Lectures on rationally connected varieties

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Contents

References: for the definition and basic theory of rationally connected varieties, the standard source is Kollár [6]. A more user friendly book is Debarre [3]. For the deformation theory, see Vistoli [8]; for the fibration theorem, see Graber-Harris-Starr [4].

1 Definition and good properties

Let X be a smooth projective variety over a algebraically field of characteristic zero—say the field of complex numbers.

The following definition is due independently to Kollár-Miyaoka-Mori [7] and Campana [1].

1.1 Definition. A variety X is called *rationally connected* if any two general points p, q on X can be connected by a chain of rational curves.

Equivalently (as we shall prove), rationally connected varieties can be characterized by the following apparently much stronger condition.

1.2 Lemma. *A variety X is rationally connected if and only if for every finite set of points $\Gamma \subset X$ there exists an immersed rational curve $C \to X$ containing Γ and with ample normal bundle.*

1.3 Remark. Being rationally connected is a birational condition: if X is rationally connected and if $Y \to X$ is a birational map then Y is also rationally connected.

For curves and surfaces, being rationally connected is equivalent to being rational.

1.4 Why is rationality and rational connectivity the same for surfaces? We will see later (2.9) that a rationally connected variety has no forms. If X is a surface without forms, it follows from the classification of surfaces that X is rational.

1.5 The thesis of these talks is that the notion of rational connectivity is a much more natural and geometric notion than rationality or unirationality (which are more algebraic of nature). (Recall that a variety X of dimension n is rational if there is a birational map $\mathbb{P}^n \dashrightarrow X$, and that it is unirational if there exists a generically finite rational map $\mathbb{P}^n \dashrightarrow X$.) Here comes a list of features of the notion of rational connectivity.

- *There is a 'local' criterion for rational connectivity.* Namely,

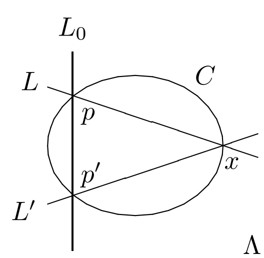
1.6 Lemma. *A smooth projective variety X is rationally connected if and only if there exists a smooth rational curve $C \subset X$ with ample normal bundle.*

1.7 Corollary. *Let $D \subset X$ be a smooth ample divisor on a smooth projective variety X. If D is rationally connected then X is also rationally connected.*

In contrast, no such result holds for rationality. For example, the cubic threefold in $\mathbb{P}^4$ is not rational, but its hyperplane sections (smooth cubic surfaces) are rational.

1.8 Why aren't cubic threefolds rational? This is a very delicate question; its answer was given in Clemens-Griffiths [2]. Roughly they show that the middle Hodge structure of a smooth cubic threefold $X \subset \mathbb{P}^4$ is not the Jacobian of any curve, and therefore X cannot be obtained from $\mathbb{P}^3$ by blowing up curves. (Slightly more precisely, $H^{1,2}(X)$ quotiented by the lattice generated by third integral cohomology is a principally polarized abelian variety J_X called the intermediate Jacobian. If one blows up X along a curve C then J_X acquires the Jacobian $J(C)$ as a direct summand. (Blowing up a point doesn't change J_X.) Now the precise statement is that J_X is not a direct sum of Jacobians of curves, and in particular X can not be obtained from $\mathbb{P}^3$ by blowing up and down. . .)

1.9 Exercise: cubic threefolds are unirational. Let $X \subset \mathbb{P}^4$ be a cubic threefold. First, one shows that there exists a line $L_0 \subset X$. (In fact there is an ∞^2 of lines, but for the argument we just need one.) Now consider the variety W whose points are pairs (p, L) where p is a point on the fixed line L_0, and L is a tangent line to X at p. Clearly W is a $\mathbb{P}^2$-bundle over L_0, and in particular a rational variety. Since each such line L is tangent to X it cuts X in a third point, so in this way we also get a rational map $\varphi : W \dashrightarrow X$. Thus X is unirational. (In fact this map $\varphi : W \dashrightarrow X$ is two-to-one. Indeed, let $x \in X$ be a general point, and consider the plane Λ spanned by x and L_0: this plane cuts X in a conic C (residual to L_0). The lines through x which are tangent to X at L_0 are exactly the two lines joining x with the intersection points $C \cap L_0$ in Λ.)



Alternatively, consider the projection from L_0 to a plane $\mathbb{P}^2$: blowing up X along L_0 yields a regular map $\widetilde{X} \to \mathbb{P}^2$ whose fibres are conics, so X is birational to a conical bundle over $\mathbb{P}^2$. (Unirationality follows from this construction via a base change argument…) Now this conical bundle has no section! Otherwise it would be birational to a $\mathbb{P}^1$-bundle and then X would be rational, contradicting the result of Clemens and Griffiths…

- *Rational connectivity is both an open and a closed condition.* Precisely:

1.10 Lemma. *If $X \to B$ is a smooth and projective morphisms, then the locus $\{b \in B \mid X_b$ is r.c.$\} \subset B$ is open and closed.*

Idea of the openness: let $X \to B$ be a family with a special fibre X_0 which is rationally connected, so X_0 contains a rational curve C with ample normal bundle. Now there is an exact sequence

$$0 \to N_{C/X_0} \to N_{C/X} \to \mathcal{O}_C^{\oplus} \to 0$$

and since we are on a rational curve this sequence splits. It follows that $N_{C/X}$ is generated by global sections and that $H^1(C, N_{C/X}) = 0$. So the deformations of C in X are unobstructed (cf. 3.1 below) and they decompose into direct sums of deformations lying inside the fibres and deformations transversal to the fibres. So we can move C into a neighbouring fibre…

Idea of closedness: suppose the fibres over a punctured analytic disc $\Delta \setminus 0$ are rationally connected, and take two points in X_0. Choose two local sections through these points to define 'neighbouring points' in neighbouring fibres X_t; inside each of these fibres, these two sections can be joined by rational curves. Now impose algebraic conditions on these curves (e.g., in some component of the relative Hilbert scheme) in order to get down to a finite number, and we can assume (if necessary applying a base change) that there is one vertical rational curve that we can follow from fibre to fibre in $\Delta \setminus 0$. Now take the flat limit to X_0. (This may not be an irreducible curve, but then use the original weak definition of rational connectivity: it's enough to have a chain).

In contrast, it is an open question whether rationality is an open (or closed) condition.

