Lectures on abelianisation of G-bundles

by William Oxbury at GAeL 2001, notes by Joachim Kock

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1 G-bundles on a curve

Let G be a complex reductive group. For example, G could be $\mathbb{C}^*$, $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{PGL}(n)$ or any of the simple groups listed in Table 2.3 below. (Good references for these groups are Fulton-Harris [11], and Adams [1].) And let X be a smooth, complex, projective curve of genus $g \geq 2$.

Definition. A *principal G-bundle* on X is a morphism $E \to X$ with a free right action of G on E, such that E is locally trivial in the étale topology. $\square$

The need to use étale topology here arises from the fact that local trivialisation can require one to take square roots over open sets of the base, as in the Gram-Schmidt process—although for some groups such as $\mathrm{SL}(n)$ and $\mathrm{Sp}(n)$ it is enough to use the Zariski topology. An excellent reference for the general theory of G-bundles on curves is Sorger [32].

Given a linear representation $G \to \mathrm{GL}(V)$, we can associate to every G-bundle E a vector bundle $F := E(V) := E \times_G V$ defined as the quotient of $E \times V$ by the equivalence relation $(e, \mathbf{v}) \sim (eg, g^{-1}\mathbf{v})$. Another way of saying this is to take the class of E in $H^1(X, G)$, then its image in $H^1(X, \mathrm{GL}(V))$ can be interpreted as transition data for the vector bundle F.

1.1 Example. If $G = \mathrm{GL}(n)$ then this functor $E \mapsto E(\mathbb{C}^n) =: F$ is an equivalence of categories, between the category of G-bundles and the category of rank-n vector bundles. For $G = \mathrm{SL}(n)$, $E \mapsto E(\mathbb{C}^n)$ gives vector bundles with trivial determinant line bundle. Conversely, to construct a G-bundle E from the vector bundle F: take E to be the 'frame bundle' of fibre bases $\subset F^{\oplus n}$, that is complement of $\det^{-1}(0)$ via:

$$E \subset F^{\oplus n} \xrightarrow{\det} \mathbb{C} \times X \to \mathbb{C}.$$

For $G = \mathrm{SO}(n)$ we get vector bundles $F = E(\mathbb{C}^n)$ equipped with a quadratic form $q : S^2F \to \mathscr{O}_X$. Conversely, E is recovered from F, q as the frame bundle of oriented orthonormal bases in the fibres. $\square$

G-bundles on a curve X are parametrised by a projective moduli space, and we can summarise the basic facts in the following theorem due to Ramanathan [27], [28]. Balaji has recently given a simplification of the GIT construction (see [2]).

1.2 Theorem. (Ramanathan, Balaji.)

(i) *There exists a normal projective variety $M_X(G)$ parametrising equivalence classes of semi-stable G-bundles on X.*

(ii) *The connected components of $M_X(G)$ are labelled by $|\pi_1(G)|$, the fundamental group of G.*

(iii) *At points corresponding to generic stable G-bundles E, we have*

$$T_E M_X(G) \simeq H^1(X, \operatorname{ad} E).$$

(iv) $\dim M_X(G) = (g-1)\dim G + \dim Z(G)$. □

1.3 Remark. (i) The notion of semi-stability for G-bundles is a natural generalisation of slope stability for vector bundles: let $P \subset G$ be a maximal parabolic subgroup and consider the bundle $E/P \to X$. A section $\sigma : X \to E/P$ is called a *reduction of structure group* of E to $P \subset G$, and E is called *semistable* if for all P, σ,

$$\deg(\sigma^* T^{\mathrm{vert}}) \geq 0.$$

Note that in the case $G = \mathrm{SL}(n)$ or $\mathrm{GL}(n)$, if we view E as a vector bundle then E/P is a Grassmann bundle of subspaces of a given dimension in the fibres. Then a reduction $\sigma : X \to E/P$ is equivalent to specifying a subbundle $F \subset E$. Moreover, the pull-back along σ of the vertical tangent bundle is just $\operatorname{Hom}(F, E/F)$, and one can easily verify that the above inequality is equivalent to

$$\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E}{\operatorname{rk} E}.$$

(ii) E is always topologically trivial on $X \smallsetminus \{\mathrm{pt}\}$. So topologically, E is determined by a loop $S^1 \to G$. This explains (ii).

(iii) At a stable bundle $E \in M_X(G)$ the moduli space locally looks like $H^1(X, \operatorname{ad} E)/\Gamma$ where $\Gamma = \operatorname{Aut}(E)/Z(G)$ is a finite group. (Here $\operatorname{ad}(E) = E(\mathfrak{g})$ is the adjoint representation.) At points corresponding to bundles E such that $Z(G) \subsetneq \operatorname{Aut} E$, the moduli space $M_X(G)$ has finite quotient singularities. □

1.4 Example. If $G = \mathbb{C}^*$, then $M_X(G) = \operatorname{Pic} X$. In this case, $T_E M = H^1(\mathscr{O}_X)$. □

1.5 Example. $G = \mathrm{SL}(n)$: $M_X(\mathrm{SL}(n))$ is the irreducible moduli space of vector bundles F with $\det F = \mathscr{O}_X$. Every stable vector bundle is simple, so the group Γ is trivial. So the moduli space $M_X(\mathrm{SL}(n))$ is smooth at all stable bundles—though this is not typical. The tangent space is $T_F M = H^1(X, \operatorname{End}_0 F)$ where $\operatorname{End}_0 F$ is the vector bundle of trace free endomorphisms. □

2 The Verlinde formula

The Verlinde formula is just the Riemann-Roch formula for $M_X(G)$. We will assume that G is simple. Then there is a machine:

$$\left\{\begin{array}{l}\text{linear representations}\\ G\times V\to V\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{line bundles}\\ \Theta(V)\in \operatorname{Pic} M_X(G)\end{array}\right\}$$

This works as follows. For each $\xi \in \operatorname{Pic}^{g-1}(X)$ define

$$D_\xi = \{E \in M_X(G) \mid H^0(X, \xi \otimes E(V)) \neq 0\}.$$

It follows from results of Drézet-Narasimhan [8] that this is a Cartier divisor, and that its definition does not depend on the choice of ξ. It therefore defines a line bundle on the moduli which we denote by

$$\Theta(V) := \mathscr{O}_M(D_\xi),$$

called the *theta line bundle* of the representation. □

2.1 Remark. In particular, this gives a map $f : \operatorname{Pic}^{g-1}(X) \to |\Theta(V)|_M$ such that $f^*\mathscr{O}(1) = \mathscr{O}_{\operatorname{Pic}^{g-1}(X)}(n\Theta)$, where $n = \dim V$, and where Θ is the Riemann theta divisor. Thus we get a linear map

$$H^0(M, \Theta(V))^* \to H^0(J_X, n\Theta). \qquad \square$$

2.2 Theorem. (Laszlo-Sorger [18]; see also [32].) *The moduli stack of G-bundles on X has infinite cyclic Picard group $\mathbb{Z}\langle\mathscr{L}\rangle$. The (ample) generator $\mathscr{L}$ does not in general descend to $M_X(G)$, but there is an injection*

$$\operatorname{Pic} M_X(G) \hookrightarrow \mathbb{Z}\langle\mathscr{L}\rangle$$

under which $\Theta(V) \xrightarrow{\sim} \mathscr{L}^{d_V}$, where d_V is the Dynkin index *of the representation V.* □

The *Dynkin index* $d_V \in \mathbb{Z}$ is a number computed from the weights of the representation. Denoting the set of weights in the weight lattice by $\mathfrak{X}(V) \subset \Lambda$ (see 5.12), this number is computed by

$$d_V = \tfrac{1}{2} \sum_{\lambda \in \mathfrak{X}(V)} m_\lambda \langle \lambda \mid \theta^\vee \rangle^2$$

where m_λ is the multiplicity of the weight λ, and $\theta^\vee$ denotes the maximal coroot. The Dynkin index has a topological interpretation as the degree of the homotopy map induced by the representation,

$$\pi_3(G) = \mathbb{Z} \to \pi_3(\mathrm{SL}(V)) = \mathbb{Z}. \qquad \square$$

2.3 Table. Here is a list of the simple groups, the dimension of their smallest representations, and Dynkin indices. (The last column indicates whether this representation is minuscule or quasi-minuscule, and will be used later (see Definition 5.12 below).)

G	$\dim G$	dimension of smallest repn	Dynkin index	
$\mathrm{SL}(n)$	n^2-1	n	1	min.
$\mathrm{Spin}(2n) \xrightarrow{2:1} \mathrm{SO}(n)$	$n(2n-1)$	$2n$	2	min.
$\mathrm{Spin}(2n+1) \xrightarrow{2:1} \mathrm{SO}(2n+1)$	$n(2n+1)$	$2n+1$	2	q.-min.
$\mathrm{Sp}(n)$	$n(2n+1)$	$2n$	1	min.
$G_2 = \mathrm{Aut}(\mathbb{O})$	14	7	2	q.-min.
F_4	52	26	6	q.-min.
E_6	78	27	6	min.
E_7	133	56	12	min.
E_8	248	248	60	q.-min.

For every $i > 0$ we have $h^i(M_X(G), \mathscr{L}^{\otimes k}) = 0$, while for $i = 0$, the *Verlinde formula* has the form:

$$h^0(M_X(G), \mathscr{L}^{\otimes k}) = \sum \left(\prod_{\text{pos. roots}} \binom{\text{trigonometric expr.}}{\text{involving weights}}^{g-1} \right)^{g-1}$$

where the sum is over a finite set of weights (depending on k). There is no need here to be more precise than this, though the interested reader can consult Beauville [3] for the derivation of the formula using fusion rings, and Oxbury-Wilson [26] for computations and some properties when G is a classical group.

The formula was proved by Faltings [10], and a useful expository account of the whole story can be found in Sorger [31].

