Recent progresses in Nonlinear Potential Theory

Giuseppe Mingione

September 2015

Advanced Course on Geometric Analysis
Centre de Recerca Matemàtica - Barcelona
Old Poland Grandeur (and legacy)
From linear to nonlinear CZ-theory
Scheme of the course

- From linear to nonlinear CZ-theory
- Parabolic problems
Scheme of the course

- From linear to nonlinear CZ-theory
- Parabolic problems
- Non-uniformly elliptic operators
Scheme of the course

- From linear to nonlinear CZ-theory
- Parabolic problems
- Non-uniformly elliptic operators
- Nonlinear potential theory
From linear to nonlinear CZ-theory
Parabolic problems
Non-uniformly elliptic operators
Nonlinear potential theory
Parabolic potential theory
Part 1.1: The classical CZ-theory
Consider the model case

$$\triangle u = f \quad \text{in} \quad \mathbb{R}^n$$
Consider the model case

\[ \triangle u = f \quad \text{in } \mathbb{R}^n \]

Then

\[ f \in L^q \quad \text{implies} \quad D^2 u \in L^q \quad 1 < q < \infty \]

with natural failure in the borderline cases \( q = 1, \infty \).
Consider the model case

\[ \triangle u = f \quad \text{in} \quad \mathbb{R}^n \]

Then

\[ f \in L^q \quad \text{implies} \quad D^2 u \in L^q \quad 1 < q < \infty \]

with natural failure in the borderline cases \( q = 1, \infty \)

As a consequence (Sobolev embedding)

\[ Du \in L^{\frac{nq}{n-q}} \quad q < n \]
Consider the model case

$$\triangle u = f \quad \text{in} \quad \mathbb{R}^n$$
Consider the model case

$$\triangle u = f \quad \text{in} \quad \mathbb{R}^n$$

Then

$$f \in L^q \quad \text{implies} \quad D^2 u \in L^q \quad 1 < q < \infty$$

with natural failure in the borderline cases $q = 1, \infty$
Consider the model case

\[ \triangle u = f \quad \text{in} \quad \mathbb{R}^n \]

Then

\[ f \in L^q \quad \text{implies} \quad D^2 u \in L^q \quad 1 < q < \infty \]

with natural failure in the borderline cases \( q = 1, \infty \).

As a consequence (Sobolev embedding)

\[ Du \in L^{\frac{nq}{n-q}} \quad q < n \]
The singular integral approach

- **Representation via Green’s function**
  \[ u(x) \approx \int G(x, y)f(y) \, dy \]

  with
  \[
  G(x, y) = \begin{cases} 
  |x - y|^{2-n} & \text{if } n > 2 \\
  -\log |x - y| & \text{if } n = 2 
  \end{cases}
  \]

- **Differentiation yields**
  \[ D^2 u(x) = \int K(x, y)f(y) \, dy \]

  and \( K(x, y) \) is a singular integral kernel, and the conclusion follows
Singular kernels with cancellations

- **Initial boundedness assumption**
  \[ \| \hat{K} \|_{L^\infty} \leq B, \]
  where \( \hat{K} \) denotes the Fourier transform of \( K(\cdot) \)

- **Hörmander cancelation condition**
  \[ \int_{|x| \geq 2|y|} |K(x - y) - K(x)| \, dx \leq B \quad \text{for every} \quad y \in \mathbb{R}^n \]
The fractional integral approach

- Again differentiating
  \[ |Du(x)| \lesssim I_1(|f|)(x) \]

- where \( I_1 \) is a fractional integral
  \[ I_\beta(g)(x) := \int \frac{g(y)}{|x - y|^{n-\beta}} \, dy \quad \beta \in [0, n) \]

- and then
  \[ I_\beta : L^q \to L^{\frac{nq}{n-\beta q}} \quad \beta q < n \]

- This is in fact equivalent to the original proof of Sobolev embedding theorem for the case \( q > 1 \), which uses that
  \[ |u(x)| \lesssim I_1(|Du|)(x) \]
Important remark: the theory of fractional integral operators substantially differs from that of singular ones.

In fact, while the latter is based on cancelation properties of the kernel, the former only considers the size of the kernel.

As a consequence all the estimates related to the operator $I_\beta$ degenerate when $\beta \to 0$. 
Another linear case

Higher order right hand side

\[ \triangle u = \text{div } Du = \text{div } F \]

Then

\[ F \in L^q \implies Du \in L^q \quad q > 1 \]

just “simplify” the divergence operator!!
Interpolation approach

- Define the operator

\[ T : F \mapsto T(F) : = \text{gradient of the solution to } \triangle u = \text{div } F \]

- Then

\[ T : L^2 \to L^2 \]

by testing with the solution, and

\[ T : L^\infty \to \text{BMO} \]

by regularity estimates (hard part).

- Campanato-Stampacchia interpolation

\[ T : L^q \to L^q \quad 1 < q < \infty \]
Define

\[(v)_{B_s} := \frac{1}{|B_s|} \int_{B_s} v \, dx\]

and

\[\omega(R) := \sup_{s \leq R} \frac{1}{|B_s|} \int_{B_s} |v - (v)_{B_s}| \, dx\]

A map \(v\) belongs to BMO iff

\[\omega(R) < \infty\]

A map \(v\) belongs to VMO iff

\[\lim_{R \to 0} \omega(R) = 0\]
Part 1.2: Basics from Nonlinear CZ-theory
The problem is now to extend the results to (potentially degenerate) nonlinear equations of the type

$$\text{div } a(Du) = D.$$ 

The main issues are two:

- to find nonlinear methods, by-passing linearity and in particular the use of fundamental solutions
- considering estimates that allow to treat also cases in which the right-hand side $D$ does not belong to the dual space of the operator considered
Theorem (Iwaniec, Studia Math. 83)

\[ \text{div} (|Du|^{p-2} Du) = \text{div} (|F|^{p-2} F) \quad \text{in} \ \mathbb{R}^n \]

Then it holds that

\[ F \in L^q \iff Du \in L^q \quad p \leq q < \infty \]
**Theorem (Iwaniec, Studia Math. 83)**

\[
\text{div } (|Du|^{p-2} Du) = \text{div } (|F|^{p-2} F) \quad \text{in } \mathbb{R}^n
\]

Then it holds that

\[
F \in L^q \implies Du \in L^q \quad p \leq q < \infty
\]

**Theorem (DiBenedetto & Manfredi, Amer. J. Math. 93)**

The previous result holds for the $p$-Laplacean system, moreover

\[
F \in BMO \implies Du \in BMO
\]
Caffarelli & Peral (CPAM 98) give an important new approach to the $L^p$-estimates for equations as

$$\text{div } a(x, Du) = 0$$

with high oscillating coefficients in the context of homogenization.
Caffarelli & Peral (CPAM 98) give an important new approach to the $L^p$-estimates for equations as

$$\text{div } a(x, Du) = 0$$

with high oscillating coefficients in the context of homogenization.

Byun & Wang, in a recent series of papers, used the above method to derive Calderón-Zygmund estimates for solutions to boundary value problems involving non-homogeneous equations, under weak assumptions on the boundary regularity. Papers by several authors like: Lee, Oh, Ok, Yao, Zhou.
The local estimate

\[
\left( \int_{B_R} |Du|^q \, dz \right)^{\frac{1}{q}} \leq c \left( \int_{B_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c \left( \int_{B_{2R}} |F|^q \, dz \right)^{\frac{1}{q}}
\]

holds for solutions of solutions of the previous problems
In the same way the non-linear result of Iwaniec extends to all elliptic equations in divergence form of the type

\[ \text{div } a(Du) = \text{div } (|F|^{p-2}F) \]

where \( a(\cdot) \) is \( p \)-monotone in the sense of the previous slides.

and to all systems with special structure

\[ \text{div } (g(|Du|)Du) = \text{div } (|F|^{p-2}F) \]

Moreover, VMO-coefficients can be considered too

\[ \text{div } [c(x)a(Du)] = \text{div } (|F|^{p-2}F) \]
Open problems

- The full range

\[ p - 1 < q < \infty \]

Compare with the linear case \( p = 2 \).

- This is the case **below the duality exponent**, when

\[ \text{div} \left( |F|^{p-2}F \right) \notin W^{-1,p'} \]

- Iwaniec & Sbordone (Crelle J., 94), Lewis (Comm. PDE 93)

\[ p - \varepsilon \leq q < \infty \quad \varepsilon \equiv \varepsilon(n, p) \]

- Parabolic case: important approach of Kinnunen & Lewis (Ark. Math 02)
Part 2: Parabolic problems
The parabolic case


\[ u_t - \text{div} \left( |Du|^{p-2} Du \right) = \text{div} \left( |F|^{p-2} F \right) \quad \text{in } \Omega \times (0, T) \]

for

\[ p > \frac{2n}{n+2} \]

Then it holds that

\[ F \in L^q_{\text{loc}} \implies Du \in L^q_{\text{loc}} \quad \text{for } p \leq q < \infty \]
The parabolic case

Theorem (Acerbi & Min., Duke Math. J. 07)

\[ u_t - \nabla \cdot (|Du|^{p-2} Du) = \nabla \cdot (|F|^{p-2} F) \quad \text{in } \Omega \times (0, T) \]

for

\[ p > \frac{2n}{n+2} \]

Then it holds that

\[ F \in L^q_{\text{loc}} \implies Du \in L^q_{\text{loc}} \quad \text{for} \quad p \leq q < \infty \]

For \( q = p + \varepsilon \) see the important work of Kinnunen & Lewis (Duke Math. J. 01)
The parabolic case

- The elliptic approach via maximal operators only works in the case $p = 2$
- The result also works for systems, that is when $u(x, t) \in \mathbb{R}^N$, $N \geq 1$
- First Harmonic Analysis free approach to non-linear Calderón-Zygmund estimates
The parabolic case

- The result is new already in the case of equations i.e. $N = 1$, the difficulty being in the lack of homogenous scaling of parabolic problems with $p \neq 2$, and not being caused by the degeneracy of the problem, but rather by the polynomial growth.