1.11 Example: Cubic fourfolds. Consider the family of all smooth cubic fourfolds. *Suspicion:* the locus of rational cubic fourfolds is neither open nor closed: it should be the countable union of subvarieties of the family, e.g., a countable union of divisors (B. Hassett [5] shows that there exists an infinite countable number of families of rational cubics—codimension-2 subvarieties…)

The idea behind this suspicion is somewhat similar to the idea of 1.8. A fourfold X is rational if and only if it can be obtained from $\mathbb{P}^4$ by a finite sequence of blow-ups and blow-downs, along points, curves and surfaces. Blowing up along points and curves does not change the middle Hodge structure whereas each blow-up along a surface will change the middle Hodge structure by adding a direct summand which is the Hodge structure of a surface… So X is rational if and only if its middle Hodge structure is a direct sum of Hodge structures of surfaces… Such cubic fourfolds form subvarieties in the space of all cubic fourfolds, so to prove the suspicion one would need an infinite (countable) list of possible Hodge structures of surfaces…

• *There is a divisor-theoretic criterion for rational connectivity:*

1.12 Theorem. (Kollár-Miayoka-Mori [7]) *Let X be a smooth projective variety, then*

$$K_X < 0 \ \Rightarrow X \text{ is rationally connected.}$$

1.13 Example. *A smooth hypersurface $X \subset \mathbb{P}^n$ of degree d is rationally connected if and only if $d \leq n$.*

In contrast, very little is known about the rationality of hypersurfaces. For example it is not known if there are any smooth rational hypersurfaces $X \subset \mathbb{P}^n$ of degree $d \geq 4$... (Mori always asks this question).

• *There is a conjectured numerical criterion for rational connectivity.* We call it Mumford's conjecture, although it is not clear when and how Mumford formulated it...

1.14 Mumford's conjecture. *A smooth projective variety X is rationally connected if and only if*

$$h^0(X, (T_X^*)^{\otimes m}) = 0, \quad \forall m > 0.$$

It is known that rational connectivity implies 'no global covariant tensor fields', the conjecture is about the converse implication.

This is related to another conjecture (which is probably due to Mori). Recall that X is called *uniruled* if through every point (or equivalently: a general point) of X there exists a smooth rational curve. (Alternatively: X (of dimension n) is uniruled if there is a dominant rational map from some $\mathbb{P}^1 \times Y$, (with $\dim Y = n - 1$).)

1.15 Mori's conjecture. *A smooth projective variety X is uniruled if and only if*

$$h^0(X, K_X^{\otimes m}) = 0, \quad \forall m > 0.$$

Again it is known that uniruledness implies $h^0(X, K_X^{\otimes m}) = 0$.

We will see later that Mori's conjecture implies Mumford's. While Mori's conjecture is well founded in birational geometry of algebraic varieties, Mumford's conjecture seems to be some strange guess—how could Mumford come up with that? (and how did he formulate it, since the notion of rational connectivity is much more recent?).

• *There is a quantitative version of rational connectivity.* Instead of just asking, for every pair of points, that there be a chain of rational curves connecting them, one could try to measure to which extent this fails. For example one could define an equivalence relation on X by declaring two points equivalent if there is such a chain, and then look at the quotient of this equivalence relation—call it Z. Now measure the size of Z, for example its dimension... However this is not a good notion. For instance, on a K3 surface there is only a countable number of rational curves, so most points would be alone in their equivalence class while all the rational curves would be contracted: Z would be extremely ugly...

1.16 Theorem-definition. (Kollár-Miyaoka-Mori [7].) *For any variety X there exists a rational map $\varphi : X \dashrightarrow Z$ (unique up to birational equivalence) characterized by the following properties:*

(i) The (general) fibres of φ are rationally connected.

(ii) Conversely, almost all rational curves in X lie in the fibres: for a very general point $z \in Z$, any rational curve in X which meets X_z is actually contained in X_z.

The variety Z is called the *maximal rationally connected quotient* of X (for short, the mrc quotient), and $\varphi : X \dashrightarrow Z$ the *maximal rationally connected fibration* of X (the mrc fibration).

'Very general' means: in the complement of a countable union of Zariski closed subsets...

1.17 Exercise. Prove that the following is equivalent to condition (ii): *For general $x \in X$, every rational curve in X through x lies in the fibre $\varphi^{-1}\varphi(x)$.*

1.18 Example. If X is a K3 surface, then the mrc quotient is X itself...

Definition. The rational dimension of X is defined as $\operatorname{rd} X := \dim X - \dim Z$.

1.19 Example. $\operatorname{rd} X > 0$ if and only if X is uniruled.

1.20 Proof. We prove the contrapositive statement: $\operatorname{rd} X = 0$ if and only if X is not uniruled. Indeed, if $\operatorname{rd} X = 0$ then the general fibres of $X \dashrightarrow Z$ are 0-dimensional, and since they are rationally connected, in particular they must be connected, so we can just take $Z = X$. In this situation, condition (ii) of the definition is equivalent to saying that any very general point has no rational curve through it. It remains to observe that the set of points through which some rational curves passes is closed, since it is the image of the projection map from the universal family over the Hilbert scheme. So having just one point without rational curves through it is equivalent to having an open set of such points...

1.21 Example. X is rationally connected if and only if $\operatorname{rd} X = \dim X$.

1.22 Remark. Let X contain a smooth rational curve C. Write the normal bundle as

$$N_{C/X} \simeq E \oplus \mathcal{O}_C^{\oplus}$$

where E is of rank r. Then the rational dimension of X is at least r.

- *Rational connectivity behaves well in fibrations:*

1.23 The fibration theorem. (Graber, Harris, Mazur, Starr [4]) *Let $X \to B$ be dominant. If B is rationally connected and the general fibre is rationally connected, then X is rationally connected.*

Again, the corresponding statement is false when 'rationally connected' is replaced by 'rational'.

1.24 The embarrassing question. Rationality implies unirationality, and unirationality implies rational connectivity. For curves and surfaces the three notions coincide. For higher dimensional varieties it is known that there are unirational varieties which are not rational (for example cubic threefolds, cf. 1.8).

But it is an embarrassing open problem whether there exists varieties which are rationally connected but not unirational...

The problem is not on the side of rational connectivity—as we have seen there are many criteria for showing that a given variety is rationally connected. The problem is with the notion of unirationality: there are no ways in practice to show that a given variety is not unirational!

