Recent progresses in Nonlinear Potential Theory

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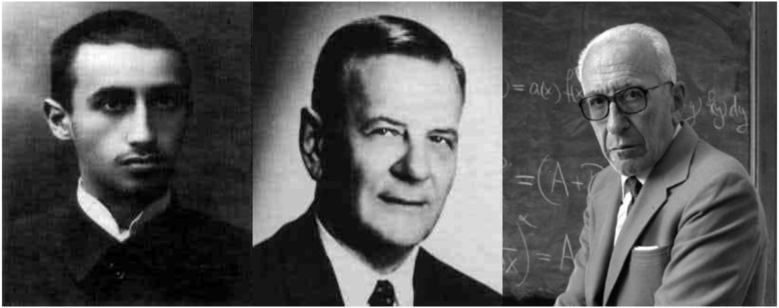
Advanced Course on Geometric Analysis
Centre de Recerca Matemàtica - Barcelona

Old Poland Grandeur (and legacy)



Scheme of the course

- **From linear to nonlinear CZ-theory**

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- **Parabolic problems**

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Part 1.1: The classical CZ-theory

The standard CZ theory

- **Consider the model case**

$$\triangle u = f \qquad \text{in} \;\; \mathbb{R}^n$$

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Then

$$f \in L^q \quad \text{implies} \quad D^2 u \in L^q \qquad 1 < q < \infty$$

with natural failure in the borderline cases $q = 1, \infty$

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- **As a consequence (Sobolev embedding)**

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Overture: The standard CZ theory

- **Consider the model case**

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with natural failure in the borderline cases $q = 1, \infty$

- **As a consequence (Sobolev embedding)**

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The singular integral approach

- **Representation via Green's function**

$$u(x) \approx \int G(x,y) f(y)\, dy$$

with

$$G(x,y) = \begin{cases} |x-y|^{2-n} & \text{if } n > 2 \\ -\log|x-y| & \text{if } n = 2 \end{cases}$$

- **Differentiation yields**

$$D^2 u(x) = \int K(x,y) f(y)\, dy$$

and $K(x,y)$ is a singular integral kernel, and the conclusion follows

Singular kernels with cancellations

- **Initial boundedness assumption**

$$\|\hat{K}\|_{L^\infty} \leq B\,,$$

where $\hat{K}$ denotes the Fourier transform of $K(\cdot)$

- **Hörmander cancelation condition**

$$\int_{|x|\geq 2|y|} |K(x-y)-K(x)|\,dx \leq B \qquad \text{for every} \quad y \in \mathbb{R}^n$$

The fractional integral approach

- **Again differentiating**

$$|Du(x)| \lesssim I_1(|f|)(x)$$

- **where I_1 is a fractional integral**

$$I_\beta(g)(x) := \int \frac{g(y)}{|x-y|^{n-\beta}}\,dy \qquad \beta \in [0,n)$$

- **and then**

$$I_\beta \colon L^q \to L^{\frac{nq}{n-\beta q}} \qquad \beta q < n$$

- **This is in fact equivalent to the original proof of Sobolev embedding theorem for the case $q > 1$, which uses that**

$$|u(x)| \lesssim I_1(|Du|)(x)$$

The fractional integral approach

- **Important remark: the theory of fractional integral operators substantially differs from that of singular ones**
- **In fact, while the latter is based on cancelation properties of the kernel, the former only considers the size of the kernel**
- **As a consequence all the estimates related to the operator I_β degenerate when $\beta \to 0$**

- **Higher order right hand side**

$$\triangle u = \text{div } Du = \text{div } F$$

Then

$$F \in L^q \Longrightarrow Du \in L^q \qquad q > 1$$

just "simplify" the divergence operator!!

Interpolation approach

- **Define the operator**

$$T\colon F \mapsto T(F) := \text{ gradient of the solution to } \triangle u = \operatorname{div} F$$

- **Then**

$$T\colon L^2 \to L^2$$

by testing with the solution, and

$$T\colon L^\infty \to \mathrm{BMO}$$

by regularity estimates (hard part).

- **Campanato-Stampacchia interpolation**

$$T\colon L^q \to L^q \qquad 1 < q < \infty$$

BMO/VMO

- **Define**

$$(v)_{B_s} := \frac{1}{|B_s|}\int_{B_s} v\,dx$$

and

$$\omega(R) := \sup_{s\le R} \frac{1}{|B_s|}\int_{B_s} |v-(v)_{B_s}|\,dx$$

- **A map v belongs to BMO iff**

$$\omega(R) < \infty$$

- **A map v belongs to VMO iff**

$$\lim_{R\to 0} \omega(R) = 0$$

Part 1.2: Basics from Nonlinear CZ-theory

From linear to nonlinear CZ-theory

The problem is now to extend the results to (potentially degenerate) nonlinear equations of the type

$$\text{div } a(Du) = \mathcal{D} \,.$$

The main issues are two:

- to find nonlinear methods, by-passing linearity and in particular the use of fundamental solutions
- considering estimates that allow to treat also cases in which the right-hand side $\mathcal{D}$ does not belong to the dual space of the operator considered

Theorem (Iwaniec, Studia Math. 83)

$$\mathrm{div}\,(|Du|^{p-2}Du) = \mathrm{div}\,(|F|^{p-2}F) \qquad \textit{in } \mathbb{R}^n$$

Then it holds that

$$F \in L^q \Longrightarrow Du \in L^q \qquad p \leq q < \infty$$

Opening

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Theorem (DiBenedetto & Manfredi, Amer. J. Math. 93)

The previous result holds for the p-Laplacean system, moreover $F \in BMO \Longrightarrow Du \in BMO$

An alternative approach

- **Caffarelli & Peral** (CPAM 98) give an important new approach to the L^p-estimates for equations as

$$\operatorname{div} a(x, Du) = 0$$

with high oscillating coefficients in the context of homogenization

An alternative approach

- **Caffarelli & Peral** (CPAM 98) give an important new approach to the L^p-estimates for equations as

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  with high oscillating coefficients in the context of homogenization
- **Byun & Wang, in a recent series of papers**, used the above method to derive Calderón-Zygmund estimates for solutions to boundary value problems involving non-homogeneous equations, under weak assumptions on the boundary regularity. Papers by several authors like: Lee, Oh, Ok, Yao, Zhou

- **The local estimate**

$$\left(\fint_{B_R} |Du|^q \, dz\right)^{\frac{1}{q}} \leq c \left(\fint_{B_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + c \left(\fint_{B_{2R}} |F|^q \, dz\right)^{\frac{1}{q}}$$

holds for solutions of solutions of the previous problems

General elliptic problems

- **In the same way the non-linear result of Iwaniec extends to all elliptic equations in divergence form of the type**

$$\text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

where $a(\cdot)$ is p-monotone in the sense of the previous slides

- **and to all systems with special structure**

$$\text{div } (g(|Du|)Du) = \text{div } (|F|^{p-2}F)$$

- **Moreover, VMO-coefficients can be considered too**

$$\text{div } [c(x)a(Du)] = \text{div } (|F|^{p-2}F)$$

Open problems

- **The full range**

$$p - 1 < q < \infty$$

Compare with the linear case $p = 2$.

- **This is the case below the duality exponent**, when

$$\text{div}\,(|F|^{p-2}F) \notin W^{-1,p'}$$

- **Iwaniec & Sbordone (Crelle J., 94), Lewis (Comm. PDE 93)**

$$p - \varepsilon \leq q < \infty \qquad \varepsilon \equiv \varepsilon(n,p)$$

- **Parabolic case: important approach of Kinnunen & Lewis (Ark. Math 02)**

Part 2: Parabolic problems

The parabolic case

Theorem (Acerbi & Min., Duke Math. J. 07)

$$u_t - \operatorname{div}\,(|Du|^{p-2}Du) = \operatorname{div}\,(|F|^{p-2}F) \qquad \textit{in } \Omega \times (0, T)$$

for

$$p > \frac{2n}{n+2}$$

Then it holds that

$$F \in L^q_{\mathrm{loc}} \Longrightarrow Du \in L^q_{\mathrm{loc}} \qquad \textit{for} \qquad p \leq q < \infty$$

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For $q = p + \varepsilon$ see the important work of Kinnunen & Lewis (Duke Math. J. 01)

The parabolic case

- **The elliptic approach via maximal operators only works in the case $p = 2$**
- **The result also works for systems, that is when $u(x, t) \in \mathbb{R}^N$, $N \geq 1$**
- **First Harmonic Analysis free approach to non-linear Calderón-Zygmund estimates**

The parabolic case

- **The result is new already in the case of equations i.e. $N = 1$, the difficulty being in the lack of homogenous scaling of parabolic problems with $p \neq 2$, and not being caused by the degeneracy of the problem, but rather by the polynomial growth.**
- **The result extends to all parabolic equations of the type**

$$u_t - \text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

with $a(\cdot)$ being a monotone operator with p-growth. More precisely we assume

$$\begin{cases} \nu(s^2 + |z_1|^2 + |z_2|^2)^{\frac{p-2}{2}}|z_2 - z_1|^2 \leq \langle a(z_2) - a(z_1), z_2 - z_1\rangle \\ |a(z)| \leq L(s^2 + |z|^2)^{\frac{p-1}{2}} \ , \end{cases}$$

- **The result also holds for systems with a special structure (sometimes called Uhlenbeck structure). This means**

$$u_t - \text{div } a(Du) = \text{div } (|F|^{p-2}F)$$

with $a(\cdot)$ being p-monotone in the sense of the previous slide, and satisfying the structure assumption

$$a(Du) = g(|Du|)Du$$

- **The p-Laplacean system is an instance of such a structure**

Elliptic vs parabolic local estimates

- **Elliptic estimate**

$$\left(\fint_{B_R} |Du|^q \, dz\right)^{\frac{1}{q}} \leq c \left(\fint_{B_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + c \left(\fint_{B_{2R}} |F|^q \, dz\right)^{\frac{1}{q}}$$

