if $y \in \Omega \cap B$ satisfies $\operatorname{dist}(y, \partial\Omega) \geqslant \frac{1}{4}\, r(B)$ and ε is small enough, then we have

$$|\nabla u(y)| \approx \partial_{n+1} u(y) \approx r(B)^{-1}\, u(y) \tag{10.5}$$

and

$$r(B)\, |\nabla^2 u(y)| + |\nabla_L u(y)| \leqslant \varepsilon\, |\nabla u(y)| \ll |\nabla u(y)|, \tag{10.6}$$

where ∇_L denotes the tangential derivative in L.

Proof. Consider an arbitrary point $y_0 \in B \cap \Omega$ such that $\operatorname{dist}(y_0, \partial\Omega) \geqslant r(B)/4$. Then we will prove (10.4) with

$$\lambda = \frac{u(y_0)}{y_{0,n+1}}.$$

Denote $v(y) = \lambda\, y_{n+1}$. For the sake of contradiction, suppose that there exists some $\varepsilon > 0$ such that for any $\delta = 1/k$ there is an NTA domain Ω_k (with some bounded NTA character independent of k), a ball B_k centered in $\partial\Omega_k$ such that $\beta_{\infty,\partial\Omega_k}(kB_k, L) \leqslant 1/k$, and a continuous function $u_k : \overline{\Omega_k \cap kB_k} \to \mathbb{R}$ vanishing identically in $\partial\Omega_k \cap kB_k \to \mathbb{R}$, positive and harmonic in $\Omega_k \cap kB_k$, such that

$$\|u_k - v_k\|_{\infty,B} > \varepsilon\, \|u_k\|_{\infty,B_\delta}, \tag{10.7}$$

with $v_k(y) = \frac{u_k(y_0)}{y_{0,n+1}}\, y_{n+1}$. By translating and dilating B_k and Ω_k if necessary, we may assume that $B_k = B_1(0)$.

Since the domains Ω_k are NTA (with constants uniform in k), we infer that for any ball $M \geqslant 1$,

$$\|u_k\|_{\infty,MB} \lesssim_M \|u_k\|_{\infty,B} \approx u_k(y_0).$$

Hence, the sequence of functions $u_k(y_0)^{-1}\, u_k$ is uniformly locally bounded in compact subset of $\mathbb{R}^{n+1}$ (we assume these functions to be extended by zero in Ω_k^c). These functions are also uniformly Hölder continuous in compact subsets of $\mathbb{R}^{n+1}$ (by Lemma 7.25). Also, since $\beta_{\infty,\partial\Omega_k}(kB_k, L) \to 0$, by the Arzela-Ascoli Theorem we infer that there is a subsequence $u_{k_j}(y_0)^{-1}\, u_{k_j}$ that convergences uniformly to some function $\widetilde{u}$ which is positive and harmonic in H and vanishes continuously in $L = \partial H$. Clearly we have $\widetilde{u}(y_0) = 1$ and so $\widetilde{u}$ does not vanish identically in H. Thus, by Lemma 10.2 we know that $\widetilde{u}(y) = \frac{1}{y_{0,n+1}}\, y_{n+1}$ in H.

On the other hand, notice also that $\frac{v_k}{u_k(y_0)} = \frac{1}{y_{0,n+1}}\, y_{n+1}$ for all k, and thus by (10.7) we get the contradiction

$$0 = \|\widetilde{u} - \widetilde{u}\|_{\infty,B} = \lim_{j\to\infty} \frac{\|u_{k_j} - v_{k_j}\|_{\infty,B}}{u_{k_j}(y_0)} \gtrsim \limsup_{j\to\infty} \frac{\|u_{k_j} - v_{k_j}\|_{\infty,B}}{\|u_{k_j}\|_{\infty,B}} \geqslant \varepsilon,$$

which proves (10.4) with $\lambda = \frac{u(y_0)}{y_{0,n+1}}$.

Our next objective is to derive (10.5) and (10.6) from (10.4) with the preceding choice of λ, and with B replaced by $2B$ (it is clear that this estimate also holds in this case, by modifying suitably δ). By the usual interior Caccioppoli estimates for harmonic functions, we deduce that for all $y \in \Omega \cap B$ satisfying $\operatorname{dist}(y, \partial\Omega) \geqslant \frac{1}{4}\, r(B)$, we have

$$|\partial_{n+1} u(y) - \lambda| + |\nabla_L u(y)| \leqslant 2\, |\nabla u(y) - \lambda e_{n+1}| \lesssim \frac{1}{r(B)}\, \|u - v\|_{\infty, \Omega \cap 2B} \leqslant \frac{\varepsilon}{r(B)}\, \|u\|_{\infty, B} \quad (10.8)$$

and

$$|\nabla^2 u(y) - 0| \lesssim \frac{1}{r(B)^2}\, \|u - v\|_{\infty, \Omega \cap 2B} \leqslant \frac{\varepsilon}{r(B)^2}\, \|u\|_{\infty, B}. \quad (10.9)$$

Notice now that

$$\lambda = \frac{u(y_0)}{y_{0,n+1}} \approx \frac{u(y)}{r(B)} \approx \frac{1}{r(B)}\, \|u\|_{\infty, B},$$

and so from (10.8) we deduce that, for ε small enough,

$$|\partial_{n+1} u(y) - \lambda| \leqslant |\nabla u(y) - \lambda e_{n+1}| \leqslant \frac{\lambda}{2},$$

and so $\partial_{n+1} u(y) \approx |\nabla u(y)| \approx \lambda$, which yields (10.5). On the other hand, from (10.8) and (10.5) we derive

$$|\nabla_L u(y)| \lesssim \frac{\varepsilon}{r(B)}\, \|u\|_{\infty, B} \approx \varepsilon\, \frac{u(y)}{r(B)} \approx \varepsilon\, |\nabla u(y)|.$$

Finally, the estimate $r(B)\, |\nabla^2 u(y)| \lesssim \varepsilon\, |\nabla u(y)|$ in (10.6) follows from (10.9) in an analogous way. □

10.1.4 A key lemma for the smooth case

As in Section 10.1.2, to prove Dahlberg's theorem, we will assume that the ball B is small enough, so that $x_0 \notin 4B$ and $4B$ is contained in $2Z$, where Z is one of the cylinders Z_j defined above. We denote by $\mathcal{D}(\partial\Omega, Z)$ the family of the following "dyadic cubes" of $\partial\Omega$ obtained as follows. Let $\mathcal{D}(\mathbb{R}^n)$ the usual dyadic lattice of $\mathbb{R}^n$. Let Π_Z be the orthogonal projection from $8Z$ to $\mathbb{R}^n \equiv \mathbb{R}^n \times \{0\}$, in the coordinate system associated with Z. Then we let

$$\mathcal{D}(\partial\Omega, Z) = \{\Pi_Z^{-1}(Q) \cap \partial\Omega : Q \in \mathcal{D}(\mathbb{R}^n), Q \subset 8Z \cap \mathbb{R}^n\}.$$

Here again we are identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$. Observe that the cubes from this family are contained in $\partial\Omega \cap 8Z$. We also denote $\ell(\Pi_Z^{-1}(Q) \cap \partial\Omega) := \ell(Q)$ and we call this the side length of $\Pi_Z^{-1}(Q) \cap \partial\Omega$. Its center is the point whose projection by Π_Z coincides with the center of Q. We let $\mathcal{D}_k(\partial\Omega, Z)$ be subfamily of the cubes from $\mathcal{D}(\partial\Omega, Z)$ with side length 2^{-k}, and given a cube $R \in \mathcal{D}(\partial\Omega, Z)$, we let $\mathcal{D}_k(\partial\Omega, Z, R)$ be the subfamily of the cubes from $\mathcal{D}(\partial\Omega, Z)$ which are contained in R and have side length $2^{-k}\ell(R)$.

Lemma 10.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a Lipschitz domain. Let $Z \subset \mathbb{R}^{n+1}$ be one of the cylinders in the definition of the Lipschitz character of Ω. Let $R \in \mathcal{D}(\partial\Omega, Z)$ such that $4R \subset 4Z$ and $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, 4R) \geqslant 4\operatorname{diam}(R)$. Suppose that $\partial\Omega$ is C^1 in a neighborhood of $4R$. Then, for any $k \geqslant 1$ big enough, we have*

$$\sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 \sigma(Q) \leqslant C \left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2 \sigma(R), \tag{10.10}$$

with C depending only on the Lipschitz character of Ω.

Notice that (10.10) can be considered as a discrete version of (10.2).

Proof. Suppose that $\partial\Omega \cap Z$ coincides with the graph of the Lipschitz function $y_{n+1} = A(y)$ in Z. For $\varepsilon > 0$, let $A_\varepsilon(y) = A(y) + \varepsilon$ and let $\Omega_\varepsilon = \{y \in \Omega : y_{n+1} > A_\varepsilon(y)\}$ (the definition of the function A away from $4Z$ does not matter).

