Notes on harmonic measure

Martí Prats* and Xavier Tolsa†

May 5, 2025

*Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Catalonia: martiprats@ub.edu
†ICREA; Departament de Matemàtiques, Universitat Autònoma de Barcelona; CRM, Catalonia: xavier.tolsa@uab.cat

Contents

1 Introduction

In these notes we provide a straightforward introduction to the topic of harmonic measure. This is an area where many advances have been obtained in the last years and we think that this book can be useful for people interested in this topic.

In the first Chapters 2-6 we have followed classical references such as [Fol95], [Car98], [GM05], [Lan72], [AG01], and [Ran95], as well as some private notes of Jonas Azzam. A large part of the content of Chapter 7 is based on Kenig's book [Ken94], and on papers by Aikawa, Hofmann, Martell, and many others. Chapter 8 is based on a paper by Jerison and Kenig [JK82]. In Chapter 9, the proof of Jones-Wolff theorem about the dimension of harmonic measure in the plane follows the presentation of [CTV18][1]. In some parts of Chapter 10 we follow the book of Caffarelli and Salsa [CS05] and some work by Mourgoglou and the second named author of these notes. A large part of Chapter 11 follows [AHM+16].

We apologize in advance for possible inaccuracies or lack of citation. Anyway, we remark that this work is still under construction and we plan to add more content as well as more accurate citations in future versions of these notes.

[1] We thank J. Cufí and J. Verdera for allowing us to reproduce a large part of the content from [CTV18].

2 Harmonic functions

2.1 Definition and basic properties

Given an open set $\Omega \subset \mathbb{R}^d$ we say that a real-valued function u is harmonic in Ω if $u \in C^2(\Omega)$ and

$$\Delta u(x) = \sum_{j=1}^{d} \partial_j^2 u(x) = 0$$

for every $x \in \Omega$ (later on we will see that the C^2 hypothesis can be replaced by just locally integrable if we consider the distributional Laplacian).

Let κ_d denote the area of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, that is,

$$\kappa_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)}$$

see [Fol95, Proposition 0.7] for instance, and $d\sigma$ denote the surface measure. Recall that the volume of the unit ball is then $|B_1(0)| = \frac{\kappa_d}{d}$ (see [Fol95, Corollary 0.8]). Below, we denote $B_r(x)$ the open ball centered at x with radius r, and $S_r(x) = \partial B_r(x)$.

Throughout the notes $|U| = m(U)$ stands for the Lebesgue measure of a set U, and $\fint_U f\, d\mu$ for the average integral of f with respect to the measure μ in U, i.e., $\frac{1}{\mu(U)} \int_U f\, d\mu$. We will use also $dx = dm(x)$ for the integration with respect to Lebesgue measure and $m_U(f)$ for the mean of f with respect to the Lebesgue measure in U.

Lemma 2.1 (Mean value theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be open. If $u \in C^2(\Omega)$ is harmonic, then*

$$u(x_0) = \fint_{B_r(x_0)} u(y)dy = \fint_{B_1(0)} u(x_0 + ry)dy \quad \text{for every } \overline{B_r(x_0)} \subset \Omega \subset \mathbb{R}^d. \tag{2.1}$$

Moreover

$$u(x_0) = \fint_{S_r(x_0)} u(y)d\sigma(y) = \fint_{S_1(0)} u(x_0 + ry)d\sigma(y) \quad \text{for every } \overline{B_r(x_0)} \subset \Omega \subset \mathbb{R}^d. \tag{2.2}$$

Proof. Changing variables, we have that

$$A(\rho) := \frac{1}{\rho^d} \int_{B_\rho(x_0)} u(x)dx = \int_{B_1} u(\rho x + x_0)dx.$$

On the other hand, set

$$\begin{aligned}\widetilde{A}(\rho) &:= \int_{B_1} \nabla u(\rho x + x_0) \cdot x\, dx \\ &= \int_{B_\rho(x_0)} \frac{\nabla u(x) \cdot (x - x_0)}{\rho} \frac{dx}{\rho^d} = \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \nabla u(x) \cdot \nabla |x - x_0|^2\, dx.\end{aligned}$$

Since u satisfies that $\Delta u = 0$ in Ω, we can apply Green's formula twice to obtain

$$\begin{aligned}\widetilde{A}(\rho) &= \frac{1}{2\rho^{d+1}} \int_{S_\rho(x_0)} |x - x_0|^2 \, \nabla u(x) \cdot \nu \, dx - \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, |x - x_0|^2 \, dx \\ &= \frac{1}{2\rho^{d-1}} \int_{S_\rho(x_0)} \nabla u(x) \cdot \nu \, dx = 0, \end{aligned} \tag{2.3}$$

where ν stands for the normal vector to the sphere pointing outward.

Since $u \in C^2(\Omega)$, for every x we have $\int_\rho^r \nabla u(tx + x_0) \cdot x \, dt = u(rx + x_0) - u(\rho x + x_0)$ by the fundamental theorem of calculus. Applying Fubini's Theorem we get

$$\begin{aligned} 0 &\overset{(2.3)}{=} \int_\rho^r \widetilde{A}(t) \, dt = \int_{B_1} \int_\rho^r \nabla u(tx + x_0) \cdot x \, dt \, dx = \int_{B_1} (u(rx + x_0) - u(\rho x + x_0)) \, dx \\ &= A(r) - A(\rho). \end{aligned} \tag{2.4}$$

So $A(r) = A(\rho)$ for all $\rho < r$.

On the other hand, taking the mean and using the continuity of u we obtain

$$\left| u(x_0) - \frac{d}{\kappa_d} \lim_{\rho \to 0} A(\rho) \right| = \lim_{\rho \to 0} \frac{1}{|B_\rho(x_0)|} \left| \int_{B_\rho(x_0)} (u(x_0) - u(x)) \, dx \right| \leqslant \lim_{\rho \to 0} o_{\rho \to 0}(1) = 0.$$

To see the coincidence with the average on spheres, note that in polar coordinates we have

$$A(\rho) = \frac{1}{\rho^d} \int_{S_1(0)} \int_0^\rho u(t\theta) t^{d-1} dt \, d\theta.$$

From this formula one can easily show that (2.2) implies (2.1), but we need to prove the converse. Let us differentiate this expression. We get that

$$\begin{aligned} 0 = A'(\rho) &= \frac{-d}{\rho^{d+1}} \int_{S_1(0)} \int_0^\rho u(t\theta) t^{d-1} dt \, d\theta + \frac{1}{\rho^d} \int_{S_1(0)} u(\rho\theta) \rho^{d-1} d\theta \\ &= \frac{-d}{\rho} A(\rho) + \frac{1}{\rho^d} \int_{S_\rho(x_0)} u(\rho\theta) d\theta. \end{aligned} \tag{2.5}$$

Since $u(x_0) = \frac{d}{\kappa_d} A(\rho)$ by (2.1), we readily get (2.2) multiplying the last expression times $\frac{\rho}{\kappa_d}$. $\square$

Remark 2.2. Arguing as above, it follows that if $u \in C^2(\Omega)$ satisfies $\Delta u \geqslant 0$ in Ω, then

$$u(x_0) \leqslant \fint_{B_r(x_0)} u(y) dy \leqslant \fint_{S_r(x_0)} u(y) d\sigma(y) \tag{2.6}$$

whenever $\overline{B_r(x_0)} \subset \Omega \subset \mathbb{R}^d$. Indeed, instead of (2.3), we have

$$\begin{aligned} \widetilde{A}(\rho) &= \frac{1}{2\rho^{d-1}} \int_{S_\rho(x_0)} \nabla u(x) \cdot \nu \, dx - \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, |x - x_0|^2 \, dx \\ &= \frac{1}{2\rho^{d-1}} \int_{S_\rho(x_0)} \Delta u(x) \, dx - \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, |x - x_0|^2 \, dx \\ &= \frac{1}{2\rho^{d+1}} \int_{B_\rho(x_0)} \Delta u(x) \, (\rho^2 - |x - x_0|^2) \, dx \geqslant 0. \end{aligned}$$

Then, as in (2.4), we deduce that

$$A(r) - A(\rho) \geqslant 0 \quad \text{if } \rho < r.$$

Then, letting $\rho \to 0$, the first inequality in (2.6) follows.

Further, notice that the preceding discussion shows that $A'(\rho) \geqslant 0$, and then by (2.5) it follows that

$$0 \leqslant \frac{-d}{\rho} A(\rho) + \frac{1}{\rho^d} \int_{S_\rho(x_0)} u(\rho\theta) d\theta,$$

which is equivalent to the last inequality in (2.6).

Theorem 2.3 (Converse of the mean value Theorem)**.** *If $u \in C(\Omega)$ satisfies (2.1) or (2.2), then $u \in C^\infty$ and it is harmonic.*

Proof. Note that we have seen that (2.1) and (2.2) are in fact equivalent. Thus, it suffices to assume that u satisfies (2.2).

Let $\psi \in C^\infty([0,1])$ be a non-negative function with $\int_0^\infty \psi(t) t^{d-1} dt = 1$. Define $\phi_\varepsilon(x) := \frac{1}{\kappa_d \varepsilon^d} \psi\left(\frac{|x|}{\varepsilon}\right)$. Then $\int \phi_\varepsilon = 1$ for every ε. Next consider the subset $\Omega_\varepsilon := \{x \in \Omega : \overline{B_\varepsilon(x)} \subset \Omega\}$. If $x \in \Omega_\varepsilon$ then we claim that

$$u(x) = \int u(y) \phi_\varepsilon(x-y)\, dy.$$

Indeed,

$$\begin{aligned} u(x) - \int u(y)\phi_\varepsilon(x-y)\,dy &= \int (u(x) - u(y))\phi_\varepsilon(x-y)\,dy \\ &= \int_0^\varepsilon \frac{\psi(\frac{\rho}{\varepsilon})}{\kappa_d \varepsilon^d} \int_{S_1(0)} (u(x) - u(x+\rho\theta))\, d\theta\, d\rho \overset{(2.2)}{=} 0. \end{aligned}$$

We can conclude that u is C^∞ in Ω_ε and, therefore, in the whole of Ω.

To get the harmonicity, note that the derivative with respect to r of $\fint_{S_1(0)} u(x+ry) d\sigma(y)$ is zero by assumption. That is

$$\begin{aligned} 0 &= \frac{d}{dr} \fint_{S_r(x)} u(y) d\sigma(y) = c \frac{d}{dr} \int_{S_1(0)} u(x+ry) d\sigma(y) = c \int_{S_1(0)} \partial_\nu u(x+ry)\, d\sigma \\ &= c \fint_{S_r(x)} \partial_\nu u\, d\sigma \overset{\text{Green Thm}}{=} \frac{c}{r^{d-1}} \int_{B_r(x)} \Delta u\, dx. \end{aligned}$$

Since the Laplacian vanishes on every ball, we deduce that it is actually zero everywhere. □

In particular, every harmonic C^2 function is C^∞. Therefore we can restate the definition of harmonic function:

Definition 2.4. We say that a function $u : \Omega \to \mathbb{R}$ is harmonic if $u \in C(\Omega)$ and it satisfies the mean value property (2.1).

As we have seen, every harmonic function satisfies also the mean value property in spheres, it is $C^\infty(\Omega)$ and $\Delta u = 0$. This self-improvement property is also true for harmonic distributions, we will see that later on.

Theorem 2.5 (The maximum principle)**.** *Let Ω be a domain (i.e. open and connected set). If u is harmonic and real-valued and $A := \sup_\Omega u < \infty$, then either $u(x) < A$ for every $x \in \Omega$ or $u(x) = A$ for every $x \in \Omega$.*

Proof. $\{x \in \Omega : u(x) = A\}$ is relatively closed by continuity and open by the mean value theorem. □

Corollary 2.6. *Let Ω be a bounded open set. If $u \in C(\Omega)$ is harmonic and real-valued, then the supremum and the infimum are attained at the boundary.*

Proof. Assume that the supremum is not attained at the boundary. Then, by compactness it must be attained in the interior. This implies that u is constant in some component of Ω, which in turn implies that the supremum is also attained at the boundary of that component, a contradiction. Also the infimum is attained at the boundary since $\inf_\Omega u = -\sup_\Omega(-u)$. □

Theorem 2.7 (Uniqueness theorem)**.** *Let Ω be a bounded open set. If $u_1, u_2 \in C(\overline{\Omega})$ are harmonic in Ω, and $u_1|_{\partial\Omega} \equiv u_2|_{\partial\Omega}$, then $u_1|_\Omega \equiv u_2|_\Omega$.*

Proof. Apply the corollary to $u_1 - u_2$. □

Theorem 2.8 (Liouville's theorem)**.** *Let u be a bounded harmonic function in $\mathbb{R}^d$. Then u is constant.*

Proof. Note that for $r > 2|x|$

$$\begin{aligned}|u(x) - u(0)| &= \left| \fint_{B_r(x)} u(y)dy - \fint_{B_r(0)} u(y)dy \right| \leqslant \frac{d}{\kappa_d r^d} \int_{B_{r+|x|}(0)\setminus B_{r-|x|}(0)} |u(y)|dy \\ &\leqslant \frac{d\|u\|_\infty}{\kappa_d} \frac{|B_{r+|x|}(0)\setminus B_{r-|x|}(0)|}{r^d} \lesssim_d \frac{|x|\|u\|_\infty}{r} \xrightarrow{r\to\infty} 0.\end{aligned}$$

□

2.2 The Caccioppoli inequality

We have shown that every harmonic function $u \in C(\Omega)$ is $C^\infty(\Omega)$. Next we turn our attention to weakly harmonic functions.

Definition 2.9. Given an open set $\Omega \subset \mathbb{R}^d$, we say that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is *weakly harmonic* if every test function $\varphi \in C_c^\infty(\Omega)$ satisfies that

$$\langle \Delta u, \varphi \rangle := -\langle \nabla u, \nabla \varphi \rangle = 0. \tag{2.7}$$

We say that $u \in D'(\Omega)$ is *distributionally harmonic* if, instead, test functions satisfy

$$\langle \Delta u, \varphi \rangle := \langle u, \Delta \varphi \rangle = 0. \tag{2.8}$$

Arguing by density, if u is weakly harmonic then equation (2.7) is verified also for every $\varphi \in W_c^{1,2}(\Omega)$. Note that every harmonic function is weakly harmonic, and every weakly harmonic function is distributionally harmonic, but the converse has not been established yet (see Proposition 2.19 below).

Lemma 2.10 (Caccioppoli Inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set, and let u be weakly harmonic in Ω. Then for every $t > 0$ and every ball B of radius r such that $(t+1)B \subset \Omega$, we have*

$$\int_B |\nabla u|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2.$$

Proof. Let η be a Lipschitz function such that $\chi_B \leqslant \eta \leqslant \chi_{(t+1)B}$ and with $|\nabla \eta| \leqslant \frac{1}{rt}$. Since u is weakly harmonic and η is compactly supported in Ω, we have that

$$0 = \int_{(t+1)B} \nabla u \cdot \nabla(u\eta^2).$$

By the Leibniz rule, the former identity can be written as

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 = -\int_{(t+1)B} 2u\eta \nabla u \cdot \nabla \eta,$$

and using Hölder's inequality we get

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \left(\int_{(t+1)B} 4u^2 |\nabla \eta|^2 \right)^{\frac{1}{2}} \left(\int_{(t+1)B} \eta^2 |\nabla u|^2 \right)^{\frac{1}{2}}.$$

Thus,

$$\int_B |\nabla u|^2 \leqslant \int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \int_{(t+1)B} 4u^2 |\nabla \eta|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2.$$

□

The Caccioppoli inequality is also valid for subharmonic functions, see Section 5.1. This inequality implies the universal control for the gradient in terms of the distance to the boundary and the L^∞ norm of u:

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^d$ be an open set, and let u be harmonic in Ω. Then*

$$|\nabla u(x)| \lesssim \frac{\|u\|_{L^\infty(\Omega)}}{d_\Omega(x)}, \tag{2.9}$$

where $d_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$.

Proof. Since the derivatives of u are harmonic, by the mean value theorem and the Caccioppoli inequality

$$\begin{aligned} |\nabla u(x)| &= \left| \fint_{B_{\frac{1}{2}d_\Omega(x)}(x)} \nabla u \, dm \right| \leqslant \left(\fint_{B_{\frac{1}{2}d_\Omega(x)}(x)} |\nabla u|^2 \, dm \right)^{\frac{1}{2}} \\ &\leqslant \left(\frac{4}{(\frac{1}{2}d_\Omega(x))^2} \fint_{B_{d_\Omega(x)}(x)} |u|^2 \, dm \right)^{\frac{1}{2}} \lesssim \frac{1}{d_\Omega(x)} \|u\|_{L^\infty(\Omega)}, \end{aligned}$$

as claimed. □

By iterating the estimate in Lemma 2.10, we immediately obtain the following.

Lemma 2.12. *Let u be a harmonic function in $B_1(0)$. Then, for all $k \geqslant 1$,*

$$\|u\|_{C^k(B_{1/2}(0))} \leqslant C(k)\,\|u\|_{L^\infty(B_1(0))}.$$

Then we deduce the following generalization of Liouville's theorem.

Proposition 2.13. *Let $\gamma > 0$ and let u be harmonic in $\mathbb{R}^d$ such that $|u(x)| \leqslant C(1+|x|)^\gamma$ for all $x \in \mathbb{R}^d$. Then u is a polynomial of degree at most $\lfloor \gamma \rfloor$.*

Proof. For $r > 0$, consider the function $u_r(x) = u(r\,x)$. Since u_r is harmonic, for any $k > 1$, by Lemma 2.12 we have

$$\begin{aligned}\|D^k u\|_{L^\infty(B_{r/2}(0))} &= \frac{1}{r^k}\,\|D^k u_r\|_{L^\infty(B_{1/2}(0))} \leqslant \frac{C(k)}{r^k}\,\|u_r\|_{L^\infty(B_1(0))} \\ &= \frac{C(k)}{r^k}\,\|u\|_{L^\infty(B_r(0))} \leqslant \frac{C'(k)(1+r)^\gamma}{r^k}.\end{aligned}$$

For $k = \lfloor \gamma \rfloor + 1$, the term on the right hand side tends to 0 as $r \to \infty$, and thus $D^k u$ vanishes identically in $\mathbb{R}^d$. Consequently, u is a polynomial of degree at most $k - 1 = \lfloor \gamma \rfloor$. □

Lemma 2.14. *Every sequence of uniformly bounded harmonic functions in an open set Ω is locally equicontinuous, it has a converging subsequence, and the limit is harmonic as well.*

Proof. Let $\{u_n\}_n$ with $\Delta u_n = 0$ in Ω and $\|u_n\|_{L^\infty(\Omega)} \leqslant C < \infty$.

By assumption u_n is a sequence of uniformly bounded and, by Lemma 2.11, uniformly locally equicontinuous functions. By the Ascoli-Arzelá theorem, u_n has a partial converging uniformly in every compact subset of Ω.

To see that the limit is also harmonic just apply the converse to the mean value theorem (see Theorem 2.3) to the limiting function. □

2.3 Harnack's inequality

Lemma 2.15 (Harnack's inequality)**.** *Let B be a ball and let $u \geqslant 0$ be a harmonic function in $2B$. Then*

$$\sup_B u \leqslant C \inf_B u.$$

Remark that the estimate above is equivalent to saying that

$$C^{-1}u(x) \leqslant u(y) \leqslant C\,u(x) \qquad \text{for all } x, y \in B.$$

Proof. Set $B = B(x_0, r)$. To prove the lemma it suffices to show that, for all $y, z \in B$, $u(y) \lesssim u(z)$, with the implicit constant depending only on d. Suppose first that $|y - z| \leqslant r/4$. Then we have $B(y, r/4) \subset B(z, r/2) \subset\subset 2B$, and so we have, by the mean value property,

$$u(y) = \fint_{B(y,r/4)} u\,dx \lesssim \fint_{B(z,r/2)} u\,dx = u(z).$$

In the case when $|y - z| > r/4$, we partition the segment $[y, z]$ into eight segments I_j with equal length and disjoint interiors. So we write

$$[y, z] = \bigcup_{0 \leqslant j \leqslant 7} [y_j, y_{j+1}],$$

and we assume that $y = y_0$, $z = y_8$. Since the length of $[y, z]$ is at most $\mathrm{diam}(B) = 2r$, it holds $|y_j - y_{j+1}| \leqslant r/4$ for each j. By the previous estimate, then we have $u(y_j) \lesssim u(y_{j+1})$ for each j. Thus,

$$u(y) = u(y_0) \lesssim u(y_1) \lesssim \cdots \lesssim u(y_8) = u(z).$$

□

Note that by modifying the argument above we can get that for every $t \geqslant 0$ there exists an optimal constant $\varepsilon(t)$ so that every harmonic function $u \geqslant 0$ in $(1 + t)B$ satisfies

$$\sup_B u \leqslant (1 + \varepsilon(t)) \inf_B u.$$

The reader can prove that ε is non-increasing and $\varepsilon(t) \xrightarrow{t \to 0} \infty$. But the interesting asymptotic behavior is for $t \to \infty$:

Lemma 2.16 (Asymptotic Harnack inequality)**.** *There exists a nonnegative function* $\varepsilon(t) \xrightarrow{t \to \infty} 0$ *so that every harmonic function* $u \geqslant 0$ *in* $(1 + t)B$ *satisfies that*

$$\sup_B u \leqslant (1 + \varepsilon(t)) \inf_B u.$$

Proof. The proof follows by an argument very similar to the one in the preceding lemma. Indeed, assume $t \geqslant 8$, say, and consider arbitrary points $x, z \in B$. Furthermore, assume without loss of generality that $r(B) = 1$. Then we have $B(x, t/2) \subset B(z, 2 + t/2) \subset (1 + t)B$ and so

$$\begin{aligned} u(x) &= \frac{1}{|B(x, t/2)|} \int_{B(x,t/2)} u\,dy \leqslant \frac{1}{|B(x, t/2)|} \int_{B(z,2+t/2)} u\,dy \\ &= \frac{|B(z, 2 + t/2)|}{|B(x, t/2)|} u(z) = \left(\frac{4 + t}{t}\right)^d u(z). \end{aligned}$$

So we may choose $\varepsilon(t) = \left(\frac{4+t}{t}\right)^d - 1$. □

Lemma 2.17. *Let* $\Omega \subset \mathbb{R}^d$ *be a domain and let* $x, y \in \Omega$*. Then there is a constant* $C_{x,y} > 0$ *depending just on* x*,* y*, and* Ω *such that for any positive harmonic function* u *in* Ω*, it holds*

$$C_{x,y}^{-1}\, u(x) \leqslant u(y) \leqslant C_{x,y}\, u(y).$$

Remark that the important fact about the estimate above is that the constant $C_{x,y}$ does not depend on the particular function u.

Proof. Let $\gamma \subset \Omega$ be a compact curve contained in Ω whose end points are x and y, and let $\delta = \operatorname{dist}(\gamma, \partial\Omega)$. By the compactness of γ, there is a finite covering of γ by open balls B_i, $i = 1, \ldots, m$, centered in γ with $r(B_i) = \delta/2$ (with m depending on Ω and γ).

We reorder the balls B_i as follows. Suppose that $x \in B_1$ without loss of generality. If $m \geqslant 2$, because of the connectivity of γ, there exists another ball B_i, call it B_2, such that $B_1 \cap B_2 \neq \varnothing$. Next, if $m \geqslant 3$, by the connectivity of γ again, there exists another ball, call it B_3, such that $(B_1 \cup B_2) \cap B_3 \neq \varnothing$, and so on. Denote $U_k = \bigcup_{1 \leqslant i \leqslant k} B_i$, so that $U_k = U_{k-1} \cup B_k$, $U_{k-1} \cap B_k \neq \varnothing$, and $\gamma \subset U_m$.

Given u harmonic and positive in Ω, by Harnack's inequality $u(z) \approx u(z')$ for all $z, z' \in B_i$ (since $2B_i \subset \Omega$). Then, by induction it follows easily that $u(z) \approx u(z')$ for all $z, z' \in U_k$ (with the implicit constant depending on k), for $k = 1, \ldots, m$. In particular, $u(x) \approx_m u(y)$. □

2.4 The fundamental solution

To conclude this chapter, we will see that every harmonic distribution (see Definition 2.9) is in fact a C^∞ function. This is a quite general fact for elliptic partial differential equations with C^∞ fundamental solutions, see [Fol95, Theorem 1.58] for the details.

Let us define

$$\mathcal{E}(x) = \begin{cases} \dfrac{|x|^{2-d}}{(d-2)\kappa_d} & \text{if } d > 2, \\[2ex] \dfrac{-\log|x|}{2\pi} & \text{if } d = 2, \end{cases} \tag{2.10}$$

Note that, since $\kappa_2 = 2\pi$, for every $n \geqslant 1$ its gradient is

$$\nabla \mathcal{E}(x) = \frac{-x}{\kappa_d |x|^d}. \tag{2.11}$$

Proposition 2.18. *The fundamental solution of $(-\Delta)$ in $\mathbb{R}^d$ is precisely $\mathcal{E}$, i.e. $-\Delta\mathcal{E}$ is the Dirac delta distribution δ_0.*

The preceding proposition must be understood in the sense that for every test function $\varphi \in D(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$, we have

$$\varphi(0) =: \langle \delta_0, \varphi \rangle = -\langle \Delta\mathcal{E}, \varphi \rangle = -\langle \mathcal{E}, \Delta\varphi \rangle.$$

Proof of Proposition 2.18. Consider $\epsilon > 0$ and let ν be the normal vector to S_ϵ pointing towards the origin. For $\varphi \in C_c^\infty$ we have

$$-\langle \mathcal{E}, \Delta\varphi \rangle = \int \nabla\mathcal{E} \cdot \nabla\varphi. \tag{2.12}$$

Indeed,

$$\begin{aligned}\left|-\langle\mathcal{E},\Delta\varphi\rangle-\int\nabla\mathcal{E}\cdot\nabla\varphi\right| &= \left|\int_{B_\epsilon}\mathcal{E}\Delta\varphi+\int_{B_\epsilon^c}\mathcal{E}\Delta\varphi+\int\nabla\mathcal{E}\cdot\nabla\varphi\right| \\ &\overset{\text{Green}}{\leqslant} \left|\int_{B_\epsilon}\mathcal{E}\Delta\varphi\right|+\left|\int_{B_\epsilon^c}\nabla\mathcal{E}\cdot\nabla\varphi-\int\nabla\mathcal{E}\cdot\nabla\varphi\right|+\left|\int_{S_\epsilon}\mathcal{E}\nabla\varphi\cdot\nu\right| \\ &\lesssim \|\Delta\varphi\|_\infty\|\mathcal{E}\|_{L^1(B_\epsilon)}+\left|\int_{B_\epsilon}\nabla\mathcal{E}\cdot\nabla\varphi\right|+\|\mathcal{E}\|_{L^\infty(S_\epsilon)}\|\nabla\varphi\|_\infty\epsilon^{d-1}.\end{aligned}$$

For $d=2$, using (2.10) we have $\|\mathcal{E}\|_{L^1(B_\epsilon)}\approx\int_0^\epsilon r|\log r|dr\xrightarrow{\epsilon\to0}0$ and $\|\mathcal{E}\|_{L^\infty(S_\epsilon)}=c|\log\epsilon|$. In case $d>2$, then using (2.10) we have $\|\mathcal{E}\|_{L^1(B_\epsilon)}\approx\int_0^\epsilon rdr\xrightarrow{\epsilon\to0}0$ and $\|\mathcal{E}\|_{L^\infty(S_\epsilon)}=c\epsilon^{2-d}$. All in all, letting $\epsilon\to0$ we get (2.12).

Moreover,

$$\begin{aligned}|-\langle\mathcal{E},\Delta\varphi\rangle-\varphi(0)| &\overset{(2.12)}{=} \left|\int\nabla\mathcal{E}\cdot\nabla\varphi-\varphi(0)\right| = \left|\int_{B_\epsilon}\nabla\mathcal{E}\cdot\nabla\varphi+\int_{B_\epsilon^c}\nabla\mathcal{E}\cdot\nabla\varphi-\varphi(0)\right| \\ &\overset{\text{Green}}{\lesssim} \|\nabla\varphi\|_\infty\int_{B_\epsilon}|x|^{1-d}+\left|\int_{S_\epsilon^c}\nabla\mathcal{E}\cdot\nu\varphi-\varphi(0)\right|+\left|\int_{B_\epsilon^c}\Delta\mathcal{E}\varphi\right|\end{aligned}$$

Now, $\int_{B_\epsilon}|x|^{1-d}\approx\epsilon\xrightarrow{\epsilon\to0}0$, and $\Delta\mathcal{E}\equiv0$ in B_ϵ^c. Moreover, for $y\in S_\epsilon$ we get

$$\nabla\mathcal{E}(y)\cdot\nu(y)=\frac{-y}{\kappa_d|y|^d}\cdot\frac{-y}{|y|}=\frac{1}{\kappa_d\epsilon^{d-1}}=\frac{1}{\sigma(S_\epsilon)}.$$

Thus,

$$|-\langle\mathcal{E},\Delta\varphi\rangle-\varphi(0)|\lesssim\left|\fint_{S_\epsilon^c}\varphi-\varphi(0)\right|\xrightarrow{\epsilon\to0}0,$$

as claimed by the continuity of φ at the origin.

□

The preceding proposition implies that for every test function $\varphi\in D(\Omega)$, we have

$$-\Delta(\mathcal{E}*\varphi)(x)=\varphi(x). \tag{2.13}$$

Note that $\mathcal{E}*\varphi\in C^\infty$ because $\mathcal{E}\in L^1_{\text{loc}}$.

In fact we obtain the following:

Proposition 2.19. *Let u be a harmonic distribution in an open set Ω. Then $u\in C^\infty(\Omega)$, and u is a harmonic function.*

Remark that a distribution is called harmonic if it is distributionally harmonic.

Proof. Given a distibution T with compact support contained in a bounded open set V, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ we can define

$$\langle \mathcal{E} * T, \varphi \rangle := \langle T, \psi(\mathcal{E} * (\varphi_-))_- \rangle,$$

where ψ is any cuttof function $\psi \in C_c^\infty$ with $\chi_{\text{supp}T} \leqslant \psi \leqslant \chi_V$, and $f_-(x) := f(-x)$. This definition does not depend on the particular choice of ψ, because the test function in the right-hand side will not vary in the support of T. Moreover, we claim that this distribution is in fact C^∞ out of the support of T. Indeed, for any test function φ with $\text{supp}\varphi \cap \text{supp}T = \varnothing$, one can consider $\varepsilon := \text{dist}(\text{supp}\varphi, \text{supp}T)$, and given a C^∞ function ϕ such that $\chi_{B_{\varepsilon/4}} \leqslant \phi \leqslant \chi_{B_{\varepsilon/2}}$, one can infer that $\langle \mathcal{E} * T, \varphi \rangle = \langle ((1-\phi)\mathcal{E}) * T, \varphi \rangle$. The latter can be shown to be a C^∞ distribution arguing as in the proof of [Gra08, Theorem 2.3.20].

When u is a distribution in an open set Ω such that $\Delta u = 0$, given a ball $B \subset \Omega$ we can define a cut-off function $\psi_B \in C^\infty$ such that $\chi_{\frac{1}{2}B} \leqslant \psi_B \leqslant \chi_B$. Then $\Delta(\psi_B u)$ is a distribution supported in $\overline{B} \backslash \frac{1}{2}B$ and therefore $\mathcal{E} * (\Delta(\psi_B u))$ is a well-defined distribution. Given $\varphi \in D(\Omega) := C_c^\infty(\Omega)$, assuming if necessary that $\psi_B \nabla \psi \equiv 0$, we have

$$\langle \mathcal{E} * (-\Delta(\psi_B u)), \varphi \rangle = \langle (-\Delta(\psi_B u)), \psi(\mathcal{E} * (\varphi_-))_- \rangle = \langle \psi_B u, -\Delta(\mathcal{E} * (\varphi_-))_- \rangle \overset{(2.13)}{=} \langle \psi_B u, \varphi \rangle,$$

i.e. $\mathcal{E} * (-\Delta(\psi_B u)) = \psi_B u$ in the distributional sense. Since the former is in fact C^∞ out of the support of $\Delta(\psi_B u)$, we conclude in particular that in $\frac{1}{2}B$, the function $u = \psi_B u$ is C^∞. Harmonicity comes by integration by parts. □

The approach above can be slightly modified in order to obtain the hypoellipticity of the laplacian:

Theorem 2.20 ([Fol95, Theorem 1.58])**.** *The laplacian Δ is hypoelliptic, i.e., if u is a distribution on a bounded open set Ω such that $\Delta u \in C^\infty(\Omega)$ then $u \in C^\infty(\Omega)$.*

Remark 2.21. Note that $\mathcal{E} \in L^p_{\text{loc}}$ for every $p < \frac{d}{d-2}$, and $\nabla \mathcal{E} \in L^p_{\text{loc}}$ for every $p < \frac{d}{d-1}$. The integrability at infinity is obtained for $p > \frac{d}{d-2}$, and $p > \frac{d}{d-1}$ respectively.

3 The Dirichlet Problem

3.1 The weak formulation

Consider the problem of finding a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ in an open set $\Omega \subset \mathbb{R}^d$ to the Dirichlet problem with boundary data $f \in C(\partial\Omega)$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

To obtain a general theory of existence and uniqueness, we can work in Sobolev spaces with only one derivative, and this requires a weak formulation of the Dirichlet problem. Assume that $u \in C^1(\overline{\Omega})$, and let $\varphi \in C_c^\infty(\Omega)$. Then Green's theorem implies that

$$0 = \int_\Omega \varphi \Delta u = -\int_\Omega \nabla u \cdot \nabla \varphi + \int_{\partial\Omega} \varphi \nabla u \cdot \nu \, d\sigma = -\int_\Omega \nabla u \cdot \nabla \varphi. \tag{3.2}$$

Equation (3.2) provides us with a weak formulation of $\Delta u = 0$. But how can we encode the boundary behavior? Set

$$H^1(\Omega) := W^{1,2}(\Omega) := \{f \in L^2(\Omega) : \partial_i f \in L^2(\Omega) \text{ for } 1 \leqslant i \leqslant n+1\},$$

and we define

$$H_0^1(\Omega) := \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$$

and the quotient space

$$H^{1/2}(\partial\Omega) := H^1(\Omega)/H_0^1(\Omega)$$

(see [Sch02, Theorem 3.13], for instance). Given $f \in H^1(\Omega)$, its class in $H^{1/2}(\partial\Omega)$ is often called "the trace of f". Now, in a bounded open set Ω, if $u = f$ in $\partial\Omega$ and $u, f \in C^2(\overline{\Omega})$, then one can show that $u - f \in H_0^1(\Omega)$. Moreover, the identity (3.2) can be extended by density to $\varphi \in H_0^1(\Omega)$.

All in all, in an open set Ω, we say that $u \in H^1(\Omega)$ is a (weak) solution to the Dirichlet problem (3.1) if

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla \varphi = 0 & \text{for every } \varphi \in H_0^1(\Omega), \text{ and} \\ f - u \in H_0^1(\Omega). \end{cases} \tag{3.3}$$

Note that if $u \in C^2(\overline{\Omega}) \cap H^1(\Omega)$ is a weak solution (3.3), then it is also a solution to (3.1) for f regular enough.

Let us write $v := u - f$. Solving (3.3) is equivalent to finding $v \in H_0^1(\Omega)$ solving

$$\int_\Omega \nabla v \cdot \nabla \varphi = -\int_\Omega \nabla f \cdot \nabla \varphi \quad \text{for every } \varphi \in H_0^1(\Omega), \tag{3.4}$$

which in the strong formulation reads as

$$\begin{cases} \Delta v = \Delta f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in H_0^1(\Omega)$ be a harmonic function. Then it is the null function.*

Proof. There exist C_c^∞ functions ψ_i such that $\psi_i \to u$ in H^1. Note that

$$\int \nabla \psi_i \cdot \nabla \psi_i = \int \nabla \psi_i \cdot \nabla(u - \psi_i) + \int \nabla \psi_i \cdot \nabla u.$$

But the last integral is null because u is harmonic. Thus, using the Cauchy-Schwartz inequality we get

$$\|\nabla \psi_i\|_{L^2}^2 \leqslant \|\nabla \psi_i\|_{L^2} \|\nabla(u - \psi_i)\|_{L^2},$$

i.e.

$$\|\nabla \psi_i\|_{L^2} \leqslant \|\nabla(u - \psi_i)\|_{L^2}.$$

Taking limits,

$$\|\nabla u\|_{L^2} = \lim_{i\to\infty} \|\nabla \psi_i\|_{L^2} \leqslant \lim_{i\to\infty} \|\nabla(u - \psi_i)\|_{L^2} = 0.$$

Thus, u is constant and has trace 0, so it is the null function. □

Remark 3.2. Note that the preceding result does not apply to $\log|x|$ in the complement of B_1, since it does not have trace 0 according to the definitions, neither to x_d in $\mathbb{R}_+^d$. Indeed, C_c^∞ functions cannot approach in L^2 norm a function which does not belong to L^2. The condition $u \in H^1(\Omega)$ is not satisfied in this case.

Theorem 3.3 (Riesz representation theorem for Hilbert spaces, see [Sch02, Theorem 2.1])**.** *Let H be a Hilbert space with inner product $(\cdot, \cdot)$, and let H^* be its dual. Then for each $u^* \in H^*$ there exists a unique $u \in H$ such that*

$$\langle u^*, v \rangle = (u, v).$$

Corollary 3.4. *Let Ω be open and let $f \in H^{\frac{1}{2}}(\partial\Omega)$. If the Dirichlet problem (3.1) has a solution $u \in H^1(\Omega)$, then this is unique and moreover $u \in C^\infty(\Omega)$. If Ω is bounded, then the solution exists.*

Proof. The uniqueness of the solution comes from Proposition 3.1 and the smoothness from hypoellipticity (see Section 2.4).

Suppose now that Ω is bounded. Then $\|\nabla v\|_{L^2(\Omega)}$ is a norm for the functions $v \in H_0^1(\Omega)$ (because of the Poincaré inequality) and the associated scalar product equals

$$(v, \varphi) = \int \nabla v \cdot \nabla \varphi \quad \text{for all } v, \varphi \in H_0^1(\Omega).$$

Let F denote a representative of f in $H^1(\Omega)$. Consider the linear functional $T_F : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$T_F(\varphi) = -\int_\Omega \nabla F \cdot \nabla \varphi \quad \text{for every } \varphi \in H_0^1(\Omega).$$

By the Riesz representation theorem, there exists a unique $v \in H_0^1(\Omega)$ such that $(v, \varphi) = T_F(\varphi)$ for every $\varphi \in H_0^1(\Omega)$. Then $u := v + F$ is weakly harmonic in Ω, since

$$\int \nabla u \cdot \nabla \varphi = \int \nabla (v + F) \cdot \nabla \varphi = -\int_\Omega \nabla F \cdot \nabla \varphi + \int_\Omega \nabla F \cdot \nabla \varphi = 0$$

for every $\varphi \in H_0^1(\Omega)$. Moreover, $u = f$ on $\partial\Omega$ in the sense that $F - u = v \in H_0^1(\Omega)$. So u solves (3.3). $\square$

3.2 The Green function

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, let $x \in \Omega$, and define the fundamental solution (to $-\Delta$) with pole at x as

$$\mathcal{E}^x(y) := \mathcal{E}(x - y),$$

see (2.10). Note that $\mathcal{E}^0 = \mathcal{E}$. The equation

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = -\mathcal{E}^x(\cdot) & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

has a unique weak solution $v^x \in H^1(\Omega)$ by Corollary 3.4. Then we define the *Green function* with pole at x as

$$G^x(y) := v^x(y) + \mathcal{E}^x(y). \tag{3.6}$$

Using Remark 2.21, we immediately obtain the following result:

Lemma 3.5. *Let Ω be a bounded open set, and let G^x be its Green function with pole $x \in \Omega$. Then $G^x \in W^{1,p}(\Omega)$ for every $p < \frac{d}{d-1}$.*

The thoughtful reader may notice that $\mathcal{E}^x$ is not an H^1 function, so (3.5) is not well defined, but this can be fixed by multiplying $\mathcal{E}$ times $\psi^x_{\partial\Omega}$, which is defined to be a C^∞ function vanishing in a neighborhood of x such that $\psi^x_{\partial\Omega} \equiv 1$ in a neighborhood of $\partial\Omega$, i.e., v^x is the weak solution to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = -\psi^x_{\partial\Omega}\mathcal{E}^x & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Definition 3.6. Given $x \in \Omega$, define $\mathrm{d}_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$ and call $U_x := B_{\frac{1}{2}\mathrm{d}_\Omega(x)}(x)$. Then, since $\overline{U_x} \cap \partial\Omega = \varnothing$, we can find a compact set K_x and open sets V_x, $\widetilde{V}_x$ such that $\partial\Omega \subset V_x \subset \widetilde{V}_x \subset K_x \subset \overline{U_x}^c$ and a bump function $\psi^x_{\partial\Omega} \in C^\infty(\mathbb{R}^d)$ satisfying

$$\chi_{V_x} \leqslant \psi^x_{\partial\Omega} \leqslant \chi_{\widetilde{V}_x}. \tag{3.8}$$

Note that for every $\varphi \in C^\infty_c(\Omega)$ one has

$$\begin{aligned} \int \nabla G^x(y) \cdot \nabla\varphi(y)\, dy &= \int \nabla v^x(y) \cdot \nabla\varphi(y)\, dy + \int \nabla\mathcal{E}^x(y) \cdot \nabla\varphi(y)\, dy \\ &= 0 + \int \nabla\mathcal{E}(z) \cdot \nabla_z\varphi(x+z)\, dz \overset{\text{P.2.18}}{=} \varphi(x). \end{aligned} \tag{3.9}$$

That is $\Delta G^x = -\delta_x$ as a distribution in $D'(\Omega)$, with "vanishing" boundary values, i.e., with $\psi^x_{\partial\Omega} G^x \in H^1_0(\Omega)$ (see (3.8) above and Remark 2.21), so we say that G^x is the weak solution to

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega, \\ G^x = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

For any given $\varphi \in C^\infty_c(\Omega)$, we can write

$$\varphi(y) = \int_\Omega \nabla\varphi(z) \cdot \nabla G^y(z)$$

by (3.9). We want to apply this identity to $\varphi = G^x$, but it is not a test function, so we need to be more careful.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Then*

$$G^x(y) = \int_\Omega \nabla G^x \cdot \nabla G^y\, dm,$$

whenever $x, y \in \Omega$ are different points. In particular,

$$G^x(y) = G^y(x).$$

In other words, the Green function is symmetric and, therefore, it is harmonic also with respect to x. As a consequence, $v^x(y) = v^y(x)$ and it is harmonic with respect to $x \in \Omega$ as well. Note that for the lemma to make sense, we need that $\nabla G^x \cdot \nabla G^y \in L^1(\Omega)$. A priori one may think that for $p < \frac{d}{d-1}$, estimate $G^x \in W^{1,p}_{\mathrm{loc}}(\Omega)$ from Lemma 3.5 is not enough to grant integrability of $\nabla G^x \cdot \nabla G^y$. However, both terms are C^∞ away from the pole, and since $x \neq y$, then integrability comes from the local boundedness of the Green function away from the pole together with the integrability of the singularities.

Proof of Lemma 3.7. In order to apply (3.9), we need to substitute the Green function by a suitable test function approximating it. Let $\psi := \psi^x_{\partial\Omega}\psi^y_{\partial\Omega}$, and consider

$$G^x = (1-\psi)G^x + \psi G^x. \tag{3.11}$$

Let $U := (\widetilde{V}_x \cup \widetilde{V}_y)\backslash\Omega^c$ (see Definition 3.6) so that $\mathrm{supp}(\psi) \cap \Omega \subset U$. Since $\psi G^x \in H_0^1(U)$, there exists $\{\varphi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(U)$ so that

$$\varphi_k \xrightarrow[H^1(\Omega)]{k\to\infty} \psi G^x, \tag{3.12}$$

which allows us to approximate the last term in (3.11). On the other hand, let $\eta \in C^\infty(\mathbb{R})$ such that $\chi_{(0,1/2)} \leqslant \eta \leqslant \chi_{(0,1)}$ and write $\eta_k(z) := \eta(k|x-z|)$, which allows us to approximate the Green function around the pole $(1-\psi)G^x$ in (3.11) by $(1-\eta_k-\psi)G^x$.

Next, we define

$$f_k(z) := (1-\eta_k(z)-\psi(z))G^x(z) + \varphi_k(z),$$

which is in $C_c^\infty(\Omega)$ for k large enough. Note that subtracting η_k skips the pole x where the Green function is not C^∞, and subtracting ψ skips the boundary, while the values of ψG^x are substituted by the approximation φ_k. Since $\psi(y) = \varphi_k(y) = \eta_k(y) = 0$, for k large enough

$$\begin{aligned} G^x(y) = f_k(y) &\overset{(3.9)}{=} \int_\Omega \nabla f_k \cdot \nabla G^y \, dm \\ &= \int_\Omega \nabla G^x \cdot \nabla G^y \, dm + \int_\Omega \nabla(f_k - G^x) \cdot \nabla G^y \, dm. \end{aligned} \tag{3.13}$$

The lemma follows if we prove that

$$\left| \int_\Omega \nabla(f_k - G^x) \cdot \nabla G^y \, dm \right| \xrightarrow{k\to\infty} 0 \tag{3.14}$$

Indeed,

$$G^x - f_k = (\eta_k + \psi)G^x - \varphi_k,$$

and

$$\nabla(G^x - f_k) = \nabla\eta_k G^x + \eta_k \nabla G^x + \nabla(\psi G^x - \varphi_k).$$

Since $y \notin \mathrm{supp}\nabla(G^x - f_k)$, ∇G^y stays bounded in the integral (3.14). For $z \in U \subset\subset \mathbb{R}^d\backslash\{x\}$ also G^x and ∇G^x stay bounded. Therefore we only need to show that

$$\boxed{1} := \int_U |\nabla(\psi G^x - \varphi_k)| \xrightarrow{k\to\infty} 0,$$

and

$$\boxed{2} := \int_{B_{1/k}(x)} |\nabla\eta_k(z)G^x(z) + \eta_k(z)\nabla G^x(z)| \xrightarrow{k\to\infty} 0.$$

By the Cauchy-Schwartz inequality, since $|U| < \infty$, using (3.12) we get the integrability of the first term:

$$\boxed{1} \leqslant |U|^{\frac{1}{2}} \|\nabla(\psi G^x - \varphi_k)\|_2 \xrightarrow{k\to\infty} 0.$$

Finally, for $d \geqslant 3$ and k large enough, we can neglect the v^x term and bound the last term by

$$\boxed{2} \lesssim \int_{B_{1/k}(x)} k|x-z|^{2-d} + |x-z|^{1-d} \leqslant k\frac{1}{k^2} + \frac{1}{k} \xrightarrow{k\to\infty} 0,$$

proving (3.14). When $d = 2$ the limit is also 0:

$$\int_{B_{1/k}(x)} k|\log(|x-z|)| + |x-z|^{-1} \lesssim k\,\frac{1}{k^2}\left(-\log(k)+\frac{1}{2}\right)+\frac{1}{k} \xrightarrow{k\to\infty} 0.$$

□

Consider $f \in C_c^\infty(\Omega)$. Then define

$$v(x) := -\int_\Omega G^x(y)f(y)\,dy = -f * \mathcal{E}(x) - \int_\Omega v^x(y)f(y)\,dy.$$

Since v^x is harmonic, $\Delta v = f$ in Ω. Moreover, if G^x is continuous up to the boundary, then $G^x(y)$ vanishes for $x \in \partial\Omega$. So v is the natural candidate to be the solution to the Dirichlet problem

$$\begin{cases} \Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

Assuming regularity on $\partial\Omega$, we can define the Poisson kernel

$$P^x(\xi) := -\partial_\nu G^x(\xi) \qquad \text{for every } x \in \Omega,\ \xi \in \partial\Omega.$$

If $u \in C(\overline{\Omega})$ is harmonic in Ω, then we can write formally

$$\begin{aligned} u(x) = \int u(z)\delta_x(z) &= \int_\Omega (u(z)(-\Delta G^x(z)) + \Delta u(z)G^x(z)) \\ &\overset{\text{Green}}{=} \int_{\partial\Omega} (-u(\zeta)\partial_\nu G^x(\zeta) + \partial_\nu u(\zeta)G^x(\zeta))d\zeta. \end{aligned}$$

If G^x vanishes continuously in the boundary, we get that

$$u(x) = \int_{\partial\Omega} u(\zeta)P^x(\zeta)d\zeta.$$

Therefore, we expect that the Dirichlet problem (3.1) may be solved by integrating the boundary values times the Poisson kernel for regular enough domains. Harmonic measure will be a generalization of the Poisson kernel to more rough domains.

Exercise 3.2.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, and let G be its Green function. Then

$$R^{d-2}G\left(\frac{x}{R},\frac{y}{R}\right)$$

is the Green function for the set $R\Omega = \{Rx : x \in \Omega\}$.

3.3 Limitations of the weak formulation

The weak solution to the Dirichlet problem exposed above is only half-satisfactory. We get existence and uniqueness for every bounded open set, but it is not quite clear what does it mean to have 0 trace. In practical applications of (3.1) we would like to prescribe boundary values f only in the boundary of the domain, and not in a neighborhood of it. Moreover, one should expect that in case f is continuous, then the solution u is continuous up to the boundary, with $u|_{\partial\Omega} \equiv f$. However, the weak solutions above may not be continuous up to the boundary.

Example 3.8. Let $\Omega = B_1\backslash\{0\} \subset \mathbb{R}^d$ with $d \geqslant 3$, and take $f = 0$ in $\partial B_1(0)$ and $f(0) = 1$. A natural candidate to "represent" f in $H^1(\Omega)$ is the function $F(x) = 1 - |x|\chi_{B_1}$ is in $H^1(\Omega)$. Let us see that its class in $H^1_0(\Omega)$ coincides with the class of $G(x) \equiv 0$, i.e., let's show that $F - G = F \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$.

Let $\eta \in C^\infty(\mathbb{R})$ such that $\chi_{(-\infty,1/2)} \leqslant \eta \leqslant \chi_{(-\infty,1)}$. Then let $\varphi_\varepsilon(x) = \eta(\varepsilon^{-1}|x|)$ and let $\psi_\varepsilon(x) = \eta(\varepsilon^{-1}(|x| - 1 + \varepsilon))$, and consider $h_\varepsilon := \psi_\varepsilon(1-\varphi_\varepsilon)F \in C_c^\infty(\Omega)$. Then we have that $F = h_\varepsilon$ in $B_1^c \cup (B_{1-\varepsilon}\backslash B_\varepsilon)$

$$\|F - h_\varepsilon\|_2 = \|(1-\psi_\varepsilon(1-\varphi_\varepsilon))(1-|x|\chi_{B_1})\|_2 \leqslant (|B_1\backslash(B_{1-\varepsilon} \cup B_\varepsilon|)^{\frac{1}{2}} \xrightarrow{\varepsilon\to 0} 0.$$

On the other hand, since

$$\|\nabla\varphi_\varepsilon\|_\infty + \|\nabla\psi_\varepsilon\|_\infty \leqslant \varepsilon^{-1}\|\eta'\|_\infty,$$

and using that the support of $F - h_\varepsilon$ is contained in $\overline{B_1}\backslash B_{1-\varepsilon} \cup \overline{B_\varepsilon}$, using the product rule we deduce that

$$\begin{aligned}\|\nabla(F - h_\varepsilon)\|_2 &= \|\nabla[(1-\psi_\varepsilon(1-\varphi_\varepsilon))(1-|x|\chi_{B_1})]\|_{L^2(B_1\backslash B_{1-\varepsilon}\cup B_\varepsilon)} \\ &\leqslant \left(\|\varepsilon\nabla\psi_\varepsilon\|^2_{L^2(B_1\backslash B_{1-\varepsilon})} + \|\nabla\varphi_\varepsilon\|^2_{L^2(B_\varepsilon)}\right)^{\frac{1}{2}} + \|\nabla(|x|\chi_{B_1})\|_{L^2(B_1\backslash B_{1-\varepsilon}\cup B_\varepsilon))} \xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

We have seen that $F \in \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}$ and therefore $F \equiv 0$ in $H^1_0(\Omega)$. Thus, the weak solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = F & \text{on } \partial\Omega \end{cases} \tag{3.16}$$

is $u = 0$.

The example above is related to the fact that a point has capacity zero in $\mathbb{R}^d$ for every $d \geqslant 2$, see Chapter 6. We will see in further chapters that, in fact, there exists no harmonic function u in $\Omega = B_1\backslash\{0\} \subset \mathbb{R}^d$ vanishing in ∂B_1 such that $\lim_{z\to 0} u(z) = 1$ for $d \geqslant 2$.

Further, is there a one-to-one relation between $H^{\frac{1}{2}}(\partial\Omega)$ and some class of functions defined in $\partial\Omega$? If the boundary of the domain is regular enough (existence of local bi-lipschitz, C^1 parameterizations should suffice, for instance), then the traces $H^{\frac{1}{2}+\varepsilon}(\partial\Omega)$ of $W^{1+\varepsilon,2}$ coincide with the Besov space $B^{1/2+\varepsilon}_{2,2}(\partial\Omega)$, with an appropriate definition using partitions of the unity and local parameterizations, see [Tri83, Section 3.3.3], for instance.

3.4 Solvability of the Dirichlet problem for continuous functions: the case of the unit ball

Definition 3.9. We say that the Dirichlet problem (3.1) in an open set Ω is *solvable for continuous functions* if there exists a function $u_f \in C(\overline{\Omega})$ for every $f \in C(\partial\Omega)$ such that $\Delta u = 0$ in Ω and $u(y) = f(y)$ for $y \in \partial\Omega$.

Note that such a solution would be unique by the Uniqueness Theorem 2.7.

Next we will study the sovability of the Dirichlet problem for continuous functions in the case Ω is the unit ball. First we will need to introduce the Green function in the unit ball, which has a nice algebraic expression.

Lemma 3.10. *Let $x, y \in \mathbb{R}^d \backslash \{0\}$. Then*

$$\left| \frac{x}{|x|} - |x|y \right| = \left| |y|x - \frac{y}{|y|} \right|.$$

Proof. Let $t \in \mathbb{R}$, $t > 0$. Then

$$\left| \frac{x}{t} - ty \right|^2 = \frac{|x|^2}{t^2} - 2x \cdot y + t^2 |y|^2.$$

Evaluating for $t = |x|$ and for $t = |y|^{-1}$ we reach the same expression. $\square$

Define

$$v^x(y) := \begin{cases} -\mathcal{E}(\frac{x}{|x|} - |x|y) & \text{if } x \neq 0, \\ -\mathcal{E}(e_1) & \text{if } x = 0. \end{cases}$$

Note that for $|\xi| = 1$, $x \neq 0$ we get that $\left| \frac{x}{|x|} - |x|\xi \right| = |x - \xi|$ from the previous lemma, so $v^x(\xi) = -\mathcal{E}(x - \xi)$. The same happens when $x = 0$ because the fundamental solution depends only on the modulus. Moreover, for fixed $x \in B_1$, v^x has no singularity in B_1, given that

$$\frac{x}{|x|} - |x|y = 0 \implies y = \frac{x}{|x|^2} \implies y \notin B_1.$$

Therefore $v^x \in C^1(\overline{B_1}) \subset H^1(\Omega)$ and $\Delta v^x = 0$ in B_1. So the Green function (3.6) in the unit ball is

$$G^x(y) := \begin{cases} \mathcal{E}(x - y) - \mathcal{E}(\frac{x}{|x|} - |x|y) & \text{if } x \neq 0, \\ \mathcal{E}(-y) - \mathcal{E}(e_1) & \text{if } x = 0. \end{cases}$$

Note that $G^x(y) = G^y(x)$ by Lemma 3.10.