- The result extends to all parabolic equations of the type

$$u_t - \text{div} \ a(Du) = \text{div} \ (|F|^{p-2} F)$$

with $a(\cdot)$ being a monotone operator with $p$-growth. More precisely we assume

$$\left\{ \begin{array}{l}
\nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_2 - z_1|^2 \leq \langle a(z_2) - a(z_1), z_2 - z_1 \rangle \\
|a(z)| \leq L(s^2 + |z|^2)^{\frac{p-1}{2}}\end{array} \right.$$
The result also holds for systems with a special structure (sometimes called Uhlenbeck structure). This means

$$u_t - \text{div} \ a(Du) = \text{div} \ (|F|^{p-2}F)$$

with $a(\cdot)$ being $p$-monotone in the sense of the previous slide, and satisfying the structure assumption

$$a(Du) = g(|Du|)Du$$

The $p$-Laplacean system is an instance of such a structure
Elliptic vs parabolic local estimates

- **Elliptic estimate**

\[
\left( \int_{B_R} |Du|^q \, dz \right)^{\frac{1}{q}} \leq c \left( \int_{B_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c \left( \int_{B_{2R}} |F|^q \, dz \right)^{\frac{1}{q}}
\]
Elliptic vs parabolic local estimates

- **Elliptic estimate**

\[
\left( \int_{B_R} |Du|^q \, dz \right)^{\frac{1}{q}} \leq c \left( \int_{B_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c \left( \int_{B_{2R}} |F|^q \, dz \right)^{\frac{1}{q}}
\]

- **Parabolic estimate - \( p \geq 2 \)**

\[
\left( \int_{Q_R} |Du|^q \, dz \right)^{\frac{1}{q}} \leq c \left[ \left( \int_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + \left( \int_{Q_{2R}} |F|^q \, dz \right)^{\frac{1}{q}} + 1 \right]^{\frac{p}{2}}
\]
Elliptic vs parabolic local estimates

- **Elliptic estimate**

\[
\left( \int_{B_R} |Du|^q \, dz \right)^{1/q} \leq c \left( \int_{B_{2R}} |Du|^p \, dz \right)^{1/p} + c \left( \int_{B_{2R}} |F|^q \, dz \right)^{1/q}
\]

- **Parabolic estimate - \( p \geq 2 \)**

\[
\left( \int_{Q_R} |Du|^q \, dz \right)^{1/q} \\
\leq c \left[ \left( \int_{Q_{2R}} |Du|^p \, dz \right)^{1/p} + \left( \int_{Q_{2R}} |F|^q \, dz \right)^{1/q} + 1 \right]^{p/2}
\]

- **Parabolic cylinders** \( Q_R \equiv B_R \times (t_0 - R^2, t_0 + R^2) \)
Elliptic vs parabolic local estimates

- **Elliptic estimate**

\[
\left( \int_{B_R} |Du|^q \, dz \right)^{\frac{1}{q}} \leq c \left( \int_{B_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c \left( \int_{B_{2R}} |F|^q \, dz \right)^{\frac{1}{q}}
\]

- **Parabolic estimate** - \( p \geq 2 \)

\[
\left( \int_{Q_R} |Du|^q \, dz \right)^{\frac{1}{q}} \leq c \left[ \left( \int_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + \left( \int_{Q_{2R}} |F|^q \, dz \right)^{\frac{1}{q}} + 1 \right]^{\frac{p}{2}}
\]

- **Parabolic cylinders** \( Q_R \equiv B_R \times (t_0 - R^2, t_0 + R^2) \)

- **The exponent** \( p/2 \) **is the scaling deficit of the system**
Interpolation nature of local estimates

- **Parabolic local estimate** - $p \geq 2$

$$\left( \int_{Q_R} |Du|^q \, dz \right)^{\frac{1}{q}}$$

$$\leq c \left[ \left( \int_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c(q) \left( \int_{Q_{2R}} |F|^q \, dz \right)^{\frac{1}{q}} + 1 \right]^{\frac{p}{2}}$$

- **Taking** $F = 0$ **and letting** $q \to \infty$ **yields**

$$\sup_{Q_R} |Du| \leq c \left[ \left( \int_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + 1 \right]^{\frac{p}{2}}$$

- **This is the original sup estimate** of DiBenedetto & Friedman (Crelles J. 84)
The local estimate in the singular case

- The singular case

\[ \frac{2n}{n + 2} < p < 2 \]

- The local estimate is

\[
\left( \int_{Q_R} |Du|^q \, dz \right)^{\frac{1}{q}} \\
\leq c \left[ \left( \int_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c(q) \left( \int_{Q_{2R}} |F|^q \, dz \right)^{\frac{1}{q}} + 1 \right]^{\frac{2p}{p(n+2) - 2n}}
\]

where \( c \equiv c(n, N, p) \)

- Observe that

\[ \frac{2p}{p(n+2) - 2n} \to \infty \quad \text{when} \quad p \to \frac{2n}{n + 2} \]
The basic analysis is the following: consider intrinsic cylinders

\[ Q^\lambda_\varrho(z_0) \equiv Q^\lambda_\varrho(x_0, t_0) = B(x_0, \varrho) \times (t_0 - \lambda^{2-p} \varrho^2, t_0) \]

where it happens that

\[ |Du| \approx \lambda \quad \text{in} \quad Q^\lambda_\varrho(x_0, t_0) \]

then the equation behaves as

\[ u_t - \lambda^{p-2} \Delta u = 0 \]

that is, scaling back in the same cylinder, as the heat equation

On intrinsic cylinders estimates "ellipticize"; in particular, they become homogeneous
The effect of intrinsic geometry

**Theorem (DiBenedetto & Friedman, Crelle J. 85)**

*There exists a universal constant $c \geq 1$ such that*

$$c \left( \int_{Q_R^\lambda(z_0)} |Du|^{p-1} \, dz \right)^{1/(p-1)} \leq \lambda$$

*then*

$$|Du(z_0)| \leq \lambda$$
Recall the estimate

\[ \int |Du|^q = q \int_0^\infty \lambda^{q-1} |\{|Du| > \lambda\}| \, d\lambda \]

Therefore we want to find a decay estimates for the level sets \( |\{|Du| > \lambda\}| \) in terms of the level sets \( |\{|F| > \lambda\}| \).
Recall the estimate

\[ \int |Du|^q = q \int_0^\infty \lambda^{q-1} |\{ |Du| > \lambda \}| \, d\lambda \]

Therefore we want to find a decay estimates for the level sets \(|\{ |Du| > \lambda \}|\) in terms of the level sets \(|\{ |F| > \lambda \}|\). We make a decomposition of CZ type of \(|\{ |Du| > \lambda \}|\) and for this we use a direct exit time argument on intrinsic cubes via the functional

\[ \int_{Q_\lambda^R} (|Du|^p + M|F|^p) \, dx \, dt \]
Sketch of the proof (lots of cheating)

If \( z_0 \in \{|Du| > \lambda \} \) then it happens that

\[
\liminf_{r \to 0} \int_{Q_r^\lambda(z_0)} (|Du|^p + M|F|^p) \, dx \, dt > \lambda
\]

therefore for every such point we find an exit time radius \( r(z_0) \) such that

\[
\int_{Q_r(z_0)}^\lambda (|Du|^p + M|F|^p) \, dx \, dt \approx \lambda
\]

and using Vitali or Besicovitch cover

\[
|\{|Du|^p > \lambda\}| \subset \bigcup_i Q_{r(z_i)/2}(z_i)
\]
This means that
\[ \int_{Q^\lambda_{r(z_0)}} |Du|^p \, dx \, dt \lesssim \lambda \quad \text{and} \quad \int_{Q^\lambda_{r(z_0)}} |F|^p \, dx \, dt \lesssim \frac{\lambda}{M} \]

therefore for every such point we find an exit time radius \( r(z_0) \) such that
\[ \lambda \lesssim \int_{Q^\lambda_{r(z_i)}} (|Du|^p + M|F|^p) \, dx \, dt \]
Then solve

\[
\begin{cases}
(v_t)_t - \text{div} \left( |Dv|^p - 2 Dv \right) = 0 & \text{in } Q^\lambda_{r(z_i)}(z_i) \\
v_i = u & \text{in } \partial_p Q^\lambda_{r(z_i)}(z_i)
\end{cases}
\]
Then solve

\[
\begin{cases}
(v_i)_t - \text{div}(|Dv_i|^{p-2}Dv_i) = 0 \quad \text{in} \quad Q^\lambda_{r(z_i)}(z_i) \\
v_i = u \quad \text{in} \quad \partial_p Q^\lambda_{r(z_i)}(z_i)
\end{cases}
\]

then

\[
\int_{Q^\lambda_{r(z_i)}(z_i)} |Dv_i|^p \, dx \, dt \lesssim \lambda
\]

and

\[
\int_{Q^\lambda_{r(z_i)}(z_i)} |Dv_i - Du|^p \, dx \, dt \lesssim \frac{\lambda}{M}
\]
The first inequality allows to assert that
\[ \sup_{Q^{\lambda}_{r(z_0)/2}} |Dv_i|^p \lesssim \lambda \]
that is
\[ |Q^{\lambda}_{r(z_0)/2} \cap \{|Dv_i|^p > \lambda\}| = 0 \]
Sketch of the proof (lots of cheating)

Then

\[
|Q_{r(z_0)/2} \cap \{|Du|^p > \lambda\}| \lesssim |Q_{r(z_0)/2} \cap \{|Du - Dv_i|^p > \lambda\}| \\
+ |Q_{r(z_0)/2} \cap \{|Dv_i|^p > \lambda\}| \\
\lesssim |Q_{r(z_0)/2} \cap \{|Du - Dv_i|^p > \lambda\}| \\
\lesssim \frac{1}{\lambda} \int_{Q_{r(z_i)/2}(z_i)} |Du - Dv_i|^p \, dx \, dt \\
\lesssim \frac{|Q_{r(z_0)}|}{M}
\]
Density information (De Giorgi style)

\[ \frac{|Q_{r(z_0)/2}^\lambda \bigcap \{|Du|^p > \lambda\}|}{|Q_{r(z_0)/2}|} \lesssim \frac{1}{M} \]

Density is small provided \(M\) is large.
Density information (De Giorgi style)

\[
\frac{|Q_r^{\lambda}(z_0)/2 \cap \{|Du|^p > \lambda\}|}{|Q_r(z_0)/2|} \lesssim \frac{1}{M}
\]

density is small provided \( M \) is large
Sketch of the proof (lots of cheating)

But then, using the exit time information

\[ |Q_{r(z_0)}| \lesssim \frac{1}{\lambda} \int_{Q_{r(z_i)}(z_i) \cap \{|Du|^p > \lambda\}} |Du|^p \, dx \, dt \]

\[ + \frac{1}{\lambda} \int_{Q_{r(z_i)}(z_i) \cap \{|F|^p > \lambda\}} M|F|^p \, dx \, dt \]
Sketch of the proof (lots of cheating)

Summarizing

\[ \lambda^{\gamma-1} |Q_{r(z_0)/2}^\lambda \cap \{|Du|^p > \lambda\}| \]

\[ \geq \frac{\lambda^{\gamma-2}}{M} \int_{Q_{r(z_i)}(z_i) \cap \{|Du|^p > \lambda\}} |Du|^p \, dx \, dt \]

\[ + \lambda^{\gamma-2} \int_{Q_{r(z_i)}(z_i) \cap \{|F|^p > \lambda\}} |F|^p \, dx \, dt \]
Sketch of the proof (lots of cheating)

Summarizing

\[
\lambda^{\gamma-1} |Q_{r(z_0)/2}^\lambda \cap \{|Du|^p > \lambda\}| \\
\leq \lambda^{\gamma-2} \int_{Q_r(z_i)^\lambda(z_i) \cap \{|Du|^p > \lambda\}} |Du|^p \, dx \, dt \\
+ \lambda^{\gamma-2} \int_{Q_r(z_i)^\lambda(z_i) \cap \{|F|^p > \lambda\}} |F|^p \, dx \, dt
\]