For every $Q \in \mathcal{D}_k(\partial\Omega, Z, R)$ consider a C^∞ bump function φ_Q which equals 1 on Q and vanishes in $\mathbb{R}^{n+1} \backslash B_{\operatorname{diam}(Q)}(x_Q)$ and in $\partial\Omega \backslash \Pi_Z^{-1}(2Q)$ (here x_Q is the center of Q). Since the function $g := g(x_0, \cdot)$ belongs to $W^{1,2}(\Omega \backslash \overline{B_r(x)})$ for any $r > 0$, we infer that

$$\omega^{x_0}(Q) \leqslant -\int_\Omega \nabla g\, \nabla\varphi_Q\, dm = -\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \nabla g\, \nabla\varphi_Q\, dm = -\lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \partial_{\nu_\varepsilon} g\, \varphi_Q\, d\sigma_\varepsilon,$$

where ν_ε and σ_ε denote the outer unit normal and the surface measure for Ω_ε, respectively. Consequently, denoting $2Q_\varepsilon = \Pi_Z^{-1}(2Q) \cap \partial\Omega_\varepsilon$,

$$\begin{aligned} \sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 \sigma(Q) &\leqslant \limsup_{\varepsilon\to 0} \sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \left(\int_{2Q_\varepsilon} \partial_{\nu_\varepsilon} g\, \varphi_Q\, d\sigma_\varepsilon\right)^2 \sigma(Q)^{-1} \\ &\lesssim \limsup_{\varepsilon\to 0} \sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \int_{2Q_\varepsilon} |\partial_{\nu_\varepsilon} g|^2\, \varphi_Q^2\, d\sigma_\varepsilon \\ &\leqslant \limsup_{\varepsilon\to 0} \int_{2R_\varepsilon} |\partial_{\nu_\varepsilon} g|^2\, \varphi_R^2\, d\sigma_\varepsilon. \end{aligned} \tag{10.11}$$

From the C^1 character of $\partial\Omega$ in a neighborhood of $4R$ and Lemma 10.3 (applied to some ball $B = B_{2\varepsilon}(y)$, $y \in 2R_\varepsilon$, and to a suitable n-plane L orthogonal to $\nu_\varepsilon(y)$), we infer that for ε small enough and all $y \in 2R_\varepsilon$,

$$|\nabla g(y)| \approx |\partial_{\nu_\varepsilon} g(y)| = -\partial_{\nu_\varepsilon} g(y) \approx \varepsilon^{-1}\, g(y) \tag{10.12}$$

and

$$\varepsilon\, |\nabla^2 g(y)| + |\nabla_{T_\varepsilon} g(y)| \leqslant C(\varepsilon)\, |\nabla g(y)| \ll |\nabla g(y)|, \tag{10.13}$$

where ∇_{T_ε} denotes the tangential derivative in $\partial\Omega_\varepsilon$ and $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\tilde{t}_\varepsilon(y)$ be the orthogonal projection of e_{n+1} on the tangent n-plane to $\partial\Omega_\varepsilon$ in y and set $t_\varepsilon(y) = |\tilde{t}_\varepsilon(y)|^{-1}\tilde{t}_\varepsilon(y)$. Writing

$$\partial_{n+1} g(y) = e_{n+1} \cdot \nabla g(y) = \langle e_{n+1}, \nu_\varepsilon(y)\rangle\, \partial_{\nu_\varepsilon} g(y) + \langle e_{n+1}, t_\varepsilon(y)\rangle\, \partial_{t_\varepsilon} g(y)$$

and taking into account (10.12) and (10.13), we derive

$$-\partial_{\nu_\varepsilon} g(y) = |\partial_{\nu_\varepsilon} g(y)| \approx \partial_{n+1} g(y) \quad \text{ for all } y \in 2R_\varepsilon.$$

Thus, for ε small enough, we have

$$\begin{aligned} I_\varepsilon := \int_{2R_\varepsilon} |\partial_{\nu_\varepsilon} g|^2 \, \varphi_R^2 \, d\sigma_\varepsilon &\approx - \int_{2R_\varepsilon} \partial_{\nu_\varepsilon} g \, \partial_{n+1} g \, \varphi_R^2 \, d\sigma_\varepsilon \\ &= - \int_{2R_\varepsilon} \partial_{\nu_\varepsilon} (g \, \varphi_R^2) \, \partial_{n+1} g \, d\sigma_\varepsilon + 2 \int_{2R_\varepsilon} g \, \varphi_R \, \partial_{\nu_\varepsilon} \varphi_R \, \partial_{n+1} g \, d\sigma_\varepsilon. \end{aligned} \tag{10.14}$$

We estimate the last integral on the right hand side above using Cauchy-Schwarz, the Hölder continuity of g in a neigborhood of $B_{\operatorname{diam}(R)}(x_R)$, (10.12), and the connection between ω^{x_0} and g:

$$\begin{aligned} \int_{2R_\varepsilon} |g \, \varphi_R \, \partial_{\nu_\varepsilon} \varphi_R \, \partial_{n+1} g| \, d\sigma_\varepsilon &\lesssim \frac{1}{\ell(R)} \sup_{2R_\varepsilon} g(y) \left(\int_{2R_\varepsilon} |\varphi_R \, \partial_{n+1} g|^2 \, d\sigma_\varepsilon \right)^{1/2} \sigma(R)^{1/2} \\ &\lesssim \frac{1}{\ell(R)} \left(\frac{\varepsilon}{\ell(R)} \right)^\alpha \sup_{y \in B_{2\operatorname{diam}(R)}(x_R)} g(y) \; I_\varepsilon^{1/2} \, \sigma(R)^{1/2} \\ &\lesssim \left(\frac{\varepsilon}{\ell(R)} \right)^\alpha \frac{\omega^{x_0}(R)}{\sigma(R)^{1/2}} \, I_\varepsilon^{1/2}. \end{aligned}$$

To estimate the first integral on the right hand side of (10.14) we use Green's formula again and we take into account that $\partial_{n+1} g$ is harmonic away from x_0 in Ω:

$$\int_{2R_\varepsilon} \partial_{\nu_\varepsilon} (g \, \varphi_R^2) \, \partial_{n+1} g \, d\sigma_\varepsilon = \int_{\Omega_\varepsilon} \Delta (g \, \varphi_R^2) \, \partial_{n+1} g \, dm - \int_{2R_\varepsilon} g \, \varphi_R^2 \, \partial_{\nu_\varepsilon} \partial_{n+1} g \, d\sigma_\varepsilon \tag{10.15}$$

The first integral on the right hand side is estimated exactly as in (10.3). Indeed, denoting by B_R some ball centered in $\partial\Omega$ that contains $\operatorname{supp}\varphi_R$ and such that $\operatorname{diam}(B_R) \approx \ell(R)$, we get

$$\begin{aligned} \int_{\Omega_\varepsilon} |\Delta (g \, \varphi_R^2) \, \partial_{n+1} g| \, dm &\leqslant \int_\Omega \left| \Delta \varphi_R^2 \, g \, \partial_{n+1} g - 2 \partial_{n+1} g \, \nabla \varphi_R^2 \cdot \nabla g \right| dm \\ &\lesssim \frac{1}{r(B_R)^2} \left(\int_{\Omega \cap B_R} g^2 \, dm \right)^{1/2} \left(\int_{\Omega \cap B_R} |\partial_{n+1} g|^2 \, dm \right)^{1/2} + \frac{1}{r(B_R)} \int_{\Omega \cap B_R} |\nabla g|^2 \, dm \\ &\lesssim \frac{1}{r(B_R)^3} \int_{\Omega \cap 2B_R} g^2 \, dm \lesssim \frac{1}{r(B_R)^3} \left(\frac{\omega^{x_0}(R)}{r(B_R)^{n-1}} \right)^2 m(B_R) \approx \frac{\omega^{x_0}(R)^2}{\sigma(R)}. \end{aligned}$$

To deal with the last integral on the right hand side of (10.15) we apply (10.12) and (10.13):

$$\begin{aligned} \int_{2R_\varepsilon} |g \, \varphi_R^2 \, \partial_{\nu_\varepsilon} \partial_{n+1} g| \, d\sigma_\varepsilon &\leqslant \int_{2R_\varepsilon} g \, \varphi_R^2 \, |\nabla^2 g| \, d\sigma_\varepsilon \\ &\lesssim \int_{2R_\varepsilon} (\varepsilon |\partial_{\nu_\varepsilon} g|) \, \varphi_R^2 \, (\varepsilon^{-1} C(\varepsilon) |\partial_{\nu_\varepsilon} g|) \, d\sigma_\varepsilon \\ &= C(\varepsilon) \int_{2R_\varepsilon} |\partial_{\nu_\varepsilon} g|^2 \, \varphi_R^2 \, d\sigma_\varepsilon = C(\varepsilon) I_\varepsilon, \end{aligned}$$

with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Altogether, we obtain

$$I_\varepsilon \lesssim \left(\frac{\varepsilon}{\ell(R)}\right)^\alpha \frac{\omega^{x_0}(R)}{\sigma(R)^{1/2}} I_\varepsilon^{1/2} + \frac{\omega^{x_0}(R)^2}{\sigma(R)} + C(\varepsilon) I_\varepsilon.$$

For ε small enough, this yields

$$I_\varepsilon \lesssim \frac{\omega^{x_0}(R)^2}{\sigma(R)}.$$

Plugging this estimate into (10.11), the lemma follows. □

10.1.5 Proof of Theorem 10.1

We assume that B is small enough so that $x_0 \notin 4B$ and $4B$ is contained in $2Z$, where Z is one of the cylinders in the definition of Lipschitz domain.

By reducing B and translating the dyadic lattice $\mathcal{D}(\partial\Omega, Z)$ if necessary, taking into account that ω^{x_0} is doubling, we may assume that $B \cap \partial\Omega$ is contained in some cube $R \in \mathcal{D}(Z, \partial\Omega)$ like the one in the statement of Lemma 10.4, so that moreover $\ell(R) \approx r(B)$. We claim that for any $k \geqslant 1$ big enough, we have

$$\sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 \sigma(Q) \leqslant C \left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2 \sigma(R), \tag{10.16}$$

which C depending only on the Lipschitz character of Ω.