Now we can compute the Poisson kernel: for $x = 0$, $|\xi| = 1$, it is

$$\partial_\nu G^0(\xi) = \xi \cdot \nabla \mathcal{E}(\xi) \overset{(2.11)}{=} \xi \cdot \frac{-\xi}{\kappa_d |\xi|^d} = -\frac{1}{\kappa_d},$$

and for $x \neq 0$, $|\xi| = 1$ we get

$$\begin{aligned}
\partial_\nu G^x(\xi) &= \xi \cdot \nabla_y \left(\mathcal{E}(x-y) - \mathcal{E}\left(\frac{x}{|x|} - |x|y\right)\right)|_{y=\xi} \\
&\overset{(2.11)}{=} \xi \cdot \left(\frac{x-\xi}{\kappa_d |x-\xi|^d} - \frac{\frac{x}{|x|} - |x|\xi}{\kappa_d \left|\frac{x}{|x|} - |x|\xi\right|^d} |x| \right) \\
&\overset{\text{L. } 3.10}{=} \xi \cdot \left(\frac{x - \xi - \left(x - |x|^2\xi\right)}{\kappa_d\, |x-\xi|^d} \right) = |\xi|^2 \frac{|x|^2-1}{\kappa_d\, |x-\xi|^d} = \frac{|x|^2-1}{\kappa_d\, |x-\xi|^d}.
\end{aligned}$$

Summing up, for $x \in B_1$ and $|\xi| = 1$ we get

$$P^x(\xi) = \frac{1-|x|^2}{\kappa_d\, |x-\xi|^d}. \tag{3.17}$$

Theorem 3.11. *Let $f \in L^1(\partial B_1)$ and define*

$$u_f(x) := \int_{\partial B_1} P^x(\zeta) f(\zeta)\, d\sigma(\zeta) \qquad \text{for } x \in B_1.$$

Then u is harmonic on B_1. If f is continuous, then $u_f \in C(\overline{B_1})$, with $u_f|_{\partial B_1} = f$. If $f \in L^p(\partial B_1)$, then $u_f(r\cdot) \to f$ in $L^p(\partial B_1)$ as $r \to 1$.

Proof. The function u_f is well defined because the Poisson kernel is bounded for x fixed. Since G is harmonic on x, P is also harmonic on x and so is u_f.

We claim that for every $x \in \partial B_1$, $P^x\, d\sigma$ is a probability measure, i.e.,

$$\int_{\partial B_1} P^x\, d\sigma = 1. \tag{3.18}$$

Indeed, for $x = 0$ it is trivial. By (3.17), the mean value theorem and Lemma 3.10 we get

$$\frac{1}{\kappa_d} \overset{(3.17)}{=} P^0\left(\frac{x}{|x|}\right) \overset{(2.2)}{=} \fint P^{|x|\xi}\left(\frac{x}{|x|}\right) d\sigma(\xi) \overset{\text{L. } 3.10}{=} \fint P^x(\xi)\, d\sigma(\xi),$$

as claimed.

If f is continuous and $\xi \in \partial B_1$, then

$$\begin{aligned}
|f(\xi) - u_f(r\xi)| &\overset{(3.18)}{=} \left| \int_{\partial B_1} P^{r\xi}(\zeta)(f(\xi) - f(\zeta))\, d\sigma(\zeta) \right| \\
&\leqslant \int_{|\zeta-\xi|\leqslant\delta} \left|P^{r\xi}(\zeta)\right| |f(\xi)-f(\zeta)|\, d\sigma(\zeta) + \int_{|\zeta-\xi|>\delta} \left|P^{r\xi}(\zeta)\right| |f(\xi)-f(\zeta)|\, d\sigma(\zeta) \\
&\overset{(3.18)}{\leqslant} \sup_{|\zeta-\xi|\leqslant\delta} |f(\xi)-f(\zeta)| + 2\|f\|_\infty \sup_{|\zeta-\xi|>\delta} \left|P^{r\xi}(\zeta)\right|.
\end{aligned}$$

The first term in the right-hand side of the last estimate can be made arbitrarily small by fixing δ small enough, and then the second term can also be made small by choosing r close enough to 1. Choices can be made independently of ξ. This shows that $u_f(r\cdot)$ converges uniformly to u_f, and this implies global continuity.

If $f \in L^p(\partial B_1)$, then we can use the density of C^∞ on L^p to find a function $f_\varepsilon \in C^\infty(\partial B_1)$ with $\|f - f_\varepsilon\|_{L^p(\partial B_1)} \leqslant \varepsilon$. Now,

$$\|f - u_f(r\cdot)\|_{L^p(\partial B_1)} \leqslant \|f - f_\varepsilon\|_{L^p(\partial B_1)} + \|f_\varepsilon - u_{f_\varepsilon}(r\cdot)\|_{L^p(\partial B_1)} + \|u_{f_\varepsilon}(r\cdot) - u_f(r\cdot)\|_{L^p(\partial B_1)}.$$

Choosing ε small enough and r close enough to 1, the two first terms can be made arbitrarily small.

Regarding the last one, we claim that $\|u_{f_\varepsilon}(r\cdot) - u_f(r\cdot)\|_{L^p(\partial B_1)} \leqslant \|f_\varepsilon - f\|_{L^p(\partial B_1)}$. Indeed, for $p = 1$ we have

$$\|u_g(r\cdot)\|_{L^1(\partial B_1)} \leqslant \int_{\partial B_1}\int_{\partial B_1} P^{r\xi}(\zeta)\,|g(\zeta)|\,d\sigma(\zeta)\,d\sigma(\xi) \leqslant \|g\|_{L^1(\partial B_1)} \int_{\partial B_1} P^{r\xi}(\zeta)\,d\sigma(\xi).$$

Note that the mean value theorem

$$\int_{\partial B_1} P^{r\xi}(\zeta)\,d\sigma(\xi) = \kappa_d P^0(\zeta) = 1,$$

so $g \mapsto u_g$ is bounded in $L^1(\partial B_1)$ with norm 1. On the other hand,

$$\|u_g(r\cdot)\|_{L^\infty(\partial B_1)} \leqslant \sup_{\xi\in\partial B_1} \int_{\partial B_1} P^{r\xi}(\zeta)\,|g(\zeta)|\,d\sigma(\zeta) \leqslant \|g\|_\infty \sup_{\xi\in\partial B_1} \int_{\partial B_1} P^{r\xi}(\zeta)\,d\sigma(\zeta) \overset{(3.18)}{=} \|g\|_\infty.$$

By interpolation we get that $f \mapsto u_f(r\cdot)$ is a bounded operator in $L^p(\partial B_1)$ with norm 1. This fact proves the claim and, therefore, the L^p convergence follows. □

Remark 3.12. For the ball $B_r(0)$, with $r > 0$, we have a similar result. In this case the Poisson kernel for $B_r(0)$ equals

$$P^x_{B_r(0)}(\xi) = \frac{r^2 - |x|^2}{\kappa_d\, r\, |x-\xi|^d}.$$

Then the same result as in Theorem 3.11 holds for $f \in L^1(\partial B_r(0))$, with $P^x(\zeta)$ replaced by $P^x_{B_r(0)}(\zeta)$. That is, the function

$$u_f(x) := \int_{\partial B_r(0)} P^x_{B_r(0)}(\zeta)\,d\sigma(\zeta) \qquad \text{for } x \in B_r(0),$$

solves the Dirichlet problem with boundary data f in $B_r(0)$ when f is continuous. Also, for $f \in L^p(\partial B_r(0))$, we have that $u_f(r\cdot) \to f$ in $L^p(\partial B_r(0))$ as $r \to 1$.

3.5 Double layer potential: exploiting the jump formulas

When a domain Ω has bounded and smooth boundary, say $\partial\Omega \in C^{1+\epsilon}$, then a usual way to solve the Dirichlet problem (3.1) for continuous functions is via the double layer potential. We will not prove here the results, but we will sketch the main ideas, which can be found for instance in [Fol95, Chapter 3].

Consider the gradient of the fundamental solution

$$\nabla \mathcal{E}^x(y) = \frac{(x-y)}{\kappa_d |x-y|^d},$$

which is the kernel of the so-called Riesz transform of homogeneity $1-d$. In particular, the normal derivative of $\mathcal{E}$ in the boundary of Ω,

$$K^x(\zeta) := \partial_\nu \mathcal{E}^x(\zeta) = \nu(\zeta) \cdot \nabla \mathcal{E}^x(\zeta) = \frac{(x-\zeta)\cdot \nu(\zeta)}{\kappa_d |x-\zeta|^d}$$

for $\zeta \in \partial\Omega$ and $x \in \mathbb{R}^d \backslash \{\zeta\}$ is well defined whenever $\partial\Omega$ has C^1 parameterizations. Then for every $g \in C(\partial\Omega)$ and every $x \in \mathbb{R}^d \backslash \partial\Omega$, we can consider the double layer potential

$$\mathcal{D}g(x) := \int_{\partial\Omega} K^x(\zeta) g(\zeta) d\sigma(\zeta),$$

which is harmonic in $(\partial\Omega)^c$.

The double layer potential is not well defined a priori in the boundary of the domain, but it makes sense to define its principal value for $\xi \in \partial\Omega$ as

$$T_K(g)(\xi) := \text{p.v.}\mathcal{D}g(\xi) = \lim_{\varepsilon\to 0} \int_{\partial\Omega\backslash B_\varepsilon(\xi)} K^x(\zeta) g(\zeta) d\sigma(\zeta). \tag{3.19}$$

This pointwise definition does not coincide with the (non-tangential) limit of the double layer potential,

$$\mathcal{D}g(\xi) := \text{n.t.}\lim_{x\to\xi} \mathcal{D}g(x) = \lim_{x\to\xi : 2\mathrm{d}_\Omega(x) \geqslant |x-\xi|} \mathcal{D}g(x),$$

where $\mathrm{d}_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$. However, they are related by the so-called jump formula:

$$\mathcal{D}g(\xi) = \frac{1}{2} g(\xi) + T_K(g)(\xi),$$

which is a consequence of the identities

$$\int K^x(\zeta)\, d\sigma(\zeta) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 1/2 & \text{if } x \in \partial\Omega, \text{ understood as a principal value,} \\ 0 & \text{if } x \in \overline{\Omega}^c. \end{cases}$$

When the boundary has parameterizations in $C^{1+\varepsilon}$, the normal vector becomes Hölder continuous and the singularity of K^x is of homogeneity below $d-1$, and it is therefore integrable with respect to the surface measure, so we can omit the principal value in (3.19).

Then the kernel K^x becomes somewhat *smoothing* in this case, in the sense that T_K maps $L^\infty(\partial\Omega)$ to $C(\partial\Omega)$ for instance, and it is compact in $L^2(\partial\Omega)$, and the operator $\frac{1}{2}\,\mathrm{I}+T_K$ is Fredholm in $L^2(\partial\Omega)$. Moreover, if $(\frac{1}{2}\,\mathrm{I}+T_K)(g) \in C(\Omega)$ with $g \in L^2(\partial\Omega)$, then $g \in C(\partial\Omega)$.

In fact, if Ω is simply connected and $C^{1+\varepsilon}$, then $\frac{1}{2}\,\mathrm{I}+T_K$ happens to be invertible in $L^2(\partial\Omega)$. Thus, given $f \in C(\Omega)$, one can find a unique solution to the Dirichlet problem by finding the unique solution to the equation $f = (\frac{1}{2}\,\mathrm{I}+T_K)(g)$. Then $u := \mathcal{D}(g)$, i.e. $u = \mathcal{D}(\frac{1}{2}\,\mathrm{I}+T_K)^{-1}(f)$ satisfies (3.1) in the sense that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \text{n.t.}\lim_{x\to\xi} u(x) = f(\xi) & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

If Ω is multiply connected, some modifications related to the connectivity of the complement need to be done in order to find an inverse operator in a suitable function space.

The Dirichlet problem in the unbounded component can also be solved in this way, and assuming a priori that the solution u_f satisfies that $u_f(x) = O_{x\to\infty}(|x|^{3-d})$ one can get also uniqueness.

4 Basic results from measure theory and weights

4.1 Measures

Following [Mat95] or [EG15], we will define a measure on a set X as a function on the parts of X, regardless of the σ-algebra of measurable sets. This is often called exterior measure in some references, but it is quite elementary to define the σ-algebra of measurable sets once the (exterior) measure is given. Conversely, every countably additive non-negative set function on a σ-algebra of subsets of X can be extended to every set, see [Mat95]. Let us assume that X is a metric space.

Definition 4.1. We say that $\mu : \{A : A \subset X\} \to \mathbb{R}$ is a measure if

1. $\mu(\varnothing) = 0$,
2. $\mu(A) \leqslant \mu(B)$ whenever $A \subset B \subset X$ and
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} \mu(A_i)$, whenever $A_i \subset X$ for every $1 \leqslant i < \infty$.

We say that $A \subset X$ is μ-measurable if

$$\mu(E) = \mu(E \cap A) + \mu(E \backslash A) \qquad \text{for every } E \subset X.$$

Definition 4.2. Given a set X, we say that a collection Σ of subsets of X is a *σ-algebra* whenever Σ is closed under complement, countable unions, and countable intersections. When X is a topological space, we define the collection of *Borel sets of X* as the minimal σ-algebra containing all the open sets in the topology.

Lemma 4.3. *The measurable sets form a σ-algebra. If $\{A_i\}_{i=1}^{\infty}$ is a collection of μ-measurable and pairwise disjoint sets, then*

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i). \tag{4.1}$$

If $\{B_i\}_{i=1}^{\infty}$ is a collection of μ-measurable sets with $B_i \nearrow B$, i.e., if $B_1 \subset B_2 \subset \cdots$ and $B = \bigcup_i B_i$, then $\mu(B) = \lim_i \mu(B_i)$.

If $\{C_i\}_{i=1}^{\infty}$ is a collection of μ-measurable sets with $C_i \searrow C$, i.e., if $C_1 \supset C_2 \supset \cdots$ and $C = \bigcap_i C_i$, and moreover $\mu(C_1) < +\infty$, then $\mu(C) = \lim_i \mu(C_i)$.

Definition 4.4. Let μ be a measure on a metric space X.

1. μ is a Borel measure if all Borel sets are μ-measurable.
2. μ is a Borel regular measure if it is a Borel measure and for every $A \subset X$ there is a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$.
3. μ is a Radon measure if it is Borel,
   a) $\mu(K) < \infty$ for every compact set $K \subset X$,
   b) $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact}\}$ for every open set $V \subset X$,
   c) $\mu(A) = \inf\{\mu(V) : V \supset A \text{ is open}\}$ for every set $A \subset X$.
4. In those cases, if the metric space is separable we say that $\operatorname{supp}\mu := \bigcap_{F=\overline{F}:\, \mu(F^c)=0} F$.

Proposition 4.5 ([EG15, Theorem 1.8])**.** *Let μ be a Radon measure in $\mathbb{R}^d$. Then, for each μ-measurable set A*

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ is compact}\}.$$

Proposition 4.6 ([Mat95, Corollary 1.11])**.** *Every locally finite Borel measure is a Radon measure.*

4.2 Integration

Let μ be a measure in $\mathbb{R}^d$. We say that $\phi : \mathbb{R}^d \to \mathbb{R}$ is a simple function whenever there exist a finite number of μ-measurable sets $\{A_j\}_{j=1}^N$ and coefficients $\{\alpha_j\}_{j=1}^N \subset \mathbb{R}$ such that

$$\phi = \sum_{j=1}^N \alpha_j \chi_{A_j}.$$

We can define its integral by

$$\int \phi \, d\mu := \sum_{j=1}^N \alpha_j \mu(A_j).$$

The set of simple functions is denoted by $\mathcal{S}_\mu$. Note that for $\phi \in \mathcal{S}_\mu$, the decomposition described above is not unique, but its choice does not change the value of the integral. Given a non-negative measurable function $f : \mathbb{R}^d \to \mathbb{R}$ (i.e., a function such that $f^{-1}(r, +\infty)$ is measurable for every $r \in \mathbb{R}$), we define its integral

$$\int f \, d\mu := \sup\left\{\int \phi \, d\mu : \ \phi \in \mathcal{S}_\mu \text{ with } 0 \leqslant \phi \leqslant f\right\}.$$

Integration in measurable subsets is defined as

$$\int_A f \, d\mu := \int f \chi_A \, d\mu.$$

Theorem 4.7 (Fubini's theorem). *Suppose that μ, ν are locally finite Borel measures on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. If f is a non-negative Borel function on $\mathbb{R}^{d_1+d_2}$, then*

$$\int\int f(x,y)\,d\mu(x)\,d\nu(y) = \int\int f(x,y)\,d\nu(y)\,d\mu(x).$$

Corollary 4.8. *Suppose that μ is a locally finite Borel measure on $\mathbb{R}^d$. If f is a non-negative Borel function on $\mathbb{R}^d$, then*

$$\int f(x)\,d\mu(x) = \int_0^\infty \mu(\{x \in \mathbb{R}^d : f(x) \geqslant t\})\,dt.$$

Given a μ-measurable function $f : \mathbb{R}^d \to \mathbb{R}$, and $0 < p < \infty$, we say that $f \in L^p(\mu)$ whenever $\|f\|_{L^p(\mu)} := \left(\int |f|^p\,d\mu\right)^{\frac{1}{p}} < +\infty$. In case $f \in L^1(\mu)$, we can define

$$\int f\,d\mu := \int f_+\,d\mu - \int f_-\,d\mu,$$

where

$$f_+ := \max\{f, 0\}, \qquad \text{and} \qquad f_- := \max\{-f, 0\}.$$

Note that $f = f_+ - f_-$, with $f_+, f_- \geqslant 0$. We say that $f \in L^1_{\mathrm{loc}}(\mu)$ if $f\chi_K \in L^1(\mu)$ for every compact set K. In this case we can define the centered Hardy-Littlewood maximal operator

$$M_\mu f(x) := \sup_{r>0} \fint_{B_r(x)} |f|\,d\mu,$$

and the uncentered maximal operator

$$M_{\mu,u} f(x) := \sup_{B \ni x} \fint_B |f|\,d\mu.$$

We say a measurable function f is in the weak space $L^{p,\infty}(\mu)$, writting $f \in L^{p,\infty}(\mu)$, whenever

$$\|f\|_{L^{p,\infty}(\mu)} := \sup_{0<t<\infty} t\,(\mu\{x : |f(x)| > t\})^{\frac{1}{p}} < \infty.$$

Jensen's inequality $\fint_A f \lesssim \left(\fint_A |f|^p\,d\mu\right)^{\frac{1}{p}}$, extends to the weak space as follows (see [Mat95, Lemma 20.24], for instance)

Lemma 4.9. *Both maximal operators M_μ and $M_{\mu,u}$ are bounded operators from L^1 to $L^{1,\infty}$, and from L^p to L^p, see [Mat95, Chapter 2] or [Gra08, Exercise 2.1.1].*

Lemma 4.10 (Kolmogorov's inequality). *Let μ be a Radon measure in $\mathbb{R}^d$, and let $g : \mathbb{R}^d$ be a Borel function such that $\|g\|_{L^{p,\infty}} < \infty$, with $1 < p < \infty$. Then for every μ-measurable set $A \subset \mathbb{R}^d$ with $\mu(A) < \infty$ we have*

$$\fint_A |g|\,d\mu \leqslant \frac{p}{p-1} \frac{\|g\|_{L^{p,\infty}(\mu)}}{\mu(A)^{\frac{1}{p}}}.$$

Exercise 4.2.1. Let μ be a Radon measure and let $E = \mathrm{supp}(\mu)$. Show that continuous functions are dense in $L^1(\mu)$. Hint: Use the density of simple functions and via regularity and Urysohn's lemma, find continuous functions approximating f in the L^1 norm.

4.3 Differentiation of measures

Definition 4.11. Let μ and ν be Radon measures on $\mathbb{R}^d$. We say that ν is differentiable with respect to μ at $x \in \operatorname{supp}(\mu)$ if the limit

$$\frac{d\nu}{d\mu}(x) := \lim_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$$

exists and is finite. We call this limit the *density* (or the *derivative*) of ν with respect to μ.

Theorem 4.12 (see [Rud87, Theorem 1.29])**.** *Whenever μ and ν are Radon measures, the density $\frac{d\nu}{d\mu}$ is a μ-measurable function well defined μ-almost everywhere.*

Definition 4.13. Let μ and ν be Borel measures on $\mathbb{R}^d$. The measure ν is *absolutely continuous* with respect to μ, written $\nu \ll \mu$ if

$$\mu(A) = 0 \implies \nu(A) = 0, \text{ for all } A \subset \mathbb{R}^d.$$

The measures are *mutually singular*, written $\nu \perp \mu$, if there exists a Borel set $B \subset \mathbb{R}^d$ so that

$$\mu(\mathbb{R}^d \backslash B) = 0 = \nu(B).$$

Theorem 4.14 (Radon-Nikodym derivative, see [Rud87, Theorem 1.30])**.** *Let μ and ν be Radon measures on $\mathbb{R}^d$, with $\nu \ll \mu$. Then $\frac{d\nu}{d\mu} \in L^1(\mu)$ and*

$$\nu(A) = \int_A \frac{d\nu}{d\mu}\, d\mu \tag{4.2}$$

for all μ-measurable sets $A \subset \mathbb{R}^d$.

Theorem 4.15 (Lebesgue decomposition theorem)**.** *Let μ and ν be Radon measures on $\mathbb{R}^d$. Then*

$$\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{s}},$$

where ν_{ac} and ν_{s} are Radon measures such that

$$\nu_{\mathrm{ac}} \ll \mu \quad \textit{and} \quad \nu_{\mathrm{s}} \perp \mu.$$

Moreover,

$$\frac{d\nu}{d\mu}(x) = \frac{d\nu_{\mathrm{ac}}}{d\mu}(x) \quad \textit{and} \quad \frac{d\nu_{\mathrm{s}}}{d\mu}(x) = 0 \qquad \mu-\text{a.e.}\, x \in \mathbb{R}^d,$$

so

$$\nu(A) = \int_A \frac{d\nu}{d\mu}\, d\mu + \nu_{\mathrm{s}}(A)$$

for all Borel sets $A \subset \mathbb{R}^d$.

Theorem 4.16 (Lebesgue differentiation theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^d$, and $f \in L^1_{\text{loc}}(\mu)$. Then*

$$\lim_{r\to 0} \fint_{B_r(x)} f\, d\mu = f(x), \quad \textit{for } \mu\textit{-a.e. } x \in \mathbb{R}^d,$$

and the same can be said replacing balls by cubes centered at x.

We say that a point x satisfying the previous equality is a *density point* of f with respect to μ.

Exercise 4.3.1. Let μ and ν be Radon measures on $\mathbb{R}^d$, with $\nu \ll \mu$ and $\nu(\mathbb{R}^d) > 0$. Show that there exists a μ-measurable set G with $\nu(G) > 0$ and $\mu|_G \ll \nu|_G \ll \mu|_G$.

Exercise 4.3.2. Let μ and ν be Radon measures on $\mathbb{R}^d$, with $\nu \ll \mu$, and let $f \in L^1(\nu)$ be a Borel function. Show that $f\frac{d\nu}{d\mu} \in L^1(\mu)$, with

$$\int f\, d\nu = \int f \frac{d\nu}{d\mu}\, d\mu.$$

4.4 Muckenhoupt weights in general measures

In this section we define quantitative versions of mutual absolute continuity such as A_∞ for measures supported in closed subsets of $\mathbb{R}^d$.

In this section we will consider a Radon measure μ, X will denote its support and we will consider balls in X:

Definition 4.17. Let $X \subset \mathbb{R}^d$. For every $x \in X$ and $r > 0$ we write the restricted ball

$$\Delta_{r,x} := \Delta_r(x) := B_r(x) \cap X.$$

Note that, in particular, we always assume that restricted balls are centered in X. We also use the classical notation for rescaled balls in the setting of restricted balls:

$$t\Delta_{r,x} := \Delta_{tr,x}$$

Definition 4.18. Let μ be a Radon measure in $\mathbb{R}^d$. Given a Radon measure ν supported in $X := \operatorname{supp} \mu$, we say that $\nu \in AB_{\delta_0,\varepsilon_0}(\mu, U)$ in an open set $U \subset X$ if for every restricted ball $\Delta \subset U$,

$$\mu(E) < \delta_0 \mu(\Delta) \implies \nu(E) \leqslant \varepsilon_0 \nu(\Delta) \quad \forall x \in U,\, E \subset \Delta \text{ Borel}. \tag{4.3}$$

We say that $\nu \in A_\infty(\mu, U)$ if for every $\delta_0 \in (0, 1)$, there exists $\varepsilon_0 \in (0, 1)$ such that (4.3) is satisfied. We say that $\nu \in B_1(\mu, U)$ if for every $\varepsilon_0 \in (0, 1)$, there exists $\delta_0 \in (0, 1)$ such that (4.3) is satisfied. If this is satisfied for $U = E$ we simply omit U.

Remark 4.19. Note that the existence of δ_0 and ε_0 satisfying (4.3) implies that $\nu|_U \ll \mu|_U$ by the dyadic Lebesgue differentiation theorem. Indeed, if $\nu(E) > 0$, there exists a point $x \in E \cap \operatorname{supp}\mu$ with

$$\lim_{r\to 0} \frac{\nu(E \cap \Delta_{r,x})}{\nu(\Delta_{r,x})} = 1.$$

Thus, for r small enough we need to have $\nu(E \cap \Delta_{r,x}) > \varepsilon_0 \nu(\Delta_{r,x})$ and $\Delta_{r,x} \subset U$, and we get $\mu(E) \geqslant \delta_0 \mu(\Delta_{r,x}) > 0$, so we have shown absolute continuity.

Moreover, note that (4.3) is equivalent to

$$\mu(E) > (1-\delta_0)\mu(\Delta) \implies \nu(E) \geqslant (1-\varepsilon_0)\nu(\Delta) \quad \forall x \in U,\ E \subset \Delta \subset U \text{ Borel}, \tag{4.4}$$

by substituting E by $\Delta \backslash E$. Thus, it is also equivalent to

$$\nu(E) < (1-\varepsilon_0)\nu(\Delta) \implies \mu(E) \leqslant (1-\delta_0)\mu(\Delta).$$

Note that this implies $\operatorname{supp}\mu = \operatorname{supp}\nu$. By symmetry, we have shown that

$$\nu \in AB_{\delta_0,\varepsilon_0}(\mu, U) \iff \mu \in AB_{1-\varepsilon_0,1-\delta_0}(\nu, U),$$

so in any case we get $\mu|_U \ll \nu|_U$. Put in other words, the $AB_{\delta,\varepsilon}$ condition is a quantitative version of mutual absolute continuity.

Note that Theorem 4.14 implies that $d\nu = w d\mu$, so we will write also $w = \frac{d\nu}{d\mu} \in AB_{\delta_0,\varepsilon_0}(\mu, U)$. Given $w \in L^1_{\text{loc}}(\mu)$, one can construct such a measure ν using (4.2).

Next we define the reverse Hölder classes of weights B_p and the Muckenhoupt classes A_p.

Definition 4.20. Let μ be a Radon measure in $\mathbb{R}^d$ and let $X := \operatorname{supp}\mu$. Given $w \in L^1_{\text{loc}}(\mu)$ and a relatively open set $U \subset X$,

- we say that $w \in B_p(\mu, U)$ whenever the following reverse Hölder inequality is satisfied

$$\left(⨍_\Delta w^p \, d\mu \right)^{\frac{1}{p}} \leqslant C ⨍_\Delta w \, d\mu \text{ for every restricted ball } \Delta \text{ with } \Delta \subset U.$$

- we say that $w \in A_p(\mu, U)$ whenever $w^{\frac{1}{1-p}} \in L^1_{\text{loc}}(\mu)$ and

$$⨍_\Delta w \, d\mu \left(⨍_\Delta w^{\frac{1}{1-p}} \, d\mu \right)^{p-1} \leqslant C \text{ for every restricted ball } \Delta \text{ with } \Delta \subset U.$$

The minimal constants satisfying these properties are called $[w]_{B_p(\mu,U)}$ and $[w]_{A_p(\mu,U)}$ respectively. If this is satisfied for $U = \mathbb{R}^d$ we omit U in the notation. If we call $\nu := w d\mu$, then we may write also $\nu \in B_p(\mu, U)$ and $\nu \in A_p(\mu, U)$ respectively.

Remark 4.21. Let μ be a Radon measure in $\mathbb{R}^d$, let $X := \operatorname{supp}\mu$ and let $F \subset X$ be a set with positive measure. Assume that $w \in L^1_{\mathrm{loc}}(\mu)$ satisfies

$$\left(\fint_F w^p \, d\mu \right)^{\frac{1}{p}} \leqslant C_1 \fint_F w \, d\mu. \tag{4.5}$$

If E is a μ-measurable subset of F, then by the Hölder inequality,

$$\int_E w \, d\mu \leqslant \left(\int_F w^p \, d\mu \right)^{\frac{1}{p}} \mu(E)^{1-\frac{1}{p}} = \left(\fint_F w^p \, d\mu \right)^{\frac{1}{p}} \mu(F)^{\frac{1}{p}} \mu(E)^{1-\frac{1}{p}}.$$

If we define ν using (4.2), by (4.5) we obtain

$$\frac{\nu(E)}{\nu(F)} \leqslant C_1 \left(\frac{\mu(E)}{\mu(F)} \right)^{1-\frac{1}{p}}.$$

Thus, for every $0 < \varepsilon_0 < 1$, writing $\delta_0 := (C_1^{-1}\varepsilon_0)^{\frac{1}{\alpha}}$ we have that

$$\mu(E) < \delta_0 \mu(F) \quad \Longrightarrow \quad \nu(E) \leqslant \varepsilon_0 \nu(F).$$

In particular, $\nu|_F \ll \mu|_F$.

Let us write $\nu \in B_1^\alpha(\mu, U)$ if $\operatorname{supp}\nu \subset \operatorname{supp}\mu$ and there exists $C_1 \geqslant 1$ such that

$$\frac{\nu(E)}{\nu(\Delta)} \leqslant C_1 \left(\frac{\mu(E)}{\mu(\Delta)} \right)^{\alpha} \quad \text{for every Borel } E \subset \Delta \subset U. \tag{4.6}$$

When we consider all restricted balls $\Delta \subset U$ in the estimates above, we get

$$B_p(\mu, U) \subset B_1^{1-\frac{1}{p}}(\mu, U) \subset B_1(\mu, U) \subset \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}(\mu, U).$$

Remark 4.22. If μ is a Radon measure and $\nu \in AB_{\delta,\varepsilon}(\mu, U)$, then $\nu \ll \mu \ll \nu$ by Remark 4.19. In particular, $0 < w^{-1} < +\infty$ μ-a.e.

In general, given a Borel set $F \subset X$, if we assume

$$\fint_F w \, d\mu \left(\fint_F w^{\frac{1}{1-p}} \, d\mu \right)^{p-1} \leqslant C,$$

then $0 < w < +\infty$ μ-a.e. in F as well. Writing $w \, d\mu = d\nu$ in the integral, see Exercise 4.3.2, the preceding estimate is equivalent to

$$\frac{\nu(F)^p}{\mu(F)^p} \left(\fint_F w^{-p'} \, d\nu \right)^{p-1} \leqslant C,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. That is, w^{-1} satisfies (4.5) with $d\nu$ in place of $d\mu$, and p' instead of p.

When we consider these condicions for all $\Delta \subset U$ in place of F, we get

$$w^{-1} \in B_{p'}(\nu, U) \iff w \in A_p(\mu, U),$$

with $[w]_{A_p(\mu,U)} = [w^{-1}]_{B_{p'}(\nu,U)}^{p'}$.

Back to Remark 4.21 we get

$$A_p(\mu,U) \subset A_\infty^{\frac{1}{p}}(\mu,U) \subset A_\infty(\mu,U) \subset \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}(\mu,U),$$

where we say that $\nu \in A_\infty^\alpha(\mu,U)$ if $\mu \in B_1^\alpha(\nu,U)$. This is a consequence of the *duality* relation

$$\begin{aligned} \nu \in A_p(\mu,U) &\iff \mu \in B_{p'}(\nu,U) \\ \nu \in A_\infty^\alpha(\mu,U) &\iff \mu \in B_1^\alpha(\nu,U) \\ \nu \in A_\infty(\mu,U) &\iff \mu \in B_1(\nu,U) \\ \nu \in AB_{\delta_0,\varepsilon_0}(\mu,U) &\iff \mu \in AB_{1-\varepsilon_0,1-\delta_0}(\nu,U). \end{aligned} \tag{4.7}$$

4.5 Dyadic grids in metric spaces with a doubling measure

To establish the weight theory for doubling measures whose support is not dense in $\mathbb{R}^d$, we will use a dyadic decomposition of $\operatorname{supp}\mu$.

Definition 4.23. Given a closed set $X \subset \mathbb{R}^d$, a dyadic grid associated to X with constants $0 < \ell_0 < 1$, $0 < a_1 < \infty$ is a collection $\mathcal{D}$ of Borel subsets of X such that $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ with $\mathcal{D}_k = \{Q^i\}_{i\in I_k}$, with the following properties:

(a) *Completeness*: for every $k \in \mathbb{Z}$ we have $X = \bigcup_{Q\in\mathcal{D}_k} Q$.

(b) *Nesting*: For every $k_0 \leqslant k_1$ and $Q_j \in \mathcal{D}_{k_j}$ for $j \in \{0,1\}$, then either $Q_1 \subset Q_0$ or $Q_1 \cap Q_0 = \varnothing$.

(c) *Tree structure*: For each $Q_1 \in \mathcal{D}_{k_1}$ and each $k_0 < k_1$ there exists a unique cube $Q_0 := Q_{k_0}(Q_1) \in \mathcal{D}_{k_0}$ such that $Q_1 \subset Q_0$. If $k_1 = k_1 - 1$, then we say that Q_0 is the parent of Q_1 write $Q_0 = \mathcal{P}(Q_1)$.

(d) *Scaling*: For $Q \in \mathcal{D}_k$ there are $z_Q \in Q$ and balls $B_Q = B(z_Q, a_1\ell_0^k)$ and $\widetilde{B}_Q = B(z_Q, \frac{1}{2}\ell_0^k)$ such that $B_Q \cap X \subset Q \subset \widetilde{B}_Q$.

Definition 4.24. We say that $Q \in \mathcal{D}_k$ is a *dyadic cube* of generation k, and write $r_Q := \frac{1}{2}\ell_0^k$, and $\ell(Q) := \ell_0^k$. We call r_Q the *exterior* radius of Q and $\ell(Q)$ its side–length. Note that we abuse notation because two dyadic cubes Q and R can have the same set of points but $r_Q \neq r_R$ because they belong to different *generations*. According to the previous definition, for every $x \in X$, there exists a unique dyadic cube $Q_k(x)$ containing x for every $k \in \mathbb{Z}$.

We say that two dyadic cubes of the same generation $Q, R \in \mathcal{D}_k$ are neighbors if $\lambda\widetilde{B}_Q \cap \lambda\widetilde{B}_R \neq \varnothing$, writing $R \in \mathcal{N}(Q)$, with $\lambda \geqslant 1$. Then we define the *triple cube* $3Q := \bigcup_{R\in\mathcal{N}(Q)} R$, by analogy with the usual dyadic grid. Note that $3Q \subset 3\lambda\widetilde{B}_Q$. We say that $3Q \in 3\mathcal{D}$.

Remark 4.25. Assume that $\ell_0 < \frac{1}{3}$ (this can be guaranteed by skipping generations). Let $\lambda = 2\min\{(1-3\ell_0)^{-1}, \ell_0^{-1}\}$. Then

$$\mathcal{F}(Q) \subset 3Q \subset 3\mathcal{F}(Q).$$

Moreover,

$$\lambda\widetilde{B}_Q \subset \lambda\widetilde{B}_{\mathcal{F}(Q)}.$$

Proof. Let us show first that $\lambda \geqslant \frac{1}{1-3\ell_0}$ implies $3Q \subset 3\mathcal{F}(Q)$. Let $x \in 3Q \cap R$ with $R, \mathcal{F}(Q) \in \mathcal{D}_k$. To show that $\lambda\widetilde{B}_{\mathcal{F}(Q)} \cap \lambda\widetilde{B}_R \neq \varnothing$, it is enough to show that $x \in \lambda\widetilde{B}_{\mathcal{F}(Q)}$. But

$$\mathrm{dist}(x, c_{\mathcal{F}(Q)}) \leqslant \mathrm{dist}(x, c_Q) + \mathrm{dist}(c_Q, c_{\mathcal{F}(Q)}) < 3\lambda r_Q + r_{\mathcal{F}(Q)} = (3\lambda\ell_0 + 1) r_{\mathcal{F}(Q)}.$$

Now, assume consider $\lambda > \ell_0^{-1}$, and let us show that $\mathcal{F}(Q) \subset 3Q$. We want to prove that whenever $\mathcal{F}(R) = \mathcal{F}(Q)$, then $\lambda\widetilde{B}_Q \cap \lambda\widetilde{B}_R \neq \varnothing$. Let z_Q and z_R be the centers of both cubes. Then we have

$$\mathrm{dist}(z_Q, z_R) \leqslant \mathrm{diam}(\widetilde{B}_{\mathcal{F}(Q)}) = 2\ell_0^{-1} r(\widetilde{B}_Q) = 2\ell_0^{-1} r(\widetilde{B}_R),$$

which implies that $\lambda\widetilde{B}_Q \cap \lambda\widetilde{B}_R \neq \varnothing$.

The last assertion comes from assuming $x \in \lambda\widetilde{B}_Q$, then

$$\mathrm{dist}(x, z_{\mathcal{F}(Q)}) \leqslant \mathrm{dist}(x, z_Q) + \mathrm{dist}(z_Q, z_{\mathcal{F}(Q)}) \leqslant \lambda r_Q + 2r_{\mathcal{F}(Q)} = (\lambda\ell_0 + 2) r_{\mathcal{F}(Q)} \leqslant \lambda r_{\mathcal{F}(Q)}.$$

□

Theorem 4.26 (see [HK12, Theorem 2.2])**.** *Let $X \subset \mathbb{R}^d$ be a closed set. There exists a dyadic grid $\mathcal{D}$ associated to X with constants $0 < \ell_0 < 1$, $0 < a_1 < \infty$ depending only on the dimension.*

Remark 4.27. Consider a Radon measure in $\mathbb{R}^d$, and let $X = \mathrm{supp}(\mu)$. There exist a dyadic grid $\mathcal{D}$ associated to X with constants $0 < \ell_0 < 1$, $0 < a_1 < \infty$, with all the constants depending on the dimension satisfying also the following hypothesis: there exists a dimensional constant $0 < \eta, C_1 < +\infty$ such that

(e) *Thin boundary*: For $Q \in \mathcal{D}_k$ we have

$$\mu(\{x \in Q : \mathrm{dist}(x, X \backslash Q) \leqslant t\ell_0^k\}) \leqslant C_1 t^\eta \mu(Q)$$

and

$$\mu(\{x \in Q^c : \mathrm{dist}(x, Q) \leqslant t\ell_0^k\}) \leqslant C_1 t^\eta \mu(Q)$$

for every $t > 0$.

See [**?**, Theorem 3.2], for instance. There is also a construction with open cubes in [Chr90, Theorem 11] which cover the support of μ modulo null sets.

Definition 4.28. Given a dyadic grid $\mathcal{D}$ associated to $X := \operatorname{supp}\mu$, we define the dyadic maximal operator by

$$M_{\mu,\mathcal{D}}f(x) := \sup_{k\in\mathbb{Z}} \fint_{Q_k(x)} |f|\,d\mu \quad \text{for every } f \in L^1_{\mathrm{loc}}(\mu).$$

Lemma 4.29. *The dyadic maximal operator $M_{\mu,\mathcal{D}}$ is a bounded operator from L^1 to $L^{1,\infty}$, and from L^p to L^p.*

The proof is left as an exercise for the reader, see for instance [Mat95, Theorem 2.19].

Lemma 4.30 (Dyadic Lebesgue differentiation theorem)**.** *Let μ be a Radon measure in $\mathbb{R}^d$ and let $\mathcal{D}$ be defined as above. For every $f \in L^1_{\mathrm{loc}}(\mu)$, we have*

$$\lim_{k\to\infty} \fint_{Q_k(x)} f\,d\mu = f(x) \textit{ for } \mu\textit{-a.e.} x \in X.$$

Proof. Let

$$E_n = \left\{ x \in X : \left| \limsup_{k\to 0} \fint_{Q_k(x)} f\,d\mu - f(x) \right| > 1/n \right\}.$$

We want to show that $\mu(E_n) \to 0$ as $n \to \infty$. To do so, pick f_δ continuous such that $\|f - f_\delta\|_{L^1(\mu)} < \delta$, see Exercise 4.2.1. Then write $E_n = A_n \cup B_n \cup C_n$ with

$$A_n = \left\{ x \in X : \limsup_{k\to 0} \fint_{Q_k(x)} |f - f_\delta|\,d\mu > 1/(3n) \right\}$$

$$B_n = \left\{ x \in X : \left| \limsup_{k\to 0} \fint_{Q_k(x)} f_\delta\,d\mu - f_\delta(x) \right| > 1/(3n) \right\}$$

$$C_n = \{ x \in X : |f_\delta(x) - f(x)| > 1/(3n) \}.$$

Note that $\mu(B_n) = 0$ by continuity. Also

$$\fint_{Q_k(x)} |f|\,d\mu \leqslant M_{\mu,\mathcal{D}}(f),$$

so

$$\mu(A_n) \leqslant 3n \|M_{\mu,\mathcal{D}}(f - f_\delta)\|_{L^{1,\infty}(\mu)} \overset{\text{L.4.29}}{\leqslant} Cn\delta,$$

while

$$\mu(C_n) \leqslant 3n \int_{C_n} |f_\delta - f|\,d\mu \leqslant 3n\delta,$$

and the lemma follows from this estimates, by picking $\delta = \frac{1}{n^2}$. $\square$

Definition 4.31. We say that μ is a doubling measure in $\mathbb{R}^d$ with constant C_μ if

$$\mu(B_{2r}(x)) \leqslant C_\mu \mu(B_r(x)) \text{ for every } x \in \operatorname{supp}(\mu) \text{ and every } r > 0.$$

Given $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$, we write

$$\mathcal{D}(Q_0) := \{Q \in \mathcal{D} : Q \subset Q_0\} \text{ and } 3\mathcal{D}(Q_0) := \{3Q \cap Q_0 : Q \in \mathcal{D}(Q_0)\}.$$

Definition 4.32. Let μ be a Radon measure in $\mathbb{R}^d$, and let $\mathcal{D}$ be a dyadic grid associated to suppμ. Given $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$, we say that μ is $\mathcal{D}$-doubling in Q_0 with constant $\widetilde{C}_\mu$ if

$$\sup_{R \in \mathcal{N}(Q) \cap \mathcal{D}(Q_0)} \frac{\mu(3Q \cap Q_0)}{\mu(R)} \leqslant \widetilde{C}_\mu \text{ for every } Q \in \mathcal{D}(Q_0).$$

Note that if μ is doubling with constant C_μ, then it is also $\mathcal{D}$-doubling with constant $\widetilde{C}_\mu \lesssim_{a_1} C_\mu$, and so from now on we will write C_μ for the greatest constant.

Next we will adapt an iterated Calderón-Zygmund decomposition (see, for instance [Gra09, Corollary 9.2.4]) to doubling Radon measures in $\mathbb{R}^d$.

Lemma 4.33 (Calderón-Zygmund decomposition)**.** *Let μ be a Radon measure in $\mathbb{R}^d$ and let $\mathcal{D}$ be a dyadic grid associated to* supp μ*. Let $Q \in \mathcal{D}_{k_0} \cup 3\mathcal{D}_{k_0}$, let $f \in L^1(Q)$ be non-negative. If μ is $\mathcal{D}$-doubling in Q with constant C_μ, and $t \geqslant C_\mu^{-1} ⨍_Q f\, d\mu$, then there exist a decomposition $Q = U_t \cup G_t \cup Z_t$ in disjoint μ-measurable sets and a family of disjoint dyadic cubes $\mathcal{F}_t \subset \bigcup_{k \geqslant k_0} \mathcal{D}_k \cup \{Q\}$ such that*

CZ1. $U_t := \bigcup_{R \in \mathcal{F}_t} R$, and these cubes satisfy $t < ⨍_R f\, d\mu \leqslant C_\mu t$.

CZ2. For $x \in G_t$ we have $f(x) \leqslant t$.

CZ3. $\mu(Z_t) = 0$.

We call this a Calderón-Zygmund decomposition *at level t.*

Proof. Consider $\mathcal{F}_k$ to be the maximal family of cubes $R \in \bigcup_{k \geqslant k_0} \mathcal{D}_k \cup \{R\}$ such that

$$⨍_R f\, d\mu > t.$$

The case $t < ⨍_Q f\, d\mu$ is trivial, so we assume $Q \notin \mathcal{F}_k$.

To prove the first claim, note that either $\mathcal{P}(R)$ is an eligible cube where the stopping condition fails, or R is one of the building blocks of Q whenever $Q \in 3\mathcal{D}$. In any case, using for a moment the convention that $\mathcal{P}(R) = Q$ in the latter case, we get

$$⨍_R f\, d\mu \leqslant C_\mu ⨍_{\mathcal{P}(R)} f\, d\mu \leqslant C_\mu t.$$

The second and third conditions follows from the dyadic Lebesgue differentiation theorem above (Z_t is the exceptional set). ☐

Lemma 4.34 (Calderón-Zygmund iteration)**.** *Let μ be a Radon measure in $\mathbb{R}^d$ and let $\mathcal{D}$ be a dyadic grid associated to* supp μ*. Let $Q \in \mathcal{D} \cup 3\mathcal{D}$, let $f \in L^1(Q)$ be non-negative. Assume that μ is $\mathcal{D}$-doubling in Q with constant C_μ, $0 < \delta < 1$ and $\alpha_0 \geqslant C_\mu^{-1} ⨍_Q f\, d\mu$. For every $j \geqslant 1$, define $\alpha_j := \left[C_\mu \delta^{-1}\right]^j \alpha_0$. Then the sequence of Calderón-Zygmund decompositions $Q = U^j \cup G^j \cup Z^j$ with $U^j := \bigcup_{R \in \mathcal{F}^j} R$ at levels α_j satisfies:*

CZ4. *If $R \in \mathcal{F}^j$ then there exists a cube $\hat{R} \in \mathcal{F}^{j-1}$ such that $R \subset \hat{R}$.*

CZ5. $\mu(U^{j+1}) \leqslant \delta\mu(U^j) \leqslant \cdots \leqslant \delta^{j+1}\mu(U^0) \xrightarrow{j\to\infty} 0.$

CZ6. *If $R \in \mathcal{F}^j$ then $\mu(R \cap U^{j+1}) < \delta\mu(R)$.*

Proof. For the first condition, note that for $R \in \mathcal{F}_{j+1}$

$$\fint_R f\, d\mu > \alpha_{j+1} > \alpha_j,$$

so R must be contained in some maximal cube satisfying this property (perhaps itself!). The first claim is proven.

On the other hand, for every $R \in \mathcal{F}^j$ we have

$$C_\mu \alpha_j \overset{\text{CZ1}}{\geqslant} \fint_R f\, d\mu \geqslant \frac{1}{\mu(R)} \int_{R \cap U^{j+1}} f\, d\mu \geqslant \frac{1}{\mu(R)} \sum_{R \subset R: R \in \mathcal{F}_{j+1}} \mu(R) \fint_R f\, d\mu$$
$$\overset{\text{CZ1, CZ4}}{>} \frac{\alpha_{j+1}}{\mu(R)} \mu(R \cap U^{j+1}).$$

Since $\alpha_{j+1} = C_\mu \delta^{-1}\alpha_j$, CZ6 follows.

CZ5 follows by summing on cubes CZ6, and using condition CZ4. □

Finally we introduce a Whitney covering for generalized dyadic grids.

Lemma 4.35 (Whitney decomposition)**.** *Let μ be a Radon measure supported in $X \subset \mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to X with constants $0 < \ell_0 < 1$, $0 < a_1 < \infty$, and let $\Omega \subset X$ be a relative open set, with non-empty relative complement $F := X\backslash\Omega$ and λ defined in Definition 4.24. Then, there exist $\alpha = \alpha(\ell_0) > 7$ and a collection $\mathcal{W} \subset \mathcal{D}$ of dyadic cubes, which we call a* Whitney decomposition (or covering) of Ω, *satisfying the following properties:*

Wh1. *The Whitney cubes cover Ω, i.e., $\Omega = \bigcup_{Q \in \mathcal{W}} Q$.*

Wh2. *Whitney cubes are disjoint, i.e.,*

$$\sum_{Q \in \mathcal{W}} \chi_Q = \chi_\Omega.$$

Wh3. *Their exterior radius is comparable to their distance to the boundary, namely,*

$$(\alpha\lambda - 1) r_Q \leqslant \operatorname{dist}(Q, F) \leqslant \ell_0^{-1}(\alpha\lambda + 1) r_Q.$$

Wh4. *If $Q, R \in \mathcal{W}$ satisfy that $3Q \cap 3R \neq \varnothing$, then*

$$\ell_0 \leqslant \frac{\ell(Q)}{\ell(R)} \leqslant \ell_0^{-1}.$$

Wh5. The triple cubes *have bounded overlapping, namely,*

$$\sum_{Q\in\mathcal{W}} \chi_{3Q} \leqslant C_{\mathcal{W}},$$

with $C_{\mathcal{W}}^{\frac{1}{d}}$ depending only on ℓ_0, a_1 and λ.

Proof. Take the maximal dyadic cubes such that

$$\alpha\lambda r_Q \leqslant \operatorname{dist}(c_Q, F). \tag{4.8}$$

Property Wh1 follows from Definition 4.23, since every $x \in X$ satisfies that $r_{Q_k(x)} = \frac{1}{2}\ell_0^k \to 0$, while $\operatorname{dist}(c_{Q_k(x)}, F) \to \operatorname{dist}(x, F) > 0$ as $k \to \infty$. Property Wh2 follows from the construction.

The bound below in Wh3 follows from construction and the triangle inequality, while the bound above follows from the stopping time condition and the triangle inequality as well. Indeed,

$$\operatorname{dist}(c_Q, F) \leqslant \operatorname{dist}(c_{\mathcal{P}(Q)}, F) + r_{\mathcal{P}(Q)} \overset{(4.8)}{<} (\alpha\lambda + 1) r_{\mathcal{P}(Q)} = (\alpha\lambda + 1)\ell_0^{-1} r_Q, \tag{4.9}$$

and Wh3 follows since $\operatorname{dist}(Q, F) \leqslant \operatorname{dist}(c_Q, F)$.

Let us show Wh4. Assume that $3R \cap 3Q \neq \varnothing$, with $r_R \leqslant r_Q$. Then, there exists $x \in B(c_Q, 3\lambda r_Q) \cap B(c_R, 3\lambda r_R)$. Thus,

$$\operatorname{dist}(c_R, F) \geqslant \operatorname{dist}(c_Q, F) - (3\lambda r_Q + 3\lambda r_R) \overset{\text{Wh3}}{\geqslant} (\alpha\lambda - 1 - 6\lambda) r_Q \geqslant (\alpha - 7)\lambda r_Q.$$

Combining with (4.9) we get

$$\frac{r_R}{r_Q} \geqslant \frac{(\alpha - 7)\lambda}{(\alpha\lambda + 1)}\ell_0 \geqslant \frac{\alpha - 7}{\alpha + 1}\ell_0 > \ell_0^2$$

if α is big enough.

To end with the last property, assume $x \in \Omega$, and let $Q \in \mathcal{W}$ be a cube of maximal r_Q such that $x \in 3Q$. Then for any other $R \in \mathcal{W}$ with $x \in 3R$, we have

$$B_R \subset B(c_Q, 3\lambda r_Q + 3\lambda r_R + r(B_R)) \subset B(c_Q, (6\lambda + 2a_1) r_Q).$$

For every such R we have

$$r(B_R) = 2a_1 r_R \overset{\text{Wh4}}{\geqslant} 2a_1 \ell_0 r_Q.$$

By the disjointness of inner balls B_R with $R \in \mathcal{D}_k$, (that is $\sum_{Q\in\mathcal{D}_k} \chi_{B_Q} \leqslant 1$), we infer that

$$\#\{P \in \mathcal{W} : 3P \cap 3Q \neq \varnothing,\, r(R) = \ell_0 r(Q)\} \leqslant \frac{|B(c_Q, (6\lambda + 2a_1) r_Q)|}{|B_R|}.$$

Arguing anagously for $r(R) = r(Q)$, Wh5 follows with $C \leqslant \left(\frac{6\lambda + 2a_1}{2a_1}\right)^d \left(1 + \ell_0^{-1}\right)^d$. $\square$

4.6 Muckenhoupt weights and doubling measures

In this section we define A_∞ weights in subsets of $\mathbb{R}^d$ equipped with doubling measures introduced in the previous section. Recall that given $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$, we write

$$\mathcal{D}(Q_0) := \{Q \in \mathcal{D} : Q \subset Q_0\} \text{ and } 3\mathcal{D}(Q_0) := \{3Q \cap Q_0 : Q \in \mathcal{D}(Q_0)\}.$$

4.6.1 Equivalent conditions in dyadic grids

Definition 4.36. Let μ be a Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to $X := \operatorname{supp}\mu$, let $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$ (see Definition 4.24), and let ν be a Radon measure supported in X.

- We say that $\nu \in AB_{\delta_0,\varepsilon_0}(\mu, \mathcal{D}, Q_0)$ if

  $\mu(E) < \delta_0\mu(Q) \implies \nu(E) \leqslant \varepsilon_0\nu(Q)$ for every Borel set $E \subset Q \in \mathcal{D}(Q_0) \cup 3\mathcal{D}(Q_0)$,

  (or equivalently, $\mu(E) > (1-\delta_0)\mu(Q) \implies \nu(E) \geqslant (1-\varepsilon_0)\nu(Q)$ for every $E \subset Q \subset Q_0$ Borel, where $Q \in \mathcal{D} \cup 3\mathcal{D}$.)

- We say that $\nu \in A_\infty(\mu, \mathcal{D}, Q_0)$ if for every $\delta_0 \in (0,1)$, there exists $\varepsilon_0 \in (0,1)$ such that $\nu \in AB_{\delta_0,\varepsilon_0}(\mu, \mathcal{D}, Q_0)$.

- We say that $\nu \in B_1(\mu, \mathcal{D}, Q_0)$ if for every $\varepsilon_0 \in (0,1)$, there exists $\delta_0 \in (0,1)$ such that $\nu \in AB_{\delta_0,\varepsilon_0}(\mu, \mathcal{D}, Q_0)$.

- We say that $\nu \in B_1^\alpha(\mu, \mathcal{D}, Q_0)$ if there exists $C \geqslant 1$ such that

$$\frac{\nu(E)}{\nu(Q)} \leqslant C\left(\frac{\mu(E)}{\mu(Q)}\right)^\alpha \text{ for every Borel set } E \subset Q \subset Q_0 \text{ with } Q \in \mathcal{D} \cup 3\mathcal{D}. \quad (4.10)$$

- We say that $\nu \in A_\infty^\alpha(\mu, \mathcal{D}, Q_0)$ if $\mu \in B_1^\alpha(\nu, \mathcal{D}, Q_0)$.

- We say that $\nu \in B_p(\mu, \mathcal{D}, Q_0)$ whenever $\nu \ll \mu$ and the density $w = \frac{d\nu}{d\mu}$ satisfies the following reverse Hölder inequality

$$\left(\fint_Q w^p \, d\mu\right)^{\frac{1}{p}} \leqslant C \fint_Q w \, d\mu \text{ for every } Q \in \mathcal{D} \cup 3\mathcal{D} \text{ with } Q \subset Q_0.$$

- We say that $\nu \in A_p(\mu, \mathcal{D}, Q_0)$ whenever $\nu \ll \mu$, the density $w = \frac{d\nu}{d\mu}$ satisfies that $w^{\frac{1}{1-p}} \in L^1_{\mathrm{loc}}(\mu)$ and

$$\fint_Q w \, d\mu \left(\fint_Q w^{\frac{1}{1-p}} \, d\mu\right)^{p-1} \leqslant C \text{ for every } Q \in \mathcal{D} \cup 3\mathcal{D} \text{ with } Q \subset Q_0.$$

The minimal constants satisfying the last two properties are called $[w]_{B_p(\mu,\mathcal{D},Q_0)}$ and $[w]_{A_p(\mu,\mathcal{D},Q_0)}$ respectively.

If any one of this conditions is satisfied for every $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$, we simply omit Q_0 in the notation.