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- **Parabolic estimate** - $p \geq 2$

$$\left(\fint_{Q_R} |Du|^q \, dz\right)^{\frac{1}{q}}$$

$$\leq c \left[\left(\fint_{Q_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{2R}} |F|^q \, dz\right)^{\frac{1}{q}} + 1 \right]^{\frac{p}{2}}$$

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- **Parabolic cylinders** $Q_R \equiv B_R \times (t_0 - R^2, t_0 + R^2)$

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- **Parabolic cylinders** $Q_R \equiv B_R \times (t_0 - R^2, t_0 + R^2)$
- **The exponent** $p/2$ **is the scaling deficit of the system**

Interpolation nature of local estimates

- **Parabolic local estimate -** $p \geq 2$

$$\left(\fint_{Q_R} |Du|^q \, dz\right)^{\frac{1}{q}} \leq c \left[\left(\fint_{Q_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + c(q) \left(\fint_{Q_{2R}} |F|^q \, dz\right)^{\frac{1}{q}} + 1\right]^{\frac{p}{2}}$$

- **Taking** $F = 0$ **and letting** $q \to \infty$ **yields**

$$\sup_{Q_R} |Du| \leq c \left[\left(\fint_{Q_{2R}} |Du|^p \, dz\right)^{\frac{1}{p}} + 1\right]^{\frac{p}{2}}$$

- **This is the original sup estimate of DiBenedetto & Friedman (Crelles J. 84)**

The local estimate in the singular case

- **The singular case**

$$\frac{2n}{n+2} < p < 2$$

- **The local estimate is**

$$\left(\fint_{Q_R} |Du|^q \, dz \right)^{\frac{1}{q}}$$
$$\leq c \left[\left(\fint_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + c(q) \left(\fint_{Q_{2R}} |F|^q \, dz \right)^{\frac{1}{q}} + 1 \right]^{\frac{2p}{p(n+2)-2n}}$$

where $c \equiv c(n, N, p)$

- **Observe that**

$$\frac{2p}{p(n+2) - 2n} \nearrow \infty \qquad \text{when} \qquad p \searrow \frac{2n}{n+2}$$

The intrinsic geometry of DiBenedetto

- **The basic analysis is the following: consider intrinsic cylinders**

  $$Q_\varrho^\lambda(z_0) \equiv Q_\varrho^\lambda(x_0, t_0) = B(x_0, \varrho) \times (t_0 - \lambda^{2-p}\varrho^2, t_0)$$

  where it happens that

  $$|Du| \approx \lambda \qquad \text{in } Q_\varrho^\lambda(x_0, t_0)$$

  then the equation behaves as

  $$u_t - \lambda^{p-2}\triangle u = 0$$

  that is, scaling back in the same cylinder, as the heat equation

- **On intrinsic cylinders estimates "ellipticize"; in particular, they become homogeneous**

Intrinsic estimate

- The effect of intrinsic geometry

Theorem (DiBenedetto & Friedman, Crelle J. 85)

There exists a universal constant $c \geq 1$ *such that*

$$c\left(\fint_{Q_R^\lambda(z_0)} |Du|^{p-1}\, dz\right)^{1/(p-1)} \leq \lambda$$

then

$$|Du(z_0)| \leq \lambda$$

Sketch of the proof (lots of cheating)

Recall the estimate

$$\int |Du|^q = q \int_0^\infty \lambda^{q-1} |\{|Du| > \lambda\}| \, d\lambda$$

Therefore we want to find a decay estimates for the level sets $|\{|Du| > \lambda\}|$ in terms of the level sets $|\{|F| > \lambda\}|$

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We make a decomposition of CZ type of $|\{|Du| > \lambda\}|$ and for this we use a direct exit time argument on intrinsic cubes via the functional

$$\int_{Q_R^\lambda} (|Du|^p + M|F|^p) \, dx \, dt$$

Sketch of the proof (lots of cheating)

If $z_0 \in \{|Du| > \lambda\}$ then it happens that

$$\liminf_{r\to 0} \fint_{Q_r^\lambda(z_0)} (|Du|^p + M|F|^p)\, dx\, dt > \lambda$$

therefore for every such point we find an exit time radius $r(z_0)$ such that

$$\fint_{Q^\lambda_{r(z_0)}(z_0)} (|Du|^p + M|F|^p)\, dx\, dt \approx \lambda$$

and using Vitali or Besicovitch cover

$$|\{|Du|^p > \lambda\}| \subset \bigcup_i Q^\lambda_{r(z_i)/2}(z_i)$$

Sketch of the proof (lots of cheating)

This means that

$$\fint_{Q^{\lambda}_{r(z_0)}} |Du|^p \, dx\, dt \lesssim \lambda \quad \text{and} \quad \fint_{Q^{\lambda}_{r(z_0)}} |F|^p \, dx\, dt \lesssim \frac{\lambda}{M}$$

therefore for every such point we find an exit time radius $r(z_0)$ such that

$$\lambda \lesssim \fint_{Q^{\lambda}_{r(z_i)}(z_i)} \left(|Du|^p + M|F|^p\right) \, dx\, dt$$

Sketch of the proof (lots of cheating)

Then solve

$$\begin{cases} (v_i)_t - \operatorname{div}\left(|Dv_i|^{p-2}Dv_i\right) = 0 & \text{in} \quad Q^{\lambda}_{r(z_i)}(z_i) \\ v_i = u & \text{in} \quad \partial_{\mathrm{p}} Q^{\lambda}_{r(z_i)}(z_i) \end{cases}$$

Sketch of the proof (lots of cheating)

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then

$$\fint_{Q^{\lambda}_{r(z_i)}(z_i)} |Dv_i|^p \, dx \, dt \lesssim \lambda$$

and

$$\fint_{Q^{\lambda}_{r(z_i)}(z_i)} |Dv_i - Du|^p \, dx \, dt \lesssim \frac{\lambda}{M}$$

The first inequality allows to assert that

$$\sup_{Q^{\lambda}_{r(z_0)/2}} |Dv_i|^p \lesssim \lambda$$

that is

$$|Q^{\lambda}_{r(z_0)/2} \cap \{|Dv_i|^p > \lambda\}| = 0$$

Sketch of the proof (lots of cheating)

Then

$$
\begin{aligned}
|Q^{\lambda}_{r(z_0)/2} \cap \{|Du|^p > \lambda\}| &\lesssim |Q^{\lambda}_{r(z_0)/2} \cap \{|Du - Dv_i|^p > \lambda\}| \\
&\quad + |Q^{\lambda}_{r(z_0)/2} \cap \{|Dv_i|^p > \lambda\}| \\
&\lesssim |Q^{\lambda}_{r(z_0)/2} \cap \{|Du - Dv_i|^p > \lambda\}| \\
&\lesssim \frac{1}{\lambda} \int_{Q^{\lambda}_{r(z_i)/2}(z_i)} |Du - Dv_i|^p \, dx \, dt \\
&\lesssim \frac{|Q_{r(z_0)}|}{M}
\end{aligned}
$$

Sketch of the proof (lots of cheating)

Density information (De Giorgi style)

$$\frac{|Q^{\lambda}_{r(z_0)/2} \cap \{|Du|^p > \lambda\}|}{|Q_{r(z_0)/2}|} \lesssim \frac{1}{M}$$

Sketch of the proof (lots of cheating)

Density information (De Giorgi style)

$$\frac{|Q^{\lambda}_{r(z_0)/2} \cap \{|Du|^p > \lambda\}|}{|Q_{r(z_0)/2}|} \lesssim \frac{1}{M}$$

density is small provided M is large

Sketch of the proof (lots of cheating)

But then, using the exit time information

$$
\begin{aligned}
&|Q_{r(z_0)}| \\
&\quad \lesssim \frac{1}{\lambda}\int_{Q^{\lambda}_{r(z_i)}(z_i)\cap\{|Du|^p>\lambda\}} |Du|^p \, dx \, dt \\
&\qquad + \frac{1}{\lambda}\int_{Q^{\lambda}_{r(z_i)}(z_i)\cap\{|F|^p>\lambda\}} M|F|^p \, dx \, dt
\end{aligned}
$$

Sketch of the proof (lots of cheating)

Summarizing

$$\lambda^{\gamma-1}|Q^{\lambda}_{r(z_0)/2} \cap \{|Du|^p > \lambda\}|$$
$$\lesssim \frac{\lambda^{\gamma-2}}{M}\int_{Q^{\lambda}_{r(z_i)}(z_i)\cap\{|Du|^p>\lambda\}} |Du|^p \, dx \, dt$$
$$+\lambda^{\gamma-2}\int_{Q^{\lambda}_{r(z_i)}(z_i)\cap\{|F|^p>\lambda\}} |F|^p \, dx \, dt$$

Sketch of the proof (lots of cheating)

Summarizing

$$\lambda^{\gamma-1}|Q^{\lambda}_{r(z_0)/2} \cap \{|Du|^p > \lambda\}|$$
$$\lesssim \frac{\lambda^{\gamma-2}}{M} \int_{Q^{\lambda}_{r(z_i)}(z_i)\cap\{|Du|^p>\lambda\}} |Du|^p \, dx \, dt$$
$$+\lambda^{\gamma-2} \int_{Q^{\lambda}_{r(z_i)}(z_i)\cap\{|F|^p>\lambda\}} |F|^p \, dx \, dt$$

Integration yields

$$\int |Du|^{p\gamma} \approx \int^{\infty} \lambda^{\gamma-1}|\{|Du|^p > \lambda\}|$$
$$\lesssim \frac{\lambda^{\gamma-2}}{M} \int_{\{|Du|^p>\lambda\}} |Du|^p \, dx \, dt + \lambda^{\gamma-2} \int_{\{|Du|^p>\lambda\}} |F|^p \, dx \, dt$$
$$\approx \frac{1}{M} \int |Du|^{p\gamma} \, dx \, dt + c(M) \int |F|^{p\gamma} \, dx \, dt$$