To prove the claim we approximate Ω by a domain Ω_δ whose boundary is C^1 in $2Z$. To this end, we consider a smooth approximation of the identity $\{\phi_\delta\}_{\delta>0}$ in $\mathbb{R}^n$, we take a bump function $\eta : \mathbb{R}^n \to 0$ which equals 1 in a neighborhood of $3Z \cap \mathbb{R}^n$ and vanishes in $\mathbb{R}^n \backslash 3.1Z$, and for $z \in \mathbb{R}^n$ we denote

$$A_\delta(z) = A * \phi_{\eta(z)\delta}(z),$$

where $\delta \ll \ell(R)$ and we understand that $A * \phi_0(z) = A(z)$. It is easy to check that A_δ is Lipschitz (uniformly in δ), with $\|\nabla A_\delta\|_\infty \lesssim \|\nabla A\|_\infty$, and that A_δ is C^∞ in a neighborhood of $3R$. We let Ω_δ be the domain whose boundary is the graph of A_δ in Z and coincides with $\partial\Omega$ in $\mathbb{R}^{n+1}\backslash Z$. We denote by $\omega_\delta^{x_0}$ the harmonic measure in Ω_δ with pole x_0, and we let $Q_\delta = \Pi_Z^{-1}(Q) \cap \partial\Omega_\delta$ for $Q \in \mathcal{D}(Z, \partial\Omega)$, so that $Q_\delta \in \mathcal{D}(Z, \partial\Omega_\delta)$.

For some δ small enough (possibly depending on k) we have

$$\omega^{x_0}(\tfrac{1}{2}Q) \leqslant 2\, \omega_\delta^{x_0}(Q_\delta) \qquad \text{for every } Q \in \mathcal{D}_k(\partial\Omega, Z, R). \tag{10.17}$$

Indeed, $\omega_\delta^{(\cdot)}(Q)$ is a function harmonic in Ω_δ, which extends continuously to 1 in $\frac{1}{2}Q_\delta$, with a Hölder modulus of continuity uniform in δ. This can be derived by applying Lemma 7.25 to the function $1 - \omega_\delta^{(\cdot)}(Q)$. Then it easily follows that there is a sequence $\delta_j \to 0$ such that

$$\liminf_{j\to\infty} \omega_\delta^{x_0}(Q_\delta) \geqslant \omega^{x_0}(\tfrac{1}{2}Q) \qquad \text{for all } Q \in \mathcal{D}_k(\partial\Omega, Z, R),$$

which proves (10.17). By a similar argument, we infer that for δ small enough we have

$$\omega^{x_0}(R) \geqslant \frac{1}{2}\,\omega_\delta^{x_0}(\tfrac{1}{2}R_\delta). \tag{10.18}$$

Now the claim (10.17) follows immediately from Lemma 10.4, (10.17), (10.18), and the doubling properties of ω and ω_δ:

$$\begin{aligned}\sum_{Q\in\mathcal{D}_k(\partial\Omega,Z,R)} \left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 \sigma(Q) &\lesssim \sum_{Q_\delta\in\mathcal{D}_k(\partial\Omega_\delta,Z,R)} \left(\frac{\omega_\delta^{x_0}(Q_\delta)}{\sigma_\delta(Q_\delta)}\right)^2 \sigma_\delta(Q_\delta)\\ &\lesssim \left(\frac{\omega_\delta^{x_0}(R_\delta)}{\sigma_\delta(R_\delta)}\right)^2 \sigma_\delta(R_\delta) \lesssim \left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2 \sigma(R).\end{aligned}$$

The theorem follows easily from (10.17). First we show that $\omega^{x_0} \in A_\infty(\sigma)$, with the A_∞ constants depending on the Lipschitz character of Ω and $\operatorname{dist}(x_0,\partial\Omega)$. To this end, it suffices to prove that there are $\delta_0,\varepsilon_0 \in (0,1)$ such that for any compact set $E\subset R$,

$$\sigma(E) \leqslant \delta_0\,\sigma(R) \quad\Rightarrow\quad \omega^{x_0}(E) \leqslant \varepsilon_0\,\omega^{x_0}(R). \tag{10.19}$$

Indeed, from the regularity of σ, we infer that for any $\delta_0\in(0,1)$ there exists some k large enough and some family $I_k \subset \mathcal{D}_k(\partial\Omega,Z,R)$ such that the set $\widetilde{E} = \bigcup_{Q\in I_k} 2Q$ satisfies

$$E\subset\widetilde{E}, \qquad \sigma(\widetilde{E}) \leqslant \sigma(E) + \delta_0\,\sigma(2R) \leqslant 2\delta_0\,\sigma(2R).$$

By Cauchy-Schwarz and (10.17), we get

$$\begin{aligned}\omega^{x_0}(E) \leqslant \omega^{x_0}(\widetilde{E}) \leqslant \sum_{Q\in I_k} \frac{\omega^{x_0}(2Q)}{\sigma(Q)}\,\sigma(Q) &\leqslant \left(\sum_{Q\in I_k}\left(\frac{\omega^{x_0}(2Q)}{\sigma(Q)}\right)^2 \sigma(Q)\right)^{1/2} \sigma(\widetilde{E})^{1/2}\\ &\leqslant C\left(\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2\sigma(R)\right)^{1/2} \delta_0^{1/2}\sigma(R) = C\delta_0^{1/2}\omega^{x_0}(R).\end{aligned}$$

So (10.19) holds if we choose δ_0 small enough. In particular, this implies that ω^{x_0} and σ are mutually absolutely continuous.

Finally we turn our attention to the estimate (10.2). Given any $\eta>0$, by the Lebesgue differentiation theorem, for σ-a.e. $y\in R$ there exists some $k_y\geqslant 1$ such that

$$\left|\frac{d\omega^{x_0}}{d\sigma}(y) - \frac{\omega^{x_0}(Q)}{\sigma(Q)}\right| \leqslant \eta \quad \text{ if } x\in Q\in\mathcal{D}(\partial\Omega,Z) \text{ and } \ell(Q)\leqslant 2^{-k_y}\ell(R).$$

Denote $R(k_0) = \{y\in R : k_y \leqslant k_0\}$ for $k_0\in\mathbb{N}$. Then, using again (10.16) we obtain

$$\begin{aligned}\int_{R(k_0)}\left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma &\leqslant 2\sum_{Q\in\mathcal{D}_{k_0}(\partial\Omega,Z,R)} \int_{R(k_0)\cap Q}\left(\frac{d\omega^{x_0}}{d\sigma} - \frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 d\sigma\\ &\quad + 2\sum_{Q\in\mathcal{D}_{k_0}(\partial\Omega,Z,R)}\left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2\sigma(Q)\\ &\leqslant 2\eta^2\sigma(R) + C\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2\sigma(R).\end{aligned}$$

Since R coincides with $\bigcup_{k_0\geqslant 1} R(k_0)$ up to a set of zero σ measure, by the monotone converge theorem we derive

$$\int_B \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \leqslant \int_R \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \leqslant 2\eta^2\sigma(R) + C\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2 \sigma(R).$$

Since η is arbitrarily small and $\omega^{x_0}(R) \approx \omega^{x_0}(B)$, clearly this yields (10.2). □

10.2 Harmonic measure in chord-arc domains

A domain $\Omega \subset \mathbb{R}^{n+1}$ whose boundary is n-AD-regular is called an Ahlfors regular domain. A chord-arc domain in $\mathbb{R}^{n+1}$ is an NTA domain whose boundary is n-AD-regular. Here we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the corkscrew condition if for all $\xi \in \partial\Omega$ and $0 < r \leqslant \operatorname{diam}(\partial\Omega)$ there[1] exists some ball $B \subset B_r(\xi) \cap \Omega$ with $\operatorname{rad}(B) \approx r$. We say that Ω is a two-sided corkscrew domain if both Ω and $\mathbb{R}^{n+1}\backslash\overline{\Omega}$ satisfy the corkscrew condition. It is clear that any chord-arc domain is also a two-sided corkscrew domain.

We will need the following geometric result, proved independently by David and Jerison [DJ90] and Semmes [Sem90]:

Theorem 10.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an Ahlfors regular and two-sided corkscrew domain. Then, for all $\xi \in \partial\Omega$ and all $r \in (0, \operatorname{diam}(\partial\Omega))$ there exists a Lipschitz domain $U_{\xi,r} \subset \Omega \cap B_r(\xi)$ such that*

$$\mathcal{H}^n(\partial\Omega \cap \partial U_{\xi,r}) \gtrsim r^n.$$

The Lipschitz character of the domains $U_{\xi,r}$ and the implicit constant above only depend on n and the parameters involved in the n-AD-regularity of $\partial\Omega$ and the two-sided corkscrew condition for Ω.

Remark that, for the theorem above to hold, the two-sided corkscrew condition can be weakened, for example, by replacing the corkscrew balls by suitable disks not intersecting $\partial\Omega$. An immediate corollary of the above result is that the boundary of an Ahlfors regular two-sided corkscrew domain is uniformly n-rectifiable (see [DS93] for the definition of uniform n-rectifiability). Another consequence is the following.