Remark 4.37. If μ is a Radon measure and $\nu \in AB_{\delta,\varepsilon}(\mu, \mathcal{D}, Q_0)$, then $\mu|_{Q_0} \ll \nu|_{Q_0} \ll \mu|_{Q_0}$ using the dyadic Lebesgue differentiation theorem as in Remark 4.19, and $\operatorname{supp}\nu = \operatorname{supp}\mu$ so $\mathcal{D}$ is associated to $\operatorname{supp}\nu$. If, instead, we assume $\nu \in A_p(\mu, \mathcal{D}, Q_0)$, then $0 < w < +\infty$ μ-a.e. as well, and $\nu \ll \mu \ll \nu$ so $\mathcal{D}$ is associated to $\operatorname{supp}\nu$ again. As in Remarks 4.19 and 4.22, we get

$$\begin{aligned} \nu \in A_p(\mu, \mathcal{D}, Q_0) &\iff \mu \in B_{p'}(\nu, \mathcal{D}, Q_0) \\ \nu \in A_\infty^\alpha(\mu, \mathcal{D}, Q_0) &\iff \mu \in B_1^\alpha(\nu, \mathcal{D}, Q_0) \\ \nu \in A_\infty(\mu, \mathcal{D}, Q_0) &\iff \mu \in B_1(\nu, \mathcal{D}, Q_0) \\ \nu \in AB_{\delta_0,\varepsilon_0}(\mu, \mathcal{D}, Q_0) &\iff \mu \in AB_{1-\varepsilon_0,1-\delta_0}(\nu, \mathcal{D}, Q_0), \end{aligned} \tag{4.11}$$

and $[w]_{A_p(\mu,\mathcal{D},Q_0)} = [w^{-1}]^{p'}_{B_{p'}(\nu,\mathcal{D},Q_0)}$.

Lemma 4.38. *Let μ is a Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$*, and let $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$. Then*

$$B_p(\mu, \mathcal{D}, Q_0) \subset B_1^{1-\frac{1}{p}}(\mu, \mathcal{D}, Q_0) \subset B_1(\mu, \mathcal{D}, Q_0) \subset \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}(\mu, \mathcal{D}, Q_0),$$

and

$$A_p(\mu, \mathcal{D}, Q_0) \subset A_\infty^{\frac{1}{p}}(\mu, \mathcal{D}, Q_0) \subset A_\infty(\mu, \mathcal{D}, Q_0) \subset \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}(\mu, \mathcal{D}, Q_0).$$

All conditions above are again quantitative versions of mutual absolute continuity.

Proof. Consider all subcubes of Q_0 in Remark 4.21 to get the first chain of inclusions. The second comes immediately as a consequence of (4.11). □

Theorem 4.39. *Let μ, ν be Radon measures in $\mathbb{R}^d$ with* $\operatorname{supp}\nu \subset \operatorname{supp}\mu$*, let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$ *and let $w := \frac{d\nu}{d\mu}$. If μ is a $\mathcal{D}$-doubling measure in $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$, then if $\nu \in AB_{\delta_0,\varepsilon_0}(\mu, \mathcal{D}, Q_0)$ with $0 < \delta_0, \varepsilon_0 < 1$, then $\nu \in B_p(\mu, \mathcal{D}, Q_0)$ for some $1 < p < p_0 < \infty$, with p_0 depending on δ_0, ε_0 and the doubling constant, and the B_p constant depending also on p.*

Proof. Consider a given dyadic cube $Q \subset Q_0$ (or $Q = Q_0 \in 3\mathcal{D}$). Apply Lemma 4.34, with $f = w$, $\alpha_0 = \fint_Q w\, d\mu = \frac{\nu(Q)}{\mu(Q)}$ and $\delta = \delta_0$. Given $R \in \mathcal{F}^j$, by CZ6, we get

$$\nu(R \cap U^{j+1}) \leqslant \varepsilon_0 \nu(R),$$

and therefore $\nu(U^{j+1}) \leqslant \varepsilon_0 \nu(U^j) \leqslant \cdots \leqslant \varepsilon_0^{j+1} \nu(U^0)$. In particular, $Q = (Q \backslash U_0) \cup (\bigcup_k U_k \backslash U_{k+1}) \cup Z$, with $\mu(Z) = \nu(Z) = 0$.

Now,

$$\begin{aligned}\int_Q w^p \, d\mu &= \int_{Q\setminus U_0} w^{p-1} w \, d\mu + \sum_{j\geqslant 0} \int_{U^j\setminus U^{j+1}} w^{p-1} w \, d\mu \\ &\overset{\text{CZ2}}{\leqslant} \alpha_0^{p-1}\nu(Q\setminus U^0) + \sum_{j\geqslant 0} \alpha_{j+1}^{p-1}\nu(U^j) \\ &\leqslant \alpha_0^{p-1}\nu(Q\setminus U^0) + \sum_{j\geqslant 0}[(C_\mu\delta_0^{-1})^{j+1}\alpha_0]^{p-1}\varepsilon_0^j\nu(U^0) \\ &\leqslant \alpha_0^{p-1}\left(1+\sum_{j\geqslant 0}(C_\mu\delta_0^{-1})^{(j+1)(p-1)}\varepsilon_0^j\right)\nu(Q) \\ &= \left(1+\frac{(C_\mu\delta_0^{-1})^{p-1}}{1-(C_\mu\delta_0^{-1})^{p-1}\varepsilon_0}\right)\frac{\nu(Q)^p}{\mu(Q)^{p-1}},\end{aligned}$$

whenever $(C_\mu\delta_0^{-1})^{p-1}\varepsilon_0 < 1$, because $\alpha_0 = \fint_Q w \, d\mu = \frac{\nu(Q)}{\mu(Q)}$.

If $p-1 = \theta\frac{-\log\varepsilon_0}{\log C_\mu - \log\delta_0}$, we get $(C_\mu\delta_0^{-1})^{p-1} = \varepsilon_0^{-\theta}$. For $0<\theta<1$ we get $\varepsilon_0^{-\theta}\varepsilon_0 < 1$, implying summability above, and the last estimate reads as

$$\fint_Q w^p \, d\mu \leqslant \left(1+\frac{\varepsilon_0^{-\theta}}{1-\varepsilon_0^{1-\theta}}\right)\frac{\nu(Q)^p}{\mu(Q)^p} = C\left(\fint_Q w \, d\mu\right)^p,$$

as claimed. □

Corollary 4.40. *Let μ be a Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$*. If μ is a $\mathcal{D}$-doubling measure in $Q_0 \in \mathcal{D}\cup 3\mathcal{D}$, then*

$$\bigcup_{1<p<\infty} B_p(\mu,\mathcal{D},Q_0) = \bigcup_{0<\alpha<1} B_1^\alpha(\mu,\mathcal{D},Q_0) = B_1(\mu,\mathcal{D},Q_0) = \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}(\mu,\mathcal{D},Q_0).$$

Proof. Combine Remark 4.37 with Theorem 4.39. □

Corollary 4.41. *Let μ be a Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$*. If we define*

$$X^d(\mu,\mathcal{D},Q_0) := \{\nu \in X(\mu,\mathcal{D},Q_0) : \nu \text{ is } \mathcal{D}\text{-doubling in } Q_0\}$$

with $X \in \{A_p, A_\infty^\alpha, A_\infty, AB_{\delta,\varepsilon}\}$, and $Q_0 \in \mathcal{D}\cup 3\mathcal{D}$, then

$$\bigcup_{1<p<\infty} A_p^d(\mu,\mathcal{D},Q_0) = \bigcup_{0<\alpha<1} A_\infty^{\alpha,d}(\mu,\mathcal{D},Q_0) = A_\infty^d(\mu,\mathcal{D},Q_0) = \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}^d(\mu,\mathcal{D},Q_0).$$

Proof. Combine Corollary 4.41 with (4.11). □

Lemma 4.42. *Let μ be a Radon measure in $\mathbb{R}^d$ and let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$*. If μ is a $\mathcal{D}$-doubling measure in $Q_0 \in \mathcal{D}\cup 3\mathcal{D}$ and $\nu \in A_\infty(\mu,\mathcal{D},Q_0)$ then ν is $\mathcal{D}$-doubling in Q_0.*

If, instead, μ is doubling in restricted balls Δ with $2\Delta \subset U$ and $\nu \in A_\infty(\mu, U)$, then ν is doubling in balls Δ such that $2\Delta \subset U$.

By duality, if ν is $\mathcal{D}$-doubling in Q_0 and $\nu \in B_1(\mu, \mathcal{D}, Q_0)$, then μ is doubling as well, and the same holds for balls instead of cubes.

Proof. We proof the first statement, the others being proved analogously. Note that μ being $\mathcal{D}$-doubling in Q_0 (see Definition 4.32) is equivalent to

$$\mu(3Q \cap Q_0 \backslash R) \leqslant \left(1 - \frac{1}{C_\mu}\right) \mu(3Q \cap Q_0) \quad \text{for every } Q \in \mathcal{D}(Q_0) \text{ and every } R \in \mathcal{N}(Q).$$

Thus, if $\nu \in A_\infty(\mu, \mathcal{D}, Q_0)$, picking $\delta_0 = 1 - \frac{1}{C_\mu}$, there exists ε_0 such that

$$\nu(3Q \cap Q_0 \backslash R) \leqslant \varepsilon_0 \nu(3Q \cap Q_0) \quad \text{for every } R \in \mathcal{N}(Q).$$

we find out that ν is $\mathcal{D}$-doubling in Q_0 as well with constant $C_\nu = \frac{1}{1-\varepsilon_0}$, that is,

$$\nu(3Q \cap Q_0) \leqslant \frac{1}{1-\varepsilon_0} \nu(R) \quad \text{for every } R \in \mathcal{N}(Q) \cap \mathcal{D}(Q_0).$$

□

Remark 4.43. In view of Lemma 4.42 and Lemma 4.38, when μ is $\mathcal{D}$-doubling in Q_0, Corollary 4.41 reads

$$\bigcup_{1<p<\infty} A_p(\mu, \mathcal{D}, Q_0) = \bigcup_{0<\alpha<1} A_\infty^\alpha(\mu, \mathcal{D}, Q_0) = A_\infty(\mu, \mathcal{D}, Q_0) = \bigcup_{0<\delta,\varepsilon<1} AB^d_{\delta,\varepsilon}(\mu, \mathcal{D}, Q_0).$$

4.6.2 From dyadic grids to balls

Lemma 4.44. *Let μ be a Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$. *Then, if μ is a doubling measure, there exists a relative open set $U \supset Q_0$ with diameter comparable to $\ell(Q_0)$ (with constants depending only on d) such that*

$$A_p(\mu, U) \subset A_p(\mu, \mathcal{D}, Q_0) \subset A_p(\mu, Q_0).$$

In particular $A_p(\mu) = A_p(\mu, \mathcal{D})$.

Proof. Let $w := \frac{d\nu}{d\mu}$. First, let us show that $A_p(\mu, \mathcal{D}, Q_0) \subset A_p(\mu, Q_0)$. If $\Delta \subset Q_0$ is a restricted ball, there exists a cube $Q_\Delta \in \mathcal{D}$ with $\Delta \subset 3Q \cap Q_0$ and $r(\Delta) \approx r(Q_\Delta)$. If $w \in A_p(\mu, \mathcal{D}, Q_0)$, we get

$$⨍_\Delta w \, d\mu \left(⨍_\Delta w^{\frac{1}{1-p}} \, d\mu \right)^{p-1} \lesssim_{C_\mu, p} ⨍_{3Q \cap Q_0} w \, d\mu \left(⨍_{3Q \cap Q_0} w^{\frac{1}{1-p}} \, d\mu \right)^{p-1} \leqslant C.$$

To show the other inclusion, $A_p(\mu, U) \subset A_p(\mu, \mathcal{D}, Q_0)$, let $U := U_{Q_0}$ be an open set containing $\widetilde{B}_Q$ for every $Q \in \mathcal{D} \cup 3\mathcal{D}(Q_0)$ (here, in case $Q \in 3\mathcal{D}$ we need to define $B_Q \cap X \subset$

$Q \subset \widetilde{B}_Q$ with comparable radius, task that we leave for the reader to complete). Then, if $w \in A_p(\mu, U)$, we get

$$\fint_Q w \, d\mu \left(\fint_Q w^{\frac{1}{1-p}} \, d\mu \right)^{p-1} \lesssim_{C_\mu,p} \fint_{\widetilde{B}_Q} w \, d\mu \left(\fint_{\widetilde{B}_Q} w^{\frac{1}{1-p}} \, d\mu \right)^{p-1} \leqslant C.$$

□

Lemma 4.45. *Using the notation in Lemma 4.44, if μ is a doubling Radon measure satisfying $\mu(\widetilde{B}_Q) \leqslant C_\mu \mu(B_Q)$ for $Q \in \mathcal{D} \cup 3\mathcal{D}$, we have*

$$AB_{1-C_\mu^{-1}\delta,1-\varepsilon}(\mu, U) \subset AB_{1-\delta,1-\varepsilon}(\mu, \mathcal{D}, Q_0),$$

and

$$AB_{1-C^{-1}\delta,1-\varepsilon}(\mu, \mathcal{D}, Q_0) \subset AB_{1-\delta,1-\varepsilon}(\mu, Q_0)$$

with $C > 1$ depending on C_μ, and the dimensional parameters involved in the definition of $\mathcal{D}$. In particular,

$$A_\infty(\mu, U) \subset A_\infty(\mu, \mathcal{D}, Q_0) \subset A_\infty(\mu, Q_0).$$

Proof. Assume $\nu \in AB_{1-C_\mu^{-1}\delta,1-\varepsilon}(\mu, U)$. Every set $E \subset Q \in \mathcal{D}(Q_0) \cup 3\mathcal{D}(Q_0)$ satisfies the implication

$$\mu(E) \geqslant \delta\mu(Q) \implies \mu(E) \geqslant \delta C_\mu^{-1} \mu(\widetilde{B}_Q) \implies \nu(E) > \varepsilon\nu(B_Q) > \varepsilon\nu(Q).$$

The other inclusion follows analogously by granting the existence of $\Delta \subset Q_\Delta \in 3\mathcal{D}(Q_0)$ with $\ell(Q_\Delta) \approx \ell(\Delta)$ for every boundary ball $\Delta \subset Q_0$ as in the preceding proof. □

Define

$$X^d(\mu) := \{\nu \in X(\mu) : \nu \text{ is doubling}\}$$

and

$$X^d(\mu, \mathcal{D}) := \{\nu \in X(\mu, \mathcal{D}) : \nu \text{ is } \mathcal{D}\text{-doubling for every } Q_0 \in \mathcal{D}\}$$

with $X \in \{A_p, A_\infty^\alpha, A_\infty, B_p, B_1^\alpha, B_1, AB_{\delta,\varepsilon}\}$.

Corollary 4.46. *Let μ be a Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$. *Following the hypothesis in the previous two lemmas (except that we allow μ to be non-doubling), we obtain*

$$B_p^d(\mu, U) \subset B_p^d(\mu, \mathcal{D}, Q_0) \subset B_p^d(\mu, Q_0),$$

and

$$B_1^d(\mu, U) \subset B_1^d(\mu, \mathcal{D}, Q_0) \subset B_1^d(\mu, Q_0).$$

In particular $B_p^d(\mu) = B_p^d(\mu, \mathcal{D})$ and $B_1^d(\mu) = B_1^d(\mu, \mathcal{D})$.

Proof. This is an immediate consequence of lemmas 4.44 and 4.45 combined with the duality relations (4.7) and (4.11). □

Lemma 4.47. *If μ is a doubling Radon measure in $\mathbb{R}^d$ and $\mathcal{D}$ is a dyadic grid associated to* $\operatorname{supp}\mu$*, then*

$$\bigcup_{1<p<\infty} A_p(\mu,\mathcal{D}) = \bigcup_{0<\alpha<1} A_\infty^\alpha(\mu,\mathcal{D}) = A_\infty(\mu,\mathcal{D}) = A_\infty(\mu) = \bigcup_{0<\alpha<1} A_\infty^\alpha(\mu) = \bigcup_{1<p<\infty} A_p(\mu).$$

Proof. By Remark 4.22 and Lemma 4.45 we get

$$A_p(\mu,U_{Q_0}) \subset A_\infty^{\frac{1}{p}}(\mu,U_{Q_0}) \subset A_\infty(\mu,U_{Q_0}) \subset A_\infty(\mu,\mathcal{D},Q_0).$$

By Remark 4.43 and Lemma 4.44 we also have

$$A_\infty(\mu,\mathcal{D},Q_0) \overset{\text{R.4.43}}{=} \bigcup_{1<p<\infty} A_p(\mu,\mathcal{D},Q_0) \subset \bigcup_{1<p<\infty} A_p(\mu,Q_0).$$

Since

$$A_p(\mu) = \bigcup_{Q_0\in\mathcal{D}} A_p(\mu,Q_0) = \bigcup_{Q_0\in\mathcal{D}} A_p(\mu,U_{Q_0}),$$

we conclude the proof. □

Corollary 4.48. *If μ is a Radon measure in $\mathbb{R}^d$ and $\mathcal{D}$ is a dyadic grid associated to* $\operatorname{supp}\mu$*, then*

$$\bigcup_{1<p<\infty} B_p^d(\mu,\mathcal{D}) = \bigcup_{0<\alpha<1} B_1^{\alpha,d}(\mu,\mathcal{D}) = B_1^d(\mu,\mathcal{D}) = B_1^d(\mu) = \bigcup_{0<\alpha<1} B_1^{\alpha,d}(\mu) = \bigcup_{1<p<\infty} B_p^d(\mu).$$

Proof. This is an immediate consequence of Lemma 4.47 combined with the duality relations (4.7) and (4.11). □

Corollary 4.49. *Let μ be a doubling Radon measure in $\mathbb{R}^d$ and $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$*, then*

$$A_\infty(\mu) = B_1^d(\mu).$$

Proof. By Lemma 4.47, Remark 4.43 we get

$$A_\infty(\mu) \overset{\text{R.4.43}}{=} \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}^d(\mu,\mathcal{D}).$$

By Corollary 4.40, if we restrict to doubling measures, we have

$$\bigcup_{1<p<\infty} B_p^d(\mu,\mathcal{D}) = \bigcup_{0<\alpha<1} B_1^{\alpha,d}(\mu,\mathcal{D}) = B_1^d(\mu,\mathcal{D}) = \bigcup_{0<\delta,\varepsilon<1} AB_{\delta,\varepsilon}^d(\mu,\mathcal{D}).$$

□

To end we check that, in case μ and ν are both assumed to be doubling *a priori*, then all the conditions studied here are equivalent. Given a cube $Q \in \mathcal{D} \cup 3\mathcal{D}$ there exists a restricted ball $\Delta_Q \subset Q$ with comparable diameter (see Definition 4.23) and given a restricted ball Δ we can define Q_Δ to be the largest dyadic cube such that $x \in Q_\Delta \subset \Delta$. It is easy to see that Q_Δ and Δ also have comparable diameter. If μ is a doubling measure, there exists a constant $\widetilde{C}_\mu \geqslant 1$ such that

$$\mu(Q) \leqslant \widetilde{C}_\mu \mu(\Delta_Q) \quad \text{and} \quad \mu(\Delta) \leqslant \widetilde{C}_\mu \mu(Q_\Delta) \tag{4.12}$$

for every restricted ball Δ and every cube $Q \in \mathcal{D} \cup 3\mathcal{D}$.

Lemma 4.50. *If μ is a doubling Radon measure, $\mathcal{D}$ is a dyadic grid associated to* $\operatorname{supp}\mu$ *and $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$, then*

$$\bigcup_{0<\varepsilon,\delta<1} AB^d_{\delta,\varepsilon}(\mu, Q_0, \mathcal{D}) = \bigcup_{0<\varepsilon,\delta<1} AB^d_{\delta,\varepsilon}(\mu, Q_0).$$

In particular,

$$\nu \in AB^d_{\delta,1-\varepsilon}(\mu, Q_0, \mathcal{D}) \implies \nu \in AB^d_{\widetilde{C}_\mu^{-1}\delta, 1-\widetilde{C}_\nu^{-1}\varepsilon}(\mu, Q_0),$$

and

$$\nu \in AB^d_{\delta,1-\varepsilon}(\mu, Q_0) \implies \nu \in AB^d_{\widetilde{C}_\mu^{-1}\delta, 1-\widetilde{C}_\nu^{-1}\varepsilon}(\mu, Q_0),$$

with C_μ and C_ν defined as in (4.12).

Proof. Assume that $\nu \in AB^d_{\delta,1-\varepsilon}(\mu, Q_0)$ and consider a set $E \subset Q \in \mathcal{D}(Q_0) \cup 3\mathcal{D}(Q_0)$. First note that

$$\nu(E \cap B_Q) \leqslant (1-\varepsilon)\nu(B_Q) \implies \nu(B_Q \backslash E) \geqslant \varepsilon\nu(B_Q) \implies \nu(Q\backslash E) \geqslant \widetilde{C}_\nu^{-1}\varepsilon\nu(Q),$$

so we get

$$\nu(E \cap B_Q) \leqslant (1-\varepsilon)\nu(B_Q) \implies \nu(E) \leqslant (1-\widetilde{C}_\nu^{-1}\varepsilon)\nu(Q),$$

or, equivalently,

$$\nu(E) > (1-\widetilde{C}_\nu^{-1}\varepsilon)\nu(Q) \implies \nu(E \cap B_Q) > (1-\varepsilon)\nu(B_Q).$$

Since $\nu \in AB^d_{\delta,1-\varepsilon}(\mu, Q_0)$, we deduce that

$$\nu(E) > (1-\widetilde{C}_\nu^{-1}\varepsilon)\nu(Q) \implies \mu(E \cap B_Q) \geqslant \delta\mu(B_Q).$$

Now using the doubling condition for μ we get

$$\nu(E) > (1-\widetilde{C}_\nu^{-1}\varepsilon)\nu(Q) \implies \mu(E) \geqslant \widetilde{C}_\mu^{-1}\delta\mu(Q),$$

that is,

$$\nu \in AB^d_{\widetilde{C}_\mu^{-1}\delta, 1-\widetilde{C}_\nu^{-1}\varepsilon}(\mu, Q_0).$$

the other statement is proven analogously.

□

Remark 4.51. Whenever μ and ν are doubling Radon measures with common support, all the conditions we have studied are equivalent. That is $\nu \in A_\infty(\mu)$ if and only if

$$\nu \in \bigcup_{1<p<\infty} A_p(\mu) \iff \nu \in \bigcup_{1<p<\infty} B_p(\mu) \iff \nu \in \bigcup_{0<\varepsilon,\delta<1} AB_{\delta,\varepsilon}(\mu) \cup \bigcup_{0<\varepsilon,\delta<1} AB_{\delta,\varepsilon}(\mu,\mathcal{D}).$$

Thus, to prove that $\nu \in A_\infty(\mu)$ we can check a reverse Hölder inequality, bound an A_p constant or find constants $0<\delta,\varepsilon<1$ such that $\nu \in AB_{\delta,\varepsilon}(\mu)$ or $\nu \in AB_{\delta,\varepsilon}(\mu,\mathcal{D})$.

Remark 4.52. Last, but nor least, assume that μ is doubling, with $\mu(\widetilde{B}_Q) \leqslant C_\mu \mu(B_Q)$ for $Q \in 3\mathcal{D}$. Then if $\nu \in A_{1-\delta_0,\varepsilon_0}(\mu)$ with $\delta_0 \leqslant C_\mu^{-1}$ we can immediately infer from Lemma 4.45 that $\nu \in A_{1-C_\mu\delta_0,\varepsilon_0}(\mu,\mathcal{D}) \subset A_\infty(\mu)$. In particular ν is also doubling.

4.6.3 Self-improvement properties

We are interested in the self-improvement properties of weights. In general, we have the inclusions $A_p(\mu) \subset A_{p+\varepsilon}(\mu)$ and $B_p(\mu) \subset B_{p-\varepsilon}(\mu)$.

In the doubling setting of Remark 4.51 above (where μ and ν are both doubling), we also have a self-improvement property for $w \in A_p(\mu)$. Namely, for ε small enough we have

$$[w]_{A_{\frac{p+\varepsilon}{1+\varepsilon}}(\mu)} \leqslant [w]_{B_{1+\varepsilon}(\mu)}[w^{1-p'}]^{p-1}_{B_{1+\varepsilon}(\mu)}[w]_{A_p(\mu)}.$$

Indeed, since $w \in A_p(\mu)$ is equivalent to $w^{1-p'} \in A_{p'}(\mu) \subset A_\infty(\mu)$, we get the existence of a reverse Hölder class for both w and its dual weight $w^{1-p'}$, say $w, w^{1-p'} \in B_{1+\varepsilon}(\mu)$ if $\varepsilon < \varepsilon_0$. An analogous self-improvement property is satisfied for reverse Hölder classes:

Lemma 4.53 (Gehring Lemma for doubling weights)**.** *Let μ and ν be mutually absolutely continuous doubling measures in $\mathbb{R}^d$ with $w = \frac{d\nu}{d\mu}$. If $w \in B_p(\mu)$, then there exists ε_0 such that for every $\varepsilon < \varepsilon_0$ we have $w \in B_{p+\varepsilon}(\mu)$.*

Proof. By Remark 4.22, $w \in B_p(\mu)$ is equivalent to $w^{-1} \in A_{p'}(\nu)$. By Remark 4.43 we infer the existence of δ_0 such that $w^{-1} \in A_{p'-\delta}(\nu)$ for $\delta < \delta_0$. But this is equivalent to $w \in B_{(p'-\delta)'}(\mu)$, with

$$(p'-\delta)' = \left(\frac{p}{p-1} - \delta\right)' = \frac{p-\delta(p-1)}{1-\delta(p-1)} > p.$$

That is, take $\varepsilon_0 = \frac{(p-1)^2\delta_0}{1-\delta_0(p-1)}$. ☐

The self-improvement property above still holds when ν is not doubling:

Lemma 4.54 (Gehring Lemma)**.** *Let μ be a doubling Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to* $\operatorname{supp}\mu$ *and let ν be an absolutely continuous measure with respect to μ, with $w = \frac{d\nu}{d\mu}$. If $[w]_{B_p(\mu,\mathcal{D},Q_0)} \leqslant C_{RH}$, where $Q_0 \in \mathcal{D} \cup 3\mathcal{D}$, then there exists ε_0 such that for every $\varepsilon < \varepsilon_0$ we have $w \in B_{p+\varepsilon}(\mu,\mathcal{D},Q_0)$, with ε_0 depending only on C_{RH}, C_μ and p; and $[w]_{B_{p+\varepsilon}(\mu,\mathcal{D},Q_0)}$ depending also on ε. Namely,*

$$\int_R w^{p+\varepsilon}\, d\mu \lesssim \left(\fint_R w\, d\mu\right)^\varepsilon \int_R w^p\, d\mu$$

for every $R \in \mathcal{D}(Q_0) \cup 3\mathcal{D}(Q_0)$.

Proof. Since we won't use restricted triple cubes, we assume without loss of generality that $R = Q_0$. Perform an iterated Calderón-Zygmund decomposition (see Lemma 4.34) with $f = w$, with δ to be fixed depending on C_μ, C_{RH} and p and with $\alpha_0 := \fint_{Q_0} w\,d\mu$. We write $\alpha_j := \rho^j \alpha_0$, with $\rho = C_\mu \delta^{-1}$. We will show that for $\varepsilon = q - p$ small enough we can find a constant $C(C_\mu, C_{RH}, p, q) > 0$ such that

$$\int_{Q_0} w^q\,d\mu \leqslant C(C_\mu, C_{RH}, p, q)\alpha_0^{q-p} \int_{Q_0} w^p\,d\mu.$$

Combining CZ1 and Hölder's inequality, for $Q \in \mathcal{F}^j$ we get

$$\alpha_j < \left(\fint_Q w^p\right)^{\frac{1}{p}} \leqslant C_{RH} \fint_Q w\,d\mu.$$

Now we define the level set $A^j := \{x \in Q_0 : w(x) > \alpha_j\} \subset U^j$ for $j \geqslant -1$ (with the convention $U^{-1} = Q_0$, $\mathcal{F}^{-1} = \{Q_0\}$). Then

$$\alpha_j \mu(Q) < C_{RH}\left(\int_{Q\cap A^{j-1}} w\,d\mu + \alpha_{j-1}\mu(Q)\right).$$

Since $\alpha_{j-1} = C_\mu^{-1}\delta\alpha_j$, if we pick $\delta \leqslant \frac{C_\mu}{2C_{RH}}$, then

$$\alpha_j \mu(Q) < 2C_{RH} \int_{Q\cap A^{j-1}} w\,d\mu.$$

Note that for $j \geqslant 0$ we have

$$\int_{U^j} w^p\,d\mu = \sum_{Q\in\mathcal{F}^j} \mu(Q) \fint_Q w^p\,d\mu \leqslant C_{RH}^p \sum_Q \mu(Q)\left(\fint_Q w\,d\mu\right)^p \overset{\text{CZ1}}{\leqslant} (C_{RH}C_\mu\alpha_j)^p \sum_Q \mu(Q).$$

All in all, for $j \geqslant 0$ we get

$$\int_{A^j} w^p\,d\mu < 2C_{RH}^{p+1}C_\mu^p\alpha_j^{p-1} \int_{A^{j-1}} w\,d\mu.$$

Trivially we also have

$$\int_{A^{j-1}\setminus A^j} w^p\,d\mu \leqslant \alpha_j^{p-1} \int_{A^{j-1}\setminus A^j} w\,d\mu,$$

so

$$\int_{A^{j-1}} w^p\,d\mu < C_p\alpha_j^{p-1} \int_{A^{j-1}} w\,d\mu,$$

with $C_p := 1 + 2C_{RH}^{p+1}C_\mu^p$.

By Lemma 4.55 below, fixing $\rho^{p-1} \geqslant 2C_p$ (that is, $\delta \leqslant C_\mu/[2(1+2C_{RH}^{p+1}C_\mu^p)]^{\frac{1}{p-1}}$), we get

$$\int_{Q_0} w^q\,d\mu \leqslant \alpha_0^{q-p}\int_{Q_0} w^p\,d\mu\left(1 + \frac{(2C_p)^{q-p}((2C_p)^{p-1}-1)}{((2C_p)^{q-1}-1) - 2C_p((2C_p)^{q-p}-1)(2C_p)^{p+q-2}}\right),$$

as claimed. ∎

Lemma 4.55. *Let μ be a Radon measures supported in $U \subset \mathbb{R}^d$, let $f \in L^p(\mu)$, let $\alpha_0 > 0$, let $\alpha_j = \rho^j \alpha_0$ with $\rho^{p-1} \geqslant 2C_0 > 2$, and consider $A^j := \{x \in U : f(x) > \alpha_j\}$. If*

$$\int_{A^{j-1}} f^p \, d\mu < C_0 \alpha_j^{p-1} \int_{A^{j-1}} f \, d\mu, \quad \text{for every } j \geqslant 0, \tag{4.13}$$

then

$$\int_U f^q \, d\mu \leqslant \alpha_0^{q-p} \int_U f^p \, d\mu \left(1 + \frac{\rho^{q-p}(\rho^{p-1}-1)}{(\rho^{q-1}-1) - 2C_0(\rho^{q-p}-1)\rho^{p+q-2}}\right),$$

for every $q > p$ such that the denominator in the right-hand side is positive.

Proof. To ensure finiteness of certain integrals below, we will need to find the same estimates for $A^j \backslash A^N$ for N large and $j \leqslant N$. Note that

$$\int_{A^N} f \, d\mu \leqslant \int_{A^N} \frac{f^p}{\alpha_N^{p-1}} \, d\mu \leqslant \int_{A^{j-1}} \frac{f^p}{\rho^{(p-1)(N-j)} \alpha_j^{p-1}} \, d\mu.$$

Thus,

$$\int_{A^{j-1} \backslash A^N} f \, d\mu \overset{(4.13)}{\geqslant} \left(\frac{1}{C_0 \alpha_j^{p-1}} - \frac{1}{\rho^{(p-1)(N-j)} \alpha_j^{p-1}} \right) \int_{A^{j-1}} f^p \, d\mu.$$

If ρ is big enough, say $\rho^{(p-1)(N-j)} \geqslant 2C_0$ for every $N \geqslant j+1$, we deduce

$$\int_{A^{j-1} \backslash A^N} f \, d\mu \geqslant \frac{1}{2C_0 \alpha_j^{p-1}} \int_{A^{j-1}} f^p \, d\mu,$$

so

$$\int_{A^{j-1} \backslash A^N} f^p \, d\mu \leqslant 2C_0 \rho^{p-1} \alpha_{j-1}^{p-1} \int_{A^{j-1} \backslash A^N} f \, d\mu. \tag{4.14}$$

In case $N = j$, estimate (4.14) holds trivially:

$$\int_{A^{j-1} \backslash A^j} f^p \, d\mu \leqslant \alpha_j^{p-1} \int_{A^{j-1} \backslash A^j} f \, d\mu = \rho^{p-1} \alpha_{j-1}^{p-1} \int_{A^{j-1} \backslash A^j} f \, d\mu.$$

Now, for $q \geqslant 1$ let us write

$$I_N(q) := \int_{A^0 \backslash A^N} f^q \, d\mu = \sum_{j=0}^{N-1} \int_{A^j \backslash A^{j+1}} f^q \, d\mu.$$

For $t \geqslant 0$ consider the *summation by parts* identity

$$\begin{aligned} \sum_{j=0}^{N-1} \alpha_{j+1}^t \int_{A^j \backslash A^{j+1}} f^q \, d\mu &= \alpha_1^t \int_{A^0 \backslash A^N} f^q \, d\mu + \sum_{k=1}^{N-1} (\alpha_{N-k+1}^t - \alpha_{N-k}^t) \int_{A^{N-k} \backslash A^N} f^q \, d\mu, \\ &= \alpha_1^t I_N(q) + \sum_{k=1}^{N-1} (\rho^t - 1) \alpha_{N-k}^t \int_{A^{N-k} \backslash A^N} f^q \, d\mu, \end{aligned} \tag{4.15}$$

Then for $q>p$ we get

$$I_N(q)\leqslant \sum_{j=0}^{N-1}\alpha_{j+1}^{q-p}\int_{A^j\setminus A^{j+1}} f^p\,d\mu \overset{(4.15)}{=} \alpha_1^{q-p}I_N(p)+\sum_{k=1}^{N-1}(\rho^{q-p}-1)\alpha_{N-k}^{q-p}\int_{A^{N-k}\setminus A^N} f^p\,d\mu$$
$$\overset{(4.14)}{\leqslant} \alpha_1^{q-p}I_N(p)+\frac{2C_0(\rho^{q-p}-1)\rho^{p-1}}{(\rho^{q-1}-1)}\sum_{k=1}^{N-1}(\rho^{q-1}-1)\alpha_{N-k}^{q-p}\alpha_{N-k}^{p-1}\int_{A^{N-k}\setminus A^N} f\,d\mu.$$
$$\overset{(4.15)}{=} \alpha_1^{q-p}I_N(p)+\frac{2C_0(\rho^{q-p}-1)\rho^{p-1}}{(\rho^{q-1}-1)}\left(\sum_{j=0}^{N-1}\alpha_{j+1}^{q-1}\int_{A^j\setminus A^{j+1}} f\,d\mu-\alpha_1^{q-1}I_N(1)\right).$$

In A_j we have $\alpha_{j+1}=\rho\alpha_j\leqslant\rho|f|$, so

$$I_N(q)\overset{(4.14)}{\leqslant}\alpha_1^{q-p}I_N(p)+\frac{2C_0(\rho^{q-p}-1)\rho^{p-1}}{(\rho^{q-1}-1)}\left(\rho^{q-1}I_N(q)-(2C_0)^{-1}\alpha_1^{q-p}I_N(p)\right),$$

which implies

$$I_N(q)\left(1-\frac{2C_0(\rho^{q-p}-1)\rho^{p+q-2}}{(\rho^{q-1}-1)}\right)\leqslant\alpha_1^{q-p}I_N(p)\left(1-\frac{(\rho^{q-p}-1)\rho^{p-1}}{(\rho^{q-1}-1)}\right).$$

If $q-p$ is small enough, the factors above are positive and we obtain

$$I_N(q)\leqslant\alpha_0^{q-p}I_N(p)\frac{\rho^{q-p}(\rho^{p-1}-1)}{(\rho^{q-1}-1)-2C_0(\rho^{q-p}-1)\rho^{p+q-2}}.$$

Since

$$\int_U f^q\,d\mu\leqslant\alpha_0^{q-p}\int_{U\setminus A^0} f^p\,d\mu+\int_{A^0} f^q\,d\mu,$$

the lemma follows letting $N\to\infty$, since $\mu(A_N)\to 0$. □

4.7 Weak conditions

Lemma 4.56 (Gehring Lemma for enlarged balls)**.** *Let $\lambda>1$. Let μ be a doubling Radon measure in $\mathbb{R}^d$, and let ν be an absolutely continuous measure with respect to μ, with $w=\frac{d\nu}{d\mu}$. Let $U\subset X$ be a relative open set with $\mu(U)<\infty$. If*

$$\left(\fint_\Delta w^p\,d\mu\right)^{\frac{1}{p}}\leqslant C_{RH}\fint_{\lambda\Delta} w\,d\mu\ \ \textit{for every boundary ball with } \lambda B\subset U,$$

given a compact set $K\subset U$ with positive measure then there exists ε_0 such that for every $\varepsilon<\varepsilon_0$,

$$\left(\fint_K w^{p+\varepsilon}\,d\mu\right)^{\frac{1}{p+\varepsilon}}\leqslant C_{RH,\varepsilon,\lambda}\left(\fint_U w^p\,d\mu\right)^{\frac{1}{p}},$$

with ε_0 depending only on C_{RH}, p, the doubling constant C_μ, λ and the Whitney constants; and $C_{RH,\varepsilon}$ depending also on ε, $\frac{\mu(K)}{\mu(U)}$ and $\frac{\operatorname{dist}(K,X\setminus U)}{\operatorname{diam}K}$.

This lemma will be deduced from the following version for dyadic cubes.

Lemma 4.57 (Gehring Lemma for enlarged cubes)**.** *Let μ be a doubling Radon measure in $\mathbb{R}^d$, let $\mathcal{D}$ be a dyadic grid associated to $X = \operatorname{supp}\mu$ and let ν be an absolutely continuous measure with respect to μ, with $w = \frac{d\nu}{d\mu}$. Let $U \subset X$ be a relative open set with $\mu(U) < \infty$. If*

$$\left(\fint_Q w^p \, d\mu\right)^{\frac{1}{p}} \leqslant C_{RH} \fint_{3Q} w \, d\mu \text{ for every } Q \in \mathcal{D} \text{ with } 3Q \subset U,$$

given a compact set $K \subset U$ with positive measure then there exists ε_0 such that for every $\varepsilon < \varepsilon_0$

$$\left(\fint_K w^{p+\varepsilon} \, d\mu\right)^{\frac{1}{p+\varepsilon}} \leqslant C_{RH,\varepsilon} \left(\fint_U w^p \, d\mu\right)^{\frac{1}{p}},$$

with ε_0 depending only on C_{RH}, p, the doubling constant C_μ and the Whitney constants; and $C_{RH,\varepsilon}$ depending also on ε, $\frac{\mu(K)}{\mu(U)}$ and $\frac{\operatorname{dist}(K, X\setminus U)}{\operatorname{diam} K}$.

Proof. Let $\mathcal{W}$ be a Whitney decomposition of U, see Lemma 4.35. If $U = X$, then just consider $\mathcal{W} = \{U\}$.

Consider the auxiliary function $\phi(x) = \sum_{Q \in \mathcal{W}} \chi_Q(x)\mu(Q)$, and let

$$\alpha_0 := \left(\fint_U (\mu(U)w)^p \, d\mu\right)^{\frac{1}{p}}.$$

Note that for $Q \in \mathcal{W}$ we get

$$\fint_Q (w\phi)^p \, d\mu = \mu(Q)^{p-1} \int_Q w^p \, d\mu \leqslant \mu(U)^{p-1} \int_U w^p \, d\mu = \alpha_0^p.$$

Write $\alpha_j := \rho^j \alpha_0$, with $\rho = C_\mu^2 \delta^{-1}$, so $\alpha_j^p = \rho^{pj}\alpha_0^p$, with δ to be fixed depending on C_{RH}. Thus, we can perform an iterated Calderón-Zygmund decomposition with $f = (\phi w)^p$ and ground levels α_j^p at every whitney cube $Q = U_Q^j \cup G_Q^j \cup Z_Q^j$ with $U_Q^j := \bigcup_{R \in \mathcal{F}_Q^j} R$, and write $U^j = \bigcup_{Q \in \mathcal{W}} U_Q^j$ and so on and so forth. Note that $\rho^p = C_\mu(C_\mu^{2p-1}\delta^{-p})$, and in particular

$$\rho^p \geqslant C_\mu$$

assuming $\delta < 1$, so the notation is different than in Lemma 4.34 (in the statement of the iterated decomposition we should now replace δ by $C_\mu^{1-2p}\delta^p$ and α_j by α_j^p).

We will show that for $\varepsilon = q-p$ small enough we can find a constant $C(C_\mu, C_{RH}, C_{\mathcal{W}}, p, q) > 0$ such that

$$\int_U (\phi w)^q \, d\mu \leqslant C(C_\mu, C_{RH}, C_{\mathcal{W}}, p, q)\alpha_0^{q-p} \int_U (\phi w)^p \, d\mu. \tag{4.16}$$

Note that picking

$$k_0 := \left\lceil \frac{C_{d,C_{\mathcal{W}}} + \log \frac{\operatorname{diam} K}{\operatorname{dist}(K,U^c)}}{-\log \ell_0} \right\rceil,$$

we can grant that for $x \in K$ and $x \in Q_x \in \mathcal{W}$ we have $3\mathcal{F}^{k_0}(Q_x) \supset K$, which in particular implies

$$\mu(K) \leqslant C_\mu^{k_0+1}\mu(Q_x) = C_\mu^{k_0+1}\phi(x).$$

This yields

$$\left(\mu(K)^{q-1}\int_K w^q\,d\mu\right)^{\frac{1}{q}} \lesssim C_\mu^{k_0+1}\left(\frac{\mu(U)}{\mu(K)}\right)^{\frac{1}{q}}\left(\fint_U (\phi w)^q\,d\mu\right)^{\frac{1}{q}}.$$

Therefore, we get

$$\left(\mu(K)^{q-1}\int_K w^q\,d\mu\right)^{\frac{1}{q}} \overset{(4.16)}{\lesssim} C_\mu^{k_0+1}\left(\frac{\mu(U)}{\mu(K)}\right)^{\frac{1}{q}} C^{\frac{1}{q}}\mu(U)^{\frac{q-p}{q}+\frac{p}{q}}\left(\fint_U w^p\,d\mu\right)^{\frac{q-p}{pq}+\frac{1}{q}},$$

and the lemma follows by fixing appropriately the constant $C_{RH,\varepsilon}$.

It remains to establish (4.16). By CZ1, for $Q \in \mathcal{W}$ and $P \in \mathcal{F}_Q^j$ we get

$$\alpha_j^p < \fint_P (\phi w)^p \leqslant C_\mu \alpha_j^p, \tag{4.17}$$

and thus

$$\alpha_j < \left(\fint_P (\phi w)^p\,d\mu\right)^{\frac{1}{p}} = \mu(Q)\left(\fint_P w^p\,d\mu\right)^{\frac{1}{p}} \leqslant C_{RH}\mu(Q)\fint_{3P} w\,d\mu.$$

Next we claim that $\mu(Q) \leqslant C_\mu^2\phi$ in $3P$ and thus we get

$$\alpha_j < C_{RH}C_\mu^2\fint_{3P}\phi w\,d\mu.$$

Indeed, note that $\fint_Q(\phi w)^p \leqslant \alpha_j^p$ implies $P \neq Q$. By Remark 4.25 we have $3P \subset 3Q$, but $3P$ may intersect a Whitney cube $\widetilde{Q} \neq Q$. Assume that $x \in \widetilde{Q} \cap 3P$, and there must exist a cube R with $x \in R \in \mathcal{N}(P)\backslash\{P\}$. Since $3P \subset 3Q$, we get $3\widetilde{Q} \cap 3Q \neq \varnothing$ and Wh4 implies if say $Q \in \mathcal{D}_{k_0}$, then

$$\widetilde{Q} \in \mathcal{D}_{k_0-1} \cup \mathcal{D}_{k_0} \cup \mathcal{D}_{k_0+1}.$$

Since $P \subsetneq Q$, we deduce $R \subset \widetilde{Q}$ and Remark 4.25 also implies $\lambda\widetilde{B}_Q \cap \lambda\widetilde{B}_{\widetilde{Q}} \neq \varnothing$. In particular, either they are neighbors or one of them is neighbor to the father of the other. In any case,

$$\phi(x) = \mu(\widetilde{Q}) \geqslant C_\mu^2\mu(Q),$$

and the claim follows.

Now we define the level set $A^j := \{x \in U : \phi(x)w(x) > \alpha_j\} \subset U^j$ for $j \geqslant -1$ (with the convention $U^{-1} = U$). Then

$$\alpha_j\mu(3P) < C_{RH}C_\mu^2\left(\int_{3P\cap A^{j-1}}\phi w\,d\mu + \alpha_{j-1}\mu(3P)\right).$$

Since $\alpha_{j-1} = C_\mu^{-2}\delta\alpha_j$, if we pick $\delta \leqslant \frac{1}{2C_{RH}}$, then

$$\alpha_j \mu(3P) < 2C_{RH}C_\mu^2 \int_{3P\cap A^{j-1}} \phi w \, d\mu.$$

Note that

$$\int_{U^j} (\phi w)^p \, d\mu = \sum_{P\in\mathcal{F}^j} \mu(P) \fint_P (\phi w)^p \, d\mu \overset{(4.17)}{\leqslant} C_\mu \alpha_j^p \sum_{P\in\mathcal{F}^j} \mu(P).$$

All in all, for $j \geqslant 0$ we get

$$\int_{A^j} (\phi w)^p \, d\mu < 2C_\mu^3 C_{RH}\alpha_j^{p-1} \sum_{P\in\mathcal{F}^j} \int_{3P\cap A^{j-1}} \phi w \, d\mu \leqslant 2C_\mu^3 C_{RH}\alpha_j^{p-1} C_{\mathcal{W}} \int_{A^{j-1}} \phi w \, d\mu.$$

Trivially we also have

$$\int_{A^{j-1}\setminus A^j} (\phi w)^p \, d\mu < \alpha_j^{p-1} \int_{A^{j-1}\setminus A^j} \phi w \, d\mu,$$

so

$$\int_{A^{j-1}} (\phi w)^p \, d\mu < C_0 \alpha_j^{p-1} \int_{A^{j-1}} \phi w \, d\mu,$$

with $C_0 := 1 + 2C_{\mathcal{W}} C_{RH} C_\mu^3$.

By Lemma 4.55, if $\rho^{p-1} \geqslant 2C_0$, i.e.

$$\delta = \min\left\{ \frac{C_\mu^2}{(2C_0)^{\frac{1}{p-1}}}, \frac{1}{2C_{RH}} \right\},$$

we infer that

$$\int_U (\phi w)^q \, d\mu \lesssim_{C_0,q,p} \alpha_0^{q-p} \int_U (\phi w)^p \, d\mu,$$

as claimed, with $q - p \leqslant C_{C_0,p}$. $\square$

Proof of Lemma 4.57. Let $Q \in \mathcal{D}$ with $3Q \subset U$. By covering Q with finitely many boundary balls Δ_i with radii comparable to $\ell(Q)$ such that $\lambda\Delta_i \subset 3Q$, we deduce that

$$\left(\fint_Q w^p \, d\mu \right)^{\frac{1}{p}} \approx \sum_i \left(\fint_{\Delta_i} w^p \, d\mu \right)^{\frac{1}{p}} \lesssim \sum_i \fint_{\lambda\Delta_i} w \, d\mu \lesssim \fint_{3Q} w \, d\mu.$$

So Lemma 4.57 implies that

$$\left(\fint_K w^{p+\varepsilon} \, d\mu \right)^{\frac{1}{p+\varepsilon}} \leqslant C'_{RH,\varepsilon} \left(\fint_U w^p \, d\mu \right)^{\frac{1}{p}}$$

for some $\varepsilon > 0$. $\square$

4.8 The Riesz representation theorem

Recall that a topologic space X is said to be *locally compact* if every point $x \in X$ has a neighborhood whose closure is compact.

Theorem 4.58 (Riesz representation Theorem)**.** *Let X be a locally compact metric space and $L : C_c(X) \to \mathbb{R}$ a positive linear functional. Then there is a unique Radon measure μ such that*

$$L_f = \int f \, d\mu \qquad \text{for } f \in C_c(X).$$

The approach presented below is based on the proof of [Rud87, Chapter 2], where the reader may find all the details and the proofs of every single lemma used here.

Proof. Given an open set $V \subset X$ we write $f \prec V$ whenever $f \in C_c(V)$, and $0 \leqslant f \leqslant \chi_V$. We define

$$\mu(V) := \sup\{L_f : f \prec V\}.$$

Note that for open sets $U \subset V$ it follows immediately that $\mu(U) \leqslant \mu(V)$. Therefore it makes sense to define for every $E \subset X$

$$\mu(E) := \inf\{\mu(V) : V \supset E \text{ and } V \text{ is open}\}.$$

We will use often the following immediate consequence of the positivity of L_f:

$$\text{If } f, g \in C_c(X) \text{ are such that } 0 \leqslant f \leqslant g, \text{ then } L_f \leqslant L_g \tag{4.18}$$

First we claim that μ is a measure.

1. Since $\varnothing$ is open, $\mu(\varnothing) = \sup\{L_f : f \prec \varnothing\} = L_0 = 0$.
2. Given sets $A \subset B \subset X$,
$$\{V : V \supset A \text{ and } V \text{ is open}\} \supset \{V : V \supset B \text{ and } V \text{ is open}\}$$
trivially, and taking infimum in a subset always increases the result, so
$$\mu(A) \leqslant \mu(B). \tag{4.19}$$
3. Let $A_i \subset X$ for $1 \leqslant i < \infty$, and let $\varepsilon > 0$. Consider open sets $V_i \supset A_i$ such that $\mu(V_i) \leqslant \mu(A_i) + \frac{\epsilon}{2^i}$, and let $f \prec V := \bigcup_i V_i$ so that $\mu(V) \leqslant L_f + \varepsilon$.
Since $K := \operatorname{supp} f$ is compactly contained in V we infer that there exist $n \in \mathbb{N}$ and a finite subcovering, i.e., a subset $\{i_j\}_{j=1}^n \subset \mathbb{N}$ so that $K \subset \bigcup_{j=1}^n V_{i_j}$.
There exists a partition of the unity in K for the covering V_{i_j}, i.e., there exist functions $h_j \prec V_{i_j}$ with $\chi_K \leqslant \sum_j h_j \leqslant 1$. Then
$$\begin{aligned}\mu\left(\bigcup_i A_i\right) &\leqslant \mu(V) \leqslant L_f + \varepsilon = L_{f \sum_j h_j} + \varepsilon = \sum_j L_{f h_j} + \varepsilon \\ &\leqslant \sum_j \mu\left(V_{i_j}\right) + \varepsilon \leqslant \sum_i \left(\mu(A_i) + \frac{\varepsilon}{2^i}\right) + \varepsilon \leqslant \sum_i \mu(A_i) + 2\varepsilon, \end{aligned} \tag{4.20}$$
concluding the proof that μ is a mesaure.

Next we show that μ is in fact a Radon measure. To show that we begin by $a) - c)$ in Definition 4.4:

a) Let $K \subset X$ be a compact set. Then K is contained in a ball B. Consider a continuous function $\chi_K \leqslant f \leqslant \chi_B$, which exists by Urysohn's lemma. Then call $V := \{x : f(x) > 1/2\}$. Every function $g \prec V$ satisfies that $g \leqslant 2f$. Therefore

$$\mu(K) \leqslant \mu(V) = \sup\{L_g : g \prec V\} \overset{(4.18)}{\leqslant} 2L_f < \infty.$$

b) Let V be an open set. We will prove that its measure coincides with the supremum of the measures of its compact subsets. Let $\varepsilon > 0$ and $f \prec V$ such that $\mu(V) \leqslant L_f + \varepsilon$. Then write $K := \operatorname{supp} f$ and consider an open set $U \supset K$. It is clear that $f \prec U$ and thus $\mu(U) > L_f$. Since this holds for every such U, passing to the infimum we can infer that $\mu(K) \geqslant L_f$. All in all,

$$\mu(V) \leqslant L_f + \varepsilon \leqslant \mu(K) + \varepsilon.$$

Since such a compact set can be obtained for every ε, we conclude that

$$\mu(V) \leqslant \sup\{\mu(K) : K \subset V\}.$$

The converse inequality follows from (4.19).

c) $\mu(E) := \inf\{\mu(V) : V \supset E \text{ and } V \text{ is open}\}$ follows by definition.

To complete the proof that μ is Radon, we will check that it is Borel regular. First of all, let K_1, K_2 be compact, disjoint subsets of X. We claim that

$$\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2). \tag{4.21}$$

Indeed, it is well known that there exist open sets $V_i \supset K_i$, such that $V_1 \cap V_2 = \varnothing$ (see [Rud87, Theorem 2.7], for instance), and also there exists an open set $W \supset K_1 \cup K_2$ such that $\mu(W) < \mu(K_1 \cup K_2) + \varepsilon$. Moreover, there exist functions $f_i \prec V_i \cap W$ so that $\mu(V_i \cap W) \leqslant L_{f_i} + \varepsilon$. Then, since the supports of f_i are disjoint, $f_1 + f_2 \prec W$ and we get

$$\begin{aligned}\mu(K_1) + \mu(K_2) &\overset{(4.19)}{\leqslant} \mu(V_1 \cap W) + \mu(V_2 \cap W) \leqslant L_{f_1} + L_{f_2} + 2\varepsilon \\ &= L_{f_1+f_2} + 2\varepsilon \leqslant \mu(W) + 2\varepsilon < \mu(K_1 \cup K_2) + 3\varepsilon,\end{aligned}$$

proving the claim.

Since the μ-measurable sets form a σ-algebra, to show that μ is a Borel measure we only need to check that every open set V is μ-measurable, i.e., every $E \subset X$ satisfies that

$$\mu(E) = \mu(E \cap V) + \mu(E \cap V^c).$$

By the subadditivity shown in (4.20), it suffices to prove that

$$\mu(E) \geqslant \mu(E \cap V) + \mu(E \cap V^c) \tag{4.22}$$

and for this we may assume that $\mu(E) < \infty$.

First let us assume that E is an open set with finite measure. Then write $\widetilde{V} = V \cap E$, so $E \cap V^c = E \cap (V^c \cup E^c) = E \cap (V \cap E)^c = E \cap \widetilde{V}^c$, i.e. we have to show that

$$\mu(E) \geqslant \mu(\widetilde{V}) + \mu(E \cap \widetilde{V}^c).$$

Let $K_1 \subset \widetilde{V}$ be a compact set such that

$$\mu(\widetilde{V}) \leqslant \mu(K_1) + \varepsilon.$$

Then consider an open set $U \supset E \cap \widetilde{V}^c$ so that $\mu(U) \leqslant \mu(E \cap \widetilde{V}^c) + \varepsilon$. Define $\widetilde{U} := U \cap E \cap K_1^c$ which is again an open set. Then

$$\mu(\widetilde{U}) \overset{(4.19)}{\leqslant} \mu(U) \leqslant \mu(E \cap \widetilde{V})^c + \varepsilon,$$

and

$$E \cap \widetilde{V}^c = U \cap E \cap \widetilde{V}^c \subset U \cap E \cap K_1^c = \widetilde{U} \subset K_1^c \cap E. \tag{4.23}$$

To end consider a compact set $K_2 \subset \widetilde{U}$ such that $\mu(\widetilde{U}) \leqslant \mu(K_2) + \varepsilon$. All in all,

$$\begin{aligned}\mu(\widetilde{V}) + \mu(E \cap \widetilde{V}^c) &\overset{(4.23)}{\leqslant} \mu(K_1) + \varepsilon + \mu(\widetilde{U}) \leqslant \mu(K_1) + \mu(K_2) + 2\varepsilon \\ &\overset{(4.21)}{=} \mu(K_1 \cup K_2) + 2\varepsilon \overset{(4.19)}{\leqslant} \mu(E) + 2\varepsilon,\end{aligned}$$

and (4.22) follows for open sets.