Here are three examples of rationally connected varieties which are probably not unirational:

(i) A hypersurface $X \subset \mathbb{P}^n$ of degree n ($n \geq 5$).

(ii) A double cover $X \xrightarrow{2:1} \mathbb{P}^n$ branched along a smooth hypersurface $B \subset \mathbb{P}^n$ of degree $2n$ ($n \geq 3$).

(iii) A hypersurface $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2, n)$ ($n >> 0$). These are conic bundles over $\mathbb{P}^2$.

1.25 Homework. Let X be a rationally connected variety.

(a) Show that X has no fixed-point-free automorphisms. (Hint: use that X has no higher cohomology…)

(b) Show that X is not an étale cover of any variety.

(c) Show that there are no étale covers of X.

(Hints for (b) and (c): The topological fundamental group π_1 has no finite quotients. In fact $\pi_1 = 0$. Another hint: the Euler characteristic of X is equal to 1.)

2 Consequences and extensions of the fibration theorem

Let us first extract two important consequences of the fibration theorem. The first is about the mrc quotient and the second states that Mori's conjecture implies Mumford's.

2.1 Corollary. *If X is any variety with mrc fibration $\varphi : X \dashrightarrow Z$, then Z is not uniruled.*

Proof. Suppose Z is uniruled: through each point $z \in Z$ there is a rational curve C. Now the fibration theorem implies that $\varphi^{-1}(C)$ is rationally connected. Therefore, every point $x \in X$ lies on a rational curve not in any fibre. □

2.2 Corollary. *Mori's conjecture implies Mumford's conjecture.*

Proof. The proof is contrapositive in the following way: assuming Mori's conjecture, we want to prove that if X is *not* rationally connected, then there is a section of some tensor power of T_X^*. So assume that X is not rationally connected. Then for the mrc fibration $X \dashrightarrow Z$ we have $\dim Z = k > 0$, and by the preceding corollary, Z is not uniruled. So by Mori's conjecture we conclude that there exists a section $\sigma \in H^0(K_Z^{\otimes m}) \subset H^0((T_Z^*)^{\otimes km})$. Now the pullback $\varphi^*\sigma$ is a section of $(T_X^*)^{\otimes km}$ □

Note: K_Z is a direct summand in $(T_Z^*)^{\otimes k}$ where k is the dimension of Z. Indeed, in a point $z \in Z$, $(T_Z^*)_z^{\otimes k}$ are the k-linear forms on $T_z Z$. While $(K_Z)_z$ are the skew-symmetric k-linear forms.

Next let us analyze the statement of the fibration theorem—this will lead to a natural generalization of it, which is what we actually will prove. We are given the map $\pi : X \to B$

where B and the general fibre are rationally connected. Take two points p and q in X and look at their images in B: since B is rationally connected, there is a rational curve $B' \subset B$ joining these two points. Now look at the inverse image $X' := \pi^{-1}(B') \subset X$. If we can prove that X' is rationally connected we are through because in that case we can join p and q by a three-component rational curve: one rational curve from p to X' (inside the fibre which we assumed rationally connected), another curve from q to X' (inside another fibre), and a middle component joining these two curves inside X'.

So to prove the theorem it is enough to prove it for the case $B = \mathbb{P}^1$. In this case we have to exhibit a rational multi-section. In fact we can do more. We will prove there is a section, and we will do that not only for $\mathbb{P}^1$ but for any smooth curve B. The fibration theorem will be a corollary of the following more general result.

2.3 Theorem. *Let $f : X \to B$ be a non-constant map to a smooth curve, such that the general fibre is rationally connected. Then f has a section.*

2.4 Sections v. rational points. To have a section to $X \to B$ is equivalent to having a point in the fibre over the generic point of B. But this in turn is the same as having a K-rational point of X, where K is the function field of B. So we can restate Theorem 2.3 in an algebraic way. (Recall that throughout, k is an algebraically closed field of characteristic zero.)

2.5 Theorem. *If K is a field of transcendence degree 1 over k, and X is a rationally connected variety over K, then X has a K-rational point (i.e., $X(K) \neq \emptyset$).*

2.6 Remark. In fact, the set of K-rational points will be dense. This will follow from the fibration theorem.

In this form, the theorem is very similar to the following classical result.

2.7 Tsen's theorem. *Let $X \subset \mathbb{P}^n_K$ be a hypersurface of degree d over a field K of transcendence degree 1 over k. Then*

$$d \leq n \;\Rightarrow\; X(K) \neq \emptyset.$$

Since we know (cf. 1.13) that a hypersurface of degree $d \leq n$ is rationally connected, we see that 2.5 implies Tsen's theorem as a special case.

We are looking for further generalizations. There are two possibilities: either we can weaken the conditions or we can strengthen the conclusion.

2.8 Question. Can we replace 'rationally connected variety' by a larger class of varieties in the statement of Theorem 2.5? (This is an ill-posed problem, because we can easily find some special isolated examples of varieties for which the conclusion holds, so we could always define the larger class to be the union of rationally connected varieties with those extra varieties we found. What we are looking for is a class of varieties described in a reasonable geometric way for which the conclusion is true. . .)

2.9 'Formless' varieties. Fact: *A rationally connected variety has no differential forms, i.e.,* $H^0(X, \Omega_X^p) = 0$ *for all* $p > 0$. So for example we might try to formulate and prove the theorem more generally for such 'formless' varieties. Examples that come to mind are Enriques surfaces (see 5.4)...

2.10 Remark. The density statement 2.6 does not generalize...

Does it work over all C_1 fields? It is actually true for number fields, for other reasons, but it is not known in general.

In his thesis (on Tsen's theorem), Serge Lang asked if one can do without the hypothesis of transcendence 1, and what other hypotheses are needed then. He found:

2.11 Lang's theorem. *Let* $X \subset \mathbb{P}^n_K$ *be a smooth hypersurface of degree* d *over a field* K *of transcendence degree* r *over* k. *Then*

$$d^r \leq n \;\Rightarrow\; X(K) \neq \emptyset.$$

2.12 Question. Can we find a larger class of varieties for which Lang's theorem holds? What geometric conditions do we need in general to deduce Lang's statement?

2.13 Homework. Find 1-parameter families without a rational section of
(i) hypersurfaces $X \subset \mathbb{P}^n$ of degree $d \geq n+1$
(ii) curves of genus $g \geq 1$.