Theorem 10.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. The harmonic measure for Ω is an A_∞ weight with respect to the surface measure σ. More precisely, there are constants $\delta, \varepsilon \in (0,1)$ such that for any ball B centered in $\partial\Omega$, any $x_0 \in \Omega\backslash 2B$, and any Borel set $E \subset \partial\Omega \cap B$, the following holds:*

$$\sigma(E) \geqslant \delta\, \sigma(B) \quad \Rightarrow \quad \omega^{x_0}(E) \geqslant \varepsilon\, \omega^{x_0}(B).$$

Proof. By Theorem 10.5, for a ball B as above there a Lipschitz domain $U \subset \Omega \cap B$ such that

$$\mathcal{H}^n(\partial\Omega \cap \partial U) \geqslant \eta\, \mathcal{H}^n(\partial U \cap B),$$

[1] Remark that in Definition 8.3 we only asked this condition to hold for $0 < r \leqslant r_0$, for a given r_0, and here we assume that $r_0 = \operatorname{diam}(\Omega)$.

where $\eta > 0$ depends on the parameters of the chord-arc domain character of Ω. We claim that if δ is close enough to 1 and $\sigma(E) \geqslant \delta\,\sigma(B)$ (for $E \subset \partial\Omega$), then $\mathcal{H}^n(E \cap \partial U \cap B) \gtrsim_{\varepsilon,\eta} \mathcal{H}^n(\partial U \cap B)$. Indeed,

$$\begin{aligned}\mathcal{H}^n(E \cap \partial U \cap B) &= \mathcal{H}^n(E \cap \partial\Omega \cap B) - \mathcal{H}^n(E \cap (\partial\Omega\backslash\partial U) \cap B)\\ &\geqslant \mathcal{H}^n(E \cap \partial\Omega \cap B) - \mathcal{H}^n((\partial\Omega\backslash\partial U) \cap B)\\ &\geqslant \delta\,\mathcal{H}^n(\partial\Omega \cap B) - (1-\eta)\,\mathcal{H}^n(\partial\Omega \cap B)\\ &\approx (\delta + \eta - 1)\,\mathrm{rad}(B)^n \approx_{\delta,\eta} \mathcal{H}^n(\partial U \cap B).\end{aligned}$$

Consider a point $x_B \in U$ such that $\mathrm{dist}(x_B, \partial U) \approx \mathrm{rad}(B)^n$. By Dahlberg's theorem, $\omega_U^{x_B}$ is an $A_\infty(\mathcal{H}^n|_U)$ weight, and taking also into account that U satisfies the CDC condition, we deduce that

$$\omega_U^{x_B}(E \cap \partial U \cap B) \gtrsim_{\delta,\eta} \omega_U^{x_B}(\partial U \cap B) \approx_{\delta,\eta} 1.$$

By the maximum principle, we obtain

$$\omega_\Omega^{x_B}(E \cap B) \geqslant \omega_\Omega^{x_B}(E \cap \partial U \cap B) \geqslant \omega_U^{x_B}(E \cap \partial U \cap B) \gtrsim_{\delta,\eta} 1 \approx \omega_\Omega^{x_B}(B).$$

Then, by the change of pole formula for NTA domains we deduce

$$\omega_\Omega^{x_0}(E \cap B) \gtrsim_{\delta,\eta} \omega_\Omega^{x_0}(B),$$

which proves the theorem. □

10.3 L^p-solvability of the Dirichlet problem in terms of harmonic measure

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and set $\sigma := \mathcal{H}^n|_{\partial\Omega}$ to be its surface measure. For $\alpha > 0$ and $x \in \partial\Omega$, we define the cone with vertex x and aperture $\alpha > 0$ by

$$\gamma_\alpha(x) = \big\{y \in \Omega : |x - y| < (1+\alpha)\mathrm{dist}(y, \partial\Omega)\big\} \tag{10.20}$$

and the non-tangential maximal function operator of a measurable function $u : \Omega \to \mathbb{R}$ by

$$\mathcal{N}_\alpha(u)(x) := \sup_{y \in \gamma_\alpha(x)} |u(y)|, \ \ x \in \partial\Omega. \tag{10.21}$$

Theorem 10.7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with such that $\partial\Omega$ is n-AD-regular. For $\alpha, \beta > 0$ and any function $u : \Omega \to \mathbb{R}$, we have*

$$\|\mathcal{N}_\alpha(u)\|_{L^p(\sigma)} \approx_{\alpha,\beta} \|\mathcal{N}_\beta(u)\|_{L^p(\sigma)}.$$

For the proof, see [HMT09], for example.

Because of the preceding result, when estimating $\|\mathcal{N}_\alpha(u)\|_{L^p(\sigma)}$, quite often we will not just write $\mathcal{N}(u)$ in place of $\mathcal{N}_\alpha(u)$. For definiteness, we can think that $\alpha = 1$, although the relevant value of α will not be important for us.

For $1 \leqslant p \leqslant \infty$, we say that the Dirichlet problem is solvable in L^p for the Laplacian (writing (D_p) is solvable) if there exists some constant $C_p > 0$ such that, for any $f \in C_c(\partial\Omega)$, the solution $u : \Omega \to \mathbb{R}$ of the continuous Dirichlet problem for the Laplacian in Ω with boundary data f satisfies

$$\|\mathcal{N}(u)\|_{L^p(\sigma)} \leqslant C_p \, \|f\|_{L^p(\sigma)}.$$

By the maximum principle, it is clear that (D_∞) is solvable. Consequently, by interpolation, if (D_p) is solvable, then (D_q) is solvable for $q > p$.

The objective of this section is to characterize the solvability of (D_p) for $1 < p < \infty$ in terms of the analytic properties of harmonic measure. We need the following result.

Lemma 10.8. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with bounded n-AD-regular boundary. Given $x \in \Omega$, denote by ω^x the harmonic measure for Ω with pole at x. Suppose that ω^x is absolutely continuous with respect to surface measure for every x. Let $p \in (1,\infty)$ and $\Lambda > 1$ and suppose that, for every ball B centered at $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\Omega)$ and all $x \in \Lambda B$ such that $\operatorname{dist}(x,\partial\Omega) \geqslant \Lambda^{-1} r(B)$, it holds*

$$\left(⨍_{\Lambda B} \left(\frac{d\omega^x}{d\sigma}\right)^p d\sigma \right)^{1/p} \leqslant \kappa \, \sigma(B)^{-1}, \tag{10.22}$$

for some $\kappa > 0$. Then, if Λ is big enough, the Dirichlet problem is solvable in L^s, for $s > p'$. Further, for all $f \in L^{p'}(\sigma) \cap C(\partial\Omega)$, its harmonic extension u to Ω satisfies

$$\|\mathcal{N}(u)\|_{L^{p',\infty}(\sigma)} \lesssim \kappa \, \|f\|_{L^{p'}(\sigma)}. \tag{10.23}$$

Proof. Let $f \in C(\partial\Omega)$ and let u the solution of the Dirichlet problem in Ω with boundary data f. Suppose that $f \geqslant 0$. Consider a point $\xi \in \partial\Omega$ and a non-tangential cone $\gamma(\xi) \subset \Omega$, with vertex ξ and with a fixed aperture. Fix a point $x \in \gamma(\xi)$ such denote $d_x = \operatorname{dist}(x,\partial\Omega)$. We intend to estimate $u(x)$, first assuming $d_x \leqslant 2\operatorname{diam}(\partial\Omega)$.

To this end, we pick a smooth function φ which equals 1 in $B_1(0)$ and vanishes in $\mathbb{R}^{n+1} \backslash B_2(0)$. For some $M > 4$ to be chosen later, we denote

$$\varphi_M(y) = \varphi\Big(\frac{y}{M d_x}\Big).$$

We set

$$f_0(y) = f(y)\, \varphi_M(y - \xi), \qquad f_1(y) = f(y) - f_0(y),$$

and we denote by u_0 and u_1 the corresponding solutions of the associated Dirichlet problems so that $u = u_0 + u_1$.

To estimate $u_0(x)$ we use (10.22) to show that

$$\begin{aligned} u_0(x) &= \int f_0 \, d\omega^x \leqslant \int_{B_{2Md_x}(\xi)} f \, \frac{d\omega^x}{d\sigma} \, d\sigma \\ &\leqslant \left(\int_{B_{2Md_x}(\xi)} |f|^{p'} \, d\sigma \right)^{1/p'} \left(\int_{B_{2Md_x}(\xi)} \left(\frac{d\omega^x}{d\sigma}\right)^p d\sigma \right)^{1/p} \\ &\leqslant \kappa \, C(M)\, \mathcal{M}_{\sigma,p'} f(\xi) \, \frac{\sigma(B_{2Md_x}(\xi))^{1/p'}}{\sigma(B_{d_x}(\xi))^{1/p'}} \lesssim \kappa \, C(M)\, \mathcal{M}_{\sigma,p'} f(\xi), \end{aligned}$$

assuming $\Lambda \geqslant 2M$.

To deal with $u_1(x)$, we first estimate $\fint_{B_{Md_x}(\xi)} u_1\,dm$. To do so, we consider the splitting of Ω into the usual family of Whitney cubes and we denote by I_B the family of those cubes that intersect $B := B_{Md_x}(\xi)$. By the properties of $\mathcal{W}(\Omega)$, the cubes $P \in I_B$ are contained in $CB := B_{CMd_x}(\xi)$, for some C depending just on n and the parameters in the construction of $\mathcal{W}(\Omega)$. Then, taking into account that $u_1 \leqslant u$, we have

$$\int_{B_{Md_x}(\xi)} u_1\,dm \leqslant \sum_{P\in I_B}\int_P u\,dm \leqslant \sum_{P\in I_B}\inf_{y\in b(P)} \mathcal{N}u(y)\,\ell(P)^{n+1} \tag{10.24}$$
$$\lesssim \sum_{Q\in\mathcal{D}_\sigma:Q\subset C'B} \ell(Q)\int_Q \mathcal{N}u\,d\sigma \lesssim M d_x \int_{C'B}\mathcal{N}u\,d\sigma,$$

where in the second inequality we took into account that $d_x \leqslant 2\mathrm{diam}(\partial\Omega)$. So we deduce

$$\fint_{B_{Md_x}(\xi)} u_1\,dm \lesssim \fint_{C'B}\mathcal{N}u\,d\sigma \lesssim \mathcal{M}_\sigma(\mathcal{N}u)(\xi).$$

Now, taking into account that f_1 vanishes in $B_{Md_x}(\xi)$, from the Hölder continuity of u_1 in $\partial\Omega \cap B_{Md_x/2}(\xi)$, we infer that

$$u_1(x) \lesssim \frac{1}{M^\alpha}\fint_{B_{Md_x}(\xi)} u_1\,dm \lesssim \frac{1}{M^\alpha}\,\mathcal{M}_\sigma(\mathcal{N}u)(\xi),$$

for some $\alpha > 0$ depending just on the AD-regularity constant of $\partial\Omega$.