Integration yields

\[
\int |Du|^{p\gamma} \approx \int_\infty^\infty \lambda^{\gamma-1} \{|Du|^p > \lambda\} \approx \lambda^{\gamma-2} \int \{|Du|^p > \lambda\} |Du|^p \, dx \, dt + \lambda^{\gamma-2} \int \{|Du|^p > \lambda\} |F|^p \, dx \, dt
\]

\[
\leq \frac{\lambda^{\gamma-2}}{M} \int \{|Du|^p > \lambda\} |Du|^p \, dx \, dt + \lambda^{\gamma-2} \int \{|Du|^p > \lambda\} |F|^p \, dx \, dt
\]

\[
\approx \frac{1}{M} \int |Du|^{p\gamma} \, dx \, dt + c(M) \int |F|^{p\gamma} \, dx \, dt
\]
Part 3: Non-uniformly elliptic operators
Classical facts

consider variational problems of the type

\[ W^{1,1} \ni v \mapsto \int_{\Omega} f(x, Dv) \, dx \quad \Omega \subset \mathbb{R}^n \]
consider variational problems of the type

\[ W^{1,1} \ni v \mapsto \int_{\Omega} f(x, Dv) \, dx \quad \Omega \subset \mathbb{R}^n \]

the standard growth conditions are

\[ |z|^p \lesssim f(x, z) \lesssim |z|^p + 1 \]

for \( p > 1 \), and the problem is well settled in \( W^{1,p} \)
consider variational problems of the type

\[ W^{1,1} \ni v \mapsto \int_{\Omega} f(x, Dv) \, dx \quad \Omega \subset \mathbb{R}^n \]

the standard growth conditions are

\[ |z|^p \lesssim f(x, z) \lesssim |z|^p + 1 \]

for \( p > 1 \), and the problem is well settled in \( W^{1,p} \)

a model example is

\[ v \mapsto \int_{\Omega} c(x)|Dv|^p \, dx \]
consider now variational problems of the type

\[ W^{1,1} \ni v \mapsto \int_{\Omega} f(x, Dv) \, dx \quad \Omega \subset \mathbb{R}^n \]

with

\[ |z|^p \lesssim f(x, z) \lesssim |z|^q + 1 \quad \text{and} \quad q > p > 1 \]
a basic condition

\[ W^{1,1} \ni v \mapsto \int_{\Omega} f(Dv) \, dx \quad \Omega \subset \mathbb{R}^n \]

with

\[ |z|^p \lesssim f(z) \lesssim |z|^q + 1 \quad \text{and} \quad q > p > 1 \]

then

\[ \frac{q}{p} < 1 + o(n) \]

is a sufficient (Marcellini) and necessary (Giaquinta and Marcellini) condition for regularity.
Several people on non-uniformly elliptic operators

- Leon Simon
- Uraltseva & Urdaletoa
- Zhikov
- Marcellini
- Hong
- Lieberman
- Fusco-Sbordone
- many, many, many others (including me, unfortunately for the subject)
Non-autonomous functionals of the type

\[ v \mapsto \int_{\Omega} f(x, Dv) \, dx \]

new phenomena appear in this situation, and the presence of \( x \) is *not any longer a perturbation*
Zhikov introduced, between the 80s and the 90s, the following functionals:

\begin{align*}
v & \mapsto \int_{\Omega} |Dv|^2 w(x) \, dx \quad w(x) \geq 0 \\
v & \mapsto \int_{\Omega} |Dv|^{p(x)} \, dx \quad p(x) \geq 1 \\
v & \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q) \, dx \quad a(x) \geq 0
\end{align*}

motivations: modelling of strongly anisotropic materials, Elasticity, Homogenization, Lavrentiev phenomenon etc
Two counterexamples

**Theorem (Esposito-Leonetti-Min. JDE 04)**

For every choice of \( n \geq 2 \), \( \Omega \subset \mathbb{R}^n \) and of \( \varepsilon > 0 \) and \( \alpha \in (0, 1) \)

there exists a non-negative function \( a(\cdot) \in C^{0,\alpha} \), a boundary datum \( u_0 \in W^{1,\infty}(B) \) and exponents \( p, q \) satisfying

\[
n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon
\]

such that the solution to the Dirichlet problem

\[
\begin{cases}
  u \mapsto \min_w \int_B (|Dv|^p + a(x)|Dv|^q) \, dx \\
  w \in u_0 + W^{1,p}_0(B)
\end{cases}
\]

does not belong to \( W^{1,q}_{\text{loc}}(B) \)
The example goes via Lavrentiev phenomenon

\[ \inf_{w \in u_0 + W_0^{1,p}(B)} \int_B (|Dv|^p + a(x)|Dv|^q) \, dx \]

\[ < \inf_{w \in u_0 + W_0^{1,p}(B) \cap W_{\text{loc}}^{1,q}(B)} \int_B (|Dv|^p + a(x)|Dv|^q) \, dx \]
Two counterexamples

Theorem (Fonseca-Malý-Min. ARMA 04)

For every choice of $n \geq 2$, $\Omega \subset \mathbb{R}^n$ and of $\varepsilon > 0$, $\alpha > 0$, there exists a non-negative function $a(\cdot) \in C[\alpha]+\{\alpha\}$, a boundary datum $u_0 \in W^{1,\infty}(B)$ and exponents $p, q$ satisfying

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases}
  u \mapsto \min_{w} \int_{B} (|Dv|^p + a(x)|Dv|^q) \, dx \\
  w \in u_0 + W^{1,p}_0(B)
\end{cases}$$

has a singular set of essential discontinuity points of Hausdorff dimension larger than $n - p - \varepsilon$
Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimiser of the functional

$$v \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q) \, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad \frac{q}{p} < 1 + \frac{\alpha}{n}$$

then

$Du$ is Hölder continuous
Theorem (Colombo-Min. ARMA 15)

Let \( u \in W^{1,p}(\Omega) \) be a bounded local minimiser of the functional

\[
v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q) \, dx
\]

and assume that

\[
0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad q \leq p + \alpha
\]

then

\( Du \) is Hölder continuous
Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$ be a bounded local minimiser of the functional

$$v \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q) \, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad q \leq p + \alpha$$

then

$Du$ is Hölder continuous

Notice the delicate borderline case $q = p + \alpha$ is achieved
Theorem 3

**Theorem (Colombo-Min. JFA 15)**

Let \( u \in W^{1,p}(\Omega) \) be a distributional solution to

\[
\text{div} \left( |Du|^{p-2}Du + a(x)|Du|^{q-2}Du \right) = \text{div} \left( |F|^{p-2}F + a(x)|F|^{q-2}F \right)
\]

and assume that

\[
0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad \frac{q}{p} \leq 1 + \frac{\alpha}{n}
\]

then

\[
(|F|^p + a(x)|F|^q) \in L^\gamma_{\text{loc}} \quad \implies \quad (|Du|^p + a(x)|Du|^q) \in L^\gamma_{\text{loc}}
\]

for every \( \gamma \geq 1 \).
Theorem 4

Theorem (Colombo-Min. JFA 15)

Let $u \in W^{1,p}(\Omega)$ be a **bounded** minimiser of the functional

$$
v \mapsto \int \left[ |Dv|^p + a(x)|Dv|^q - (|F|^{p-2} + a(x)|F|^{q-2}) \langle F, Dv \rangle \right]
$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad q \leq p + \alpha$$

and

$$\sup_{B_\rho} \rho^{p_0} \int_{B_\rho} \left[ |F|^p + a(x)|F|^q \right] dx < \infty \quad \text{for some} \quad p_0 < p$$

then

$$(|F|^p + a(x)|F|^q) \in L^\gamma_{\text{loc}} \implies (|Du|^p + a(x)|Du|^q) \in L^\gamma_{\text{loc}}$$
The general viewpoint

is to consider functionals as

$$v \mapsto \int_{\Omega} f(x, v, Dv) \, dx$$

where

$$H(x, |z|) \lesssim f(x, u, z) \lesssim H(x, |z|) + 1$$

with

$$H(x, |z|) = |z|^p + a(x)|z|^q$$

being a replacement of

$$|z|^p$$
the Euler equation of the functional is

\[ \text{div } a(x, Du) = \text{div } (|Du|^{p-2}Du + (q/p)a(x)|Du|^{q-2}Du) = 0 \]

then

\[
\frac{\text{highest eigenvalue of } \partial_z a(x, Du)}{\text{lowest eigenvalue of } \partial_z a(x, Du)} \approx 1 + a(x)|Du|^{q-p} \\
\approx 1 + R^\alpha|Du|^{q-p}
\]
Heuristic explanation - the bound $q \leq p + \alpha$

consider the usual $p$-capacity for $p < n$

$$\operatorname{cap}_p(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |Dv|^p \, dx : f \in W^{1,p}, f \geq 1 \text{ on } B_r \right\}$$

we have

$$\operatorname{cap}_p(B_r) \approx r^{n-p}$$

then consider the weighted capacity

$$\operatorname{cap}_{q,\alpha}(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |x|^\alpha |Dv|^q \, dx : f \in C_0^\infty(\mathbb{R}^n), f \geq 1 \text{ on } B_r \right\}$$

we then have (the ball is centered at the origin)

$$\operatorname{cap}_{q,\alpha}(B_r) \approx r^{n-q+\alpha}$$
We then ask for

$$\text{cap}_{q,\alpha}(B_r) \lesssim \text{cap}_p(B_r)$$

that is

$$r^{n-q+\alpha} \leq r^{n-p}$$

for $r$ small enough, so that

$$q \leq p + \alpha$$
A parallel with Muckenhoupt weights

A maximal theorem holds

$$\int_{\Omega} \left[ H(x, |M(f)|) \right]^t \, dx \lesssim \int_{\Omega} \left[ H(x, |f|) \right]^t \, dx$$

where $Mf$ is the usual (localised) Hardy-Littlewood maximal operator, together with a Sobolev-Poincaré type inequality

$$\left( \int_{B_R} \left[ H \left( x, \left| \frac{f - (f)_{B_R}}{R} \right| \right) \right]^d \, dx \right)^{1/d} \leq c \int_{B_R} \left[ H(x, |Df|) \right] \, dx$$

for $d > 1$
A parallel with Muckenhoupt weights

A non-negative function $w \in L^p$ is said to be of class $A_p$ if

$$\sup_{B_R} \left( \int_{B_R} |w| \, dx \right) \left( \int_{B_R} |w|^{1/(1-p)} \, dx \right)^{1/(p-1)} < \infty$$

then it follows

$$\int_{\Omega} |M(f)|^t w(x) \, dx \lesssim \int_{\Omega} |f|^t w(x) \, dx$$

holds for $t > 1$ and

$$\left( \int_{B_R} \left[ H \left( x, \left| \frac{f - (f)_{B_R}}{R} \right| \right) \right]^d \, dx \right)^{1/d} \leq c \int_{B_R} H(x, |Df|) \, dx$$

holds for $d > 1$
Questions

- Study more general conditions for which such abstract results hold in connection to regularity theorems, for instance
- Define the quantity

\[
\text{cap}_H(B_r) = \inf \left\{ \int_{\mathbb{R}^n} H(x, Dv) \, dx : f \in C_0^\infty(\mathbb{R}^n), f \geq 1 \text{ on } B_r \right\}
\]

and prove it is a capacity in the usual sense when \( q \leq p + \alpha \);
also consider the condition \( q/p < 1 + \alpha/n \)
- Consider removability of singularities problems using this capacity, and in connection obstacle problems
Minima of functionals of the type

\[ v \rightarrow \int f(x, v, Dv) \, dx \]

with

\[ f(x, v, z) \approx |z|^p w(x) \equiv H(x, |z|) \]

are locally Hölder continuous provided

Questions

- Study more general conditions for which such abstract results hold in connection to regularity theorems, for instance
- Define the quantity

\[
\text{cap}_H(B_r) = \inf \left\{ \int_{\mathbb{R}^n} H(x, Dv) \, dx : f \in C_0^\infty(\mathbb{R}^n), f \geq 1 \text{ on } B_r \right\}
\]

and prove it is a capacity in the usual sense when \( q \leq p + \alpha \);
also consider the condition \( q/p < 1 + \alpha/n \)

