Spherical cone structures on 2-bridge knots and links

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Abstract

We study spherical cone structures on 2-bridge knots and links. It is known that such structures exist for cone angle $\alpha \in (\alpha_0, \pi]$, and that they become Euclidean when α approaches α_0. Here we prove that these structures exist for cone angle $\alpha \in [\pi, 2\pi - \alpha_0)$. When L is a hyperbolic link and $\alpha \to 2\pi - \alpha_0$, the singular locus crosses transversally with itself along the tunnels. When L is a torus link, the crossing is not transverse and it is described by means of the Seifert fibration.

1 Introduction

A 2-bridge link $L \subset S^3$ is a link of one or two components such that the pair (S^3, L) is a union of two trivial tangles T_1 and T_2 along the boundary, where each T_i is a pair consisting of a ball and two unknotted arcs. Each tangle T_i has a tunnel τ_i, which is a path from one arc to the other, so that the union of τ_i with the arcs has a letter H shape.

A spherical cone structure on (S^3, L) is a metric on S^3 singular on L, which is a smooth Riemannian metric on $S^3 - L$ of constant curvature $+1$, and on a neighborhood of each point of L has the expression in cylindrical coordinates:

$$d\,r^2 + (\frac{\alpha}{2\pi}\sin r)^2\, d\,\theta^2 + \cos^2 r\; d\,h^2$$

where $\alpha > 0$ is the cone angle, $r > 0$ denotes the distance to the singular locus, θ is the angle parameter and h is the height or length parameter of L. A similar definition applies to hyperbolic and Euclidean cone structures, by replacing trigonometric functions by hyperbolic trigonometric and affine ones respectively.

A two bridge link is either hyperbolic or a torus link. By [1], [8], [6] and [10], in the hyperbolic case, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ depending on L such that (S^3, L) has a unique cone structure with cone angle α of the following kind:

- Euclidean for $\alpha = \alpha_0$,
- hyperbolic for $\alpha \in (0, \alpha_0)$, and
- spherical for $\alpha \in (\alpha_0, \pi]$.

When $\alpha = \pi$, this structure corresponds to an orbifold doubly covered by a lens space.

By a deformation argument, the cone angle α can be increased slightly beyond π by keeping the spherical structure, and we may ask till which angle it can be increased. The main theorem of this paper shows that it can be increased up to $2\pi - \alpha_0$, and at this angle the singular locus intersects with itself along the tunnels, whose length converges to zero. This is one of the few explicit examples of accident for cone angles larger that π.

Theorem 1.1 *Let $L \subset S^3$ be a hyperbolic two-bridge link and let $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ be the Euclidean angle as above. For every $\alpha \in [\pi, 2\pi - \alpha_0)$ there exist a spherical cone structure on (S^3, L) with cone angle α, denoted by $C(\alpha)$.*

As $\alpha \to 2\pi - \alpha_0$, $C(\alpha)$ converges to the suspension of a sphere with four cone points, and the length of the tunnels converge to zero.

For the figure eight knot, this family of spherical cone structures has been constructed by Mednykh and Rasskazov [7]. Their construction uses fundamental polyhedra, and they obtain an explicit formula for the volumes.

In Section 2 we illustrate the phenomenon of collapse after π for toric two-bridge links, complementary to Theorem 1.1. In this case the singular locus intersects with itself in a different way from the hyperbolic case, not in transverse directions but in parallel ones, so that for a link the two components become one, and for a knot the singular locus is mapped onto a circle by a map of degree 2.

The proof of Theorem 1.1 has two main steps. Firstly, in Section 3 we explicitly construct a family of representations. Secondly, the argument to show that these are in fact holonomy representations is a standard connectedness argument, the only difficult think is to prove that the singular locus does not cross with itself when we deform (i.e. that there is a tube of the singularity with radius bounded below away from zero). This is proved in Section 4 by means of a volume estimate. Finally in Section 5 we analyze the crossing of the singular locus.