Consider a set $E \subset X$ (without the openness assumption). Then there exists an open set $V_E \supset E$ such that $\mu(V_E) \leqslant \mu(E) + \varepsilon$. Then

$$\mu(E \cap V) + \mu(E \cap V^c) \overset{(4.19)}{\leqslant} \mu(V_E \cap V) + \mu(V_E \cap V^c) = \mu(V_E) \leqslant \mu(E) + \varepsilon,$$

proving (4.22) for general sets.

To end we have to check that $L_f = \int f \, d\mu$ for every $f \in C_c(X)$. For simplicity we may assume that f is real valued. Moreover, it suffices to show

$$L_f \leqslant \int f \, d\mu, \tag{4.24}$$

since we can apply the same inequality to $-f$ to obtain the converse estimate.

Let $[a, b] \cup \{0\}$ be the range of f. For every n consider $\{y_i\}_{i=0}^{n+1}$ with $y_0 < a$, $y_{n+1} = b$ and $0 < y_{i+1} - y_i \leqslant (b-a)/n =: \varepsilon$ for every $i \leqslant n$. Let $E_i := f^{-1}((y_{i-1}, y_i]) \cap \operatorname{supp} f$, which are Borel sets and, thus, measurable. Consider open sets $V_i \supset E_i$ with $\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{n+1}$ and such that $f(x) < y_i + \varepsilon$ for every $x \in V_i$; and let h_i be a partition of the unity of $\operatorname{supp} f$ with respect to the covering $\{V_i\}$, that is $h_i \prec V_i$ with $\chi_{\{\operatorname{supp} f\}} \leqslant \sum_i h_i \leqslant 1$. Then

$$\begin{aligned}L_f &= \sum_i L_{h_i f} \overset{(4.18)}{\leqslant} \sum_i (y_i + \varepsilon) L_{h_i} \leqslant \sum_i (y_i + \varepsilon)\mu(V_i) \leqslant \sum_i (y_i - \varepsilon + 2\varepsilon) \left(\mu(E_i) + \frac{\varepsilon}{n+1} \right) \\ &= \sum_i \mu(E_i)(y_i - \varepsilon) + 2\varepsilon \sum_i \mu(E_i) + \frac{\varepsilon}{n+1} \sum_i y_i + \varepsilon^2 \overset{(4.1)}{\leqslant} \int f \, d\mu + \varepsilon(2\mu(\operatorname{supp} f) + b + \varepsilon)\end{aligned}$$

and (4.24) follows choosing ε arbitrarily small.

As for uniqueness, assume that μ_1, μ_2 are Radon measures satisfying the hypotheses of the Theorem. Since Radon measures are determined by their values on compact sets, we only need to check that $\mu_1(K) = \mu_2(K)$ for every compact set $K \subset X$. Consider such a compact set, and let $V \supset K$ be an open set such that $\mu_2(V) \leqslant \mu_2(K) + \varepsilon$. By Urysohn's lemma, there exists $f \prec V$ such that $\chi_K \leqslant f$. Then

$$\mu_1(K) = \int \chi_K \, d\mu_1 \leqslant \int f \, d\mu_1 = L_f = \int f \, d\mu_2 \leqslant \int \chi_V \, d\mu_2 = \mu_2(V) \leqslant \mu_2(K) + \varepsilon.$$

□

4.8.1 Image measure

Definition 4.59. The image of a measure μ under a mapping $f : X \to Y$ (also known as *push-forward measure*) is defined by $f_\# \mu(A) = \mu(f^{-1}(A))$ for $A \subset Y$.

Theorem 4.60. *If X, Y are separable metric spaces, f is continuous and μ is a compactly supported Radon measure, then $f_\# \mu$ is a Radon measure, with* $\operatorname{supp} f_\# \mu = f(\operatorname{supp} \mu)$.

Theorem 4.61. *If X, Y are metric spaces, f is a Borel mapping, μ is a Borel measure and g is a nonnegative Borel function, then*

$$\int g \, df_\# \mu = \int (g \circ f) \, d\mu.$$

4.8.2 Weak convergence

Let $\{\mu_i\}_{i=0}^{\infty}$ be a collection of Radon measures in a metric space X. We say that μ_i converge weakly to μ_0, and write

$$\mu_i \rightharpoonup \mu_0,$$

if

$$\lim_{i \to \infty} \int \varphi \, d\mu_i = \int \varphi \, d\mu_0 \quad \text{for every } \varphi \in C_c(X).$$

As a consequence of the Riesz representation theorem, one can prove that a uniformly locally finite collection of measures has a weakly convergent subsequence:

Theorem 4.62. *If $\{\mu_i\}_{i=1}^{\infty}$ is a collection of Radon measures in $\mathbb{R}^d$, with*

$$\sup_i \mu_i(K) < +\infty,$$

for every compact set $K \subset \mathbb{R}^d$, then there is a weakly convergent subsequence $\{\mu_{i_k}\}_{k=1}^{\infty}$, and a Radon measure μ with

$$\mu_{i_k} \rightharpoonup \mu.$$

Consider the Dirac delta measure δ_i in $i \in \mathbb{N}$. Note that the sequence $\delta_i \rightharpoonup 0$. This example shows that the weak convergence of measures does not imply the convergence of the measure of a particular set. However, the following semicontinuity properties hold:

Theorem 4.63. *Let $\{\mu_i\}_{i=0}^{\infty}$ be a collection of Radon measures in a locally compact metric space X. If $\mu_i \rightharpoonup \mu_0$, $K \subset X$ is compact and $G \subset X$ is open, then*

$$\mu_0(K) \geqslant \limsup_{i\to\infty} \mu_i(K),$$

and

$$\mu_0(G) \leqslant \liminf_{i\to\infty} \mu_i(G).$$

4.9 Hausdorff measure and dimension

For every subset $A \subset \mathbb{R}^d$, $0 \leqslant s < +\infty$ and $0 < \delta \leqslant +\infty$, define

$$\mathcal{H}^s_\delta(A) := \inf\left\{\sum_i \operatorname{diam}(E_i)^s : A \subset \bigcup_i E_i \text{ with } \operatorname{diam}(E_i) \leqslant \delta\right\},$$

and let

$$\mathcal{H}^s(A) := \lim_{\delta\searrow 0} \mathcal{H}^s(A)$$

be the *s-dimensional Hausdorff measure of A*. The quantity $\mathcal{H}^s_\infty(A)$ also plays an important role and is called *s-dimensional Hausdorff content of A*. The Hausdorff measure happens to be a Radon measure. The 0-dimensional Hausdorff measure is the counting measure, the 1-dimensional measure is a generalization of the length measure in $\mathbb{R}^d$, and the d-dimensional measure is a multiple of the Lebesgue measure.

If A is a set with $\mathcal{H}^s(A) < +\infty$, then $\mathcal{H}^s|_A$ is locally finite and, in fact, it happens to be a Radon measure (see [Mat95, chapter 4]).

Another interesting fact is that although

$$\mathcal{H}^s_\infty(A) \leqslant \mathcal{H}^s_\delta(A) \nearrow \mathcal{H}^s(A),$$

having null Hausdorff content is equivalent to having zero Hausdorff measure:

$$\mathcal{H}^s_\infty(A) = 0 \iff \mathcal{H}^s(A) = 0.$$

Theorem 4.64. *For $0 \leqslant s < t < \infty$ and $A \subset \mathbb{R}^d$,*

1. *$\mathcal{H}^s(A) < +\infty$ implies $\mathcal{H}^t(A) = 0$, and*
2. *$\mathcal{H}^t(A) > 0$ implies $\mathcal{H}^s(A) = +\infty$.*

This leads to the concept of Hausdorff dimension:

Definition 4.65. The *Hausdorff dimension* of a set $A \subset \mathbb{R}^d$ is

$$\dim_{\mathcal{H}} A = \sup\{s : \mathcal{H}^s(A) > 0\}.$$

Equivalently,

$$\dim_{\mathcal{H}} A = \sup\{s : \mathcal{H}^s_\infty(A) > 0\}.$$

From the previous theorem, one can infer that

$$\dim_{\mathcal{H}} A = \sup\{s : \mathcal{H}^s(A) = +\infty\} = \inf\{s : \mathcal{H}^s(A) < +\infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

4.10 Frostman's lemma

The following result is Frostman's Lemma, which is a fundamental tool in geometric measure theory and in potential theory.

Theorem 4.66. *Let E be a Borel set in $\mathbb{R}^d$. Then $\mathcal{H}^s(E) > 0$ if and only if there exists a non-zero finite Radon measure μ compactly supported in E such that*

$$\mu(B_r(x)) \leqslant r^s \quad \text{for every } x \in \mathbb{R}^d \text{ and } r > 0.$$

Further,

$$\mathcal{H}^s_\infty(E) \approx \sup\big\{\mu(E) : \operatorname{supp}\mu \subset E,\, \mu(B_r(x)) \leqslant r^s \text{ for every } x \in \mathbb{R}^d \text{ and } r > 0\big\},$$

with the implicit constant depending only on d.

Below we provide a proof for the case when E is a compact set. The case when E is σ-compact is easily deduced from this. These two cases suffice for the purposes of these notes.

Proof. Suppose first that such a measure μ exists, and let us see that $\mathcal{H}^s_\infty(E) \geqslant \mu(E)$. Indeed, consider a covering $\bigcup_i A_i \supset E$, and take for each i a point $x_i \in A_i$. Since the union of the balls $B_{\operatorname{diam}(A_i)}(x_i)$ covers E, we get

$$\sum_i \operatorname{diam}(A_i)^s \geqslant \sum_i \mu\big(B_{\operatorname{diam}(A_i)}(x_i)\big) \geqslant \mu(E).$$

Taking the infimum over all possible coverings of E, we obtain $\mathcal{H}^s_\infty(E) \geqslant \mu(E)$.

For the converse implication of the theorem, assume that E is contained in a dyadic cube Q_0. The measure μ will be constructed as a weak limit of measures μ_n, $n \geqslant 0$. The first measure is

$$\mu_0 = \mathcal{H}^s_\infty(E)\,\frac{\mathcal{L}^d|_{Q_0}}{\mathcal{L}^d(Q_0)}.$$

For $n \geqslant 1$, each measure μ_n vanishes in $\mathbb{R}^d \backslash Q_0$, it is absolutely continuous with respect to Lebesgue measure, and in each cube from $\mathcal{D}_n(Q_0)$ (this is the family of dyadic n-descendants of Q_0), it has constant density with respect to Lebesgue measure. It is defined from μ_{n-1} as follows. If $P \in \mathcal{D}_n(Q_0)$ and P is a dyadic child of $Q \in \mathcal{D}_{n-1}(Q_0)$ (then we write $P \in \mathcal{C}h(Q)$), we set

$$\mu_n(P) = \frac{\mathcal{H}^s_\infty(P \cap E)}{\sum_{R \in \mathcal{C}h(Q)} \mathcal{H}^s_\infty(R \cap E)}\,\mu_{n-1}(Q). \tag{4.25}$$

Observe that

$$\sum_{P \in \mathcal{C}h(Q)} \mu_n(P) = \mu_{n-1}(Q) \qquad \text{for all } Q \in \mathcal{D}_{n-1}(Q_0),$$

and thus $\mu_n(\mathbb{R}^d) = \mu_{n-1}(\mathbb{R}^d)$.

As said above, μ is just a weak limit of the measures μ_n. The fact that μ is supported on E is easy to check: from the definition of μ_n in (4.25), $\mu_n(P) = 0$ if $P \in \mathcal{D}_n(Q_0)$ does not intersect E. As a consequence, $\mu_k(P) = 0$ for all $k \geqslant n$ too, and thus,

$$\mathrm{supp}(\mu_k) \subset \mathcal{U}_{2^{-n+1}\mathrm{diam}(Q_0)}(E) \qquad \text{for all } k \geqslant n, \tag{4.26}$$

where $\mathcal{U}_t(E)$ stands for the t-neighborhood of E, that is,

$$\mathcal{U}_t(E) = \{x \in \mathbb{R}^d : \mathrm{dist}(x, E) < t\}.$$

From (4.26) one gets that $\mathrm{supp}(\mu) \subset \mathcal{U}_{2^{-n+1}\mathrm{diam}(Q_0)}(E)$, for all $n \geqslant 0$, which proves the claim.

Next we will show that

$$\mu_n(P) \leqslant \mathcal{H}^s_\infty(P \cap E) \qquad \text{for all } P \in \mathcal{D}_n(Q_0).$$

This follows easily by induction: it is clear for $n = 0$, and if it holds for $n-1$ and Q is the dyadic parent of P, then

$$\mu_{n-1}(Q) \leqslant \mathcal{H}^s_\infty(Q \cap E) \leqslant \sum_{R \in \mathcal{C}h(Q)} \mathcal{H}^s_\infty(R \cap E).$$

Thus, from (4.25), we infer that $\mu_n(P) \leqslant \mathcal{H}^s_\infty(P \cap E)$, as claimed. As a consequence, for all $j \geqslant n$,

$$\mu_j(P) \leqslant \mathcal{H}^s_\infty(P \cap E) \qquad \text{for all } P \in \mathcal{D}_n(Q_0).$$

Moreover, by construction, all the dyadic cubes which do not intersect Q_0 have zero measure μ_j.

Since every open ball B_r of radius r with $2^{-n-1}\ell(Q_0) \leqslant r < 2^{-n}\ell(Q_0)$ (where $\ell(Q_0)$ stands for the side length of Q_0) is contained in a union of at most 2^d dyadic cubes P_k with side length $2^{-n}\ell(Q_0)$, we get

$$\mu_j(B_r) \leqslant \sum_{k=1}^{2^d} \mu_j(P_k) \leqslant \sum_{k=1}^{2^d} \mathcal{H}^s_\infty(P_k \cap E) \leqslant 2^d\,\mathrm{diam}(P_k)^s \leqslant c\,r^s,$$

for all $j \geqslant n$. Letting $j \to \infty$, we infer that $\mu(B_r) \leqslant c\,r^s$.

So we have constructed a measure μ supported on E such that $\mu(E) = \mathcal{H}^s_\infty(E)$ with $\mu(B_r(x)) \leqslant c\,r^s$ for all $x \in \mathbb{R}^d$ and all $r > 0$, which implies

$$\mathcal{H}^s_\infty(E) \lesssim \sup\big\{\mu(E) : \mathrm{supp}\mu \subset E,\, \mu(B_r(x)) \leqslant r^s \;\forall x \in \mathbb{R}^d, r > 0\big\}.$$

$\square$

5 Harmonic measure via Perron's method

To solve the Dirichlet problem for a very general class of open sets, it is convenient to use harmonic measure. Before introducing this notion, we will introduce subharmonic functions and we will show the solution of the Dirichlet problem via Perron's method.

5.1 Subharmonic functions

Definition 5.1. For $\Omega \subset \mathbb{R}^d$ open, we say that $u : \Omega \to [-\infty, \infty)$ is *subharmonic* if it is upper semicontinuous in Ω and

$$u(x) \leqslant \fint_{B_r(x)} u \tag{5.1}$$

whenever $B_r(x) \subset\subset \Omega$.

On the other hand, $u : \Omega \to (-\infty, +\infty]$ is *superharmonic* if it is lower semicontinuous and $u(x) \geqslant \fint_{B_r(x)} u$ whenever $B_r(x) \subset\subset \Omega$.

Recall that u is called upper semicontinuous at $x \in \Omega$ if $\limsup_{y \to x} u(y) \leqslant u(x)$, and it is lower semicontinuous if $\liminf_{y \to x} u(y) \geqslant u(x)$. It is easily checked that, if K is compact and $u : K \to [-\infty, \infty)$ is upper semicontinuous, then u attains the maximum on K. Analogously, if $u : K \to (-\infty, \infty]$ is lower semicontinuous, then u attains the minimum on K. Note that upper semicontinuity does not imply local Lebesgue integrability. However, the function is locally bounded above and therefore, the average $\fint_{B_r(x)} u$ in the previous definition is in $[-\infty, +\infty)$.

Of course, any function that is harmonic in Ω is both subharmonic and superharmonic. Further, u is subharmonic if and only if $-u$ is superharmonic. Other immediate properties are stated below.

Lemma 5.2. *If u, v are subharmonic in Ω, then $u+v$ and $\max(u, v)$ are both subharmonic in Ω. On the other hand, if u, v are superharmonic in Ω, then $u + v$ and $\min(u, v)$ are both superharmonic in Ω.*

Proof. This is immediate. □

Subharmonicity condition can be checked in spheres instead of balls:

Lemma 5.3. *If u is upper semicontinuous in Ω and $u(x) \leqslant \fint_{\partial B_r(x)} u$ whenever $B_r(x) \subset\subset \Omega$, then u is subsharmonic.*

Proof. We can assume that $-\infty < \fint_{B_r} u\, dm$ (otherwise there is nothing to prove). By upper semicontinuity, we have a bound above and therefore u is in $L^1(B)$ and we can apply Fubini's theorem to recover the solid means as

$$\fint_{B_r} u\, dm = \frac{d}{\kappa_d r^d} \int_0^r \int_{\partial B_t} u\, d\sigma\, dt = \frac{d}{r^d} \int_0^r \fint_{\partial B_t} u\, d\sigma\, t^{d-1} dt \geqslant \frac{du(x)}{r^d} \int_0^r t^{d-1}\, dt = u(x).$$

□

Subharmonic functions satisfy the maximum principle (and superharmonic functions satisfy the minimum principle):

Lemma 5.4 (Maximum principle)**.** *If u is a subharmonic function in a bounded open set Ω such that*

$$\limsup_{x\to\xi} u(x) \leqslant 0 \qquad \text{for every } \xi \in \partial\Omega,$$

then $u \leqslant 0$ in Ω. If moreover Ω is connected, then either $u \equiv 0$ or $u < 0$ in Ω.

Proof. By considering each component of Ω separately, we can assume that Ω is connected and it is enough to prove the second statement of the lemma. Suppose first that u does not achieve a supremum in Ω. If $x_j \in \Omega$ is such that $\lim_j u(x_j) = \sup_\Omega u$, then $\lim_j \operatorname{dist}(x_j, \partial\Omega) = 0$, for otherwise we could extract a subsequence converging to a point inside Ω and obtain a contradiction. Using that Ω is bounded, by passing to a subsequence we may assume that $x_j \to \xi \in \partial\Omega$. By assumption, this implies that every $x \in \Omega$ satisfies

$$u(x) < \sup_\Omega u = \lim_j u(x_j) \leqslant \limsup_{y\to\xi} u(y) \leqslant 0.$$

If u achieves the supremum at some $x \in \Omega$, then there exists r such that $B_r(x) \subset \Omega$. Assume that there exists $y \in B_r(x)$ such that $u(y) < u(x) = \sup_\Omega u$. Then, by upper semicontinuity we would get

$$\sup_\Omega u = u(x) \leqslant \fint_{B_r(x)} u < \sup_\Omega u,$$

reaching a contradiction. Therefore, the function is constant in the ball $B_r(x)$. This implies that the set where the supremum is achieved is open. But it is also relatively closed in Ω by semicontinuity and so u is constant in Ω. □

Next we give a couple of characterizations of subharmonicity under certain a priori regularity conditions. First, we check the behavior of the Laplacian when a subharmonic function has two derivatives, and then we use it to show that the fundamental solution to $-\Delta$, see (2.10), is an example of superharmonic function.

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in C^2(\Omega)$. The function u is subharmonic in Ω if and only if $\Delta u \geqslant 0$ in Ω.*

Proof. The fact that $\Delta u \geqslant 0$ in Ω implies the subharmonicity of u is a direct consequence of Remark 2.2. To prove the converse implication, we have to show that $\Delta u(x) \geqslant 0$ for every $x \in \Omega$. To this end, consider the function

$$v(y) = u(y) - u(x) - \nabla u(x)\,(y-x).$$

Since u is subharmonic and any affine function is harmonic, it follows that v is also subharmonic. The Taylor expansion of v in x equals

$$v(y) = \frac{1}{2}\,(y-x)^T\,D^2u(x)\,(y-x) + o(|y-x|^2),$$

where $D^2u(x)$ is the Hessian matrix of u. For any ball $\overline{B_r(x)} \subset \Omega$, we have

$$\begin{aligned} 0 = v(x) &\leqslant \fint_{B_r(x)} v\,dy = \frac{1}{2}\fint_{B_r(x)} (y-x)^T\,D^2u(x)\,(y-x)\,dy + o(r^2) \\ &= \frac{1}{2}\sum_{i,j}\partial_{i,j}u(x)\fint_{B_r(x)} (y_i-x_i)\,(y_j-x_j)\,dy + o(r^2) \\ &= c\,\Delta u(x)\,r^2 + o(r^2), \end{aligned}$$

where we took into account that $\int_{B_r(x)} (y_i-x_i)\,(y_j-x_j)\,dy$ vanishes if $i \neq j$ and is positive otherwise. Dividing by cr^2, we deduce

$$\Delta u(x) + o(1) \geqslant 0,$$

with $o(1) \to 0$ as $r \to 0$. This implies that $\Delta u(x) \geqslant 0$, and the proof of the lemma is concluded. □

Lemma 5.6. *The fundamental solution of $-\Delta$ is harmonic in $\mathbb{R}^d\backslash\{0\}$ and superharmonic in $\mathbb{R}^d$.*

Proof. Harmonicity can be easily checked. To prove superharmonicity, notice first that $\mathcal{E}$ is lower semicontinuous. Next, for every $\varepsilon > 0$ let φ_ε be a C^∞, positive, radially decreasing, function supported on $B_\varepsilon(0)$ with $\int \varphi_\varepsilon = 1$. Then $\mathcal{E} * \varphi_\varepsilon \in C^\infty(\mathbb{R}^d)$. Further,

$$\Delta(\mathcal{E} * \varphi_\varepsilon) = -\varphi_\varepsilon \leqslant 0.$$

Thus, by Lemma 5.5, $\mathcal{E} * \varphi_\varepsilon$ is superharmonic in $\mathbb{R}^d$. Consequently, for any ball B centered in $x_0 \neq 0$ and any $\varepsilon > 0$,

$$\fint_B \mathcal{E} * \varphi_\varepsilon \leqslant \mathcal{E} * \varphi_\varepsilon(x_0).$$

Letting $\varepsilon \to 0$, we deduce

$$\fint_B \mathcal{E} \leqslant \mathcal{E}(x_0).$$

In case $x_0 = 0$, we have $\mathcal{E}(x_0) = +\infty$ and the last inequality is satisfied trivially. □

Now we turn our attention to continuous subharmonic functions. Although the maximum of two subharmonic is subharmonic in their common domain of definition, in some occasions we want to extend the domain. Here we check a particularly easy case which consists in extending a subharmonic function out of its domain of definition as a constant function.

Lemma 5.7. *Let Ω be an open set, let u be a subharmonic, continuous function in Ω, let V be a connected component of $\overline{\Omega}^c$ and let $t \in \mathbb{R}$ such that $t \geqslant \sup_{\xi \in \partial V \cap \partial\Omega} \limsup_{x \to \xi} u(x)$. Then*

$$\widetilde{u} = \begin{cases} \max(u,t) & \text{in } \Omega, \\ t & \text{in } \overline{V}. \end{cases}$$

is subharmonic and continuous in $\Omega \cup \overline{V}$.

Proof. Without loss of generality, we may assume $t = 0$. Continuity is left as an exercise for the reader. To establish subharmonicity in $U = \Omega \cup \overline{V}$, we will check that $\widetilde{u}(x) \leqslant \fint_{\partial B} \widetilde{u}$ whenever $B = B_r(x) \subset\subset U$. This is rather trivial when $\widetilde{u}(x) = 0$ because $\widetilde{u}$ is non-negative. Thus, we may assume that $\widetilde{u}(x) = u(x) > 0$ and, in particular, $x \in \Omega$. Let v be the solution to the Dirichlet problem in B with boundary values $f(y) := \widetilde{u}(y)$ for $y \in \partial B$, given in Theorem 3.11.

Since v is harmonic in B, continuous in $\overline{B}$ and it has non-negative boundary values, by the maximum principle we get that v is non-negative in $A = B \cap \Omega$, and moreover it is continuous up to the boundary in A. In particular, $u - v$ is subharmonic in A.

Note that $\partial A = (\partial B \cap \Omega) \cup (\partial\Omega \cap \overline{B})$. Consider now $y \in \partial A$. If $y \in \partial B \cap \Omega$, then $\widetilde{u}(y) - v(y) = \widetilde{u}(y) - f(y) = 0$ by definition. Otherwise, $y \in \partial\Omega \cap \overline{B}$ and by assumption $\limsup_{z \to y}(\widetilde{u}(z) - v(z)) \leqslant 0 - v(y) \leqslant 0$. All in all, by the maximum principle we obtain $\widetilde{u} \leqslant v$ in B, implying in particular that

$$\widetilde{u}(x) \leqslant v(x) = \fint_{\partial B} v = \fint_{\partial B} \widetilde{u},$$

and subharmonicity follows by Lemma 5.3. □

Next we characterize continuous subharmonic functions as those functions whose interior values in balls lie below the solution to the Dirichlet problem with the same boundary values.

Lemma 5.8. *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in C(\Omega)$. Then u is subharmonic if and only if for every ball $B \subset\subset \Omega$ and every harmonic function v such that $u(x) \leqslant v(x)$ for every $x \in \partial B$, it holds either $v > u$ or $v \equiv u$ in B.*

Proof. The only if implication follows by the maximum principle to the subharmonic function $u - v$. To see the converse, let $B_r(x) \subset\subset \Omega$ and let v be the harmonic function in B_r continuous up to the boundary that agrees with u on ∂B_r (see Theorem 3.11). Then

$$\fint_{\partial B_r} u \, d\sigma = \fint_{\partial B_r} v \, d\sigma = v(x) \geqslant u(x).$$

The proof is completed by Lemma 5.3. □

Let $u \in C(\Omega)$ be subharmonic in a ball B. Let $\tilde{u}$ be the harmonic function in B that agrees with u on ∂B and set $U := \chi_{\Omega\backslash B} u + \chi_B \tilde{u}$. Note that $U \geqslant u$ by Lemma 5.8. This is called the *harmonic lift* of u in B.

Lemma 5.9. *Let $\Omega \subset \mathbb{R}^d$ be open. If $u \in C(\Omega)$ is subharmonic in Ω, $x \in \Omega$ and $B = B_r(x) \subset\subset \Omega$, then the harmonic lift of u in B is also subharmonic in Ω.*

Proof. Let U be the harmonic lift of u in B. Consider v harmonic in a ball $B' \subset \Omega$ with $B' \cap B \neq \varnothing$ and $v \geqslant U$ in the boundary of B'. We want to prove that either $v > U$ or $v \equiv U$ in B'.

Case 1: $\partial B \cap B' = \varnothing$, that is $B' \subset B$ and U is harmonic in B'. Then the claim follows by Lemma 5.8 applied to U.

Case 2: $\partial B \cap B' \neq \varnothing$ and $v(y) > U(y)$ in $\partial B \cap B'$. Using the continuity of U and the maximum principle applied to $U - v$ in $B' \backslash B$ and $B' \cap B$ separately, we get that $v > U$ in B'.

Case 3: $\partial B \cap B' \neq \varnothing$ and there exists $y \in \partial B \cap B'$ such that $v(y) \leqslant U(y) = u(y)$. In this case, since $v \geqslant u$ in $\partial B'$, Lemma 5.8 implies that $v \equiv u$ in B'. If $\partial B' \cap B \neq \varnothing$, the identity $v \equiv u$ in B' implies the existence of a point in $\partial B' \cap B \neq \varnothing$ where $u(y) \leqslant U(y) \leqslant v(y) = u(y)$ and therefore $U \equiv u$ by Lemma 5.8. If, instead, $\partial B' \cap B = \varnothing$, that is if $B \subset B'$, then u is harmonic in B and, therefore, $U \equiv u$ as well and the claim follows. $\square$

Next we provide a couple of properties of subharmonic functions, again under certain a priori conditions. First we see that subharmonicity is preserved by an approximation of the identity. Then we use this fact to show that subharmonic Sobolev functions are weakly subharmonic, see Remark 5.14 below. This properties will be used to show the Caccioppoli inequality for subharmonic functions.

Lemma 5.10. *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in L^1_{\text{loc}}(\Omega)$ be subharmonic. For $\rho > 0$, denote $\Omega_\rho = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > \rho\}$. Then following holds:*

(a) If μ is a (non-negative) Radon measure supported in $B_\rho(0)$ and $u * \mu$ is upper semi-continuous in Ω_ρ, then $u * \mu$ is subharmonic in Ω_ρ.

(b) If φ is a continuous non-negative function supported in $B_\rho(0)$, then $u * \varphi$ is subharmonic in Ω_ρ.

Proof. Clearly, the statement (b) is a consequence of (a), since $u * \varphi$ is continuous because φ is continuous and compactly supported. To prove (a), we have to check that for any $x \in \Omega_\rho$ and $r > 0$ such that $B_r(x) \subset \Omega_\rho$, we have $u * \mu(x) \leqslant \fint_{B_r(x)} u * \mu \, dm$. Without loss of generality, assume that $x = 0$ and $B_r(0) \subset \Omega_\rho$. Denoting $\widetilde{u}(y) = u(-y)$ and $\widehat{\chi}_{B_r(0)} = m(B_r(0))^{-1} \chi_{B_r(0)}$, we have

$$\fint_{B_r(0)} u * \mu \, dm = \left\langle u * \mu, \widehat{\chi}_{B_r(0)} \right\rangle = \left\langle \mu, \widetilde{u} * \widehat{\chi}_{B_r(0)} \right\rangle.$$

Notice now that for any $y \in \operatorname{supp}\mu$, $B_r(y) \subset B_{r+\rho}(0) \subset \Omega$ (because $\operatorname{supp}\mu \subset B_\rho(0)$ and $B_r(0) \subset \Omega_\rho$) and so

$$\widetilde{u} * \widehat{\chi}_{B_r(0)}(y) = \fint_{B_r(y)} \widetilde{u} \, dm \geqslant \widetilde{u}(y).$$

Consequently,

$$\fint_{B_r(0)} u * \mu \, dm \geqslant \langle \mu, \tilde{u} \rangle = u * \mu(0).$$

□

Lemma 5.11 (Locality of subharmonicity)**.** *Let $u \in L^1_{loc}(\Omega)$ be an upper semicontinuous function in Ω satisfying (5.1) whenever $\bar{B}_r(x) \subset \Omega$, with $r < \rho$, then u is subharmonic in Ω.*

From here, it is possible to show as well that if $u \in L^1_{loc}(U \cup V)$ is subharmonic on two open sets U and V, then it is also subharmonic in the union $U \cup V$.

Proof of Lemma 5.11. First we will show that, if $\varphi \in C^1$ is a non-negative, non-increasing radial function supported in $B_1(0)$ with $\int \varphi = 1$, then

$$u * \varphi_\varepsilon(x) \xrightarrow{\varepsilon \to 0} u(x) \text{ for every } x \in \Omega. \tag{5.2}$$

Let $x \in \Omega$ and $\varepsilon < \rho$. Abusing notation we write $\varphi(|x|) := \varphi(x)$. Using that $\varphi(|x|) = -\int_{|x|}^\infty \varphi'(t)\, dt$, we get

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi(\varepsilon^{-1}|x|) = -\frac{1}{\varepsilon^d} \int_0^\infty \chi_{[0,t]}(\varepsilon^{-1}|x|) \varphi'(t)\, dt,$$

Then, by Fubini's Theorem,

$$u * \varphi_\varepsilon(x) = \int_{\mathbb{R}^d} u(y) \varphi_\varepsilon(|x-y|)\, dm(y) = -\frac{1}{\varepsilon^d} \int_0^\infty \varphi'(t) m(B_{\varepsilon t}(x)) \fint_{B_{\varepsilon t}(x)} u(y)\, dm(y)\, dt.$$

Since $\varphi' \leqslant 0$, by (5.1), we obtain

$$u * \varphi_\varepsilon(x) \geqslant -u(x) \int_0^\infty \varphi'(t) m(B_t(0))\, dt.$$

Note that

$$-\int_0^\infty \varphi'(t) m(B_t(0))\, dt = -\int_{\mathbb{R}^d} \int_{|y|}^\infty \varphi'(t)\, dt\, dm(y) = \|\varphi\|_{L^1},$$

so that we get

$$u * \varphi_\varepsilon(x) \geqslant u(x). \tag{5.3}$$

Next we show (5.2) arguing by contradiction. Assume that $u * \varphi_\varepsilon(x)$ does not converge to $u(x)$, i.e., there exists a $\delta > 0$ and a sequence $\varepsilon_n \to 0$ such that

$$|u * \varphi_{\varepsilon_n}(x) - u(x)| \geqslant \delta.$$

By (5.3) we have $u * \varphi_{\varepsilon_n}(x) - u(x) \geqslant 0$, so we necessarily have

$$u * \varphi_{\varepsilon_n}(x) - u(x) \geqslant \delta.$$

Since φ_{ε_n} has integral one, is non-negative, and is supported on the ball $B_{\varepsilon_n}(0)$, there exists a set with positive measure in $B_{\varepsilon_n}(x)$ where $u(y) \geqslant u(x) + \delta$. In particular, we can fix $y_n \in B_{\varepsilon_n}(x) \backslash \{x\}$ with

$$u(y_n) \geqslant u(x) + \delta.$$

Since $y_n \to x$, we get

$$\limsup_{y \to x} u(y) \geqslant \limsup_{n \to \infty} u(y_n) \geqslant u(x) + \delta,$$

contradicting upper semicontinuity, so (5.2) has been established.

To prove the lemma, note that it is enough to show that u satisfies (5.1) in every ball B compactly contained in Ω. Consider a function $\varphi \in C^2(B_1(0))$ as above. By Lemma 5.10, the function $u * \varphi_\varepsilon$ is subharmonic in balls of radius $r < \rho - \varepsilon$ contained in Ω_ε, which using Lemma 5.5 implies that $\Delta(u * \varphi_\varepsilon) \geqslant 0$ in Ω_ε. Using Lemma 5.5 again, we derive the subharmonicity of $u * \varphi_\varepsilon$ in a neighborhood of B for ε small enough. In particular

$$u * \varphi_\varepsilon(x) \leqslant \fint_B u * \varphi_\varepsilon \, dm. \tag{5.4}$$

Now, since $u \in L^1(B)$, by standard properties about approximations of the identity (see [Gra08, Theorem 1.2.19], for instance), we infer that

$$\int_B u * \varphi_\varepsilon \, dm \to \int_B u \, dm$$

as $\varepsilon \to 0$. Recalling also that $u * \varphi_\varepsilon(x) \to u(x)$, passing to the limit in both sides of (5.4) we recover (5.1) for the ball B, as wanted. □

Remark 5.12. Note that we have shown than given a subharmonic function $u \in L^1_{\text{loc}}(\Omega)$, if $\varphi \in C^1$ is a non-negative, non-increasing radial function supported in $B_1(0)$ with $\int \varphi = 1$, then

$$u * \varphi_\varepsilon(x) \xrightarrow{\varepsilon \to 0} u(x) \text{ for every } x \in \Omega.$$

Lemma 5.13. *Let $\Omega \subset \mathbb{R}^d$ be open, let $u \in L^1_{\text{loc}}(\Omega)$ be subharmonic in Ω, and $\varphi \in C_c^\infty(\Omega)$, with $\varphi \geqslant 0$. Then, its distributional gradient satisfies*

$$\langle \nabla u, \nabla \varphi \rangle \leqslant 0.$$

Consequently, if $u \in W^{1,p}_{\text{loc}}(\Omega)$ with $1 < p \leqslant \infty$ and $\varphi \in W^{1,p'}_c(\Omega)$ with $\varphi \geqslant 0$, we have

$$\int \nabla u \cdot \nabla \varphi \leqslant 0. \tag{5.5}$$

Proof. For every $\varepsilon > 0$, let ψ_ε be a C^∞, positive, radially decreasing, function supported on $B_\varepsilon(0)$ with $\int \psi_\varepsilon = 1$. Let $\Omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \Omega^c) > \varepsilon\}$ and take ε small enough such that $\mathrm{supp}\varphi \subset \Omega_\varepsilon$. Then we have

$$\langle \nabla u, \nabla \varphi \rangle = -\int u \,\Delta\varphi \, dx = -\lim_{\varepsilon\to 0} \int (u * \psi_\varepsilon)\, \Delta\varphi \, dx = -\lim_{\varepsilon\to 0} \int \Delta(u * \psi_\varepsilon)\, \varphi \, dx.$$

Since $u * \psi_\varepsilon$ is C^∞ and subharmonic in Ω_ε, it follows that $\Delta(u * \psi_\varepsilon) \geqslant 0$ in Ω_ε, see Lemmas 5.5 and 5.10. Thus,

$$\int \Delta(u * \psi_\varepsilon)\, \varphi \, dx \geqslant 0$$

for any $\varepsilon > 0$ small enough, and so $\langle \nabla u, \nabla \varphi \rangle \leqslant 0$.

The second statement in the lemma follows easily by a density argument. $\square$

Remark 5.14. A function $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ satisfying (5.5) is called weakly subharmonic. Note that we don't ask for semicontinuity in this definition. What we call weakly subharmonic is sometimes called a *subsolution* to $\Delta u = 0$, see [Ken94, Section 1.1], for instance.

Lemma 5.15 (Caccioppoli Inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be weakly subharmonic in Ω and non-negative. Then for every ball $B \subset \Omega$ of radius r we have*

$$\int_B |\nabla u|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2,$$

where $t = \mathrm{dist}(B, \partial\Omega)$

Proof. The arguments are very similar to the ones in Lemma 2.10. Let η be a Lipschitz function such that $\chi_B \leqslant \eta \leqslant \chi_{(t+1)B}$ and with $|\nabla \eta| \leqslant \frac{1}{rt}$. Since u is weakly subharmonic, η is compactly supported, and $u\eta^2 \geqslant 0$, by Leibniz' rule we have

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 = \int_{(t+1)B} \nabla u \cdot \nabla (u\eta^2) - \int_{(t+1)B} 2u\eta \nabla u \cdot \nabla \eta \leqslant -\int_{(t+1)B} 2u\eta \nabla u \cdot \nabla \eta.$$

By Hölder's inequality we get

$$\int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \left(\int_{(t+1)B} 4u^2 |\nabla \eta|^2 \right)^{\frac{1}{2}} \left(\int_{(t+1)B} \eta^2 |\nabla u|^2 \right)^{\frac{1}{2}},$$

and so

$$\int_B |\nabla u|^2 \leqslant \int_{(t+1)B} \eta^2 |\nabla u|^2 \leqslant \int_{(t+1)B} 4u^2 |\nabla \eta|^2 \leqslant \frac{4}{(rt)^2} \int_{(t+1)B \setminus B} u^2.$$

$\square$

5.2 Perron classes and resolutive functions

Throughout this section we assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set (not necessarily connected).

For $f \in C(\partial\Omega)$, the Perron method, that we will describe below, associates a harmonic function $u_f : \Omega \to \mathbb{R}$ to f. Even if f is continuous, the function u_f may not extend continuously to the boundary to coincide with f, see Example 3.8. However, We will see that if Ω is regular enough in some sense, then u_f extends continuously to $\partial\Omega$ and its boundary values coincide with f.

Definition 5.16. Given a bounded function $f : \partial\Omega \to \mathbb{R}$, define the lower Perron class as

$$\mathcal{L}_f = \left\{u \in C(\Omega) : \text{ is subharmonic and } \limsup_{x\to\xi} u(x) \leqslant f(\xi) \text{ for all } \xi \in \partial\Omega\right\},$$

and the upper Perron class as

$$\mathcal{U}_f = \left\{u \in C(\Omega) : u \text{ is superharmonic and } \liminf_{x\to\xi} u(x) \geqslant f(\xi) \text{ for all } \xi \in \partial\Omega\right\}.$$

Note that the constant function $x \mapsto \sup_{\partial\Omega} f$ is an element of $\mathcal{U}_f$ (and $x \mapsto \inf_{\partial\Omega} f$ is an element of $\mathcal{L}_f$). Therefore, $\mathcal{U}_f$ and $\mathcal{L}_f$ are non-empty and we can define the real-valued functions

$$\underline{H}_f(x) = \sup_{u\in\mathcal{L}_f} u(x), \qquad \overline{H}_f(x) = \inf_{u\in\mathcal{U}_f} u(x)$$

for $x \in \Omega$, which we call *lower Perron solution* and *upper Perron solution* respectively.

Remark 5.17. If $f \in C(\overline{\Omega})$ is harmonic in Ω, for every $u \in \mathcal{L}_f$ we can apply the maximum principle (see Lemma 5.4) to $u - f$ to infer that $u \leqslant f$ in Ω. In particular, we deduce that $f = \underline{H}_f = \overline{H}_f$. So if the solution of the Dirichlet problem with continuous boundary data exists, then it coincides with the lower and upper Perron solutions.

Lemma 5.18. *For every bounded function $f : \partial\Omega \to \mathbb{R}$, the functions $\underline{H}_f$ and $\overline{H}_f$ are harmonic.*

Proof. We will show only the case $\underline{H}_f$. The other follows by noting that $\overline{H}_f = -\underline{H}_{-f}$.

Fix $x \in \Omega$ and $B = B_r(x)$ with $\overline{B} \subset \Omega$. Let $\{u_j\}_{j=1}^\infty \subset \mathcal{L}_f$ be a sequence of subharmonic functions so that $u_j(x) \xrightarrow{j\to\infty} \underline{H}_f(x)$. By replacing u_j by $\max(u_j, \inf_{\partial\Omega} f)$ if necessary (see Lemma 5.2), we may assume that the sequence of functions u_j is uniformly bounded from below.

Let U_j be the harmonic lift of u_j in B, which is subharmonic by Lemma 5.9 and therefore $U_j \leqslant \underline{H}_f$. This sequence is uniformly bounded above by $\sup_{\partial\Omega} f$ by the maximum principle and it is also bounded below since the u_j's are uniformly bounded from below. Thus, passing to a subsequence if necessary, we may assume that U_j converges pointwise in B to a harmonic function U (see Lemma 2.14). As we have seen, $u_j \leqslant U_j \leqslant \underline{H}_f$ and, therefore, $U(x) = \underline{H}_f(x)$.

We claim that $U \equiv \underline{H}_f$ in B. Assume not. Then there is $y \in B$ so that $U(y) < \underline{H}_f(y)$, and by definition of $\underline{H}_f$, there must be $v \in \mathcal{L}_f$ so that $U(y) < v(y) \leqslant \underline{H}_f(y)$. Set

$v_j = \max\{U_j, v\}$ (which is again subharmonic by Lemma 5.2) and let V_j be the harmonic lift of v_j in B, so now V_j is harmonic in B. Passing to a subsequence, we may assume V_j converges pointwise to a harmonic function V in B. Since $U_j \leqslant V_j$, we have that $U \leqslant V \leqslant \underline{H}_f$ in B, and so $U(x) = V(x) = \underline{H}_f(x)$, which implies $U = V$ in B by the maximum principle. However, $U(y) < v(y) \leqslant V_j(y)$ which implies $U(y) < V(y)$, a contradiction.

□

Lemma 5.19. *Every bounded function $f : \partial\Omega \to \mathbb{R}$ satisfies $\underline{H}_f \leqslant \overline{H}_f$.*

Proof. Let $u \in \mathcal{U}_f$ and $v \in \mathcal{L}_f$. Then $v - u$ is subharmonic with $\limsup_{x\to\xi}(v - u) \leqslant f(\xi) - f(\xi) = 0$ for all $\xi \in \partial\Omega$, and so by the maximum principle, $v \leqslant u$. Taking infimum and supremum over $\mathcal{U}_f$ and $\mathcal{L}_f$ respectively, we get $\underline{H}_f \leqslant \overline{H}_f$. □

Definition 5.20. We say that a bounded function $f : \partial\Omega \to \mathbb{R}$ is resolutive if $\underline{H}_f = \overline{H}_f$.

Lemma 5.21. *If f, g are resolutive so are $-f$, $f + g$, and λf for any $\lambda \in \mathbb{R}$. Further,*

$$H_{f+g} = H_f + H_g \qquad \text{and} \qquad H_{\lambda f} = \lambda H_f.$$

Proof. Note that if $u \in \mathcal{U}_f$ and $v \in \mathcal{U}_g$, then $u+v \in \mathcal{U}_{f+g}$, and so $\overline{H}_{f+g} \leqslant u+v$. Therefore, $\overline{H}_{f+g} \leqslant \overline{H}_f + \overline{H}_g$. Similarly, $\underline{H}_{f+g} \geqslant \underline{H}_f + \underline{H}_g = \overline{H}_f + \overline{H}_g$. Therefore $\overline{H}_{f+g} \leqslant \underline{H}_{f+g}$ and the converse inequality follows from Lemma 5.19.

Also being f resolutive implies that $\underline{H}_{-f} = -\overline{H}_f = -\underline{H}_f = \overline{H}_{-f}$. For $\lambda \geqslant 0$, $\underline{H}_{\lambda f} = \lambda \underline{H}_f$ and $\overline{H}_{\lambda f} = \lambda \overline{H}_f$ and thus $H_{\lambda f} = \lambda H_f$. For $\lambda < 0$, we write $H_{\lambda f} = H_{(-\lambda)(-f)} = -\lambda H_{(-f)} = \lambda H_f$. □

Lemma 5.22. *If $f \in C(\overline{\Omega})$ is subharmonic in Ω, then $f|_{\partial\Omega}$ is resolutive.*

Proof. Since f is subharmonic and continuous up to the boundary, we have $f \in \mathcal{L}_f$, and so $f \leqslant \underline{H}_f$. Note that $\underline{H}_f$ is harmonic (hence superharmonic) and $\liminf_{x\to\xi} \underline{H}_f(x) \geqslant \liminf_{x\to\xi} f(x) = f(\xi)$, so $\underline{H}_f \in \mathcal{U}_f$, hence $\underline{H}_f \geqslant \overline{H}_f$. □

Lemma 5.23. *Polynomials are resolutive in every bounded open set.*

Proof. Let u be a polynomial. Note that the function $v(x) = |x|^2$ satisfies $\Delta v = 2d > 0$. In particular v is subharmonic in $\mathbb{R}^d$ by Lemma 5.5. Since Δu is a polynomial, it is bounded in any bounded open set Ω. Thus, for $k > 0$ large enough, $\Delta(u + kv) > 0$ in Ω. So both v and $u + kv$ are subharmonic in Ω and continuous in $\overline{\Omega}$. Hence they are resolutive, and therefore $u = (u + kv) - kv$ is resolutive too. □

Theorem 5.24 (Wiener). *$C(\partial\Omega)$ functions are resolutive.*

Proof. Let $f \in C(\partial\Omega)$ and $\varepsilon > 0$. By the Stone-Weierstrass theorem [Sto48], we may find a polynomial u such that $|f - u| < \varepsilon$ on $\partial\Omega$. Thus,

$$\overline{H}_f \leqslant \overline{H}_{u+\varepsilon} = \overline{H}_u + \varepsilon = \underline{H}_u + \varepsilon \leqslant \underline{H}_f + 2\varepsilon,$$

and letting $\varepsilon \to 0$ gives that f is resolutive. □

In this way, we can associate to a continuous function f a harmonic function $H_f := \underline{H}_f = \overline{H}_f$. The fact that f is resolutive is not the reason we can pick an association. For example, we could just associate to any bounded function f on the boundary the harmonic function $\overline{H}_f$. The property of being resolutive is significant for us not because it helps us to define a harmonic function for f, but because the fact that $\underline{H}_f$ and $\overline{H}_f$ agree (for resolutive functions) will be useful in maximum principle arguments when trying to prove continuity at the boundary. Further, as shown above, the set of resolutive functions is a vector space and the map $f \mapsto H_f$ is linear in this vector space, as shown in Lemma 5.21.

As mentioned earlier, H_f may not coincide with f at the boundary, even if f is continuous. To give an example, consider $\Omega = B_1(0)\backslash\{0\} \subset \mathbb{R}^d$, and let $f(\xi) = 0$ for $\xi \in \partial B_1(0)$, $f(0) = 1$. Define

$$u_\varepsilon(x) := \frac{\varepsilon}{|x|^{d-2}}$$

for $d \geqslant 3$ (for $d = 2$ use the logarithm). Since $u_\varepsilon > 0$ is harmonic and goes to $+\infty$ at the origin, we immediately get $u_\varepsilon \in \mathcal{U}_f$, so

$$H_f(x) \leqslant \frac{\varepsilon}{|x|^{d-2}} \xrightarrow{\varepsilon \to 0} 0.$$

Since $0 \in \mathcal{L}_f$ trivially, we get that $H_f(x) \geqslant 0$ and Lemma 5.19 implies that $H_f(x) = 0$. That is, H_f is the same for $\Omega = B_1(0)$ and for $\Omega = B_1(0)\backslash\{0\}$.

5.3 Harmonic measure via Perron's method

Throughout this section we assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set, unless otherwise stated. Next we provide the definition of harmonic measure via the so-called *Perron's method*.

Definition 5.25. Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $x \in \Omega$. The harmonic measure for Ω based at x (or with pole in x) is the unique Radon measure ω^x on $\partial\Omega$ such that

$$H_f(x) = \int_{\partial\Omega} f(\xi)d\omega^x(\xi) \qquad \text{for all } f \in C(\partial\Omega).$$

The existence and uniqueness of ω^x is ensured by the Riesz representation theorem, i.e. Theorem 4.58, and the linearity of the map $f \mapsto H_f$, implied by Theorem 5.24 and Lemma 5.21. Abusing notation we extend ω^x by 0 to the whole $\mathbb{R}^d$, that is $\omega^x(\mathbb{R}^d\backslash\partial\Omega) := 0$.

Remark 5.26. Note that $1 \in \mathcal{L}_1 \cap \mathcal{U}_1$, so $H_1(x) = 1$ regardless of any consideration on the geometry of Ω by Lemma 5.19. Therefore

$$\omega^x(\partial\Omega) = \int 1 d\omega^x = H_1(x) = 1.$$

So ω^x is a probability measure.

Example 5.27. Consider the case of the unit ball B_1. We showed in Theorem 3.11 that the Dirichlet problem is solvable in B_1 and that, for any $f \in C(\partial B_1)$, its harmonic extension equals

$$u_f(x) = \int_{\partial B_1} P^x(\zeta) f(\zeta)\, d\sigma(\zeta) \qquad \text{for } x \in B_1,$$

where $P^x(\xi)$ is the Poisson kernel:

$$P^x(\xi) = \frac{1 - |x|^2}{\kappa_d\, |x - \xi|^d}.$$

Since $u_f = H_f$ for all $f \in C(\partial B_1)$, by the uniqueness of ω^x it follows that

$$d\omega^x(\xi) = P^x(\xi)\, d\sigma(\xi).$$

In the case $x = 0$, we have

$$d\omega^0(\xi) = \frac{1}{\kappa_d}\, d\sigma(\xi).$$

That is, ω^0 is the normalized surface measure on the unit sphere.

In many geometric and qualitative analytic properties of harmonic measure, the choice of the pole plays no role. This is due to the fact that harmonic measures with different poles are mutually absolutely continuous in (connected) domains. To prove this fact, we start by checking the harmonicity with respect to the pole of the harmonic measure of a given compact set.

Lemma 5.28. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let ω^x be the harmonic measure for Ω. Let $K \subset \partial\Omega$ be compact. Then the function $u(x) := \omega^x(K)$ is harmonic in Ω.*

Proof. For each $n \geqslant 1$, let U_n be the $(1/n)$-neighborhood of K, i.e. $U_n = \{x : \operatorname{dist}(x, K) < 1/n\}$. Consider a sequence of functions $f_n \in C(\partial\Omega)$ such that $\chi_K \leqslant f_n \leqslant \chi_{U_n \cap \partial\Omega}$, so that $f_n \to \chi_k$ pointwise in $\partial\Omega$.

By dominated convergence theorem, it follows that, for any fixed $x \in \Omega$,

$$u(x) = \omega^x(K) = \lim_{n\to\infty} \int f_n\, d\omega^x \leqslant \omega^x(U_1) \leqslant 1.$$

Since $u_n(x) := \int f_n d\omega^x$, with $n \geqslant 1$, is a uniformly bounded sequence of harmonic functions, the limit is also harmonic (see Lemma 2.14). ∎

Lemma 5.29. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let ω^x be the harmonic measure for Ω. For all $x, y \in \Omega$, the measures ω^x and ω^y are mutually absolutely continuous.*

Proof. By the inner regularity of Radon measures, it suffices to show that $\omega^x(K) \approx \omega^y(K)$ for any compact set K, with the implicit constant depending only on Ω, x, y, but not on K. This is an immediate consequence of Lemma 2.17, as $u(x) := \omega^x(K)$ is a positive harmonic function in Ω, □

As a matter of fact, the harmonicity with respect to the pole is also satisfied when the set is Borel regular. The proof in this case is a bit more technical, since the approximating open sets given by Borel regularity in Definition 4.4 depend on the particular pole.

Remark 5.30. There may be sets which are not Borel, but which are measurable for certain ω^{x_0}, however mesurability for other poles should may not be obvious. Fortunately, measurability for ω^x is immediate from absolute continuity and Borel regularity. Indeed, if A is measurable for ω^{x_0}, then there exists a Borel set $B \supset A$ such that $\omega^{x_0}(A) = \omega^{x_0}(B)$, that is $\omega^{x_0}(B\backslash A) = 0$. Given another pole $x \in \Omega$ we get $\omega^x(B\backslash A) = 0$, which implies that $B\backslash A$ is measurable also for ω^x. The ω^x-measurability of $A = B \cap (B\backslash A)^c$ follows from this fact. Thus we can define ω-measurable set without specifying the particular pole.

Lemma 5.31. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, let ω^x be the harmonic measure for Ω, and let $A \subset \partial\Omega$ be a ω-measurable set. Then the function $u(x) := \omega^x(A)$ is harmonic in Ω.*

Proof. If A is compact, this has already been shown in Lemma 5.28. If A is open, then $\omega^x(A^c)$ is harmonic and we write $u(x) = \omega^x(A) = 1 - \omega^x(A^c)$. So u is harmonic in Ω.

Let $A \subset \Omega$ be now an arbitrary ω-measurable set A and fix $x \in \Omega$. By the regularity of ω^x, there exists a sequence of open sets $U_n \supset A$ such that $\omega^x(U_n\backslash A) \leqslant 1/n$. Moreover, we can take $U_{n+1} \subset U_n$ by redefining the sequence suitably. Then, letting $G = \bigcap_{n\geqslant 1} U_n$, we have $\omega^x(G\backslash A) = 0$. By the mutual absolute continuity of all the harmonic measures ω^y, with $y \in \Omega$, it follows that $\omega^y(G\backslash A) = 0$ for all $y \in \Omega$. Thus, since A is ω-measurable (and therefore, it is ω^y-measurable), we get

$$\omega^y(G) = \omega^y(G\backslash A) + \omega^y(G \cap A) = \omega^y(A) = u(y)$$

for all $y \in \Omega$.

Now it just remains to notice that $\omega^y(G)$ is a harmonic function, since it equals a pointwise limit of uniformly bounded harmonic functions, because Lemma 4.3 implies

$$\omega^y(G) = \lim_{n\to\infty} \omega^y(U_n).$$

□

The next result will be useful in other chapters when studying the properties of harmonic measure.

Lemma 5.32. *Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}^d$ be bounded open sets such that $\widetilde{\Omega} \subset \Omega$ and $\partial\Omega \cap \partial\widetilde{\Omega} \neq \varnothing$. Denote by ω_Ω and $\omega_{\widetilde{\Omega}}$ the respective harmonic measures for Ω and $\widetilde{\Omega}$. For any $x \in \widetilde{\Omega}$ and any Borel set $A \subset \partial\Omega \cap \partial\widetilde{\Omega}$, it holds*

$$\omega^x_{\widetilde{\Omega}}(A) \leqslant \omega^x_\Omega(A).$$

Proof. To simplify notation we write $\omega = \omega_\Omega$ and $\widetilde{\omega} = \omega_{\widetilde{\Omega}}$. By the regularity properties of harmonic measure, it suffices to prove that $\widetilde{\omega}^x(A) \leqslant \omega^x(A)$ for any compact subset $A \subset \partial\Omega \cap \partial\widetilde{\Omega}$. Consider an arbitrary function $\varphi \in C(\partial\Omega)$ such that $\varphi = 1$ on A. To illustrate the main idea of the proof, suppose first that Dirichlet problem is solvable in Ω for any continuous boundary data, so that the Perron solution $v = H_\varphi$ in Ω of the Dirichlet problem with boundary data φ extends continuously to $\partial\Omega$ and $v|_{\partial\Omega} = \varphi$. Then,

$$\widetilde{\omega}^x(A) \leqslant \int_{\partial\widetilde{\Omega}} v \, d\widetilde{\omega}^x = v(x) = \int_{\partial\Omega} \varphi \, d\omega^x.$$

Then taking the infimum over all the functions $\varphi \in C(\partial\Omega)$ as above, we deduce that $\widetilde{\omega}^x(A) \leqslant \omega^x(A)$.