- Consider removability of singularities problems using this capacity, and in connection obstacle problems
- Consider weights with respect to this new norm
There exists a universal threshold \( M \equiv M(n, p, q, \alpha) \) such that if on the ball \( B_R \)

\[
a_i(R) := \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha
\]

Then our functional is essentially equivalent to

\[
v \mapsto \int_{B_R} |Dv|^p \, dx
\]
there exists a universal threshold $M \equiv M(n, p, q, \alpha)$ such that if on the ball $B_R$

$$a_i(R) := \inf_{x \in B_R} a(x) > M[a]_0,\alpha R^\alpha$$

then our functional is essentially equivalent to

$$v \mapsto \int_{B_R} (|Dv|^p + a_i(R)|Dv|^q) \, dx$$

Implementation of this is very delicate and goes through a delicate analysis involving an exit time argument
Lemma

Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional

$$v \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q) \, dx$$

and let $B_R$ be a ball such that

$$\inf_{x \in B_R} a(x) \leq M[a] \alpha R^\alpha$$

and

$$\frac{q}{p} < 1 + \frac{\alpha}{n}$$

hold. then there exists a positive constant $c \equiv c(M)$ such that

$$\left( \int_{B_{R/2}} |Du|^{2q-p} \, dx \right)^{1/(2q-p)} \leq c \left( \int_{B_{R}} |Du|^p \, dx \right)^{1/p}$$
Lemma

Let $u \in W^{1,p}(\Omega)$ be a bounded local minimiser of the functional

$$v \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q) \, dx$$

and let $B_R$ be a ball such that

$$\inf_{x \in B_R} a(x) \leq M[a]_\alpha R^\alpha \quad \text{and} \quad q \leq p + \alpha$$

hold. Then there exists a positive constant $c \equiv c(M)$ such that

$$\int_{B_{R/2}} |Du|^p \, dx \leq c \int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^p \, dx$$
Theorem (Colombo-Min. ARMA 15)

Let \( u \in W^{1,p}(\Omega) \) be a bounded local minimiser of the functional

\[
v \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q) \, dx
\]

and assume that

\[
0 \leq a(\cdot) \in C^{0,\alpha}(\Omega)\quad \text{and} \quad q \leq p + \alpha
\]

then

\( Du \) is Hölder continuous
Theorem (Colombo-Min. ARMA 15)

Let \( u \in W^{1,p}(\Omega) \) be a \textbf{bounded} local minimiser of the functional

\[
v \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q) \, dx
\]

and assume that

\[
0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad q \leq p + \alpha
\]

then

\( Du \) is Hölder continuous

A parabolic theorem is on its way
Proof goes in ten different Steps

- Step 1: Low Hölder continuity (to treat the borderline case $q = p + \alpha$)
- Step 2: $p$-harmonic approximation to handle the $p$-phase
- Step 3: Decay estimate on all scales in the $(p, q)$-phase
- Step 4: Exit time argument implies $u \in C^{0,\gamma}$ for every $\gamma < 1$
- Step 5: Previous Step implies that $Du$ is in every Morrey space
- Step 6: Morrey space regularity of the gradient implies absence of Lavrentiev phenomenon
- Step 7: Gradient fractional Sobolev regularity
- Step 8: Upgraded Caccioppoli inequality via interpolation inequalities in fractional Sobolev spaces
- Step 9: Higher integrability of the gradient implies a better $p$-harmonic approximation in the $p$-phase
- Step 10: Hölder gradient continuity via weighted separation of phases
I will consider for simplicity the case $p \geq 2$
I will consider for simplicity the case $p \geq 2$

$$E(u; x_0, R) := \left( \int_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^p \, dx \right)^{1/p}$$

You want to prove that

$$E(u; x_0, \tau^k R) \leq \tau^{k\gamma} E(u; x_0, R)$$

and this implies that

$$u \in C^{0, \gamma}$$
Step 1: Preliminary microscopic Hölder continuity

\( u \) is locally Hölder continuous with some potentially microscopic exponent \( \gamma_0 \in (0, 1) \). This essentially serve to catch the borderline case \( q = p + \alpha \).
Step 2: $p$-phase

assume

$$\inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha$$

holds for some number $M \geq 1$. then for every $\gamma \in (0, 1)$ there exists a positive radius $R_* \equiv R_*(M, \gamma)$ and $\tau \equiv \tau(M, \gamma) \in (0, 1/4)$ such that the decay estimate

$$E(u; x_0, \tau R) \leq \tau^\gamma E(u; x_0, R)$$

holds whenever $0 < R \leq R_*$
Step 2: $p$-phase

→ Caccioppoli inequality in the $p$-phase becomes

$$
\int_{B_{R/2}} |Du|^p \, dx \leq c \int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^p \, dx = \left( \frac{E(u; x_0, R)}{R} \right)^p,
$$

→ then define

$$
v(x) := \frac{u(x_0 + Rx)}{E(u; x_0, R)}, \quad x \in B_1
$$

so that

$$
\int_{B_{1/2}} |Dv|^p \, dx \leq c
$$
→ moreover, \( \nu \) solves, for every \( \varphi \in C_0^\infty (B_1) \)

\[
\int_{B_1} \langle |D\nu|^{p-2} D\nu + (q/p)\tilde{a}(x) R^{p-q}[E(u; x_0, R)]^{q-p} |D\nu|^{q-2} D\nu, D\varphi \rangle \, dx = 0
\]

this means that

\[
\left| \int_{B_1} \langle |D\nu|^{p-2} D\nu, D\varphi \rangle \, dx \right| \\
\leq c M R^{p+\alpha-q} [E(u; x_0, R)]^{q-p} \| D\varphi \|_{L^\infty (B_{1/2})} \int_{B_{1/2}} |D\nu|^{q-1} \, dx \\
\leq c R^{p+\alpha-q+\gamma_0(q-p)} \| D\varphi \|_{L^\infty (B_{1/2})} \left( \int_{B_{1/2}} |D\nu|^p \, dx \right)^{\frac{q-1}{p}} \\
\leq C_* R_*^{p+\alpha-q+\gamma_0(q-p)} \| D\varphi \|_{L^\infty (B_{1/2})}
\]
we conclude that

$$\left| \int_{B_1} \langle |Dv|^{p-2} Dv, D\varphi \rangle \, dx \right| \leq \varepsilon \|D\varphi\|_{L^\infty(B_{1/2})}$$

by taking $R_*$ suitably small
Step 2: $p$-phase

→ apply the $p$-harmonic approximation lemma

**Theorem (Duzaar - Min. Calc. Var. 04)**

Given $\varepsilon > 0$ and $L > 0$, there exists $\delta \in (0, 1]$ such that whenever $v \in W^{1,p}(B_{1/2})$ satisfies

$$\int_{B_{1/2}} |Dv|^p \, dx \leq L$$

and

$$\int_{B_{1/2}} \left\langle |Dv|^{p-2}Dv, D\varphi \right\rangle \, dx \leq \delta \|D\varphi\|_{L^\infty(B_{1/2})}$$

holds for all $\varphi \in C^1_0(B_{1/2})$. There exists a $p$-harmonic map $h \in W^{1,p}(B_{1/2})$, that is $\text{div} \left( |Dh|^{p-2}Dh \right) = 0$, such that

$$\int_{B_{1/2}} |v - h|^p \, dx \leq \varepsilon^p$$
Step 2: $p$-phase

→ we conclude that

$$\left| \int_{B_1} \langle |Dv|^{p-2} Dv, D\varphi \rangle \, dx \right| \leq \varepsilon \|D\varphi\|_{L^\infty(B_{1/2})}$$

by taking $R_*$ suitably small

→ find a $p$-harmonic map $h$ such that

$$\int_{B_{1/2}} |v - h|^p \, dx \leq \varepsilon^p$$

→ for harmonic maps you know that you have a good excess decay, and therefore, since $v$ and $h$ are close, then also $v$ has the same property; scaling back, the same property holds for $u$
Step 3: \((p, q)\)-phase

assume

\[
\inf_{x \in B_R} a(x) > M[a]_{0, \alpha} R^\alpha
\]

holds for some number \(M \geq 1\). Fix \(\gamma \in (0, 1)\); there exist positive constants \(M_1 \geq 4\) and \(\tau \in (0, 1/4)\), with depending on \(\gamma\), such that if \(M \geq M_1\), then the decay estimate

\[
E(u; x_0, \tau^k R) \lesssim \tau^{k \gamma} R \left[ \int_{B_2R} \left( \left| \frac{u - (u)_{B_2R}}{R} \right|^p + a(x) \left| \frac{u - (u)_{B_2R}}{R} \right|^q \right) \, dx \right]^{1/p}
\]

holds for every integer \(k \geq 0\).
Step 4: Separation of phases via exit time

→ choose $\gamma \in (0, 1)$
→ Find $M \geq 1$ and $\tau_2$ from Step 2
→ Use this $M$ in Step 1 and find $R_*$ and $\tau_1$ from Step 1
→ consider the sequence of balls

$$\ldots \subset B_{R_k} \subset B_{R_1} \subset B_R,$$

and the condition

$$\inf_{x \in B_{R_k}} a(x) \leq MR_k^\alpha \quad (1)$$

the exit time index is

$$m := \min \{ k \in \mathbb{N} \cup \{\infty\} : (1) \text{ fails} \} .$$
→ keep on using Step 1 as long as the exit time is not reached, this yields

\[ E(u; x_0, \tau_1^k R_0) \leq \tau_1^{k\gamma} E(u; x_0, R_0) \quad \text{for every } k \in \{0, \ldots, m\} . \]

→ after the exit time you can use Step 2 to get

\[
E(u; x_0, \tau_1^k \tau_1^m R_0) \lesssim \tau_2^{k\gamma} E(u; x_0, 2\tau_1^m R_0) \\
+ \tau_2^{k\gamma} \tau_1^m R_0 \left( \int_{B_{2\tau_1^m R_0}} a(x) \left| \frac{u - (u)_{B_{2\tau_1^m R_0}}}{\tau_1^m R_0} \right|^q \, dx \right)^{1/p}
\]