Altogether, we have

$$u(x) \leqslant \kappa\,C(M)\,\mathcal{M}_{\sigma,p'}f(\xi) + \frac{C}{M^\alpha}\,\mathcal{M}_\sigma(\mathcal{N}u)(\xi) \quad \text{for all } x\in\gamma(\xi) \text{ with } d_x \leqslant 2\mathrm{diam}(\partial\Omega). \tag{10.25}$$

In case that Ω is unbounded, it turns out that the closure of $A := \{x\in\Omega : d_x > 2\mathrm{diam}(\partial\Omega)\}$ is contained in the cone $\gamma(\xi)$ if the aperture of $\gamma(\xi)$ is assumed to be big enough. Thus, by the maximum principle, since (10.25) holds for $x \in \partial A$ and u vanishes at ∞, it follows that the same estimate is also valid for $x\in\gamma(\xi)\cap A$. Hence (10.25) holds for all $x\in\gamma(\xi)$ in any case. So we obtain

$$\mathcal{N}u(\xi) \leqslant \kappa\,C(M)\,\mathcal{M}_{\sigma,p'}f(\xi) + \frac{C}{M^\alpha}\,\mathcal{M}_\sigma(\mathcal{N}u)(\xi) \quad \text{for all } \xi\in\partial\Omega. \tag{10.26}$$

Thus, for $s > p'$,

$$\|\mathcal{N}u\|_{L^s(\sigma)} \leqslant \kappa\,C(M)\,\|\mathcal{M}_{\sigma,p'}f\|_{L^s(\sigma)} + \frac{C}{M^\alpha}\,\|\mathcal{M}_\sigma(\mathcal{N}u)\|_{L^s(\sigma)}$$
$$\leqslant \kappa\,C'(M)\,\|f\|_{L^s(\sigma)} + \frac{C'}{M^\alpha}\,\|\mathcal{N}u\|_{L^s(\sigma)}.$$

Since f is continuous $\partial\Omega$ is bounded, $\|\mathcal{N}u\|_{L^s(\sigma)} < \infty$, and hence, choosing M (and thus Λ) big enough, we get

$$\|\mathcal{N}u\|_{L^s(\sigma)} \leqslant \kappa\,C'(M)\,\|f\|_{L^s(\sigma)}.$$

Regarding the last statement of the lemma, recall that $\mathcal{M}_{\sigma,p'}$ is bounded from $L^{p'}(\sigma)$ to $L^{p',\infty}(\sigma)$ and that $\mathcal{M}_\sigma$ is bounded in $L^{p',\infty}(\sigma)$. Then, from (10.26) we infer that

$$\begin{aligned}\|\mathcal{N}u\|_{L^{p',\infty}(\sigma)} &\leqslant \kappa\, C(M)\, \|\mathcal{M}_{\sigma,p'}f\|_{L^{p',\infty}(\sigma)} + \frac{C}{M^\alpha}\, \|\mathcal{M}_\sigma(\mathcal{N}u)\|_{L^{p',\infty}(\sigma)}\\ &\lesssim \kappa\, C(M)\, \|f\|_{L^{p'}(\sigma)} + \frac{C}{M^\alpha}\, \|\mathcal{N}u\|_{L^{p',\infty}(\sigma)}.\end{aligned}$$

Since $\|\mathcal{N}u\|_{L^{p',\infty}(\sigma)} < \infty$, the latter gives (10.23) for M and Λ big enough. □

Theorem 10.9. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with bounded n-AD-regular boundary. Given $x \in \Omega$, denote by ω^x the harmonic measure for Ω with pole at x. For $p \in (1,\infty)$, the following are equivalent:*

(a) $(D_{p'})$ is solvable for Ω.

(b) The harmonic measure ω is absolutely continuous with respect to σ and for every ball B centered in $\partial\Omega$ and for all $x \in \Omega \cap 3B\backslash 2B$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$, it holds

$$\left(\fint_B \left(\frac{d\omega^x}{d\sigma}\right)^p d\sigma\right)^{1/p} \lesssim \sigma(B)^{-1}.$$

(c) The harmonic measure ω is absolutely continuous with respect to σ and there is some $\Lambda > 1$ big enough such that, for every ball B centered in $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$ and all $x \in \Lambda B$ such that $\operatorname{dist}(x,\partial\Omega) \geqslant \Lambda^{-1} r(B)$, it holds

$$\left(\fint_{\Lambda B} \left(\frac{d\omega^x}{d\sigma}\right)^p d\sigma\right)^{1/p} \lesssim_\Lambda \sigma(B)^{-1}.$$

Proof. (a) ⇒ (b). By duality, it is enough to show that for every ball B centered in $\partial\Omega$, for all $x \in \Omega \cap 3B\backslash 2B$, and all $f \in C_c(\partial\Omega \cap B)$,

$$\left|\int_B f\, d\omega^x\right| \lesssim \|f\|_{L^{p'}(\sigma)}\sigma(B)^{-1/p'}.$$

Denoting by u the harmonic extension of f to Ω, the preceding inequality can be rewritten as

$$|u(x)| \lesssim \|f\|_{L^{p'}(\sigma)}\sigma(B)^{-1/p'}.$$

To prove the latter inequality, by standard arguments (as in (10.24), say) and the $L^{p'}$ solvability of the Dirichlet problem, it follows that

$$\fint_{4B} |u|\, dm \lesssim \fint_{CB\cap\partial\Omega} |\mathcal{N}(u)|\, d\sigma \leqslant \left(\fint_{CB\cap\partial\Omega} |\mathcal{N}(u)|^{p'}\, d\sigma\right)^{1/p'} \lesssim \|f\|_{L^{p'}(\sigma)}\sigma(B)^{-1/p'}.$$

By the subharmonicity of $|u|$ (extended by 0 in Ω^c) in $4B\backslash B$, we have

$$|u(x)| \lesssim \fint_{4B} |u|\, dm \quad \text{for all } x \in \Omega \cap 3B\backslash 2B.$$

Together with the previous estimate, this implies (b).

(a) $\Rightarrow$ (c). The arguments are almost the same as the ones in the proof of (a) $\Rightarrow$ (b), just replacing the condition $x \in \Omega \cap 3B\backslash 2B$ by $x \in \Omega \cap \Lambda B$, $\operatorname{dist}(x, \partial\Omega) \geqslant \Lambda^{-1} r(B)$. We leave the details for the reader.

(b) $\Rightarrow$ (a). First we will show that there exists some $\varepsilon > 0$ such that for any ball B centered in $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$ and for all $x \in \Omega\backslash 6B$,

$$\left(\fint_B \left(\frac{d\omega^x}{d\sigma} \right)^{p+\varepsilon} d\sigma \right)^{1/(p+\varepsilon)} \lesssim \sigma(B)^{-1}, \tag{10.27}$$

To this end, notice first that, for all $x \in \Omega \cap \partial(2B)$, by the so-called Bourgain's estimate,

$$\omega^x(8B) \gtrsim 1.$$

Then, for any function $f \in C_c(\partial\Omega)$, the assumption in (b) and the preceding estimate give

$$|u(x)| \leqslant C\, \|f\|_{L^{p'}(\sigma)} \sigma(B)^{-1/p'} \leqslant C\, \|f\|_{L^{p'}(\sigma)} \frac{\omega^x(8B)}{\sigma(B)^{1/p'}} \quad \text{for all } x \in \Omega \cap \partial(2B),$$

where, as above, u is the harmonic extension of f to Ω. By the maximum principle we infer that the above inequality also holds for all $y \in \Omega\backslash 2B$. By duality it follows that

$$\left(\fint_B \left(\frac{d\omega^y}{d\sigma} \right)^{p} d\sigma \right)^{1/p} \lesssim \frac{\omega^y(8B)}{\sigma(B)} \quad \text{for all } y \in \Omega\backslash 2B.$$

So for any given ball B_0 centered in $\partial\Omega$ with $\operatorname{diam}(B_0) \leqslant 2\operatorname{diam}(\partial\Omega)$ and $y \in \Omega\backslash 6B_0$ and any ball B' centered at $1.1B_0 \cap \partial\Omega$ with $r(B') \leqslant 2r(B_0)$, we have

$$\left(\fint_{B'} \left(\frac{d\omega^y}{d\sigma} \right)^{p} d\sigma \right)^{1/p} \lesssim \frac{\omega^y(8B')}{\sigma(B')}.$$

By Gehring's lemma (see [GM12, Theorem 6.38], for example) adapted to n-AD-regular sets, there exists some $\varepsilon > 0$ such that

$$\left(\fint_{B_0} \left(\frac{d\omega^y}{d\sigma} \right)^{p+\varepsilon} d\sigma \right)^{1/(p+\varepsilon)} \lesssim \frac{\omega^y(8B_0)}{\sigma(B_0)},$$

which yields (10.27).

Next we intend to apply Lemma 10.8 with $p+\varepsilon$ in place of p. To this end, given $\Lambda > 1$, a ball B centered in $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$, and $x \in \Lambda B$ with $\operatorname{dist}(x, \partial\Omega) \geqslant \Lambda^{-1} r(B)$, we cover $B \cap \partial\Omega$ with a family of balls B_i, $i \in I_B$, with $r(B_i) = (100\Lambda)^{-1} r(B)$, so that the balls B_i are centered at $B \cap \partial\Omega$, $x \notin 6B_i$ for any $i \in I_B$, and $\#I_B \leqslant C(\Lambda)$. Applying (10.27) to each of the balls B_i and summing over $i \in I_B$, we infer that

$$\left(\fint_{\Lambda B} \left(\frac{d\omega^x}{d\sigma}\right)^{p+\varepsilon} d\sigma \right)^{1/(p+\varepsilon)} \leqslant C(\Lambda)\,\sigma(B)^{-1}.$$

From Lemma 10.8 we deduce that (D_s) is solvable for $s > (p+\varepsilon)'$, and thus in particular for $s = p'$.