Part 3: Non-uniformly elliptic operators

Classical facts

consider variational problems of the type

$$W^{1,1} \ni v \mapsto \int_{\Omega} f(x, Dv)\, dx \qquad \Omega \subset \mathbb{R}^n$$

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the standard growth conditions are

$$|z|^p \lesssim f(x,z) \lesssim |z|^p + 1$$

for $p > 1$, and the problem is well settled in $W^{1,p}$

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for $p > 1$, and the problem is well settled in $W^{1,p}$
a model example is

$$v \mapsto \int_\Omega c(x)|Dv|^p\, dx$$

Non-standard growth conditions

consider now variational problems of the type

$$W^{1,1} \ni v \mapsto \int_\Omega f(x, Dv)\, dx \qquad \Omega \subset \mathbb{R}^n$$

with

$$|z|^p \lesssim f(x,z) \lesssim |z|^q + 1 \qquad \text{and } q > p > 1$$

a basic condition

$$W^{1,1} \ni v \mapsto \int_\Omega f(Dv)\,dx \qquad \Omega \subset \mathbb{R}^n$$

with

$$|z|^p \lesssim f(z) \lesssim |z|^q + 1 \qquad \text{and } q > p > 1$$

then

$$\frac{q}{p} < 1 + o(n)$$

is a sufficient (Marcellini) and necessary (Giaquinta and Marcellini) condition for regularity

Several people on non-uniformly elliptic operators

- Leon Simon
- Uraltseva & Urdaletova
- Zhikov
- Marcellini
- Hong
- Lieberman
- Fusco-Sbordone
- many, many, many others (including me, unfortunately for the subject)

Non-atonomous functionals

Non-autonomous functionals of the type

$$v \mapsto \int_\Omega f(x, Dv)\, dx$$

new phenomena appear in this situation, and the presence of x *is not any longer a perturbation*

Three functionals of Zhikov

Zhikov introduced, between the 80s and the 90s, the following functionals:

$$v \mapsto \int_\Omega |Dv|^2 w(x)\,dx \qquad w(x) \geq 0$$

$$v \mapsto \int_\Omega |Dv|^{p(x)}\,dx \qquad p(x) \geq 1$$

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\,dx \qquad a(x) \geq 0$$

motivations: modelling of strongly anisotropic materials, Elasticity, Homogenization, Lavrentiev phenomenon etc

Two counterexamples

Theorem (Esposito-Leonetti-Min. JDE 04)

For every choice of $n \geq 2$, $\Omega \subset \mathbb{R}^n$ *and of*

$$\varepsilon > 0 \qquad \text{and} \qquad \alpha \in (0,1)$$

there exists a non-negative function $a(\cdot) \in C^{0,\alpha}$, *a boundary datum* $u_0 \in W^{1,\infty}(B)$ *and exponents* p, q *satisfying*

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min\limits_{w} \int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\ w \in u_0 + W^{1,p}_0(B) \end{cases}$$

does not belong *to* $W^{1,q}_{\text{loc}}(B)$

The example goes via Lavrentiev phenomenon

$$\inf_{w \in u_0 + W_0^{1,p}(B)} \int_B (|Dv|^p + a(x)|Dv|^q)\, dx$$

$$< \inf_{w \in u_0 + W_0^{1,p}(B) \cap W_{\mathrm{loc}}^{1,q}(B)} \int_B (|Dv|^p + a(x)|Dv|^q)\, dx$$

Two counterexamples

Theorem (Fonseca-Malý-Min. ARMA 04)

For every choice of $n \geq 2$, $\Omega \subset \mathbb{R}^n$ *and of* $\varepsilon > 0$, $\alpha > 0$, *there exists a non-negative function* $a(\cdot) \in C^{[\alpha]+\{\alpha\}}$, *a boundary datum* $u_0 \in W^{1,\infty}(B)$ *and exponents* p, q *satisfying*

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min\limits_{w} \displaystyle\int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\ \qquad w \in u_0 + W^{1,p}_0(B) \end{cases}$$

has a singular set of essential discontinuity points of Hausdorff dimension larger than $n - p - \varepsilon$

Theorem 1

Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimiser of the functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad \frac{q}{p} < 1 + \frac{\alpha}{n}$$

then

Du is Hölder continuous

Theorem 2

Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$ *be a* **bounded** *local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad q \leq p + \alpha$$

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$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \text{and} \qquad q \leq p + \alpha$$

then

Du is Hölder continuous

Notice the the delicate borderline case $q = p + \alpha$ is achieved

Theorem 3

Theorem (Colombo-Min. JFA 15)

Let $u \in W^{1,p}(\Omega)$ *be a distributional solution to*

$$\operatorname{div}\left(|Du|^{p-2}Du + a(x)|Du|^{q-2}Du\right) = \operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F\right)$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad \frac{q}{p} \leq 1 + \frac{\alpha}{n}$$

then

$$(|F|^p + a(x)|F|^q) \in L^{\gamma}_{\mathrm{loc}} \Longrightarrow (|Du|^p + a(x)|Du|^q) \in L^{\gamma}_{\mathrm{loc}}$$

for every $\gamma \geq 1$

Theorem 4

Theorem (Colombo-Min. JFA 15)

Let $u \in W^{1,p}(\Omega)$ be a **bounded** *minimiser of the functional*

$$v \mapsto \int [|Dv|^p + a(x)|Dv|^q - (|F|^{p-2} + a(x)|F|^{q-2})\langle F, Dv\rangle]$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \text{and} \qquad q \leq p + \alpha$$

and

$$\sup_{B_\varrho} \varrho^{p_0} \fint_{B_\varrho} [|F|^p + a(x)|F|^q] \, dx < \infty \qquad \text{for some} \quad p_0 < p$$

then

$$(|F|^p + a(x)|F|^q) \in L^\gamma_{\text{loc}} \Longrightarrow (|Du|^p + a(x)|Du|^q) \in L^\gamma_{\text{loc}}$$

The general viewpoint

is to consider functionals as

$$v \mapsto \int_\Omega f(x, v, Dv)\, dx$$

where

$$H(x, |z|) \lesssim f(x, u, z) \lesssim H(x, |z|) + 1$$

with

$$H(x, |z|) = |z|^p + a(x)|z|^q$$

being a replacement of

$$|z|^p$$

Heuristic explanations - dependence on α of the bound

the Euler equation of the functional is

$$\operatorname{div} a(x, Du) = \operatorname{div}\left(|Du|^{p-2}Du + (q/p)a(x)|Du|^{q-2}Du\right) = 0$$

then

$$\frac{\text{highest eigenvalue of } \partial_z a(x, Du)}{\text{lowest eigenvalue of } \partial_z a(x, Du)} \approx 1 + a(x)|Du|^{q-p}$$

$$\approx 1 + R^{\alpha}|Du|^{q-p}$$

Heuristic explanation - the bound $q \leq p + \alpha$

consider the usual p-capacity for $p < n$

$$\mathrm{cap}_p(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |Dv|^p \, dx \; : \; f \in W^{1,p}, f \geq 1 \text{ on } B_r \right\}$$

we have

$$\mathrm{cap}_p(B_r) \approx r^{n-p}$$

then consider the weighted capacity

$$\mathrm{cap}_{q,\alpha}(B_r) = \inf \left\{ \int_{\mathbb{R}^n} |x|^\alpha |Dv|^q \, dx \; : \; f \in C_0^\infty(\mathbb{R}^n), f \geq 1 \text{ on } B_r \right\}$$

we then have (the ball is centered at the origin)

$$\mathrm{cap}_{q,\alpha}(B_r) \approx r^{n-q+\alpha}$$

We then ask for

$$\operatorname{cap}_{q,\alpha}(B_r) \lesssim \operatorname{cap}_p(B_r)$$

that is

$$r^{n-q+\alpha} \leq r^{n-p}$$

for r small enough, so that

$$q \leq p + \alpha$$

A parallel with Muckenhoupt weights

a maximal theorem holds

$$\int_\Omega [H(x,|M(f)|)]^t \, dx \lesssim \int_\Omega [H(x,|f|)]^t \, dx$$

where Mf is the usual (localised) Hardy-Littlewood maximal operator, together with a Sobolev-Poincaré type inequality

$$\left(\fint_{B_R} \left[H\left(x, \left|\frac{f-(f)_{B_R}}{R}\right|\right)\right]^d dx\right)^{1/d} \leq c \fint_{B_R} [H(x,|Df|)] dx$$

for $d > 1$

A parallel with Muckenhoupt weights

A non-negative function $w \in L^p$ is said to be of class A_p if

$$\sup_{B_R} \left(\fint_{B_R} |w| \, dx \right) \left(\fint_{B_R} |w|^{1/(1-p)} \, dx \right)^{1/(p-1)} < \infty$$

then it follows

$$\int_\Omega |M(f)|^t w(x) \, dx \lesssim \int_\Omega |f|^t w(x) \, dx$$

holds for $t > 1$ and

$$\left(\fint_{B_R} \left[H \left(x, \left| \frac{f - (f)_{B_R}}{R} \right| \right) \right]^d dx \right)^{1/d} \leq c \fint_{B_R} H(x, |Df|) dx$$

holds for $d > 1$

Questions

- Study more general conditions for which such abstract results hold in connection to regularity theorems, for instance
- Define the quantity

$$\begin{aligned}&\mathrm{cap}_H(B_r)\\&\quad=\inf\left\{\int_{\mathbb{R}^n} H(x,Dv)\,dx\,:\,f\in C_0^\infty(\mathbb{R}^n), f\geq 1 \text{ on } B_r\right\}\end{aligned}$$

  and prove it is a capacity in the usual sense when $q \leq p+\alpha$; also consider the condition $q/p < 1+\alpha/n$
- Consider removability of singularities problems using this capacity, and in connection obstacle problems

Minima of functionals of the type

$$v \to \int f(x, v, Dv)\, dx$$

with

$$f(x, v, z) \approx |z|^p w(x) \equiv H(x, |z|)$$

are locally Hölder continuous provided
Fabes-König-Serapioni (Comm. PDE 1982) - Modica (Ann. Mat. Pura Appl. 1985)