→ match the two inequalities using the exit time condition and ones again the bound \( q \leq p + \alpha \)
Step 5: Morrey space regularity of the gradient

this tells that

$$\int_{B_R} |Du|^p \ dx \lesssim R^{n-\theta} \quad \forall \ \theta > 0$$
there exists a sequence of smooth functions \( \{u_n\} \) such that

\[
\int_B (|Du_n|^p + a(x)|Du_n|^q) \, dx \\
\rightarrow \int_B (|Du|^p + a(x)|Du|^q) \, dx
\]

for every ball \( B \subset \Omega \)
We get suitable uniform estimates in

\[ Du \in W^{\beta/p, p} \quad \text{for every } \beta < \alpha \]
Step 7: Fractional differentiability

We get suitable uniform estimates in

$$Du \in W^{\beta/p, p}$$

for every $\beta < \alpha$

we recall that this means

$$\int_{\Omega'} \int_{\Omega'} \frac{|Du(x) - Du(y)|^p}{|x - y|^{n+\beta}} < \infty$$

for every $\Omega' \subset \Omega$
Step 7: Fractional differentiability

the proof goes via approximation

\[ v_n \mapsto \min_{w} \int_{B} \left( |Dv|^p + [a(x) + \sigma_n]|Dv|^q \right) dx \]

where \( 0 < \sigma_n \to 0 \)

\[ \int_{B} \left( |Du_n|^p + a(x)|Du_n|^q \right) dx \to \int_{B} \left( |Du|^p + a(x)|Du|^q \right) dx \]

and

\( u_n \in C^\infty(B) \)
Step 7: Fractional differentiability

the proof goes via approximation

\[
\begin{cases}
    v_n \mapsto \min_w \int_B \left( |Dv|^p + [a(x) + \sigma_n]|Dv|^q \right) \, dx \\
    w \in u_n + W^{1,q}_0(B)
\end{cases}
\]

where \(0 < \sigma_n \to 0\)

\[
\int_B \left( |Du_n|^p + a(x)|Du_n|^q \right) \, dx \to \int_B \left( |Du|^p + a(x)|Du|^q \right) \, dx
\]

and

\[u_n \in C^\infty(B)\]

this implies \(v_n \to u\)
the following improved Caccioppoli type inequality holds:

$$
\int_{B_{R/2}} |Du|^{2q-p} \, dx \\
\lesssim \frac{1}{R^{\alpha/2}} \left[ \int_{B_{2R}} \left( \left| \frac{u - (u)_{B_R}}{R} \right|^p + a(x) \left| \frac{u - (u)_{B_{2R}}}{R} \right|^q \right) \, dx + 1 \right]^b
$$
we use the fractional interpolation inequality

\[ \| f \|_{W^{\tilde{s}, t}} \leq c \| f \|^{\theta}_{W^{s_1, p_1}} \| f \|^{1-\theta}_{W^{s_2, p_2}} \]

with

\[ \tilde{s} = \theta s_1 + (1 - \theta) s_2 \quad \quad \frac{1}{t} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2} \]
Step 8: Upgraded Caccioppoli inequality

we apply as

$$\| Dv_n \|_{L^t} \leq c [v_n]_{s,p_1}^{\theta} \| Dv_n \|_{W^{\beta/p,p}}^{1-\theta}$$

with exponents

$$1 = \theta s + (1 - \theta) \left( 1 + \frac{\beta}{p} \right) \quad \quad \frac{1}{t} = \frac{\theta}{p_1} + \frac{1 - \theta}{p}$$

and

$$[v_n]_{s,p_1} := \left( \int \int \frac{|v_n(x) - v_n(y)|^{p_1}}{|x - y|^{n + sp_1}} \, dx \, dy \right)^{1/p_1}$$

and take $s$ close to 1 as you please and $p_1$ as large as you like
Step 9: Improved estimate in the $p$-phase

if for some $M \geq 1$

$$a_i(R) = \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha$$

then solve

$$\begin{cases} 
v \mapsto \min_w \int_{B_R} |Dv|^p \ dx \\
  w \in u + W_{0}^{1,p}(B_R) \end{cases}$$

and find

$$\int_{B_R} |Du - Dv|^p \ dx \leq M^2 R^\alpha$$
Step 9: Improved estimate in the $p$-phase

if for some $M \geq 1$

$$a_i(R) = \inf_{x \in B_R} a(x) \leq M[a]_0,\alpha R^\alpha$$

then solve

$$\begin{cases}
  v_R \mapsto \min_w \int_{B_R} (|Dv|^p + a_i(R)|Dv|^q) \, dx \\
  w \in u + W^{1,p}_0(B_R)
\end{cases}$$

and get

$$\int_{B_R} |Du - Dv|^p \, dx \lesssim \frac{1}{M} \int_{B_{2R}} \left( \left| \frac{u - (u)_{B_R}}{R} \right|^p + a(x) \left| \frac{u - (u)_{B_{2R}}}{R} \right|^q \right) \, dx$$
Step 10: Final gradient continuity

→ take $B_R$ and $M > 0$ and consider the functionals

$$v \mapsto \int_{B_R} \left( |Dv|^p + a_i(R)|Dv|^q \right) \, dx$$

where

$$a_i(R) := \begin{cases} 
0 & \text{if } \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha \\
\inf_{x \in B_R} a(x) & \text{if } \inf_{x \in B_R} a(x) > M[a]_{0,\alpha} R^\alpha 
\end{cases}$$

→ solve

$$\begin{cases} 
\nu_R \mapsto \min_w \int_{B_R} \left( |Dv|^p + a_i(R)|Dv|^q \right) \, dx \\
w \in u + W_{0,1,p}^1(B_R)
\end{cases}$$
Part 4: Nonlinear potential theory
Consider the model case

\[-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n\]
Consider the model case

\[-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n\]

We have

\[u(x) = \int G(x, y)\mu(y)\]
Consider the model case

$$-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n$$

We have

$$u(x) = \int G(x, y)\mu(y)$$

where

$$G(x, y) \approx \begin{cases} 
|x - y|^{2-n} & \text{se } n > 2 \\
- \log |x - y| & \text{se } n = 2
\end{cases}$$
Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-2}} = l_2(|\mu|)(x)$$
Previous formula gives

\[ |u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-2}} = I_2(|\mu|)(x) \]

while, after differentiation, we obtain

\[ |Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-1}} = I_1(|\mu|)(x) \]
In bounded domains one uses

\[ I_\beta^\mu(x, R) := \int_0^R |\mu|(B_\varrho(x)) \frac{d\varrho}{\varrho^{n-\beta}} \quad \beta \in (0, n] \]

since

\[ I_\beta^\mu(x, R) \lesssim \int_{B_R(x)} \frac{d|\mu|(y)}{|x - y|^{n-\beta}} = I_\beta(|\mu|_{B_R(x)})(x) \leq I_\beta(|\mu|)(x) \]

for non-negative measures.
What happens in the nonlinear case?

- For instance for nonlinear equations with linear growth
  \[- \text{div} \, a(Du) = \mu\]
  that is equations well posed in $W^{1,2}$ ($p$-growth and $p = 2$)
  that is
  \[|\partial a(z)| \leq L \quad \nu|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda \rangle\]

- And degenerate ones like
  \[- \text{div} \, (|Du|^{p-2}Du) = \mu\]

- To be short, we shall concentrate on the case $p \geq 2$
Nonlinear potentials

- The nonlinear Wolff potential is defined by

\[
W_{\beta,p}^\mu(x, R) := \int_0^R \left( \frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \quad \beta \in (0, n/p]
\]

which for \( p = 2 \) reduces to the usual Riesz potential

\[
I_{\beta}^\mu(x, R) := \int_0^R \frac{\mu(B_\rho(x))}{\rho^{n-\beta}} \frac{d\rho}{\rho} \quad \beta \in (0, n]
\]

- The nonlinear Wolff potential plays in nonlinear potential theory the same role the Riesz potential plays in the linear one.
The first nonlinear potential estimate

**Theorem (Kilpeläinen & Malý, Acta Math. 94)**

If $u$ solves

$$- \text{div} \left( |Du|^{p-2} Du \right) = \mu$$

then

$$|u(x)| \lesssim \mathcal{W}_{1,p}^\mu(x, R) + \left( \int_{B_R(x)} |u|^{p-1} \, dy \right)^{1/(p-1)}$$

holds.
The first nonlinear potential estimate

**Theorem (Kilpeläinen & Malý, Acta Math. 94)**

If $u$ solves

$$-\text{div} \left( |Du|^{p-2} Du \right) = \mu$$

then

$$|u(x)| \lesssim W_{1,p}^\mu(x, R) + \left( \int_{B_R(x)} |u|^{p-1} \, dy \right)^{1/(p-1)}$$

holds

where

$$W_{1,p}^\mu(x, R) := \int_0^R \left( \frac{|\nu|(B_\varrho(x))}{\varrho^{n-p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

For $p = 2$ we are back to the Riesz potential $W_{1,2} = I\mu$, the above estimate is non-trivial already in this situation.
The first nonlinear potential estimate

**Theorem (Kilpeläinen & Malý, Acta Math. 94)**

If \( u \) solves

\[
- \text{div} \left( |Du|^{p-2} Du \right) = \mu
\]

then

\[
|u(x)| \lesssim \mathcal{W}^{\mu}_{1,p}(x, R) + \left( \int_{B_R(x)} |u|^{p-1} \, dy \right)^{1/(p-1)}
\]

holds

where

\[
\mathcal{W}^{\mu}_{1,p}(x, R) := \int_0^R \left( \frac{\mu(B_{\varrho}(x))}{\varrho^{n-p}} \right)^{1/(p-1)} \, d\varrho
\]

For \( p = 2 \) we are back to the Riesz potential \( \mathcal{W}^{\mu}_{1,2} = I^{\mu}_2 \) - the above estimate is non-trivial already in this situation.
Indeed

\[ \mu \in L^q \iff \mathcal{W}^{\mu}_{\beta,p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \quad q \in (1, n) \]

and more in general estimates in rearrangement invariant function spaces
Corollary: optimal integrability

Indeed

\[ \mu \in L^q \implies \mathcal{W}^{\mu}_{\beta,p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \quad q \in (1, n) \]

and more in general estimates in rearrangement invariant function spaces

This property follows by another pointwise estimate

\[ \int_0^\infty \left( \frac{|\mu| (B_\varrho(x))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho} \lesssim I_{\beta} \left\{ [I_{\beta}(|\mu|)]^{1/(p-1)} \right\} (x) \]
Corollary: optimal integrability

Indeed

\[ \mu \in L^q \implies W^{\mu}_{\beta, p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \quad q \in (1, n) \]

and more in general estimates in rearrangement invariant function spaces

This property follows by another pointwise estimate

\[ \int_0^\infty \left( \frac{\mu (B_{\varrho} (x))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \varrho^{n-\beta p} \leq I_{\beta} \left\{ [I_{\beta} (|\mu|)]^{1/(p-1)} \right\} (x) \]

The quantity in the right-hand side is usually called Havin-Mazya potential
NON-LINEAR POTENTIAL THEORY