2.4 Example. For $G = \mathrm{SL}(2)$ the Verlinde formula was proved using more traditional algebro-geometric methods than for the general case by Thaddeus [33]. In this case it reads

$$h^0(M_X(G), \mathscr{L}^k) = \sum_{j=1}^{k+1} \left(\frac{k+2}{2\sin^2\left(j\pi/(k+2)\right)} \right)^{g-1}.$$

(As an exercise, the reader may care to check that when $g = 2$, for which $M \simeq \mathbb{P}^3$ (cf. Narasimhan-Ramanan [21] and Remark 4.2 below), this reduces to

$$h^0(M, \mathscr{L}^k) = \binom{k+3}{k}.$$

Despite appearances the formula always gives an integer (as it must!). Zagier [34] explains this as follows. Rewrite $h^0(M_X(\mathrm{SL}(2)), \mathscr{L}^k) = \left(\frac{k+2}{2}\right) V_g(k+2)$ where

$$V_g(m) := \sum_{j=1}^{m-1} \left(\frac{1}{\sin(j\pi/m)} \right)^{2g-2}.$$

These numbers are the coefficients of the following generating function.

$$1 - \sum_{g=2}^{\infty} V_g(m) \sin^{2g-2}(x) = \frac{m \tan(x)}{\tan(mx)},$$

and from this one can extract an expression in terms of Bernoulli numbers:

$$V_g(m) = \sum_{s=0}^{g-1} \left\{ \frac{(-1)^{s-1} 2^{2s} B_{2s}}{(2s)!} \operatorname*{Res}_{z=0} \left(\frac{z^{2s-1}}{\sin^{2g-2}(z)} \right) \right\} m^{2s}.$$

It now follows that $V_g(m)$ is a polynomial in m, and that it is in fact an integer. □

2.5 Table. Here is a list of Verlinde numbers for the simple, simply-connected groups at level $k = 1$, and we take as the focus of these lectures the problem of understanding some of these numbers.

G	$h^0(M, \mathscr{L})$
$\mathrm{SL}(n)$	n^g
$\mathrm{Spin}(2n)$	4^g
$\mathrm{Spin}(2n+1)$	$2^{g-1}(2^g + 1)$
$\mathrm{Sp}(n)$	$\sum_{j=1}^{n+1} \left(\frac{n+2}{2 \sin^2(j\pi/(n+2))} \right)^{g-1}$
G_2	$\left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}$
F_4	$\left(\frac{5+\sqrt{5}}{2}\right)^{g-1} + \left(\frac{5-\sqrt{5}}{2}\right)^{g-1}$
E_6	3^g
E_7	2^g
E_8	1

There are some obvious remarks. The number n^g for $\mathrm{SL}(n)$ (the reader may care to check that for $n = 2$ this agrees with Example 2.4) coincides with the number of level-n theta functions on the Jacobian of the curve, and this will be explained in Section 4. For the spin groups the numbers that appear are the numbers of theta characteristics and even theta characteristics, respectively, of the curve. This was explained in Obxury [23], though we shall return to the even spin case in the last lecture. The symplectic Verlinde number the reader will recognise from Example 2.4: this is an example of a reciprocity relation (as is the equality of the G_2 and F_4 numbers) which is a common feature of the Verlinde formula (see Oxbury-Wilson [26]).

But the most striking fact which appears in the table is the simplicity of the Verlinde numbers for E_6, E_7 and E_8, and our goal should be to interpret these spaces as spaces of theta functions on polarised abelian varieties.

3 A digression on abelian varieties

3.1 Principal polarisation. Let $A = \mathbb{C}^g/\Gamma$ be an abelian variety, where $\Gamma \subset \mathbb{C}^g$ is a lattice. Line bundles on A are described by Appel-Humbert data:

$$0 \to \operatorname{Pic}^0 A \to \operatorname{Pic} A \xrightarrow{c_1} \operatorname{NS}(A) \to 0$$

where $\operatorname{Pic}^0 A = \operatorname{Hom}(\Gamma, S^1)$ and $\operatorname{NS}(A)$ is the Néron-Severi group whose elements are represented by Hermitian forms H on $\mathbb{C}^g$ with $\operatorname{im} H$ integral on Γ.

Given $L \in \operatorname{Pic} A$, define

$$\begin{array}{rcl} \varphi_L : A & \longrightarrow & \widehat{A} := \operatorname{Pic}^0 A \\ x & \longmapsto & t_x^* L \otimes L^{-1}, \end{array}$$

and let $K(L) = \ker \varphi_L$. This is a finite subgroup of A and is uniquely expressible as

$$K(L) \simeq (\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_g})^2, \qquad d_1 \mid d_2 \mid \cdots \mid d_g.$$

Equivalently, $c_1(L)$ is represented by the skew form

$$\left(\begin{array}{ccc|ccc} & & & d_1 & & \\ & 0 & & & \ddots & \\ & & & & & d_g \\ \hline -d_1 & & & & & \\ & \ddots & & & 0 & \\ & & -d_g & & & \end{array}\right)$$

on Γ. The sequence $(d_1, \dots, d_g)$ is called the *type of the polarisation* $c_1(L) \in \operatorname{NS}(A)$. By Riemann-Roch,

$$h^0(A, L) = d_1 \cdots d_g.$$

The line bundle L gives a *principal polarisation* if $K(L) = 0$, or equivalently, the polarisation type is $(1, \dots, 1)$, or $h^0(A, L) = 1$.

If L is not a principal polarisation then φ_L does not have an inverse in $\operatorname{End}_{\mathbb{Z}} A$. However, it is always invertible in $\operatorname{End}_{\mathbb{Q}} A = (\operatorname{End}_{\mathbb{Z}} A) \otimes_{\mathbb{Z}} \mathbb{Q}$; indeed there exists $\psi_L : \widehat{A} \to A$ such that $\varphi_L \psi_L = \psi_L \varphi_L = e \operatorname{id}$, where $e = e(L)$ is the exponent of the group $K(L)$ (the least common multiple of the orders of its elements). We therefore write $\varphi_L^{-1} = \frac{1}{e}\psi_L \in \operatorname{End}_{\mathbb{Q}} A$. $\square$

3.2 Norm map. Now let $i : B \hookrightarrow A$ be an abelian subvariety, and define its *norm map* $\operatorname{Nm}_B : A \to B$ by the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_L"] \arrow[d, "\operatorname{Nm}_B"'] & \widehat{A} \arrow[d, "\widehat{i}"] \\ B & \widehat{B}. \arrow[l, "\psi_{i^*L}"] \end{tikzcd}$$

(Here $\widehat{i} = i^*$ is restriction of line bundles.) This satisfies

$$(\operatorname{Nm}_B)^2 = e(i^*L) \operatorname{Nm}_B,$$

where $e(i^*L)$ is the exponent of the induced polarisation. $\square$

3.3 Complementary subvarieties. We define the *complementary abelian subvariety* of $B \subset A$ to be

$$P := \mathrm{im}\,\big(e(i^*L)\,\mathrm{id}_A - \mathrm{Nm}_B\,\big).$$

Now assume L is a principal polarisation on A. Then the induced polarisation on P has the same exponent $e = e(i^*L)$ and

$$\mathrm{Nm}_B = \mathrm{Nm}_P = e \cdot \mathrm{id} \in \mathrm{End}_{\mathbb{Z}}\, A.$$

Moreover (see Lange-Birkenhake [17], p. 366), P can be expressed equivalently as

$$P = (\ker \mathrm{Nm}_B)_0 = \ker \widehat{i} = \widehat{(A/B)}. \qquad \square$$

3.4 Corollary. *If (A, L) is a principally polarised abelian variety, and $B, P \subset A$ are complementary subvarieties, then*

$$K(i^*L) = B \cap P.$$

Proof. We have $K(i^*L) = \ker \varphi_{i^*L}$, where we can identify φ_{i^*L} with $\widehat{i} \circ i : B \to A \xrightarrow{\sim} \widehat{A} \to \widehat{B}$. So $K(i^*L) = \ker(\widehat{i} \circ i) = i^{-1}(\ker \widehat{i}) = i^{-1}(P) = B \cap P$. $\square$

It follows immediately if A is principally polarised, that $B \subset A$ and $P \subset A$ have induced polarisation types

$$(d_1, \dots, d_r) \qquad (1, \dots, 1, d_1, \dots, d_r).$$

3.5 Remark. The spaces of sections $H^0(B, i_B^*L)$ and $H^0(P, i_P^*L)$ not only have the same dimension, but are canonically dual, via the multiplication map

$$\begin{array}{ccc} B \times P & \xrightarrow{m} & A \\ H^0(B, i_B^*L) \otimes H^0(P, i_P^*L) & \xleftarrow{m^*} & H^0(A, L) = \mathbb{C}. \end{array} \qquad \square$$

We now restrict to the case where $(A, L) = (J_Y, \Theta_Y)$, the Jacobian of a curve Y, and we consider a morphism of curves $\pi : Y \to X$.

3.6 Remark. The map on Jacobians $\pi^* : J_X \to J_Y$ fails to be injective if and only if π factorises as $\pi : Y \to Y' \to X$, where the second map is cyclic étale of degree $d \geq 2$. Such maps are in 1–1 correspondence with (primitive) d-torsion points $\eta \in J_X[d]$, and then (if $Y \to Y'$ does not factorise further) $\ker \pi^* = \langle \eta \rangle \subset J_X$. (See Lange-Birkenhake [17], p. 337.) $\square$

3.7 Prym variety and trace correspondence. We let $B = \pi^* J_X \subset J_Y$. The complementary subvariety $P \subset J_Y$ is in this case called the *Prym variety* of the cover and denoted $P = \mathrm{Prym}(Y/X)$.