2.14 Solution: Example of a 1-parameter family of hypersurfaces which does not allow a section. Let $X_0 \subset \mathbb{P}^n$ be the Fermat hypersurface of degree d; it is given by the homogeneous polynomial

$$z_0^d + z_1^d + \cdots + z_n^d.$$

Let ξ be a d'th root of unity and consider the automorphism

$$\begin{aligned} \sigma : X_0 &\longrightarrow X_0 \\ [z_0 : \ldots : z_n] &\longmapsto [\xi^0 z_0 : \ldots : \xi^n z_n] \end{aligned}$$

Now consider the product $X_0 \times \mathbb{P}^1$ considered as a family over $\mathbb{P}^1$; equip $\mathbb{P}^1$ with the automorphism

$$\begin{aligned} \tau : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ t &\longmapsto \xi\, t. \end{aligned}$$

Now look at the quotient $X = (X_0 \times \mathbb{P}^1)/(\sigma, \tau) \longrightarrow \mathbb{P}^1/\tau \simeq \mathbb{P}^1$. The fibres over 0 and ∞ have multiplicity d.

Now in general, if in a family $\mathfrak{X} \to B$ there is a non-reduced fibre and the total space $\mathfrak{X}$ is smooth, then there can never be any section, not even locally: any subvariety of $\mathfrak{X}$ will intersect this multiple fibre with multiplicity > 1, so it cannot be a section.

So in order for our example to work we must ensure that the total space X is smooth. It will be so if σ has no fixpoints. This is the case if and only if all the roots ξ^i are distinct, and clearly this is the case if and only if $d > n$. So for $d > n$ the constructed family has no sections. This also shows that the statement of Tsen's theorem is sharp, in the sense that there exists 1-parameter families of hypersurfaces of degree $d > n$ which have no sections.

Note however that this family is very special: it is isotrivial...

2.15 Another idea. Let $\mathbb{P}^N$ be the space of all hypersurfaces of $\mathbb{P}^n$ of degree $d \geq n+1$ (so $N = \binom{d+n}{d} - 1$). Consider the universal family

$$\begin{array}{l} \mathfrak{X} \subset \mathbb{P}^N \times \mathbb{P}^n \\ \big\downarrow \\ \mathbb{P}^N \end{array}$$

Here $\mathfrak{X}$ is given as the zero locus of the general polynomial of degree d.

Now take a curve B (for instance $\mathbb{P}^1$), map it into $\mathbb{P}^N$, and pull back the universal family. If B is a line the induced family will have a base locus and therefore it will have sections. Similarly, if B is contained is a linear subspace.

But what if B is a rational normal curve of degree $\geq n$? Then probably there will be no sections... ???

2.16 1-parameter family of curves without a section. Start with any (non-isotrivial) 1-parameter family $S \to B$ of curves of genus $g \geq 2$. We know from the theorem of Manin (cf. Lucia Caporaso's lectures) that there is only a finite number of sections. Now draw a smooth curve R in the surface S in such a way that it is transversal to each of the sections. (Actually it is enough that for each section there exists a transversal intersection with R.) Now let $\widetilde{S}$ be the double cover of S branched along R—for this to make sense, the class of R in $\operatorname{Pic} S$ must be divisible by 2). Now the family $\widetilde{S} \to B$ will be a family of hyperelliptic curves. Each of the original sections σ will induce a 2-section $\widetilde{\sigma}$ to this family but they will not be sections. The reason why we needed transversality of the original sections with R is that if in all intersection points there were a tangency then the cover of the section would be reducible and then it would be possible to find a section from σ to $\widetilde{\sigma}$...

2.17 Pencils with a non-reduced member. We have remarked that pencils of hypersurfaces always have sections, and we have seen that a non-reduced fibre prevents the existence of sections. So what about the a pencil with a non-reduced fibre? This apparent contradiction is resolved by the observation that such a pencil will always have *singular* total space! and then you can sneak in a section even though the fibre is non-reduced...

2.18 Instructive dimension count which suggests Tsen's theorem. (This is not a proof, but it is a good way to guess the statement.) Let $\mathfrak{X} \subset \mathbb{P}^n \times \mathbb{P}^1$ be a 1-parameter family of hypersurfaces of degree d. So $\mathfrak{X}$ is the zero locus of a bihomogeneous polynomial $F(t_0, t_1, z_0, \ldots, z_n)$ of degree d in the variables z and of some degree e in the variables t. We ask ourselves if this family has any sections?

First we look at the space of sections to $\mathbb{P}^n \times \mathbb{P}^1$ of some degree k; this is just the space of maps $s : \mathbb{P}^1 \to \mathbb{P}^n$ of degree k, and this space is of dimension $(k+1)(n+1) - 1$. The image of this section is contained in $\mathfrak{X}$ when $F(t, s(t)) = 0$. This is now a polynomial in t of degree $e + kd$, so there are $e + kd + 1$ coefficients to kill to make it vanish.

So we expect to find sections to $\mathfrak{X} \to \mathbb{P}^1$ when

$$e + kd + 1 \leq (k+1)(n+1) - 1.$$

We are looking for the largest d such that for every e there exists a k satisfying the inequality. We can suppose $k \geq e$; then the inequality is implied by

$$d \leq n,$$

and in this case we expect to find sections.

(For $d = n + 1$, the inequality is equivalent to $e \leq n - 1$, so except for families with that small e we cannot expect any sections.)

2.19 Generalization. Exercise: generalize the dimension count to a base of arbitrary dimension r, to arrive at the inequality of Lang's theorem. The number of sections will depend on numerical invariants of B through Riemann-Roch. The leading term on the left-hand side will have a factor d^r while the right hand side will have $(n+1)$...

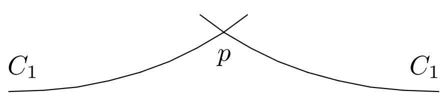
3 Some deformation theory

Reference: Vistoli [8]

We are only concerned about nodal curves on smooth varieties. So throughout let X be a smooth projective variety, and let $C \subset X$ be a nodal curve (of any genus).

3.1 Two basic facts.
(a) The space of first-order deformations of C in X is naturally isomorphic to $H^0(C, N_{C/X})$.
(b) If $H^1(C, N_{C/X}) = 0$ then every first-order deformation extends to a true deformation.
Of course these statements are true in much more general situations.