2 Two bridge torus links

We first discuss the toric case, as complementary to the hyperbolic one, to illustrate a different kind of collapse of spherical structures.

A two bridge knot is toric if it is the regular fiber of a Seifert fibration of S^3 with singular fibers of order 2 and n, with $n > 1$ odd. As torus knot, it is denoted by $t(2, n)$. A 2-component link is toric and has two bridges when it is the union of two regular fibres of a Seifert fibration of S^3 with a single singular fibre of order $m > 1$, and as torus link it is denoted by $t(2, 2m)$.

Using the Seifert fibration one can easily construct geometric cone structures on S^3 with singular locus the torus knot $t(2, n)$. For $\alpha < \pi - \frac{2\pi}{n}$ there is a $\widetilde{SL_2(\mathbf{R})}$ structure, and for $\alpha = \pi - \frac{2\pi}{n}$ there exists a Nil one. The same assertion holds for the torus link $t(2, 2m)$ for angle $\pi - \frac{\pi}{2m}$.

Proposition 2.1 *Let $L = t(2, n)$ be a torus knot ($n > 1$ odd). For every $\alpha \in (\pi - \frac{2\pi}{n}, \pi + \frac{2\pi}{n})$ there exist a spherical cone structure on (S^3, L) with cone angle α, denoted by $C(\alpha)$.*

In addition, as $\alpha \to \pi + \frac{2\pi}{n}$, the singular locus collapses by a map of degree 2, in the same way as in a Seifert fibration a regular fiber approaches a singular fiber of order 2. The result is a suspension of a sphere with two cone points of angle $\frac{2\pi}{n}$.

Proposition 2.2 *Let $L = t(2, 2m)$ be a toric link with two components, with $m > 1$. For every $\alpha \in (\pi - \frac{\pi}{m}, \pi + \frac{\pi}{m})$ there exist a spherical cone structure on (S^3, L) with cone angle α, denoted by $C(\alpha)$.*

In addition, as $\alpha \to \pi + \frac{\pi}{m}$, the components of the singular locus collapse to a single one, and the limit is a suspension of a sphere with two cone points of angle $\frac{2\pi}{m}$.

Propositions 2.1 and 2.2 can be easily proved using the Seifert fibration and analyzing the behavior of the basis. We do not give details of the proof, just mention few remarks about the geometry of spheres with three cone points.

Firstly, a spherical surface with three cone points is rigid. Here the angles of the three cone points are $\frac{2\pi}{n}$, π and α for knots and $\frac{2\pi}{m}$, α and α for 2-components links.

Secondly, in the knot case, when $\alpha \to \pi + \frac{2\pi}{n}$, the cone point of angle α approaches the one of cone angle π, and the result is a single cone point of cone angle $\left(\pi + \frac{2\pi}{n}\right)+\pi-2\pi = \frac{2\pi}{n}$. Thus the limit is a sphere with two cone points of angle $\frac{2\pi}{n}$. Similarly, in the link case, when $\alpha \to \pi + \frac{\pi}{m}$ both cone points of angle α approach, and the limit is a single cone point of angle $2(\pi + \frac{\pi}{m}) - 2\pi = \frac{2\pi}{m}$.

3 The variety of representations

In order to deform the cone structures, we work with incomplete spherical structures on $M = S^3 - L$. In addition, instead of $SO(4)$, we work in $Spin(4)$, which is isomorphic to the product $SU(2) \times SU(2)$, as every holonomy representation lifts to $Spin$ [3]. Thus we work with pairs of conjugacy classes of representations in $SU(2)$, i.e. points in

$$X(M, SU(2)) \times X(M, SU(2)),$$

where $X(M, SU(2))$ denotes the variety of characters of $\pi_1 M$. Elements in $X(M, SU(2))$ are conjugacy classes of representations.