V. G. Maz'ya and V. P. Khavin

Contents

Introduction .............................................. 71

Part I .................................................. 73
§ 1. The space \( \mathcal{L}^t_p \) .................................. 73
§ 2. \((p, l)\)-potentials of generalized functions with finite \((p, l)\)-energy .............................. 74
§ 3. The maximum principle. A generalization of a theorem of Evans–Vasilesko. Lemmas on sequences of potentials ........................................ 75
§ 4. The \((p, l)\)-capacity of a compact set. The capacity potential ........................................ 76
§ 5. The capacity potential of an analytic set. The measurability of an analytic set relative to \((p, l)\)-capacity and \((p, l)\)-complete functions .............................. 77
§ 6. An estimate of the potential \( \mathcal{Z}^t_{p,l} \) in terms of the modulus of continuity of the measure \( \mu \) .............................................. 78
§ 7. Metric properties of sets of zero \((p, l)\)-capacity .............................................. 79
§ 8. Use of Bessel potentials (the case \( pl = n \)). The capacity \( \gamma_{p,l} \) ....................... 80
§ 9. Guide to the literature ........................................... 81

Part II .................................................. 84
§ 1. Auxiliary information on the \( \mathcal{L}^t_p \) spaces .............................................. 84
§ 2. Generalized functions with finite \((p, l)\)-capacity ........................................... 85
A first gradient potential estimate

**Theorem (Min., JEMS 11)**

When $p = 2$, if $u$ solves

$$-\text{div} \ a(Du) = \mu$$

then

$$|Du(x)| \lesssim I_1^{|\mu|}(x, R) + \int_{B_R(x)} |Du| \, dy$$

holds
A first gradient potential estimate

**Theorem (Min., JEMS 11)**

When $p = 2$, if $u$ solves

$$-\text{div} \ a(Du) = \mu$$

then

$$|Du(x)| \lesssim l_1^{\mu}(x, R) + \int_{B_R(x)} |Du| \, dy$$

holds

For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-1}} = l_1(|\mu|)(x)$$
The $p \neq 2$ case: a long path towards optimality

**Theorem (Duzaar & Min., AJM 11)**

When $p \geq 2$, if $u$ solves

$$-\text{div } a(Du) = \mu$$

then

$$|Du(x)| \lesssim W_{1/p,p}^\mu(x, R) + \int_{B_R(x)} |Du| \, dy$$

holds
The $p \neq 2$ case: a long path towards optimality

**Theorem (Duzaar & Min., AJM 11)**

When $p \geq 2$, if $u$ solves

$$-\text{div } a(Du) = \mu$$

then

$$|Du(x)| \lesssim W_{1/p, p}^\mu(x, R) + \int_{B_R(x)} |Du| \, dy$$

holds

where

$$W_{1/p, p}^\mu(x, R) = \int_0^R \left( \frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$
The $p \neq 2$ case: a long path towards optimality

Theorem (Duzaar & Min., JFA 10)

When $2 - 1/n < p < 2$, if $u$ solves

$$-\text{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \left[ I_{1/|\mu|}(x, R) \right]^{1/(p-1)} + \int_{B_R(x)} |Du| \, dy$$

holds.
The $p \neq 2$ case: a long path towards optimality

When $p < 2$ it holds that

$$W^{\mu}_{1/p,p}(x, R) \lesssim \left[ I^{\mu}_1(x, R) \right]^{1/(p-1)}$$
The $p \neq 2$ case: a long path towards optimality

When $p < 2$ it holds that

$$
W_{1/p,p}^\mu(x, R) \lesssim \left[ I_1^{\mu}(x, R) \right]^{1/(p-1)}
$$

Indeed

$$
W_{1/p,p}^\mu(x, R) = \int_0^R \left( \frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}
\approx \sum_i \left[ \frac{|\mu|(B_{\varrho_i}(x))}{\varrho_i^{n-1}} \right]^{1/(p-1)}
\approx \sum_i \left[ \frac{|\mu|(B_{\varrho_i}(x))}{\varrho_i^{n-1}} \right]^{1/(p-1)}
\lesssim \left[ I_1^{\mu}(x, R) \right]^{1/(p-1)}
$$
Consider

\[- \text{div} \, \nu = \mu\]

with

\[\nu = |Du|^{p-2} Du\]
Indeed

Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If $u$ solves

$$- \text{div} (|Du|^{p-2} Du) = \mu$$

then

$$|Du(x)|^{p-1} \sim \| \mu \| (x, R) + \left( \int_{B_R(x)} |Du| \, dy \right)^{p-1}$$

holds.
Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If $u$ solves

$$-\text{div} (|Du|^{p-2} Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim I_1^\mu(x, R) + \left(\int_{B_R(x)} |Du| \, dy\right)^{p-1}$$

holds

The theorem still holds for general equations of the type

$$-\text{div} a(Du) = \mu$$
Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If \( u \in W^{1,1}(\mathbb{R}^n) \) solves

\[
- \text{div} \left( |Du|^{p-2} Du \right) = \mu
\]

then

\[
|Du(x)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-1}} = l_1(|\mu|)(x).
\]
Part 4.2: Estimates in the vectorial case
The vectorial case

Theorem (Kuusi & Min., Preprint 15)

If \( u : \Omega \rightarrow \mathbb{R}^N \) solves

\[
-\text{div} \left( |Du|^{p-2} Du \right) = \mu
\]

then

\[
|u(x) - (u)_{B_R(x)}| \lesssim W_{1,p}^\mu(x, R) + \int_{B(x, R)} |u - (u)_{B_R(x)}| \, dy
\]

holds whenever the right hand sides are finite.
Theorem (Kuusi & Min., Preprint 15)

If $u : \Omega \to \mathbb{R}^N$ solves

$$-\text{div} (|Du|^{p-2} Du) = \mu$$

then

$$|Du(x) - (Du)_{B(x,R)}| \lesssim \left[ \mathbb{I}_{1} |\mu| (x, R) \right]^{1/(p-1)}$$

$$+ \int_{B(x,R)} |Du - (Du)_{B(x,R)}| \, dy$$

holds whenever the right hand sides are finite.
Potential characterisation of Lebesgue points

**Theorem (Kuusi & Min. BMS 14)**

*If $x$ is a point such that

$$W_{1,p}^\mu(x, R) < \infty$$

for some $R > 0$ then $x$ is a Lebesgue point of $u$ that is, the following limit

$$\lim_{\rho \to 0} \int_{B_\rho(x)} u(y) \, dy$$

exists.*
Theorem (Kuusi & Min. BMS 14)

If $x$ is a point such that

$$\left| \mu \right|_1 \left( x, R \right) < \infty$$

for some $R > 0$ then $x$ is a Lebesgue point of $Du$ that is, the following limit

$$\lim_{\varrho \to 0} \int_{B_\varrho(x)} Du(y) \, dy$$

exists.
Part 4.3: Oscillation bounds
The general continuity criterion

**Theorem (Kuusi & Min. ARMA 13)**

If $u$ solves

$$-\text{div} \left( |Du|^{p-2} Du \right) = \mu$$

and

$$\lim_{R \to 0} \mathbf{1}_{|\mu|}(x, R) = 0 \text{ uniformly w.r.t. } x$$

then

$Du$ is continuous
A classical theorem of Stein

Theorem (Stein, Ann. Math. 81)

\[ Dv \in L(n, 1) \implies v \text{ is continuous} \]
A classical theorem of Stein

**Theorem (Stein, Ann. Math. 81)**

\[ Dv \in L(n, 1) \implies v \text{ is continuous} \]

We recall that

\[ g \in L(n, 1) \iff \int_0^\infty \left| \left\{ x : |g(x)| > \lambda \right\} \right|^{1/n} d\lambda < \infty \]
A classical theorem of Stein

Theorem (Stein, Ann. Math. 81)

\[ Dv \in L(n, 1) \implies v \text{ is continuous} \]

We recall that

\[ g \in L(n, 1) \iff \int_0^\infty \left| \left\{ x : |g(x)| > \lambda \right\} \right|^{1/n} d\lambda < \infty \]

It follows that

\[ \triangle u = \mu \in L(n, 1) \implies Du \text{ is continuous} \]

Giuseppe Mingione

Recent progresses in Nonlinear Potential Theory
A classical theorem of Stein

**Theorem (Stein, Ann. Math. 81)**

\[ Dv \in L(n, 1) \iff v \text{ is continuous} \]

We recall that

\[ g \in L(n, 1) \iff \int_0^\infty \left| \left\{ x : |g(x)| > \lambda \right\} \right|^{1/n} d\lambda < \infty \]

An example of \( L(n, 1) \) function is given by

\[ \frac{1}{|x| \log^\beta (1/|x|)} \quad \beta > 1 \]

in the ball \( B_{1/2} \)
A nonlinear Stein theorem

Theorem (Kuusi & Min., ARMA 13)

If $u$ solves the $p$-Laplacean equation

$$-\text{div} \left( |Du|^{p-2} Du \right) = \mu \in L(n,1)$$

then

$Du$ is continuous
Part 4.4: A fully fractional approach
The setting

We take $p = 2$ and consider

$$\left\{ \begin{array}{l}
|a(z)| + |\partial a(z)||z| \leq L|z|
\vspace{1em}
\nu^{-1}|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda \rangle
\end{array} \right.$$
A first gradient potential estimate

**Theorem (Min., JEMS 11)**

*When \( p = 2 \), if \( u \) solves

\[
- \text{div} \ a(Du) = \mu
\]

then

\[
|D_{\xi}u(x)| \leq c I_1(\mu)(x, R) + c \int_{B(x, R)} |D_{\xi}u| \, dx
\]

for every \( \xi \in \{1, \ldots, n\} \)
A first gradient potential estimate

**Theorem (Min., JEMS 11)**

*When \( p = 2 \), if \( u \) solves

\[
-\text{div} \, a(Du) = \mu
\]

then

\[
|D_\xi u(x)| \leq c I_1(\mu)(x, R) + c \int_{B(x, R)} |D_\xi u| \, dx
\]

for every \( \xi \in \{1, \ldots, n\} \)

For solutions in \( W^{1,1}(\mathbb{R}^N) \) we have

\[
|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-1}} = I_1(|\mu|)(x)
\]
Classical Gradient estimates

- **Consider energy solutions to** $\text{div } a(Du) = 0$ for $p = 2$
- **First prove** $Du \in W^{1,2}$
- **Then use that** $v = D_\xi u$ solves

$$\text{div}(A(x)Dv) = 0 \quad A(x) := a_z(Du(x))$$

- **The boundedness of** $D_\xi u$ **follows by Standard DeGiorgi’s theory**
- **This is a consequence of Caccioppoli’s inequalities of the type**

$$\int_{B_{R/2}} |D(D_\xi u - k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} |(D_\xi u - k)_+|^2 \, dy$$

where

$$(D_\xi u - k)_+ := \max\{D_\xi u - k, 0\}$$
Recall the definition

We have

\[ v \in W^{\sigma,1}(\Omega') \]

iff \( v \in L^1(\Omega') \) and

\[ [v]_{\sigma,1;\Omega'} = \int_{\Omega'} \int_{\Omega'} \frac{|v(x) - v(y)|}{|x - y|^{n+\sigma}} \, dx \, dy < \infty \]
There is a differentiability problem