This can be described in terms of the *trace correspondence*

$$\begin{array}{rcl} T : Y & \vdash & Y \\ y & \longmapsto & \pi^{-1}(\pi y). \end{array}$$

If $\pi : Y \to X$ has degree n then T satisfies

$$T^2 = nT. \tag{1}$$

If we view T as determining an endomorphism $T \in \mathrm{End}_{\mathbb{Z}} J_Y$, then clearly

$$\mathrm{im}\, T = \pi^* J_X.$$

It follows from this and the *norm-endomorphism criterion* (Lange-Birkenhake [17], p. 126) that $T = \mathrm{Nm}_{\pi^* J_X}$. Thus we have $P = (\ker T)_0 = \mathrm{im}(T - n)$. Its induced polarisation is given by

$$K(i^*\Theta_Y) = P \cap \pi^* J_X = \pi^* J_X[n].$$

If $\pi^* : J_X \to J_Y$ is injective then it follows that P has induced polarisation of type $(1, \dots, 1, n, \dots, n)$ and that

$$H^0(P, i^*\Theta_Y) = H^0(J_X, n\Theta_X)^*. \tag{2}$$

4 The case $G = \mathrm{SL}(n)$

Let us now return to the discussion of Section 2. Recall that $h^0(M_X(\mathrm{SL}(n)), \mathscr{L}) = n^g$, equal to the dimension of $H^0(J_X, n\Theta)$, the space of level-n theta functions on the Jacobian. Here $\mathscr{L} = \Theta(\mathbb{C}^n)$, the theta line bundle of the standard representation, since the Dynkin index of this representation is $d_V = 1$.

4.1 Theorem. (Beauville-Narasimhan-Ramanan [4].) *The linear map*

$$H^0(M_X(\mathrm{SL}(n)), \mathscr{L})^* \to H^0(J_X, n\Theta)$$

introduced in Remark 2.1 is an isomorphism. □

Recall the definition of the map. Inside $M_X(\mathrm{SL}(n)) \times \mathrm{Pic}^{g-1} X$, let $\mathscr{D}$ denote the set of pairs (E, ξ) such that $h^0(X, \xi \otimes E) > 0$. (Then the divisor D_ξ is the intersection of $\mathscr{D}$ with the fibre over $\xi \in \mathrm{Pic}^{g-1}$.) Now $\mathscr{D}$ is considered an element in $H^0(M \times \mathrm{Pic}^{g-1}, \Theta(V) \boxtimes \mathscr{O}(n\Theta)) = H^0(M, \Theta(V)) \otimes H^0(\mathrm{Pic}^{g-1}, \mathscr{O}(n\Theta))$. This defines the linear map $H^0(M, \Theta(V))^* \to H^0(\mathrm{Pic}^{g-1}, \mathscr{O}(n\Theta))$.

4.2 Remark. Geometrically this theorem says that the rational map to projective space determined by the complete linear series $|\mathscr{L}|$ on $M_X(\mathrm{SL}(n))$ can be identified with

$$M_X(\mathrm{SL}(n)) \dashrightarrow |n\Theta| \cong \mathbb{P}^{2^g - 1},$$

taking a vector bundle E to the divisor D_E supported on $\xi \in \mathrm{Pic}^{g-1}$ such that $h^0(X, \xi \otimes E) > 0$.

For $n = 2$ this map is quite well understood, and is an embedding when X is non-hyperelliptic (or when $g = 2$). When $g = 2$, in fact it is an isomorphism to $\mathbb{P}^3$; when $g = 3$ it is an isomorphism to the *Coble quartic* in $\mathbb{P}^7$ (the unique Heisenberg invariant quartic singular along the Kummer variety); and when $g = 4$ its image lies in the singular locus of another unique Heisenberg invariant quartic in $\mathbb{P}^{15}$. (See Narasimhan-Ramanan [22] and [21], and Oxbury-Pauly [24].) □

The proof of the theorem is by expressing vector bundles as direct images of line bundles. Let $\pi : Y \to X$ be some degree-n cover. Then taking direct images of line bundles defines a rational map $\pi_* : \operatorname{Pic} Y \dashrightarrow M_X(\mathrm{GL}(n))$. To determine the subvariety mapping to $M_X(\mathrm{SL}(n)) \subset M_X(\mathrm{GL}(n))$ we require that the determinant of the direct image be trivial. If $\mathrm{Nm} : \operatorname{Pic} Y \longrightarrow \operatorname{Pic} X$ is the norm map induced by $p \mapsto \pi(p)$ on points, then

$$\det \pi_* L = \mathrm{Nm}\, L \otimes \det \pi_* \mathscr{O}_Y. \tag{3}$$

(This is an exercise for the reader: use induction from $L = \mathscr{O}_Y$, adding and subtracting points.) Now we have $\mathrm{Nm} : J_Y \to J_X$ and $\pi^* : J_X \to J_Y$. If $\pi : Y \to X$ is sufficiently ramified then π^* is an injection (cf. 3.6), so we think of J_X as an abelian subvariety of J_Y. Then according to 3.7, $\mathrm{Nm} = \mathrm{Nm}_{\pi^* J_X}$ is induced by the trace correspondence on Y, which we view as an endomorphism

$$\begin{array}{rcl} T : J_Y & \longrightarrow & J_Y \\ p & \longmapsto & \pi^{-1}(\pi(p)). \end{array}$$

Hence under π_*,

$$\begin{array}{ccc} \operatorname{Pic} Y & \dashrightarrow & M_X(\mathrm{GL}(n)) \\ \cup & & \cup \\ P & \dashrightarrow & M_X(\mathrm{SL}(n)) \end{array}$$

where P is the Prym variety $T^{-1}(\xi)$ where $\xi = \det^{-1} \pi_* \mathscr{O}_Y \in \operatorname{Pic} X$ is determined by (3). This is a translate in $\operatorname{Pic} Y$ of the identity component $(\ker T)_0$.

Let μ denote the map $\pi_* : J_Y \dashrightarrow M_X(\mathrm{GL}(n))$. For fixed $\xi \in \operatorname{Pic}^{g-1} X$, and for $L \in P \subset J_Y$, consider the $\mathrm{SL}(n)$-bundle $\xi \otimes \pi_* L \in M_X(\mathrm{SL}(n))$. We have

$$0 \neq H^0(X, \xi \otimes \pi_* L) = H^0(Y, \pi^* \xi \otimes L).$$

So under μ we have $\mu^* \mathscr{L} = \mathscr{O}_P(\Theta_Y)$. Hence

$$H^0(M_X(\mathrm{SL}(n)), \mathscr{L}) \xrightarrow{\mu^*} H^0(P, \Theta_Y) \xrightarrow{\sim} H^0(J_X, n\Theta_X)^*,$$

where the last isomorphism comes from (2) on page 8. The remaining problem is to arrange for μ^* to be an isomorphism too.

4.3 Key point. This is the observation of Hitchin [12] (and further elaborated in [13], [14]) that there exists a choice of a n-sheeted covering $\pi : Y \to X$ with the property that

$$\dim P = \dim M_X(\mathrm{SL}(n))$$

and that $\mu : P \dashrightarrow M_X(\mathrm{SL}(n))$ is a finite dominant rational map. This is what is meant by *abelianisation* of the moduli space. □

In the present situation, using the Hitchin curve guarantees that μ^* is an isomorphism, and the theorem is proved.

4.4 Résumé. In general, the question of abelianisation is to find an abelian variety P with a dominant rational map to $M_X(G)$. P should be an abelian subvariety of J_Y (a sort of Prym variety) for a finite cover $\pi : Y \to X$. The map $P \dashrightarrow M_X(G)$ should be the restriction of π_*. We want to choose $Y \to X$ in such a way that we get a finite dominant map. The theory that results was worked out in full generality by Faltings [9] and Donagi [6] and [7]. In these lectures I will follow the approach of Donagi.

5 Construction of covers with Galois group $W(G)$

5.1 Invariant polynomials. Let $\mathfrak{g}$ be the Lie algebra of G and consider the action on polynomials $G \times \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{g}]$ induced by the adjoint representation on $\mathfrak{g}$. We shall consider the (finitely generated) subring of invariant polynomials $\mathbb{C}[\mathfrak{g}]^G$ under this action. (See for example Kobayashi-Nomizu [16]).

5.2 Example. If $G = \mathrm{GL}(n)$ then $\mathfrak{g} = \mathrm{End}(\mathbb{C}^n)$ and an element $g \in G$ acts on a matrix $A \in \mathrm{End}(\mathbb{C}^n)$ by $A \mapsto g^{-1}Ag$. Then $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\operatorname{tr} A, \operatorname{tr} A^2, \dots, \det A]$, where $\operatorname{tr} A \dots$ are the coefficients of the characteristic polynomial $\chi_A(t)$ of A. It can also be described as $\mathbb{C}[\sigma_1, \dots, \sigma_n]$, where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric functions in the eigenvalues of A. □

More generally, let $\mathfrak{t} \subset \mathfrak{g}$ be a maximal abelian subalgebra, coming from a maximal torus $T \subset G$. Recall that the Weyl group $W = N(T)/T$ acts on $\mathfrak{t}$ as a finite reflection group.