For every $\gamma \in \pi_1 M$ we consider the map

$$\begin{array}{rcl} I_\gamma : X(M, SU(2)) & \to & \mathbf{C} \\ [\rho] & \mapsto & \mathrm{trace}(\rho(\gamma)) \end{array}$$

that can be used to describe the action of $(\rho_1(\gamma), \rho_2(\gamma)) \in SU(2) \times SU(2)$ as isometry in $SO(4)$. For instance $(\rho_1(\gamma), \rho_2(\gamma)) \in SU(2) \times SU(2)$ is a rotation of angle α iff $I_\gamma(\rho_1) = I_\gamma(\rho_2) = \pm 2 \cos\frac{\alpha}{2}$. Thus, if $\mu_1 \in \pi_1 M$ is a representative of a meridian (μ_1 and μ_2 for a 2-components link), we are interested in pairs of representations

$$([\rho_1], [\rho_2]) \in X(M, SU(2)) \times X(M, SU(2)) \quad \text{such that} \quad I_{\mu_i}([\rho_1]) = I_{\mu_i}([\rho_2]).$$

The existence of an Euclidean structure with angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ is proved in [8] an appendix of [1]. Then the existence of the spherical cone structures in $(\alpha_0, \pi]$ is proved in [6]. In particular it follows from [10] and [6] that there is an explicit description of a subset $\mathcal{C} \subset X(M, SU(2))$ corresponding to those structures.

The curve $\mathcal{C}$. Let $\rho_{\alpha_0} : \pi_1 M \to SU(2)$ be a lift of the rotational part of the Euclidean holonomy at cone angle α_0 (by [3] such a lift exists). We know that $I_{\mu_i}(\rho_{\alpha_0}) = \pm 2 \cos\frac{\alpha_0}{2}$, and we may assume that $I_{\mu_i}(\rho_{\alpha_0}) = 2 \cos\frac{\alpha_0}{2}$, up to changing some signs.

Proposition 3.1 *The connected component $\mathcal{C} \subset X(M, SU(2))$ of $I_\mu^{-1}[0, 2\cos\frac{\alpha_0}{2}]$ (for a link we add the restriction that $I_{\mu_1} = I_{\mu_2}$) is diffeomorphic to an interval.*

In addition the restriction $I_\mu|_{\mathcal{C}} : \mathcal{C} \to [0, 2\cos\frac{\alpha_0}{2}]$ is onto. The fiber of each point in $[0, 2\cos\frac{\alpha_0}{2})$ has two elements and the fiber of $2\cos\frac{\alpha_0}{2}$ has a single one.

This proposition is proved in [6]. For each $\alpha \in [\alpha_0, \pi]$ the elements of the fibre $I_\mu^{-1}(2\cos\frac{\alpha}{2})$ are denoted by $[\rho_\alpha^\pm]$, so that each branch $[\rho_\alpha^+]$ and $[\rho_\alpha^-]$ is continuous on α and $[\rho_\alpha^+] = [\rho_\alpha^-]$ iff $\alpha = \alpha_0$. In [6] it is proved that for $\alpha \neq \alpha_0$, the pair $(\rho_\alpha^+, \rho_\alpha^-)$ is the holonomy representation of the spherical cone structure with cone angle α. Cf. Figure 1

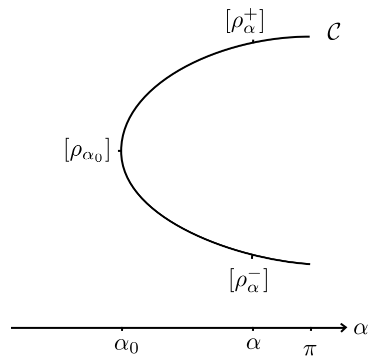

Figure 1: The curve $\mathcal{C}$ from Proposition 3.1.

The involution. To construct more representations, we use an involution on $X(M, SU(2))$. Let $\theta : \pi_1 M \to \mathbf{Z}/2\mathbf{Z}$ be the unique surjection such that $\theta(\mu_i)$ is not trivial for each meridian μ_i. Given $\rho\colon \pi_1 M \to SU(2)$, we consider the representation $\iota(\rho)$ defined as:

$$(\iota(\rho))(\gamma) = (-1)^{\theta(\gamma)}\rho(\gamma), \qquad \forall \gamma \in \pi_1 M.$$

Thus ι induces an involution on $X(M, SU(2))$, also denoted by ι.