(c) $\Rightarrow$ (b). We will argue in the same way as in the proof of (a) $\Rightarrow$ (b), using the estimate (10.23) instead of the solvability of $(D_{p'})$. Again by duality, it suffices to show that for every ball B centered in $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$, for all $x \in \Omega \cap 3B\backslash 2B$ and all $f \in C_c(\partial\Omega \cap B)$, the harmonic extension u of f to Ω satisfies

$$|u(x)| \lesssim \|f\|_{L^{p'}(\sigma)}\sigma(B)^{-1/p'}. \tag{10.28}$$

By standard arguments, the Kolmogorov inequality, and (10.23), we have

$$\fint_{4B} |u|\,dm \lesssim \fint_{CB} \mathcal{N}(u)\,d\sigma \lesssim \|\mathcal{N}(u)\|_{L^{p',\infty}(\sigma)}\,\sigma(B)^{-1/p'} \lesssim \|f\|_{L^{p'}(\sigma)}\,\sigma(B)^{-1/p'}.$$

Since f vanishes in $\partial\Omega\backslash B$, by the subharmonicity of $|u|$ (extended by 0 to Ω^c) in $4B\backslash B$ we have

$$|u(x)| \lesssim \fint_{4B} |u|\,dm \quad \text{for all } x \in \Omega \cap 3B\backslash 2B,$$

which, together with the previous estimate, implies (10.28). □

Remark 10.10. The arguments in the above proof of (b) $\Rightarrow$ (a) show that solvability of $(D_{p'})$ for some $p' \in (1,\infty)$ implies solvability of $(D_{p'-\varepsilon})$ for some $\varepsilon > 0$.

Remark 10.11. The above theorem also holds if $\partial\Omega$ is unbounded. Indeed, the only place where the boundedness of $\partial\Omega$ is used is in Lemma 10.8, to ensure that $\|\mathcal{N}u\|_{L^s(\mu)} < \infty$ and $\|\mathcal{N}u\|_{L^{p',\infty}(\sigma)} < \infty$. A way of circumventing this technical problem is the following. For $r > 0$, consider the open set $\Omega_r := \Omega \cap B_r(0)$. It is easy to check that $\partial\Omega_r$ is n-AD-regular and that an estimate such as (10.22) also holds for the harmonic measure ω_{Ω_r}, with bounds uniform on r, so that (D_s) is solvable for Ω_r, with $s > p'$, and (10.23) also holds. Given $f \in C(\partial\Omega)$ with compact support, let $r > 0$ be big enough so that $\operatorname{supp} f \subset B_r(0)$, and let $f_r : \partial\Omega_r \to \mathbb{R}$ be such that $f_r = f$ in $\partial\Omega \cap B_r(0)$ and $f_r = 0$ in $\partial\Omega_r \cap \Omega$. The we apply Lemma 10.8 to the solution u_r of the Dirichlet problem with data f_r in Ω_r. Letting $r \to \infty$, then one easily deduces that $\|\mathcal{N}u\|_{L^s(\sigma)} \lesssim \kappa\|f\|_{L^s(\sigma)}$, as well as the related estimate (10.23). We leave the details for the reader.

Theorem 10.12. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain. Then we have:*

(a) *If Ω is a Lipschitz domain, then there exists some $\varepsilon_0 > 0$ depending just on the Lipschitz character of Ω such that (D_p) is solvable for $p \geqslant 2 - \varepsilon_0$.*

(b) *If Ω is chord-arc domain, then there exists some $p_0 > 1$ depending just on the chord-arc character of Ω such that (D_p) is solvable for $p \geqslant p_0$.*

Proof. Suppose that Ω is a Lipschitz domain. Let $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, \partial\Omega) \approx \operatorname{diam}(\partial\Omega)$. By Dahlberg's theorem, the density function $\frac{d\omega^{x_0}}{d\sigma}$ satisfies the reverse Hölder inequality (10.2) with exponent 2. By Gehring's lemma we deduce that an analogous reverse Hölder inequality holds for some exponent $q_0 > 2$. That is, for any ball B centered in $\partial\Omega$,

$$\left(⨍_{B\cap\partial\Omega} \left(\frac{d\omega^{x_0}}{d\sigma}\right)^{q_0} d\sigma \right)^{1/q_0} \leqslant C ⨍_{B\cap\partial\Omega} \frac{d\omega^{x_0}}{d\sigma}\, d\sigma = C\,\frac{\omega^{x_0}(B)}{\sigma(B)}, \tag{10.29}$$

Consequently, by the change of pole formula for NTA domains, the condition (b) in Theorem 10.9 is satisfied, with exponent q_0, which implies that $(D_{q_0'})$ is solvable, where q_0' is the conjugate exponent of q_0. By interpolation, (D_p) is solvable for $p \geqslant q_0'$, with $q_0' < 2$.

In case that Ω is assumed to be just a chord-arc domain, by Theorem 10.6 we know that $\frac{d\omega^{x_0}}{d\sigma}$ is an $A_\infty(\sigma)$ weight, and thus there exists some $q_0 > 1$ such that a reverse Hölder inequality such as (10.29) holds. As above, by the change of pole formula and by Theorem 10.9 we infer that $(D_{q_0'})$ is solvable, and by interpolation, (D_p) is solvable for $p \geqslant q_0'$, with $q_0' \in (1, \infty)$. $\square$

11 Rectifiability of harmonic measure

A set $E \subset \mathbb{R}^{n+1}$ is called n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $i = 1, 2, \dots$, such that

$$\mathcal{H}^n\Big(E \setminus \bigcup_i f_i(\mathbb{R}^n)\Big) = 0. \tag{11.1}$$

A set $F \subset \mathbb{R}^{n+1}$ is called purely n-unrectifiable if $\mathcal{H}^n(F \cap E) = 0$ for every n-rectifiable set E. As for sets, one can define a notion of rectifiabilty also for measures: a measure μ is said to be n-rectifiable if it vanishes outside an n-rectifiable set $E \subset \mathbb{R}^{n+1}$ and, moreover, it is absolutely continuous with respect to $\mathcal{H}^n|_E$.

In this section we will prove the following result.

Theorem 11.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open set and let $p \in \Omega$. Suppose that there exists a set $E \subset \partial\Omega$ such that $0 < \mathcal{H}^n(E) < \infty$ and that the harmonic measure $\omega^p_\Omega|_E$ is absolutely continuous with respect to $\mathcal{H}^n|_E$. Then E is n-rectifiable.*

Of course, in the theorem above, saying that E is n-rectifiable is equivalent to saying that $\omega^p_\Omega|_E$ is n-rectifiable. Remark that the theorem also holds for unbounded open sets with compact boundary. In fact, the theorem for this type of domains can be easily derived from the case when Ω is bounded. We leave the details for the reader.

11.1 The Riesz transform and harmonic measure and the reduction to Wiener regular domains

The proof of Theorem 11.1 relies on the solution of David-Semmes problem from [NTV14b] and [NTV14c] about the connection between the L^2 boundedness of the Riesz transform and rectifiability. Given a measure μ in $\mathbb{R}^{n+1}$, its (n-dimensional) Riesz transform equals

$$\mathcal{R}\mu(x) = \int \frac{x-y}{|x-y|^{n+1}}\, d\mu(y),$$

whenever the integral makes sense. For $\varepsilon > 0$, we also consider the ε-truncared version, defined by

$$\mathcal{R}_\varepsilon\mu(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}}\, d\mu(y).$$

The maximal Riesz transform of μ is defined by

$$\mathcal{R}_*\mu(x) = \sup_{\varepsilon>0} |\mathcal{R}_\varepsilon\mu(x)|.$$

We also consider the maximal radial operator $\mathcal{M}_n$, defined by

$$\mathcal{M}_n\mu(x) = \sup_{r>0} \frac{\mu(B_r(x))}{r^n}.$$

For a given function $f \in L^1_{loc}(\mu)$, we denote

$$\mathcal{R}_\mu f(x) = \mathcal{R}(f\,\mu)(x), \quad \mathcal{R}_{\varepsilon,\mu} f(x) = \mathcal{R}_\varepsilon(f\,\mu)(x), \quad \mathcal{R}_{*,\mu} f(x) = \mathcal{R}_*(f\,\mu)(x).$$

We say that $\mathcal{R}_\mu$ is bounded in $L^2(\mu)$ if the operators $\mathcal{R}_{\varepsilon,\mu}$ are bounded in $L^2(\mu)$ uniformly on $\varepsilon > 0$.

The connection between the Riesz transform and harmonic measure stems from the fact that the Riesz kernel K equals the gradient of the fundamental solution $\mathcal{E}$ modulo a constant factor. That is,

$$K(x) = \frac{x}{|x|^{n+1}} = c_n \nabla\mathcal{E}(x).$$

Consequently, from the identity (7.2), we deduce

$$c_n \nabla_y G(x,y) = K(y-x) - \int_{\partial\Omega} K(y-z)\, d\omega^x(z) = K(y-x) - \mathcal{R}\omega^x(y) \quad \text{for } x \notin \operatorname{supp}\omega^x.$$

Next we show that it suffices to prove Theorem 11.1 for Wiener regular domains.