The crux is this: if $C \subset X$ is a smooth curve then $N_{C/X}$ is a vector bundle, and the fibre of $N_{C/X}$ at a point $p \in C$ is just the space T_pX/T_pC of tangent directions pointing out of the curve. If $C \subset X$ is nodal it is still true that $N_{C/X}$ is a vector bundle, but there is no good geometric description of the fibre of $N_{C/X}$ at a node. But there is a description of the sheaf of sections. To give this description let us look at the case of two smooth curves C_1 and C_2 meeting transversely in a single point p which is then the unique node of $C := C_1 \cup C_2$:



Once we understand this case, the general case can easily be understood as well.

We want to compare the normal bundle of C with the normal bundles $N_{C_1/X}$ and $N_{C_2/X}$. Let us compare $N_{C/X}|_{C_1}$ with $N_{C_1/X}$. Away from p they agree. The sections of $N_{C/X}|_{C_1}$ are going to be described as sections of $N_{C_1/X}$ with some singularity at p.

Let us make the statement and then draw some conclusions. The basic fact is that the sections of $N_{C/X}|_{C_1}$ correspond to the sections of $N_{C_1/X}$ with possibly a simple pole at p in direction of T_pC_2.

3.2 Notation. In general, let E be a vector bundle on a curve C, consider a point $p \in C$, and let $\xi \subset E_p$ denote a 1-dimensional subspace. Define $E(\xi)$ to be the sheaf of sections of E with at most a simple pole at p in direction ξ. (Note that $c_1(E(\xi)) = c_1(E) + 1$.)

Returning to the case of $C = C_1 \cup C_2$, we can now formulate the result as

$$N_{C/X}|_{C_1} = N_{C_1/X}\big((T_pC_1 + T_pC_2)/T_pC_1\big),$$

and a section $\sigma \in H^0(C, N_{C/X})$ smoothes the node at p if and only if its restriction to C_1 is not a global section of $N_{C_1/X}$. (Similarly of course for C_2.)

Taking the view-point of C_1, we see that when we attach the curve C_2 we increase the normal bundle: the first Chern class is incremented by 1. If we attach sufficiently many curves, we achieve ampleness (precisely we kill H^1).

3.3 Lemma. *Let E be a vector bundle on a curve C. For any n there exists m such that for any m general points $p_1, \ldots, p_m \in C$ and m general 1-dimensional subspaces $\xi_i \subset E_{p_i}$, then for any divisor $D \subset C$ of degree n we have*

$$H^1(C, E'(-D)) = 0,$$

where $E' := E(\xi_1 + \cdots + \xi_m)$.

Returning to our rationally connected variety X we can now draw the conclusion: If X is rationally connected and $C \subset X$ is any curve (even completely rigid), we can loosen it up: we can attach rational curves $C_1, \ldots, C_m \subset X$ such that the union $C \cup C_1 \cup \cdots \cup C_m$ can be deformed to some curve C' with ample normal bundle — in practice what we obtain is $H^1(C', N_{C'/X}) = 0$.

3.4 Homework. Consider a smooth plane space curve $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ of degree d. Now attach general lines $L_1, \ldots, L_m$ to it. The question is: when can $C \cup L_1 \cup \cdots \cup L_m$ be smoothened? (Hint: see if the normal bundle has global sections...

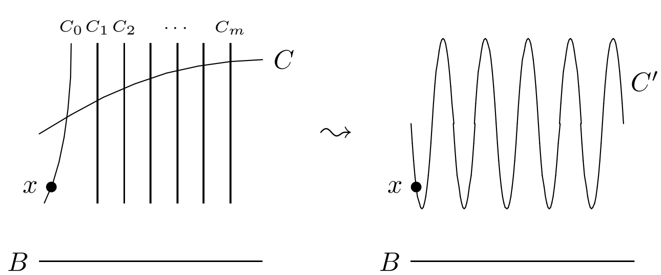
As an application of these ideas, let us prove the statement made in 2.6 about the density of the rational points—or as we shall put it geometrically: that for rationally connected varieties, the existence of one section implies there are lots of sections.

3.5 Lemma. *Suppose $f : X \to B$ has rationally connected fibres, and let $C \subset X$ be a section. Then through every point $x \in X$ there exists a section of f.*

Proof. Since the fibre containing x is rationally connected, there is a smooth rational curve C_0 in it with ample normal bundle, joining x to C. In m other fibres, attach to C a smooth rational curve C_i with ample normal bundle—again this is possible because the fibres are rationally connected. For m sufficiently big, by Lemma 3.3, the union

$$\widetilde{C} = C \ \cup \ C_0 \cup C_1 \cup \cdots \cup C_m$$

can be deformed into a smooth curve C', while keeping x fixed:



Precisely, since $N_{\widetilde{C}/X}$ is generated by global sections we can find such a first-order deformation, and since furthermore we have $H^1(\widetilde{C}, N_{\widetilde{C}/X}) = 0$, it follows that the first order deformation actually extends to a true deformation. The new curve C' will still be a section since its intersection with each fibre is still 1: all the attached curves C_i have $F \cdot C_i = 0$ for a general fibre F. □

4 Proof of the fibration theorem

Given a map $\pi : X \to B$ where B is a smooth curve and the general fibre is rationally connected, we want to produce a section. (This will prove Theorem 2.3, and thus the fibration theorem.) For simplicity we assume that X is smooth.

First we do it for $B = \mathbb{P}^1$, but we continue to call it B since there are too many other rational curves in play... Afterwards we'll show the general statement follows from this case.

Program: our starting point will be any curve $C \subset X$, possibly of large degree over B. Then we want to degenerate it into a union of curves of degree 1 over B, as in the figure on page 13.

To use deformation theory we need some ambient parameter space of curves in X. We could use the Hilbert scheme or the Chow variety, but the space we will use is the Kontsevich space: $\overline{M}_g(X, \gamma)$ is the space of stable maps $f : C \to X$, where C is a nodal curve of genus g, and $f_*[C] = \gamma \in H_2(X)$. The stability requirement is that the automorphism group of f is finite (in other words there is a finite number of automorphisms of C compatible with the map f). In the case where the target space is just a curve B, we simply write $\overline{M}_g(B, d)$ for the space of stable maps of degree $d[B]$.

So we will start at some moduli point in $\overline{M}_g(X, \gamma)$ and wander around until we get to a curve of the desired type. We know how to move a curve around, but we don't know how to degenerate it. But for the spaces $\overline{M}_g(B, d)$ of branched coverings of a curve, we know a lot (and a lot has been known for a hundred years): For $B = \mathbb{P}^1$, the space $\overline{M}_g(B, d)$ has a unique component whose general point corresponds to a simply branched cover $C \to B$, and this component does contain the sort of curves we are looking for, namely $f : C \to B$ whose restriction to each irreducible component C_i is of degree at most 1.