In the general case, we need a more careful argument. For φ as above and any $\varepsilon > 0$, let $u \in \mathcal{U}^\Omega_\varphi$ (the upper Perron class for φ in Ω) be such that

$$\int_{\partial\Omega} \varphi \, d\omega^x \geqslant u(x) - \varepsilon.$$

By the definition of $\mathcal{U}^\Omega_\varphi$, we have

$$\liminf_{y \to \xi} u(y) \geqslant \varphi(\xi) = 1 \quad \text{for all } \xi \in A.$$

Then, by the compactness of A, there exists δ-neighborhood $U_\delta(A)$ such that $u(y) \geqslant 1 - \varepsilon$ for all $y \in U_\delta(A) \cap \Omega$. Consider now a function $\widetilde{\varphi} \in C(\partial\widetilde{\Omega})$ supported on $U_\delta(A) \cap \partial\widetilde{\Omega}$ which equals 1 on A and is bounded above uniformly by 1. Then we claim that $u|_{\widetilde{\Omega}} \in \mathcal{U}^{\widetilde{\Omega}}_{(1-\varepsilon)\widetilde{\varphi}}$ (the upper Perron class for $(1-\varepsilon)\widetilde{\varphi}$ in $\widetilde{\Omega}$). Indeed, u is superharmonic in $\widetilde{\Omega}$ and

$$\liminf_{y \to \xi} u(y) \geqslant 0 = \widetilde{\varphi}(\xi) \quad \text{for all } \xi \in \partial\widetilde{\Omega} \backslash U_\delta(A),$$

and

$$\liminf_{y \to \xi} u(y) \geqslant 1 - \varepsilon \geqslant (1-\varepsilon)\widetilde{\varphi}(\xi) \quad \text{for all } \xi \in \partial\widetilde{\Omega} \cap U_\delta(A).$$

Therefore,

$$(1-\varepsilon)\, \widetilde{\omega}^x(A) \leqslant \int_{\partial\widetilde{\Omega}} (1-\varepsilon)\widetilde{\varphi} \, d\widetilde{\omega}^x \leqslant u(x) \leqslant \int_{\partial\Omega} \varphi \, d\omega^x + \varepsilon.$$

Since ε is arbitrarily small, we have $\widetilde{\omega}^x(A) \leqslant \int_{\partial\Omega} \varphi \, d\omega^x$. Taking the infimum over all the functions $\varphi \in C(\partial\Omega)$ such that $\varphi = 1$ on A, we derive $\widetilde{\omega}^x(A) \leqslant \omega^x(A)$. $\square$

5.4 Wiener regularity

In this section we continue to assume that $\Omega \subset \mathbb{R}^d$ is a bounded open set, unless stated otherwise. In view of Lemma 5.31 it is tempting to refer to the harmonic measure of any set $A \subset \partial\Omega$ as the harmonic function in Ω having boundary values χ_A. Unfortunately, χ_A is not a continuous function, and it is not clear what does it mean to have a discontinuous function as trace, for instance, when A is a dense subset with null harmonic measure. If the boundary is regular enough, this limit may be understood in the L^p sense, for instance, see Theorem 3.11, but the limit would be defined almost everywhere in some sense. We could expect, however, that $\lim_{x\to\xi}\omega^x(A) = 1$ if $\operatorname{dist}(\xi, \partial\Omega \cap A^c) > 0$, and $\lim_{x\to\xi}\omega^x(A) = 0$ if $\operatorname{dist}(\xi, A) > 0$. Unfortunately, we cannot grant yet that $H_f|_{\partial\Omega} \equiv f$ for continuous functions. We need to describe when this happens, that is, we need to study regular points.

Definition 5.33. We say that $\xi \in \partial\Omega$ is a *regular point* if whenever $f \in C(\partial\Omega)$, $H_f(x) \to f(\xi)$ as $\Omega \ni x \to \xi$, i.e.

$$\int_{\partial\Omega} f(\zeta)d\omega^x(\zeta) \xrightarrow{\Omega\ni x\to\xi} f(\xi). \tag{5.6}$$

We say that Ω is *Wiener regular* if every point in the boundary is regular.

From the definition above, it follows easily that if a domain Ω is Wiener regular, then the support of harmonic measure is the whole boundary of Ω.

A method for proving regularity at a point $\xi \in \partial\Omega$ consists in showing the existence of a barrier function for ξ, that is, a function $v : \Omega \to \mathbb{R}$ such that

1. v is superharmonic in Ω.
2. $\liminf_{y\to\zeta} v(y) > 0$ for all $\zeta \in \partial\Omega\backslash\{\xi\}$.
3. $\lim_{y\to\xi} v(y) = 0$.

Notice that, by the minimum principle applied to each component of Ω, $v > 0$ in Ω.

Theorem 5.34. *If $\xi \in \partial\Omega$ has a barrier function, then for any bounded function f on $\partial\Omega$ which is continuous at ξ, we have*

$$\lim_{x\to\xi} \underline{H}_f(x) = \lim_{x\to\xi} \overline{H}_f(x) = f(\xi).$$

In particular, ξ is a regular point.

Proof. Let v be a barrier for ξ and let $\varepsilon > 0$. Since f is continuous in ξ, there is $\delta > 0$ so that $|\zeta - \xi| \leqslant \delta$ implies $|f(\zeta) - f(\xi)| < \varepsilon$. Since v is superharmonic, the infimum of v in $\Omega_\delta := \Omega\backslash\bar{B}_\delta(\xi)$ is attained in $\partial\Omega_\delta$, see Lemma 5.4. That is, there exists some $y \in \partial\Omega_\delta$ such that

$$\inf_{\Omega_\delta} v = \liminf_{z\to y} v(z).$$

If $y \in \partial\Omega$, then $\liminf_{z\to y} v(z) > 0$ by the definition of barrier, and if $y \in \Omega \cap \partial B_\delta(\xi)$, then $\liminf_{z\to y} v(z) \geqslant v(y) > 0$ too, by the lower semicontinuity of v and the fact that $v > 0$ in Ω. Thus $\inf_{\Omega_\delta} v > 0$. So we can pick $k > 0$ such that

$$k \liminf_{z\to\zeta} v(z) > 2\sup|f| \qquad \text{for every } \zeta \in \partial\Omega\backslash\bar{B}_\delta(\xi)$$

(we can do this because f is bounded).

Now, since $f(\zeta) < f(\xi) + \varepsilon$ on $\bar{B}_\delta(\xi) \cap \partial\Omega$ and $f(\zeta) \leqslant 2\sup|f| + f(\xi)$ on $\partial\Omega\backslash\bar{B}_\delta(\xi)$, we have

$$f(\zeta) \leqslant k \liminf_{z\to\zeta} v(z) + f(\xi) + \varepsilon \quad \text{ for all } \zeta \in \partial\Omega.$$

Thus, $k\,v + f(\xi) + \varepsilon \in \mathcal{U}_f$ and therefore $\overline{H}_f(x) \leqslant kv(x) + f(\xi) + \varepsilon$ in Ω and so

$$\limsup_{x\to\xi} \overline{H}_f(x) \leqslant \limsup_{x\to\xi} k\,v(x) + f(\xi) + \varepsilon \leqslant 0 + f(\xi) + \varepsilon.$$

Letting $\varepsilon \to 0$ we get $\limsup_{x\to\xi} \overline{H}_f(x) \leqslant f(\xi)$, and arguing analogously we can also prove that $\liminf_{x\to\xi} \underline{H}_f(x) \geqslant f(\xi)$. The theorem is an immediate consequence of this fact, by Lemma 5.19. □

The preceding theorem asserts that the existence of a barrier for $\xi \in \partial\Omega$ implies that ξ is a regular point. The converse result is also true:

Theorem 5.35. *Let Ω be a bounded open set and let $\xi \in \partial\Omega$ be a regular point. Then there exists a barrier for ξ. This barrier can be chosen to be harmonic in Ω.*

Proof. Let $u(x) = |x - \xi|^2$. Obviously, $f := u|_{\partial\Omega} \in C(\partial\Omega)$. We claim that $v = H_f$ is a barrier for ξ. Indeed, this is harmonic in Ω and $\lim_{y\to\xi} H_f(y) = f(\xi)$ by the regularity of ξ. Also, u is subharmonic (because $\Delta u > 0$) and so $u \in \mathcal{L}_f$ and then $u \leqslant \underline{H}_f = H_f = v$ in Ω. Therefore, for all $\zeta \in \partial\Omega\backslash\{\xi\}$,

$$\liminf_{y\to\zeta} v(y) \geqslant \liminf_{y\to\zeta} u(y) = u(\zeta) > 0.$$

□

As a consequence, the harmonic measure of any open set with pole approaching to a boundary point interior to this set tends to 1.

Corollary 5.36. *Let Ω be a bounded open set and let $\xi \in \partial\Omega$ be a regular point. For every open set $U \subset \mathbb{R}^d$ containing ξ,*

$$\lim_{\Omega\ni x\to\xi} \omega^x(U) = 1.$$

Also

$$\lim_{\Omega\ni x\to\xi} \omega^x(\overline{U}^c) = 0.$$

Proof. By Urysohn's lemma, there exists a continuous function $f : \partial\Omega \to \mathbb{R}$ such that $f(\xi) = 1$ and $f|_{U^c \cap \partial\Omega} \equiv 0$. Then we have

$$H_f(x) = \int f \, d\omega^x \leqslant \int \chi_U \, d\omega^x = \omega^x(U)$$

by the monotonicity of integration. Since ξ is a regular point we have

$$1 \geqslant \limsup_{\Omega \ni x \to \xi} \omega^x(U) \geqslant \liminf_{\Omega \ni x \to \xi} \omega^x(U) \geqslant \lim_{\Omega \ni x \to \xi} H_f(x) = f(\xi) = 1.$$

The other estimate follows by an analogous reasoning assuming $f(\xi) = 0$ and $f|_{\overline{U}^c \cap \partial\Omega} \equiv 1$. □

Remark 5.37. There is a *thickness* property described in terms of capacity which characterizes regularity as well, see Chapter 6 for more details.

Remark 5.38. One easy criterion for ξ to have a barrier is the existence of an exterior tangent ball, that is, the existence of $B = B_r(y) \subset \Omega^c$ so that $\partial\Omega \cap \partial B = \{\xi\}$. In this way, the function $w(x) = \mathcal{E}^y(\xi) - \mathcal{E}^y(x)$ is a barrier function at ξ.

Note that harmonic measure associates a function $H_f(x)$ to each continuous function f on the boundary, although we don't necessarily know if it is a "true" extension in the sense that it is continuous up to the boundary and coincides with f there; all we know is that it is a harmonic function. If it happens that Ω is Wiener regular, then $\int f d\omega^x = H_f(x)$ is a harmonic function continuous up to the boundary with boundary values f.

5.5 The Dirichlet problem in unbounded domains with compact boundary

In order to study the properties of harmonic measure it is convenient to extend the study of the Dirichlet problem to unbounded open sets with compact boundary and to define the harmonic measure for this type of domains too. This the objective of this section.

Let $\Omega \subsetneq \mathbb{R}^d$ be an unbounded open set with compact boundary. Solving the Dirichlet problem in Ω for a function $f \in C(\partial\Omega)$ consists in finding a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the following:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega, \\ \|u\|_{\infty,\Omega} < \infty, \\ \text{when } d \geqslant 3, \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{5.7}$$

Proposition 5.39. *Let $\Omega \subsetneq \mathbb{R}^d$ be un unbounded open set with compact boundary and let $f \in C(\partial\Omega)$. If there exists a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying (5.7), then it is unique.*

Proof. Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ be two solutions of (5.7) and let us check that they are equal. Suppose first that $d \geqslant 3$. For $r > 0$, denote $\Omega_r = \Omega \cap B_r(0)$. Let r be large enough so that $\partial\Omega \subset B_r(0)$. For $0 < r_0 < r$, by the maximum principle, taking into account that $u = v$ on $\partial\Omega$,

$$\|u - v\|_{\infty,\Omega_{r_0}} \leqslant \|u - v\|_{\infty,\Omega_r} = \|u - v\|_{\infty,\partial\Omega_r} = \|u - v\|_{\infty,S_r(0)} \leqslant \|u\|_{\infty,S_r(0)} + \|v\|_{\infty,S_r(0)}.$$

By the last condition in (5.7), $\|u\|_{\infty,S_r(0)} + \|v\|_{\infty,S_r(0)} \to 0$ as $r \to \infty$, and so $u = v$ in Ω_{r_0}, with r_0 arbitrarily large.

Next we consider the case $d = 2$. Without loss of generality, we assume that $\partial\Omega \subset B_{1/4}(0)$. Let $\xi \in \partial\Omega$, and for a given $\delta > 0$, consider the function

$$h_\delta(x) = u(x) - v(x) - \delta \log|x - \xi|.$$

By the continuity of u and v at ξ, for any $\varepsilon > 0$ there exists some $\rho \in (0, 1/4)$ such that

$$|u(x) - v(x)| \leqslant \varepsilon \quad \text{for all } x \in \Omega \text{ such that } |x - \xi| \leqslant \rho.$$

For $r \gg \rho$, consider the domain $\Omega_{\rho,r} = \Omega \cap B_r(\xi)\backslash\overline{B}_\rho(\xi)$. We assume r large enough so that $\partial\Omega \subset B_r(\xi)$. Notice that

$$\partial\Omega_{\rho,r} \subset \partial\Omega \cup (\Omega \cap S_\rho(\xi)) \cup S_r(\xi).$$

Notice that $|u - v| \leqslant \varepsilon$ and $|\log|\cdot - \xi|| \leqslant |\log\rho|$ in $(\partial\Omega\backslash B_\rho(\xi)) \cup (\Omega \cap S_\rho(\xi)) \subset B_{1/2}(0)$. Thus,

$$|h_\delta| \leqslant \varepsilon + \delta|\log\rho| \quad \text{in } (\partial\Omega\backslash B_\rho(\xi)) \cup (\Omega \cap S_\rho(\xi)).$$

On the other hand, for $x \in S_r(\xi)$, $\log|x - \xi| = \log r$. So for a given $\delta > 0$, if r is large enough taking into account also that u and v are bounded, we have

$$h_\delta \leqslant 0 \quad \text{in } S_r(\xi).$$

From the last estimates and the maximum principle, we deduce that

$$h_\delta \leqslant \varepsilon + \delta|\log\rho| \quad \text{in } \Omega_{\rho,r},$$

Letting $r \to \infty$, we get infer that the same estimate is valid in $\Omega\backslash\overline{B}_\rho(\xi)$. That is,

$$u(x) - v(x) - \delta \log|x - \xi| \leqslant \varepsilon + \delta|\log\rho(\varepsilon)| \quad \text{for all } x \in \Omega_{\rho(\varepsilon)},$$

where we wrote $\rho(\varepsilon)$ to emphasize the dependence of ρ on ε. Since this inequality holds for all $\delta > 0$, we derive that $u \leqslant v + \varepsilon$ in $\Omega_{\rho(\varepsilon)}$. Finally, letting $\varepsilon \to 0$ and $\rho(\varepsilon) \to 0$, it follows that $u \leqslant v$ in Ω. Interchanging the roles of u and v in the arguments above, we deduce $v \leqslant u$ in Ω, and so we are done. □

Definition 5.40. Let Ω be an unbounded open set with bounded boundary. We say that Ω is Wiener regular if for $r > 0$ such that $\partial\Omega \subset B_r(0)$, the set $\Omega_r := \Omega \cap B_r(0)$ is Wiener regular. Also, we say that $\xi \in \partial\Omega$ is a regular point for Ω if it is regular for Ω_r.

Let us check that the definition does not depend on the precise $r > 0$ such that $\partial\Omega \subset B_r(0)$. Notice first that $\partial\Omega_r = \partial\Omega \cup \partial B_r(0)$. By the exterior tangent ball criterion in Remark 5.38 it follows all the points $\xi \in \partial B_r(0)$ are Wiener regular (for the open set Ω_r). To deal with the points from $\partial\Omega$, let $0 < r_1 < r_2$ be such that $\partial\Omega \subset B_{r_1}(0)$. If v_2 is barrier for $\xi \in \partial\Omega$ in Ω_{r_2}, then it is also a barrier in Ω_{r_1}, and so the Wiener regularity of ξ in Ω_{r_2} implies the Wiener regularity in Ω_{r_1}. Conversely, let v_1 be a harmonic barrier for ξ in Ω_{r_1} (see Theorem 5.35) and consider $r_0 < r_1$ such that we still have $\partial\Omega \subset B_{r_0}(0)$. Then

$$m_{r_0} := \inf_{\partial B_{r_0}(0)} v_1(x) > 0$$

because of the superharmonicity of v_1, the other properties in the definition of a barrier, and the minimum principle. Then we define

$$v_2(x) = \begin{cases} \min(v_1(x), m_{r_0}) & \text{in } \Omega \cap B_{r_0}(0), \\ m_{r_0} & \text{in } \mathbb{R}^d \backslash B_{r_0}(0), \end{cases}$$

which is superharmonic in Ω_{r_2} by Lemma 5.7 and moreover it is a barrier for this set at ξ. Thus the Wiener regularity of ξ in Ω_{r_1} implies the Wiener regularity in Ω_{r_2}.

Remark 5.41. Note that, arguing as above, we also see that ξ is regular for an unbounded set $\Omega \subset \mathbb{R}^d$ with compact boundary if and only if there exists a barrier at ξ.

We will show below that if $\Omega \subsetneq \mathbb{R}^d$ is an unbounded open set with compact boundary which is Wiener regular, then the Dirichlet problem in (5.7) is solvable for all $f \in C(\partial\Omega)$. The main step is contained in the following theorem.

Theorem 5.42. *Let $\Omega \subsetneq \mathbb{R}^d$ be an unbounded open set with compact boundary and let $f \in C(\partial\Omega)$. For $r > 0$ such that $\partial\Omega \subset B_r(0)$, denote $\Omega_r = \Omega \cap B_r(0)$ and let H_f^r be the Perron solution of the Dirichlet problem in Ω_r with boundary data equal to f in $\partial\Omega$ and equal to 0 in $S_r(0)$. Then the following holds:*

(a) The functions H_f^r converge uniformly in bounded subsets of Ω to a function harmonic and bounded in Ω as $r \to \infty$. Further, $H_f^r(x) \leqslant H_f^R(x)$ if $0 < r \leqslant R$ and $x \in \Omega_r$.

(b) In the case $d \geqslant 3$, the limiting function H_f satisfies $\lim_{x\to\infty} H_f(x) = 0$.

(c) If $\xi \in \partial\Omega$ is a regular point, then $\lim_{\Omega\ni x\to\xi} H_f(x) = f(\xi)$.

Remark that (a) asserts that the convergence of the functions H_f^r to H_f is uniform in $\Omega \cap B_{r_1}(0)$ for any $r_1 > 0$. This a stronger statement than just asking for the local uniform convergence in compact subsets of Ω.

By the theorem above, it is clear that if $\Omega \subsetneq \mathbb{R}^d$ is a Wiener regular unbounded open set with compact boundary, then H_f is the solution of the Dirichlet problem stated in (5.7).

Proof of Theorem 5.42. We claim that it suffices to prove the theorem for $f \geqslant 0$. Indeed, for an arbitrary function $f \in C(\partial\Omega)$, we can write $f = f^+ - f^-$, so that the functions $f^\pm$ are non-negative and continuous. Then we have

$$H_f^r = H_{f^+}^r - H_{f^-}^r,$$

and it is enough to prove the statements (a), (b), (c) for $f^{\pm}$.

(a) Let $r_0 > 0$ be such that $\partial\Omega \subset B_{r_0/2}(0)$. The fact that $0 \leqslant f \leqslant \sup_{\partial\Omega} f$, ensures that

$$0 \leqslant H_f^r \leqslant \sup_{\partial\Omega} f \quad \text{ in } \Omega_r, \text{ for all } r \geqslant r_0. \tag{5.8}$$

Next we will show that, for $r_0 < r < R$,

$$H_f^r \leqslant H_f^R \quad \text{ in } \Omega_r. \tag{5.9}$$

This is an easy consequence of the maximum principle. Indeed, for $s > r_0$ denote by $\mathcal{L}_f^s$ and $\mathcal{U}_f^s$ the respective lower and upper Perron classes in Ω_s for the function f_s which equals f on $\partial\Omega$ and vanishes in $S_s(0)$. Given $u \in \mathcal{L}_f^r$, let $\tilde{u} : \Omega_R \to \mathbb{R}$ be defined by

$$\tilde{u} = \begin{cases} \max(u, 0) & \text{in } \Omega_r, \\ 0 & \text{in } B_R(0) \backslash B_r(0). \end{cases}$$

By Lemma 5.7, $\tilde{u}$ is subharmonic in Ω_R and so that $\tilde{u} \in \mathcal{L}_f^R$. So for all $x \in \Omega_r$ we have

$$u(x) \leqslant \tilde{u}(x) \leqslant \underline{H}_f^R(x) = H_f^R(x).$$

Taking the supremum over all $u \in \mathcal{L}_f^r$, we deduce $H_f^r(x) \leqslant H_f^R(x)$, so that (5.9) holds.

From the monotonicity of the family of functions $\{H_f^r\}_{r>0}$ ensured by (5.9) and the bound in (5.8), we infer that the limit $\lim_{r\to\infty} H_f^r(x)$ exists for all $x \in \Omega$ and that the limit function H_f is bounded. Since the functions H_f^r, for $r > 0$, are harmonic in Ω_r and uniformly bounded, it follows that the preceding limit is uniform on compact subsets of Ω.

Next we will show that for any $r_1 > r_0$, the functions H_f^r converge uniformly on Ω_{r_1}. Observe first that they converge uniformly in $S_{r_1}(0)$ since this is a compact subset of Ω. So given $\varepsilon > 0$, there exists $r_2 > r_1$ such that

$$\|H_f^s - H_f\|_{\infty, S_{r_1}(0)} < \varepsilon \quad \text{ for all } s > r_2.$$

For $R > r > r_2$, consider now two arbitrary functions $u_r \in \mathcal{U}_f^r$ and $u_R \in \mathcal{L}_f^R$. Notice that

$$\limsup_{\Omega \ni x \to \xi} u_R(x) \leqslant f(\xi) \leqslant \liminf_{\Omega \ni x \to \xi} u_r(x) \quad \text{ on } \partial\Omega.$$

Since $\|H_f^r - H_f^R\|_{\infty, S_{r_1}(0)} < \varepsilon$, we also have

$$u_R \leqslant H_f^R \leqslant H_f^r + \varepsilon \leqslant u_r + \varepsilon \quad \text{ in } S_{r_1}(0).$$

Using that $u_R - u_r$ is subharmonic in Ω_{r_1} and the maximum principle, it follows that

$$u_R \leqslant u_r + \varepsilon \quad \text{ in } \Omega_{r_1}.$$

Taking the supremum over all $u_R \in \mathcal{L}_f^R$ and the infimum over all $u_r \in \mathcal{U}_f^r$ and using that continuous functions are resolutive, we deduce that

$$H_f^R \leqslant H_f^r + \varepsilon \quad \text{in } \Omega_{r_1}.$$

Together with (5.9), this implies $\|H_f^r - H_f^R\|_{\infty,\Omega_{r_1}} \leqslant \varepsilon$. Letting $R \to \infty$, it follows that

$$\|H_f^r - H_f\|_{\infty,\Omega_{r_1}} \leqslant \varepsilon \quad \text{for all } r > r_2,$$

which proves (a).

(b) Suppose $d \geqslant 3$. Let $M > 0$ be large enough so that

$$f(\xi) \leqslant M\,\mathcal{E}(\xi) \quad \text{for all } \xi \in \partial\Omega.$$

By the maximum principle, using that $\mathcal{E}$ is superharmonic in $\mathbb{R}^d$, we easily infer that $u \leqslant M\,\mathcal{E}$ in Ω_r for all $u \in \mathcal{L}_f^r$, for $r > r_0$. This implies that $H_f^r \leqslant M\,\mathcal{E}$ in Ω_r. Letting $r \to \infty$, it follows that $H_f \leqslant M\,\mathcal{E}$ in Ω, and so

$$\limsup_{x\to\infty} H_f(x) \leqslant \limsup_{x\to\infty} \mathcal{E}(x) = 0.$$

Since H_f is non-negative, this implies that H_f vanishes at infinity.

(c) For all $r > r_0$, since $\xi \in \partial\Omega$ is regular point for Ω_r, then $\lim_{\Omega\ni x\to\xi} H_f^r(x) = f(\xi)$. Together with the uniform convergence of H_f^r to H_f in Ω_{r_1} for any given $r_1 > r_0$, this easily yields $\lim_{\Omega\ni x\to\xi} H_f(x) = f(\xi)$. □

Under the assumptions and notation of Theorem 5.42, it is immediate to check that, for any $x \in \Omega$, the functional $C(\partial\Omega) \ni f \mapsto H_f(x)$ is linear and bounded. Indeed, the linearity is due to the linearity of $C(\partial\Omega) \ni f \mapsto H_f^r(x)$ and the boundedness follows from the fact that $\inf_{\partial\Omega} f \leqslant H_f^r \leqslant \sup_{\partial\Omega} f$ for all $r \geqslant r_0$, which yields

$$\|H_f\|_{\infty,\Omega} \leqslant \|f\|_{\infty,\partial\Omega} \tag{5.10}$$

letting $r \to \infty$.

Definition 5.43. Let $\Omega \subset \mathbb{R}^d$ be an unbounded open set with compact boundary and let $x \in \Omega$. The harmonic measure for Ω with pole at x is the unique Radon measure ω^x on $\partial\Omega$ such that

$$H_f(x) = \int_{\partial\Omega} f(\xi)d\omega^x(\xi) \qquad \text{for all } f \in C(\partial\Omega),$$

where is H_f defined as in Theorem 5.42. The existence and uniqueness of ω^x is ensured by the Riesz representation theorem, i.e. Theorem 4.58. Abusing notation we extend ω^x by 0 to the whole $\mathbb{R}^d$, that is $\omega^x(\mathbb{R}^d \backslash \partial\Omega) := 0$.

Remark 5.44. By the definition, for any unbounded open set with compact boundary $\Omega \subset \mathbb{R}^d$, for any $f \in C(\partial\Omega)$, and any $x \in \Omega$, we have

$$\int_{\partial\Omega} f(\xi)d\omega^x(\xi) = \lim_{r\to\infty} \int_{\partial\Omega} f(\xi)d\omega^x_{\Omega_r}(\xi).$$

By Theorem 5.42, the convergence is uniform in bounded subsets of Ω.

Lemma 5.45. *Let $\Omega \subset \mathbb{R}^d$ be an unbounded open set with compact boundary, let $x \in \Omega$ be the pole of $\omega := \omega^x_\Omega$. Let $\Omega_r = \Omega \cap B_r$ and for $r > |x|$ let $\omega_r := \omega^x_{\Omega_r}$. For any Borel set $A \subset \partial\Omega$ we have*

$$\omega_r(A) \nearrow \omega(A) \qquad \textit{as } r \to \infty.$$

Proof. Let us assume that $A = U$ is a bounded open set in $\partial\Omega$ and let us consider only values of r such that $U \subset\subset B_r$. The preceding remark states that $\omega_r \rightharpoonup \omega$, and by Theorem 4.63 we only need to check that $\omega(U) \geqslant \limsup \omega_r(U)$. By the inner regularity of Radon measures we have

$$\omega(U) = \sup_{f\leqslant\chi_U} \int f d\omega = \sup_{f\leqslant\chi_U} H_f(x),$$

where the supremum is taken over all functions $f \in C(\partial\Omega)$ supported in U. By Theorem 5.42, for every r large enough we have

$$\omega(U) \geqslant \sup_{f\leqslant\chi_U} H^r_f(x) = \sup_{f\leqslant\chi_U} \int f d\omega_r = \omega_r(U).$$

For general bounded Borel sets, note that $\omega_r(A)$ is an increasing sequence by Lemma 5.32, so let $s := \lim_{r\to\infty} \omega_r(A)$. We need to check that $s = \omega(A)$.

We claim that $s \leqslant \omega(A)$. Indeed, by the definition of s, for every $\varepsilon > 0$ there exists $r = r_\varepsilon$ so that $s - \varepsilon \leqslant \omega_r(A)$. By outer regularity, there exists a bounded open set $U_\varepsilon \supset A$ so that $\omega(U_\varepsilon) \leqslant \omega(A) + \varepsilon$. Thus,

$$s \leqslant \omega_r(A) + \varepsilon \leqslant \omega_r(U_\varepsilon) + \varepsilon \overset{\text{L.5.32}}{\leqslant} \omega(U_\varepsilon) + \varepsilon \leqslant \omega(A) + 2\varepsilon.$$

Since ε is arbitrarily small, the claim follows.

To prove that $s \geqslant \omega(A)$, we apply the preceding estimate to $\partial\Omega\backslash A$ and we take into account that $\lim_{r\to\infty} \omega_r(\partial\Omega) = \omega(\partial\Omega)$, because $\partial\Omega$ is relatively in open in $\partial\Omega$ [1]. Then we get

$$s = \lim_{r\to\infty} \omega_r(A) = \lim_{r\to\infty} \omega_r(\partial\Omega) - \lim_{r\to\infty} \omega_r(\partial\Omega\backslash A) \geqslant \omega(\partial\Omega) - \omega(\partial\Omega\backslash A) = \omega(A),$$

and thus $s = \omega(A)$, as wished. □

[1] In general, for unbounded open sets Ω, the harmonic measure ω for Ω is not a probability measure, that is, $\omega(\partial\Omega) \neq 1$. See Proposition 5.48 below.

Remark 5.46. By lemmas 5.45, 5.31 and 2.14 we obtain that $u(x) = \omega^x(A)$ is harmonic in Ω in the setting of the previous results, and Lemma 2.17 implies the mutual absolute continuity of ω^x and ω^y for x and y in the unbounded component of Ω.

We can also recover the monotonicity of harmonic measure when the domain increases in Lemma 5.32 for unbounded open sets.

Lemma 5.47. *Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}^d$ be open sets with compact boundary such that $\widetilde{\Omega} \subset \Omega$ and $\partial\Omega \cap \partial\widetilde{\Omega} \neq \varnothing$. Denote by ω_Ω and $\omega_{\widetilde{\Omega}}$ the respective harmonic measures for Ω and $\widetilde{\Omega}$. For any $x \in \widetilde{\Omega}$ and any Borel set $A \subset \partial\Omega \cap \partial\widetilde{\Omega}$, it holds*

$$\omega^x_{\widetilde{\Omega}}(A) \leqslant \omega^x_{\Omega}(A).$$

Proof. The only relevant case here is when both domains are unbounded and x belongs to the unbounded component of $\widetilde{\Omega}$. Then, following the notation of the previous results, we get

$$\omega_{\widetilde{\Omega}}(A) \overset{\text{L.5.45}}{=} \lim_r \omega_{\widetilde{\Omega}_r}(A) \overset{\text{L.5.32}}{\leqslant} \lim_r \omega_{\Omega_r}(A) \overset{\text{L.5.45}}{=} \omega_\Omega(A).$$

□

Observe that, by (5.10) it follows that

$$0 \leqslant \omega^x(\partial\Omega) \leqslant 1 \qquad \text{for all } x \in \Omega. \tag{5.11}$$

The following proposition provides additional information.

Proposition 5.48. *Let $\Omega \subsetneq \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary and let $x \in \Omega$. In the case $d = 2$, $\omega^x(\partial\Omega) = 1$, that is, ω^x is a probability measure. In the case $d = 3$, if x belongs to the unbounded component of Ω, then $0 < \omega^x(\partial\Omega) < 1$.*

In particular, the proposition implies that the statement (b) in Theorem 5.42 may fail in the case $d = 2$. Without the Wiener regular assumption on Ω, further information will be obtained later in Proposition 6.36.

Proof. Since Ω is Wiener regular, in the case $d = 2$ the function identically 1 in Ω solves the Dirichlet problem (5.7) for $f = 1$ in $\partial\Omega$. By the uniqueness of the solution, $H_f = 1$ identically in Ω and thus $\omega^x(\partial\Omega) = 1$.

In the case $d \geqslant 3$, again we have $\omega^x(\partial\Omega) = H_1(x)$ by Theorem 5.42. On the other hand, the statement (b) in the same theorem asserts that $H_1(x) \to 0$ as $x \to \infty$. So H_1 is a non constant non negative harmonic function in the unbounded component of Ω which is bounded above by 1, by (5.10). By the strong maximum principle (applied to $\Omega \cap B_r(0)$ and r large enough) it follows that $0 < \omega^x(\partial\Omega) = H_1(x) < 1$. □

Example 5.49. Let $\Omega = \mathbb{R}^d \backslash \bar{B}_1(0)$ for $d \geqslant 3$. The solution of the Dirichlet problem for $f \equiv 1$ in $\partial\Omega$ is the function $u(x) = |x|^{2-d}$. Thus,

$$\omega^x(\partial\Omega) = \frac{1}{|x|^{d-2}} \qquad \text{for all } x \in \Omega.$$

Corollary 5.36 also has a counterpart in unbounded open sets with compact boundary.

Corollary 5.50. *Let Ω be an open set with compact boundary and let $\xi \in \partial\Omega$ be a regular point. For every open set $U \subset \mathbb{R}^d$ containing ξ,*

$$\lim_{\Omega \ni x \to \xi} \omega^x(U) = 1.$$

Also

$$\lim_{\Omega \ni x \to \xi} \omega^x(\overline{U}^c) = 0.$$

Proof. We have already seen that $\omega^x(U) \leqslant 1$ so we need to show

$$\lim_{x \to \xi} \omega^x(U) \geqslant 1.$$

To prove this, we can assume that U is bounded by intersecting with a bounded ball containing $\partial\Omega$. Choose any radius r big enough. Using Corollary 5.36, we can pick δ so that

$$\omega_r^x(U) \geqslant 1 - \varepsilon \qquad \text{for every } x \in B_\delta(\xi) \cap \Omega.$$

Then

$$\omega^x(U) \geqslant \omega_r^x(U) \geqslant 1 - \varepsilon \qquad \text{for every } x \in B_\delta(\xi) \cap \Omega,$$

showing the first statement.

On the other hand, note that the first statement implies that

$$\lim_{\Omega \ni x \to \xi} \omega^x(\partial\Omega) = 1.$$

Thus,

$$0 \leqslant \omega^x(\overline{U}^c) = \omega^x(\partial\Omega) - \omega^x(\overline{U}) \leqslant \omega^x(\partial\Omega) - \omega^x(U) \xrightarrow{\Omega \ni x \to \xi} 1 - 1 = 0,$$

establishing the second claim. □

Next we wish to show that, in the case $d = 2$, we can easily define the notion of harmonic measure with pole at ∞. First we need the following auxiliary result, which has its own interest.

Proposition 5.51. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $x_0 \in \Omega$. Let $u : \Omega \backslash \{x_0\} \to \mathbb{R}$ be a harmonic function such that $u(x) = o(\mathcal{E}(x - x_0))$ as $x \to x_0$. Then u extends as a harmonic function to the whole Ω.*

Of course, the proposition applies to the particular case where u is bounded and harmonic in $\Omega \backslash \{x_0\}$. See also Theorem 6.35 for a related result.

Proof. Let $\bar{B}_r(x_0)$ be a closed ball contained in Ω, with $r < 1$, and let v be the solution of the Dirichlet problem in $B_r(x_0)$ with boundary data $u|_{S_r(x_0)}$. For any $\varepsilon > 0$, consider the function

$$h_\varepsilon(x) = u(x) - v(x) - \varepsilon\, \mathcal{E}(x - x_0), \qquad \text{for } x \in B_r(x_0) \backslash \{x_0\}.$$

This is harmonic in $B_r(x_0)\backslash\{x_0\}$ and $\lim_{x\to x_0} h_\varepsilon(x) = -\infty$. By the maximum principle applied to any annulus $A_{s,r}(x_0)$ with s sufficiently small, we deduce that $h_\varepsilon \leqslant 0$ in $B_r(x_0)\backslash\{x_0\}$. Since this holds for any $\varepsilon > 0$, we get $u \leqslant v$ in $B_r(x_0)\backslash\{x_0\}$. Reversing the roles of u and v, we obtain the opposite inequality. Thus $u = v$ in $B_r(x_0)\backslash\{x_0\}$ and so the proposition follows just letting $u = v$ in the whole $B_r(x_0)$. □

Corollary 5.52. *For some $r > 0$, let $u : \mathbb{C}\backslash\bar{B}_r(0) \to \mathbb{R}$ be a harmonic and bounded function. Then $\lim_{z\to\infty} u(z)$ exists and the function defined by $v(z) := u(1/z)$ can be extended to a harmonic function in $B_{1/r}(0)$.*

Proof. The function $v(z) := u(1/z)$ is harmonic and bounded in $B_{1/r}(0)\backslash\{0\}$. So it extends to a harmonic function in $B_{1/r}(0)$ by the preceding proposition. Thus,

$$\lim_{z\to\infty} u(z) = \lim_{z\to 0} v(z)$$

exists. □

Now we can define harmonic measure with pole at ∞ for an unbounded open set with compact boundary in the plane as in Definition 5.43, just putting $x = \infty$ there:

Definition 5.53. Let $\Omega \subset \mathbb{R}^2$ be an unbounded open set with compact boundary. The harmonic measure for Ω with pole at ∞ is the unique Radon measure ω^∞ on $\partial\Omega$ such that

$$\lim_{x\to\infty} H_f(x) = \int_{\partial\Omega} f(\xi)d\omega^\infty(\xi) \qquad \text{for all } f \in C(\partial\Omega),$$

where H_f is defined as in Theorem 5.42. The existence and uniqueness of ω^∞ is ensured by the Riesz representation theorem.

Obviously, for any function $f \in C(\partial\Omega)$ (and Ω as in the definition),

$$\int_{\partial\Omega} f(\xi)d\omega^\infty(\xi) = \lim_{z\to\infty} \int_{\partial\Omega} f(\xi)d\omega^z(\xi).$$

Observe that for any z belonging to the unbounded component of Ω, the measures ω^z and ω^∞ are mutually absolutely continuous. Indeed, for any Borel set $E \subset \partial\Omega$, it follows easily from the strong maximum principle applied to the function $v(z) = \omega^{1/z}(E)$ in a neighborhood of the origin that $v(0) = 0$ if and only if v vanishes identically, see Exercise 5.5.1.

In the case $d \geqslant 3$, one can also the define the notion of harmonic measure with pole at ∞ for unbounded open set with compact boundary in $\mathbb{R}^d$, at least under the assumption of Wiener regularity, following a different approach. We postpone this task to Chapter 7.

Exercise 5.5.1. Given an unbounded domain $\Omega \subset \mathbb{C}$ with compact boundary, show that for every Borel set $E \subset \partial\Omega$, we have

$$\lim_{z\to\infty} \omega^z(E) = \omega^\infty(E).$$

Exercise 5.5.2. Let $\Omega \subset \mathbb{R}^d$ be open with compact boundary and suppose that Ω is not connected. Let U be a connected component of Ω and let $x \in U$. Show that

$$\operatorname{supp} \omega_\Omega^x \subset \partial U.$$

5.6 A Markov type property of harmonic measure

In this section we will show the following result.

Theorem 5.54. *Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}^d$ be open sets with compact boundary such that $\widetilde{\Omega} \subset \Omega$. Suppose also that $\widetilde{\Omega}$ is Wiener regular and that the points from $\partial\Omega \cap \partial\widetilde{\Omega}$ are regular for Ω. Denote by ω_Ω and $\omega_{\widetilde{\Omega}}$ the respective harmonic measures for Ω and $\widetilde{\Omega}$. Then, for every $x \in \widetilde{\Omega}$ and every Borel set $A \subset \partial\Omega$, it holds*

$$\omega_\Omega^x(A) = \omega_{\widetilde{\Omega}}^x(A) + \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega_\Omega^y(A)\, d\omega_{\widetilde{\Omega}}^x(y). \tag{5.12}$$

This result can be deduced from the connection between harmonic measure and Brownian motion, using the strong Markov property of Brownian motion. However, below we provide an analytic proof.

Proof. To shorten notation, we write $\omega = \omega_\Omega$ and $\widetilde{\omega} = \omega_{\widetilde{\Omega}}$. Suppose first that A is compact. For any $\varepsilon > 0$, let $f_\varepsilon : \partial\Omega \to \mathbb{R}$ be a continuous function which equals 1 on A and vanishes away from an ε-neighborhood of A. Denote

$$u_\varepsilon(x) = \int_{\partial\Omega} f_\varepsilon\, d\omega^x, \qquad v_\varepsilon(x) = \int_{\partial\widetilde{\Omega}} u_\varepsilon\, d\omega^x.$$

In the above definition of v_ε we identify $u_\varepsilon|_{\partial\widetilde{\Omega}\cap\partial\Omega} \equiv f_\varepsilon|_{\partial\widetilde{\Omega}\cap\partial\Omega}$. In this way, from the regularity of the points from $\partial\Omega \cap \partial\widetilde{\Omega}$ for Ω, it follows that $u_\varepsilon|_{\partial\widetilde{\Omega}}$ is a continuous function.

We claim now that

$$u_\varepsilon(x) = v_\varepsilon(x) \quad \text{for all } x \in \widetilde{\Omega}. \tag{5.13}$$

Indeed, by the Wiener regularity of $\widetilde{\Omega}$ and the regularity of the points from $\partial\Omega \cap \partial\widetilde{\Omega}$ for Ω, we have

$$\lim_{x\to\xi} u_\varepsilon(x) = \lim_{x\to\xi} v_\varepsilon(x) \quad \text{for all } \xi \in \partial\widetilde{\Omega}.$$

and, when $\widetilde{\Omega}$ is unbounded and $n \geqslant 2$, by the definition of u_ε and v_ε,

$$\lim_{x\to\infty} u_\varepsilon(x) = \lim_{x\to\infty} v_\varepsilon(x) = 0.$$

Then, by the maximum principle, our claim follows.

From the identity (5.13), for $x \in \widetilde{\Omega}$, we deduce

$$u_\varepsilon(x) = v_\varepsilon(x) = \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} u_\varepsilon\, d\widetilde{\omega}^x + \int_{\partial\widetilde{\Omega}\cap\partial\Omega} u_\varepsilon\, d\widetilde{\omega}^x = \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} u_\varepsilon\, d\widetilde{\omega}^x + \int_{\partial\widetilde{\Omega}\cap\partial\Omega} f_\varepsilon\, d\widetilde{\omega}^x. \tag{5.14}$$

By dominated convergence, for every $y \in \Omega$, $u_\varepsilon(y) \to \omega^y(A)$ as $\varepsilon \to 0$. So the left hand side of (5.14) converges to $\omega^x(A)$ and the first integral on the right hand side to $\int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^y(A)\, d\widetilde{\omega}^x(y)$. Again by dominated convergence, the last integral on the right hand side tends to $\widetilde{\omega}^x(A)$. Thus the identity (5.12) holds when A is compact.

Suppose now that A is an arbitrary Borel set. By the inner regularity of Radon measures, there is a sequence of compact sets $E_k \subset A$, with $E_k \subset E_{k+1}$, such that $\omega^x(E_k) \to \omega^x(A)$ and $\widetilde{\omega}^x(E_k) \to \widetilde{\omega}^x(A)$ as $k \to \infty$. Let U and $\widetilde{U}$ be the respective connected components of Ω and $\widetilde{\Omega}$ that contain x, so that $\widetilde{U} \subset U$ and $\operatorname{supp} \widetilde{\omega}^x \subset \partial\widetilde{U} \subset \overline{U}$ (see Exercise 5.5.2). Then, for every k we have

$$\omega^x(E_k) = \widetilde{\omega}^x(E_k) + \int_{\partial\widetilde{\Omega}\cap\overline{U}\setminus\partial\Omega} \omega^y(E_k)\, d\widetilde{\omega}^x(y) = \widetilde{\omega}^x(E_k) + \int_{\partial\widetilde{\Omega}\cap U} \omega^y(E_k)\, d\widetilde{\omega}^x(y). \tag{5.15}$$

Since $w_k(y) := \omega^y(A\setminus E_k)$ is a positive harmonic function in U, by connectedness we have (see Lemma 2.17):

$$w_k(y) \approx_{x,y} w_k(x) \to 0 \qquad \text{as } k \to \infty, \text{ for all } y \in U.$$

Equivalently, $\omega^y(E_k) \to \omega^y(A)$ for all $y \in U$. Therefore, passing to the limit in (5.15) and using dominated convergence, we obtain

$$\omega^x(A) = \widetilde{\omega}^x(A) + \int_{\partial\widetilde{\Omega}\cap U} \omega^y(A)\, d\widetilde{\omega}^x(y) = \widetilde{\omega}^x(A) + \int_{\partial\widetilde{\Omega}\setminus\partial\Omega} \omega^y(A)\, d\widetilde{\omega}^x(y).$$

$\square$

6 Potential theory

6.1 Potentials

Recall that the fundamental solution of the minus Laplacian in $\mathbb{R}^d$ equals

$$\mathcal{E}(x) = \begin{cases} \dfrac{|x|^{2-d}}{(d-2)\kappa_d} & \text{if } d \geqslant 3, \\[2ex] \dfrac{-\log|x|}{2\pi} & \text{if } d = 2, \end{cases}$$

For a Radon measure μ in $\mathbb{R}^d$, we consider the potential U_μ defined by

$$U_\mu(x) = \mathcal{E} * \mu(x) = \int \mathcal{E}(x-y)\, d\mu(y), \tag{6.1}$$

and the energy integral

$$I(\mu) := \iint \mathcal{E}(x-y) d\mu(y) d\mu(x). \tag{6.2}$$

For $d \geqslant 3$, U_μ is called the Newtonian potential of μ, and for $d = 2$, the logarithmic or Wiener potential of μ.

Lemma 6.1 (Semicontinuity properties)**.** *For non-negative Radon measures $\mu_n \rightharpoonup \mu$ with compact support we have:*

(a) $\liminf_{y \to x} U_\mu(y) \geqslant U_\mu(x)$ *for all $x \in \mathbb{R}^d$. So the potential U_μ is lower semicontinuous in $\mathbb{R}^d$.*

(b) $\liminf_{n \to \infty} U_{\mu_n}(x) \geqslant U_\mu(x)$ *for all $x \in \mathbb{R}^d$.*

(c) $\liminf_{n \to \infty} I(\mu_n) \geqslant I(\mu)$.

(d) The potential U_μ is superharmonic.

The proof of this lemma is an easy exercise that we leave for the reader. The superharmonicity of U_μ is a consequence of the lower semicontinuity of U_μ, the superharmonicity of $\mathcal{E}$, and Lemma 5.10 (a). For more details, alternatively, the reader may have a look at [Lan72] or [Ran95].

Theorem 6.2 (Continuity principle for potentials)**.** *Given a compactly supported Radon measure μ in $\mathbb{R}^d$, if $U_\mu \in C(\operatorname{supp}\mu)$, then $U_\mu \in C(\mathbb{R}^d)$.*

Proof. In the case $d = 2$, by a suitable contraction we can assume that $\operatorname{diam}(\operatorname{supp}\mu) \leqslant 1/2$, so that $\mathcal{E}(x-y) > 0$ for all $x, y \in \operatorname{supp}\mu$.

Since U_μ is continuous in $\mathbb{R}^d \setminus \operatorname{supp}\mu$ we only have to check the continuity in $\operatorname{supp}\mu$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a radial continuous function such that $\chi_{\mathbb{R}^d \setminus B_1(0)} \leqslant \varphi \leqslant \chi_{\mathbb{R}^d \setminus B_{1/2}(0)}$ and, for each $\delta > 0$, let

$$f_\delta(x) = \int_{|x-y|\geqslant\delta} \mathcal{E}(x-y)\,d\mu(y), \quad \tilde{f}_\delta(x) = \int \mathcal{E}(x-y)\,\varphi\Big(\frac{x-y}{\delta}\Big)\,d\mu(y).$$

Since $\{\tilde{f}_\delta\}$ is a monotone family of continuous functions and $U_\mu|_{\operatorname{supp}\mu}$ is continuous, the convergence of $\tilde{f}_\delta$ to U_μ is uniform in $\operatorname{supp}\mu$, by Dini's theorem. In turn, since $\tilde{f}_\delta \leqslant f_\delta \leqslant U_\mu$, this implies the uniform convergence of f_δ to U_μ in $\operatorname{supp}\mu$. Equivalently, $U_{\chi_{B_\delta(x)}\mu}(x) \to 0$ uniformly on $x \in \operatorname{supp}\mu$ as $\delta \to 0$.

To prove the continuity of U_μ at a given $x \in \operatorname{supp}(\mu)$, fix $\varepsilon > 0$, and take $\delta \in (0, 1/4)$ such that $U_{\chi_{B_\delta(z)}\mu}(z) < \varepsilon$ for all $z \in \operatorname{supp}\mu$ and such that $\mu(B_\delta(x)) < \varepsilon$ (that the latter condition holds for δ small enough is due to the fact that μ has no point masses, because $U_\mu(z) < \infty$ for all $z \in \operatorname{supp}\mu$). For $y \in B_{\delta/4}(x)$, we write

$$\begin{aligned} |U_\mu(x) - U_\mu(y)| \leqslant &\int_{|x-z|<\delta/2} \mathcal{E}(x-z)\,d\mu(z) + \int_{|x-z|<\delta/2} \mathcal{E}(y-z)\,d\mu(z) \\ &+ \left| \int_{|x-z|\geqslant\delta/2} (\mathcal{E}(x-z) - \mathcal{E}(y-z))\,d\mu(z) \right|. \end{aligned}$$

The first integral on the right hand side is bounded above by ε. The third one tends to 0 as $y \to x$, because for a fixed $\delta > 0$, the function $g(y) = \int_{|x-z|\geqslant\delta/2} \mathcal{E}(y-z)\,d\mu(z)$ is continuous in $B_{\delta/4}(x)$. To estimate the second integral on the right hand side, let y' be the closest point to y from $\operatorname{supp}\mu$. Notice that $|y'-y| \leqslant |x-y| \leqslant \delta/4$, and thus $y' \in B_{\delta/2}(x)$. It is immediate to check that then

$$|z-y'| \lesssim |z-y| \quad \text{ for all } z \in \operatorname{supp}\mu.$$

Thus, in the case $d \geqslant 3$, $\mathcal{E}(y-z) \lesssim \mathcal{E}(y'-z)$, and so, using that $y' \in \operatorname{supp}\mu$,

$$\int_{|x-z|<\delta/2} \mathcal{E}(y-z)\,d\mu(z) \lesssim \int_{B_{\delta/2}(x)} \mathcal{E}(y'-z)\,d\mu(z) \lesssim \int_{B_\delta(y')} \mathcal{E}(y'-z)\,d\mu(z) \lesssim \varepsilon.$$

In the case $d = 2$, we have $|y-z| \geqslant |y'-z|$ for $z \in B_{|y-y'|}(y')$ and so $\mathcal{E}(y-z) \leqslant \mathcal{E}(y'-z)$ for such z. On the other hand, for $z \in \operatorname{supp}\mu \setminus B_{|y-y'|}(y')$, we have $|y-z| \approx |y'-z|$ and thus

$$\mathcal{E}(y-z) = \mathcal{E}(y'-z) + \frac{1}{2\pi}\log\frac{|y'-z|}{|y-z|} \leqslant \mathcal{E}(y'-z) + C.$$

Therefore,

$$\begin{aligned} \int_{|x-z|<\delta/2} \mathcal{E}(y-z)\,d\mu(z) &\leqslant \int_{B_{\delta/2}(x)} \mathcal{E}(y'-z)\,d\mu(z) + C\,\mu(B_{\delta/2}(x)) \\ &\leqslant \int_{B_\delta(y')} \mathcal{E}(y'-z)\,d\mu(z) + C\,\mu(B_{\delta/2}(x)) \lesssim \varepsilon. \end{aligned}$$

So for any dimension, we have

$$\limsup_{y\to x} |U_\mu(x) - U_\mu(y)| \lesssim \varepsilon + \limsup_{y\to x} \left| \int_{|x-z|\geqslant \delta/2} (\mathcal{E}(x-z) - \mathcal{E}(y-z))\, d\mu(z) \right| \approx \varepsilon.$$

Since ε is arbitrary, we have that $U_\mu(y) \to U_\mu(x)$ as $y \to x$. □

Theorem 6.3 (Maximum principle for potentials)**.** *Given a compactly supported Radon measure μ in $\mathbb{R}^d$, if $U_\mu(x) \leqslant 1$ μ-a.e., then $U_\mu(x) \leqslant 1$ everywhere in $\mathbb{R}^d$.*

Proof. Again, by contracting suitably suppμ, we can assume that diam(suppμ) $\leqslant 1/2$ in the case $d = 2$, see Exercise 6.1.2.

Let $E = \operatorname{supp}\mu$. By the semicontinuity property in Theorem 6.1(a), it holds that $U_\mu(x) \leqslant 1$ for all $x \in E$. Thus, it suffices to show $U_\mu(x) \leqslant 1$ for all $x \in \mathbb{R}^d \backslash E$.

For any $\tau > 0$, by Egorov's theorem, there is a compact subset $F = F_\tau \subset E$ such that $\mu(E\backslash F) < \tau$ and so that $U_{\chi_{B_\varepsilon(x)}\mu}(x)$ converges uniformly to 0 in F as $\varepsilon \to 0$. We claim that $U_{\chi_F\mu}$ is continuous in $\mathbb{R}^d$. Indeed, by the preceding theorem, if suffices to show that $U_{\chi_F\mu} \in C(F)$. To prove this, for any $\varepsilon \in (0, 1/2)$ and $x, x' \in F$ such that $|x - x'| \leqslant \varepsilon^d$, we write

$$\begin{aligned} |U_{\chi_F\mu}(x) - U_{\chi_F\mu}(x')| \leqslant & \int_{|x-y|\leqslant\varepsilon} \mathcal{E}(x-y) d\mu|_F(y) + \int_{|x-y|\leqslant\varepsilon} \mathcal{E}(x'-y) d\mu|_F(y) \\ & + \int_{|x-y|>\varepsilon} \big|\mathcal{E}(x-y) - \mathcal{E}(x'-y)\big|\, d\mu|_F(y) \end{aligned}$$

The first integral on the right hand side tends to 0 as $\varepsilon \to 0$ (uniformly on $x \in F$), and the same happens with the second one, taking into account that $\{y : |x - y| \leqslant \varepsilon\} \subset \{y : |x' - y| \leqslant 2\varepsilon\}$. For the last one, in the case $d \geqslant 3$, for $y, x, x' \in F$ such that $|x - y| > \varepsilon$ and $|x - x'| \leqslant \varepsilon^d$ (in particular $|x - x'| \leqslant \varepsilon/2$), we have

$$\big|\mathcal{E}(x-y) - \mathcal{E}(x'-y)\big| = \left| \frac{c}{|x-y|^{d-2}} - \frac{c}{|x'-y|^{d-2}} \right| \lesssim \frac{|x-x'|}{|x-y|^{d-1}} \lesssim \varepsilon.$$

In the case $d = 2$, observe that

$$\left| \frac{|x'-y|}{|x-y|} - 1 \right| \leqslant \frac{|x'-x|}{|x-y|} \leqslant \varepsilon, \quad \text{for } y, x, x' \text{ such that } |x-y| > \varepsilon \text{ and } |x-x'| \leqslant \varepsilon^2,$$

and thus, for some constant $C > 0$,

$$\big|\mathcal{E}(x-y) - \mathcal{E}(x'-y)\big| \approx \left| \log \frac{|x'-y|}{|x-y|} \right| \lesssim \varepsilon.$$

Then, for any dimension d,

$$\int_{|x-y|>\varepsilon} \big|\mathcal{E}(x-y) - \mathcal{E}(x'-y)\big|\, d\mu|_F(y) \lesssim \varepsilon\mu(F).$$

Therefore,

$$\lim_{\varepsilon\to 0} \sup_{x,x'\in F:|x-x'|\leqslant\varepsilon^2} |U_{\chi_F\mu}(x) - U_{\chi_F\mu}(x')| = 0,$$

and thus the claim holds.