Questions

- Study more general conditions for which such abstract results hold in connection to regularity theorems, for instance
- Define the quantity

$$\operatorname{cap}_H(B_r) = \inf\left\{\int_{\mathbb{R}^n} H(x,Dv)\,dx \,:\, f\in C_0^\infty(\mathbb{R}^n), f\geq 1 \text{ on } B_r\right\}$$

  and prove it is a capacity in the usual sense when $q \leq p+\alpha$; also consider the condition $q/p < 1+\alpha/n$
- Consider removability of singularities problems using this capacity, and in connection obstacle problems
- Consider weights with respect to this new norm

There exists a universal threshold $M \equiv M(n, p, q, \alpha)$ such that if on the ball B_R

$$a_i(R) := \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^{\alpha}$$

Then our functional is essentially equivalent to

$$v \mapsto \int_{B_R} |Dv|^p \, dx$$

there exists a universal threshold $M \equiv M(n, p, q, \alpha)$ such that if on the ball B_R

$$a_i(R) := \inf_{x \in B_R} a(x) > M[a]_{0,\alpha} R^{\alpha}$$

then our functional is essentially equivalent to

$$v \mapsto \int_{B_R} (|Dv|^p + a_i(R)|Dv|^q)\, dx$$

Implementation of this is very delicate and goes though a delicate analysis involving an exit time argument

Tool 1: reverse Hölder inequality

Lemma

Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and let B_R be a ball such that

$$\inf_{x \in B_R} a(x) \leq M[a]_\alpha R^\alpha \qquad \text{and} \qquad \frac{q}{p} < 1 + \frac{\alpha}{n}$$

hold. then there exists a positive constant $c \equiv c(M)$ such that

$$\left(\fint_{B_{R/2}} |Du|^{2q-p}\, dx \right)^{1/(2q-p)} \leq c \left(\fint_{B_R} |Du|^p\, dx \right)^{1/p}$$

Tool 2: Caccioppoli type inequality

Lemma

Let $u \in W^{1,p}(\Omega)$ *be a* **bounded** *local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and let B_R *be a ball such that*

$$\inf_{x \in B_R} a(x) \leq M[a]_\alpha R^\alpha \qquad \text{and} \qquad q \leq p + \alpha$$

hold. then there exists a positive constant $c \equiv c(M)$ *such that*

$$⨍_{B_{R/2}} |Du|^p\, dx \leq c ⨍_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^p dx$$

Theorem on bounded minimisers

Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$ *be a* **bounded** *local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and assume that

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \textit{and} \qquad q \leq p + \alpha$$

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Du is Hölder continuous

Theorem on bounded minimisers

Theorem (Colombo-Min. ARMA 15)

Let $u \in W^{1,p}(\Omega)$ *be a* **bounded** *local minimiser of the functional*

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx$$

and assume that

$$0 \le a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \text{and} \qquad q \le p + \alpha$$

then

Du is Hölder continuous

A parabolic theorem is on its way

Proof goes in ten different Steps

- Step 1: Low Hölder continuity (to treat the borderline case $q = p + \alpha$)
- Step 2: p-harmonic approximation to handle the p-phase
- Step 3: Decay estimate on all scales in the (p, q)-phase
- Step 4: Exit time argument implies $u \in C^{0,\gamma}$ for every $\gamma < 1$
- Step 5: Previous Step implies that Du is in every Morrey space
- Step 6: Morrey space regularity of the gradient implies absence of Lavrentiev phenomenon
- Step 7: Gradient fractional Sobolev regularity
- Step 8: Upgraded Caccioppoli inequality via interpolation inequalities in fractional Sobolev spaces
- Step 9: Higher integrability of the gradient implies a better p-harmonic approximation in the p-phase
- Step 10: Hölder gradient continuity via weighted separation of phases

The excess functional

I will consider for simplicity the case $p \geq 2$

The excess functional

I will consider for simplicity the case $p \geq 2$

$$E(u; x_0, R) := \left(\fint_{B_R(x_0)} |u - (u)_{B_R(x_0)}|^p \, dx \right)^{1/p}$$

You want to prove that

$$E(u; x_0, \tau^k R) \leq \tau^{k\gamma} E(u; x_0, R)$$

and this implies that

$$u \in C^{0,\gamma}$$

u is locally Hölder continuous with some potentially microscopic exponent $\gamma_0 \in (0, 1)$. This essentially serve to catch the borderline case $q = p + \alpha$.

Step 2: p-phase

assume

$$\inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^{\alpha}$$

holds for some number $M \geq 1$. then for every $\gamma \in (0,1)$ there exists a positive radius $R_* \equiv R_*(M,\gamma)$ and $\tau \equiv \tau(M,\gamma) \in (0,1/4)$ such that the decay estimate

$$E(u; x_0, \tau R) \leq \tau^{\gamma} E(u; x_0, R)$$

holds whenever $0 < R \leq R_*$

→ Caccioppoli inequality in the p-phase becomes

$$\int_{B_{R/2}} |Du|^p \, dx \leq c \int_{B_R} \left| \frac{u-(u)_{B_R}}{R} \right|^p dx = \left(\frac{E(u;x_0,R)}{R} \right)^p ,$$

→ then define

$$v(x) := \frac{u(x_0+Rx)}{E(u;x_0,R)} , \qquad x \in B_1$$

so that

$$\fint_{B_{1/2}} |Dv|^p \, dx \leq c$$

Step 2: p-phase

$\rightarrow$ moreover, v solves, for every $\varphi \in C_0^\infty(B_1)$

$$\int_{B_1} \langle |Dv|^{p-2}Dv+(q/p)\tilde{a}(x)R^{p-q}[E(u;x_0,R)]^{q-p}|Dv|^{q-2}Dv, D\varphi\rangle \, dx = 0$$

this means that

$$\begin{aligned}
&\left| \fint_{B_1} \langle |Dv|^{p-2}Dv, D\varphi\rangle \, dx \right| \\
&\qquad \leq cMR^{p+\alpha-q}[E(u;x_0,R)]^{q-p}\|D\varphi\|_{L^\infty(B_{1/2})} \fint_{B_{1/2}} |Dv|^{q-1}\, dx \\
&\qquad \leq cR^{p+\alpha-q+\gamma_0(q-p)}\|D\varphi\|_{L^\infty(B_{1/2})} \left(\fint_{B_{1/2}} |Dv|^{p}\, dx \right)^{\frac{q-1}{p}} \\
&\qquad \leq C_* R_*^{p+\alpha-q+\gamma_0(q-p)}\|D\varphi\|_{L^\infty(B_{1/2})}
\end{aligned}$$

Step 2: p-phase

$\rightarrow$ we conclude that

$$\left| \fint_{B_1} \langle |Dv|^{p-2} Dv, D\varphi \rangle \, dx \right| \leq \varepsilon \|D\varphi\|_{L^\infty(B_{1/2})}$$

by taking R_* suitably small

Step 2: p-phase

$\rightarrow$ apply the p-harmonic approximation lemma

Theorem (Duzaar - Min. Calc. Var. 04)

Given $\varepsilon > 0$ and $L > 0$, there exists $\delta \in (0,1]$ such that whenever $v \in W^{1,p}(B_{1/2})$ satisfies

$$\fint_{B_{1/2}} |Dv|^p \, dx \leq L$$

and

$$\fint_{B_{1/2}} \langle |Dv|^{p-2} Dv, D\varphi \rangle \, dx \leq \delta \|D\varphi\|_{L^\infty(B_{1/2})}$$

holds for all $\varphi \in C^1_0(B_{1/2})$. there exists a p-harmonic map $h \in W^{1,p}(B_{1/2})$, that is $\operatorname{div}(|Dh|^{p-2}Dh) = 0$, such that

$$\fint_{B_{1/2}} |v - h|^p \, dx \leq \varepsilon^p$$

Step 2: p-phase

$\rightarrow$ we conclude that

$$\left| \fint_{B_1} \langle |Dv|^{p-2}Dv, D\varphi \rangle \, dx \right| \leq \varepsilon \|D\varphi\|_{L^\infty(B_{1/2})}$$

by taking R_* suitably small
$\rightarrow$ find a p-harmonic map h such that

$$\fint_{B_{1/2}} |v - h|^p \, dx \leq \varepsilon^p$$

$\rightarrow$ for harmonic maps you know that you have a good excess decay, and therefore, since v and h are close, then also v has the same property; scaling back, the same property holds for u

Step 3: (p, q)-phase

assume

$$\inf_{x \in B_R} a(x) > M[a]_{0,\alpha} R^\alpha$$

holds for some number $M \geq 1$. Fix $\gamma \in (0,1)$; there exist positive constants $M_1 \geq 4$ and $\tau \in (0, 1/4)$, with depending on γ, such that if $M \geq M_1$, then the decay estimate

$$E(u; x_0, \tau^k R) \lesssim \tau^{k\gamma} R \left[\fint_{B_{2R}} \left(\left| \frac{u - (u)_{B_{2R}}}{R} \right|^p + a(x) \left| \frac{u - (u)_{B_{2R}}}{R} \right|^q \right) dx \right]^{1/p}$$

holds for every integer $k \geq 0$

Step 4: Separation of phases via exit time

$\rightarrow$ choose $\gamma \in (0,1)$

$\rightarrow$ Find $M \geq 1$ and τ_2 from Step 2

$\rightarrow$ Use this M in Step 1 and find R_* and τ_1 from Step 1

$\rightarrow$ consider the sequence of balls

$$....B_{R_{k+1}} \subset B_{R_k} ... \subset B_{R_1} \subset B_R \,, \qquad\qquad R_k = \tau_1^k R_0$$

and the condition

$$\inf_{x \in B_{R_k}} a(x) \leq M R_k^{\alpha} \tag{1}$$

the exit time index is

$$m := \min \left\{ k \in \mathbb{N} \cup \{\infty\} \; : \; (1) \, \text{fails} \right\} \, .$$