We can relate to this knowledge: if we let d be the intersection number of γ with a fibre of π, then there is a morphism

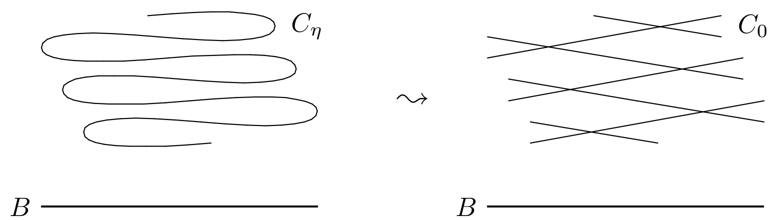
$$\overline{M}_g(X, \gamma) \to \overline{M}_g(B, d)$$

which is given by composition with π and stabilising components that become unstable (i.e. components contained in a fibre of π). So we need a locus in $\overline{M}_g(X, \gamma)$ dominating the good component of $\overline{M}_g(B, d)$.

4.1 Lemma-definition. Let $C \subset X$ be a smooth curve of genus g and degree d over B, and only simply branched over B, and let γ be the homology class of C. Then the corresponding component of $\overline{M}_g(X, \gamma)$ dominates the good component of $\overline{M}_g(B, d)$. We call such curves *flexible*.

Another characterization of flexible curves is that their deformations correspond to deformations of the branch points: the flexible curves $C \subset X$ are those which are simply branched over B and whose branch points can be deformed freely and independently in arbitrary directions.

Since the component in $\overline{M}_g(X, \gamma)$ of flexible curves dominates the good component of $\overline{M}_g(X, \gamma)$, in particular any flexible curve can be degenerated into a union of sections (and possibly some vertical curves).



So our theorem will be proved if we can produce a flexible curve in X.

4.2 First construction. Start with any smooth curve $C \subset X$, of degree d over B; attach rational curves in the fibres, each with ample normal bundle, and each meeting C transversely in one point. If we attach sufficiently many such curves, the normal bundle of the union $C \cup C_1 \cup \cdots \cup C_m$ will be generated by global section and with vanishing first cohomology, so by the results in Section 3 there is a smooth deformation C'. This new curve has the same genus as the original curve, and the same degree over B (since the attached curves are just vertical rational curves).

Now by Lemma 3.3 we have $H^1(N_{C'/X}(-D)) = 0$ for any divisor D of degree n on C': any n points on C' move in arbitrary directions: find a section of the normal bundle which is zero at all branch points except x: this gives a way to move x (so take n to be the number of branch points). So we can move all of them, one by one. The branch points move independently...

This only applies when the branch points are at smooth points of the fibres! If there is a double fibre, then curves through this fibre cannot be deformed... In fact, multiple fibre are the only obstruction to deforming.

Definition. Say that a a curve $C \subset X$ is *preflexible* if $C \to B$ is simply branched and all ramification points are at smooth points of fibres of π.

Since in any case a curve meeting $\mathrm{Sing}(\pi)$ will have a branch point there, we can say equivalently that a preflexible curve is just a curve simply branched over B and disjoint from $\mathrm{Sing}(\pi)$.

Construction 1 implies that any preflexible curve can be made flexible: attach rational vertical curves and smoothe. We already argued that a flexible curve can be degenerated into a union of sections, so the crux is to produce preflexible curves.

What happens if we start with a general curve? embed X in some big $\mathbb{P}^N$; intersect with n general hypersurfaces to get a smooth curve $C \subset X$. It will be simply branched over B away from the singular points of the fibres. By choosing those hypersurfaces generally enough we can make C miss any codimension-2 locus. The locus of isolated singular points of fibres is clearly of codimension at least 2, so we can produce curves avoiding those singularities. The problem is the fibres which have a multiple component.

Let M denote the subset of B consisting of those points $b_1, \ldots, b_m$ such that the fibre X_{b_i} has multiple component. Then C will be simply branched over B with ramification at smooth points of the fibres, away from $\pi^{-1}(M)$.

This is what we have to deal with! How to get rid of those extra bad branch points!

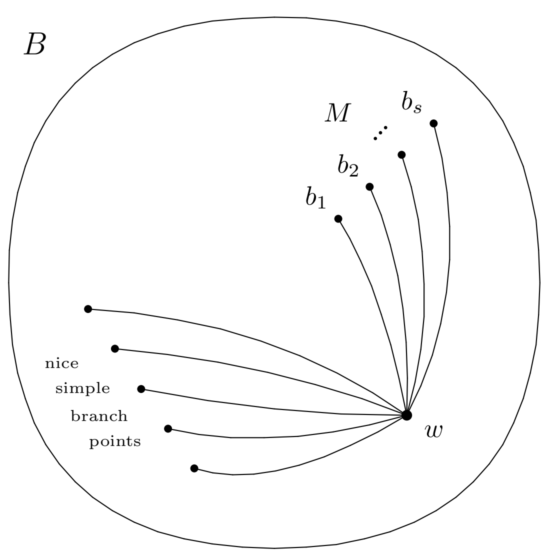
In order to do that we have to modify the first construction slightly to get a more general construction.

4.3 Second construction. Start with a smooth curve $C \subset X$. Pick a fibre X_0 and two points p and q in the intersection $C \cap X_0$. Take a rational curve $C_0 \subset X_0$ which passes through both p and q and does not intersect C in other points (we can assume that these intersections are transverse). Now we also attach rational curves $C_1, \ldots, C_m$ in other fibres, just as in Construction 1. All this produces a curve C' which is the smooth deformation of the union $C \cup C_0 \cup C_1 \cup \cdots \cup C_m$. The normal bundle of this new curve $N_{C'/X}$ has lots of global sections, with preassigned zeros at p and q. The degree of C' over B is the same as the degree of the original curve C, because we only attached vertical curves. But the genus has augmented by 1. We can easily see this by the Hurwitz-Riemann formula: the smoothening introduces two new branch points p and q with simple monodromy just exchanging the sheets near p and q. Now since $N_{C'/X}$ is nice, the new branch points will be independent too.

Now we are ready for the

4.4 Proof of the fibration theorem. Draw an analytical picture of B. There are some nasty points $b_1, \ldots, b_s$ corresponding to fibres with multiple components, and there are a lot of other ramification points, which are all simple and which can be moved around freely by deforming C.