We shall show that the conjugacy classes of $\rho_\pi^\pm$ are invariant by ι, and we shall construct our representations in $\iota(\mathcal{C})$.

Lemma 3.2 *The conjugacy classes $[\rho_\pi^\pm]$ are invariant by ι.*

Proof: It suffices to check that $I_\gamma([\rho_\pi^\pm]) = I_\gamma(\iota[\rho_\pi^\pm])$ for every $\gamma \in \pi_1 M$, as the conjugacy class of a representation in $SU(2)$ is determined by the value of its traces. For $\gamma \in \ker\theta$ the equality is obvious. The representations $\rho_\pi^\pm$ are binary dihedral, which means that under the natural projection $SU(2) \to SO(3)$ they project to a dihedral one. In particular, if $\theta(\gamma) \neq 0$, then $\rho_\pi^\pm(\gamma)$ projects to a rotation of angle π in $SO(3)$ (as every element in $O(2) - SO(2)$). Thus if $\theta(\gamma) = 0$ then $\mathrm{trace}(\rho_\pi^\pm(\gamma)) = \pm 2\cos\frac{\pi}{2} = 0$, and the equality is proved. $\square$

Thus $\iota(\mathcal{C})$ can be connected to $\mathcal{C}$ along $[\rho_\pi^\pm]$. In addition we define

$$[\rho_\alpha^\pm] = \iota[\rho_{2\pi-\alpha}^\pm] \qquad \forall \alpha \in [\pi, 2\pi - \alpha_0].$$

Let λ denote the translation length of the holonomy of a peripheral element (it is zero for meridians and the length of the singular locus for longitudes).

Lemma 3.3 *For every $\alpha \in [\pi, 2\pi - \alpha_0]$, $(\rho_\alpha^+(\mu_i), \rho_\alpha^-(\mu_i))$ is a rotation of angle α.*

In addition:

(a) If L is a knot and l represents a longitude, then

$$\lambda(\rho_\alpha^+(l), \rho_\alpha^-(l)) = 4\pi - \lambda(\rho_{2\pi-\alpha}^+(l), \rho_{2\pi-\alpha}^-(l)).$$

(b) If L has 2 components, and l_1 and l_2 represent the respective longitudes, then

$$\lambda(\rho_\alpha^+(l_i), \rho_\alpha^-(l_i)) = 2\pi - \lambda(\rho_{2\pi-\alpha}^+(l_i), \rho_{2\pi-\alpha}^-(l_i)).$$

Proof: The proof consists in choosing explicit representatives $\rho_\alpha^\pm$ of conjugacy classes and lift the involution ι to an involution $\tilde\iota$ in the variety of representations, so that $\tilde\iota(\rho_\pi^\pm)=\rho_\pi^\pm$.

We choose representatives $\rho_\alpha^\pm$ so that for $\alpha\in[\alpha_0,\pi]$,

$$\rho_\alpha^\pm(\mu_1)=\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2}\end{pmatrix}$$

In particular $\rho_\pi^\pm(\mu_1)=\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}$. We also choose $\rho_\pi^\pm$ so that its image is contained in the group generated by $\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}$ and $\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta\end{pmatrix}$, with $\beta\in\mathbb{R}$, so that $\rho_\pi^\pm(\gamma)$ has real coefficients iff $\gamma\in\ker\theta$.

The lift $\tilde\iota$ on the variety of representations of ι is:

$$\tilde\iota(\rho)(\gamma)=(-1)^{\theta(\gamma)}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\rho(\gamma)\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\qquad \forall\gamma\in\pi_1 M.$$

With this construction $\tilde\iota(\rho_\pi^\pm)=\rho_\pi^\pm$. In addition

$$\rho_{2\pi-\alpha}^\pm(\mu_1)=\tilde\iota\rho_\alpha^\pm(\mu_1)=\begin{pmatrix} -e^{-i\alpha/2} & 0 \\ 0 & -e^{i\alpha/2}\end{pmatrix}=\begin{pmatrix} e^{i(2\pi-\alpha)/2} & 0 \\ 0 & e^{-i(2\pi-\alpha)/2}\end{pmatrix}$$

Which proves the first assertion.