Step 4: Separation of phases via exit time

$\rightarrow$ keep on using Step 1 as long as the exit time is not reached, this yields

$$E(u; x_0, \tau_1^k R_0) \leq \tau_1^{k\gamma} E(u; x_0, R_0) \qquad \text{for every } k \in \{0, \ldots, m\}\ .$$

$\rightarrow$ after the exit time you can use Step 2 to get

$$E(u; x_0, \tau_2^k \tau_1^m R_0) \lesssim \tau_2^{k\gamma} E(u; x_0, 2\tau_1^m R_0) + \tau_2^{k\gamma} \tau_1^m R_0 \left(\fint_{B_{2\tau_1^m R_0}} a(x) \left| \frac{u - (u)_{B_{2\tau_1^m R_0}}}{\tau_1^m R_0} \right|^q dx \right)^{1/p}$$

$\rightarrow$ match the two inequalities using the exit time condition and ones again the bound $q \leq p + \alpha$

Step 5: Morrey space regularity of the gradient

this tells that

$$\int_{B_R} |Du|^p \, dx \lesssim R^{n-\theta} \qquad \forall \ \theta > 0$$

there exists a sequence of smooth functions $\{u_n\}$ such that

$$\int_B (|Du_n|^p + a(x)|Du_n|^q)\,dx \to \int_B (|Du|^p + a(x)|Du|^q)\,dx$$

for every ball $B \subset \Omega$

Step 7: Fractional differentiability

We get suitable uniform estimates in

$$Du \in W^{\beta/p,p} \qquad \text{for every } \beta < \alpha$$

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We get suitable uniform estimates in

$$Du \in W^{\beta/p,p} \qquad \text{for every } \beta < \alpha$$

we recall that this means

$$\int_{\Omega'} \int_{\Omega'} \frac{|Du(x) - Du(y)|^p}{|x-y|^{n+\beta}} < \infty$$

for every $\Omega' \Subset \Omega$

Step 7: Fractional differentiability

the proof goes via approximation

$$\begin{cases} v_n \mapsto \min_w \int_B (|Dv|^p + [a(x) + \sigma_n]|Dv|^q)\, dx \\ \qquad\qquad w \in u_n + W^{1,q}_0(B) \end{cases}$$

where $0 < \sigma_n \to 0$

$$\int_B (|Du_n|^p + a(x)|Du_n|^q)\, dx \to \int_B (|Du|^p + a(x)|Du|^q)\, dx$$

and

$$u_n \in C^\infty(B)$$

Step 7: Fractional differentiability

the proof goes via approximation

$$\begin{cases} v_n \mapsto \min\limits_{w} \displaystyle\int_B (|Dv|^p + [a(x) + \sigma_n]|Dv|^q)\, dx \\ \qquad\qquad w \in u_n + W_0^{1,q}(B) \end{cases}$$

where $0 < \sigma_n \to 0$

$$\int_B (|Du_n|^p + a(x)|Du_n|^q)\, dx \to \int_B (|Du|^p + a(x)|Du|^q)\, dx$$

and

$$u_n \in C^\infty(B)$$

this implies $v_n \to u$

Step 8: Upgraded Caccioppoli inequality

the following improved Caccioppoli type inequality holds:

$$
\begin{aligned}
& \fint_{B_{R/2}} |Du|^{2q-p} \, dx \\
& \lesssim \frac{1}{R^{\alpha/2}} \left[\fint_{B_{2R}} \left(\left| \frac{u-(u)_{B_R}}{R} \right|^p + a(x) \left| \frac{u-(u)_{B_{2R}}}{R} \right|^q \right) \, dx + 1 \right]^b
\end{aligned}
$$

we use the fractional interpolation inequality

$$\|f\|_{W^{\tilde{s},t}} \leq c\|f\|^{\theta}_{W^{s_1,p_1}}\|f\|^{1-\theta}_{W^{s_2,p_2}}$$

with

$$\tilde{s} = \theta s_1 + (1-\theta)s_2 \qquad \frac{1}{t} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$$

Step 8: Upgraded Caccioppoli inequality

we apply as

$$\|Dv_n\|_{L^t} \leq c[v_n]_{s,p_1}^{\theta}\|Dv_n\|_{W^{\beta/p,p}}^{1-\theta}$$

with exponents

$$1 = \theta s + (1-\theta)\left(1+\frac{\beta}{p}\right) \qquad \frac{1}{t} = \frac{\theta}{p_1} + \frac{1-\theta}{p}$$

and

$$[v_n]_{s,p_1} := \left(\int\int \frac{|v_n(x)-v_n(y)|^{p_1}}{|x-y|^{n+sp_1}}\,dx\,dy\right)^{1/p_1}$$

and take s close to 1 as you please and p_1 as large as you like

Step 9: Improved estimate in the p-phase

if for some $M \geq 1$

$$a_i(R) = \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha$$

then solve

$$\begin{cases} v \mapsto \min_w \int_{B_R} |Dv|^p \, dx \\ w \in u + W_0^{1,p}(B_R) \end{cases}$$

and find

$$\fint_{B_R} |Du - Dv|^p \, dx \leq M^2 R^\alpha$$

Step 9: Improved estimate in the p-phase

if for some $M \geq 1$

$$a_i(R) = \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^{\alpha}$$

then solve

$$\begin{cases} v_R \mapsto \min_w \int_{B_R} (|Dv|^p + a_i(R)|Dv|^q) \, dx \\ \qquad\qquad w \in u + W_0^{1,p}(B_R) \end{cases}$$

and get

$$\fint_{B_R} |Du - Dv|^p \, dx \lesssim \frac{1}{M} \fint_{B_{2R}} \left(\left| \frac{u - (u)_{B_R}}{R} \right|^p + a(x) \left| \frac{u - (u)_{B_{2R}}}{R} \right|^q \right) dx$$

Step 10: Final gradient continuity

$\to$ take B_R and $M > 0$ and consider the functionals

$$v \mapsto \int_{B_R} (|Dv|^p + a_i(R)|Dv|^q) \, dx$$

where

$$a_i(R) := \begin{cases} 0 & \text{if} \quad \inf_{x \in B_R} a(x) \leq M[a]_{0,\alpha} R^\alpha \\ \inf_{x \in B_R} a(x) & \text{if} \quad \inf_{x \in B_R} a(x) > M[a]_{0,\alpha} R^\alpha \end{cases}$$

$\to$ solve

$$\begin{cases} v_R \mapsto \min_w \int_{B_R} (|Dv|^p + a_i(R)|Dv|^q) \, dx \\ w \in u + W_0^{1,p}(B_R) \end{cases}$$

Part 4: Nonlinear potential theory

The classical potential estimates

- **Consider the model case**

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

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- **We have**

$$u(x) = \int G(x,y)\mu(y)$$

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$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n$$

- **We have**

$$u(x) = \int G(x,y)\mu(y)$$

where

$$G(x,y) \approx \begin{cases} |x-y|^{2-n} & \text{se } n > 2 \\ -\log|x-y| & \text{se } n = 2 \end{cases}$$

- Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

Estimates via Riesz potentials

- Previous formula gives

$$|u(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-2}} = I_2(|\mu|)(x)$$

- while, after differentiation, we obtain

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

Local versions

- **In bounded domains one uses**

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho} \qquad \beta\in(0,n]$$

since

$$\begin{aligned}\mathbf{I}^{\mu}_{\beta}(x,R) &\lesssim \int_{B_R(x)} \frac{d|\mu|(y)}{|x-y|^{n-\beta}}\\ &= I_\beta(|\mu|\llcorner B_R(x))(x)\\ &\leq I_\beta(|\mu|)(x)\end{aligned}$$

for non-negative measures

What happens in the nonlinear case?

- **For instance for nonlinear equations with linear growth**

$$-\text{div } a(Du) = \mu$$

that is equations well posed in $W^{1,2}$ (p-growth and $p = 2$) that is

$$|\partial a(z)| \leq L \qquad \nu|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda\rangle$$

- **And degenerate ones like**

$$-\text{div } (|Du|^{p-2}Du) = \mu$$

- To be short, we shall concentrate on the case $p \geq 2$

Nonlinear potentials

- **The nonlinear Wolff potential is defined by**

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p]$$

which for $p = 2$ reduces to the usual Riesz potential

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{\mu(B_\varrho(x))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n]$$

- **The nonlinear Wolff potential** plays in nonlinear potential theory the same role the Riesz potential plays in the linear one

The first nonlinear potential estimate

Theorem (Kilpeläinen & Malý, Acta Math. 94)

If u *solves*

$$-\text{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|u(x)| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \left(\fint_{B_R(x)} |u|^{p-1}\,dy\right)^{1/(p-1)}$$

holds

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where

$$\mathbf{W}^{\mu}_{1,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

For $p = 2$ we are back to the Riesz potential $\mathbf{W}^{\mu}_{1,p} = \mathbf{I}^{\mu}_2$ - the above estimate is non-trivial already in this situation

Corollary: optimal integrability

- **Indeed**

$$\mu \in L^q \Longrightarrow \mathbf{W}^{\mu}_{\beta,p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \qquad q \in (1,n)$$

and more in general estimates in rearrangement invariant function spaces

- **Indeed**

$$\mu \in L^q \Longrightarrow \mathbf{W}^{\mu}_{\beta,p} \in L^{\frac{nq(p-1)}{n-qp\beta}} \qquad q \in (1,n)$$

and more in general estimates in rearrangement invariant function spaces

- **This property follows by another pointwise estimate**

$$\int_0^\infty \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} \lesssim I_\beta \left\{[I_\beta(|\mu|)]^{1/(p-1)}\right\}(x)$$

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- **This property follows by another pointwise estimate**

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- The quantity in the right-hand side is usually called Havin-Mazya potential

NON-LINEAR POTENTIAL THEORY

V. G. Maz'ya and V. P. Khavin

Contents

A first gradient potential estimate

Theorem (Min., JEMS 11)

When $p = 2$, if u solves

$$-\operatorname{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{I}_1^{|\mu|}(x, R) + \fint_{B_R(x)} |Du| \, dy$$

holds

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For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)$$