5.3 Example. If G is $\mathrm{SL}(n)$ or $\mathrm{GL}(n)$ then T is the corresponding group of diagonal matrices, and W is just the symmetric group S_n which acts by permuting the eigenvalues. □

5.4 Theorem. (Chevalley, Shepherd-Todd. See Bourbaki [5].)

(i) *The restrictioning map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{t}]^W$ is an isomorphism.*

(ii) *We have $\mathbb{C}[\mathfrak{t}]^W \simeq \mathbb{C}[\sigma_1, \dots, \sigma_r]$, where r is the dimension of $\mathfrak{t}$, and $\sigma_1, \dots, \sigma_r$ are algebraically independent polynomials.*

(iii) *The degrees $d_1, \dots, d_r$ of $\sigma_1, \dots, \sigma_r$ are independent of choice of generating invariants $\sigma_1, \dots, \sigma_r$.* □

5.5 Example. For $G = \mathrm{SO}(2n)$, the Lie algebra is $\mathfrak{so}(2n)$ consisting of skew symmetric matrices. The characteristic polynomial is

$$\chi_A(t) = t^{2n} - (\operatorname{tr} \textstyle\bigwedge^2 A)t^{2n-2} + \cdots + (\operatorname{Pf} A)^2,$$

where Pf is the Pfaffian. So letting $\sigma_i = \operatorname{tr} \bigwedge^{2n-i} A$ for $i = 2, 4, \dots, 2n-2$, we have

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\sigma_2, \sigma_4, \dots, \sigma_{2n-2}, \mathrm{Pf}]$$

with degrees $d_i = \deg \sigma_i = i$, and $\deg \mathrm{Pf} = n$. □

5.6 Remark. The invariant degrees carry important topological information about the group G. Their product equals the order of the Weyl group while their sum gives the number of reflections in W—that is, the number of positive roots R of G:

$$\prod d_i = |W| , \qquad \sum (d_i - 1) = |R| .$$

In particular this implies that $\sum(2d_i - 1) = \dim G$. The Betti numbers of G are also computable from the d_i. $\square$

5.7 Example. The exceptional group G_2 is the automorphism group of the 8-dimensional complex algebra $\mathbb{O}$ of octonions. It has a natural 7-dimensional irreducible representation $\operatorname{im}\mathbb{O}$ and a subgroup $\mathrm{SL}(3) \subset G_2$ which stabilises an isotropic vector in $\operatorname{im}\mathbb{O}$. A maximal torus $T \subset \mathrm{SL}(3)$ is also maximal in G_2, so the two groups share the same weight lattice $\Lambda \subset \mathfrak{t}^*$. Therefore

$$\mathbb{C}[\mathfrak{t}]^{W(G_2)} \subset \mathbb{C}[\mathfrak{t}]^{W(\mathrm{SL}(3))} = \mathbb{C}[\sigma_2, \sigma_3],$$

where $W(\mathrm{SL}(3) = S_3$ and σ_2, σ_3 are elementary symmetric polynomials in three variables (with $\operatorname{tr}\sigma_1 = 0$).

$W(G_2)$ is the dihedral group D_6, and has invariant degrees $2, 6$ (for example by Molien's theorem). We can identify the invariant polynomials by considering the characteristic polynomial $\chi_A(t)$ of $A \in \mathfrak{t}$ in the 7-dimensional representation $\operatorname{im}\mathbb{O}$. Under SL(3) this decomposes as $\mathbb{C} \oplus V \oplus V^*$, where $V = \mathbb{C}^3$ is the standard representation of SL(3). So

$$\begin{aligned}
\chi_{A,\operatorname{im}\mathbb{O}}(t) &= t\,\chi_{A,V}(t)\chi_{A,V^*}(t) \\
&= t(t^3 + \sigma_2 t - \sigma_3)(t^3 + \sigma_2 t + \sigma_3) \\
&= t(t^6 + 2\sigma_2 t^4 + \sigma_2^2 t^2 - \sigma_3^2).
\end{aligned}$$

This shows that $\mathbb{C}[\mathfrak{t}]^{W(G_2)} = \mathbb{C}[\sigma_2, \sigma_3^2]$. $\square$

5.8 Table. Here is a list of the simple groups, their Weyl groups, and invariant degrees.

G	$W(G)$	invariant degrees
$\mathrm{SL}(n)$	S_n	$2, 3, \ldots, n$
$\mathrm{Spin}(2n)$	$\mathbb{Z}^{n-1} \rtimes S_n$	$2, 4, \ldots, 2n-2,\ n$
$\mathrm{Spin}(2n+1)$	$\mathbb{Z}^n \rtimes S_n$	$2, 4, \ldots, 2n$
$\mathrm{Sp}(n)$	$\mathbb{Z}^n \rtimes S_n$	$2, 4, \ldots, 2n$
G_2	D_6 dihedral group	$2, 6$
F_4	extension of S_3 by $W(\mathrm{Spin}(8))$	$2, 6, 8, 12$
E_6	symmetry group of the 27 lines on a cubic surface, order $72 \cdot 6!$	$2, 5, 6, 8, 9, 12$
E_7	lines on quartic double plane	$2, 6, 8, 10, 12, 14, 18$
E_8	lines on a degree 1 del Pezzo	$2, 8, 12, 14, 18, 20, 24, 30$

5.9 Cameral covers. We have $\mathbb{C}[\mathfrak{t}] \supset \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\sigma_1, \ldots, \sigma_r]$. Taking Spec we get an affine quotient map

$$(\sigma_1, \ldots, \sigma_r) : \mathbb{A}^r \twoheadrightarrow \mathfrak{t}/W \simeq \mathbb{A}^r.$$

(This is the geometric meaning of Theorem 5.4.) For any line bundle $K \in \operatorname{Pic} X$, form the map

$$K \otimes \mathfrak{t} \xrightarrow{\sigma_1, \ldots, \sigma_r} \bigoplus_{i=1}^{r} K^{d_i}.$$

Given a section $s \in H^0(\bigoplus K^{d_i}) = \bigoplus H^0(K^{d_i})$ we construct the fibre product

$$\begin{tikzcd} Z_s \arrow[r] \arrow[d] & K \otimes \mathfrak{t} \arrow[d, "{(\sigma_1, \ldots, \sigma_r)}"] \\ X \arrow[r, "s"'] & \bigoplus K^{d_i} \end{tikzcd}$$

Here the right-hand side is the quotient map by the natural action of W covering the trivial action on X. The Weyl group W therefore acts on Z_s with quotient X. Donagi calls this Galois cover a *cameral cover* of X.

5.10 Remark. For a given line bundle $K \in \operatorname{Pic} X$, the cameral covers constructed above form a family over the vector space $\bigoplus H^0(K^{d_i})$. When $K = \omega_X$ is the canonical line bundle, Hitchin observed the beautiful coincidence of dimensions:

$$\dim \bigoplus H^0(K^{d_i}) = \dim M_X(G).$$

This follows from Riemann-Roch and the formula $\sum (2d_i - 1) = \dim G$. Now consider the cotangent bundle $T^*M_X(G) \to M_X(G)$. The fibre of this bundle is $H^1(\operatorname{ad} E)^*$, which by Serre duality is isomorphic to $H^0(K \otimes \operatorname{ad} E)$. (Here $\operatorname{ad} E$ is the vector bundle with fibre $\mathfrak{g}$.) Hitchin showed that the components of the map

$$T^*M_X(G) \xrightarrow{(\sigma_1, \ldots, \sigma_r)} \bigoplus H^0(K^{d_i})$$

are Poisson commuting functions on the (holomorphic) symplectic manifold $T^*M_X(G)$ (away from the singular points), and that we thus obtain an algebraically completely integrable Hamiltonian system. By Liouville's theorem in mechanics it then follows that the fibres are all complex tori with dimension equal to that of $M_X(G)$. This is the origin of the idea of abelianisation. □

5.11 Ramification. Let us describe the ramification points of the cameral cover. Recall that W is generated by reflections; we denote the set of these by $R \subset W$. (This is the set of positive roots.) We have seen in Remark 5.6 that

$$|R| = \sum_{i=1}^{r} (d_i - 1).$$

The fixed-point set of a reflection $\alpha \in R$ in the space $\mathfrak{t} \otimes K$ is a divisor $D_\alpha \in |\pi^* K|$. We will assume that all ramification points of $\pi : Z \to X$ are simple. Equivalently the curve $Z \subset \mathfrak{t} \otimes K$ and the divisor $\sum_{\alpha \in R} D_\alpha$ have intersection multiplicity one at all intersection points.

Let $B \subset X$ be the branch locus. Over a point $x \in B$ there are $|W|/2$ ramification points on which W acts transitively with stabiliser $\langle \alpha \rangle$ for some $\alpha \in R$. Thus $(|W|/2) \times \deg B = |D_\alpha| \times |R| = |R||W| \deg K$, and hence

$$\deg B = 2|R| \deg K. \tag{4}$$

By Riemann-Hurwitz this gives the genus of the cover:

$$g(Z) = 1 + |W| \left(g - 1 + \frac{1}{2}|R| \deg K \right), \tag{5}$$

where $g = g(X)$. □

5.12 Minuscule weights. Let us recall a little representation theory. Given a representation $G \times V \to V$ we can restrict to a maximal torus $T \subset G$ to get a representation $T \times V \to V$, under which V decomposes into eigenspaces $V = \bigoplus_{\lambda \in \mathfrak{X}(V)} L_\lambda$, where the eigenvalues are $e^{2\pi i \lambda}$ for linear forms $\lambda : \mathfrak{t} \to \mathbb{R}$ integral on the kernel of the exponential map $\mathfrak{t} \to T$. The set $\Lambda \subset \mathfrak{t}^*$ of such linear forms is called the *weight lattice* of the pair G, T. Now the Weyl group $W = N(T)/T$ acts on Λ preserving the finite set of weights $\mathfrak{X}(V)$ of the representation V. The set $\mathfrak{X}(V) \subset \Lambda$ is therefore some union of W-orbits, and this set determines the representation V up to isomorphism. In some cases it consists of a single W-orbit $W\lambda$, and then V is called a *minuscule* representation. The representation is called *quasi-minuscule* if zero is weight as well.

Table 2.3 lists some minuscule representations for the simple groups, and in fact the only other examples are the spinor representations of the spin groups. □

5.13 Example. The adjoint representation $\mathrm{End}_0(\mathbb{C}^3)$ of $\mathrm{SL}(3)$ is an 8-dimensional quasi-minuscule representation. The root system is a hexagon, with a double weight at the origin (since $\mathfrak{t}$ is a 2-dimensional invariant subspace under T). □

5.14 Strategy. Abelianisation of $M_X(G)$ makes use of subcovers $Y \to X$ where $Y = Z/H$ for some subgroup $H < W$. If we take $H = \mathrm{Stab}(\lambda)$ where λ is a weight of a minuscule (or quasi-minuscule) representation (of dimension n say) then we can hope for a diagram of the form

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, dashed] & J_Y \arrow[d, dashed, "\pi_*"] \\ M_X(G) \arrow[r, "\mathrm{repr}"'] & M_X(\mathrm{GL}(n)) \end{tikzcd}$$

where $P \subset J_Y$ is some abelian subvariety which has to be determined.