By commutativity, the image of a peripheral element γ is

$$\rho_\alpha^\pm(\gamma)=\begin{pmatrix} z_\alpha^\pm(\gamma) & 0 \\ 0 & 1/z_\alpha^\pm(\gamma)\end{pmatrix}$$

If $\lambda_\alpha(\gamma)$ denotes the translation length of $(\rho_\alpha^+(\gamma),\rho_\alpha^-(\gamma))$, then $e^{i\,\lambda_\alpha(\gamma)}=z_\alpha^+(\gamma)/z_\alpha^-(\gamma)$. From the explicit description of $\tilde\iota$, we have $z_{2\pi-\alpha}^\pm(\gamma)=(-1)^{\theta(\gamma)}/z_\alpha^\pm(\gamma)$. Hence

$$e^{i\lambda_{2\pi-\alpha}(\gamma)}=e^{-i\lambda_\alpha(\gamma)}$$

which means that $\lambda_{2\pi-\alpha}(\gamma)=2\pi k-\lambda_\alpha(\gamma)$, with $k\in\mathbf{Z}$. Statements (a) and (b) follow from continuity and the fact that, at angle $\alpha=\pi$, the length of the singular locus is 2π for a knot, and π for each component of the link. □

4 Volume estimations

In previous section we have constructed a family of representations in $Spin(4)\cong SU(2)\times SU(2)$. We can show that they are holonomy representations of spherical cone structures by a connectedness argument. For that we need to guarantee that the singular locus does not cross with itself.

By continuity, we know that there exists a family of cone manifolds $C(\alpha)$ with holonomy $(\rho_\alpha^+,\rho_\alpha^-)$ for $\alpha\in[\pi,\pi+\varepsilon]$. In the connectedness argument, we may assume that ε is any value in $(0,\pi-\alpha_0)$. Let $r(\alpha)$ denote the normal radius of the singular locus of $C(\alpha)$ (i.e. $r(\alpha)$ is the supremum of all r such that the metric tube of radius r of the singular locus of $C(\alpha)$ is embedded).

Lemma 4.1 *For $\alpha\in[\pi,\pi+\varepsilon]$, $\operatorname{vol}(C(\alpha))\le 2\pi r(\alpha)+2\pi(\alpha-\pi)$.*

Proof: Let σ be the shortest segment with end-points in the singular locus of $C(\alpha)$, which it is not itself contained in the singular locus. By construction, it is perpendicular to the singular locus and its length equals $2r(\alpha)$. We consider the Dirichlet domain centered at σ, i.e. the set of points with a unique minimizing segment to σ. By construction, this domain is star-shaped and has the same volume as $C(\alpha)$. In addition, it embeds isometrically in the union of the following pieces:

(i) A lens of with $2r(\alpha)$ (i.e. bounded by two planes with dihedral angles $2r(\alpha)$).

(ii) Four sectors (which are again lenses) of angle $(\alpha-\pi)/2$. The edge of these sectors is viewed as the singular locus, and those sectors share a face with the lens.

The volume of the first piece is $2\pi r(\alpha)$, and the volume of the second one is $4\pi(\alpha-\pi)/2$, hence the lemma is proved. □

Lemma 4.2 *For* $\alpha \in [\pi, \pi+\varepsilon]$, $\mathrm{vol}(C(\alpha)) = \mathrm{vol}(C(2\pi-\alpha)) + 2\pi(\alpha-\pi)$.