The $p \neq 2$ case: a long path towards optimality

Theorem (Duzaar & Min., AJM 11)

When $p \geq 2$, if u solves

$$-\text{div}\, a(Du) = \mu$$

then

$$|Du(x)| \lesssim \mathbf{W}^{\mu}_{1/p,p}(x,R) + \fint_{B_R(x)} |Du|\, dy$$

holds

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where

$$\mathbf{W}^{\mu}_{1/p,p}(x,R) = \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

The $p \neq 2$ case: a long path towards optimality

Theorem (Duzaar & Min., JFA 10)

When $2 - 1/n < p < 2$, *if* u *solves*

$$-\operatorname{div} a(Du) = \mu$$

then

$$|Du(x)| \lesssim \left[\mathbf{I}_1^{|\mu|}(x, R) \right]^{1/(p-1)} + \fint_{B_R(x)} |Du| \, dy$$

holds

The $p \neq 2$ case: a long path towards optimality

When $p < 2$ it holds that

$$\mathbf{W}^{\mu}_{1/p,p}(x,R) \lesssim \left[\mathbf{I}^{|\mu|}_{1}(x,R)\right]^{1/(p-1)}$$

The $p \neq 2$ case: a long path towards optimality

When $p < 2$ it holds that

$$\mathbf{W}^{\mu}_{1/p,p}(x,R) \lesssim \left[\mathbf{I}_1^{|\mu|}(x,R)\right]^{1/(p-1)}$$

Indeed

$$\begin{aligned}
\mathbf{W}^{\mu}_{1/p,p}(x,R) &= \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-1}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} \\
&\approx \sum_i \left[\frac{|\mu|(B_{\varrho_i}(x))}{\varrho_i^{n-1}}\right]^{1/(p-1)} \\
&\lesssim \left[\sum_i \frac{|\mu|(B_{\varrho_i}(x))}{\varrho_i^{n-1}}\right]^{1/(p-1)} \\
&\approx \left[\mathbf{I}_1^{|\mu|}(x,R)\right]^{1/(p-1)}
\end{aligned}$$

- Consider

$$-\text{div}\, v = \mu$$

with

$$v = |Du|^{p-2}Du$$

Indeed

Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B_R(x)} |Du|\, dy\right)^{p-1}$$

holds

Indeed

Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If u solves

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \mathbf{I}_1^{|\mu|}(x,R) + \left(\fint_{B_R(x)} |Du|\, dy \right)^{p-1}$$

holds

The theorem still holds for general equations of the type $-\mathrm{div}\, a(Du) = \mu$

A global estimate

Theorem (Kuusi & Min., CRAS 11 + ARMA 13)

If $u \in W^{1,1}(\mathbb{R}^n)$ *solves*

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} = I_1(|\mu|)(x)\,.$$

Part 4.2: Estimates in the vectorial case

Theorem (Kuusi & Min., Preprint 15)

If $u\colon \Omega \to \mathbb{R}^N$ *solves*

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|u(x) - (u)_{B_R(x)}| \lesssim \mathbf{W}^{\mu}_{1,p}(x,R) + \fint_{B(x,R)} |u - (u)_{B_R(x)}|\, dy$$

holds whenever the right hand sides are finite.

The vectorial case

Theorem (Kuusi & Min., Preprint 15)

If $u\colon \Omega \to \mathbb{R}^N$ *solves*

$$-\mathrm{div}\,(|Du|^{p-2}Du) = \mu$$

then

$$|Du(x) - (Du)_{B(x,R)}| \lesssim \left[\mathbf{I}_1^{|\mu|}(x,R)\right]^{1/(p-1)} + \fint_{B(x,R)} |Du - (Du)_{B(x,R)}|\,dy$$

holds whenever the right hand sides are finite.

Potential characterisation of Lebesgue points

Theorem (Kuusi & Min. BMS 14)

If x is a point such that

$$\mathbf{W}^{\mu}_{1,p}(x,R) < \infty$$

for some $R > 0$ then x is a Lebesgue point of u that is, the following limit

$$\lim_{\varrho\to 0} \fint_{B_\varrho(x)} u(y)\, dy$$

exists

Potential characterisation of Lebesgue points

Theorem (Kuusi & Min. BMS 14)

If x is a point such that

$$\mathbf{I}_1^{|\mu|}(x,R) < \infty$$

for some $R > 0$ then x is a Lebesgue point of Du that is, the following limit

$$\lim_{\varrho \to 0} \fint_{B_\varrho(x)} Du(y)\, dy$$

exists

Part 4.3: Oscillation bounds

The general continuity criterion

Theorem (Kuusi & Min. ARMA 13)

If u solves

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu$$

and

$$\lim_{R\to 0} \mathbf{I}_1^{|\mu|}(x,R) = 0 \text{ uniformly w.r.t. } x$$

then

Du is continuous

A classical theorem of Stein

Theorem (Stein, Ann. Math. 81)

$$Dv \in L(n,1) \Longrightarrow v \text{ is continuous}$$

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We recall that

$$g \in L(n,1) \Longleftrightarrow \int_0^\infty |\{x : |g(x)| > \lambda\}|^{1/n} \, d\lambda < \infty$$

A classical theorem of Stein

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We recall that

$$g \in L(n,1) \Longleftrightarrow \int_0^\infty |\{x : |g(x)| > \lambda\}|^{1/n} \, d\lambda < \infty$$

It follows that

$$\triangle u = \mu \in L(n,1) \Longrightarrow Du \text{ is continuous}$$

A classical theorem of Stein

Theorem (Stein, Ann. Math. 81)

$$Dv \in L(n,1) \Longrightarrow v \textit{ is continuous}$$

We recall that

$$g \in L(n,1) \Longleftrightarrow \int_0^\infty |\{x : |g(x)| > \lambda\}|^{1/n}\, d\lambda < \infty$$

An example of $L(n,1)$ function is given by

$$\frac{1}{|x| \log^\beta(1/|x|)} \qquad \beta > 1$$

in the ball $B_{1/2}$

A nonlinear Stein theorem

Theorem (Kuusi & Min., ARMA 13)

If u solves the p-Laplacean equation

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu \in L(n,1)$$

then

$$Du \text{ is continuous}$$

Part 4.4: A fully fractional approach

The setting

We take $p = 2$ and consider

$$\begin{cases} |a(z)| + |\partial a(z)||z| \leq L|z| \\ \nu^{-1}|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda\rangle \end{cases}$$

A first gradient potential estimate

Theorem (Min., JEMS 11)

When $p = 2$*, if* u *solves*

$$-\operatorname{div} a(Du) = \mu$$

then

$$|D_\xi u(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c \fint_{B(x,R)} |D_\xi u|\, dx$$

for every $\xi \in \{1, \ldots, n\}$

A first gradient potential estimate

Theorem (Min., JEMS 11)

When $p = 2$, if u solves

$$-\text{div}\, a(Du) = \mu$$

then

$$|D_\xi u(x)| \leq c\mathbf{I}_1^{|\mu|}(x, R) + c \fint_{B(x,R)} |D_\xi u| \, dx$$

for every $\xi \in \{1, \ldots, n\}$

For solutions in $W^{1,1}(\mathbb{R}^N)$ we have

$$|Du(x)| \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x - y|^{n-1}} = I_1(|\mu|)(x)$$

Classical Gradient estimates

- **Consider energy solutions to** $\operatorname{div} a(Du) = 0$ for $p = 2$
- **First prove** $Du \in W^{1,2}$
- **Then use that** $v = D_\xi u$ solves

$$\operatorname{div}(A(x)Dv) = 0 \qquad\qquad A(x) := a_z(Du(x))$$

- **The boundedness of** $D_\xi u$ follows by Standard DeGiorgi's theory
- **This is a consequence of Caccioppoli's inequalities of the type**

$$\int_{B_{R/2}} |D(D_\xi u - k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} |(D_\xi u - k)_+|^2 \, dy$$

where

$$(D_\xi u - k)_+ := \max\{D_\xi u - k, 0\}$$

Recall the definition

- **We have**

$$v \in W^{\sigma,1}(\Omega')$$

iff $v \in L^1(\Omega')$ and

$$[v]_{\sigma,1;\Omega'} = \int_{\Omega'}\int_{\Omega'} \frac{|v(x)-v(y)|}{|x-y|^{n+\sigma}}\, dx\, dy < \infty$$

There is a differentiability problem

For solutions to

$$\operatorname{div} a(Du) = \mu \quad \text{in general} \quad Du \notin W^{1,1}$$

but nevertheless it holds

Theorem (Min., Ann. SNS Pisa 07)

$$Du \in W^{1-\varepsilon,1}_{\text{loc}}(\Omega, \mathbb{R}^n) \qquad \text{for every } \varepsilon \in (0,1)$$

This means that

$$[Du]_{1-\varepsilon,1;\Omega'} = \int_{\Omega'} \int_{\Omega'} \frac{|Du(x) - Du(y)|}{|x-y|^{n+1-\varepsilon}} \, dx \, dy < \infty$$

holds for every $\varepsilon \in (0,1)$, and every subdomain $\Omega' \Subset \Omega$

Step 1: A non-local Caccioppoli inequality

Theorem (Min., JEMS 11)

Let

$$w = D_\xi u \quad \textit{with} \quad -\operatorname{div} a(Du) = \mu$$

where $\xi \in \{1, \ldots, n\}$ *then*

$$[(|w| - k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c}{R^\sigma} \int_{B_R} (|w| - k)_+ \, dy + \frac{cR|\mu|(B_R)}{R^\sigma}$$

holds for every $\sigma < 1/2$

Step 1: A non-local Caccioppoli inequality

Theorem (Min., JEMS 11)