5.15 Example. Let $G = \mathrm{SL}(n)$. The weight lattice

$$\Lambda \subset \mathfrak{t}^* = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$$

is spanned over $\mathbb{Z}$ by the orthogonal projections of the standard basis in $\mathbb{R}^n$, that is:

$$\Lambda = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$$

where $\varepsilon_j = \frac{1}{n}(-1, \ldots, -1, n-1, -1, \ldots, -1)$. The vectors $\varepsilon_1, \ldots, \varepsilon_n$ are the weights of the standard representation $V = \mathbb{C}^n$, and the Weyl group S_n acts on Λ by permutation of coordinates. The stabiliser $\mathrm{Stab}(\lambda)$ of $\lambda = \varepsilon_1$ is the subgroup $S_{n-1} < S_n$. Letting $Y := Z/\mathrm{Stab}(\lambda)$ we get a configuration

$$\begin{tikzcd} Z \arrow[dd, "W = S_n"'] \arrow[dr, "S_{n-1}"] & \\ & Y \arrow[dl, "n:1"] \\ X & \end{tikzcd}$$

A point of Z is an ordering of the corresponding fibre $Y \to X$; while Y is (choosing $K = \omega_X$) the curve used in the proof of Theorem 4.1. (See Example 6.12 below.) $\square$

5.16 Example. Let $G = E_6$. This group can be described as the automorphism group of the octonionic projective plane $\mathbb{OP}^2$. (See Lazarsfeld-van de Ven [19].) This can be described in its 'quadratic Veronese' embedding as the variety of rank-1 Hermitian 3×3 matrices over the octonion algebra $\mathbb{O}$, and is a 16-dimensional projective variety in $\mathbb{P}^{26}_{\mathbb{C}}$. It is the singular locus of its secant variety, which is a cubic hypersurface defined by the vanishing of the 3×3 determinant (the rank-2 matrices). After a change of coordinates this cubic can be written as

$$\mathrm{tr}\, abc - \det a - \det b - \det c, \qquad (a, b, c) \in (\mathfrak{gl}_3)^{\oplus 3} \cong \mathbb{C}^{27}.$$

The group E_6 is then the subgroup of $\mathrm{SL}(27)$ which preserves this cubic form. (This is the description used by Adams [1].) It has maximal torus of dimension 6, and weight lattice $\Lambda \subset \mathfrak{t}^*$ isomorphic to the primitive cohomology $K_S^{\perp} \subset H^2(S, \mathbb{Z})$ of a smooth cubic surface $S \subset \mathbb{P}^3$. (See Manin [20], Theorem 23.9.) Let $\lambda \in \Lambda$ be a weight of the (minuscule) 27-dimensional representation $V = \mathbb{C}^{27}$. Then we have a configuration

$$\begin{tikzcd} Z \arrow[dd, "W(E_6)"'] \arrow[dr] & \\ & Y = Z/\mathrm{Stab}\,\lambda \arrow[dl, "27:1"] \\ X & \end{tikzcd}$$

The fibre of $Y \to X$ can be identified with the set of the 27 lines on S. The Weyl group has order $72 \cdot 6!$ and acts on the 27 lines by permuting ordered 'Schläfli double-sixes', consisting of sets of lines $\{a_1, \ldots, a_6, b_1, \ldots, b_6\}$ which contract to 6 points in $\mathbb{P}^2$ and 6 residual conics respectively. (See Hunt [15].) A point of the cameral curve Z is therefore a choice of ordered double-six in the corresponding fibre of Y. $\square$

6 Decomposition of J_Z

We are given data consisting of a line bundle $K \in \operatorname{Pic} X$ and a section $s \in \bigoplus H^0(K^{d_i})$. From this we have constructed configurations

$$\begin{tikzcd} Z \arrow[dd, "W"'] \arrow[dr, "\pi"] & \\ & Y = Z/\operatorname{Stab}\lambda \arrow[dl] \\ X & \end{tikzcd}$$

We are looking for a natural abelian subvariety $P \subset J_Y \xrightarrow{\pi^*} J_Z$. Now W acts on $J_Z = H^1(\mathscr{O}_Z)/H^1(Z, \mathbb{Z})$ and we are going to decompose it under this action and find out which components lie in J_Y.

6.1 Example. For $G = \mathrm{SL}(2)$, the Weyl group is $W = \{1, \sigma\}$, where $\sigma^2 = 1$. So in this case our map $Z \to X$ is just a double cover and σ interchanges the sheets. We get

$$J_Z = J_X + P = \operatorname{im}(1+\sigma) + \operatorname{im}(1-\sigma)$$

where $J_X = \operatorname{im}(1+\sigma)$ is the $(+1)$-eigenspace (invariant line bundles) and $P = \operatorname{im}(1-\sigma)$ is the (-1)-eigenspace (anti-invariant line bundles). (We use additive notation since we are dealing with abelian varieties, so $1+\sigma$ on a bundle L denotes $L \otimes \sigma^* L$.) Note that

$$\mathbb{Q}[W] = \mathbb{Q}\{1+\sigma\} \oplus \mathbb{Q}\{1-\sigma\},$$

and that this decomposition of the regular representation $\mathbb{Q}[W]$ of W is responsible for the decomposition of abelian varieties. (Note that we need rational coefficients here for $1+\sigma$ and $1-\sigma$ to be the generators. For example, $\sigma = \frac{1}{2}(1+\sigma) + \frac{1}{2}(1-\sigma)$.) $\square$

More generally for any finite group W let $\widehat{W}$ denote its group of characters, and let $\mathbb{S}_i$ be the irreducible representation of W corresponding to the character $i \in \widehat{W}$. Then

$$\mathbb{C}[W] = \sum_{i \in \widehat{W}} (\dim \mathbb{S}_i)\, \mathbb{S}_i = \bigoplus_{i \in \widehat{W}} \operatorname{End} \mathbb{S}_i.$$

6.2 Fact. For Weyl groups, all representations $\mathbb{S}_i$ are defined over $\mathbb{Z}$, so we can work with a $\mathbb{Z}[W]$-module $\Lambda_i \subset \mathbb{S}_i$ (i.e., a lattice). However, the decomposition $\mathbb{Z}[W] = \bigoplus_{i \in \widehat{W}} (\Lambda_i \otimes \Lambda_i^*)$ holds *only after tensoring* with $\mathbb{Q}$ (or $\mathbb{C}$). $\square$

Hence we get a decomposition (up to isogeny)

$$J_Z \sim \prod_{i \in \widehat{W}} \Lambda_i \otimes_{\mathbb{Z}} P_i$$

where we define

$$P_i = \operatorname{Hom}_W(\Lambda_i, J_Z)_0.$$

(As usual the subscript zero indicates that we are taking only the connected component of the identity. It's not difficult to check that this is an abelian variety. Think of P_i as sitting inside a product of copies of J_Z.)

6.3 Example. For $G = \mathrm{SL}(2)$ we have $\widehat{W} = \{\mathbf{1}, \varepsilon\}$, and then $P_{\mathbf{1}} = \pi^* J_X$ while $P_\varepsilon = \ker(1+\sigma)_0 = (\ker T)_0$ is the classical Prym variety of Section 3. □

6.4 Remark. An element $\phi \in \mathrm{Hom}_W(\Lambda_i, J_Z)$ can be viewed as a torus bundle

$$\underline{L} = L_1 \oplus \cdots \oplus L_r, \qquad w^*\underline{L} = \underline{L}^w \quad \text{for all } w \in W,$$

where $L_j = \phi(e_j)$ after choosing a $\mathbb{Z}$-basis $e_1, \ldots, e_r \in \Lambda_i$. The action on the left-hand side of the relation $w^*\underline{L} = \underline{L}^w$ comes from that on Z, and on the right-hand side it is induced by the action on Λ_i. □

6.5 Dimension of the Pryms. The dimension of P_i is the multiplicity of $\mathbb{S}_i$ as subrepresentation of $H^1(\mathscr{O}_Z)$. That is, as a $\mathbb{C}[W]$-module,

$$H^1(Z, \mathscr{O}) = \bigoplus_{i \in \widehat{W}} a_i \mathbb{S}_i, \qquad a_i = \dim P_i. \tag{6}$$

The character of this representation can be computed using the Atiyah-Bott fixed point theorem. Namely, for any nontrivial element $\alpha \in W$ the difference $1 - \mathrm{tr}\,\alpha|_{H^1(Z,\mathscr{O})}$ is equal to the sum over fixed points z of α in Z of $\mathrm{tr}\,\alpha|_{\mathscr{O}_z} / \det(1 - d\alpha_z)$. By hypothesis 5.11, the only group elements with fixed points are the reflections, and for a reflection $\alpha \in R$ we obtain

$$1 - \mathrm{tr}\,\alpha|_{H^1(Z,\mathscr{O})} = \frac{1}{2}|\mathrm{Fix}(\alpha)| = \frac{1}{2}|W| \deg K.$$

We therefore have the character of the left-hand side of (6), and combining with the decomposition on the right-hand side we obtain simultaneous equations for the a_i:

$$\sum_{i \in \widehat{W}} a_i \; i(\alpha) = \begin{cases} \mathrm{genus}(Z) & \text{if } \alpha = 1, \\ 1 - \frac{1}{2}|W| \deg K & \text{if } \alpha \text{ is a reflection}, \\ 1 & \text{if } \alpha \in W \text{ is any other nontrivial element.} \end{cases} \tag{7}$$