Proof: By Schläfli's formula, if $l(\alpha)$ is the length of the singular locus of $C(\alpha)$, we have:

$$\mathrm{vol}(C(\alpha)) = \frac{1}{2}\int_{\alpha_0}^{\alpha} l(\theta)d\theta = \frac{1}{2}\int_{\alpha_0}^{\pi} l(\theta)d\theta + \frac{1}{2}\int_{\pi}^{\alpha} l(\theta)d\theta.$$

On the second integral, we make the change of variable $w = 2\pi - \theta$. Notice that by Lemma 3.3, $l(w) = 4\pi - l(\theta)$. Thus $\mathrm{vol}(C(\alpha))$ equals

$$\begin{aligned}\frac{1}{2}\int_{\alpha_0}^{\pi} l(\theta)d\theta - \frac{1}{2}\int_{\pi}^{2\pi-\alpha}(4\pi - l(\omega))d\omega &= \frac{1}{2}\int_{\alpha_0}^{\pi} l(\theta)d\theta - \frac{1}{2}(\pi-\alpha)4\pi + \frac{1}{2}\int_{\pi}^{2\pi-\alpha} l(\theta)d\theta \\ &= \frac{1}{2}\int_{\alpha_0}^{2\pi-\alpha} l(\theta)d\theta + 2\pi(\pi-\alpha).\end{aligned}$$

Since $\mathrm{vol}(C(2\pi-\alpha)) = \frac{1}{2}\int_{\alpha_0}^{2\pi-\alpha} l(\theta)d\theta$, the lemma is proved. □

From Lemmas 4.1 and 4.2, we get:

Corollary 4.3 *For* $\alpha \in [\pi, 2\pi-\alpha_0)$,

$$r(\alpha) \geq \frac{1}{2\pi}\mathrm{vol}(C(2\pi-\alpha)).$$

Thus $r(\alpha)$ *is uniformly bounded for* α *is a compact subset of* $[\pi, 2\pi-\alpha_0)$.

Proof of the existence of structures in Theorem 1.1. Let $A = [\pi, \pi+\varepsilon) \subset [\pi, 2\pi-\alpha_0)$ be the subinterval such that for every $\alpha \in A$ there exists a spherical cone manifold structure $C(\alpha)$ on (S^3, L) with holonomy $(\rho_\alpha^+, \rho_\alpha^-)$. The set A is open, by the fact that deformation of the holonomy implies deformation of the structure. To prove that A is closed, we take a sequence $\alpha_n \in A$ converging to $\alpha_\infty \in [\pi, 2\pi-\alpha_0)$. We know that the diameter of $C(\alpha_n)$ is bounded above by π, and that $r(\alpha_n)$ is bounded below away from zero, by Corollary 4.3. From these bounds and using the arguments of [2], it follows that $C(\alpha_n)$ converges to $C(\alpha_\infty)$, a spherical cone structure on (S^3, L) with cone angle α_∞ and holonomy $(\rho_{\alpha_\infty}^+, \rho_{\alpha_\infty}^-)$. □

5 Crossing of the singular locus

Let $L_{\tau_i}(\alpha)$ denote the length of the tunnel τ_i at $C(\alpha)$. We first prove:

Lemma 5.1 $\lim_{\alpha \to 2\pi-\alpha_0} L_{\tau_i}(\alpha) = 0.$

Proof: The element t_i of the fundamental group representing τ_i is the product of two meridians, this means that $\theta(t_i) = 0$, where $\theta \colon \pi_1 M \to \mathbb{Z}/2\mathbb{Z}$ is the surjection of Section 3. Thus $(\rho^+_\alpha(t_i), \rho^-_\alpha(t_i))$ and $(\rho^+_{2\pi-\alpha}(t_i), \rho^-_{2\pi-\alpha}(t_i))$ are conjugate. In particular, $L_{\tau_i}(2\pi - \alpha) = L_{\tau_i}(\alpha) + 2\pi k$, with $k \in \mathbb{Z}$. Since $L_{\tau_i}(\pi) = \frac{\pi}{n}$, it follows that $L_{\tau_i}(2\pi - \alpha) = L_{\tau_i}(\alpha)$, hence $\lim_{\alpha \to 2\pi-\alpha_0} L_{\tau_i}(\alpha) = \lim_{\alpha \to \alpha_0} L_{\tau_i}(\alpha) = 0$. □

To understand the crossing of the singular locus, we start with the suspension of 4 cone points, and construct developing maps according to the deformation of the holonomy, being careful on the neighborhood of the tunnels.