Let

$$w = D_\xi u \quad \textit{with} \quad -\operatorname{div} a(Du) = \mu$$

where $\xi \in \{1, \ldots, n\}$ *then*

$$[(|w|-k)_+]_{\sigma,1;B_{R/2}} \leq \frac{c}{R^\sigma} \int_{B_R} (|w|-k)_+ \, dy + \frac{cR|\mu|(B_R)}{R^\sigma}$$

holds for every $\sigma < 1/2$

Compare with the usual one for $\operatorname{div} a(Du) = 0$, that is

$$[(w-k)_+]^2_{1,2;B_{R/2}} \equiv \int_{B_{R/2}} |D(w-k)_+|^2 \, dy \leq \frac{c}{R^2} \int_{B_R} (w-k)_+^2 \, dy$$

Step 1: A non-local Caccioppoli inequality

- **This approach reveal the robustness of energy inequalities, which hold below the natural growth exponent 2, and for fractional order of differentiability, although the equation has integer order**
- **Classical VS fractional**

	classical	fractional
spaces	$L^2 - L^2$	$L^1 - L^1$
differentiability	$0 \longrightarrow 1$	$0 \longrightarrow \sigma$

Step 2: Fractional De Giorgi's iteration

Theorem (Min., JEMS 11)

Let w be an L^1-function w satisfying the fractional Caccioppoli's inequality

$$[(|w|-k)_+]_{\sigma,1;B_{R/2}} \leq \frac{L}{R^\sigma}\int_{B_R}(|w|-k)_+\,dy + \frac{LR|\mu|(B_R)}{R^\sigma}$$

for some $\sigma > 0$ and every $k \geq 0$. Then it holds that

$$|w(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c\fint_{B(x,R)}|w|\,dy$$

for every Lebesgue point x of w

Part 4.5: Fully nonlinear interlude

A fully nonlinear Stein theorem

Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)

If u solves the uniformly elliptic fully nonlinear equation

$$F(D^2u) = f \in L(n,1)$$

then

$$Du \text{ is continuous}$$

A fully nonlinear Stein theorem

Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)

If u solves the uniformly elliptic fully nonlinear equation

$$F(D^2u) = f \in L(n,1)$$

then

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Previous results of Caffarelli (Ann. Math. 1989) claimed that

$$f \in L^{n+\varepsilon} \Longrightarrow Du \in C^{0,\alpha}$$

A fully nonlinear Stein theorem

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then

Du is continuous

Previous results of Caffarelli (Ann. Math. 1989) claimed that

$$f \in L^{n+\varepsilon} \Longrightarrow Du \in C^{0,\alpha}$$

Notice that

$$L^{n+\varepsilon} \subset L(n,1) \qquad \varepsilon > 0$$

Modified potentials

Key to the proof, a new potential estimate

Modified potentials

Key to the proof, a new potential estimate

$$\mathbf{I}_1^f(x,r) := \int_0^r \fint_{B_\varrho(x)} |f(y)|\, dy\, d\varrho$$

The relevant role of $L(n,1)$

Key to the proof, a new potential estimate

$$
\begin{aligned}
\mathbf{I}_1^f(x,r) &:= \int_0^r \int_{B_\varrho(x)} |f(y)|\,dy\,\frac{d\varrho}{\varrho} \\
&:= \int_0^r ⨍_{B_\varrho(x)} |f(y)|\,dy\,d\varrho \\
&\leq \int_0^r \left(⨍_{B_\varrho(x)} |f(y)|^p\,dy \right)^{1/p} d\varrho =: \mathbf{II}_1^f(x,r)\,.
\end{aligned}
$$

Modified potentials

Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)

If u solves the uniformly elliptic fully nonlinear equation

$$F(D^2u) = f \in L(n,1)$$

then

$$|Du(x)| \leq c\, \mathbf{I}_1^f(x,r) + c\left(\fint_{B_r(x)} |Du|^q\, dy\right)^{1/q}$$

for $p \geq n - \varepsilon$ and $q > n$

Modified potentials

Theorem (Daskalopoulos & Kuusi & Min., Comm. PDE 14)

If u solves the uniformly elliptic fully nonlinear equation

$$F(D^2 u) = f \in L(n,1)$$

then

$$|Du(x)| \leq c\,\mathbf{I}_1^f(x,r) + c\left(\fint_{B_r(x)} |Du|^q\,dy\right)^{1/q}$$

for $p \geq n-\varepsilon$ and $q > n$

$n - \varepsilon$ is the Escauriaza exponent, and is universal

Consequences

- It holds, with $n - \varepsilon < p$ that

$$\sup_{B_r(x)} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy < \infty \Longrightarrow Du \in \mathrm{BMO}$$

Consequences

- It holds, with $n - \varepsilon < p$ that

$$\sup_{B_r(x)} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy < \infty \Longrightarrow Du \in \mathrm{BMO}$$

- In particular

$$f \in \mathcal{M}^n \equiv L(n, \infty) \Longrightarrow Du \in \mathrm{BMO}$$

Consequences

- It holds, with $n - \varepsilon < p$ that

$$\sup_{B_r(x)} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy < \infty \Longrightarrow Du \in \mathrm{BMO}$$

- In particular

$$f \in \mathcal{M}^n \equiv L(n, \infty) \Longrightarrow Du \in \mathrm{BMO}$$

- Moreover

$$\lim_{r \to 0} r^{p-n} \int_{B_r(x_0)} |f|^p \, dy = 0 \Longrightarrow Du \in \mathrm{VMO}$$

Comparisons

- Borderline case of a theorem of Caffarelli, who proved

$$\sup_{B_r(x)} r^{n(1-\alpha)-n} \int_{B_r(x)} |f|^n \, dy < \infty \Longrightarrow Du \in C^{0,\alpha}$$

- In particular, a recent result of Teixeira (ARMA 14) who proved

$$f \in L^n \Longrightarrow u \text{ is Log-Lipschitz}$$

that is

$$|u(x) - u(y)| \leq -|x-y| \log \left(\frac{1}{|x-y|} \right)$$

follows as a corollary as

$$Du \in BMO \Longrightarrow u \text{ is Log-Lipschitz}$$

Part 4.6: Universal potential estimates

Leet us go back to

$$-\triangle u = \mu \qquad \text{in} \quad \mathbb{R}^n, \quad n \geq 3$$

Universal potential estimates

Leet us go back to

$$-\triangle u = \mu \qquad \text{in} \ \ \mathbb{R}^n, \quad n \geq 3$$

and observe the following elementary inequality:

$$\left||x-\xi|^{2-n} - |y-\xi|^{2-n}\right| \lesssim \left||x-\xi|^{2-n-\alpha} + |y-\xi|^{2-n-\alpha}\right| |x-y|^{\alpha}$$

Leet us go back to

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n, \quad n \geq 3$$

and observe the following elementary inequality:

$$\left||x-\xi|^{2-n} - |y-\xi|^{2-n}\right| \lesssim \left||x-\xi|^{2-n-\alpha} + |y-\xi|^{2-n-\alpha}\right| |x-y|^{\alpha}$$

that in turn implies

$$|u(x)-u(y)| \lesssim \left[I_{2-\alpha}(|\mu|)(x) + I_{2-\alpha}(|\mu|)(y)\right] |x-y|^{\alpha}$$

for $0 \leq \alpha \leq 1$

Calderón spaces of DeVore & Sharpley

- **The following definition is due to DeVore & Sharpley (Mem. AMS, 1982)**
- Let $\alpha \in (0,1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded open subset. A measurable function v, finite a.e. in Ω, belongs to the Calderon space $C_q^\alpha(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x - y|^\alpha$$

holds for almost every couple $(x, y) \in \Omega \times \Omega$

- In other words

$$m(x) \approx \partial^\alpha v(x)$$

First universal potential estimate

Theorem (Kuusi & Min. JFA 12)

The estimate

$$
\begin{aligned}
&|u(x)-u(y)| \\
&\lesssim \left[\mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R)+\mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(y,R)\right]|x-y|^{\alpha} \\
&\qquad +c\fint_{B_R}|u|\,d\tilde{x}\cdot\left(\frac{|x-y|}{R}\right)^{\alpha}
\end{aligned}
$$

holds uniformly in $\alpha \in [0,1]$, *whenever* $x, y \in B_{R/4}$

- The cases $\alpha = 0$ and $\alpha = 1$ give back the two known Wolff potential estimates as endpoint cases

The homogeneous case

- The estimate tells that

$$\text{“}\partial^{\alpha} u(x) \lesssim \mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R)\text{”}$$

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$$\text{“}\partial^{\alpha} u(x) \lesssim \mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R)\text{”}$$

- The case $\mu = 0$ reduces to the classical estimate

$$|u(x) - u(y)| \lesssim \fint_{B_R} |u| \, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha}$$

The homogeneous case

- The estimate tells that

$$\text{“}\partial^{\alpha} u(x) \lesssim \mathbf{W}^{\mu}_{1-\frac{\alpha(p-1)}{p},p}(x,R)\text{”}$$

- The case $\mu = 0$ reduces to the classical estimate

$$|u(x) - u(y)| \lesssim \fint_{B_R} |u| \, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha}$$

- In the case $p = 2$ we have

$$|u(x) - u(y)| \lesssim \left[\mathbf{I}^{|\mu|}_{2-\alpha}(x,R) + \mathbf{I}^{|\mu|}_{2-\alpha}(y,R)\right] |x-y|^{\alpha}$$
$$+c \fint_{B_R} |u| \, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha}$$

which in the classical case $-\triangle u = \mu$ can be derived directly from the standard representation formula via potentials

Second universal estimate

Theorem (Kuusi & Min., BMS 14)

The estimate

$$
\begin{aligned}
&|u(x)-u(y)| \\
&\leq \frac{c}{\alpha}\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,R)+\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(y,R)\right]^{1/(p-1)}|x-y|^{\alpha} \\
&\qquad +\frac{c}{\alpha}\fint_{B_R}(|u|+Rs)\,d\tilde{x}\cdot\left(\frac{|x-y|}{R}\right)^{\alpha}
\end{aligned}
$$

holds uniformly for $\alpha \in [0,1]$

Second universal estimate

Theorem (Kuusi & Min., BMS 14)