□

6.6 Example. The Weyl group $W(G_2)$ is the dihedral group of a hexagon and has characters $\widehat{W} = \{\psi_1, \psi_2, \psi_3, \psi_4, \chi_1, \chi_2\}$ (we follow the notation of Donagi [6], Serre [30] §5.3) where $\psi_1 = \mathbf{1}$, the trivial character, $\psi_2 = \varepsilon$, the sign representation, and χ_1 is the reflection representation. A cameral cover $Z \to X$ with respect to a line bundle $K \in \mathrm{Pic}\, X$ has genus $g(Z) = 1 + 12(g - 1 + 3\deg K)$. There are six Pryms, and we compute their dimensions by the method of 6.5. This means solving the system of equations (7), whose coefficients are just the entries of the character table of W (see Donagi [6] or Serre [30] §5.3 for notation):

$$\begin{array}{r c}
\begin{array}{r} 1 \\ r \\ r^2 \\ r^3 \\ \text{reflections}\left\{\begin{array}{l} s \\ sr \end{array}\right. \end{array} &
\left(\begin{array}{rrrrrr|l}
1 & 1 & 1 & 1 & 2 & 2 & 1 + 12(g - 1 + 3\deg K) \\
1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -2 & 2 & 1 \\
1 & -1 & 1 & -1 & 0 & 0 & 1 - 6\deg K \\
1 & -1 & -1 & 1 & 0 & 0 & 1 - 6\deg K
\end{array}\right) \\
 & \begin{array}{rrrrrr} \psi_1 & \psi_2 & \psi_3 & \psi_4 & \chi_1 & \chi_2 \end{array}
\end{array}$$

Solving this system yields at once:

$i \in \widehat{W}$	$\dim P_i$
$\psi_1 = \mathbf{1}$	g
$\psi_2 = \varepsilon$	$g - 1 + 6 \deg K$
ψ_3, ψ_4	$g - 1 + 3 \deg K$
χ_1, χ_2	$2g - 2 + 6 \deg K$

□

The following general dimension formula (Theorem 6.7) is proved in Oxbury-Pauly [25]. The notation is the following. A reflection $\alpha \in R$, if it acts nontrivially in Λ_i, acts as an involution, and the dimension of its -1 eigenspace depends only on its conjugacy class in W. If we write $R = \coprod_{j=1}^{c} R_j$ for the partition of R into conjugacy classes, then for $j = 1, \dots, c$ we let

$$\dim_j^- \Lambda_i = \text{dimension of the } (-1)\text{-eigenspace of a reflection } \alpha \in R_j.$$

Thus $\dim \Lambda_i - 2 \dim_j^- \Lambda_i = i(\alpha)$ for $\alpha \in R_j$.

6.7 Theorem. *If Z is connected then for each nontrivial character $i \in \widehat{W}$ the corresponding Prym variety has dimension*

$$\dim P_i = (g-1) \dim \Lambda_i + \deg K \sum_{j=1}^{c} \dim_j^- \Lambda_i \, |R_j|.$$

□

6.8 Corollary. (Scognamillo [29].) *Suppose that G is semisimple, $K = \omega_X$ is the canonical line bundle and $\Lambda_i = \Lambda$ is the weight lattice. Then*

$$\dim P_i = \dim M_X(G).$$

Proof. For each j we have $\dim_j^- \Lambda = 1$ and so $\dim P_\Lambda = \operatorname{rk} G(g-1) + (2g-2)|R| = (g-1) \dim G$, which is the dimension of $M_X(G)$ by Theorem 1.2. □

6.9 Subcovers. Now consider factorisations

$$\begin{tikzcd} Z \arrow[dd, "W"'] \arrow[dr, "\pi"] & \\ & Y = Z/H \arrow[dl] \\ X & \end{tikzcd}$$

where $H \subset W$ is a subgroup of W. Then we have an injection $\pi^* : J_Y \hookrightarrow J_Z$, and the isogeny decomposition restricts to

$$J_Y \sim \prod_{i \in \widehat{W}} \Lambda_i^H \otimes_{\mathbb{Z}} P_i,$$

where Λ_i^H denotes the H-invariant part of Λ_i. In particular, if H is our favourite subgroup $H = \operatorname{Stab}(\lambda)$ where λ is a weight, then $\Lambda^H \neq 0$ (since it contains λ). So in this case the Prym variety corresponding to the weight lattice occurs in the decomposition; it is called the *distinguished Prym variety* and we denote it P_Λ:

$$P_\Lambda = \operatorname{Hom}_W(\Lambda, J_Z)_0.$$

□

6.10 Remark. The dimension of Λ_i^H is the multiplicity of $\mathbf{1}$ in Λ_i (for H);

$$\dim \Lambda_i^H = \mathrm{mult}(\mathbf{1}, \Lambda_i)_H = \mathrm{mult}(\Lambda_i, \mathbf{1}_H^W)_W$$

by Frobenius reciprocity—here $\mathbf{1}_H^W$ denotes the representation of W induced by the trivial representation of H. Hence:

$$\left\{\begin{array}{l}\text{isogeny decomposition}\\ \text{of the Jacobian } J_Y\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\mathbb{Z}[W]\text{-decomposition of the}\\ \text{induced representation } \mathbf{1}_H^W\end{array}\right\} \qquad \square$$

Combining this isogeny decomposition of J_Y with the dimension formula 6.7, we obtain the following genus formula for quotients of the cameral curve.

6.11 Corollary. *For any subgroup $H < W$ the quotient curve Z/H has genus:*

$$g(Z/H) = 1 + (g-1)|W/H| + \deg K \sum_{j=1}^{c} |R_j| \dim_j^-(\mathbf{1}_H^W),$$

where $\dim_j^-$ *is the dimension of the (-1)-eigenspace of a reflection in conjugacy class R_j.* $\square$

6.12 Example. Let $G = \mathrm{SL}(n)$, and let's return to Example 5.15. Recall that $\lambda = \varepsilon_1 = \frac{1}{n}(n-1, -1, \ldots, -1) \in \Lambda$ is a weight of the standard representation $\mathbb{C}^n$, and that $H := \mathrm{Stab}(\lambda) = S_{n-1}$. Thus the induced representation $\mathbf{1}_H^W$, given by the action of S_n on cosets of H, is just the permutation representation on $\mathbb{Z}^n$. The diagonal vector $(1, \ldots, 1)$ is invariant, as is the linear condition $x_1 + \cdots + x_n = 0$. Hence

$$\mathbf{1}_H^W = \mathbf{1} \oplus \Lambda.$$

Accordingly the Jacobian of $Y \xrightarrow{n:1} X$ has isogeny decomposition $J_Y \sim P_{\mathbf{1}} + P_\Lambda$ where $P_{\mathbf{1}}$ is the Jacobian of X, and P_Λ is the distinguished Prym. But by definition this is the identity component of

$$\begin{aligned}\mathrm{Hom}_W(\Lambda, J_Z) = \{\underline{L} = (L_1, L_2, \ldots, L_n) \quad | \quad & L_1 \otimes \cdots \otimes L_n = \mathcal{O}_Z \\ & \text{and } w^*\underline{L} = \underline{L}^w \text{ for all } w \in S_n\} \subset (J_Z)^n.\end{aligned}$$

Evaluation of $\underline{L}$ at $\lambda = \varepsilon_1$ gives a line bundle $L := L_1^{n-1} \otimes L_2^{-1} \otimes \cdots \otimes L_n^{-1}$ which is invariant under the action of H and therefore descends to a line bundle on Y. By construction this line bundle is in the image of $\frac{1}{n}(n-T)$ where T is the trace correspondence on J_Y. Hence

$$P_\Lambda = \mathrm{im}(n-T) = (\ker T)_0 = \mathrm{Prym}(Y/X),$$

the Prym variety in the sense of Section 3.7. $\square$

6.13 Evaluations and correspondences. As illustrated in this example, the isogeny from $P_\Lambda \subset \mathrm{Hom}_W(\Lambda, J_Z)$ to an abelian subvariety in J_Y, where $Y = Z/\mathrm{Stab}(\lambda)$, is given by the natural evaluation map $\mathrm{ev}_\lambda : \phi \mapsto \phi(\lambda) \in \mathrm{Pic}\, Z$. By construction this line bundle is invariant under $\mathrm{Stab}(\lambda)$ and therefore lies in the image of J_Y in J_Z. In Example 6.12 the evaluation map was injective and its image was the image of a correspondence $n-T$:

$J_Y \to J_Y$. In general, the image and kernel of ev_λ are described by the next proposition (see Donagi [6] or Oxbury-Pauly [25]).

First, consider the group ring element

$$C_\lambda = \sum_{w\in W} \langle\, \lambda \,|\, w\lambda \,\rangle\, w \in \mathbb{Q}[W],$$

where $\langle\ |\ \rangle$ is the Killing form on Λ. This element projects the regular representation $\mathbb{Q}[W]$ onto $\lambda \otimes \mathbb{S}$ and therefore projects J_Z onto the image of P_Λ under the evaluation map ev_λ. This projection is determined by the ($\mathbb{Q}$-valued) correspondence on Z, which descends to Y as

$$C_\lambda : y \mapsto \sum_{w\in W/\,\mathrm{Stab}(\lambda)} \langle\, \lambda \,|\, w\lambda \,\rangle\, y^w.$$

Second, consider the quotient $\Lambda/\mathbb{Z}[W]\lambda$, let $e \in \mathbb{N}$ be its exponent, and $J_Z[e]$ the group of e-torsion points in the Jacobian of Z.

6.14 Proposition. *The evaluation map* $\mathrm{ev}_\lambda : P_\Lambda \to J_Z$ *has the following properties.*

(i) $\ker \mathrm{ev}_\lambda \cong \mathrm{Hom}_W(\Lambda/\mathbb{Z}[W]\lambda, J_Z[e])$. *In particular,* ev_λ *is injective if* $\Lambda = \mathbb{Z}[W]\lambda$.

(ii) $\mathrm{im}\, \mathrm{ev}_\lambda$ *is equal to the image of the correspondence* $C = C_\lambda : J_Y \to J_Y \subset J_Z$.