Lemma 11.2. *To prove Theorem 11.1 we can assume that Ω is Wiener regular.*

Proof. Let $E \subset \partial\Omega$ be as in Theorem 11.1. By an exhaustion argument, it suffices to show that there exists a subset $F \subset E$ with $\mathcal{H}^n(F) > 0$ which is n-rectifiable (see for example the argument below near (11.2)).

For any $\varepsilon > 0$, let $\widetilde{\Omega}_\varepsilon \subset \Omega$ be the Wiener regular open set constructed in Proposition 6.36 and Lemma 6.37. For E as above, let $E_\varepsilon = E \cap \partial\widetilde{\Omega}_\varepsilon$, so that by Lemma 6.37,

$$\lim_{\varepsilon\to 0} \omega^p_{\widetilde{\Omega}_\varepsilon}(E_\varepsilon) = \lim_{\varepsilon\to 0} \omega^p_{\widetilde{\Omega}_\varepsilon}(E) = \omega^p_\Omega(E).$$

Let $\varepsilon > 0$ be small enough so that $\omega^p_{\widetilde{\Omega}_\varepsilon}(E_\varepsilon) > 0$. By Lemma 5.28, we have

$$\omega^p_{\widetilde{\Omega}_\varepsilon}(A) \leqslant \omega^p_\Omega(A) \quad \text{for any Borel set } A \subset \partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon.$$

So $\omega^p_{\widetilde{\Omega}_\varepsilon}$ is absolutely continuous with respect to ω^p_Ω in $\partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon$. Consequently, there exists a subset $F \subset E_\varepsilon$ where $\omega^p_{\widetilde{\Omega}_\varepsilon}$ are mutually absolutely continuous and both $\omega^p_{\widetilde{\Omega}_\varepsilon}(F) > 0$, $\omega^p_\Omega(F) > 0$. Since F is a subset of E, $\omega^p_{\widetilde{\Omega}_\varepsilon}$ is also mutually absolutely continuous with $\mathcal{H}^n|_F$ and $\mathcal{H}^n(F) > 0$. By Theorem 11.1 applied to the Wiener regular domain $\widetilde{\Omega}_\varepsilon$, then we deduce that F is n-rectifiable, and so we are done. $\square$

11.2 Rectifiability of harmonic measure when it is absolutely continuous with respect to surface measure

To prove Theorem 11.1 we will use the following result.

Theorem 11.3. *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ and $E \subset \operatorname{supp}\mu$ such that $0 < \mathcal{H}^n(E) < \infty$ and $\mu|_E$ is absolutely continuous with respect to $\mathcal{H}^n|_E$. If $\mathcal{R}_*\mu(x) < \infty$ for μ-a.e. $x \in E$, then $\mu|_E$ is n-rectifiable.*

This theorem follows from the following deep result from [NTV14c]:

Theorem 11.4. *Let $E \subset \operatorname{supp}\mu$ such that $0 < \mathcal{H}^n(E) < \infty$. Suppose that $\mathcal{R}_{\mathcal{H}^n|_E}$ is bounded in $L^2(\mathcal{H}^n|_E)$. Then E is n-rectifiable.*

The next result can be proved using a sophisticated *Tb* theorem of Nazarov, Treil, and Volberg [NTV14a], [Vol03] in combination with the methods in [Tol00]. For the detailed proof in the case of the Cauchy transform, see [Tol14, Theorem 8.13].

Theorem 11.5. *Let μ be a Radon measure with compact support in $\mathbb{R}^{n+1}$ and consider a μ-measurable set G with $\mu(G) > 0$ such that*

$$G \subset \{x \in \mathbb{R}^{n+1} : \mathcal{M}_n\mu(x) < \infty \text{ and } \mathcal{R}_*\mu(x) < \infty\}.$$

Then there exists a Borel subset $G_0 \subset G$ with $\mu(G_0) > 0$ such that $\sup_{x\in G_0} \mathcal{M}_n\mu|_{G_0}(x) < \infty$ and $\mathcal{R}_{\mu|_{G_0}}$ is bounded in $L^2(\mu|_{G_0})$.

We will prove neither Theorem 11.5 nor Theorem 11.4, since both results are out of the scope of these notes. Instead, we will outline how one can deduce Theorem 11.3 from Theorems 11.4 and 11.5.

Proof of Theorem 11.3 using Theorems 11.4 and 11.5. This follows by a standard exhaustion argument. Indeed, let μ and E satisfy the assumptions in Theorem 11.3. We can assume E to be compact, so that $\mu(E) < \infty$. Let

$$\beta = \sup\{\mu(F) : \; F \subset E \text{ is Borel } n\text{-rectifiable}\}. \tag{11.2}$$

It is is immediate to check that the supremum is attained, that is, there exists a Borel n-rectifiable set $F \subset E$ such that $\mu(F) = \beta$.

We have to check that $\beta = \mu(E)$. Suppose that this is not the case, and let $G = E \backslash F$. By assumption, we have $\mathcal{R}_*\mu(x) < \infty$ for μ-a.e. $x \in G$. Also, for $x \in G$, we have

$$\limsup_{r\to 0} \frac{\mu(B_r(x))}{r^n} \leqslant \limsup_{r\to 0} \frac{\mu(B_r(x))}{\mathcal{H}^n(B_r(x)\cap E)} \; \limsup_{r\to 0} \frac{\mathcal{H}^n(B_r(x)\cap E)}{r^n}. \tag{11.3}$$

The first lim sup on the right hand side is finite μ-a.e. in G because of the absolute continuity of μ with respect to $\mathcal{H}^n$ in E, while the last one is also finite by the classical density bounds for Hausdorff measure. Hence the left hand side is also finite μ-a.e. in G, or equivalently,

$$\mathcal{M}_n\mu(x) < \infty \quad \text{ for } \mu\text{-a.e. } x \in G.$$

Then, by Theorem 11.5, there exists a Borel subset $G_0 \subset G$ with $\mu(G_0) > 0$ such that $\mathcal{R}_{\mu|_{G_0}}$ is bounded in $L^2(\mu|_{G_0})$. Denote by ρ the density of $\mu|_{G_0}$ with respect to $\mathcal{H}^n|_{G_0}$, so that $\mu|_{G_0} = \rho\,\mathcal{H}^n|_{G_0}$, and let $\tau > 0$ be such that the set

$$G_{0,\tau} = \{x \in G_0 : \rho(x) > \tau\}$$

has postive measure μ. It is immediate to check that $\mathcal{R}_{\mathcal{H}^n|_{G_{0,\tau}}}$ is bounded in $L^2(\mathcal{H}^n|_{G_{0,\tau}})$, and thus $G_{0,\tau}$ is n-rectifiable, by Theorem 11.4. As a consequence, the set $F' = F \cup G_{0,\tau}$ is n-rectifiable and $\mu(F') > \mu(F) = \beta$, which contradicts the definition of F and β. □

To prove Theorem 11.1, recall that Lemma 6.19 asserts the following: If $E \subset \mathbb{R}^{n+1}$ is compact and $n - 1 < s \leqslant n + 1$, in the case $n > 1$, we have

$$\mathrm{Cap}(E) \gtrsim_{s,n} \mathcal{H}^s_\infty(E)^{\frac{n-1}{s}}.$$

In the case $n = 1$,

$$\mathrm{Cap}_L(E) \gtrsim_s \mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

Proof of Theorem 11.1. Let Ω, E, and p be as in Theorem 11.1, with Ω Wiener regular, and write ω instead of ω_Ω. We will show that

$$\mathcal{R}_*\omega^p(x) < \infty \quad \text{ for } \omega^p\text{-a.e. } x \in E,$$

which implies that $\omega^p|_E$ is n-rectifiable, by Theorem 11.3. For simplicity, in this proof we will assume that all the balls denoted by $B_s(\xi)$ are closed (this is not essential, but it will ease some calculations because many lemmas in the preceding sections about the relationship between harmonic measure and the Green function are stated in terms of closed balls).

By the same argument as in (11.3), it follows that $\mathcal{M}_n\omega^p(x) < \infty$ for ω^p-a.e. $x \in E$. For $k \geqslant 1$, let

$$E_k = \{x \in E : \mathcal{M}_n\omega^p(x) \leqslant k\},$$

so that $E = \bigcup_{k\geqslant 1} E_k$, up to a set of ω^p-measure zero. For a fixed $k \geqslant 1$, let $x \in E_k$ be a density point of E_k, and let r_0 be small enough so that

$$\frac{\omega^p(B_r(x) \cap E_k)}{\omega^p(B_r(x))} \geqslant \frac{1}{2} \quad \text{ for } 0 < r \leqslant r_0,$$

with $r_0 \leqslant |x - p|/100$. Observe that, since $\omega^p(B_\rho(z) \cap E_k) \leqslant k\rho^n$ for all $z \in E_k$ and all $\rho > 0$, by Frostman's Lemma we have

$$\mathcal{H}^n_\infty(B_r(x) \cap \partial\Omega) \geqslant \mathcal{H}^n_\infty(B_r(x) \cap E_k) \geqslant C(k)\,\omega^p(B_r(x) \cap E_k) \geqslant \frac{C(k)}{2}\,\omega^p(B_r(x)), \quad (11.4)$$

for $0 < r \leqslant r_0$.