The estimate

$$
\begin{aligned}
&|u(x)-u(y)| \\
&\leq \frac{c}{\alpha}\left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,R)+\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(y,R)\right]^{1/(p-1)}|x-y|^{\alpha} \\
&\qquad +\frac{c}{\alpha}\fint_{B_R}(|u|+Rs)\,d\tilde{x}\cdot\left(\frac{|x-y|}{R}\right)^{\alpha}
\end{aligned}
$$

holds uniformly for $\alpha \in [0,1]$

Natural blow-up of the estimate as $\alpha \to 0$, with a linear behaviour

The fractional maximal operator

$$M_{\beta,R}(f)(x) := \sup_{0<r\leq R} r^{\beta} \frac{|f|(B(x,r))}{|B(x,r)|}$$

Maximal operators

The fractional maximal operator

$$M_{\beta,R}(f)(x) := \sup_{0<r\leq R} r^{\beta} \frac{|f|(B(x,r))}{|B(x,r)|}$$

The fractional sharp maximal operator

$$M^{\#}_{\beta,R}(f)(x) := \sup_{0<r\leq R} r^{-\beta} \fint_{B(x,r)} |f-(f)_{B(x,r)}|\, d\tilde{x}$$

Third universal estimate

Theorem (Kuusi & Min., BMS 14)

The estimate

$$M_{1-\alpha,R}(Du)(x) + M^{\#}_{\alpha,R}(u)(x)$$
$$\lesssim \left[\mathbf{I}^{|\mu|}_{p-\alpha(p-1)}(x,R)\right]^{1/(p-1)} + \frac{1}{R^{\alpha}} \fint_{B_R} |u| \, d\tilde{x}$$

holds uniformly for $\alpha \in [0,1]$

A lemma of Campanato-DeVore & Sharpley (revisited)

- Let $\alpha \in (0,1]$, then

$$|v(x) - v(y)| \leq \frac{c}{\alpha}\left[M^{\#}_{\alpha,R}(f)(x) + M^{\#}_{\alpha,R}(f)(y))\right]|x-y|^{\alpha}$$

  holds for all points x and y for which the right hand side is finite
- As a corollary, the second estimate follows from the third one

Part 4.7: Evolution

Parabolicities

- **The model case is here given by**

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu\,, \qquad \text{in } \Omega \times (-T,0) \subset \mathbb{R}^{n+1}$$

more in general we consider

$$u_t - \operatorname{div} a(Du) = \mu\,.$$

- The basic reference for existence and a priori estimates in the setting of SOLA is the work of Boccado, Dall'Aglio, Galloüet and Orsina, JFA, 1997

Degenerate equations - basic results

Theorem (Boccardo, Dall'Aglio, Gallouët & Orsina, JFA, 1997)

$$|Du| \in L^q(\Omega \times (-T, 0)), \quad 1 \leq q < p - 1 + \frac{1}{N-1}$$

$N = n + 2$ is the parabolic dimension

Consider the caloric Riesz potential

$$\mathbf{I}_1^{\mu}(x,t;r) := \int_0^r \frac{|\mu|(Q_\varrho(x,t))}{\varrho^{N-1}}\,\frac{d\varrho}{\varrho}\,, \quad N := n+2\,,$$

Consider the caloric Riesz potential

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then for solutions to

$$u_t - \triangle u = \mu$$

we have

$$|Du(x,t)| \leq c\mathbf{I}_1^{\mu}(x,t;r) + c\,⨍_{Q_r(x,t)} |Du|\,dz$$

The heat equation

Consider the caloric Riesz potential

$$\mathbf{I}_1^{\mu}(x,t;r) := \int_0^r \frac{|\mu|(Q_\varrho(x,t))}{\varrho^{N-1}} \frac{d\varrho}{\varrho}, \quad N := n+2,$$

then for solutions to

$$u_t - \triangle u = \mu$$

we have

$$|Du(x,t)| \leq c\mathbf{I}_1^{\mu}(x,t;r) + c \fint_{Q_r(x,t)} |Du|\, dz$$

we recall that

$$Q_r(x,t) := B_R(x) \times (t-r^2, t)$$

Inhomogeneous a priori estimates

Theorem (DiBenedetto & Friedman, Crelle J. 85)

$$\sup_{Q_{r/2}(x_0,t_0)} |Du| \leq c(n,p) \fint_{Q_r(x_0,t_0)} (|Du|+1)^{p-1}\, dz$$

The intrinsic geometry of DiBenedetto

- **The basic analysis is the following: consider intrinsic cylinders**

$$Q_\varrho^\lambda(x,t) = B_\varrho(x) \times (t - \lambda^{2-p}\varrho^2, t)$$

where it happens that

$$|Du| \approx \lambda \qquad \text{in } Q_\varrho^\lambda(x,t)$$

then the equation behaves as

$$u_t - \lambda^{p-2}\triangle u = 0$$

that is, scaling back in the same cylinder, as the heat equation

- **On intrinsic cylinders estimates "ellipticize"; in particular, they become homogeneous**

DiBenedetto's intrinsic estimate

- The homogenizing effect of intrinsic geometry

Theorem (DiBenedetto & Friedman, Crelle J. 85)

There exists a universal constant $c \geq 1$ *such that*

$$c\left(\fint_{Q_r^{\lambda}(x,t)} |Du|^{p-1}\, dz\right)^{1/(p-1)} \leq \lambda$$

then

$$|Du(x,t)| \leq \lambda$$

Intrinsic Riesz potentials

- **Define the intrinsic Riesz potential** such that

$$\mathbf{I}^{\mu}_{1,\lambda}(x,t;r) := \int_0^r \frac{|\mu|(Q^{\lambda}_{\varrho}(x,t))}{\varrho^{N-1}} \, \frac{d\varrho}{\varrho}$$

with

$$Q^{\lambda}_{\varrho}(x,t) = B_{\varrho}(x) \times (t - \lambda^{2-p}\varrho^2, t)$$

Intrinsic Riesz potentials

- **Define the intrinsic Riesz potential** such that

$$\mathbf{I}^{\mu}_{1,\lambda}(x,t;r) := \int_0^r \frac{|\mu|(Q^{\lambda}_{\varrho}(x,t))}{\varrho^{N-1}}\,\frac{d\varrho}{\varrho}$$

with

$$Q^{\lambda}_{\varrho}(x,t) = B_{\varrho}(x) \times (t - \lambda^{2-p}\varrho^2, t)$$

- **Note that**

$$\mathbf{I}^{\mu}_{1,\lambda}(x,t;r) = \mathbf{I}^{|\mu|}_{1}(x,t;r) \qquad \text{when } p = 2 \text{ or when } \lambda = 1$$

The parabolic Riesz gradient bound

Theorem (Kuusi & Min., JEMS, ARMA 14)

There exists a universal constant $c \geq 1$ such that

$$c\mathbf{I}^{\mu}_{1,\lambda}(x,t;r) + c\left(\fint_{Q_r^{\lambda}(x,t)} |Du|^{p-1}\,dz\right)^{1/p-1} \leq \lambda$$

then

$$|Du(x,t)| \leq \lambda$$

The parabolic Riesz gradient bound

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There exists a universal constant $c \geq 1$ *such that*

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then

$$|Du(x,t)| \leq \lambda$$

- **When $\mu \equiv 0$ this reduces to the sup estimate of DiBenedetto & Friedman (Crelles J. 84)**

Sharpness

- Consider the equation

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \delta,$$

where δ denotes the Dircac unit mass charging the origin

- The so called Barenblatt (fundamental solution) is

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-\frac{n}{\theta}}\left(c_b - \theta^{\frac{1}{1-p}}\left(\frac{p-2}{p}\right)\left(\frac{|x|}{t^{1/\theta}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} & t > 0 \\ 0 & t \leq 0\,. \end{cases}$$

for $\theta = n(p-2)+p$ and a suitable constant c_b such that

$$\int_{\mathbb{R}^n} \mathcal{B}_p(x,t)\,dx = 1 \qquad \forall\ t > 0$$

Sharpness

- A direct computation shows the following upper optimal upper bound
$$|D\mathcal{B}_p(x,t)| \leq ct^{-(n+1)/\theta}$$
- The intrinsic estimate above **exactly reproduces this upper bound**
- This decay estimate is indeed reproduced for all those solutions **that are initially compactly supported**

Intrinsic bounds imply explicit bounds

- The previous bound always implies a priori estimates on standard parabolic cylinders

Theorem (Kuusi & Min., JEMS, ARMA 14)

$$|Du(x,t)| \lesssim \mathbf{I}_1^{\mu}(x,t;r) + \fint_{Q_r(x,t)} (|Du|+1)^{p-1}\, dz$$

holds for every standard parabolic cylinder Q_r

Gradient continuity via potentials

Theorem (Kuusi & Min., ARMA 14)

Assume that

$$\lim_{r\to 0} \mathbf{I}_1^{\mu}(x,t;r) = 0 \qquad \textit{uniformly w.r.t. } (x,t)$$

then

$$Du \textit{ is continuous in } Q_T$$

Gradient continuity via potentials

Theorem (Kuusi & Min., ARMA 14)

Assume that

$$\lim_{r\to 0} \mathbf{I}_1^{\mu}(x,t;r) = 0 \qquad \textit{uniformly w.r.t. } (x,t)$$

then

$$Du \textit{ is continuous in } Q_T$$

Theorem (Kuusi & Min., ARMA 14)

Assume that

$$|\mu|(Q_\varrho) \lesssim \varrho^{N-1+\delta}$$

holds, then thtere exists α, depending on δ, such that

$$Du \in C^{0,\alpha} \qquad \textit{locally in } \ Q_T$$

A nonlinear parabolic Stein theorem

Theorem (Kuusi & Min., ARMA 14)

Assume that

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu \in L(N, 1)$$

that is

$$\int_0^\infty |\{|\mu| > \lambda\}|^{1/N}\, d\lambda < \infty$$

then Du is continuous in Q_T

DiBenedetto proved that Du is continuous when $\mu \in L^{N+\varepsilon}$


THANKS FOR THE ATTENTION