(iii) If λ *is a weight of a (quasi)minuscule representation* V *of* G *then the correspondence* C *on* Y *satisfies*

$$C^2 = d_V C$$

where d_V *is the Dynkin index of the representation.* □

6.15 Example. In the situation of Example 6.12, the weight lattice is $\Lambda = \mathbb{Z}[W]\lambda$, so $\mathrm{ev}_\lambda : P_\Lambda \hookrightarrow J_Y$. The ($\mathbb{Q}$-valued) correspondence is $C = (1 - \frac{1}{n}T)$, and this satisfies $C^2 = C$, where $d_V = 1$ is the Dynkin index of the standard representation of $\mathrm{SL}(n)$. □

7 Abelianisation

We can now describe the rational map $P_\Lambda \dashrightarrow M_X(G)$. In order to simplify the discussion we will assume $Z \to X$ is unramified. Note that this is equivalent to the condition $\deg K = 0$. More generally one should modify the definition of $\mathrm{Hom}_W(\Lambda, Z)$ using the ramification divisor of $Z \to X$. (See Donagi [7].)

Suppose that $\underline{L} \in P_\Lambda$ is a W-equivariant T-bundle on Z, and consider the composition

$$F_{\underline{L}} : \underline{L} \xrightarrow[T]{} Z \xrightarrow[W]{} X.$$

The labels on the maps indicate that they are principal T and W-bundles, respectively. (Note that this requires Z to be unramified over X.) The composition is an N-bundle, where N is some extension of W by T,

$$1 \to T \to N \to W \to 1.$$

In other words we have a natural map

$$\mathrm{Hom}_W(\Lambda, J_Z) \xrightarrow[\mu]{} H^2(W, T),$$

where $H^2(W,T)$ is the group classifying extensions as above. Of course, this group has a distinguished element $[n] \in H^2(W,T)$ which represents the normaliser $N(T) \subset G$ of T.

7.1 Example. In the $G = \mathrm{SL}(2)$ case of Example 6.1, $\mathrm{Hom}_W(\Lambda, J_Z) = \{L \in J_Z \mid \sigma^* L = L^{-1}\}$ has four connected components mapping onto the group $H^2(W,T) = H^2(\mathbb{Z}_2, \mathbb{C}^*) = \mathbb{Z}_2$. The trivial element represents the group

$$\mathrm{O}(2) = \langle \mathbb{C}^*, \sigma \mid \sigma^2 = 1,\ \sigma z = z^{-1}\sigma \text{ for } z \in \mathbb{C}^* \rangle,$$

and its preimage under μ is $\mathrm{Nm}^{-1}(\mathcal{O}_X) = \ker T \subset J_Z$. The nontrivial element of $H^2(W,T)$ represents the group

$$\mathrm{Pin}(2) = \langle \mathbb{C}^*, \tau \mid \tau^2 = -1,\ \tau z = z^{-1}\tau \text{ for } z \in \mathbb{C}^* \rangle,$$

which is isomorphic to the subgroup of $\mathrm{SL}(2) = \mathrm{Spin}(3)$ (double covering $\mathrm{O}(2) \subset \mathrm{SO}(3)$) consisting of matrices

$$\begin{pmatrix} z & \\ & z^{-1} \end{pmatrix}, \quad \begin{pmatrix} & z \\ -z^{-1} & \end{pmatrix}.$$

This subgroup is the normaliser of $T \subset \mathrm{Spin}(3) = \mathrm{SL}(2)$. The preimage under μ of the nontrivial element of $H^2(W,T)$, from which direct image gives $\mathrm{SL}(2)$ bundles, is $\mathrm{Nm}^{-1}(\eta) = T^{-1}(\eta) \subset J_Z$ where $\eta \in J_X[2]$ corresponds to the double cover $Z \xrightarrow{2:1} X$ (defined by $\pi_* \mathcal{O}_Z = \mathcal{O}_X \oplus \eta$). Note that this is consistent with relation (3) in Section 4. □

Up to now we have, for convenience defined $P_\Lambda \subset \mathrm{Hom}_W(\Lambda, J_Z)$ to be the connected component containing the identity. However, it is clearly more correct to define it, as in the above example, to be the component(s) which map via μ to the element $[n]$. Then we obtain a rational map (of course, one needs to show that a general element $\underline{L} \in P_\Lambda$ gives a semistable bundle)

$$P_\Lambda \to M_X(N(T)) \subset M_X(G).$$

One can also describe this map in terms of vector bundles. $\underline{L}$ is a principal T-bundle on Z, and given a weight $\lambda \in \Lambda$, or equivalently a character $e^{2\pi i \lambda} : T \to \mathbb{C}^*$, we can construct a line bundle

$$\lambda(\underline{L}) := \underline{L} \times_\lambda \mathbb{C} \ \in \mathrm{Pic}\, Z.$$

This line bundle descends to the quotient $Y = Z/\operatorname{Stab}(\lambda)$, and we shall view $\lambda(\underline{L})$ as an element of $\mathrm{Pic}\, Y$.

7.2 Proposition. *Suppose $\lambda \in \Lambda$ is a weight of a minuscule representation V of G, and let $\underline{L} \in P_\Lambda$. Then*

$$\pi_* \lambda(\underline{L}) = F_{\underline{L}}(V),$$

where $\pi : Y = Z/\operatorname{Stab}(\lambda) \to X$.

Proof. We consider the pull-back $\pi_Z^* F_{\underline{L}}$ via $\pi_Z : Z \to X$. By definition of $F_{\underline{L}}$, this pull-back is a principal T-bundle. Under the action of T the representation V decomposes into 1-dimensional eigenspaces

$$V = \bigoplus_{\mu \in \mathfrak{X}(V)} V_\mu = \bigoplus_{\mu \in W\lambda} V_\mu$$

and hence the vector bundle $\pi_Z^* F_{\underline{L}}(V)$ splits into line bundles

$$\pi_Z^* F_{\underline{L}}(V) = \bigoplus_{\mu \in W\lambda} V_\mu.$$

The line bundles V_μ are conjugate under the W-action (by W-equivariance of $\underline{L} \to Z$) to $V_\lambda = \lambda(\underline{L})$, and hence

$$\pi_Z^* F_{\underline{L}}(V) = \bigoplus_{w \in W/\operatorname{Stab}\lambda} w^*(\lambda(\underline{L})).$$

Since Z is étale over X, this is equivalent to the proposition. $\square$

7.3 Abelianisation by Hitchin Pryms. In the ramified case $\deg K > 0$ it is still possible to construct a rational map $P_\Lambda \dashrightarrow M_X(G)$, but this requires 'twisting' the definition of P_Λ (see [7]). We will just describe here two examples where, using the Hitchin Prym coming from $K = \omega_X$, this abelianisation can be used along the same lines as for Theorem 4.1.

7.4 Example. Let $G = \mathrm{Spin}(2n)$. We will interpret the Verlinde number 4^g in Table 2.5.

The weight lattice Λ is contained in $\frac{1}{2}\mathbb{Z}^n$ with the standard inner product, spanned by $\mathbb{Z}^n$ and the vector $\frac{1}{2}(1, \ldots, 1)$. (See Fulton-Harris [11] §19.2 or Adams [1], Chapter 4.) The Weyl group is the semidirect product

$$W = \mathbb{Z}_2^{n-1} \rtimes S_n$$

where $\mathbb{Z}_2^{n-1}$ is a normal subgroup acting by even numbers of sign changes, and S_n permutes the coordinates. The set of weights $\mathfrak{X}(\mathbb{C}^{2n})$ of the standard orthogonal representation is the orbit of $\lambda = (1, 0, \ldots, 0)$, of order $2n$. We let $H = \operatorname{Stab}(\lambda)$ and $Y = Z/\operatorname{Stab}(\lambda) \xrightarrow{2n:1} X$. Then the isogeny decomposition of the Jacobian J_Y corresponds to the $\mathbb{Z}[W]$-module decomposition of the induced representation

$$\mathbf{1}_H^W = \mathbf{1} \oplus \mathbb{Z}^{n-1} \oplus \Lambda.$$

This decomposition can be seen as follows. $\mathbf{1}_H^W$ has a basis $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ where the α_j are cosets corresponding to the S_n-orbit of λ and the β_j correspond to the S_n-orbit of $-\lambda$. Under the W-action $\mathbf{1}_H^W$ has a trivial $\mathbf{1}$ summand spanned by $\sum(\alpha_j + \beta_j)$, a summand $\mathbf{1} \oplus \mathbb{Z}^{n-1}$ spanned by $\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n$ (so $\mathbb{Z}^{n-1}$ is isomorphic to the weight lattice of $\mathrm{SL}(n)$), and a summand Λ spanned by $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$, isomorphic as a $\mathbb{Z}[W]$-module to the weight lattice. Hence 6.10 implies that (up to isogeny) we get

$$J_Y = P_{\mathbf{1}} \times P_{\mathbb{Z}^{n-1}} \times P_\Lambda.$$

On the other hand, the natural involution on the orbit $\mathfrak{X}(\mathbb{C}^n)$ induces a fixed-point-free involution on the curve Y so that the map to X factorises as:

$$\begin{tikzcd}
Y \arrow[dd] \arrow[dr, "2:1\ \text{étale}"] & \\
 & Y' \arrow[dl, "n:1"] \\
X &
\end{tikzcd}$$

One can check that while $P_{\mathbf{1}}$ is the Jacobian of X, $P_{\mathbb{Z}^{n-1}}$ is the Prym variety $\mathrm{Prym}(Y'/X)$. To see that $P_\Lambda = \mathrm{Prym}(Y/Y')$, consider the correspondence $C = C_\lambda$ on Y (Proposition 6.14). This is defined by the group-ring element

$$C = \sum_{w \in W/\operatorname{Stab}(\lambda)} \langle\, \lambda \,|\, w\lambda \,\rangle\, w = 1 - \sigma$$

where $\sigma \in W$ is the involution $\lambda \leftrightarrow -\lambda$. (All other orbit vectors are orthogonal to λ.) But σ acts on Y as the sheet-interchange over Y', and hence by Proposition 6.14 (ii) we have

$$\mathrm{ev}_\lambda P_\Lambda = \mathrm{im}(1-\sigma) = \mathrm{Prym}(Y/Y') \subset J_Y.$$

Note, incidentally, that $C^2 = 2C$, and that $d_V = 2$ is the Dynkin index of the standard representation—as it should be by Proposition 6.14 (iii).