Choose a base point w and draw a (real) arc from w to each of the ramification points, (in a way so they don't intersect—except at w of course).



Now the complement of the union of all the arcs has no monodromy, so we can label the sheets. Thus we can identify the monodromy of each branch point as a permutation of the sheets: at the simple branch points the permutation will be simple while at the bad points the permutation will be complicated. (For example at a branch point corresponding to an m-fold fibre such that C meets the support with multiplicity k, the permutation will be a k disjoint m-cycles…)

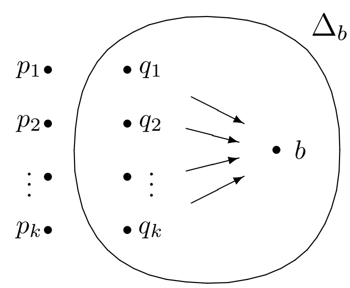
We will now treat one of the bad branch points, which for simplicity we just call b. In fact what we do should be done simultaneously for all the bad points. Let S_d denote the symmetric group on d letters, and let $\tau \in S_d$ be the monodromy of b. Write τ as a product of transpositions

$$\tau = \tau_1 \dots \tau_k.$$

For each of these transpositions τ_i we are going to create a new pair of branch points p and q (using Construction 2); each of these points will have monodromy τ_i, and one of them we will move into b in order to annihilate the monodromy at b.

Precisely, draw a small disc Δ_b which contains b and all the q_i's corresponding to the transpositions τ_i, but not the p_i's (and neither any of the other ramification points). Since the monodromy of q_i is τ_i, the total monodromy around Δ_b is trivial.

Now recall that all the simple ramification points can be moved around independently, including the new ones created by Construction 2. Fix all the original branch points and fix the p_i's but move the q_i's to b. In other words, deform the curve C in such a way that all the original branch points are fixed, as well as the p_i's, and such that each q_i moves into b.



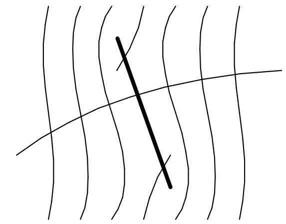
Now take the limit C_0 of this family and see what it looks like. There may be (in fact will be) vertical components, but we can just discard those. The rest will be a curve over Δ_b whose only possible ramification is over b, but since all together there is no monodromy around Δ_b in fact there can be no ramification over b either! So the new curve in X is unramified over Δ_b.

Recalling that we really do this simultaneously for every b in M, this completes the proof of the fibration theorem in the base of $B = \mathbb{P}^1$.

4.5 Remark. Now the above procedure does not modify the total space X, so if there were a everywhere nonreduced fibre it would still be there afterwards. So what we have actually proved is that *there can be no everywhere multiple fibre!*

This apparently is a contradiction: the original curve had positive intersection with the multiple fibre, whereas the new curve C_0 has intersection zero! The answer to this problem is that we have split off vertical components!

4.6 Example. There is a concrete example (given in great detail in [4]) of a family of curves whose special fibre is the union of three curves, the middle one of which has multiplicity 2, and a curve C which intersects this nonreduced component:



(So while in the picture C looks like a section it is actually a double cover!) Now move C up (or down) such that its intersection with the central fibre approaches one of the reduced components, say X'. When it hits X' it will split it off as a component.

So as a consequence of these considerations we have:

4.7 Corollary. *Suppose the general fibre of $\pi : X \to B$ is rationally connected, and suppose X is smooth. Then every fibre has a smooth point. (In other words, there are no everywhere multiple fibres.)*

4.8 Question. Is there a way to prove this directly?

4.9 Homework, and solution. *Let $S \to B$ be a 1-parameter family of curves, where B and S are smooth. If the fibre $S_0 = \sum_{\alpha=1}^{k} m_\alpha C_\alpha$ is a rational curve with k components, then there is at least one m_α equal to* 1. Proof: Since S_0 is a fibre we have $S_0^2 = 0$, and similarly $C_\beta \cdot S_0 = \sum_\alpha m_\alpha(C_\alpha \cdot C_\beta) = 0$. Now suppose $k \geq 2$. Then each component C_α must have negative self-intersection. Now by adjunction for C_α, we must have $K_S \cdot C_\alpha \geq -1$. On the other hand we have adjunction for S_0:

$$\sum_\alpha m_\alpha(K_S \cdot C_\alpha) = -2$$

so at least one of the components, say C_k, must actually have

$$K_S \cdot C_k = -1$$

and thus (by adjunction), $C_k^2 = -1$. So now we can blow down this curve, and the resulting surface will still be smooth, and we haven't changed any of the multiplicities. This gives an induction on the number of components, so we reduce to the case where $k = 1$: there is only one irreducible component, $S_0 = m_1 C_1$.

This case is easy: adjunction on the fibre yields $m_1(K_S \cdot C_1) = -2$ and adjunction on C_1 gives $K_S \cdot C_1 = -2$, so $m_1 = 1$.

This completes the proof of the case $B = \mathbb{P}^1$. What about general B?

In proving the case $B = \mathbb{P}^1$ the place where we used genus zero was when we invoked the irreducibility of the Hurwitz scheme. The corresponding result is false in higher genus. For example, by taking some unramified covers, you can easily construct a covering which does not admit a degeneration into a union of sections...

There are two ways to get around this. The first way is using a general result about the moduli spaces of higher genus coverings. The second way is easier, and depends on a trick of Johan de Jong.

4.10 First way. One can prove that for any B, if the genus g is sufficiently high, then there is a unique component of $\overline{M}_g(B, d)$ whose general point corresponds to a simply branched cover whose monodromy is the full symmetric group. We can use this result, since we can always increase the genus (by Construction 2), and we can always increase the monodromy group...

4.11 Second way. Given the family $\pi : X \to B$, choose randomly a branched covering $f : B \to \mathbb{P}^1$. Now consider the norm variety $Y := \mathrm{Norm}_f(X)$ (the geometric equivalent of the notion of norm for a finite extension of number fields (or function fields)): the fibre of $Y \to \mathbb{P}^1$ over a point p is the product of all the inverse images under π, precisely

$$Y_p = \prod_{q \in f^{-1}(p)} X_q.$$

Now the observation is that if $Y \to \mathbb{P}^1$ has a section then also $X \to B$ has a section.