Notice that $U_{\chi_F\mu}(x) \leqslant U_\mu(x) \leqslant 1$ for all $x \in F$. Further, in the case $d \geqslant 3$, $U_{\chi_F\mu}(x) \to 0$ when $x \to \infty$, while in the case $d = 2$ we get $U_{\chi_F\mu}(x) \to -\infty$. Since $U_{\chi_F\mu}$ is harmonic in $\mathbb{R}^d\backslash F$ and continuous in $\mathbb{R}^d$, by the maximum principle (applied to $\Omega_R = B_R(0)\backslash F$ and letting $R \to \infty$), we deduce that $U_{\chi_F\mu}(x) \leqslant 1$ for all $x \in \mathbb{R}^d\backslash E \subset \mathbb{R}^d\backslash F$. Now we just have to write

$$U_\mu(x) = U_{\chi_F\mu}(x) + U_{\chi_{E\backslash F}\mu}(x) \leqslant 1 + U_{\chi_{E\backslash F}\mu}(x),$$

and note that $U_{\chi_{E\backslash F}\mu}(x) \to 0$ for any $x \in \mathbb{R}^d\backslash E$, as $\tau \to 0$ (recall that $\mu(E\backslash F) \leqslant \tau$). $\square$

Exercise 6.1.1. Given a compactly supported Radon measure μ, show that U_μ is μ-measurable.

Exercise 6.1.2. Given a compactly supported Radon measure μ in $\mathbb{C}$ such that $U_\mu(x) \leqslant 1$ for μ-a.e. $x \in \mathbb{C}$, find c and λ such that $\operatorname{supp} cT_{\lambda,\#}\mu \subset \frac{1}{4}\mathbb{D}$ and $U_{cT_{\lambda,\#}\mu} \leqslant 1$.

6.2 Capacity

Definition 6.4. Given a bounded set $E \subset \mathbb{R}^d$, we define its capacity $\operatorname{Cap}(E)$ by

$$\operatorname{Cap}(E) = \frac{1}{\inf_{\mu\in M_1(E)} I(\mu)}, \tag{6.3}$$

where the infimum is taken over the collection $M_1(E)$ of all *probability* Radon measures μ supported on E. When $d \geqslant 3$, $\operatorname{Cap}(E)$ is also called the Newtonian capacity of E, and for $d = 2$, the Wiener capacity of E.

In the case $d = 2$, quite often we will write $\operatorname{Cap}_W(E)$ instead of $\operatorname{Cap}(E)$. Remark that $\operatorname{Cap}_W(E)$ may be negative, and we allow this to be infinite too. However, since E is assumed to be bounded, we have $\inf_{\mu\in M_1(E)} I(\mu) > -\infty$, so having zero capacity is equivalent to having $I(\mu) = +\infty$ for every $\mu \in M_1(E)$. On the other hand, if $\operatorname{diam}(E) < 1$, then $\mathcal{E}(x-y) \geqslant (2\pi)^{-1} \log \frac{1}{\operatorname{diam}(E)} > 0$ for all $x, y \in E$, and it follows that $\inf_{\mu\in M_1(E)} I(\mu) > 0$, and so $0 \leqslant \operatorname{Cap}_W(E) < \infty$.[1]

Definition 6.5. Given a set $E \subset \mathbb{R}^2$, we define its logarithmic capacity by

$$\operatorname{Cap}_L(E) = e^{-2\pi \inf_{\mu\in M_1(E)} I(\mu)} = e^{-\frac{2\pi}{\operatorname{Cap}_W(E)}}.$$

[1] We will see below that this also holds if $\overline{E}$ is contained in $B_1(0)$.

It is immediate to check that if $E \subset F$, then $\mathrm{Cap}(E) \leqslant \mathrm{Cap}(F)$ for $d \geqslant 3$ and $\mathrm{Cap}_L(E) \leqslant \mathrm{Cap}_L(F)$ for $d = 2$.[2] Another trivial property is that the capacities Cap, Cap_W, and Cap_L are invariant by translations. Further, the Newtonian capacity is homogeneous of degree $d-2$ when $d \geqslant 3$. That is, for a given $\lambda > 0$ and $E \subset \mathbb{R}^d$, we have

$$\mathrm{Cap}(\lambda E) = \lambda^{d-2}\,\mathrm{Cap}(E).$$

This follows easily from the fact that the fundamental solution $\mathcal{E}$ is homogeneous of degree $2-d$ in $\mathbb{R}^d$, $d \geqslant 3$. In the case $d = 2$, $\mathcal{E}$ is not homogeneous, and the behavior of Cap_W under dilations is more complicated. To study this, denote $T_\lambda(x) = \lambda\, x$, so that if μ is a probability measure supported on E, then the image measure $T_{\lambda\#}\mu$ (see Definition 4.59) is another probability measure supported on λE. Then, by Theorem 4.61 we have

$$\begin{aligned} I(T_{\lambda\#}\mu) &= \frac{1}{2\pi}\iint \log\frac{1}{|x-y|}\, dT_{\lambda\#}\mu(x)\, dT_{\lambda\#}\mu(y) \\ &= \frac{1}{2\pi}\iint \log\frac{1}{|\lambda x - \lambda y|}\, d\mu(x)\, d\mu(y) = I(\mu) - \frac{1}{2\pi}\log\lambda. \end{aligned}$$

Taking the infimum, we derive

$$\inf_{\eta\in M_1(\lambda E)} I(\eta) = \inf_{\mu \in M_1(E)} I(\mu) - \frac{1}{2\pi}\log\lambda,$$

So we get

$$\mathrm{Cap}_W(\lambda E) = \frac{1}{\dfrac{1}{\mathrm{Cap}_W(E)} - \frac{1}{2\pi}\log\lambda}.$$

In particular, notice that for λ big enough we have $\mathrm{Cap}_W(\lambda E) < 0$ [3]. On the contrary, in the case $d \geqslant 3$, Newtonian capacity is always non-negative. The rather strange behavior of the Wiener capacity under dilations and other related technical issues is one of the motivations for the introduction of logarithmic capacity. Clearly, $\mathrm{Cap}_L(E) \geqslant 0$ for any compact set E, and moreover for any $\lambda > 0$,

$$\mathrm{Cap}_L(\lambda E) = e^{-\frac{2\pi}{\mathrm{Cap}_W(E)} + \log\lambda} = \lambda\,\mathrm{Cap}_L(E). \tag{6.4}$$

So the logarithmic capacity is homogeneous of degree 1.

Remark 6.6. Note that given a bounded set E, the potential of the Lebesgue measure restricted to E is bounded. In particular, if E has positive Lebesgue measure then its capacity is not zero.

Given a compactly supported Radon measure μ, one can also check that if U_μ is a bounded potential, then $\mu(Z) = 0$ for every set Z of capacity zero.

[2] In the case $d = 2$, the inequality $\mathrm{Cap}_W(E) \leqslant \mathrm{Cap}_W(F)$ fails if $\mathrm{Cap}_W(F) < 0$, and it holds if $\mathrm{Cap}_W(F) > 0$, and in particular if $\mathrm{diam}(F) < 1$.

[3] Also, formally, $\mathrm{Cap}_W(\lambda E) = \infty$ in case that $\dfrac{1}{\mathrm{Cap}_W(E)} = \frac{1}{2\pi}\log\lambda$.

Lemma 6.7 (Outer regularity of capacity). *Let $E \subset \mathbb{R}^d$ be a compact set and let $\{V_n\}_{n\geqslant 1}$ be a decreasing sequence (i.e., $V_n \supset V_{n+1}$) of bounded open sets such that and $E = \bigcap_n V_n$. Then*

$$\lim_{n\to\infty} \mathrm{Cap}(V_n) = \mathrm{Cap}(E) \quad \text{for } d \geqslant 3$$

and

$$\lim_{n\to\infty} \mathrm{Cap}_L(V_n) = \mathrm{Cap}_L(E) \quad \text{for } d = 2.$$

Proof. This is a straightforward consequence of the semicontinuity property of the energies $I(\mu_n)$ in Lemma 6.1 and Theorems 4.62 and 4.63. We leave the details for the reader. □

Exercise 6.2.1. Show Remark 6.6 and Lemma 6.7.

Exercise 6.2.2. Let $U \subset \mathbb{R}^d$ be an open bounded set $\{V_n\}_{n\geqslant 1}$ be an increasing sequence (i.e., $V_n \subset V_{n+1}$) of bounded open sets such that and $U = \bigcup_n V_n$. Then

$$\lim_{n\to\infty} \mathrm{Cap}(V_n) = \mathrm{Cap}(U) \quad \text{for } d \geqslant 3$$

and

$$\lim_{n\to\infty} \mathrm{Cap}_L(V_n) = \mathrm{Cap}_L(U) \quad \text{for } d = 2.$$

6.3 The equilibrium measure

We say that a property holds *q.e.* (quasi everywhere) if it holds except on a set of capacity zero.

Theorem 6.8 (Existence of equilibrium measure). *Let $E \subset \mathbb{R}^d$ be a compact set with* $\mathrm{Cap}(E) > 0$. *There exists a Radon probability measure μ supported on E such that*

$$\mathrm{Cap}(E) = \frac{1}{I(\mu)}.$$

Further, any such measure satisfies $U_\mu(x) = (\mathrm{Cap}\, E)^{-1}$ q.e. $x \in E$ and $U_\mu(x) \leqslant (\mathrm{Cap}\, E)^{-1}$ for all $x \in E$.

Proof. Remark first that, for the case $d = 2$, by contracting E suitably, we can assume that $\mathrm{diam}(E) \leqslant 1/2$, so that $\mathcal{E}(x - y) > 0$ for all $x, y \in E$.

Let

$$\gamma := \inf\{I(\mu) : \mathrm{supp}\mu \subset E \text{ and } \mu(E) = 1\}. \tag{6.5}$$

By the lower semicontinuity of I, see Lemma 6.1 c), there exists a measure μ realizing this infimum. Since all the measures in the infimum are supported in the compact set E, so is the minimizer μ, which is also a probability measure, see Theorems 4.62 and 4.63.

Next we claim that

$$U_\mu(x) \geqslant \gamma \text{ q.e. } x \in E. \tag{6.6}$$

We prove this claim by contradiction. Let

$$T_\varepsilon := \{x \in E : U_\mu(x) < \gamma - \varepsilon\}$$

and assume that $\mathrm{Cap}(T_\varepsilon) > 0$. Then there exists a probability measure τ supported on T_ε with $I(\tau) < \infty$. By Chebyshev and restricting τ if necessary, we may assume that $U_\tau(x) \leqslant K < \infty$ for a suitable $K > 0$. For $\delta \in (0,1)$, let

$$\mu_\delta := (1-\delta)\mu + \delta\tau,$$

which is also a probability measure. Note that

$$\begin{aligned} I(\mu_\delta) &= \iint \mathcal{E}(x-y)\,((1-\delta)d\mu(y) + \delta d\tau(y))\,((1-\delta)\,d\mu(x) + \delta\,d\tau(x)) \\ &= (1-\delta)^2 I(\mu) + 2\delta(1-\delta)\iint \mathcal{E}(x-y)\,d\mu\,d\tau + \delta^2 I(\tau) \\ &= \gamma - 2\delta\gamma + 2\delta\int U_\mu d\tau + o(\delta^2) \leqslant \gamma - 2\delta\gamma + 2\delta(\gamma-\varepsilon) + o(\delta^2) < \gamma \end{aligned}$$

for δ small enough. This contradicts the fact that μ minimizes (6.5). Therefore, $\mathrm{Cap}(T_\varepsilon) = 0$ for every $\varepsilon > 0$, that is, the claim (6.6) holds, see Exercise 6.2.2.

We also claim that

$$U_\mu(x) \leqslant \gamma \text{ for every } x \in E. \tag{6.7}$$

Let $\nu := \mu|_{T_\varepsilon}$. Then $U_\nu(x) \leqslant U_\mu(x) < \gamma - \varepsilon$ for $x \in T_\varepsilon$. By the maximum principle U_ν is bounded and therefore $\nu(T_\varepsilon) = 0$ (see Remark 6.6), i.e., $\mu(T_\varepsilon) = 0$. Since $T_\varepsilon \nearrow T_0$, by Lemma 4.3 we get that $\mu(T_0) = 0$. We have that

$$\gamma = I(\mu) = \int_{\{U_\mu > \gamma\}} U_\mu\,d\mu + \int_{\{U_\mu = \gamma\}} U_\mu\,d\mu + \int_{\{U_\mu < \gamma\}} U_\mu\,d\mu.$$

The third integral is zero and therefore, since μ is a probability measure, we infer that the first integral must be zero as well, so $\mu(\{U_\mu > \gamma\}) = 0$ and therefore (6.7) holds μ-almost everywhere. The lower semicontinuity property of U_μ (see Lemma 6.1 a)) implies that (6.7) holds everywhere in the support of μ and by the maximum principle it holds everywhere. □

We will show soon that, for a compact set E with positive capacity, the probability measure μ supported on E such that $\mathrm{Cap}(E) = \frac{1}{I(\mu)}$ is unique. This probability measure μ is called the equilibrium measure of E, and its potential U_μ, the equilibrium potential of E.

Corollary 6.9. *Let E be compact with $\mathrm{Cap}(E) > 0$ and let μ be an equilibrium measure of E. Let ν be another Radon measure and let $A = \{x \in E : U_\nu(x) < \infty\}$. Then U_μ equals $(\mathrm{Cap}\,E)^{-1}$ ν-a.e. in A.*

Proof. In the case $d = 2$, we assume that $E \subset B_{1/2}(0)$. For $k > 1$, let $A_k = \{x \in E : U_\nu(x) \leqslant k \text{ and } U_\mu(x) < (\mathrm{Cap}(E))^{-1}\}$. If $\nu(A_k) > 0$, then the (non-zero) measure $\tau = \nu|_{A_k}$ satisfies

$$U_\tau(x) \leqslant U_\nu(x) \leqslant k \quad \text{ for all } x \in A_k.$$

So we deduce that $I(\tau) < +\infty$ and so $\mathrm{Cap}(A_k) > 0$. This contradicts the fact that $U_\mu(x) = (\mathrm{Cap}(E))^{-1}$ q.e. in E. □

Before proving the uniqueness of the equilibrium measure, we need to prove the following positivity result for the energy of signed measures. Remark that for a signed measure, its potential and its energy are defined in the same way as in (6.1) and (6.2), as soon as the corresponding integrals make sense.

Theorem 6.10. *Let ν be a compactly supported Radon signed measure in $\mathbb{R}^d$ such that $I(|\nu|) < \infty$. Assume also that $\nu(\mathbb{R}^d) = 0$ in the case $d = 2$. Then*

$$I(\nu) \geqslant 0.$$

Further, $I(\nu) > 0$ unless $\nu = 0$.

The fact that $I(\nu)$ is always non-negative (under the assumptions above) is quite remarkable. Observe that in the case $d = 2$ the assumption that $\nu(\mathbb{R}^d) = 0$ cannot be eliminated. Indeed, if E is a compact set with $\mathrm{Cap}_L(E) > 1$, then its equilibrium measure μ satisfies $I(\mu) < 0$.

Proof. Assume first that, besides satisfying the assumptions in the theorem, ν is of the form $\nu = g\,m$, where m is the Lebesgue measure in $\mathbb{R}^d$ and $g \in C_c^\infty(\mathbb{R}^d)$. Then $\mathcal{E} * g$ is a C^∞ function and we have

$$g = -\Delta(\mathcal{E} * g).$$

In the case $d \geqslant 3$, since $0 \leqslant \mathcal{E}(x) \lesssim |x|^{2-d}$, we have

$$|\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d-2}} \quad \text{and} \quad |\nabla\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d-1}} \tag{6.8}$$

as $x \to \infty$. Then, by integrating by parts, it easily follows that

$$I(gm) = \int (\mathcal{E} * g)\, g\, dm = -\int (\mathcal{E} * g)\, \Delta(\mathcal{E} * g)\, dm \overset{(6.8)}{=} \int |\nabla(\mathcal{E} * g)|^2\, dm \tag{6.9}$$

(notice that all the integrals above make sense because of (6.8). In the case $d = 2$, since $\nu(\mathbb{R}^d) = 0$, it is immediate to check that we have the improved decay

$$|\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d-1}} \quad \text{and} \quad |\nabla\mathcal{E} * g(x)| \lesssim_g \frac{1}{|x|^{d}} \tag{6.10}$$

as $x \to \infty$. Then we can integrate by parts again to deduce that (6.9) also holds. In any case, in particular, the identity (6.9) shows that $I(gm) \geqslant 0$.

Consider now an arbitrary signed measure satisfying the assumptions of the theorem. Consider a radial non-increasing C^∞ bump function φ such that $0 \leqslant \varphi \leqslant \chi_{B_2(0)}$ with $\int \varphi = 1$ and, for $\varepsilon > 0$, set $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(\varepsilon^{-1}x)$. Then the measure $\nu_\varepsilon = \varphi_\varepsilon * \nu$ is of the form $\nu_\varepsilon = g_\varepsilon\, m$, with $g_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, and has zero mean in the case $d = 2$. So by (6.9) it holds

$$I(\nu_\varepsilon) = \int |\nabla\mathcal{E} * \nu_\varepsilon|^2\, dm \geqslant 0. \tag{6.11}$$

So to prove that $I(\nu) \geqslant 0$ it suffices to show that $I(\nu_\varepsilon) \to I(\nu)$ as $\varepsilon \to 0$. To this end, applying Fubini we write

$$I(\nu_\varepsilon) = \int (\varphi_\varepsilon * \mathcal{E} * \nu)\, \varphi_\varepsilon * \nu \, dm = \int (\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E} * \nu)\, d\nu.$$

Observe now that, for any $x \in \mathbb{R}^d$, since $\varphi_\varepsilon * \varphi_\varepsilon$ is C^∞ with unitary mass, radial non-increasing, and compactly supported, then it is a convex combination of functions of the form $\frac{1}{m(B_r(0))}\chi_{B_r(0)}$ (see the proof of Lemma 5.10). Since $\mathcal{E}$ is superharmonic, by Lemma 5.10,

$$\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E}(x) \leqslant \mathcal{E}(x) \quad \text{for all } x \in \mathbb{R}^d \tag{6.12}$$

(this could also be checked by a direct computation), and also $\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E}(x) \to \mathcal{E}(x)$ as $\varepsilon \to 0$ for all $x \neq 0$.

We claim that in the case $d = 2$ we can assume that $\operatorname{supp}\nu \subset B_{1/4}(0)$. Indeed, for any $\lambda > 0$, consider the dilation $T_\lambda x = \lambda x$. Then, for a suitable $\lambda > 0$, it turns out that the image measure $(T_\lambda)_\# \nu$ is supported on $B_{1/4}(0)$ and it satisfies

$$\begin{aligned} I((T_\lambda)_\#\nu) &= \frac{1}{2\pi} \iint \log \frac{1}{|x-y|}\, d(T_\lambda)_\#\nu(x)\, d(T_\lambda)_\#\nu(y) \\ &= \frac{1}{2\pi} \iint \log \frac{1}{|\lambda x - \lambda y|}\, d\nu(x)\, d\nu(y) = I(\nu) - \frac{1}{2\pi}\nu(\mathbb{R}^d)^2 \log\lambda = I(\nu), \end{aligned}$$

which yields the claim.

So for any $d \geqslant 2$ and ε small enough we can assume that $\mathcal{E}(x-y) > 0$ for all $x, y \in \operatorname{supp}\nu \cup \operatorname{supp}\nu_\varepsilon$. Then, by the dominated convergence theorem, for all $x \in \operatorname{supp}\nu$ such that $\mathcal{E} * |\nu|(x) < \infty$, taking into account (6.12) and the fact that $\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E}(x) \to \mathcal{E}(x)$ for all $x \neq 0$, it follows that

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E} * \nu(x) = \mathcal{E} * \nu(x),$$

and moreover $\mathcal{E} * \nu(x) \leqslant \mathcal{E} * |\nu|(x)$. By another application of dominated convergence, since $I(|\nu|) < \infty$, we infer that

$$\lim_{\varepsilon \to 0} I(\nu_\varepsilon) = \lim_{\varepsilon \to 0} \int (\varphi_\varepsilon * \varphi_\varepsilon * \mathcal{E} * \nu)\, d\nu = I(\nu), \tag{6.13}$$

which concludes the proof of the fact that $I(\nu) \geqslant 0$.

Next suppose that $I(\nu) = 0$. From (6.11) and (6.13), we deduce that

$$\lim_{\varepsilon \to 0} \int |\nabla \mathcal{E} * \nu_\varepsilon|^2 \, dm = 0.$$

By an easy application of Fubini's theorem, it follows that $\mathcal{E} * \nu \in L^1_{loc}(\mathbb{R}^d)$. Now, we can compute the distributional Laplacian of the induced distribution, which happens to be precisely $\Delta(\mathcal{E} * \nu) = -\nu$. On the other hand, it is well known that $\mathcal{E} * \nu_\varepsilon = \varphi_\varepsilon * \mathcal{E} * \nu$

tends to $\mathcal{E} * \nu$ in $L^1_{loc}(\mathbb{R}^d)$, that is in $L^1(B_r(0))$ for any $r > 0$. Together with the Poincaré inequality, denoting by $m_{B_r(0)}(\mathcal{E} * \nu)$ the mean of $\mathcal{E} * \nu$ in $B_r(0)$, this implies

$$\begin{aligned}\fint_{B_r(0)} |\mathcal{E} * \nu - m_{B_r(0)}(\mathcal{E} * \nu)|\, dm &= \lim_{\varepsilon \to 0} \fint_{B_r(0)} |\mathcal{E} * \nu_\varepsilon - m_{B_r(0)}(\mathcal{E} * \nu_\varepsilon)|\, dm \\ &\lesssim \lim_{\varepsilon \to 0} \left(\fint_{B_r(0)} |\nabla(\mathcal{E} * \nu_\varepsilon)|^2\, dm \right)^{1/2} r = 0.\end{aligned}$$

So we deduce that $\mathcal{E} * \nu$ is constant a.e. with respect to Lebesgue measure. Since this happens for any ball $B_r(0)$ and $\mathcal{E} * \nu$ tends to 0 at ∞, it turns out that $\mathcal{E} * \nu$ vanishes a.e. Then, from the fact that $\nu = -\Delta(\mathcal{E} * \nu)$ in the sense of distributions, we infer that $\nu = 0$. □

Theorem 6.11. *Let $E \subset \mathbb{R}^d$ be a compact set with* $\mathrm{Cap}(E) > 0$. *Then the equilibrium measure for E is unique.*

From now on we will usually refer to the equilibrium measure for E as μ_E.

Proof. Aiming for a contradiction, suppose that there are two equilibrium measures μ and ν for E. For $t \in (0, 1)$, consider the measure

$$\sigma_t = t\, \mu + (1 - t)\, \nu.$$

Obviously, σ_t is a probability measure. Let us see that $I(\sigma_t) < I(\mu)$ for t small enough. Indeed, we have

$$\begin{aligned}I(\sigma_t) = \int \mathcal{E} * \sigma_t\, d\sigma_t &= t^2\, I(\mu) + t(1-t) \int \mathcal{E} * \mu\, d\nu + t(1-t) \int \mathcal{E} * \nu\, d\mu + (1-t)^2\, I(\nu) \\ &= (1 - 2t)\, I(\nu) + t \int \mathcal{E} * \mu\, d\nu + t \int \mathcal{E} * \nu\, d\mu + O(t^2).\end{aligned}$$

The sum of the two integrals on the right hand side can be rewritten as

$$\begin{aligned}\int \mathcal{E} * \mu\, d\nu + \int \mathcal{E} * \nu\, d\mu &= \int \mathcal{E} * (\mu - \nu)\, d\nu + I(\nu) + \int \mathcal{E} * (\nu - \mu)\, d\mu + I(\mu) \\ &= 2I(\nu) - \int \mathcal{E} * (\mu - \nu)\, d(\mu - \nu) = 2\, I(\nu) - I(\mu - \nu)\end{aligned}$$

From the identities above, we deduce

$$I(\sigma_t) = (1 - 2t)\, I(\nu) + 2t\, I(\nu) - tI(\mu - \nu) + O(t^2) = I(\nu) - tI(\mu - \nu) + O(t^2).$$

By Theorem 6.10, if $\mu \neq \nu$, then $I(\mu - \nu) > 0$, and so $I(\sigma_t) < I(\nu) = I(\mu)$ for t small enough, which yields the desired contradiction. □

From now on, $M_+(E)$ stands for the set of (non-negative) Radon measure supported on E.

Theorem 6.12. *Let $E \subset \mathbb{R}^d$ be compact, and suppose also that* $\operatorname{diam}(E) < 1$ *in the case* $d = 2$*. Then we have*

$$\operatorname{Cap}(E) = \sup\Big\{\mu(E) : \mu \in M_+(E),\ \sup_{\mathbb{R}^d} U_\mu \leqslant 1\Big\}. \tag{6.14}$$

Proof. The fact that $\operatorname{diam}(E) < 1$ in the case $d = 2$ implies that $\mathcal{E}(x-y) \geqslant \frac{1}{2\pi}\log\frac{1}{\operatorname{diam}(E)} > 0$ for all $x, y \in E$, which in turn implies that $I(\mu)$ is positive and bounded away from 0 for any measure μ supported on E, and so $\operatorname{Cap}_W(E) = \operatorname{Cap}(E) \geqslant 0$.

Denote by S_E the supremum in (6.14). In case $\operatorname{Cap}(E) = 0$, then every $\mu \in M_+(E)$ satisfies $I(\mu) = +\infty$. In particular, we infer that the potential U_μ is not bounded above in the support of μ. Thus, the only measure in the left-hand side of (6.14) is the null measure and $S_E = 0 = \operatorname{Cap}(E)$.

Let us assume $\operatorname{Cap}(E) > 0$. The fact that $\operatorname{Cap}(E) \geqslant S_E$ is immediate: for $\varepsilon > 0$, let μ be supported on E such that $\sup_{\mathbb{R}^d} U_\mu \leqslant 1$ and such that $\mu(E) + \varepsilon \geqslant S_E$. Consider the probability measure $\nu = \mu(E)^{-1}\mu$. Then

$$I(\nu) = \mu(E)^{-2}\, I(\mu) = \mu(E)^{-2}\int U_\mu(x)\, d\mu(x) \leqslant \mu(E)^{-1}.$$

Therefore,

$$\operatorname{Cap}(E) \geqslant I(\nu)^{-1} \geqslant \mu(E) \geqslant S_E - \varepsilon.$$

For the converse inequality, consider the equilibrium measure μ_E of E, so that $U_{\mu_E}(x) \leqslant \operatorname{Cap}(E)^{-1}$ for all $x \in \mathbb{R}^d$, by Theorem 6.8 and Theorem 6.3. Then the measure $\mu = \operatorname{Cap}(E)\,\mu_E$ satisfies $\sup_{\mathbb{R}^d} U_\mu \leqslant 1$ in $\mathbb{R}^d$ and thus $S_E \geqslant \mu(E) = \operatorname{Cap}(E)$. □

Remark 6.13. Note that the supremum in (6.14) is attained for E uniquely by the measure $\operatorname{Cap}(E)\,\mu_E$, where μ_E stands for the equilibrium measure of E. This can be shown arguing as in Theorem 6.12.

Corollary 6.14 (Subadditivity of capacity)**.** *For Borel sets $E_n \subset \mathbb{R}^d$, with* $\operatorname{diam}(\bigcup_n E_n) < 1$ *in the case $d = 2$, we have*

$$\operatorname{Cap}\Big(\bigcup_n E_n\Big) \leqslant \sum_n \operatorname{Cap}(E_n).$$

Proof. Let $F \subset \bigcup_n E_n$ be compact and let μ be supported on $\bigcup_n E_n$ be such that $\|U_\mu\|_\infty \leqslant 1$ in $\mathbb{R}^d$ and $\mu(F) = \operatorname{Cap}(F)$. Then $\|U_{\chi_{E_n \cap F}\mu}\|_\infty \leqslant \|U_\mu\|_\infty \leqslant 1$ for any n, and thus $\mu(E_n \cap F) \leqslant \operatorname{Cap}(E_n \cap F) \leqslant \operatorname{Cap}(E_n)$. Therefore,

$$\operatorname{Cap}(F) = \mu(F) \leqslant \sum_n \mu(E_n \cap F) \leqslant \sum_n \operatorname{Cap}(E_n).$$

Since this holds for any compact set $F \subset \bigcup_n E_n$, we are done since, by the definition of capacity,

$$\operatorname{Cap}(E) = \sup_{F \subset E : F \text{ is compact}} \operatorname{Cap} F.$$

□

Lemma 6.15. *For any Radon measure μ in $\mathbb{R}^d$ with compact support and let $\lambda > 0$. In the case $d \geqslant 3$ we have*

$$\operatorname{Cap}\left(\left\{x \in \mathbb{R}^d : U_\mu(x) \geqslant \lambda\right\}\right) \leqslant \frac{\|\mu\|}{\lambda}.$$

In the case $d = 2$,

$$\operatorname{Cap}\left(\left\{x \in B_{1/2}(0) : U_\mu(x) \geqslant \lambda\right\}\right) \leqslant \frac{\|\mu\|}{\lambda}.$$

Proof. Consider a compact set $E \subset \left\{x \in \mathbb{R}^d : U_\mu(x) \geqslant \lambda\right\}$ (in the case $d = 2$, $E \subset \left\{x \in B_{1/2}(0) : U_\mu(x) \geqslant \lambda\right\}$) and let ν be supported on E be such that $\sup_{\mathbb{R}^d} U_\nu \leqslant 1$ and $\operatorname{Cap}(E) = \nu(E)$. Then we have

$$\operatorname{Cap}(E) = \nu(E) \leqslant \frac{1}{\lambda}\int U_\mu \, d\nu = \frac{1}{\lambda}\int U_\nu \, d\mu \leqslant \frac{\|\mu\|}{\lambda}.$$

Taking the supremum on such sets E, the lemma follows. $\square$

Proposition 6.16. *For a ball $\bar{B} \subset \mathbb{R}^d$, we have*

$$\operatorname{Cap}(\bar{B}) = (d-2)\kappa_d \, r(\bar{B})^{d-2} \qquad \text{if } d \geqslant 3,$$

and

$$\operatorname{Cap}_L(\bar{B}) = r(\bar{B}) \qquad \text{if } d = 2.$$

Proof. Without loss of generality, assume that $\bar{B}$ is centered in the origin and that it is closed. In the case $d = 2$, by homogeneity we can assume $r(\bar{B}) < 1/2$. Let $x \in \bar{B}^c$ and notice that $\mathcal{E}^x(y) := \mathcal{E}(x-y)$ is harmonic in the interior of $\bar{B}$. Let σ be the surface measure on $\partial\bar{B}$. Then by the mean value theorem,

$$U_\sigma(x) = \int_{\partial\bar{B}} \mathcal{E}(x-y)\, d\sigma(y) = \sigma(\partial\bar{B})\, \mathcal{E}(x-0) = \sigma(\partial\bar{B})\, \mathcal{E}(x).$$

Note that U_σ is constant in $\partial\bar{B}$ by symmetry, and therefore it is continuous in $\mathbb{R}^d$ by the continuity principle. Thus, the same identity holds on $\partial\bar{B}$. Therefore, using also the maximum principle, in the case $d \geqslant 3$, we get

$$\sup_{\mathbb{R}^d} U_\sigma = \sup_{\partial\bar{B}} U_\sigma = \sigma(\partial\bar{B})\, \mathcal{E}(r(\bar{B})) = \frac{\kappa_d \, r(\bar{B})^{d-1}}{(d-2)\kappa_d \, r(\bar{B})^{d-2}} = \frac{r(\bar{B})}{d-2}.$$

Therefore, the measure $\mu = (d-2)r(\bar{B})^{-1}\sigma$ satisfies $\sup_{\mathbb{R}^d} U_\mu = 1$ and so

$$\operatorname{Cap}(\bar{B}) \geqslant \mu(\bar{B}) = (d-2)r(\bar{B})^{-1}\sigma(\bar{B}) = (d-2)\kappa_d \, r(\bar{B})^{d-2}.$$

For the converse estimate, remark that in fact the measure μ satisfies $U_\mu \equiv 1$ in $\partial\bar{B}$. Since μ is supported on $\partial\bar{B}$ and U_μ is harmonic in the interior of $\bar{B}$ and continuous in its closure, by the maximum principle it is identically 1 in the whole $\bar{B}$. Then, from Lemma

6.15 we deduce that $\mathrm{Cap}(\bar{B}) \leqslant \mu(\bar{B}) = (d-2)\kappa_d\, r(\bar{B})^{d-2}$, which proves the lemma in the case $d \geqslant 3$.

In the case $d = 2$ we argue analogously. Indeed, it is straightforward to check that, for all $x \in \partial\bar{B}$ we have we have $U_\sigma(x) = r(\bar{B}) \log \frac{1}{r(\bar{B})}$. Then, by the same arguments as before, it follows that

$$\mathrm{Cap}_W(\bar{B}) = \frac{2\pi}{\log \frac{1}{r(\bar{B})}},$$

and so $\mathrm{Cap}_L(\bar{B}) = r(\bar{B})$. □

As a corollary of the preceding estimate for the logarithmic capacity, we obtain:

Corollary 6.17. *Let μ be Radon measure supported on the (open) ball $B_1(0) \subset \mathbb{R}^2$. Then $I(\mu) > 0$.*

Proof. Let $E = \mathrm{supp}\mu$. Since $E \subset B_1(0)$, there exists some $\rho \in (0,1)$ such that $E \subset B_\rho(0)$. Consequently, $\mathrm{Cap}_L(E) \leqslant \mathrm{Cap}_L(\bar{B}_\rho(0)) = \rho < 1$. Thus, $e^{-2\pi I(\mu)} < 1$, which implies that $I(\mu) > 0$. □

A quick inspection of the arguments above shows that $\mathrm{Cap}(\bar{B}) = \mathrm{Cap}(\partial\bar{B})$ for any ball. This also holds for any arbitrary compact set. In fact, we show below that the capacity of a compact set equals the capacity of its outer boundary. For $E \subset \mathbb{R}^d$ compact, its outer boundary, denoted by $\partial_o E$, is the boundary of the unbounded component of $\mathbb{R}^d \backslash E$.

Theorem 6.18. *For any compact set $E \subset \mathbb{R}^d$, we have $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$ (and so $\mathrm{Cap}_L(E) = \mathrm{Cap}_L(\partial_o E)$ in the case $d = 2$).*

Proof. First we show that $\mathrm{Cap}(E) = \mathrm{Cap}(\partial E)$. To this end, it suffices to show that the equilibrium measure μ of E is supported on ∂E (in the case $d = 2$, if necessary, we can assume that $E \subset B_{1/2}(0)$). To prove this, recall that by Theorem 6.8 $U_\mu(x) = (\mathrm{Cap}\, E)^{-1}$ q.e. $x \in E$. In particular, this holds a.e. in the interior of E with respect to Lebesgue measure, see Remark 6.6. Since $-\Delta U_\mu = \mu$ in the sense of distributions, for any C^∞ function φ supported on the interior of E, it holds

$$\int \varphi\, d\mu = -\langle U_\mu, \Delta\varphi \rangle = -(\mathrm{Cap}\, E)^{-1} \int_{\mathrm{supp}\varphi} \Delta\varphi = 0.$$

Thus μ vanishes identically on the interior of E, which shows that $\mathrm{supp}\mu \subset \partial E$.

To show that $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$, let Ω be the unbounded component of $\mathbb{R}^d \backslash E$ and let $\widehat{E} = \mathbb{R}^d \backslash \Omega$ (so that $\widehat{E}$ coincides with the union of E and the bounded components of $\mathbb{R}^d \backslash E$). Then we have $\partial_o E = \partial \widehat{E}$ and

$$\partial_o E \subset \partial E \subset E \subset \widehat{E}.$$

Since $\mathrm{Cap}(\widehat{E}) = \mathrm{Cap}(\partial_o E)$, we also have $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$. □

Remark 6.19. From the uniqueness of the equilibrium measure and the fact that $\mathrm{Cap}(E) = \mathrm{Cap}(\partial_o E)$, it follows that the equilibrium measure of E is supported on $\partial_o E$.

6.4 Relationship between Hausdorff content and capacity

Lemma 6.20. *Let $E \subset \mathbb{R}^d$ be compact and $d-2 < s \leqslant d$. In the case $d \geqslant 3$, we have*

$$\mathcal{H}^s_\infty(E)^{\frac{d-2}{s}} \lesssim_{s,d} \operatorname{Cap}(E) \lesssim_d \mathcal{H}^{d-2}_\infty(E).$$

In the case $d = 2$, we have

$$\operatorname{Cap}_L(E) \gtrsim_s \mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

Proof. First we consider the case $d \geqslant 3$. To check that $\operatorname{Cap}(E) \lesssim \mathcal{H}^{d-2}_\infty(E)$, for any $\varepsilon > 0$ we consider a covering of E by a family of open balls B_i, $i \geqslant 1$, such that

$$\sum_i r(B_i)^{d-2} \lesssim_d \mathcal{H}^{d-2}_\infty(E) + \varepsilon.$$

Since E is compact, we may assume that the family of balls B_i is finite. Then, using the subadditivity of the Newtonian capacity (see Corollary 6.14) and Proposition 6.16, we get

$$\operatorname{Cap}(E) \leqslant \sum_i \operatorname{Cap}(\bar{B}_i) \approx \sum_i r(B_i)^{d-2} \lesssim_d \mathcal{H}^{d-2}_\infty(E) + \varepsilon,$$

which shows that $\operatorname{Cap}(E) \lesssim_d \mathcal{H}^{d-2}_\infty(E)$.

To see that $\operatorname{Cap}(E) \gtrsim_{s,d} \mathcal{H}^s_\infty(E)^{\frac{d-2}{s}}$, we apply Frostman's Lemma 4.66. This tells us that there exists some Borel measure μ supported on E such that

$$\mathcal{H}^s_\infty(E) \approx_d \mu(E) \tag{6.15}$$

and

$$\mu(B_r(x)) \leqslant r^s \qquad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0. \tag{6.16}$$

Then, for all $x \in \mathbb{R}^d$ we have

$$\begin{aligned} c\,U_\mu(x) &= \int \frac{1}{|x-y|^{d-2}}\,d\mu(y) = \int_0^\infty \mu\big(\{y : |x-y|^{2-d} > t\}\big)\,dt \\ &= \int_0^\infty \mu\big(B\big(x, t^{\frac{1}{2-d}}\big)\big)\,dt \overset{(6.16)}{\leqslant} \int_0^{\mu(E)^{\frac{2-d}{s}}} \mu(E)\,dt + \int_{\mu(E)^{\frac{2-d}{s}}}^\infty t^{\frac{s}{2-d}}\,dt \approx_{s,d} \mu(E)^{1-\frac{d-2}{s}}. \end{aligned}$$

Therefore,

$$\operatorname{Cap}(E) \overset{(6.14)}{\geqslant} \frac{\mu(E)}{\|U_\mu\|_\infty} \gtrsim_{s,d} \frac{\mu(E)}{\mu(E)^{1-\frac{d-2}{s}}} = \mu(E)^{\frac{d-2}{s}} \overset{(6.15)}{\approx}_d \mathcal{H}^s_\infty(E)^{\frac{d-2}{s}}.$$

In the case $d = 2$, we may and will assume that $\operatorname{diam}(E) < 1$ since, for any $\lambda > 0$.

$$\operatorname{Cap}_L(\lambda E) = \lambda \operatorname{Cap}_L(E) \quad \text{and} \quad \mathcal{H}^s_\infty(\lambda E)^{\frac{1}{s}} = \lambda\,\mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

We apply again Frostman's Lemma to get a measure μ supported on E satisfying (6.15) and (6.16). Then, for any $\tau \geqslant 0$ for $x \in \operatorname{supp}\mu$ we have

$$\begin{aligned} 2\pi\, U_\mu(x) &= \int \log \frac{1}{|x-y|}\, d\mu(y) = \int_0^\infty \mu\Big(\Big\{y : \log \frac{1}{|x-y|} > t\Big\}\Big)\, dt \\ &= \int_0^\infty \mu\big(B\big(x, e^{-t}\big)\big)\, dt \overset{(6.16)}{\leqslant} \int_0^\tau \mu(E)\, dt + \int_\tau^\infty e^{-ts}\, dt = \tau\, \mu(E) + \frac{1}{s} e^{-\tau s}. \end{aligned}$$

We choose $\tau = -\frac{1}{s} \log \mu(E)$ (notice that $\tau \geqslant 0$ because $\mu(E) \overset{(6.16)}{<} 1$, since $\operatorname{diam}(E) < 1$), and then we obtain

$$2\pi\, U_\mu(x) \leqslant \frac{\mu(E)}{s} \left(\log \frac{1}{\mu(E)} + 1 \right).$$

Hence, for the probability measure $\sigma = \mu(E)^{-1}\mu$, we have

$$2\pi\, I(\sigma) \leqslant \frac{1}{s} \left(\log \frac{1}{\mu(E)} + 1 \right).$$

Therefore,

$$\operatorname{Cap}_W(E) \geqslant \frac{1}{I(\sigma)} \geqslant \frac{2\pi s}{\log \frac{1}{\mu(E)} + 1},$$

or equivalently,

$$\operatorname{Cap}_L(E) \geqslant e^{\frac{\log \mu(E) - 1}{s}} = C(s)\, \mu(E)^{\frac{1}{s}} \overset{(6.15)}{\approx}_s \mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

□

Comparing the previous lemma with definition 4.65, we get the criticallity of dimension $d-2$.

Corollary 6.21. *Let $E \subset \mathbb{R}^d$ be a compact set. If $\operatorname{Cap} E > 0$, then $\dim_{\mathcal{H}} E \geqslant d-2$. Instead, if $\operatorname{Cap} E = 0$, then $\dim_{\mathcal{H}} E \leqslant d-2$.*

Remark 6.22. It can be shown that if $\mathcal{H}^{d-2}(E) < \infty$ for a bounded set $E \subset \mathbb{R}^d$, then $\operatorname{Cap}(E) = 0$. See [Mat95, Theorem 8.7], for example.

6.5 Wiener's criterion

Given a bounded open set $\Omega \subset \mathbb{R}^d$, by Theorem 5.34 and Theorem 5.35, a point $\xi \in \partial\Omega$ is regular (for the Dirichlet problem) if and only if there is a barrier function for ξ in Ω. In this section we show a characterization of more metric-geometric type. This is the so called Wiener's criterion.

Theorem 6.23 (Wiener's criterion)**.** *For $d \geqslant 2$, let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $\xi \in \partial\Omega$. The following are equivalent:*

(a) ξ is a regular point.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty.$

Here $\bar{A}(\xi, r_1, r_2)$ denotes the closed annulus centered at ξ with inner radius r_1 and outer radius r_2. Recall also that in the case $d \geqslant 3$, $\mathrm{Cap}(\bar{B}(\xi, 2^{-k})) \approx 2^{-k(d-2)}$, and in the case $d = 2$, $\mathrm{Cap}(\bar{B}(\xi, 2^{-k})) = \mathrm{Cap}_W(\bar{B}(\xi, 2^{-k})) \approx 1/k$. Thus, in the latter case, the condition (b) is equivalent to

(b') $\displaystyle\sum_{k=1}^{\infty} k \ \mathrm{Cap}_W(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash\Omega) = \infty.$

Remark 6.24. In the case $d \geqslant 3$, the condition (b) is equivalent to

(b”) $\displaystyle\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{B}(\xi, 2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty.$

Indeed, it is trivial that (b) $\Rightarrow$ (b”). To see that (b”) $\Rightarrow$ (b) we use the subadditivity of Newtonian capacity to write

$$\begin{aligned}\sum_{k\geqslant 1} \frac{\mathrm{Cap}(\bar{B}(\xi, 2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} &\lesssim \sum_{k\geqslant 1}\sum_{j\geqslant k} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-j-1}, 2^{-j})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} \\ &= \sum_{j\geqslant 1} \mathrm{Cap}(\bar{A}(\xi, 2^{-j-1}, 2^{-j})\backslash\Omega) \sum_{k\leqslant j} \frac{1}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))}.\end{aligned}$$

Now observe that the last sum on the right hand side is comparable to $\sum_{k\leqslant j} 2^{k(d-2)} \approx 2^{j(d-2)} \approx \mathrm{Cap}(\bar{B}(\xi, 2^{-j}))^{-1}$. Thus,

$$\sum_{k\geqslant 1} \frac{\mathrm{Cap}(\bar{B}(\xi, 2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} \lesssim \sum_{j\geqslant 1} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-j-1}, 2^{-j})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-j}))},$$

which yields the desired implication.

6.5.1 Sufficiency of the criterion for Wiener regularity

Proof of (b) $\Rightarrow$ (a) in Theorem 6.23 in the case $d \geqslant 3$. We will construct a barrier $\widetilde{w} : \Omega \to \mathbb{R}$ for the point ξ. We will show that there exists a harmonic function $w : \Omega \to \mathbb{R}$ satisfying:

(i) $\lim_{\Omega\ni x\to\xi} w(x) = 1$.

(ii) $\limsup_{x\to\zeta} w(x) < 1$ for all $\zeta \in \partial\Omega\backslash\{\xi\}$.

Then we just have to take $\widetilde{w} = 1 - w$ to get the desired barrier.

To shorten notation, write $\bar{A}_k = \bar{A}(\xi, 2^{-k-1}, 2^{-k})$, $B_k = B(\xi, 2^{-k})$, and $\bar{B}_k = \overline{B_k}$. For a fixed large constant $\Lambda \geqslant 10$ to be chosen below and for any $n_0 > 1$, the condition (b) ensures the existence of natural numbers N, M, with $n_0 \leqslant N < M$ such that

$$\Lambda \leqslant \sum_{N \leqslant k \leqslant M} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} \leqslant \Lambda + 1$$

(notice that each summand in the sum above is at most 1). For each $k \geqslant n_0$, if $\mathrm{Cap}(\bar{A}_k \backslash \Omega) = 0$, define $\mu_k \equiv 0$ and if $\mathrm{Cap}(\bar{A}_k \backslash \Omega) > 0$ let μ_k be the equilibrium measure for $\bar{A}_k \backslash \Omega$. Consider the function

$$u_k(x) = \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, U_{\mu_k}(x);$$

and set

$$v(x) = \sum_{N \leqslant k \leqslant M} u_k(x).$$

Claim 6.25. *Let $d \geqslant 3$. For any $\varepsilon > 0$, if $\Lambda = \Lambda(\varepsilon)$ is chosen large enough, the function v satisfies*

$$v(\xi) \approx \Lambda, \tag{6.17}$$

$$v(x) \leqslant (1 + \varepsilon)\, v(\xi) \quad \textit{for all } x \in \Omega, \tag{6.18}$$

$$|v(x) - v(\xi)| \leqslant C \frac{|x - \xi|}{r(\bar{B}_M)}\, v(\xi) \quad \textit{for all } x \in \Omega \cap \bar{B}_M, \tag{6.19}$$

and

$$v(x) \leqslant \frac{1}{10}\, v(\xi) \quad \textit{for all } x \in \Omega \backslash \bar{B}_{N-k_0} \textit{ if } k_0 \geqslant 2 \textit{ is large enough.} \tag{6.20}$$

Remark that the constant k_0 in the last estimate does not depend on ε. In the case $N - k_0 \leqslant 0$, we understand that $\bar{B}_{N-k_0} = 2^{k_0} \bar{B}_N$.

Proof of the Claim. The estimate (6.17) is easy: for each $k \in [N, M]$ we have

$$u_k(\xi) = \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, U_{\mu_k}(\xi) \approx \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \mathcal{E}(r(B_k)) \approx \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)}.$$

Thus,

$$v(\xi) \approx \sum_{N \leqslant k \leqslant M} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} \approx \Lambda. \tag{6.21}$$

Next we turn our attention to (6.18), which is the most delicate part of the claim. Notice first that, by the maximum principle, it suffices to prove this for $x \in \bar{B}_N \backslash B_M = \bigcup_{N \leqslant i \leqslant M} \bar{A}_i$. So fix $x \in A_i$, with $N \leqslant i \leqslant M$. For some $h \geqslant 1$ to be chosen soon, we write

$$v(x) = \sum_{k=N}^{i-h-1} u_k(x) + \sum_{k=N \vee i-h}^{M \wedge i+h} u_k(x) + \sum_{k=i+h+1}^{M} u_k(x) =: v_a(x) + v_b(x) + v_c(x).$$

To estimate $v_b(x)$ we just take into account that

$$u_k(y) \leqslant \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, U_{\mu_k}(y) \leqslant 1 \quad \text{for all } y \in \mathbb{R}^d,$$

by Theorem 6.8. So we deduce

$$v_b(x) \leqslant 2h + 1.$$

To deal with $v_a(x)$, we will use the fact that, $|x-\xi| \leqslant r(\bar{B}_i) < 2^{-k}2^{-h}$ for $k < i - h$, implying

$$\begin{aligned} u_k(x) &= u_k(\xi) + (u_k(x) - u_k(\xi)) = u_k(\xi) + \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, (U_{\mu_k}(x) - U_{\mu_k}(\xi)) \qquad (6.22) \\ &\leqslant u_k(\xi) + C\, \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \frac{|x-\xi|}{\mathrm{dist}(\xi, \bar{A}_k)^{d-1}} \\ &\leqslant u_k(\xi) + C\, \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \frac{r(\bar{B}_i)}{r(\bar{B}_k)^{d-1}} \\ &\leqslant u_k(\xi) + C\, 2^{-h}\, \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)}. \end{aligned}$$

For $v_c(x)$, we take into account that for $k > i + h$ we get $r(\bar{B}_i) > 2^h r(\bar{B}_k)$, so

$$u_k(x) \leqslant C\, \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{dist}(x, \bar{A}_k)^{d-2}} \leqslant C\, \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{r(\bar{B}_i)^{d-2}} \leqslant C\, 2^{-h(d-2)} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)}.$$

Consequently, gathering the estimates obtained for $k < i - h$ and for $k > i + h$ and using also (6.21), we get

$$v_a(x) + v_c(x) \leqslant \sum_{N \leqslant k \leqslant M} u_k(\xi) + C\, 2^{-h} \sum_{N \leqslant k \leqslant M} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} \leqslant v(\xi) + C\, 2^{-h}\, v(\xi).$$

Thus,

$$v(x) = v_a(x) + v_b(x) + v_c(x) \leqslant v(\xi) + (2h+1) + C\, 2^{-h}\, v(\xi) \leqslant v(\xi)\Big(1 + \frac{C\, h}{\Lambda} + C\, 2^{-h}\Big).$$

So choosing h large enough and then Λ large enough as well, (6.18) follows.

To prove (6.19), we can assume $x \in \frac{1}{2}\bar{B}_M$ because of (6.18). Arguing as in (6.22), we obtain

$$|u_k(x) - u_k(\xi)| \leqslant C\, \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \frac{|x-\xi|}{\mathrm{dist}(\xi, \bar{A}_k)^{d-1}} \lesssim C\, \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)}\, \frac{|x-\xi|}{r(\bar{B}_M)}.$$

Summing over $k \in [N, M]$ and using (6.21), we deduce (6.19).

Finally we deal with (6.20). So we take $x \in \Omega \backslash \bar{B}_{N-k_0}$, for $k_0 \geqslant 2$. Then we have

$$u_k(x) \approx \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{dist}(x, \bar{B}_k)^{d-2}} \leqslant \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{2^{(d-2)k_0}\, r(\bar{B}_k)^{d-2}} \approx 2^{(2-d)k_0}\, u_k(\xi).$$

Hence, summing on $k \in [N, M]$, we obtain

$$v(x) \lesssim 2^{(2-d)k_0} \sum_{N \leqslant k \leqslant M} u_k(\xi) = 2^{(2-d)k_0}\, v(\xi).$$

□

Applying the preceding claim, we construct sequences of natural numbers N_j, M_j, and functions v_j, for $j \geqslant 1$, as follows. We choose $N_0 = 1$, $M_0 = 2$. Assuming that $N_{j-1} < M_{j-1}$ have already been chosen, by applying Claim 6.25 with some $\varepsilon \in (0, 1/2)$ to be fixed below and $n_0 = M_{j-1} + k_0$, for some $k_0 \geqslant 2$ to be fixed below too, we find $M_j > N_j \geqslant n_0$ so that the function

$$v_j(x) = \sum_{N_j \leqslant k \leqslant M_j} u_k(x)$$

satisfies (6.17), (6.18), (6.19), and (6.20) (with v_j in place of v). Now we define

$$w(x) = \sum_{j \geqslant 1} 2^{-j} \frac{v_j(x)}{v_j(\xi)}. \tag{6.23}$$

Obviously, $w(\xi) = 1$ and it is easy to check that w is superharmonic in $\mathbb{R}^d$ (since each function v_j is superharmonic by Lemma 6.1). Consequently,

$$\liminf_{y \to \xi} w(y) \geqslant w(\xi) = 1. \tag{6.24}$$

Our next objective is to show that

$$\limsup_{y \to \zeta} w(y) < 1 \text{ for all } \zeta \in \partial\Omega \backslash \{\xi\} \;\text{ and }\; w(y) < 1 \text{ for all } y \in \Omega. \tag{6.25}$$

Observe that the latter condition together with (6.24) implies the condition (i) above, i.e., $\lim_{\Omega \ni y \to \xi} w(y) = 1$. To prove (6.25) it suffices to show that for any $h \geqslant 1$ there exists $\delta_h > 0$ such that

$$w(x) \leqslant 1 - \delta_h \quad \text{ for all } x \in \bar{B}_{M_h} \backslash \bar{B}_{M_{h+1}}. \tag{6.26}$$

To prove this, for a given $x \in \bar{B}_{M_h} \backslash \bar{B}_{M_{h+1}}$, we split

$$w(x) = \sum_{j=1}^{h-1} 2^{-j} \frac{v_j(x)}{v_j(\xi)} + 2^{-h} \frac{v_h(x)}{v_h(\xi)} + 2^{-h-1} \frac{v_{h+1}(x)}{v_{h+1}(\xi)} + \sum_{j \geqslant h+2} 2^{-j} \frac{v_j(x)}{v_j(\xi)} =: S_1 + S_2 + S_3 + S_4. \tag{6.27}$$

By (6.19), the first sum satisfies

$$\begin{aligned} S_1 &= \sum_{j=1}^{h-1} 2^{-j} \frac{v_j(x)}{v_j(\xi)} \leqslant \sum_{j=1}^{h-1} 2^{-j} + \sum_{j=1}^{h-1} 2^{-j} \frac{|v_j(x) - v_j(\xi)|}{v_j(\xi)} \\ &\leqslant (1 - 2^{-h+1}) + C \sum_{j=1}^{h-1} 2^{-j} \frac{r(\bar{B}_{M_h})}{r(\bar{B}_{M_j})} \leqslant (1 - 2^{-h+1}) + C \sum_{j=1}^{h-1} 2^{-j}\, 2^{k_0(j-h)}, \end{aligned}$$

where we took into account that $r(\bar{B}_{M_{j+1}}) \leqslant 2^{-k_0} r(\bar{B}_{M_j})$ for each j, by the construction of the sequence M_j. For $k_0 \geqslant 3$, we have

$$\sum_{j=1}^{h-1} 2^{-j} 2^{k_0(j-h)} = \frac{2^{-h}}{2^{k_0-1} - 1} \leqslant \frac{2^{-h}}{2^{k_0-2}} = 2^{-h-k_0+2}.$$

Thus,

$$S_1 \leqslant (1 - 2^{-h+1}) + C2^{-h-k_0}.$$

For S_2 and S_3 we apply (6.18):

$$S_2 + S_3 \leqslant (1+\varepsilon)(2^{-h} + 2^{-h-1}).$$

Finally we estimate S_4. For this term we use the fact that if $x \notin \bar{B}_{M_{h+1}}$ and $j \geqslant h+2$, then by (6.20) we have $v_j(x) \leqslant \frac{1}{10}\, v_j(\xi)$, assuming k_0 large enough. Therefore,

$$S_4 \leqslant \frac{1}{10} \sum_{j \geqslant h+2} 2^{-j} = \frac{1}{10}\, 2^{-h-1}. \tag{6.28}$$

Gathering the estimates for $S_1, \ldots, S_4$, we obtain

$$\begin{aligned} w(x) &\leqslant (1 - 2^{-h+1}) + C2^{-h-k_0} + (1+\varepsilon)(2^{-h} + 2^{-h-1}) + \frac{1}{10}\, 2^{-h-1} \\ &= 1 - 2^{-h}\left(\frac{9}{20} - C2^{-k_0} - \frac{3\varepsilon}{2}\right). \end{aligned}$$

Then, choosing ε small enough and k_0 large, we derive $w(x) \leqslant 1 - 2^{-h-2}$, which proves (6.26) and completes the proof of (b) $\Rightarrow$ (a). $\square$

Proof of (b) $\Rightarrow$ (a) in Theorem 6.23 in the case $d = 2$. The proof is very similar to the one above for $d \geqslant 3$ and so we only point out the differences in the argument. Given $1 < n_0 \leqslant N < M$, we define the functions u_k and v as above. Then the estimates (6.17), (6.18), and (6.19) in Claim 6.25 also hold if Λ is chosen large enough, while for (6.20) we require now that $k_0 \geqslant 10N/11$ and N large enough.