The evaluation map ev_λ is in this case not an isomorphism. This is because $\Lambda/\mathbb{Z}[W]\lambda \cong \mathbb{Z}_2$ (with trivial W-action), so that by Proposition 6.14 (i) the kernel of ev_λ is isomorphic to $J_X[2]$, the group of 2-torsion points in the Jacobian of X. We therefore arrive at a diagram:

$$\begin{tikzcd} P_\Lambda \arrow[r, "J_X[2]"] \arrow[d] & \mathrm{Prym}(Y/Y') \subset J_Y \\ M_X(\mathrm{Spin}(2n)) & \end{tikzcd}$$

By the projection formula, as in the proof of Theorem 4.1, the theta line bundle $\Theta(\mathbb{C}^{2n}) = \mathscr{L}^2$ on the moduli space $M_X(\mathrm{Spin}(2n))$ pulls back to $\mathrm{ev}_\lambda^*\Theta_Y$. But the fact that $Y \to Y'$ is étale implies that

$$\Theta_Y|_{\mathrm{Prym}} = 2\Xi$$

where Ξ is a principal polarisation on $\mathrm{Prym}(Y/Y')$. (This is a consequence of Corollary 3.4 and the remark following.) Moreover, it is an exercise using the theory of Section 3.1 that the principal polarisation Ξ pulls back under the isogeny ev_λ, given that the kernel is $J_X[2]$, to a polarisation of type $(1,\dots,1,4,\dots,4)$ (where 4 appears g times) on P_Λ.

In conclusion, then, pull-back of sections under the direct image map defines an injective linear map

$$H^0(M_X(\mathrm{Spin}(2n)), \mathscr{L}) \longrightarrow H^0(P_\Lambda, \mathrm{ev}_\lambda^*\Xi) \cong \mathbb{C}^{4^g}.$$

By the Verlinde formula the first space also has dimension 4^g, and so the map is an isomorphism. $\square$

7.5 Example. Let $G = E_6$. The Verlinde number in Table 2.5 is 3^g in this case. We return to the set-up of Example 5.16.

The weight lattice Λ is isomorphic to the primitive cohomology in $H^2(S,\mathbb{Z})$ of a generic cubic surface $S \subset \mathbb{P}^3$ equipped with the intersection form, and under this isomorphism the set of weights $\mathfrak{X}(\mathbb{C}^{27})$ of the standard representation becomes identified with the Weyl group orbit of the class $\lambda \in \Lambda$ of a line $\ell \subset S$. Let $H := \operatorname{Stab}(\lambda)$. Then Donagi [6] checks that the induced representation $\mathbf{1}_H^W$ decomposes as

$$\mathbf{1}_H^W = \mathbf{1} \oplus U \oplus \Lambda,$$

where $\dim U = 20$ and (the weight lattice) $\dim \Lambda = 6$. It follows that up to isogeny we have

$$J_Y \sim J_X \times P_U \times P_\Lambda.$$

We have a diagram

$$\begin{tikzcd} P_\Lambda \arrow[r] \arrow[d] & P \subset J_Y \\ M_X(E_6) & \end{tikzcd}$$

The horizontal map is an isomorphism because $\Lambda = \mathbb{Z}[W]\lambda$. Its image $P \subset J_Y$ is defined by a $\mathbb{Q}$-valued correspondence C on Y which can be calculated using the intersection form on Λ, and satisfies

$$3C = 4\ \mathrm{id} + \mathrm{Skew} - 2\ \mathrm{Inc}$$

where Inc assigns to $\ell \subset S$ the 10 incident lines and Skew assigns the 16 skew lines. An exercise is to show that this correspondence satisfies

$$C^2 = 6\ C.$$

(The Dynkin index is in this case $d_V = 6$.) It can be shown, using this relation, that $\Theta_Y|_P = 6\Xi$ where Ξ is a polarisation of type $(1, \dots, 1, 3, \dots, 3)$ (g times). On the other hand, the theta line bundle on $M_X(E_6)$ is $\Theta(\mathbb{C}^{27}) = \mathscr{L}^6$, and hence pull-back under the direct image map $P_\Lambda \to M_X(E_6)$ induces an injective linear map

$$H^0(M_X(E_6), \mathscr{L}) \to H^0(P_\lambda, \Xi).$$

By the Verlinde formula both spaces have the same dimension 3^g, and so the map is an isomorphism. □

References

[1] J. Frank Adams. *Lectures on exceptional Lie groups.* University of Chicago Press, Chicago, IL, 1996.

[2] V. Balaji. *Notes on semistable principal bundles.*

[3] Arnaud Beauville. *Conformal blocks, fusion rules and the Verlinde formula.* In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, pp. 75–96, Ramat Gan, 1996. Bar-Ilan Univ. (alg-geom/9405001).

[4] Arnaud Beauville, M. S. Narasimhan, and S. Ramanan. *Spectral curves and the generalised theta divisor.* J. Reine Angew. Math. **398** (1989), 169–179.

[5] Nicolas Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Hermann, Paris, 1968.

[6] Ron Donagi. *Decomposition of spectral covers.* Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). Astérisque **218** (1993), 145–175.

[7] Ron Donagi. *Spectral covers.* In *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, pp. 65–86. Cambridge Univ. Press, Cambridge, 1995. (alg-geom/9505009).

[8] Jean-Marc Drézet and M. S. Narasimhan. *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques.* Invent. Math. **97** (1989), 53–94.

[9] Gerd Faltings. *Stable G-bundles and projective connections.* J. Alg. Geom. **2** (1993), 507–568.

[10] Gerd Faltings. *A proof for the Verlinde formula.* J. Alg. Geom. **3** (1994), 347–374.

[11] William Fulton and Joe Harris. *Representation Theory, A first course.* No. 129 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.

[12] Nigel J. Hitchin. *The self-duality equations on a Riemann surface.* Proc. London Math. Soc. (3) **55** (1987), 59–126.

[13] Nigel J. Hitchin. *Stable bundles and integrable systems.* Duke Math. J. **54** (1987), 91–114.

[14] Nigel J. Hitchin. *Flat connections and geometric quantization.* Comm. Math. Phys. **131** (1990), 347–380.

[15] Bruce Hunt. *The Geometry of some special Arithmetic Quotients.* No. 1637 in Lecture Notes in Mathematics. Springer-Verlag, New York, 1996.

[16] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*, vol. 1+2. Interscience Publishers, New York, 1963.

[17] Herbert Lange and Christina Birkenhake. *Complex abelian varieties.* Springer-Verlag, Berlin, 1992.

[18] Yves Laszlo and Christoph Sorger. *The line bundles on the moduli of parabolic G-bundles over curves and their sections.* Ann. Sci. École Norm. Sup. (4) **30** (1997), 499–525. (alg-geom/9507002).

[19] Robert Lazarsfeld and A. van de Ven. *Topics in the Geometry of Projective Space – recent work of F.L. Zak.* DMV Seminar Band 4, Birkhäuser 1984.

[20] Yuri I. Manin. *Cubic Forms: Geometry, Algebra, Arithmetic.* North-Holland Mathematical Library, vol 4, North-Holland Publishing Co., Amsterdam, 1974.

[21] M. S. Narasimhan and S. Ramanan. *Moduli of vector bundles on a compact Riemann surface.* Ann. of Math. (2) **89** (1969), 14–51.

[22] M. S. Narasimhan and S. Ramanan. *2θ-linear systems on abelian varieties.* In *Vector bundles on algebraic varieties (Bombay, 1984)*, pp. 415–427. Tata Inst. Fund. Res., Bombay, 1987.

[23] William M. Oxbury. *Spin Verlinde spaces and Prym theta functions.* Proc. London Math. Soc. (3) **78** (1999), 52-76. (alg-geom/9602008).

[24] William M. Oxbury and Christian Pauly. *Heisenberg invariant quartics and $SU_C(2)$ for a curve of genus four.* Math. Proc. Cambridge Philos. Soc. **125** (1999), 295–319. (alg-geom/9703026).

[25] William M. Oxbury and Christian Pauly. *Polarisation of Prym varieties for Weyl groups.* In preparation.

[26] William M. Oxbury and S. M. J. Wilson. *Reciprocity laws in the Verlinde formulae for the classical groups.* Trans. Amer. Math. Soc. **348** (1996), 2689–2710.

[27] A. Ramanathan. *Moduli for principal bundles over algebraic curves. I.* Proc. Indian Acad. Sci. Math. Sci. **106** (1996), 301–328.

[28] A. Ramanathan. *Moduli for principal bundles over algebraic curves. II.* Proc. Indian Acad. Sci. Math. Sci. **106** (1996), 421–449.

[29] Renata Scognamillo. *An elementary approach to the abelianization of the Hitchin system for arbitrary reductive groups.* Comp. Math. **110** (1998), 17–37.

[30] Jean-Pierre Serre. *Représentations linéaires des groupes finis.* Hermann, Paris, 1978. English translation: Linear representations of finite groups, GTM 42, Springer-Verlag, New York, 1977.

[31] Christoph Sorger. *La formule de Verlinde.* Séminaire Bourbaki, 1994/95, Exp. No. 794. Astérisque **237** (1996), 87–114.

[32] Christoph Sorger. Lectures on moduli of principal G-bundles over algebraic curves. In *School on Algebraic Geometry (Trieste, 1999)*, pp. 1–57. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2000. Available at http://www.ictp.trieste.it/~pub_off/lectures/lns001/Sorger/.

[33] Michael Thaddeus. *Stable pairs, linear systems and the Verlinde formula.* Invent. Math. **117** (1994), 317–353. (alg-geom/9210007).

[34] Don Zagier. *Elementary aspects of the Verlinde formula and of the Harder-Narasimhan-Atiyah-Bott formula.* In *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, pp. 445–462, Ramat Gan, 1996. Bar-Ilan Univ.