Now the full argument runs like this: if the general fibre of $X \to B$ is rationally connected then also the general fibre of $Y \to \mathbb{P}^1$ is rationally connected, and then we know it has a section. Finally by the last observation, this section induces a section of $X \to B$.

5 A converse of the fibration theorem

Let us return to the question whether there is a larger class of varieties for which the statement of Theorem 2.5 holds. In particularly we can try to weaken the requirement that the base B be of dimension 1, and ask:

5.1 Question. Is it true that every map $X \to B$ whose general fibre is rationally connected admits a section? for example in the case where B is a smooth surface. The answer is no, and a counter example is provided by the cubic threefold V: V is birational to a conical bundle $X \to \mathbb{P}^2$ (cf. 1.9), but this bundle has no sections—if it had then the total space would be rational...

But here is a sort of converse statement.

5.2 Theorem. (Graber-Harris-Mazur-Star) *Let $X \to B$ be a dominant map with B of any dimension. If for sufficiently general (or all) curves $C \to B$ the pullback $X_C \to C$ has a section, then for a subvariety $Z \subset X$ the general fibre of Z over B is rationally connected.*

As an application of this result we are going to prove the existence of a 1-parameter family of Enriques surfaces without a section.

5.3 Homework. Try to construct this explicitly. The question can be posed like this: does there exist a family $X \to B$ with X smooth and where B is a curve, whose general fibre is an Enriques surface and with a fibre which is everywhere nonreduced?

5.4 Recall that an Enriques surface is a surface S with $p_g = p_a = 0$ and $2K_S = 0$. The classical example is the normalization of a sextic surface in $\mathbb{P}^3$ with six double lines forming a tetrahedron. More generally, Enriques surfaces can be obtained as quotients of K3 surfaces by a fixpoint free involution (when such exists. . .).

6 %Structure theorem for rationally connected varieties

7 Open problems

We mentioned in the first lecture the embarrassing question of whether there exists rationally connected varieties which are not unirational. In order to get closer to an answer to that question, let us ask what is different about the two notions, in particular: what properties do unirational varieties have that we don't expect for rationally connected varieties?

One thing is this: By definition, a variety X of dimension n is unirational if it admits a dominant generically finite rational map $\mathbb{P}^n \dashrightarrow X$. It follows that X contains rational subvarieties of any dimension $\leq n-1$. (Indeed, a general hyperplane will map one-to-one to X.)

On the other hand, while rational connectivity implies the existence of lots of rational curves there is no guarantee for the existence of surfaces!

7.1 Question: Do rationally connected varieties necessarily contain rational surfaces? — and of course we could ask similarly if they contain higher dimensional rational varieties?

Probably the answer is no, and then we would know that rational connectivity is not the same as unirationality. Why should the answer be no? On reason is a simple dimension count, similar to the one we have seen in some of the homeworks. . .

7.2 Example of such a dimension count. Do we expect a generic hypersurface $X \subset \mathbb{P}^n$ of degree $n \geq 5$ to contain rational surfaces?

To see how it works let us first go back and have a look again at the question of when a hypersurface $X \subset \mathbb{P}^n$ is expected to contain rational curves. The space of degree-e rational curves in $\mathbb{P}^n$ is of dimension $(n+1)(e+1)-4$. It is given by an $(n+1)$-tuple of homogeneous polynomials of degree e, say $[s_0, \ldots, s_n]$. This curve lies inside X when the polynomial F defining X vanishes: $F(s(t)) = 0$. This polynomial is of degree de, so there are $de+1$ coefficients to kill. So the expected dimension of the family of rational curves on X is the difference between these two numbers,

$$(n+1)(e+1)-4 \; - \; (de+1)$$

For $d \leq n$ this number is always positive... (For $d = n+1$, the number is positive but not enough to fill the space... perhaps there are only some few rational curves of low degree e...)

Now let us do the same count for maps $f : \mathbb{P}^2 \to \mathbb{P}^n$. Say of degree e in the sense that $f^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{O}_{\mathbb{P}^2}(e)$. The dimension of the space of such maps is

$$(n+1)\binom{e+2}{2} - 9$$

and the number of conditions to make the image land inside X is

$$\binom{de+2}{2}.$$

Let us just look at the inequality asymptotically: the first is $\sim \frac{n+1}{2}e^2 + O(e)$ while the number of conditions goes like $\frac{d^2}{2}e^2 + O(e)$. So we expect a general hypersurface of degree d to contain images of maps of type $\mathbb{P}^2 \to \mathbb{P}^n$ when $d^2 \leq n$. (Again for $d^2 = n+1$ there might be some but not enough??)

Now there are three reason for this argument not being convincing. First of all it treats only rational surfaces which are the image of a $\mathbb{P}^2$; we ought to do the same dimension count also for blow-ups of $\mathbb{P}^2$, etc.

Second, how about singular images?, how can we exhaust all possibilities?

And finally, the most important draw-back of all this is that anyway it is nothing than a dimension count, and does not prove the existence or non-existence of anything...

A second approach to the problem is this: why not take a 1-parameter family of rational curves in X and let it sweep out a surface? In other words, find a rational curve in the parameter space of all rational curves! So, for X smooth and irreducible, look at the space of maps

$$M = M_0(X, \gamma)$$

and ask for rational curves in here.

Note that there is no bar over this space: we only consider maps whose source is $\mathbb{P}^1$. (The Kontsevich space where we allow reducible source curve may have more than one irreducible component...) So we can say: M contains no rational curves if and only if X contains no rational surfaces.

Or in the other direction: could M itself be rationally connected? And if not, what is the mrc quotient of M?

There is an interesting parallel between these questions and homotopy theory. In homotopy theory the first question is whether two points can be connected, and whether a given space is connected; next one asks whether it is simply connected—this is equivalent to asking whether the loop space is connected—and so on: Each homotopy group π_k is the set of connected components of loop spaces of loop spaces of loop spaces...

Topology	Algebraic geometry	$X_d \subset \mathbb{P}^n$	$X(K) \neq \emptyset$
connected	X rationally con.	$d \leq n$	tr. deg. ≤ 1
simply connected	$M_0(X, \gamma)$ rat. con.	$d^2 \leq n$	tr. deg. ≤ 2
$\vdots$			

Note that the space $M_0(X, \gamma)$ depends on the degree γ, so it is not really an invariant of X, but one can show that its rational connectivity properties stabilizes for high γ...

7.3 Theorem. *(Harris, Starr.) Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. If $d^2 \geq n$ then $M_0(X, e)$ is rationally connected for all e.*

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