For solutions to

\[ \text{div } a(Du) = \mu \quad \text{in general} \quad Du \notin W^{1,1} \]

but nevertheless it holds

**Theorem (Min., Ann. SNS Pisa 07)**

\[ Du \in W^{1-\varepsilon,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \quad \text{for every } \varepsilon \in (0, 1) \]

This means that

\[ [Du]_{1-\varepsilon,1;\Omega'} = \int_{\Omega'} \int_{\Omega'} \frac{|Du(x) - Du(y)|}{|x - y|^{n+1-\varepsilon}} \, dx \, dy < \infty \]

holds for every \( \varepsilon \in (0, 1) \), and every subdomain \( \Omega' \subset \Omega \)
Step 1: A non-local Caccioppoli inequality

Theorem (Min., JEMS 11)

Let
\[ w = D_\xi u \quad \text{with} \quad - \div a(Du) = \mu \]
where \( \xi \in \{1, \ldots, n\} \) then

\[
[(|w| - k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c}{R^{\sigma}} \int_{B_R} (|w| - k)_+ \, dy + \frac{cR|\mu|(B_R)}{R^{\sigma}}
\]

holds for every \( \sigma < 1/2 \)
Step 1: A non-local Caccioppoli inequality

Theorem (Min., JEMS 11)

Let

$$w = D_\xi u \quad \text{with} \quad - \text{div } a(Du) = \mu$$

where $\xi \in \{1, \ldots, n\}$ then

$$\left[ (|w| - k)_+ \right]_{\sigma, 1; B_{R/2}} \leq \frac{c}{R^\sigma} \int_{B_R} (|w| - k)_+ \, dy + \frac{cR|\mu|(B_R)}{R^\sigma}$$

holds for every $\sigma < 1/2$

Compare with the usual one for $\text{div } a(Du) = 0$, that is

$$\left[ (w - k)_+ \right]_{2, 1; B_{R/2}} \equiv \int_{B_{R/2}} |D(w - k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} (w - k)_+^2 \, dy$$
Step 1: A non-local Caccioppoli inequality

- This approach reveals the robustness of energy inequalities, which hold below the natural growth exponent $2$, and for fractional order of differentiability, although the equation has integer order.

- Classical VS fractional

\[
\begin{align*}
\text{spaces} & : L^2 - L^2 \quad L^1 - L^1 \\
\text{differentiability} & : 0 \quad 1 \quad 0 \quad \sigma
\end{align*}
\]
Theorem (Min., JEMS 11)

Let $w$ be an $L^1$-function $w$ satisfying the fractional Caccioppoli’s inequality

$$
[ (|w| - k)_+ ]_{\sigma,1;B_{R/2}} \leq \frac{L}{R^\sigma} \int_{B_R} (|w| - k)_+ \, dy + \frac{LR|\mu|(B_R)}{R^\sigma}
$$

for some $\sigma > 0$ and every $k \geq 0$. Then it holds that

$$
|w(x)| \leq c |\mu|_1(x, R) + c \int_{B(x,R)} |w| \, dy
$$

for every Lebesgue point $x$ of $w$.
Part 4.5: Fully nonlinear interlude
Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)

If $u$ solves the uniformly elliptic fully nonlinear equation

$$F(D^2 u) = f \in L(n, 1)$$

then

$Du$ is continuous
If $u$ solves the uniformly elliptic fully nonlinear equation

$$F(D^2 u) = f \in L(n, 1)$$

then

$Du$ is continuous

Previous results of Caffarelli (Ann. Math. 1989) claimed that

$$f \in L^{n+\epsilon} \implies Du \in C^{0,\alpha}$$
A fully nonlinear Stein theorem

**Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)**

*If u solves the uniformly elliptic fully nonlinear equation*

\[ F(D^2 u) = f \in L(n, 1) \]

*then*

\[ Du \text{ is continuous} \]

Previous results of Caffarelli (Ann. Math. 1989) claimed that

\[ f \in L^{n+\varepsilon} \implies Du \in C^{0,\alpha} \]

Notice that

\[ L^{n+\varepsilon} \subset L(n, 1) \quad \varepsilon > 0 \]
Key to the proof, a new potential estimate
Key to the proof, a new potential estimate

\[ I_1^f(x, r) := \int_0^r \int_{B_{\varrho}(x)} |f(y)| \, dy \, d\varrho \]
The relevant role of $L(n, 1)$

Key to the proof, a new potential estimate

$$I_{1}^{f}(x, r) := \int_{0}^{r} \int_{B_{\varrho}(x)} |f(y)| \, dy \, \frac{d\varrho}{\varrho}$$

$$:= \int_{0}^{r} \int_{B_{\varrho}(x)} |f(y)| \, dy \, d\varrho$$

$$\leq \int_{0}^{r} \left( \int_{B_{\varrho}(x)} |f(y)|^{p} \, dy \right)^{1/p} \, d\varrho =: II_{1}^{f}(x, r).$$
Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)

If $u$ solves the uniformly elliptic fully nonlinear equation

$$F(D^2 u) = f \in L(n, 1)$$

then

$$|Du(x)| \leq c \|f\|_1(x, r) + c \left(\int_{B_r(x)} |Du|^q \, dy\right)^{1/q}$$

for $p \geq n - \varepsilon$ and $q > n$
Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)

If $u$ solves the uniformly elliptic fully nonlinear equation

$$F(D^2 u) = f \in L(n, 1)$$

then

$$|Du(x)| \leq c \|f\|_t(x, r) + c \left( \int_{B_r(x)} |Du|^q \, dy \right)^{1/q}$$

for $p \geq n - \varepsilon$ and $q > n$

$n - \varepsilon$ is the Escauriaza exponent, and is universal
It holds, with $n - \varepsilon < p$ that

$$\sup_{B_r(x)} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy < \infty \implies Du \in \text{BMO}$$
Consequences

- It holds, with $n - \varepsilon < p$ that

$$\sup_{B_r(x)} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy < \infty \implies Du \in \text{BMO}$$

- In particular

$$f \in \mathcal{M}^n \equiv L(n, \infty) \implies Du \in \text{BMO}$$
Consequences

- It holds, with \( n - \varepsilon < p \) that

\[
\sup_{B_r(x)} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy < \infty \implies Du \in \text{BMO}
\]

- In particular

\[
f \in \mathcal{M}^n \equiv L(n, \infty) \implies Du \in \text{BMO}
\]

- Moreover

\[
\lim_{r \to 0} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy = 0 \implies Du \in \text{VMO}
\]
Borderline case of a theorem of Caffarelli, who proved
\[
\sup_{B_r(x)} r^{n(1-\alpha)-n} \int_{B_r(x)} |f|^n \, dy < \infty \implies Du \in C^{0,\alpha}
\]

In particular, a recent result of Teixeira (ARMA 14) who proved
\[
f \in L^n \implies u \text{ is Log-Lipschitz}
\]
that is
\[
|u(x) - u(y)| \leq -|x-y| \log \left( \frac{1}{|x-y|} \right)
\]
follows as a corollary as
\[
Du \in BMO \implies u \text{ is Log-Lipschitz}
\]
Part 4.6: Universal potential estimates
Let us go back to

$$-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3$$
Leet us go back to

\[-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3\]

and observe the following elementary inequality:

\[
||x - \xi|^{2-n} - |y - \xi|^{2-n}| \lesssim ||x - \xi|^{2-n-\alpha} + |y - \xi|^{2-n-\alpha}| |x - y|^\alpha
\]
Leet us go back to

\[-\Delta u = \mu \quad \text{in} \quad \mathbb{R}^n, \quad n \geq 3\]

and observe the following elementary inequality:

\[\left| |x - \xi|^{2-n} - |y - \xi|^{2-n} \right| \lesssim \left| |x - \xi|^{2-n-\alpha} + |y - \xi|^{2-n-\alpha} \right| |x - y|^\alpha\]

that in turn implies

\[|u(x) - u(y)| \lesssim [l_{2-\alpha}(|\mu|)(x) + l_{2-\alpha}(|\mu|)(y)] |x - y|^\alpha\]

for \(0 \leq \alpha \leq 1\)
The following definition is due to DeVore & Sharpley (Mem. AMS, 1982)

Let $\alpha \in (0, 1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. A measurable function $v$, finite a.e. in $\Omega$, belongs to the Calderón space $C_q^\alpha(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x - y|^{\alpha}$$

holds for almost every couple $(x, y) \in \Omega \times \Omega$

In other words

$$m(x) \approx \partial^\alpha v(x)$$
First universal potential estimate

Theorem (Kuusi & Min. JFA 12)

The estimate

$$|u(x) - u(y)| \lesssim \left[ W_{1-\frac{\alpha(p-1)}{p},p}(x, R) + W_{1-\frac{\alpha(p-1)}{p},p}(y, R) \right] |x - y|^{\alpha}$$

$$+ c \int_{B_R} |u| \tilde{x} \cdot \left( \frac{|x - y|}{R} \right)^{\alpha}$$

holds uniformly in $\alpha \in [0, 1]$, whenever $x, y \in B_{R/4}$

- The cases $\alpha = 0$ and $\alpha = 1$ give back the two known Wolff potential estimates as endpoint cases
The homogeneous case

- The estimate tells that

\[ \partial^\alpha u(x) \lesssim W^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R) \]

The case \( \mu = 0 \) reduces to the classical estimate

\[ |u(x) - u(y)| \lesssim -\int_{B_R} |u| \, d\tilde{x} \cdot (|x-y| R)^\alpha \]

In the case \( p = 2 \) we have

\[ |u(x) - u(y)| \lesssim \left[ I \mid \mu \mid^2(x,R) + I \mid \mu \mid^2(y,R) \right] |x-y| \alpha + c - \int_{B_R} |u| \, d\tilde{x} \cdot (|x-y| R)^\alpha \]

which in the classical case \( -\Delta u = \mu \) can be derived directly from the standard representation formula via potentials.
The homogeneous case

- The estimate tells that

\[
\partial^\alpha u(x) \lesssim W_1^{\mu\alpha(p-1)/p} (x, R)
\]

- The case \( \mu = 0 \) reduces to the classical estimate

\[
|u(x) - u(y)| \lesssim \int_{B_R} |u| d\tilde{x} \cdot \left( \frac{|x - y|}{R} \right)^\alpha
\]
The homogeneous case

- The estimate tells that
  \[ \partial^\alpha u(x) \lesssim W^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x, R) \]

- The case \( \mu = 0 \) reduces to the classical estimate
  \[ |u(x) - u(y)| \lesssim \int_{B_R} |u| \, d\tilde{x} \cdot \left( \frac{|x - y|}{R} \right)^\alpha \]

- In the case \( p = 2 \) we have
  \[ |u(x) - u(y)| \lesssim \left[ \mathbf{l}^{\mu}_{2-\alpha}(x, R) + \mathbf{l}^{\mu}_{2-\alpha}(y, R) \right] |x - y|^{\alpha} \]
  \[ + c \int_{B_R} |u| \, d\tilde{x} \cdot \left( \frac{|x - y|}{R} \right)^\alpha \]

  which in the classical case \(-\triangle u = \mu\) can be derived directly from the standard representation formula via potentials.
Theorem (Kuusi & Min., BMS 14)