The proof of this variant of Claim 6.25 for the case $d = 2$ is very similar to the one for $d = 3$. Indeed, (6.17) has the same proof. Regarding (6.18), we split $v(x) = v_a(x) + v_b(x) + v_c(x)$ as in the case $d \geqslant 3$. We have $v_b(x) \leqslant 2h + 1$ by the same arguments as for $d \geqslant 3$. To deal with $v_a(x)$ we estimate the functions u_k for $k < i - h$ by arguments quite similar to the ones in (6.22). Indeed, notice that

$$|U_{\mu_k}(x) - U_{\mu_k}(\xi)| \lesssim \int \left| \log \frac{|x-y|}{|\xi - y|} \right| d\mu_k(y)$$

Writing

$$\left| \log \frac{|x-y|}{|\xi-y|} \right| = \left| \log\left(1 + \frac{|x-y| - |x-\xi|}{|\xi - y|}\right) \right| \leqslant \frac{|x-\xi|}{|x-y|},$$

we deduce

$$|U_{\mu_k}(x) - U_{\mu_k}(\xi)| \lesssim \frac{|x-\xi|}{\operatorname{dist}(\xi, \bar{A}_k)}.$$

Thus,

$$\begin{aligned} u_k(x) &= u_k(\xi) + \mathrm{Cap}(\bar{A}_k \backslash \Omega)\,(U_{\mu_k}(x) - U_{\mu_k}(\xi)) \\ &\leqslant u_k(\xi) + C\ \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \frac{|x-\xi|}{\mathrm{dist}(\xi, \bar{A}_k)} \\ &\leqslant u_k(\xi) + C\ \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, \frac{r(\bar{B}_i)}{r(\bar{B}_k)} \\ &\leqslant u_k(\xi) + C\,2^{-h}\ \mathrm{Cap}(\bar{A}_k \backslash \Omega)\, U_{\mu_k}(\xi), \end{aligned} \tag{6.29}$$

where we used the trivial bound $U_{\mu_k}(\xi) \geqslant 1$ in the last inequality for N large enough. For $v_c(x)$, we take into account that for $k > i+h$ we have

$$\begin{aligned} u_k(x) &\leqslant \mathrm{Cap}(\bar{A}_k \backslash \Omega)\,\mathcal{E}(\mathrm{dist}(x, \bar{A}_k)) \leqslant \mathrm{Cap}(\bar{A}_k \backslash \Omega)\,\mathcal{E}(c\,r(\bar{B}_i)) \\ &\leqslant \mathrm{Cap}(\bar{A}_k \backslash \Omega) \int \mathcal{E}(\xi - y)\,d\mu_k(y)\ \frac{\mathcal{E}(c\,r(\bar{B}_i))}{\inf_{y\in\bar{A}_k} \mathcal{E}(\xi - y)} \leqslant u_k(\xi), \end{aligned}$$

since $\mathcal{E}(c\,r(\bar{B}_i)) \leqslant \inf_{y\in\bar{A}_k} \mathcal{E}(\xi - y)$ for $k > i+h$ with h large enough.

Consequently, gathering the estimates obtained for $k < i-h$ and for $k > i+h$ and using also (6.17) and (6.21), we get

$$v_a(x) + v_c(x) \leqslant (1 + C2^{-h}) \sum_{N\leqslant k\leqslant M} u_k(\xi) = (1 + C2^{-h})\, v(\xi).$$

Thus,

$$v(x) = v_a(x) + v_b(x) + v_c(x) \leqslant v(\xi) + (2h+1) + C\,2^{-h}\, v(\xi) \leqslant v(\xi)\Big(1 + \frac{C\,h}{\Lambda} + C\,2^{-h}\Big).$$

So choosing h large enough and then Λ large enough, we get (6.18).

The proof of (6.19) also follows by arguments very similar to the ones for the case $d = 2$ and so we skip them.

Finally we deal with (6.20). So we take $x \in \Omega \backslash \bar{B}_{N-k_0}$, for $k_0 \geqslant 10N/11$ and N large enough. For $x \in B_{1/2}(\xi)$, then we have

$$\begin{aligned} U_{\mu_k}(x) &= \int \mathcal{E}(x-y)\,d\mu_k(y) \leqslant \int \mathcal{E}(\xi - y)\,d\mu_k(y)\, \frac{\sup_{y\in\bar{A}_k} \mathcal{E}(x-y)}{\inf_{y\in\bar{A}_k} \mathcal{E}(\xi-y)} \\ &\leqslant U_{\mu_k}(\xi)\, \frac{\log(c\,2^{k_0}\, r(\bar{B}_N))}{\log(c'\, r(\bar{B}_N))} \leqslant U_{\mu_k}(\xi)\, \frac{C + N - k_0}{C' + N}. \end{aligned}$$

From the condition that $k_0 \geqslant 10N/11$ we deduce that $N - k_0 \leqslant N/11$, and thus for N large enough it holds $\frac{C+N-k_0}{C'+N} \leqslant \frac{1}{10}$. Hence, multiplying by $\mathrm{Cap}(\bar{A}_k \backslash \Omega)$ and summing on $k \in [N, M]$, we obtain

$$v(x) \leqslant \frac{1}{10} \sum_{N\leqslant k\leqslant M} u_k(\xi) = \frac{1}{10}\, v(\xi) \quad \text{ for all } x \in \Omega \backslash \bar{B}_{N-k_0}.$$

To complete the proof of (b) ⇒ (a) we choose sequences N_j and M_j as in the case $d \geqslant 3$, but with the additional requirement that $N_j \geqslant 20M_{j-1}$ for each j, say. This condition ensures that we will be able to apply (6.20) to estimate the term S_4 in (6.27) arguing as in (6.28). Then almost the same arguments as the ones for the case $d \geqslant 3$ show that the function w defined in (6.23) is barrier for ξ. We leave the details for the reader. □

6.5.2 Necessity of the criterion for Wiener regularity

Recall that in Definition 5.40 we introduced the notion of Wiener regularity for unbounded open sets with compact boundary. Before proving the necessity part in Theorem 6.23, i.e., the implication (a) ⇒ (b), we need the following auxiliary result.

Lemma 6.26. *Let $E \subset \mathbb{R}^d$ be compact with $\mathrm{Cap}(E) > 0$ and let Ω_E be the unbounded component of $\mathbb{R}^d \backslash E$. Suppose that Ω_E is Wiener regular and let μ be the equilibrium measure for E. Then the equilibrium potential U_μ is continuous in $\mathbb{R}^d$ and $U_\mu = (\mathrm{Cap}(E))^{-1}$ identically on E.*

Proof. Without loss of generality, we assume that $E \subset B_{1/2}(0)$. For $r > 2$ we denote $\Omega_{E,r} = \Omega_E \cap B_r(0)$ and we let u_r be the solution of the Dirichlet problem in $\Omega_{E,r}$ with boundary data:

$$u_r = \begin{cases} (\mathrm{Cap}(E))^{-1} & \text{in } \partial\Omega_E, \\ U_\mu & \text{in } \partial B_r(0). \end{cases}$$

We extend u_r to $\widehat{E} = \mathbb{R}^d \backslash \Omega_E$ by setting $u_r(x) = (\mathrm{Cap}(E))^{-1}$ for $x \in \widehat{E}$, so that u_r is continuous in $B_r(0)$, by the Wiener regularity of $\Omega_{E,r}$.

Observe that, for all $\xi \in \partial\Omega_E$,

$$0 \leqslant \limsup_{x \to \xi} (u_r(x) - U_\mu(x)) \leqslant (\mathrm{Cap}(E))^{-1}.$$

Therefore, since $u_r = U_\mu$ in $\partial B_r(0)$, by the maximum principle we get

$$\|u_r - U_\mu\|_{\infty,\Omega_{E,r}} \leqslant (\mathrm{Cap}(E))^{-1}.$$

As this estimate is uniform in r, we deduce that there exists a sequence $r_k \to \infty$ such that u_{r_k} converges locally uniformly on compact subsets of Ω_E to some function u harmonic in Ω_E. In particular, it converges uniformly on $\partial B_1(0)$. Since u_{r_k} equals $(\mathrm{Cap}(E))^{-1}$ in $\partial\Omega_E$ for all k, by the maximum principle it follows that the convergence is also uniform in $\overline{\Omega_E} \cap \bar{B}_1(0)$. Then we deduce that u is continuous in $\overline{\Omega_E}$ and so it extends continuously to the whole $\mathbb{R}^d$. Further, u equals $(\mathrm{Cap}(E))^{-1}$ in $\widehat{E}$, $u \leqslant (\mathrm{Cap}(E))^{-1}$ in Ω_E, and together with the fact that u is continuous in $\mathbb{R}^d$ and harmonic in Ω_E, this implies that u is superharmonic in $\mathbb{R}^d$. Notice also that

$$\|u - U_\mu\|_{\infty,\mathbb{R}^d} \leqslant (\mathrm{Cap}(E))^{-1}.$$

The preceding estimate implies that u is non-constant in the case $d = 2$, since $U_\mu(x) \to -\infty$ as $|x| \to \infty$. In the case $d \geqslant 3$, it is also easy to check that u is non-constant. Indeed, let $\widetilde{u}_r : \bar{A}_{1,r}(0) \to \mathbb{R}$ be defined by

$$\widetilde{u}_r(x) = \mathrm{Cap}(E)^{-1}\, \mathcal{E}(1)^{-1}\, \mathcal{E}(x) + \max_{\partial B_r(0)} U_\mu,$$

where, abusing notation, we wrote $\mathcal{E}(1) = \mathcal{E}(y)$ for $|y| = 1$. It is immediate to check that $u_r \leqslant \widetilde{u}_r$ in $\partial \bar{A}_{1,r}(0)$, and thus also in $A_{1,r}(0)$ by the maximum principle. Then, letting $r \to \infty$, it follows that $u(x) \leqslant \mathrm{Cap}(E)^{-1}\, \mathcal{E}(1)^{-1}\, \mathcal{E}(x)$ for $|x| > 1$, which implies that u is non-constant.

The superharmonicity of u in $\mathbb{R}^d$ implies that $-\Delta u$ is a non-negative measure in the sense of distributions. This is an immediate consequence of Lemma 5.13 and the Riesz representation theorem. The fact that u is non-constant and the maximum principle ensures that Δu is not the zero measure.

Now we claim that there exists some constant $c_0 \in \mathbb{R}$ such that

$$u = -\mathcal{E} * \Delta u + c_0 \tag{6.30}$$

in the $L^1_{loc}(\mathbb{R}^d)$ sense. To prove this, observe first that the function $v := u + \mathcal{E} * \Delta u$ is harmonic in $\mathbb{R}^d$, and for $|x| \gg 1$ it satisfies

$$|v(x)| \leqslant |u(x)| + |\mathcal{E} * \Delta u(x)| \leqslant (\mathrm{Cap}(E))^{-1} + U_\mu(x) + |\mathcal{E} * \Delta u(x)| \leqslant C_0 + C_1\, |\mathcal{E}(|x|)|,$$

where C_0 and C_1 depend on u. In the case $d \geqslant 3$, this implies that v is bounded and so it is constant, by Liouville's theorem. In the case $d = 2$, we also deduce that v is constant. This follows easily from Lemma 2.11 applied to v in $B_R(0)$, letting $R \to \infty$:

$$\|\nabla v\|_{\infty, B_{R/2}(0)} \lesssim \frac{\|v\|_{\infty, B_R(0)}}{R} \lesssim \frac{C_0 + C_1 \log R}{R} \to 0.$$

So in any case (6.30) holds.

Let us see now that the pointwise identity

$$u(x) = -\mathcal{E} * \Delta u(x) + c_0 \tag{6.31}$$

holds for *all* $x \in \mathbb{R}^d$. Indeed, this holds in Ω_E by the continuity of $\mathcal{E} * \Delta u$ and u in Ω_E. So it remains to show that

$$(\mathrm{Cap}(E))^{-1} = -\mathcal{E} * \Delta u(x) + c_0 \quad \text{ for all } x \in \widehat{E}.$$

To this end, notice that for each $t > 0$, by the identity (6.30) in the L^1_{loc} sense and the continuity of u,

$$c_0 + \fint_{B_t(x)} \mathcal{E} * (-\Delta u)\, dm = \fint_{B_t(x)} u\, dm \xrightarrow{t \to 0} u(x).$$

On the other hand, by the superharmonicity of $\mathcal{E} * (-\Delta u)$ (recall that $-\Delta u$ is a positive measure), $\fint_{B_t(x)} \mathcal{E} * (-\Delta u)\, dm \leqslant \mathcal{E} * (-\Delta u)(x)$, and so

$$\mathrm{Cap}(E)^{-1} = u(x) = c_0 + \limsup_{t \to 0} \fint_{B_t(x)} \mathcal{E} * (-\Delta u)\, dm \leqslant c_0 + \mathcal{E} * (-\Delta u)(x).$$

For the converse inequality, we take into account that $c_0 + \mathcal{E} * (-\Delta u) \leqslant \mathrm{Cap}(E)^{-1}$ a.e. in $\mathbb{R}^d$, and thus the same estimate happens everywhere in $\mathbb{R}^d$ by the lower semicontinuity of $\mathcal{E} * (-\Delta u)$ (see Lemma 6.1(a)). So (6.31) holds for all $x \in \mathbb{R}^d$.

From (6.31) we deduce that

$$\mathcal{E} * (-\Delta u)(x) = (\mathrm{Cap}(E))^{-1} - c_0 =: c_1 \qquad \text{for all } x \in \widehat{E}.$$

Since $-\Delta u$ is a non-zero positive measure supported on $\widehat{E} \subset B_{1/2}(0)$, it follows that $c_1 > 0$. So letting $k = (c_1 \,\mathrm{Cap}(E))^{-1}$, it turns out that $\mathcal{E} * (-k\Delta u)(x) = (\mathrm{Cap}(E))^{-1}$ for all $x \in E$. Next we will show that this implies that $-k\Delta u = \mu$. To this end, by Theorem 6.10 it suffices to prove that $-k\Delta u$ is a probability measure and that $I(\mu + k\Delta u) = 0$.

To prove that $-k\Delta u$ is a probability measure we first apply Theorem 6.12, taking into account that $\|\mathcal{E} * (-k\,\mathrm{Cap}(E)\,\Delta u)\|_\infty = 1$, and then we derive $\mathrm{Cap}(E) \geqslant \|-k\,\mathrm{Cap}(E)\Delta u\|$, or equivalently, $\|-k\Delta u\| \leqslant 1$. For the converse inequality we apply Lemma 6.15 and we obtain $\mathrm{Cap}(E) \leqslant \|-k\,\mathrm{Cap}(E)\,\Delta u\|$, so that $\|-k\Delta u\| = 1$.

Next we will show that $I(\mu + k\Delta u) = 0$. Notice first that $I(|\mu + k\Delta u|) < +\infty$ because both $\mathcal{E} * \mu$ and $\mathcal{E} * (-k\Delta u)$ are uniformly bounded in E. We write

$$I(\mu + k\Delta u) = \int U_{(\mu+k\Delta u)}\, d(\mu + k\Delta u) = \int \big(U_\mu - U_{(-k\Delta u)}\big)\, d\mu + k\int \big(U_\mu - U_{(-k\Delta u)}\big)\, d(\Delta u).$$

Both integrals on the right hand side vanish because $U_{(-k\Delta u)}$ equals identically $(\mathrm{Cap}\,E)^{-1}$ in $E \supset \mathrm{supp}\mu$, while U_μ equals $(\mathrm{Cap}\,E)^{-1}$ μ-a.e. and $(-k\Delta u)$-a.e. by Corollary 6.9. Hence, $I(\mu + k\Delta u) = 0$ and thus $\mu = -k\Delta u$. In turn, this implies that $U_\mu = -k\,\mathcal{E} * \Delta u$, and so U_μ is continuous in $\mathbb{R}^d$ and identically equal to $(\mathrm{Cap}\,E)^{-1}$ in $\widehat{E}$. □

Proof of (a) ⇒ (b) in Theorem 6.23. As above, we write $\bar{A}_k = \bar{A}(\xi, 2^{-k-1}, 2^{-k})$, $B_k = B_{2^{-k}}(\xi)$, and $\bar{B}_k = \overline{B_k}$. To get a contradiction, suppose that $\xi \in \partial\Omega$ is a regular point such that

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} < \infty.$$

Without loss of generality, assume also that $\Omega \subset B_{1/2}(0)$.

We will replace Ω by an auxiliary Wiener regular open subset $\widetilde{\Omega} \subset \Omega$ so that $\xi \in \partial\Omega \cap \partial\widetilde{\Omega}$. We define $\widetilde{\Omega}$ as follows. For each $k \geqslant 1$ such that $\bar{A}_k \backslash \Omega \neq \varnothing$, let $\rho_k \in (0, 2^{-k-3})$ be such that

$$\mathrm{Cap}(\mathcal{U}_{\rho_k}(\bar{A}_k \backslash \Omega)) \leqslant \mathrm{Cap}(\bar{A}_k \backslash \Omega) + 2^{-k}\, \mathrm{Cap}(\bar{B}_k),$$

where $\mathcal{U}_\ell(G)$ stands for the ℓ-neighborhood of G. We cover $\bar{A}_k \backslash \Omega$ by a finite number of closed balls $B_{k,j}$ centered in $\bar{A}_k \backslash \Omega$ with the same radius ρ_k, and we let $E_k = \bigcup_j B_{j,k}$. In case that $\bar{A}_k \backslash \Omega = \varnothing$, then we let $E_k = \varnothing$ be a closed ball $B_{k,1}$ contained in $\bar{A}_k$ such that $\mathrm{Cap}(B_{k,1}) = 2^{-k}\, \mathrm{Cap}(\bar{B}_k)$. Finally, we let

$$\widetilde{\Omega} = \Omega \backslash \bigcup_{k \geqslant 1} E_k.$$

It is easy to check that $\widetilde{\Omega}$ is open. Further,

$$\sum_{k \geqslant 1} \frac{\mathrm{Cap}(\bar{A}_k \backslash \widetilde{\Omega})}{\mathrm{Cap}(\bar{B}_k)} \leqslant \sum_{k \geqslant 1} \frac{\mathrm{Cap}(E_{k-1} \cup E_k \cup E_{k+1})}{\mathrm{Cap}(\bar{B}_k)}.$$

Using that $\mathrm{Cap}(E_{k-1} \cup E_k \cup E_{k+1}) \leqslant \mathrm{Cap}(E_{k-1}) + \mathrm{Cap}(E_k) + \mathrm{Cap}(E_{k+1})$ and that $\mathrm{Cap}(\bar{B}_{k-1}) \approx \mathrm{Cap}(\bar{B}_k) \approx \mathrm{Cap}(\bar{B}_{k+1})$, it follows that

$$\sum_{k\geqslant 1} \frac{\mathrm{Cap}(\bar{A}_k \backslash \widetilde{\Omega})}{\mathrm{Cap}(\bar{B}_k)} \lesssim \sum_{k\geqslant 1} \frac{\mathrm{Cap}(E_k)}{\mathrm{Cap}(\bar{B}_k)} \leqslant \sum_{k\geqslant 1} \frac{\mathrm{Cap}(\bar{A}_k \backslash \Omega)}{\mathrm{Cap}(\bar{B}_k)} + \sum_{k\geqslant 1} 2^{-k} < \infty. \tag{6.32}$$

Also $\xi \in \partial\widetilde{\Omega}$ because the preceding estimate implies that, for k large enough, $\mathrm{Cap}(\bar{A}_k \backslash \widetilde{\Omega}) \ll \mathrm{Cap}(\bar{B}_k) \approx \mathrm{Cap}(\bar{A}_k)$, so that $\bar{A}_k \cap \widetilde{\Omega} \neq \varnothing$.

To check that $\widetilde{\Omega}$ is Wiener regular, notice first that ξ is a Wiener regular point for $\widetilde{\Omega}$, because if $v : \Omega \to \mathbb{R}$ is a barrier for ξ in Ω, then $v|_{\widetilde{\Omega}}$ is a barrier of ξ in $\widetilde{\Omega}$. Further, it is immediate to check that any other point $\zeta \in \partial\widetilde{\Omega}$ with $\zeta \neq \xi$ belongs to the boundary of some ball $B_{k.j}$, and so ζ is Wiener regular because of the existence of an outer tangent ball in ζ (namely, $B_{k.j}$). So $\widetilde{\Omega}$ satisfies the required properties.

For $k \geqslant 1$ we denote

$$F_k = \{\xi\} \cup \bigcup_{j\geqslant k} E_j.$$

Notice that F_k is a compact set such that $F_k \subset \bar{B}_{k-1}$, and by the same arguments as above, it follows easily that $\mathbb{R}^d \backslash F_k$ is Wiener regular and that $\xi \in \partial F_k$.

Next we will derive a contradiction from the fact that ξ is a regular point for $\widetilde{\Omega}$ and the condition (6.32). For $0 < \varepsilon < 1/4$, let $N \geqslant 2$ be such that

$$\sum_{k\geqslant N} \frac{\mathrm{Cap}(E_k)}{\mathrm{Cap}(\bar{B}_k)} < \varepsilon. \tag{6.33}$$

Because of the Wiener regularity of $\widetilde{\Omega}$, there exists a function $f \in C\big(\overline{\widetilde{\Omega}}\big)$, harmonic in $\widetilde{\Omega}$, with $0 \leqslant f \leqslant 1$, with $f(\xi) = 0$ and $f = 0$ in $\partial\widetilde{\Omega} \backslash \bar{B}_{N+1}$. By the continuity of f, there exists $s < 2^{-N-1}$ such that $f(x) > 1 - \varepsilon$ in $\overline{\widetilde{\Omega}} \cap \bar{B}_s(\xi)$.

Let us see that there exists $M \geqslant 1$ large enough such that $2^{-M} < s/4$ and such that the equilibrium potential U_{F_M} for F_M satisfies

$$\mathrm{Cap}(F_M)\, U_{F_M}(x) \leqslant \varepsilon \quad \text{ for all } x \in \mathbb{R}^d \backslash \bar{B}_s(\xi).$$

Indeed, we have

$$\mathrm{Cap}(F_M)\, U_{F_M}(x) \leqslant \mathrm{Cap}(\bar{B}_{M-1})\, \mathcal{E}(\mathrm{dist}(F_M, \partial B_s(\xi)) \lesssim \frac{\mathcal{E}(s)}{\mathcal{E}(2^{-M+1})},$$

which tends to 0 as $M \to \infty$. We denote $V_{F_M} = \mathrm{Cap}(F_M)\, U_{F_M}$.

Let $A_{N,M} = \bigcup_{N\leqslant k\leqslant M} E_k$. Again, $\mathbb{R}^d \backslash A_{N,M}$ is Wiener regular because because $A_{N,M}$ is the union of a finite number of balls, and we can apply the criterion of the outer tangent ball. Let $U_{A_{N,M}}$ be the equilibrium potential of $A_{N,M}$ and denote $V_{A_{N,M}} = \mathrm{Cap}(A_{N,M})\, U_{A_{N,M}}$. By Lemma 6.26, it turns out that V_{F_M} and $V_{A_{N,M}}$ are continuous and $V_{F_M} + V_{A_{N,M}} \geqslant 1$ on $F_M \cup A_{N,M}$. Then, by the definition of f and the maximum principle it follows that $V_{F_M} + V_{A_{N,M}} \geqslant f$ in $\widetilde{\Omega}$. Therefore,

$$V_{A_{N,M}} \geqslant f - V_{F_M} \geqslant 1 - 2\varepsilon \quad \text{ in } \partial B_s(\xi) \cap \widetilde{\Omega}.$$

We also have $V_{A_{N,M}} = 1 > 1 - 2\varepsilon$ in $A_{N,M}$, and so by the maximum principle applied to the set $B_s(\xi)\backslash A_{N,M}$ (recall that $2^{-M+2} < s < 2^{-N-1}$), it follows that

$$V_{A_{N,M}}(\xi) \geqslant 1 - 2\varepsilon. \tag{6.34}$$

Now we intend to contradict this estimate. To this end, notice that for $x \in \partial B_{1/2}(\xi)$,

$$\begin{aligned} V_{A_{N,M}}(x) &= \mathrm{Cap}(A_{N,M})\, U_{A_{N,M}}(x) \\ &\leqslant \mathrm{Cap}(B_{N-1})\, \mathcal{E}(\mathrm{dist}(x, A_{N,M})) \lesssim \mathrm{Cap}(B_{N-1}) \approx \mathcal{E}(2^{-N})^{-1}. \end{aligned}$$

In $A_{N,M}$ we also have

$$V_{A_{N,M}}(x) = 1 \leqslant \sum_{N \leqslant k \leqslant M} V_{E_k}(x) = \sum_{N \leqslant k \leqslant M} \mathrm{Cap}(E_k)\, U_{E_k}(x).$$

Then, by the maximum principle and by (6.33),

$$\begin{aligned} V_{A_{N,M}}(\xi) &\leqslant \sum_{N \leqslant k \leqslant M} \mathrm{Cap}(E_k)\, U_{E_k}(\xi) + C\, \mathcal{E}(2^{-N})^{-1} \\ &\approx \sum_{N \leqslant k \leqslant M} \frac{\mathrm{Cap}(E_k)}{\mathrm{Cap}(\bar{B}_k)} + \mathcal{E}(2^{-N})^{-1} \lesssim \varepsilon + \mathcal{E}(2^{-N})^{-1}, \end{aligned}$$

which contradicts (6.34). □

6.6 Kellogg's theorem

A set $E \subset \mathbb{R}^d$ is called polar if $\mathrm{Cap}(E) = 0$. Of course, in the case $d = 2$, this is equivalent to saying that $\mathrm{Cap}_L(E) = 0$. Kellogg's theorem asserts that, for any bounded open set $\Omega \subset \mathbb{R}^d$, the set of (Wiener) irregular points is polar. In order to prove this, we will need some auxiliary results, which have their own interest.

Recall that in Section 5.4 we introduced the notion of barrier functions, whose existence characterizes the regularity of boundary points. Next we introduce the weaker notion of generalized barrier, which also can be used to characterize regular points, as we will see below. Given an open set $\Omega \subset \mathbb{R}^d$, a function $v : \Omega \to \mathbb{R}$ is called a generalized barrier for Ω at $\xi \in \partial\Omega$ if

1. v is superharmonic in Ω,
2. $v > 0$ in Ω, and
3. $\lim_{x \to \xi} v(x) = 0$.

It is immediate to check that a barrier for ξ is also a generalized barrier. The converse statement is not true. However, we have the following key result.

Theorem 6.27. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. A point $\xi \in \partial\Omega$ is regular for Ω if and only if there exists a generalized barrier for Ω at ξ.*

To prove this theorem, we will use the following simple result:

Lemma 6.28. *For $r > 0$, let $V \subset S_r(0)$ be relatively open in $S_r(0)$, and for any $x \in B_r(0)$ let*

$$g(x) = \int_{S_r(0)} P^x_{B_r(0)}(\zeta)\, \chi_V(\zeta)\, d\sigma(\zeta),$$

where σ is the surface measure on $S_r(0)$. Then,

$$\lim_{B_r(0) \ni x \to \xi} g(x) = 1 \quad \text{for all } \xi \in V.$$

Recall that $P^x_{B_r(0)}$ is the Poisson kernel for the ball $B_r(0)$, which was introduced in Remark 3.12.

Proof. This is an immediate consequence of Example 5.27 and Corollary 5.36. □

Proof of Theorem 6.27. The statement in the theorem is equivalent to saying that there exists a barrier at $\xi \in \partial\Omega$ for Ω if and only if there exists a generalized barrier. Since any barrier is also a generalized barrier, we are left with showing that the existence of a generalized barrier at $\xi \in \partial\Omega$ for Ω implies the existence of a "usual" barrier. To this end, consider the function $\varphi : \overline{\Omega} \to \mathbb{R}$ defined by $\varphi(x) = |x - \xi|^2$. The fact that $\Delta\varphi \geqslant 0$ away from ξ ensures that φ is subharmonic in Ω. The function $f := \varphi|_{\partial\Omega}$ is continuous in $\partial\Omega$, and thus it is also resolutive. Further, since $\varphi \in \mathcal{L}_f$ (recall that this is the lower Perron class for Ω, introduced in Definition 5.16), we have $v := H_f = \underline{H}_f \geqslant \varphi$ in Ω. Thus, v is a positive harmonic function in Ω such that for all $\zeta \in \partial\Omega\backslash\{\xi\}$,

$$\liminf_{\Omega \ni x \to \zeta} v(x) \geqslant f(\zeta) > 0.$$

Hence to show that v is a "usual" barrier for ξ, it suffices to prove that

$$\lim_{\Omega \ni x \to \xi} v(x) = 0. \tag{6.35}$$

To prove (6.35), without loss of generality, assume that $\xi = 0$. Let u be a generalized barrier at 0 for Ω and let $r > 0$ be such that $S_r(0) \cap \Omega \neq \varnothing$. For a given $\varepsilon > 0$, consider a compact subset $E_{r,\varepsilon} \subset S_r(0) \cap \Omega$ such that $\sigma((S_r(0) \cap \Omega)\backslash E_{r,\varepsilon}) \leqslant \varepsilon\, \sigma(S_r(0))$, where σ is the surface measure on $S_r(0)$. Notice that $\gamma_{r,\varepsilon} = \inf_{E_{r,\varepsilon}} u > 0$ (recall that u is lower semicontinuous in Ω and so the infimum on any compact subset of Ω is attained in that compact subset). Consider the set $V_{r,\varepsilon} = (S_r(0) \cap \Omega)\backslash E_{r,\varepsilon}$, which is relatively open in $S_r(0)$. Let $g : S_r(0) \to \mathbb{R}$ be defined by the "harmonic extension" of $\chi_{V_{r,\varepsilon}}$ to $B_r(0)$, that is,

$$g(x) = \int_{S_r(0)} P^x_{B_r(0)}(\zeta)\, \chi_{V_{r,\varepsilon}}(\zeta)\, d\sigma(\zeta),$$

where $P^x_{B_r(0)}$ is the Poisson kernel for $B_r(0)$ with pole at x. Let $h : \Omega \cap B_r(0) \to \mathbb{R}$ be the function defined by

$$h = r^2 + \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2\, u + \operatorname{diam}(\Omega)^2\, g,$$

Notice that h is superharmonic in $\Omega \cap B_r(0)$. We claim that for any function $s \in \mathcal{L}_f$ (recall that this means that $s \in C(\Omega)$ is a subharmonic function such that $\limsup_{x\to\eta} s(x) \leqslant f(\eta)$ for all $\eta \in \partial\Omega$), it holds that

$$\liminf_{x\to\eta} h(x) \geqslant \limsup_{x\to\eta} s(x) \qquad \text{for all } \eta \in \partial(\Omega \cap B_r(0)). \tag{6.36}$$

Indeed, if $\eta \in \overline{B}_r(0) \cap \partial\Omega$, then

$$\liminf_{x\to\eta} h(x) \geqslant r^2 \geqslant f(\eta) \geqslant \limsup_{x\to\eta} s(x).$$

On the other hand, if $\eta \in E_{r,\varepsilon}$, since u is lower semicontinuous in Ω,

$$\liminf_{x\to\eta} h(x) \geqslant \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2 \liminf_{x\to\eta} u \geqslant \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2\, u(\eta) \geqslant \operatorname{diam}(\Omega)^2 \geqslant \sup_{\partial\Omega} f.$$

Finally, for $\eta \in V_{r,\varepsilon} = S_r(0) \cap \Omega \backslash E_{r,\varepsilon}$, by Lemma 6.28,

$$\liminf_{x\to\eta} h(x) \geqslant \operatorname{diam}(\Omega)^2 \liminf_{x\to\eta} g(x) = \operatorname{diam}(\Omega)^2 \geqslant \sup_{\partial\Omega} f.$$

Our claim holds since, in the last two cases, we can use that $s \in \mathcal{L}_f$ implies $\|s\|_\infty \leqslant \sup_{\partial\Omega} f$.

From the superharmonicity of $h - s$ and the maximum principle in Lemma 5.4 (applied to $s - h$) and (6.36), we deduce that

$$s(x) \leqslant h(x) \qquad \text{for all } x \in B_r(0) \cap \Omega.$$

Since this estimate holds for all $s \in \mathcal{L}_f$, we deduce that $H_f(x) \leqslant h(x)$ for all $x \in B_r(0) \cap \Omega$. Thus,

$$\begin{aligned}\limsup_{x\to 0} H_f(x) &\leqslant r^2 + \gamma_{r,\varepsilon}^{-1} \operatorname{diam}(\Omega)^2 \limsup_{x\to 0} u + \operatorname{diam}(\Omega)^2 \limsup_{x\to 0} g \\ &= r^2 + 0 + g(0) = r^2 + \operatorname{diam}(\Omega)^2 \frac{\sigma(V_{r,\varepsilon})}{\sigma(S_r(0))} \leqslant r^2 + \operatorname{diam}(\Omega)^2 \varepsilon.\end{aligned}$$

Choosing $\varepsilon = r^2 \operatorname{diam}(\Omega)^{-2}$, we get $\limsup_{x\to 0} H_f(x) \leqslant 2r^2$. Since r can be taken arbitrarily small and H_f is positive, we deduce that

$$\lim_{x\to 0} v(x) = \lim_{x\to 0} H_f(x) = 0,$$

as wished. □

Theorem 6.29. *Let $E \subset \mathbb{R}^d$ be compact with $\operatorname{Cap}(E) > 0$ and let Ω_E be the unbounded component of $\mathbb{R}^d \backslash E$. Let μ be the equilibrium measure for E. If a point $\xi \in \partial\Omega_E$ is irregular for Ω_E, then $U_\mu(\xi) < \operatorname{Cap}(E)^{-1}$. In particular, the set of irregular points for Ω_E is polar, and moreover it is contained in a polar F_σ set.*

Recall that a set $E \subset \mathbb{R}^d$ is called F_σ if it can be written as a countable union of closed sets.

Proof. Let us see that if $U_\mu(\xi) \geqslant \mathrm{Cap}(E)^{-1}$, then ξ is regular. Remark that the inequality $U_\mu(\xi) \geqslant \mathrm{Cap}(E)^{-1}$ is equivalent to $U_\mu(\xi) = \mathrm{Cap}(E)^{-1}$ because $\|U_\mu\|_{\infty,\mathbb{R}^d} \leqslant \mathrm{Cap}(E)^{-1}$. We claim that the function $v = \mathrm{Cap}(E)^{-1} - U_\mu$ is a generalized barrier at ξ for Ω_E (i.e., for $\Omega_E \cap B_r(0)$ for any $r > 0$ such that $E \subset B_r(0)$). To check this, notice first that v is harmonic and that $v > 0$ in Ω_E. The latter assertion follows from the fact that v is non-constant and non-negative in Ω_E and Ω_E is connected. By the semicontinuity property (a) in Lemma 6.1, we know that $\liminf_{y\to\xi} U_\mu(y) \geqslant U_\mu(\xi)$. Consequently, $\limsup_{y\to\xi} v(y) \leqslant v(\xi) = 0$. So v is a generalized barrier at ξ for Ω_E, and by Theorem 6.27 ξ is a regular point for Ω_E.

To prove the second statement of the theorem observe that, by what we have just proved, the set of irregular points for Ω_E is contained in the set

$$S = \{x \in E : U_\mu(x) < \mathrm{Cap}(E)^{-1}\},$$

which is a polar set, by Theorem 6.8. Therefore, the set of irregular points for Ω_E is also polar. Further, writing $S = \bigcup_{j\geqslant 1} S_j$, with

$$S_j = \left\{x \in E : U_\mu(x) \leqslant \mathrm{Cap}(E)^{-1} - \tfrac{1}{j}\right\},$$

by the lower semicontinuity of U_μ it is clear that S is an F_σ set, since each S_j is closed. $\square$

Remark 6.30. In fact, the converse of the first statement in Theorem 6.29 also holds. That is, for Ω_E and μ as in Theorem 6.29, a point $\xi \in \partial\Omega_E$ is irregular if and only if $U_\mu(\xi) < \mathrm{Cap}(E)^{-1}$. However, we will not need this result and so we skip the proof.

Theorem 6.31. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. A point $\xi \in \partial\Omega$ is irregular for Ω if and only if there exists some component Ω_0 of Ω such that $\xi \in \partial\Omega_0$ and x is irregular for Ω_0. In particular, if x is not in the boundary of any component of Ω, then it is regular for Ω.*

Proof. Denote by $\{\Omega_j\}_{j\in J}$ the family of components of Ω. If $\xi \in \partial\Omega_j$ and ξ is irregular for Ω_j, then there is not any barrier at ξ for Ω_j, which it readily implies that there is not any barrier at ξ for Ω. Thus, ξ is irregular for Ω.

In the converse direction, suppose that there is not any Ω_j such that ξ is irregular for Ω_j. To prove that ξ is regular for Ω, we intend to define a generalized barrier v at ξ for Ω. For any Ω_j such that $\xi \in \partial\Omega_j$, since ξ is regular for Ω_j, there exists a barrier v_j at ξ for Ω. For such Ω_j, we define $v = \min(v_j, 1/j)$. For the components Ω_j such that $\xi \notin \partial\Omega_j$, we let $v = 1/j$ on Ω_j.

To check that v is a generalized barrier at ξ for Ω, notice first that v is superharmonic and positive in Ω. To see that $\lim_{x\to\xi} v(x) = 0$, let $\varepsilon > 0$ and consider the finite set $J_\varepsilon = \{j \in J : j \leqslant \varepsilon^{-1}\}$. If $J_\varepsilon = \varnothing$, then $u \leqslant \varepsilon$ on Ω. Otherwise, for each $j \in J_\varepsilon$ there exists an open neighborhood V_j of ξ such that either $V_j \cap \Omega_j = \varnothing$ or $v \leqslant \varepsilon$ in $V_j \cap \Omega_j$. So letting $V = \bigcup_{j\in J_\varepsilon} V_j$ it turns out that V is an open neighborhood of y where $v \leqslant \varepsilon$ on V. So $\lim_{x\to\xi} v(x) = 0$ as wished, and thus v is the desired generalized barrier. $\square$

Theorem 6.32 (Kellogg's theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then the set of irregular points for Ω is polar. Further, this is contained in an F_σ polar set.*

Proof. By Theorem 6.31, it suffices to show that the set of irregular points for any component of Ω is irregular, taking into account that the number of components is at most countable and that a finite or countable union of polar sets is polar. So to prove the theorem we can assume that Ω is connected.

Given a bounded connected set Ω, for any $\xi \in \partial\Omega$ let B_ξ be an open ball centered in ξ such that $\Omega \cap \partial B_\xi \neq \varnothing$. Consider the domain $\Omega_\xi = \Omega \cup (\mathbb{R}^d \backslash \overline{B_\xi})$. Notice that Ω_ξ is an unbounded connected set with bounded boundary, and then by Theorem 6.29 the set of irregular points for Ω_ξ is polar (we can assume that $\mathrm{Cap}(\partial\Omega_\xi) > 0$ because otherwise any subset of $\partial\Omega_\xi$ is polar) and it is contained in an F_σ polar set. Now remark that $B_\xi \cap \partial\Omega \subset \partial\Omega_\xi$ and that any point from $B_\xi \cap \partial\Omega$ which is irregular for Ω is also irregular for Ω_ξ. This follows immediately from Wiener's criterion for regularity (although it could be also easily deduced from the characterization of regularity in terms of existence of barriers). Therefore, the subset of irregular points for Ω that belong to $B_\xi \cap \partial\Omega$ is polar and it is contained in an F_σ polar set.

Finally, since $\partial\Omega$ is compact, there exists a finite covering of $\partial\Omega$ with balls B_{ξ_i}, for a finite subset of points $\xi_i \in \partial\Omega$. By the preceding discussion, the set of irregular points for Ω that belong to $B_{\xi_i} \cap \partial\Omega$ is polar. Since a finite union of polar sets is also polar and a finite unions of F_σ sets is an F_σ set, the theorem follows. □

Exercise 6.6.1. Prove that the set of irregular points for an open set $\Omega \subset \mathbb{R}^d$ with compact boundary is itself an F_σ set.

Hint: This follows from Wiener's criterion. Indeed, using subadditivity and Proposition 6.16, one can check that an equivalent form of the criterion is the following. A point $\xi \in \partial\Omega$ is regular for the Dirichlet problem in Ω if and only if

$$S(x) = \sum_{k=1}^{\infty} \frac{\mathrm{Cap}(A(x, 2^{-k-2}, 2^{-k+1}) \backslash \Omega)}{\mathrm{Cap}(\bar{B}(x, 2^{-k}))} = +\infty,$$

that is, we may pick open enlarged annuli instead of closed. Now, $\mathrm{Cap}(A(x, 2^{-k-2}, 2^{-k+1}) \backslash \Omega)$ is lower semicontinuous, so $S(x)$ can be shown to be lower semicontinuous as well. Thus, the set $\{x \in \mathbb{R}^{n+1} : S(x) > \lambda\}$ is open and thus the set of Wiener regular point is a G_δ set (relative to $\partial\Omega$), and the set of the irregular points from $\partial\Omega$ is an F_σ set.

6.7 Removability of polar sets

Theorem 6.33. *Let $\Omega \subset \mathbb{R}^d$ be bounded and open, and let $Z \subset \partial\Omega$ be a Borel polar set. Then, for any $x \in \Omega$,*

$$\omega^x(Z) = 0.$$

Proof. In the case $d = 2$, we will assume that $\Omega \subset B_{1/2}(0)$. The measure ω^x is Radon and thus it is inner regular. Then it is enough to prove the theorem for Z being a compact (polar) set. Under this assumption, by the outer regularity of capacity (see Lemma 6.7), for any $\varepsilon > 0$ there is an open set $V \supset Z$ such that $\mathrm{Cap}(V) < \varepsilon$. By the compactness of Z, we can find finitely many open balls B_i, $i = 1, \ldots, m$, centered on Z such that $2B_i \subset V \cap B_{1/2}(0)$ and

$$Z \subset \bigcup_{1 \leqslant i \leqslant m} B_i.$$

Consider the compact set $E = \bigcup_{1 \leqslant i \leqslant m} \overline{B_i}$ and let $\Omega_E = \mathbb{R}^d \backslash E$. Since E consists of a union of finitely many balls, it follows either by Wiener's criterion or by the exterior ball criterion in Remark 5.38 that Ω_E is Wiener regular. Then, by Lemma 6.26, if μ stands for the equilibrium measure for E, the potential U_μ is continuous in $\mathbb{R}^d$ and $U_\mu = (\mathrm{Cap}(E))^{-1}$ identically on E.

Consider now the function $f(x) = \mathrm{Cap}(E)\, U_\mu(x)$, and notice that it is superharmonic and continuous in $\mathbb{R}^d$, and it equals 1 on E. Also, it is positive in $\overline{\Omega}$ since $\Omega \subset B_{1/2}(0)$ in the planar case. So we have

$$\omega^x(Z) \leqslant \omega^x(E) \leqslant \int f \, d\omega^x. \tag{6.37}$$

By definition, letting $g = f|_{\partial\Omega}$, the last integral above equals $H_g(x)$. Since f belongs to the upper Perron class for g, we have $H_g(x) \leqslant f(x)$. Thus,

$$\omega^x(Z) \leqslant f(x) = \mathrm{Cap}(E)\, U_\mu(x) \leqslant \mathrm{Cap}(V)\, U_\mu(x) \leqslant \varepsilon\, U_\mu(x). \tag{6.38}$$

As μ is a probability measure supported on E,

$$U_\mu(x) = \int \mathcal{E}(x-y)\, d\mu(y) \leqslant \sup_{y \in E} \mathcal{E}(x-y) \to \sup_{y \in Z} \mathcal{E}(x-y) \quad \text{as } \varepsilon \to 0.$$

Since $\sup_{y \in Z} \mathcal{E}(x-y) < \infty$, letting $\varepsilon \to 0$ in (6.38), we deduce that $\omega^x(Z) = 0$. □

Definition 6.34. Let Ω be a bounded open set and let $E \subset \Omega$ be a compact set. We say that E is removable for bounded harmonic functions in Ω if every function $f : \Omega \backslash E \to \mathbb{R}$ which is harmonic and bounded can be extended to the whole Ω as a harmonic function.

Theorem 6.35. *Let Ω be a bounded open set and let $E \subset \Omega$ be a compact set. Then E is removable for bounded harmonic functions in Ω if and only if E is polar.*

Notice that, in particular, the removability of a compact set E for bounded harmonic functions does not depend on the bounded open set Ω containing E.

Proof. First we show that if $\mathrm{Cap}(E) > 0$ then E is not removable. To this end, let μ be the equilibrium measure of E and U_μ the corresponding equilibrium potential. Then U_μ is a bounded harmonic function in $\Omega \backslash E$. Further, it is easy to check that U_μ cannot be extended harmonically to a function f harmonic in the whole Ω. Otherwise, f would be a function continuous in $\overline{\Omega}$ and harmonic in Ω such that $\max_{\overline{\Omega}} f$ is not attained in $\partial\Omega$, because $\sup_E f = \mathrm{Cap}(E)^{-1} > \max_{\partial\Omega} f$. So we get a contradiction.

To prove the converse implication, let $\Omega \subset \mathbb{R}^d$ be bounded and open and let $E \subset \Omega$ be a compact polar set. Without loss of generality we can assume that $\overline{\Omega} \subset B_{1/2}(0)$ in the case $d = 2$. We claim that there exists a Wiener regular open set $\widetilde{\Omega}$ which contains E and such that $\overline{\widetilde{\Omega}} \subset \Omega$. For example $\widetilde{\Omega}$ can be constructed as the interior of the union of finitely many dyadic cubes of the same size in a suitable way. We leave the details for the reader.

Given $\varepsilon > 0$, let V_ε be an open set such that $E \subset V_\varepsilon$ and $\mathrm{Cap}(V_\varepsilon) < \varepsilon$. By the compactness of E, we can find finitely many open balls B_i, $i = 1, \ldots, m$, centered on Z such that $3B_i \subset V_\varepsilon \cap \widetilde{\Omega}$ and

$$E \subset \bigcup_{1 \leqslant i \leqslant m} B_i.$$

Consider the compact set $F_\varepsilon = \bigcup_{1\leqslant i\leqslant m} 2\overline{B_i}$ and let $\widetilde{\Omega}_\varepsilon = \widetilde{\Omega} \backslash F_\varepsilon$. Notice that

$$\partial\widetilde{\Omega}_\varepsilon = \partial\widetilde{\Omega} \cup \partial F_\varepsilon.$$

For $x \in \widetilde{\Omega}_\varepsilon$, we bound $\omega^x_{\widetilde{\Omega}_\varepsilon}(\partial F_\varepsilon)$ as in Theorem 6.33: by considering the equilibrium measure μ of F_ε, as in (6.38) we deduce that

$$\omega^x_{\widetilde{\Omega}_\varepsilon}(\partial F_\varepsilon) \leqslant \mathrm{Cap}(F_\varepsilon)\, U_\mu(x) \leqslant \varepsilon\, U_\mu(x) \leqslant C(x)\, \varepsilon,$$

with $C(x)$ independent of ε (assuming ε small enough).

Next we will show that if $f : \Omega\backslash E \to \mathbb{R}$ is harmonic and bounded, then f extends to the whole Ω as a harmonic function. To this end, let g be the harmonic extension of $f|_{\partial\widetilde{\Omega}}$ to $\widetilde{\Omega}$ and fix $x \in \widetilde{\Omega}$. Take $\varepsilon > 0$ small enough such that $x \in \widetilde{\Omega}_\varepsilon$. Observe that both f and g are harmonic in $\widetilde{\Omega}_\varepsilon$ and continuous in $\overline{\widetilde{\Omega}_\varepsilon}$ and their boundary values coincide in $\partial\widetilde{\Omega}$. So we have

$$f(x) - g(x) = \int_{\partial\widetilde{\Omega}_\varepsilon} (f-g)\, d\omega^x_{\widetilde{\Omega}_\varepsilon} = \int_{\partial F_\varepsilon} (f-g)\, d\omega^x_{\widetilde{\Omega}_\varepsilon} \leqslant \|f-g\|_{\infty,\overline{\widetilde{\Omega}}}\, \omega^x_{\widetilde{\Omega}_\varepsilon}(F_\varepsilon) \lesssim \|f\|_{\infty,\Omega}\, C(x)\, \varepsilon.$$

Since ε is a positive constant which can be taken arbitrarily small, we infer that $f(x) = g(x)$. So we deduce that $f = g$ in $\widetilde{\Omega}$. That is, f extends harmonically to the whole $\widetilde{\Omega}$, just defining $f = g$ in E. □

Next we will apply some of the results obtained in this chapter to prove an enhanced version of Proposition 5.48 about the harmonic measure for unbounded open set with compact boundary.

Proposition 6.36. *Let $\Omega \subset \mathbb{R}^d$ be an unbounded open set with compact boundary and let $x \in \Omega$. Then the following holds:*

(a) If $\mathrm{Cap}(\partial\Omega) = 0$, then $\omega^x(\partial\Omega) = 0$.

(b) If $\mathrm{Cap}(\partial\Omega) > 0$ *and* $d = 2$*, then* $\omega^x(\partial\Omega) = 1$*, that is,* ω^x *is a probability measure.*

(c) If $\mathrm{Cap}(\partial\Omega) > 0$ *and* $d \geqslant 3$*, then* $0 < \omega^x(\partial\Omega) < 1$ *whenever* x *belongs to the unbounded component of* Ω*.*

Proof. **(a)** Suppose that $\mathrm{Cap}(\partial\Omega) = 0$. Recall that

$$\omega^x(\partial\Omega) = \lim_{r\to\infty} H_f^r(x) =: H_f(x),$$

where H_f^r is the Perron solution of the Dirichlet problem in $\Omega_r := \Omega \cap B_r(0)$ with boundary data equal to 1 in $\partial\Omega$ and to 0 in $S_r(0)$. So $H_f^r(x) = \omega_{\Omega_r}^x(\partial\Omega)$. For r large enough so that $\partial\Omega \subset B_r(0)$, we have $\omega_{\Omega_r}^x(\partial\Omega) = 0$, by Theorem 6.33. Thus, $H_f^r(x) = 0$ for any r large enough and so $\omega^x(\partial\Omega) = 0$.

(b) Suppose now that $\mathrm{Cap}(\partial\Omega) > 0$ and $d = 2$. By (5.11), $\omega^x(\partial\Omega) \leqslant 1$, so we only have to show the converse inequality. Consider the function

$$u_\varepsilon = 1 + \varepsilon\, U_\mu,$$

where μ is the equilibrium measure for $\partial\Omega$. Since $U_\mu(x) \to -\infty$ as $x \to \infty$, for any r large enough we have $\partial\Omega \subset B_r(0)$ and moreover $u_\varepsilon < 0$ on $S_r(0)$. Notice also that $u_\varepsilon \leqslant 1 + \varepsilon\ \mathrm{Cap}(\partial\Omega)^{-1}$ on $\mathbb{R}^2$. So the function

$$v_\varepsilon = \frac{1}{1 + \varepsilon\ \mathrm{Cap}(\partial\Omega)^{-1}}\, u_\varepsilon$$

belongs to the class $\mathcal{L}_f^r$, the lower Perron class in Ω_r for the function f_r which equals f on $\partial\Omega$ and vanishes on $S_r(0)$. Thus, for any $x \in \Omega_r$,

$$H_f^r(x) \geqslant v_\varepsilon(x) = \frac{1}{1 + \varepsilon\ \mathrm{Cap}(\partial\Omega)^{-1}}\,(1 + \varepsilon\, U_\mu(x)).$$

Recalling that this holds for any r large enough, we can take the limit as $r \to \infty$ to deduce that the same estimate holds for $H_f(x)$. That is,

$$\omega^x(\partial\Omega) \geqslant \frac{1}{1 + \varepsilon\ \mathrm{Cap}(\partial\Omega)^{-1}}\,(1 + \varepsilon\, U_\mu(x)).$$

Letting $\varepsilon \to 0$, we infer that $\omega^x(\partial\Omega) \geqslant 1$, which completes the proof of (b).

(c) In this case $\mathrm{Cap}(\partial\Omega) > 0$ and $d \geqslant 3$. Denote by Ω_o the unbounded component of Ω. The same arguments as in Proposition 5.48 show that $\omega^x(\partial\Omega) < 1$ for $x \in \Omega_o$. So we only have to check that $\omega^x(\partial\Omega) > 0$. By Theorem 5.42 (c), if $\xi \in \partial\Omega$ is a regular point, then

$$\lim_{\Omega \ni x \to \xi} \omega^x(\partial\Omega) = \lim_{\Omega \ni x \to \xi} H_f(x) = 1. \tag{6.39}$$

By Theorem 6.18,

$$\mathrm{Cap}(\partial\Omega_o) = \mathrm{Cap}(\mathbb{R}^2 \backslash \Omega_o) \geqslant \mathrm{Cap}(\partial\Omega) > 0.$$

By Kellogg's theorem, the set of irregular points is polar, and thus there exists some regular point $\xi \in \partial\Omega_o$. Therefore, (6.39) holds for this point ξ, and thus $\omega^x(\partial\Omega)$ does not vanish identically in Ω_o. Since $\omega^x(\partial\Omega) \geqslant 0$ for all $x \in \Omega$, by the strong maximum principle it follows that $\omega^x(\partial\Omega) > 0$ in the whole Ω_o. $\square$

6.8 Reduction to Wiener regular open sets

In this section we show some results which will be used later in these notes to reduce the proof of some properties for harmonic measure in general open sets to the case when these sets are Wiener regular. More precisely, the results in this section will be used to prove the Jones-Wolff theorem about the dimension of harmonic measure in the plane and to show the rectifiability of harmonic measure when it is absolutely continuous with respect to Hausdorff measure of codimension 1 in $\mathbb{R}^d$.

Proposition 6.37. *Let $\Omega \subset \mathbb{R}^d$ be open with compact boundary and let $p \in \Omega$. Let $Z \subset \partial\Omega$ be the family of irregular points of Ω. For any $\varepsilon > 0$, then there exists a covering of Z by a countable or finite family of closed balls $\{\bar{B}_i\}_{i\in I}$ satisfying the following properties:*

(i) *The balls $\bar{B}_i$ are centered in $\partial\Omega$ and they have bounded overlap.*

(ii) $\mathrm{Cap}(\bigcup_{i\in I} 2\bar{B}_i) \leqslant \varepsilon$.

(iii) $\widetilde{\Omega} := \Omega \setminus \bigcup_{i\in I} \bar{B}_i$ *is open.*

(iv) $\partial\widetilde{\Omega} \subset \left(\partial\Omega \setminus \bigcup_{i\in I} \bar{B}_i\right) \cup \bigcup_{i\in I} \partial\bar{B}_i$.