To show that $\mathcal{R}_*\omega^p(x) < \infty$ for $x \in E_k$ as above, clearly it suffices to show that

$$\sup_{0<r\leqslant r_0} |\mathcal{R}_r\omega^p(x)| < \infty. \tag{11.5}$$

To estimate $\mathcal{R}_r\omega^p(x)$ for $0 < r \leqslant r_0$, first we assume that

$$\omega^p(B_{40r}(x)) \leqslant 50^n \omega^p(B_r(x)). \tag{11.6}$$

We consider a radial C^∞ function $\varphi : \mathbb{R}^{n+1} \to [0,1]$ which vanishes in $B_1(0)$ and equals 1 on $\mathbb{R}^{n+1}\backslash B_2(0)$, and for $r > 0$ and $z \in \mathbb{R}^{n+1}$ we denote $\varphi_r(z) = \varphi\left(\frac{z}{r}\right)$ and $\psi_r = 1 - \varphi_r$. We set

$$\widetilde{\mathcal{R}}_r\omega^p(x) = \int K(x-y)\,\varphi_r(x-y)\,d\omega^p(y).$$

Note that

$$\begin{aligned}|\mathcal{R}_r\omega^p(x)| &\leqslant \left|\int \varphi_r(x-y)K(x-y)\,d\omega^p(y)\right| + \int \big|\chi_{|x-y|>r} - \varphi_r(x-y)\big|\,\big|K(x-y)\big|\,d\omega^p(y) \\ &\leqslant |\widetilde{\mathcal{R}}_r\omega^p(x)| + C\,\mathcal{M}_n\omega^p(x).\end{aligned} \tag{11.7}$$

For a fixed $x \in E_k$ and $z \in \mathbb{R}^{n+1}\backslash\big[\operatorname{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\big]$, consider the function

$$u_r(z) = \mathcal{E}(z-p) - \int \mathcal{E}(z-y)\,\varphi_r(x-y)\,d\omega^p(y), \tag{11.8}$$

so that, by Lemma 7.4,

$$G(z,p) = u_r(z) - \int \mathcal{E}(z-y)\,\psi_r(x-y)\,d\omega^p(y) \qquad \text{for } m\text{-a.e. } z \in \mathbb{R}^{n+1}. \tag{11.9}$$

Differentiating (11.8) with respect to z, we obtain

$$\nabla u_r(z) = \nabla\mathcal{E}(z-p) - \int \nabla\mathcal{E}(z-y)\,\varphi_r(x-y)\,d\omega^p(y).$$

In the particular case $z = x$ we get

$$c_n\,\nabla u_r(x) = K(x-p) - \widetilde{\mathcal{R}}_r\omega^p(x),$$

and thus

$$|\widetilde{\mathcal{R}}_r\omega^p(x)| \lesssim \frac{1}{\operatorname{dist}(p,\partial\Omega)^n} + |\nabla u_r(x)|. \tag{11.10}$$

Since u_r is harmonic in $\mathbb{R}^{n+1}\backslash\big[\operatorname{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\big]$ (and so in $B_r(x)$), we have

$$|\nabla u_r(x)| \lesssim \frac{1}{r} ⨍_{B_r(x)} |u_r(z) - \alpha|\,dz, \tag{11.11}$$

for any constant $\alpha \in \mathbb{R}$, possibly depending on x and r. From the identity (11.9) we deduce that

$$|\nabla u_r(x)| \lesssim \frac{1}{r} \fint_{B_r(x)} G(z,p)\,dz + \frac{1}{r} \fint_{B_r(x)} \left| \int \left(\mathcal{E}(z-y) - \alpha'\right) \psi_r(x-y)\,d\omega^p(y)\right| dz$$
$$=: I + II,$$

for any constant $\alpha' \in \mathbb{R}$, possibly depending on x and r. To estimate the term II we use Fubini and the fact that supp$\psi_r \subset B_{2r}(x)$:

$$II \lesssim \frac{1}{r^{n+2}} \int_{y\in B_{2r}(x)} \int_{z\in B_r(x)} |\mathcal{E}(z-y) - \alpha'|\,dz\,d\omega^p(y).$$

In the case $n \geqslant 2$ we choose $\alpha' = 0$, and we get

$$II \lesssim \frac{1}{r^{n+2}} \int_{y\in B_{2r}(x)} \int_{z\in B_r(x)} \frac{1}{|z-y|^{n-1}}\,dz\,d\omega^p(y) \lesssim \frac{\omega^p(B_{2r}(x))}{r^n} \lesssim \mathcal{M}_n\omega^p(x).$$

In the case $n = 1$ we take $\alpha' = \frac{1}{2\pi}\log\frac{1}{4r}$, and then we obtain

$$II \lesssim \frac{1}{r^3} \int_{y\in B_{2r}(x)} \int_{z\in B_r(x)} \log\frac{4r}{|z-y|}\,dz\,d\omega^p(y)$$
$$\leqslant \frac{1}{r^3} \int_{y\in B_{2r}(x)} \int_{z\in B_{3r}(y)} \log\frac{4r}{|z-y|}\,dz\,d\omega^p(y) \lesssim \frac{1}{r} \int_{y\in B_{2r}(x)} r^2\,d\omega^p(y) \lesssim \mathcal{M}_1\omega^p(x).$$

Next we want to show that $I \lesssim_k 1$. Clearly it is enough to prove that

$$\frac{1}{r}\,|G(y,p)| \lesssim_k 1 \qquad \text{for all } y \in B_r(x) \cap \Omega \tag{11.12}$$

(still under the assumptions $x \in E_k$, $0 < r \leqslant r_0/2$, and (11.6)). To prove this, observe that, in the case $n \geqslant 2$, by Lemma 7.18,

$$G(y,p) \lesssim \frac{\omega^p(B_{8r}(x))}{\mathrm{Cap}(B_r(x)\backslash\Omega)} \qquad \text{for all } y \in B_r(x) \cap \Omega.$$

Notice now that, by Lemma 6.19 and (11.4), we have

$$\mathrm{Cap}(B_r(x)\backslash\Omega) \gtrsim \mathcal{H}^n_\infty(B_r(x) \cap \partial\Omega)^{\frac{n-1}{n}} \gtrsim_k \omega^p(B_r(x))^{\frac{n-1}{n}}.$$

Thus, by (11.6) and the fact that $\mathcal{M}_n\omega^p(x) \lesssim_k 1$,

$$\frac{1}{r}\,G(y,p) \lesssim_k \frac{\omega^p(B_{8r}(x))}{r\,\omega^p(B_r(x))^{\frac{n-1}{n}}} = \left(\frac{\omega^p(B_{8r}(x))}{r^n}\right)^{\frac{1}{n}} \left(\frac{\omega^p(B_{8r}(x))}{\omega^p(B_r(x))}\right)^{\frac{n-1}{n}} \lesssim_k 1,$$

which proves (11.12). Almost the same arguments work in the case $n = 1$. Indeed, by Lemma 7.22,

$$G(y,p) \lesssim \omega^p(B_{40r}(x)) \left(\log\frac{r}{\mathrm{Cap}_L(B_r(x)\backslash\Omega)}\right)^2$$
$$\lesssim \omega^p(B_{40r}(x))\,\frac{r}{\mathrm{Cap}_L(B_r(x)\backslash\Omega)} \qquad \text{for all } y \in B_r(x) \cap \Omega.$$

By Lemma 6.19 and (11.4), we have

$$\mathrm{Cap}_L(B_r(x)\backslash\Omega) \gtrsim \mathcal{H}^1_\infty(B_r(x)\cap\partial\Omega) \gtrsim_k \omega^p(B_r(x)),$$

and thus, by (11.6),

$$\frac{1}{r}\,G(y,p) \lesssim_k \frac{\omega^p(B_{40r}(x))}{\omega^p(B_r(x))} \lesssim_k 1,$$

which proves again (11.12). So in any case we deduce that

$$|\mathcal{R}_r\omega^p(x)| \leqslant |\widetilde{\mathcal{R}}_r\omega^p(x)| + C\,\mathcal{M}_n\omega^p(x) \lesssim_k \frac{1}{\mathrm{dist}(p,\partial\Omega)^n} + 1 \tag{11.13}$$

for $x \in E_k$ and $0 < r \leqslant r_0/2$ satisfying (11.6).

In the case when (11.6) does not hold, we consider the smallest $r' > r$ of the form $r' = 40^j r$, $j > 0$, such that either $r' > r_0$ or (11.6) holds with r' replacing r. Let $j_0 \geqslant 1$ be such that $r' = 40^{j_0} r$ and write

$$|\mathcal{R}_r\omega^p(x)| \leqslant |\mathcal{R}_{r'}\omega^p(x)| + \int_{r<|x-y|\leqslant r'} |K(x-y)|\,d\mu(y) \leqslant |\mathcal{R}_{r'}\omega^p(x)| + C\sum_{j=1}^{j_0} \frac{\omega^p(B_{40^j r}(x))}{(40^j r)^n}.$$

To estimate the last sum, notice that, for all $1 \leqslant j \leqslant j_0 - 1$,

$$\omega^p(B_{40^j r}(x)) < 50^{-n}\omega^p(B_{40^{j+1}r}(x)),$$

and thus, by iterating this estimate,

$$\sum_{j=1}^{j_0} \frac{\omega^p(B_{40^j r}(x))}{(40^j r)^n} \leqslant \sum_{j=1}^{j_0} \frac{50^{-n(j_0-j)}\omega^p(B_{40^{j_0} r}(x))}{40^{(j-j_0)n}\,(40^{j_0} r)^n} \lesssim \frac{\omega^p(B_{r'}(x))}{(r')^n} \leqslant \mathcal{M}_n\omega^p(x).$$

On the other hand, in case that $r' < r_0$, then (11.13) holds (with r replaced by r'), and in case that $r' \geqslant r_0$, then we have $r' \approx r_0$ and we write

$$|\mathcal{R}_{r'}\omega^p(x)| \lesssim \frac{\omega^p(\partial\Omega)}{(r')^n} \lesssim \frac{1}{r_0^n}.$$

So in any case we deduce that

$$|\mathcal{R}_r\omega^p(x)| \lesssim_k \frac{1}{r_0^n} + \frac{1}{\mathrm{dist}(p,\partial\Omega)^n} + 1,$$

which yields (11.5). □

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