Since $\rho^+_{2\pi-\alpha_0} = \rho^-_{2\pi-\alpha_0}$, $(\rho^+_{2\pi-\alpha_0}, \rho^-_{2\pi-\alpha_0})$ preserves a totally geodesic 2-sphere in S^3, and both points at distance $\pi/2$ from it.

Lemma 5.2 $(\rho^+_{2\pi-\alpha_0}, \rho^-_{2\pi-\alpha_0})$ *is the holonomy representation of a suspension of a sphere with 4 cone points.*

Proof: We proceed first to construct the metric on each tangle with totally geodesic boundary, that we shall glue later. Let μ_1 and μ_2 be representatives of the meridians of a tangle, so that μ_1 and μ_2 freely generate the group of the exterior of the tangle. The rotation axis of $(\rho^+_{\alpha_0}(\mu_1), \rho^-_{\alpha_0}(\mu_1))$ and $(\rho^+_{\alpha_0}(\mu_2), \rho^-_{\alpha_0}(\mu_2))$ are different (otherwise, since μ_1 and μ_2 generate $\pi_1 M$, the rotational part of the Euclidean holonomy would be contained in $SO(3)$, contradicting hyperbolicity by [8]). In particular the rotation axis of $(\rho^+_{2\pi-\alpha_0}(\mu_1), \rho^-_{2\pi-\alpha_0}(\mu_1))$ and $(\rho^+_{2\pi-\alpha_0}(\mu_2), \rho^-_{2\pi-\alpha_0}(\mu_2))$ are also different. Thus we construct the geometric structure in one tangle by taking a half sphere with geodesic boundary (i.e. a ball in S^3 of radius $\pi/2$), by removing two sectors of angle α_0, each one with edge the rotation edge of $(\rho^+_{2\pi-\alpha_0}(\mu_1), \rho^-_{2\pi-\alpha_0}(\mu_1))$ and $(\rho^+_{2\pi-\alpha_0}(\mu_2), \rho^-_{2\pi-\alpha_0}(\mu_2))$, and by identifying the half disks in the boundary of the sectors by a rotation around its edge.

We do the same construction for the other tangle, and we have to check that the totally geodesic boundaries of both tangles match. They do match because the holonomy determines the structure of a sphere with 4 cone points and fixed cone angles. More precisely, such a sphere is the union of two discs with totally geodesic boundary and 2 cone points. The length of the boundary and the cone angles is determined by the holonomy, and such three numbers determine completely the metric on the disc. □

Once we have made the explicit construction of the structures on each tangle, we make the standard argument of "deformation of the holonomy implies deformation of the structure". We perturb those structures to smaller cone angles, being careful with the crossing and the tunnels (a explicit model can be constructed in a neighborhood of the tunnels, by removing again sectors from a small ball, but this time the sectors are disjoint). It only remains to check that those structures correspond in fact to $C(\alpha)$ constructed in previous section. Thus we keep decreasing the cone angle. By an argument similar to Lemma 4.2, we can prove that the singular set does not cross with itself anymore when we decrease the cone angle (notice that the volume of the suspension is $2\pi(\alpha_0 - \pi)$ and the computation of integrals in the proof of Lemma 4.2 applies). Once we have reached cone angle π we use the fact that spherical orbifolds are rigid (i.e. the structure is unique, by de Rham's theorem [9, 4]). De Rham's global rigidity and local rigidity of the variety of representations, imply that the structures are in fact $C(\alpha)$.

As a final remark, we notice that even if these structures are locally rigid, we do not know how to prove global rigidity, as we do not have an argument to control the normal radius of the singular locus of a give cone manifold in general.

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