The estimate

\[
|u(x) - u(y)| 
\leq \frac{c}{\alpha} \left[ |\mu|_{p-\alpha(p-1)}(x, R) + |\mu|_{p-\alpha(p-1)}(y, R) \right]^{1/(p-1)} |x - y|^{\alpha} 
+ \frac{c}{\alpha} \int_{B_R} (|u| + Rs) \, d\tilde{x} \cdot \left( \frac{|x - y|}{R} \right)^{\alpha}
\]

holds uniformly for \( \alpha \in [0, 1] \)
Second universal estimate

**Theorem (Kuusi & Min., BMS 14)**

The estimate

\[ |u(x) - u(y)| \leq \frac{c}{\alpha} \left[ \|\mu\|_{p-\alpha(p-1)}(x, R) + \|\mu\|_{p-\alpha(p-1)}(y, R) \right]^{1/(p-1)} |x - y|^{\alpha} \]

\[ + \frac{c}{\alpha} \int_{B_R} (|u| + Rs) d\tilde{x} \cdot \left( \frac{|x - y|}{R} \right)^{\alpha} \]

holds uniformly for \( \alpha \in [0, 1] \)

Natural blow-up of the estimate as \( \alpha \to 0 \), with a linear behaviour
The fractional maximal operator

\[ M_{\beta, R}(f)(x) := \sup_{0 < r \leq R} r^\beta \frac{|f|(B(x, r))}{|B(x, r)|} \]
The fractional maximal operator

\[ M_{\beta,R}(f)(x) := \sup_{0 < r \leq R} r^\beta \frac{|f|(B(x, r))}{|B(x, r)|} \]

The fractional sharp maximal operator

\[ M_{\beta,R}^\#(f)(x) := \sup_{0 < r \leq R} r^{-\beta} \int_{B(x, r)} |f - (f)_{B(x, r)}| d\tilde{x} \]
Theorem (Kuusi & Min., BMS 14)

The estimate

\[ M_{1-\alpha, R}(Du)(x) + M^\#_{\alpha, R}(u)(x) \]
\[ \lesssim \left[ \mathbf{I}_{p-\alpha(p-1)}(\mu, R) \right]^{1/(p-1)} + \frac{1}{R^\alpha} \int_{B_R} |u| \, d\tilde{x} \]

holds uniformly for \( \alpha \in [0, 1] \)
Let $\alpha \in (0, 1]$, then

$$|\nu(x) - \nu(y)| \leq \frac{c}{\alpha} \left[ M^#_{\alpha,R}(f)(x) + M^#_{\alpha,R}(f)(y) \right] |x - y|^{\alpha}$$

holds for all points $x$ and $y$ for which the right hand side is finite.

As a corollary, the second estimate follows from the third one.
Part 4.7: Evolution
The model case is here given by

\[ u_t - \text{div} (|Du|^{p-2} Du) = \mu, \quad \text{in } \Omega \times (-T, 0) \subset \mathbb{R}^{n+1} \]

more in general we consider

\[ u_t - \text{div} a(Du) = \mu. \]

The basic reference for existence and a priori estimates in the setting of SOLA is the work of Boccado, Dall’Aglio, Galloüet and Orsina, JFA, 1997
Degenerate equations - basic results

Theorem (Boccardo, Dall’Aglio, Gallouët & Orsina, JFA, 1997)

\[ |Du| \in L^q(\Omega \times (-T, 0)), \quad 1 \leq q < p - 1 + \frac{1}{N - 1} \]

\[ N = n + 2 \quad \text{is the parabolic dimension} \]
Consider the caloric Riesz potential

$$I^{\mu}_{1}(x, t; r) := \int_{0}^{r} \frac{|\mu|(Q_\varrho(x, t))}{\varrho^{N-1}} \frac{d\varrho}{\varrho}, \quad N := n + 2,$$

where $Q_\varrho(x, t)$ is defined as

$$Q_\varrho(x, t) := B_{R}(x) \times (t - r^2, t).$$
Consider the caloric Riesz potential
\[ I_1^\mu(x, t; r) := \int_0^r |\mu|(|Q_\varrho(x, t)| \varrho^{N-1} \varrho, \quad N := n + 2, \]
then for solutions to
\[ u_t - \Delta u = \mu \]
we have
\[ |Du(x, t)| \leq c I_1^\mu(x, t; r) + c \int_{Q_r(x, t)} |Du| \, dz \]
Consider the caloric Riesz potential

\[ I_1^\mu(x, t; r) := \int_0^r \frac{|\mu(Q_0(x,t))|}{\varrho^{N-1}} \frac{d\varrho}{\varrho}, \quad N := n + 2, \]

then for solutions to

\[ u_t - \Delta u = \mu \]

we have

\[ |Du(x, t)| \leq cI_1^\mu(x, t; r) + c \int_{Q_r(x,t)} |Du| \, dz \]

we recall that

\[ Q_r(x, t) := B_R(x) \times (t - r^2, t) \]
Inhomogeneous a priori estimates

Theorem (DiBenedetto & Friedman, Crelle J. 85)

\[
\sup_{Q_{r/2}(x_0, t_0)} |Du| \leq c(n, p) \int_{Q_{r}(x_0, t_0)} (|Du| + 1)^{p-1} \, dz
\]
The basic analysis is the following: consider intrinsic cylinders

\[ Q_0^\lambda(x,t) = B_0(x) \times (t - \lambda^{2-p}q^2, t) \]

where it happens that

\[ |Du| \approx \lambda \quad \text{in} \quad Q_0^\lambda(x,t) \]

then the equation behaves as

\[ u_t - \lambda^{p-2} \Delta u = 0 \]

that is, scaling back in the same cylinder, as the heat equation.

On intrinsic cylinders estimates “ellipticize”; in particular, they become homogeneous.
The homogenizing effect of intrinsic geometry

**Theorem (DiBenedetto & Friedman, Crelle J. 85)**

*There exists a universal constant $c \geq 1$ such that*

$$c \left( \int_{Q_r^\lambda(x,t)} |Du|^{p-1} \, dz \right)^{1/(p-1)} \leq \lambda$$

*then*

$$|Du(x, t)| \leq \lambda$$
Define the intrinsic Riesz potential such that

$$I_{1,\lambda}^{\mu}(x, t; r) := \int_0^r \frac{|\mu|(Q_{\varrho}^{\lambda}(x, t))}{\varrho^{N-1}} \frac{d\varrho}{\varrho}$$

with

$$Q_{\varrho}^{\lambda}(x, t) = B_{\varrho}(x) \times (t - \lambda^{2-p}\varrho^2, t)$$
Define the intrinsic Riesz potential such that

\[ I_{1,\lambda}^\mu(x, t; r) := \int_0^r \frac{|\mu|(Q_\varrho^\lambda(x, t))}{\varrho^{N-1}} \frac{d\varrho}{\varrho} \]

with

\[ Q_\varrho^\lambda(x, t) = B_\varrho(x) \times (t - \lambda^{2-p}\varrho^2, t) \]

Note that

\[ I_{1,\lambda}^\mu(x, t; r) = I_1^{\mu |}(x, t; r) \quad \text{when } p = 2 \text{ or when } \lambda = 1 \]
The parabolic Riesz gradient bound

Theorem (Kuusi & Min., JEMS, ARMA 14)

There exists a universal constant \( c \geq 1 \) such that

\[
cl_{1,\lambda}(x, t; r) + c \left( \int_{Q_{r}^{\lambda}(x, t)} |Du|^{p-1} \, dz \right)^{1/p-1} \leq \lambda
\]

then

\[ |Du(x, t)| \leq \lambda \]
The parabolic Riesz gradient bound

**Theorem (Kuusi & Min., JEMS, ARMA 14)**

There exists a universal constant $c \geq 1$ such that

$$c |I_{1,\lambda}^\mu(x, t; r) + c \left( \int_{Q_{r}^\lambda(x,t)} |Du|^{p-1} \, dz \right)^{1/(p-1)} \leq \lambda$$

then

$$|Du(x, t)| \leq \lambda$$

- When $\mu \equiv 0$ this reduces to the sup estimate of DiBenedetto & Friedman (Crelles J. 84)
Sharpness

- Consider the equation

\[ u_t - \text{div} (|Du|^{p-2}Du) = \delta, \]

where \( \delta \) denotes the Dirac unit mass charging the origin.

- The so-called Barenblatt (fundamental solution) is

\[ B_p(x, t) = \begin{cases} 
 t^{-\frac{n}{\theta}} \left( c_b - \theta^{\frac{1}{1-p}} \left( \frac{p-2}{p} \right) \left( \frac{|x|}{t^{1/\theta}} \right)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p-2}} & \text{if } t > 0 \\
 0 & \text{if } t \leq 0. 
\end{cases} \]

for \( \theta = n(p-2) + p \) and a suitable constant \( c_b \) such that

\[ \int_{\mathbb{R}^n} B_p(x, t) \, dx = 1 \quad \forall \ t > 0 \]
A direct computation shows the following upper optimal upper bound

$$|DB_p(x, t)| \leq ct^{-(n+1)/\theta}$$

The intrinsic estimate above exactly reproduces this upper bound

This decay estimate is indeed reproduced for all those solutions that are initially compactly supported
Intrinsic bounds imply explicit bounds

- The previous bound always implies a priori estimates on standard parabolic cylinders

**Theorem (Kuusi & Min., JEMS, ARMA 14)**

\[ |Du(x, t)| \lesssim I_1^{\mu}(x, t; r) + \int_{Q_r(x, t)} (|Du| + 1)^{p-1} \, dz \]

holds for every standard parabolic cylinder \( Q_r \)
Theorem (Kuusi & Min., ARMA 14)

Assume that

\[
\lim_{r \to 0} \mu_1(x, t; r) = 0 \quad \text{uniformly w.r.t. } (x, t)
\]

then

Du is continuous in $Q_T$
Theorem (Kuusi & Min., ARMA 14)

Assume that
\[
\lim_{r \to 0} \mathbb{I}_1^\mu(x, t; r) = 0 \quad \text{uniformly w.r.t. } (x, t)
\]
then
\[Du \text{ is continuous in } Q_T\]

Theorem (Kuusi & Min., ARMA 14)

Assume that
\[
|\mu|(Q_\varrho) \lesssim \varrho^{N-1+\delta}
\]
holds, then there exists \( \alpha \), depending on \( \delta \), such that
\[Du \in C^{0,\alpha} \text{ locally in } Q_T\]
Theorem (Kuusi & Min., ARMA 14)

Assume that

\[ u_t - \text{div} (|Du|^{p-2} Du) = \mu \in L(N, 1) \]

that is

\[ \int_0^\infty \left\{ |\mu| > \lambda \right\}^{1/N} d\lambda < \infty \]

then \( Du \) is continuous in \( Q_T \)

DiBenedetto proved that \( Du \) is continuous when \( \mu \in L^{N+\varepsilon} \)
THANKS FOR THE ATTENTION