(v) $\widetilde{\Omega}$ *is Wiener regular.*

(vi) *For any $x \in \widetilde{\Omega}$, if either $d = 2$ with $\Omega \subset B_{1/2}(0)$, or $d \geqslant 3$, we have*

$$\omega^x_{\widetilde{\Omega}}\Big(\bigcup_{i\in I} 2\bar{B}_i\Big) \leqslant \varepsilon \sup_{y\in\partial\widetilde{\Omega}} \mathcal{E}(x-y). \tag{6.40}$$

In the case when $d = 2$ and Ω is unbounded, suppose that $\mathrm{Cap}_L(\partial\Omega) > 0$, that x belongs to the unbounded component of Ω, and that ε is small enough. Then,

$$\omega^x_{\widetilde{\Omega}}\Big(\bigcup_{i\in I} 2\bar{B}_i\Big) \leqslant C\varepsilon, \tag{6.41}$$

with C depending on $\mathrm{dist}(x, \partial\Omega)$.

Proof. Let $Z \subset \partial\Omega$ be the subset of irregular points of $\partial\Omega$. By Kellogg's theorem $\mathrm{Cap}(Z) = 0$, and moreover Z is contained in an F_σ set Z_0 such that $\mathrm{Cap}(Z_0) = 0$. By the outer regularity of capacity for compact sets and the fact that Z_0 is an F_σ set, we deduce that there exists an open set U containing Z_0 with $\mathrm{Cap}(U) \leqslant \varepsilon$. Now, for each $x \in Z_0$ we consider a closed ball $\bar{B}_x$ contained in U, and by Besicovitch covering theorem we find a subamily $\{\bar{B}_i\}_{i\in I} \subset \{\bar{B}_x\}_{x\in Z_0}$ with bounded overlap which covers Z_0, so that the properties (i) and (ii) in the lemma hold.

Next we will show that the set $\widetilde{\Omega} = \Omega \setminus \bigcup_{i\in I} \bar{B}_i$ is open. Indeed, we claim that

$$\overline{\bigcup_{i\in I} \bar{B}_i} \setminus \bigcup_{i\in I} \bar{B}_i \subset \partial\Omega. \tag{6.42}$$

This inclusion implies that

$$\Omega\backslash\overline{\bigcup_{i\in I}\bar{B}_i}=\Omega\backslash\left[\left(\overline{\bigcup_{i\in I}\bar{B}_i}\backslash\bigcup_{i\in I}\bar{B}_i\right)\cup\bigcup_{i\in I}\bar{B}_i\right]=\Omega\backslash\bigcup_{i\in I}\bar{B}_i=\widetilde{\Omega},$$

and thus ensures that $\widetilde{\Omega}$ is open.

To show the claim (6.42) consider $x\in\overline{\bigcup_{i\in I}\bar{B}_i}\backslash\bigcup_{i\in I}\bar{B}_i$ and recall that, by construction each ball $\bar{B}_i$ is closed. Then x must be the limit of a sequence of points belonging to infinitely many different balls $\bar{B}_{i_k}$, $i_k\in I$. It turns out that then we have $r(\bar{B}_{i_k})\to 0$. This is a straightforward consequence of the fact that any family of balls $\bar{B}_j$, $j\in J\subset I$, such that $\operatorname{dist}(\bar{B}_j,x)\leqslant 1$ and $0<\varepsilon\leqslant r(\bar{B}_j)\leqslant 1$ must be finite, by the finite overlap of the family $\{\bar{B}_i\}_{i\in I}$. The fact that $r(\bar{B}_{i_k})\to 0$ implies that $x\in\partial\Omega$, since the balls $\bar{B}_{i,k}$ are centered in $\partial\Omega$.

To prove (iv), write

$$\begin{aligned}\partial\widetilde{\Omega}=\partial\left(\Omega\backslash\bigcup_{i\in I}\bar{B}_i\right)\subset\partial\Omega\cup\overline{\bigcup_{i\in I}\bar{B}_i}&=\partial\Omega\cup\left(\overline{\bigcup_{i\in I}\bar{B}_i}\backslash\bigcup_{i\in I}\bar{B}_i\right)\cup\bigcup_{i\in I}\bar{B}_i\\&=\partial\Omega\cup\bigcup_{i\in I}\bar{B}_i=\left(\partial\Omega\backslash\bigcup_{i\in I}\bar{B}_i\right)\cup\bigcup_{i\in I}\bar{B}_i.\end{aligned}$$

On the other hand, by construction the interior of each ball $\bar{B}_i$ lies in the exterior of $\widetilde{\Omega}$, and thus

$$\partial\widetilde{\Omega}=\partial\widetilde{\Omega}\backslash\operatorname{ext}(\widetilde{\Omega})\subset\left[\left(\partial\Omega\backslash\bigcup_{i\in I}\bar{B}_i\right)\cup\bigcup_{i\in I}\bar{B}_i\right]\backslash\operatorname{ext}(\widetilde{\Omega})\subset\left(\partial\Omega\backslash\bigcup_{i\in I}\bar{B}_i\right)\cup\bigcup_{i\in I}\partial\bar{B}_i,$$

which proves (iv).

Next we check that $\widetilde{\Omega}$ is Wiener regular. That is, all the points $x\in\partial\widetilde{\Omega}$ are Wiener regular for $\widetilde{\Omega}$. We have to show that

$$\sum_{k=1}^{\infty}\frac{\operatorname{Cap}(\bar{A}(\xi,2^{-k-1},2^{-k})\backslash\widetilde{\Omega})}{\operatorname{Cap}(\bar{B}(\xi,2^{-k}))}=\infty$$

for all $x\in\partial\widetilde{\Omega}$. By (iv) we know that either $x\in\left(\partial\Omega\backslash\bigcup_{i\in I}\bar{B}_i\right)$ or $x\in\partial\bar{B}_i$ for some $i\in I$. In the latter case we have

$$\sum_{k=1}^{\infty}\frac{\operatorname{Cap}(\bar{A}(\xi,2^{-k-1},2^{-k})\backslash\widetilde{\Omega})}{\operatorname{Cap}(\bar{B}(\xi,2^{-k}))}\geqslant\sum_{k=1}^{\infty}\frac{\operatorname{Cap}(\bar{A}(\xi,2^{-k-1},2^{-k})\cap\bar{B}_i)}{\operatorname{Cap}(\bar{B}(\xi,2^{-k}))}=\infty,$$

since the complement of any ball $\bar{B}_i$ is Wiener regular. If $x\in\partial\Omega\backslash\bigcup_{i\in\bar{B}_i}\bar{B}_i$, then we know that x is Wiener regular for Ω, because $Z\subset\bigcup_{i\in I}\bar{B}_i$. Thus, using just that $\widetilde{\Omega}^c\supset\Omega^c$, we obtain

$$\sum_{k=1}^{\infty}\frac{\operatorname{Cap}(\bar{A}(\xi,2^{-k-1},2^{-k})\backslash\widetilde{\Omega})}{\operatorname{Cap}(\bar{B}(\xi,2^{-k}))}\geqslant\sum_{k=1}^{\infty}\frac{\operatorname{Cap}(\bar{A}(\xi,2^{-k-1},2^{-k})\backslash\Omega)}{\operatorname{Cap}(\bar{B}(\xi,2^{-k}))}=\infty.$$

So the proof that $\widetilde{\Omega}$ is Wiener regular is concluded.

The arguments to prove (vi) are quite similar to the ones for Theorem 6.33. For any $d \geqslant 2$ we consider any finite subfamily $J \subset I$ of the closed balls $\bar{B}_i$, and we let $E = \bigcup_{i \in J} \bar{B}_i$, so that E is compact and $\mathrm{Cap}(E) \leqslant \varepsilon$, by (ii). Since E consists of a union of finitely many closed balls, it follows either by Wiener's criterion or by the exterior ball criterion in Remark 5.38 that Ω_E is Wiener regular. Then, by Lemma 6.26, if μ_E stands for the equilibrium measure for E, the potential U_{μ_E} is continuous in $\mathbb{R}^d$ and $U_{\mu_E} = (\mathrm{Cap}(E))^{-1} \geqslant \varepsilon^{-1}$ in E.

Suppose first that $d \geqslant 3$ or $d = 2$ with $\Omega \subset B_{1/2}(0)$. Consider the function $f(x) = \mathrm{Cap}(E)\, U_{\mu_E}(x)$, and notice that it is superharmonic and continuous in $\mathbb{R}^d$, and it equals 1 on E. Also, it is positive in $\overline{\Omega}$ since $\Omega \subset B_{1/2}(0)$ in the planar case. So we have

$$\omega^x_{\widetilde{\Omega}}(E) \leqslant \int f \, d\omega^x_{\widetilde{\Omega}}. \tag{6.43}$$

By definition, letting $g = f|_{\partial\widetilde{\Omega}}$, the last integral above equals $H_g(x)$. Since f belongs to the upper Perron class for g in $\widetilde{\Omega}$, we have $H_g(x) \leqslant f(x)$. Thus,

$$\omega^x_{\widetilde{\Omega}}(E) \leqslant f(x) = \mathrm{Cap}(E)\, U_{\mu_E}(x) \leqslant \varepsilon\, U_{\mu_E}(x) \leqslant \varepsilon \sup_{y \in E} \mathcal{E}(x-y), \tag{6.44}$$

using that μ is a probability measure supported on E for the last inequality. Since the estimate above holds for any finite subfamily $J \subset I$, (6.40) holds.

In the case when $d = 2$ and Ω is unbounded, we can assume that $\mathrm{Cap}(\partial\Omega) > 0$. Then consider the function

$$g(x) = U_{\mu_E}(x) - U_{\mu_{\partial\Omega}}(x),$$

where $\mu_{\partial\Omega}$ is the equilibrium measure for $\partial\Omega$. Notice that g is superharmonic in Ω and

$$g(x) \geqslant \frac{1}{\mathrm{Cap}(E)} - \frac{1}{\mathrm{Cap}(\partial\Omega)} \geqslant \frac{1}{\varepsilon} - \frac{1}{\mathrm{Cap}(\partial\Omega)} \qquad \text{for } x \in E.$$

Then for ε small enough, $g(x) \geqslant \frac{1}{2\varepsilon} > 0$ on E, and since g vanishes at ∞, by the maximum principle g is positive in the unbounded component of Ω. Thus, for x in this component,

$$\begin{aligned}
\omega^x_{\widetilde{\Omega}}(E) &\leqslant 2\varepsilon\, g(x) = 2\varepsilon(U_{\mu_E}(x) - U_{\mu_{\partial\Omega}}(x)) \\
&= \frac{\varepsilon}{\pi} \int \log \frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x, \partial\Omega)}{|x-y|}\, d\mu_E(y) - \frac{\varepsilon}{\pi} \int \log \frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x, \partial\Omega)}{|x-y|}\, d\mu_\Omega(y) \\
&\leqslant \frac{\varepsilon}{\pi} \int \log \frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x, \partial\Omega)}{|x-y|}\, d\mu_E(y) \leqslant \frac{\varepsilon}{\pi} \log \frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x, \partial\Omega)}{\mathrm{dist}(x, E)},
\end{aligned}$$

where in the before to last inequality we took into account that $\log \frac{\mathrm{diam}\partial\Omega + \mathrm{dist}(x,\partial\Omega)}{|x-y|}$ is positive in $\partial\Omega$. For ε small enough, $\mathrm{dist}(x, E) \geqslant \frac{1}{2}\mathrm{dist}(x, \partial\Omega)$, and then (6.41) follows. $\square$

Lemma 6.38. *Let $\Omega \subset \mathbb{R}^d$ be open with compact boundary and let $p \in \Omega$. For any $\varepsilon > 0$, denote by $\widetilde{\Omega}_\varepsilon$ the Wiener regular set $\widetilde{\Omega}$ constructed in Proposition 6.37. In the case $d = 2$ suppose that $\mathrm{Cap}_L(\partial\Omega) > 0$. Then, for any Borel set $A \subset \partial\Omega$,*

$$\lim_{\varepsilon\to 0} \omega^p_{\widetilde{\Omega}_\varepsilon}(A) = \omega^p_\Omega(A). \tag{6.45}$$

Proof. In the case $d = 2$ we can assume that $\partial\Omega \subset B_{1/2}(0)$ by a suitable dilation. Let $A \subset \partial\Omega$ be a Borel set. Then, by Lemma 5.32,

$$\omega^p_{\widetilde{\Omega}_\varepsilon}(A) = \omega^p_{\widetilde{\Omega}_\varepsilon}(A \cap \partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon) \leqslant \omega^p_\Omega(A \cap \partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon) \leqslant \omega^p_\Omega(A). \tag{6.46}$$

To estimate $\omega^p_\Omega(A)$ in terms of $\omega^p_{\widetilde{\Omega}_\varepsilon}(A)$, observe first that

$$\omega^p_\Omega(\partial\Omega) \leqslant \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon). \tag{6.47}$$

Indeed, either if $d = 2$ or Ω is bounded, then both terms above equal 1, and in the case when $d = 3$ and Ω is unbounded observe that the function

$$u(x) = \begin{cases} \omega^x_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon) & \text{if } x \in \widetilde{\Omega}_\varepsilon, \\ 1 & \text{if } x \in \mathbb{R}^d \backslash \widetilde{\Omega}_\varepsilon, \end{cases}$$

is continuous in $\mathbb{R}^d$ (because $\widetilde{\Omega}_\varepsilon$ is Wiener regular), it is superharmonic in $\mathbb{R}^d$, and it tends to 0 at ∞. Then, from the definition of harmonic measure in unbounded domains with compact boundary, it follows easily that $\omega^x_\Omega(\partial\Omega) \leqslant u(x)$ for all $x \in \Omega$, which gives (6.47).

Applying (6.46) to $\partial\Omega\backslash A$, using (6.47) and Lemma (5.47), we get

$$\begin{aligned}
\omega^p_\Omega(A) = \omega^p_\Omega(\partial\Omega) - \omega^p_\Omega(\partial\Omega\backslash A) &\leqslant \omega^p_\Omega(\partial\Omega) - \omega^p_\Omega(\partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon\backslash A) \\
&\leqslant \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon) - \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon\backslash A) \\
&= \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega) + \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \cap \partial\Omega) - \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \cap \partial\Omega\backslash A) \\
&= \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega) + \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon \cap \partial\Omega \cap A) \\
&= \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega) + \omega^p_{\widetilde{\Omega}_\varepsilon}(A).
\end{aligned}$$

Hence,

$$|\omega^p_{\widetilde{\Omega}_\varepsilon}(A) - \omega^p_\Omega(A)| \leqslant \omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega). \tag{6.48}$$

Since $\partial\widetilde{\Omega}_\varepsilon$ is contained in the union of the balls B_i, $i \in I$, in Proposition 6.37, by the property (vi) in the proposition $\omega^p_{\widetilde{\Omega}_\varepsilon}(\partial\widetilde{\Omega}_\varepsilon\backslash\partial\Omega)$ tends to 0 as $\varepsilon \to 0$. □

Notice that, by (6.48), the convergence in (6.45) is uniform with respect to the set $A \subset \partial\Omega$.

7 Harmonic measure and Green function in Wiener regular open sets

In this section we will assume that Ω is an open Wiener regular set.

7.1 The Green function in terms of harmonic measure in bounded open sets

For a bounded open Wiener regular set $\Omega \subset \mathbb{R}^d$, we may write the Green function in terms of harmonic measure. Let us see how.

Given $x \in \Omega$, define the harmonic extension

$$v^x(y) := -\int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) \quad \text{for } y \in \Omega, \tag{7.1}$$

where $\mathcal{E}^x$ is the fundamental solution of the minus Laplacian with pole at x. Note that $\mathcal{E}^x$ is continuous in $z \in \partial\Omega$ and Ω is Wiener regular, so $v^x \in C(\overline{\Omega})$ and its boundary values are opposite to those of the fundamental solution. Thus,

$$G^x(y) = \begin{cases} \mathcal{E}^x(y) + v^x(y) & \text{for } y \in \Omega\backslash\{x\}, \\ 0 & \text{otherwise}, \end{cases} \tag{7.2}$$

is continuous away from the pole, and harmonic in $\mathbb{R}^d\backslash(\partial\Omega \cup \{x\})$.

Thus, in a sense G is the continuous solution to the Dirichlet problem

$$\begin{cases} -\Delta G^x = \delta_x & \text{in } \Omega, \\ G^x = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular bounded open set. The Green function for Ω is non-negative in Ω, and positive in the component of Ω that contains x. Further, it is subharmonic in $\mathbb{R}^d\backslash\{x\}$.*

Proof. To prove the first statement, notice that $G^x \equiv 0$ in any component V of Ω which does not contain x, by the maximum principle, since G^x is harmonic in V and vanishes continuously in ∂V. If V_x is the component of Ω that contains x, we consider any $\varepsilon > 0$ small enough such that $\bar{B}_{2\varepsilon}(x) \subset V_x$, and we set $V_{x,\varepsilon} = V_x\backslash\bar{B}_\varepsilon(x)$. For ε small enough, $G^x > 0$ in $\partial\bar{B}_\varepsilon(x)$, and then by the maximum principle, it follows that $G^x > 0$ in $V_{x,\varepsilon}$. So $G^x > 0$ in V_x.

Regarding the second statement, using the maximum principle for harmonic functions, one can check that the Green function satisfies the condition in Lemma 5.8, implying the subharmonicity of the Green function (7.2) away from the pole. □

Here there is a small trouble. We have defined the Green function in two different ways, solving the Dirichlet problem in the Sobolev sense and in the continuous sense. Fortunately, both definitions coincide in Wiener regular open sets:

Lemma 7.2. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular bounded open set. Let v^x and G^x be defined as in (7.1) and (7.2), and let ψ^x be a bump function satisfying $\chi_{B_{2t}(x)^c} \leqslant \psi^x \leqslant \chi_{B_t(x)^c}$ for $t < \frac{1}{3}\mathrm{dist}(x, \partial\Omega)$. Then $v^x \in H^1(\Omega)$, and $\psi^x G^x \in H^1_0(\Omega)$. So G^x coincides with the other Green function defined in Section 3.2. In particular the Green function is symmetric and $G^x \in W^{1,p}(\Omega)$ for every $p < \frac{d}{d-1}$.*

Proof. First we will check that $G^x \in H^1(\Omega \backslash B_{3t}(x))$. Since Ω is bounded, it is enough to check that $\|G^x\|_{H^1(B \cap \Omega)} < +\infty$ for every ball B such that $2B \cap B_{2t}(x) = \varnothing$. To show this fact we will use Caccioppoli inequality, but in order to apply it, we need to know *a priori* the finiteness of the L^2 norm of the gradient. To avoid a circular argument, we need to define

$$u_\varepsilon(y) := \max\{G^x(y) - \varepsilon, 0\} \qquad \text{for } y \in B_{2t}(x)^c. \tag{7.3}$$

Let us check the properties of u_ε. First, since $G^x \in C^\infty(\Omega \backslash B_{2t}(x))$, we can infer that $u_\varepsilon \in H^1(2B)$ (see [EG15, Theorem 4.4]). On the other hand, since G^x is subharmonic away from the pole (see Lemma 5.7), also u_ε is subharmonic. Moreover, it is non-negative. Finally, we can apply the Caccioppoli inequality and the maximum principle to get

$$\int_B |\nabla u_\varepsilon|^2 \lesssim r(B)^{-2} \int_{2B} |u_\varepsilon|^2 \leqslant r(B)^{-2} \int_{2B} (G^x)^2 \lesssim r(B)^{d-2} \max_{\partial B_{2t}(x)} (G^x)^2,$$

which is independent of ε.

By the monotone convergence theorem, we get

$$\int_{B \cap \Omega} |\nabla G^x|^2 = \lim_{\varepsilon \to 0} \int_B |\nabla u_\varepsilon|^2 \lesssim r(B)^{d-2} \max_{\partial B_{2t}(x)} (G^x)^2 < +\infty,$$

i.e.,

$$G^x \in H^1(\Omega \backslash B_{3t}(x)),$$

and thus $v^x = G^x - \mathcal{E}^x \in H^1(\Omega \backslash B_{3t}(x))$ as well. Since it is C^∞ in a neighborhood of the pole, we get $v^x \in H^1(\Omega)$.

It remains to check $\psi^x G^x \in H^1_0(\Omega)$. For every $y \in \Omega$ define $u_\varepsilon(y) := \max\{\psi^x(y) G^x(y) - \varepsilon, 0\}$. Then

$$\lim_{\varepsilon \to 0} u_\varepsilon(y) = \psi^x(y) G^x(y), \quad \text{and} \quad \lim_{\varepsilon \to 0} \nabla u_\varepsilon(y) = \nabla(\psi^x G^x)(y).$$

Moreover, by the triangle inequality

$$\|u_\varepsilon - \psi^x G^x\|_{H^1(\Omega)} \leqslant \|u_\varepsilon\|_{H^1(\Omega)} + \|\psi^x G^x\|_{H^1(\Omega)} \leqslant 2\|\psi^x G^x\|_{H^1(\Omega)}.$$

Thus, by the dominated convergence theorem, we get

$$\|u_\varepsilon - \psi^x G^x\|_{H^1} \xrightarrow{\varepsilon \to 0} 0.$$

Note that u_ε is compactly supported in $\Omega\backslash B_t(x)$, and it is Lipschitz. Thus, we have shown the existence of Lipschitz functions (not C^∞ in general) with compact support converging to $\psi^x G^x$ in the Sobolev norm. Proving that this implies that $\psi^x G^x \in H^1_0(\Omega)$ is an exercise left for the reader.

Now, v^x is the harmonic extension of a continuous function, and hence weakly harmonic by Theorem 2.3 and integration by parts. Moreover,

$$v^x + \psi^x \mathcal{E}^x = v^x(1-\psi^x) + \psi^x G^x \in H^1_0(\Omega).$$

Thus it is the unique weak solution to (3.7) in the sense of (3.3), see Corollary 3.4. That is, both definitions (3.7) and (7.1) of v^x coincide. Therefore, both definitions of Green function coincide as well, and Lemmas 3.5 and 3.7 apply. □

Remark 7.3. In fact, when a Sobolev function vanishes continuously in the boundary, its gradient can be extended by zero in the complement of the open set, the proof is similar to [EG15, Theorem 4.4]. Thus, we have shown that $G^x \in H^1(\mathbb{R}^d\backslash B_\varepsilon(x))$, with $\nabla G^x(y) \equiv 0$ for $y \in \Omega^c$.

For $x \in \mathbb{R}^d\backslash\Omega$ and $y \in \Omega$, we will also set

$$G^x(y) = 0. \tag{7.4}$$

This choice, together with Lemmas 3.7 and 7.2 implies that

$$G^x(y) = G^y(x) \qquad \text{for all } (x,y) \in \mathbb{R}^d \times \mathbb{R}^d\backslash(\Omega^c \times \Omega^c) \text{ with } x \neq y. \tag{7.5}$$

Note that the equation (7.2) is still valid for $x \in \mathbb{R}^d\backslash\overline{\Omega}$ and $y \in \Omega$. The case when $x \in \partial\Omega$ and $y \in \Omega$ is more delicate and the identity (7.2) may fail. However, we have the following partial result:

Lemma 7.4. *Let $\Omega \subset \mathbb{R}^d$ be bounded and Wiener regular and let $y \in \Omega$. For m-almost all $x \in \Omega^c$ we have*

$$\mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) = 0. \tag{7.6}$$

Clearly, in the particular case where $m(\partial\Omega) = 0$, this result is a consequence of the aforementioned fact that (7.2) also holds for all $x \in \mathbb{R}^d\backslash\overline{\Omega}$, $y \in \Omega$, with $G^x(y) = 0$.

Proof. Let $A \subset \Omega^c$ be a compact set with $m(A) > 0$. Observe that the potential $U_A := U_{\chi_A m} = \mathcal{E} * \chi_A$ is continuous, bounded in $\mathbb{R}^d$, and harmonic in A^c, see Remark 6.6. Then, by Fubini we have for all $y \in \Omega$,

$$\begin{aligned}\int_A \left(\mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z)\right) dm(x) &= U_A(y) - \int_{\partial\Omega}\int_A \mathcal{E}^x(z)\, dm(x)\, d\omega^y(z)\\ &= U_A(y) - \int_{\partial\Omega} U_A(z)\, d\omega^y(z) = 0,\end{aligned}$$

using that U_A is harmonic in $\Omega \subset A^c$ and bounded on $\partial\Omega$ for the last identity. Since the compact set $A \subset \Omega^c$ is arbitrary, the lemma follows. □

Remark 7.5. As a corollary of the preceding lemma we deduce that

$$G^x(y) = \mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) \qquad \text{for } m\text{-a.e. } x \in \mathbb{R}^d.$$

Lemma 7.6. *For all $x \in \Omega$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\int \varphi\, d\omega^x - \varphi(x) = \int_\Omega \Delta\varphi\, G^x\, dm = -\int_\Omega \nabla\varphi \cdot \nabla G^x\, dm.$$

Proof. The first identity follows from Lemma 3.7 and (7.4), the preceding remark, and Fubini. Indeed,

$$\begin{aligned}\int_\Omega \Delta\varphi(y)\, G^x(y)\, dy &= \int_{\mathbb{R}^d} \Delta\varphi(y)\, G^y(x)\, dy = \int \Delta\varphi(y) \left(\mathcal{E}^y(x) - \int_{\partial\Omega} \mathcal{E}^y(z)\, d\omega^x(z) \right) dy \\ &= (\Delta\varphi * \mathcal{E})(x) - \int_{\partial\Omega} (\Delta\varphi * \mathcal{E})(z)\, d\omega^x(z) \\ &= -\varphi(x) + \int_{\partial\Omega} \varphi(z)\, d\omega^x(z).\end{aligned}$$

The last identity in the lemma follows integrating by parts and a density argument if $\varphi \in C_c^\infty(\Omega)$. Thus we can reduce to the case $x \notin \operatorname{supp}\varphi$. Replacing G^x by u_ε as in (7.3), we get

$$\int \nabla \cdot (u_\varepsilon \nabla\varphi) = 0$$

by the divergence theorem. Thus, the last identity follows by letting $\varepsilon \to 0$, since

$$\left| \int \nabla \cdot [(G^x - u_\varepsilon)\nabla\varphi]\, dm \right| \leqslant \int_{\{y\in\Omega : G^x(y) \leqslant \varepsilon\}} |\nabla G^x \cdot \nabla\varphi|\, dm + \varepsilon \int_\Omega |\Delta\varphi|\, dm \xrightarrow{\varepsilon\to 0} 0.$$

□

Notice that, by the preceding lemma, in the sense of distributions, that is in the dual space $\mathcal{D}'(\mathbb{R}^d)$ (here, as in the literature in functional analysis, $\mathcal{D}$ stands for C^∞ functions with compact support, equipped with a certain topology, see [Rud91, Chapter 6]), we have

$$\Delta G^x = \omega^x - \delta_x \qquad \text{for all } x \in \Omega.$$

For smooth domains with smooth Green function, we have the following:

Proposition 7.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1 domain, $x \in \Omega$ and suppose that $G^x \in C^1(\overline{\Omega} \backslash B_t(x))$ for some $t > 0$. Then*

$$\omega^x = -(\partial_\nu G^x)\, \sigma,$$

where ν is the unit outer normal to $\partial\Omega$ and σ is the surface measure on $\partial\Omega$.

Proof. It suffices to show that for any $\varphi \in \mathcal{D} = C_c^\infty(\mathbb{R}^d)$ it holds

$$\int_{\partial\Omega} \varphi\, d\omega^x(y) = -\int_{\partial\Omega} \varphi(y)\, \partial_\nu G^x(y)\, d\sigma(y).$$

We may assume that φ vanishes in a neighborhood of x by modifying suitably φ far away from $\partial\Omega$, since the domain of integration in both integrals above is $\partial\Omega$. So consider $r > 0$ such that $B_{2r}(x) \subset \Omega$ and $\operatorname{supp}\varphi \subset \mathbb{R}^d \backslash B_{2r}(x)$. Denote $\Omega^r = \Omega \backslash \bar{B}_r(x)$. Using that G^x is harmonic in Ω^r and that φ vanishes in $B_{2r}(x)$, by Lemma 7.6 and Green's formula we have

$$\begin{aligned}\int \varphi\, d\omega^x(y) &= \int_\Omega \Delta\varphi(y)\, G^x(y)\, dy = \int_{\Omega^r} \Delta\varphi(y)\, G^x(y)\, dy \\ &= -\int_{\partial\Omega^r} \varphi(y)\, \partial_\nu G^x(y)\, d\sigma(y) = -\int_{\partial\Omega} \varphi(y)\, \partial_\nu G^x(y)\, d\sigma(y).\end{aligned}$$

□

Lemma 7.8. *Let B be a ball centered in $\partial\Omega$ and let $x \in \Omega \backslash 2B$. Then,*

$$\omega^x(B) \lesssim r(B)^{d-2} \fint_{2B} G^x(y)\, dy.$$

Proof. Let φ be a bump function such that $\chi_B \leqslant \varphi \leqslant \chi_{2B}$ with $\|D^2\varphi\| \lesssim \frac{1}{r(B)^2}$. By Lemma 7.6, we have

$$\omega^x(B) \leqslant \int \varphi\, d\omega^x = \int \Delta\varphi(y)\, G^x(y)\, dy \lesssim \frac{1}{r(B)^2} \int_{2B} G^x(y)\, dy = r(B)^{d-2} \fint_{2B} G^x(y)\, dy.$$

□

As we shall see in further chapters, when Ω is an NTA or CDC uniform domain, for x and B as in the preceding lemma, we have

$$\omega^x(B) \approx r(B)^{d-2}\, G^x(X_B),$$

where X_B is an interior corkscrew point for B. One can view the result in the preceding lemma as a weak version of the estimate $\omega^x(B) \lesssim r(B)^{d-2}\, G^x(X_B)$. In the next sections we will obtain some estimates in the converse direction.

7.2 The Green function in unbounded open sets with compact boundary

Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact[1] boundary. In the case $d \geqslant 3$, we defined the Green function for Ω in the same we did for bounded open sets.

[1]We assume compact sets to be non-empty.

That is, given $x \in \Omega$, we consider the harmonic extension

$$v^x(y) := -\int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega^y(z) \quad \text{for } y \in \Omega, \tag{7.7}$$

Then we define the Green function with pole at x as follows:

$$G^x(y) = \begin{cases} \mathcal{E}^x(y) + v^x(y) & \text{for } y \in \Omega \backslash \{x\}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.8}$$

Notice that G^x is continuous away from the pole, harmonic in $\mathbb{R}^d \backslash \partial\Omega$, and $G^x(y) \to 0$ as $y \to \infty$.

In the case $d = 2$ we cannot define G^x as above because otherwise this will have a pole at ∞, which is not convenient. Instead we want G^x to be bounded at ∞. If Ω is not dense in $\mathbb{R}^d$, we can take a point $\xi \in \mathbb{R}^2 \backslash \overline{\Omega}$ and we can define G^x as above, replacing $\mathcal{E}^x$ in (7.7) and (7.8) by $\mathcal{E}^x - \mathcal{E}^\xi$. Notice that $\mathcal{E}^x - \mathcal{E}^\xi$ has a logarithmic singularity (i.e., a pole) at x, it is continuous in $\partial\Omega$, and it is bounded at ∞. Then it easily follows that the Green function G^x defined in this way has a pole at x, it is bounded at ∞, and vanishes continuously on $\partial\Omega$.

For an arbitrary Wiener regular unbounded open set with compact boundary in the plane, we define G^x as in (7.7) and (7.8), replacing $\mathcal{E}^x$ by $\mathcal{E}^x - U_\mu$, where μ is the equilibrium measure for $\partial\Omega$. Again it turns out that the Green function G^x defined in this way has a pole at x, it is bounded at ∞, and vanishes continuously on $\partial\Omega$. Indeed, recall that the equilibrium potential is continuous in $\mathbb{R}^d$ when Ω is Wiener regular by Lemma 6.26. Further, this can be written as follows, for $y \in \Omega$,

$$\begin{aligned} G^x(y) &= \mathcal{E}^x(y) - U_\mu(y) - \int_{\partial\Omega} (\mathcal{E}^x - U_\mu)\, d\omega^y \\ &= \frac{1}{2\pi} \int_{\partial\Omega} \log \frac{|y-\xi|}{|y-x|}\, d\mu(\xi) - \frac{1}{2\pi} \int_{\partial\Omega} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega^y(z). \end{aligned} \tag{7.9}$$

The analog of Lemma 7.1 holds for unbounded domains with compact boundary:

Lemma 7.9. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. The Green function for Ω is non-negative in Ω, and positive in the component of Ω that contains x. Further, it is subharmonic in $\mathbb{R}^d \backslash \{x\}$. In the case $d \geqslant 3$, G^x vanishes at ∞, and in the case $d = 2$, it is bounded at ∞*

The proof is similar to the one of Lemma 7.1 and we leave this for the reader.

Next we show that the Green function G^x is "locally" in the Sobolev space $H^1_0(\Omega)$. More precisely:

Lemma 7.10. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary and let $x \in \Omega$. Let G^x be defined as in (7.8) in the case $d \geqslant 3$ and as in (7.9) in the case $d = 2$. For $0 < t < \frac{1}{3}\mathrm{dist}(x, \partial\Omega)$, let ψ^x be a bump function satisfying $\chi_{B_{2t}(x)^c} \leqslant \psi^x \leqslant \chi_{B_t(x)^c}$. For any $r > 0$ such that $\partial\Omega \subset B_r(0)$, let ψ_r be a bump function such that $\chi_{B_r(x)} \leqslant \psi_r \leqslant \chi_{B_{2r}(x)}$. Then $\psi^x\, \psi_r\, G^x \in H^1_0(\Omega)$.*

The arguments for this lemma are similar to the ones for Lemma 7.2 and so we omit them again.

Lemma 7.11. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. For $r > 0$ such that $\partial\Omega \subset B_r(0)$, let $\Omega_r = \Omega \cap B_r(0)$. For $x \in \Omega$ and $r > |x|$, let G^x and G_r^x be the respective Green functions for Ω and Ω_r with pole at x. Then $G_r^x \to G^x$ as $r \to \infty$ uniformly on bounded sets.*

Proof. In the the case $d \geqslant 3$, for $x, y \in \Omega$ with $x \neq y$, we have

$$G^x(y) = \mathcal{E}^x(y) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_\Omega^y(z).$$

The same identity holds for G_r^x, replacing $\partial\Omega$ and ω_Ω by $\partial\Omega_r$ and ω_{Ω_r}, respectively. Thus,

$$\begin{aligned} G_r^x(y) - G^x(y) &= \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_\Omega^y(z) - \int_{\partial\Omega_r} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z) \\ &= \left(\int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_\Omega^y(z) - \int_{\partial\Omega} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z) \right) - \int_{\partial B_r(0)} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z). \end{aligned}$$

By Remark 5.44, the term in parentheses on the right hand side tends to 0 as $r \to \infty$. On the other hand, the second term can be bounded as follows:

$$\left| \int_{\partial B_r(0)} \mathcal{E}^x(z)\, d\omega_{\Omega_r}^y(z) \right| \lesssim \frac{1}{\operatorname{dist}(x, \partial B_r(0))^{d-2}}\, \omega_{\Omega_r}^y(\partial B_r(0)) \leqslant \frac{1}{\operatorname{dist}(x, \partial B_r(0))^{d-2}},$$

which also tends to 0 uniformly on bounded subsets of Ω.

In the case $d = 2$, the Green function G^x for Ω can be written as in (7.9). The Green function G_r^x for Ω_r can be written in a similar fashion, for $y \in \Omega_r$:

$$G_r^x(y) = \frac{1}{2\pi} \int_{\partial\Omega} \log \frac{|y-\xi|}{|y-x|}\, d\mu(\xi) - \frac{1}{2\pi} \int_{\partial\Omega_r} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega_{\Omega_r}^y(z). \tag{7.10}$$

Here μ is the equilibrium measure for $\partial\Omega$. To check the preceding identity, notice that μ is a probability measure and we have

$$\frac{1}{2\pi} \int_{\partial\Omega} \log|y-\xi|\, d\mu(\xi) - \frac{1}{2\pi} \int_{\partial\Omega_r} \int_{\partial\Omega} \log|z-\xi|\, d\mu(\xi)\, d\omega_{\Omega_r}^y(z) = 0,$$

because the $U_\mu(y) = -\frac{1}{2\pi} \int_{\partial\Omega} \log|y-\xi|\, d\mu(\xi)$ is harmonic and continuous in $\overline{\Omega_r}$. Then, by (7.9) and (7.10), we get

$$\begin{aligned} 2\pi(G_r^x(y) - G^x(y)) &= \int_{\partial\Omega} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega_\Omega^y(z) - \int_{\partial\Omega_r} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega_{\Omega_r}^y(z) \\ &= \left[\int_{\partial\Omega} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega_\Omega^y(z) - \int_{\partial\Omega} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega_{\Omega_r}^y(z) \right] \\ &\quad - \int_{\partial B_r(0)} \int_{\partial\Omega} \log \frac{|z-\xi|}{|z-x|}\, d\mu(\xi)\, d\omega_{\Omega_r}^y(z). \end{aligned}$$

By Remark 5.44 (applied with $f(z) := \mathcal{E}^x(z) - U_\mu(z) = \frac{1}{2\pi}\int_{\partial\Omega}\log\frac{|z-\xi|}{|z-x|}\,d\mu(\xi)$), it follows that the first term in brackets tends to 0 uniformly in bounded subsets of Ω. Using the fact that $f(z) \to 0$ as $z \to \infty$, we also get easily that that the last term tends to 0 uniformly in bounded subsets of Ω. □

Thanks to the preceding lemma, many of the results obtained in the previous section for the Green function in Wiener regular bounded open sets can be extended to the case of unbounded open sets with compact boundaries. First, we easily get that the Green function is symmetric:

Lemma 7.12. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. For all $x, y \in \Omega$, with $x \neq y$, the Green function for Ω satisfies $G^x(y) = G^y(x)$.*

Proof. Let $\Omega_r = \Omega \cap B_r(0)$, with $r > 0$ big enough so that $\partial\Omega \subset B_r(0)$ and $x, y \in \Omega_r$. Let G_r denote the Green function for Ω_r. Then we have

$$G^x(y) = \lim_{r\to\infty} G_r^x(y) = \lim_{r\to\infty} G_r^y(x) = G^y(x).$$

□

From now on, quite often we will write

$$G(x, y) = G^x(y) = G^y(x).$$

Lemma 7.13. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. For all $x \in \Omega$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\int \varphi\, d\omega^x(y) - \varphi(x) = \int_\Omega \Delta\varphi(y)\, G^x(y)\, dy = -\int_\Omega \nabla\varphi(y)\cdot\nabla G^x(y)\, dy.$$

Proof. The first identity follows from the one derived for bounded open sets in Lemma 7.6 and from the uniform convergence of G_r^x to G^x in bounded subsets of Ω (by Lemma 7.11) and the weak convergence of $\omega^x_{\Omega_r}$ to ω^x (by Remark 5.44). The second one follows from the first one by integration by parts. □

Proposition 7.14. *Let $\Omega \subset \mathbb{R}^d$ be a domain with compact boundary $\partial\Omega = E \cup \gamma$ where E is either compact or empty, γ is a C^1 curve and $E \cap \gamma = \varnothing$, and let $x \in \Omega$. If $G^x \in C^1(\Omega \cup \gamma)$, then*

$$\omega^x|_\gamma = -(\partial_\nu G^x)\,\sigma,$$

where ν is the unit outer normal to $\gamma \subset \partial\Omega$ and σ is the surface measure on γ.

Proof. This follows from the preceding lemma, arguing as in Proposition 7.7. □

Lemma 7.15. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular unbounded open set with compact boundary. Let B be a ball centered in $\partial\Omega$ and let $x \in \Omega \backslash 2B$. Then,*

$$\omega^x(B) \lesssim r(B)^{d-2} \fint_{2B} G^x(y)\, dy.$$

Proof. This is proven in the same way as Lemma 7.8 for the case of bounded open sets. □

7.3 Newtonian capacity, harmonic measure, and Green's function in the case $d \geqslant 3$

In this whole section we assume that Ω is a Wiener regular open set with compact boundary in $\mathbb{R}^d$, with $d \geqslant 3$ (Ω either bounded or unbounded).

Lemma 7.16. *Let $d \geqslant 3$ and $\Omega \subset \mathbb{R}^d$ be an open Wiener regular set with compact boundary. Let $\bar{B}$ be a closed ball interesecting centered at $\partial\Omega$. Then*

$$\omega^x(\bar{B}) \geqslant c(d)\, \frac{\mathrm{Cap}(\frac{1}{4}\bar{B}\backslash\Omega)}{r(\bar{B})^{d-2}} \qquad \text{for all } x \in \tfrac{1}{4}\bar{B} \cap \Omega,$$

with $c(d) > 0$.

Proof. We can assume that Ω is bounded. Otherwise, the estimate above follows from the analogous estimate applied to $\Omega_r = \Omega \cap B_r(0)$ letting $r \to \infty$.

Let $\mu_{\frac{1}{4}\bar{B}\backslash\Omega}$ be the equilibrium measure for $\frac{1}{4}\bar{B}\backslash\Omega$, and let $\mu = \mathrm{Cap}(\frac{1}{4}\bar{B}\backslash\Omega)\, \mu_{\frac{1}{4}\bar{B}\backslash\Omega}$, so that $\|U_\mu\|_\infty \leqslant 1$ and $\|\mu\| = \mathrm{Cap}(\frac{1}{4}\bar{B}\backslash\Omega)$. Notice that, for all $x \in B^c$,

$$U_\mu(x) = \int \frac{c_d}{|x-y|^{d-2}}\, d\mu(y) \leqslant \frac{c_d\|\mu\|}{(\frac{3}{4}r(\bar{B}))^{d-2}}.$$

Consider the function $f(x) = U_\mu(x) - \frac{c_d\|\mu\|}{(\frac{3}{4}r(\bar{B}))^{d-2}}$. Using that $f(x) \leqslant 0$ in B^c, $f(x) \leqslant 1$ in $\bar{B}$, and that f is harmonic and bounded in Ω, by Corollary 5.36 and the maximum principle we deduce that, for all $x \in \Omega$,

$$\omega^x(\bar{B}) \geqslant \omega^x(B) \geqslant f(x).$$

In particular, for $x \in \frac{1}{4}\bar{B} \cap \Omega$ we have

$$\begin{aligned}
\omega^x(\bar{B}) &\geqslant \int \frac{c_d}{|x-y|^{d-2}}\, d\mu(y) - \frac{c_d\|\mu\|}{(\frac{3}{4}r(\bar{B}))^{d-2}} \\
&\geqslant \frac{c_d\|\mu\|}{(\frac{1}{2}r(\bar{B}))^{d-2}} - \frac{c_d\|\mu\|}{(\frac{3}{4}r(\bar{B}))^{d-2}} = c_d\big(2^{d-2} - (\tfrac{4}{3})^{d-2}\big)\, \frac{\mathrm{Cap}(\frac{1}{4}\bar{B}\backslash\Omega)}{r(\bar{B})^{d-2}},
\end{aligned}$$

which proves the lemma. □

Remark 7.17. In fact, a quick inspection of above proof shows that Lemma 7.16 also holds assuming that $\frac{1}{4}\bar{B} \cap \Omega \neq \varnothing$ instead of assuming that B is centered at $\partial\Omega$. Notice also that the lemma is trivially true if $\frac{1}{4}\bar{B}\backslash\Omega = \varnothing$.

Lemma 7.18. *Let $d \geqslant 3$ and $\Omega \subset \mathbb{R}^d$ be an open Wiener regular set with compact boundary. Let $\bar{B}$ be a closed ball centered at $\partial\Omega$. Then, for all $a > 2$,*

$$\omega^x(a\bar{B}) \gtrsim \inf_{z \in 2\bar{B}\cap\Omega} \omega^z(a\bar{B})\, r(\bar{B})^{d-2}\, G^x(y) \quad \text{for all } x \in \Omega\backslash 2\bar{B} \text{ and } y \in \bar{B} \cap \Omega, \tag{7.11}$$

with the implicit constant independent of a.

Proof. We can assume that Ω is bounded. Otherwise, the estimate above follows from the one applied to $\Omega_r = \Omega \cap B_r(0)$ letting $r \to \infty$.

Fix $y \in \bar{B}\cap\Omega$ and note that for every $x \in \partial(2\bar{B})\cap\Omega$ we have $\inf_{z\in 2\bar{B}\cap\Omega} \omega^z(a\bar{B}) \leqslant \omega^x(a\bar{B})$ and, therefore

$$G^x(y) \leqslant \mathcal{E}^x(y) \approx \frac{1}{|x-y|^{d-2}} \leqslant \frac{c}{r(\bar{B})^{d-2}} \leqslant \frac{c\,\omega^x(a\bar{B})}{r(\bar{B})^{d-2}\,\inf_{z\in 2\bar{B}\cap\Omega} \omega^z(a\bar{B})}. \tag{7.12}$$

Let us observe that the two non-negative functions

$$u(x) = c^{-1}\, G^x(y)\, r(\bar{B})^{d-2} \inf_{z\in 2\bar{B}\cap\Omega} \omega^z(a\bar{B}) \qquad \text{and} \qquad v(x) = \omega^x(a\bar{B})$$

are harmonic, hence continuous, in $\Omega\backslash\bar{B}$. Note that (7.12) says that $u \leqslant v$ in $\partial(2\bar{B}) \cap \Omega$ and hence $\lim_{\Omega\backslash 2\bar{B}\ni z\to x}(v-u)(z) = (v-u)(x) \geqslant 0$ for every $x \in \partial(2\bar{B}) \cap \Omega$. On the other hand, for a fixed $y \in \bar{B} \cap \Omega$, one has that $\lim_{\Omega\ni z\to x} G^z(y) = 0$ for every $x \in \partial\Omega$. Gathering all these we conclude that

$$\liminf_{\Omega\backslash 2\bar{B}\ni z\to x} (v-u)(z) \geqslant 0$$

for every $x \in \partial(\Omega\backslash 2\bar{B})$. The lemma follows by the maximum principle. □

Combining the two preceding lemmas, choosing $a = 8$, we obtain:

Lemma 7.19. *Let $d \geqslant 3$ and $\Omega \subset \mathbb{R}^d$ be an open Wiener regular set with compact boundary. Let $\bar{B}$ be a closed ball centered at $\partial\Omega$. Then,*

$$\omega^x(8\bar{B}) \gtrsim_d \mathrm{Cap}(2\bar{B}\backslash\Omega)\, G^x(y) \quad \text{for all } x \in \Omega\backslash 2\bar{B} \text{ and } y \in \bar{B} \cap \Omega. \tag{7.13}$$

We will show in Chapter 8 that, in the case when Ω is an NTA domain, we have $\omega^x(8\bar{B}) \approx \omega^x(\bar{B})$ and $\mathrm{Cap}(2\bar{B}\backslash\Omega) \approx \mathrm{Cap}(\bar{B}) = r(\bar{B})^{d-2}$, so that we recover the estimate

$$\omega^x(\bar{B}) \gtrsim r(\bar{B})^{d-2}\, G^x(y),$$

for $y \in \frac{1}{4}\bar{B}$. Thus, Lemma 7.19 can be considered as a weak version of the converse inequality to the one in Lemma 7.8.

7.4 Logarithmic capacity, harmonic measure, and Green's function in the plane

Lemma 7.20. *Let $\Omega \subset \mathbb{R}^2$ be a Wiener regular open set with compact boundary and let $\bar{B}$ be a closed ball centered at $\partial\Omega$. Then*

$$\omega^x(\bar{B}) \gtrsim \frac{1}{\log \dfrac{\mathrm{Cap}_L(\bar{B})}{\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega)}} = \frac{1}{\log \dfrac{r(\bar{B})}{\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega)}} \qquad \text{for all } x \in \tfrac{1}{4}\bar{B} \cap \Omega.$$

Remark the estimate in the lemma is equivalent to saying that

$$\omega^x(\bar{B}) \gtrsim \frac{1}{\dfrac{1}{\mathrm{Cap}_W(\frac{1}{4}\bar{B}\backslash\Omega)} - \dfrac{1}{\mathrm{Cap}_W(\bar{B})}} \qquad \text{for all } x \in \tfrac{1}{4}\bar{B} \cap \Omega.$$

Proof. We can assume that Ω is bounded by proving first the estimate above for $\Omega_t = \Omega \cap B_t(0)$ and then letting $t \to \infty$. We denote $r = r(\bar{B})$. Replacing Ω by $\frac{1}{4r}\,\Omega$ if necessary, we can assume that $\mathrm{diam}(\bar{B}) < 1$. Then, denoting $E = \frac{1}{4}\bar{B}\backslash\Omega$, identity (6.14) holds.

Let μ be the optimal measure for the supremum in (6.14), so that $\mathrm{supp}\mu \subset E$, $\mu(E) = \mathrm{Cap}_W(E)$, and the potential $U_\mu = \mathcal{E} * \mu$ is harmonic out of E and it satisfies $\|U_\mu\|_\infty \leqslant 1$. For all $z \in \frac{1}{4}\bar{B}$ and all $y \in E$ we have $|z-y| \leqslant \frac{1}{2}\,r$. Therefore,

$$U_\mu(z) = \frac{1}{2\pi}\int \log\frac{1}{|z-y|}\,d\mu(y) \geqslant \frac{1}{2\pi}\int \log\frac{2}{r}\,d\mu(y) = \frac{\mu(E)}{2\pi}\,\log\frac{2}{r} \qquad \text{for all } z \in \tfrac{1}{4}\bar{B}.$$

Also, for $z \in B^c$, we have $\mathrm{dist}(z, E) \geqslant \frac{3}{4} r(\bar{B})$, and thus

$$U_\mu(z) \leqslant \frac{1}{2\pi}\int \log\frac{4}{3r}\,d\mu(y) = \frac{\mu(E)}{2\pi}\,\log\frac{4}{3r} \qquad \text{for all } z \in B^c.$$

Consider now the function

$$f = U_\mu - \frac{\mu(E)}{2\pi}\,\log\frac{4}{3r}.$$

Observe that

$$f(z) \geqslant \frac{\mu(E)}{2\pi}\,\log\frac{2}{r} - \frac{\mu(E)}{2\pi}\,\log\frac{4}{3r} = \frac{\mu(E)}{2\pi}\,\log\frac{3}{2} \qquad \text{for all } z \in \tfrac{1}{4}\bar{B}$$

and

$$f(z) \leqslant 0 \qquad \text{for all } z \in B^c.$$

Combining the maximum principle with Corollary 5.36, and using the fact that $x \in \frac{1}{4}\bar{B} \cap \Omega$ we deduce that

$$\omega^x(\bar{B}) \geqslant \frac{f(x)}{\sup f} \geqslant \frac{\mu(E)}{2\pi\,\sup f}\,\log\frac{3}{2} = c\,\frac{\mathrm{Cap}_W(E)}{\sup f}.$$

Regarding $\sup f$, taking into account that $\|U_\mu\|_\infty \leqslant 1$, it is clear that

$$\sup f \leqslant 1 - \frac{1}{2\pi} \log \frac{4}{3r} \mu(E) = 1 - \frac{1}{2\pi} \log \frac{4}{3r} \operatorname{Cap}_W(E) \leqslant 1 - \frac{1}{2\pi} \log \frac{1}{r} \operatorname{Cap}_W(E).$$

Therefore,

$$\omega^x(\bar{B}) \geqslant c \frac{\operatorname{Cap}_W(E)}{1 - \frac{1}{2\pi} \log \frac{1}{r} \operatorname{Cap}_W(E)} = c' \frac{1}{\log \dfrac{1}{\operatorname{Cap}_L(E)} - \log \dfrac{1}{r}} = c' \frac{1}{\log \dfrac{r}{\operatorname{Cap}_L(E)}}.$$

□

Remark 7.21. It is easy to check that the constant $1/4$ in the preceding lemma can be replaced by any constant $\alpha \in (1/4, 1/3)$, with the implicit constant depending on α.

Lemma 7.22. *Let $\Omega \subset \mathbb{R}^2$ be an open Wiener regular set with compact boundary and let $\bar{B}$ be a closed ball centered at $\partial\Omega$. Then, for all $a > 2$,*

$$\omega^x(a\bar{B}) \gtrsim \inf_{z \in 2\bar{B} \cap \Omega} \omega^z(a\bar{B}) \fint_{\bar{B}} |G^x(y) - m_{\bar{B}}(G^x)|\, dy \qquad \textit{for all } x \in \Omega \backslash 2\bar{B}. \tag{7.14}$$

Proof. We can assume that Ω is bounded by proving first the estimate above for $\Omega_t = \Omega \cap B_t(0)$ and then letting $t \to \infty$.

Let $f(x) = \frac{\omega^x(a\bar{B})}{\inf_{z \in 2\bar{B} \cap \Omega} \omega^z(a\bar{B})}$. Then (7.14) can be written as

$$\fint_{\bar{B}} |G^x(y) - m_{\bar{B}}(G^x)|\, dy \leqslant f(x).$$

Consider a continuous function φ_B such that $\chi_{\frac{3}{2}\bar{B}} \leqslant \varphi_B \leqslant \chi_{\frac{7}{4}\bar{B}}$. For $x \in \Omega \backslash 2B$, we write using (7.5)

$$2\pi\, G^x(y) = 2\pi\, G^y(x) = \log \frac{1}{|x-y|} - \int \log \frac{1}{|\xi - y|}\, d\omega^x(\xi) = g_1(y) + g_2(y),$$

with

$$g_1(y) = \log \frac{1}{|x-y|} - \int (1 - \varphi_B(\xi)) \log \frac{1}{|\xi - y|}\, d\omega^x(\xi)$$

and

$$g_2(y) = -\int \varphi_B(\xi) \log \frac{1}{|\xi - y|}\, d\omega^x(\xi),$$

for every fixed x. We will treat separately the local and the non-local parts:

$$2\pi \fint_{\bar{B}} |G^x(y) - m_{\bar{B}}(G^x)|\, dy \leqslant \fint_{\bar{B}} |g_1 - m_{\bar{B}} g_1|\, dy + \fint_{\bar{B}} |g_2 - m_{\bar{B}} g_2|\, dy =: I_1 + I_2.$$

First we will estimate the *local* term I_2. To this end, let r denote the radius of $\bar{B}$ and let

$$\widetilde{g}_2(y) = -\int \varphi_B(\xi)\,\log\frac{4r}{|\xi-y|}\,d\omega^x(\xi),$$

so that $\widetilde{g}_2 = g_2 - C(B,r)$, for a suitable constant $C(B,r)$. Then we have

$$\begin{aligned} I_2 &= \fint_{\bar{B}} |\widetilde{g}_2 - m_{\bar{B}}\widetilde{g}_2|\,dy \leqslant 2\,m_{\bar{B}}|\widetilde{g}_2| = 2\fint_{\bar{B}}\int \varphi_B(\xi)\,\log\frac{4r}{|\xi-y|}\,d\omega^x(\xi)\,dy \\ &\leqslant 2\int_{2\bar{B}}\fint_{\bar{B}} \log\frac{4r}{|\xi-y|}\,dy\,d\omega^x(\xi) \lesssim \int_{2\bar{B}}\fint_{\bar{B}(\xi,3r)} \log\frac{4r}{|\xi-y|}\,dy\,d\omega^x(\xi), \end{aligned}$$

By a change of variable, we have

$$\fint_{\bar{B}(\xi,3r)} \log\frac{4r}{|\xi-y|}\,dy = \fint_{\bar{B}(0,3)} \log\frac{4}{|y|}\,dy = C,$$

and thus

$$I_2 \lesssim \omega^x(2\bar{B}) \leqslant \omega^x(a\bar{B}) \leqslant \frac{\omega^x(a\bar{B})}{\inf_{z\in 2\bar{B}\cap\Omega}\omega^z(a\bar{B})} = f(x)$$

for any $a \geqslant 2$.

To deal with the *non-local* term I_1, we write

$$\begin{aligned} I_1 &\leqslant \fint_{\bar{B}}\fint_{\bar{B}} |g_1(y) - g_1(z)|\,dy\,dz \\ &\leqslant \fint_{\bar{B}}\fint_{\bar{B}} \left|\log\frac{|x-z|}{|x-y|} - \int (1-\varphi_B(\xi))\,\log\frac{|\xi-z|}{|\xi-y|}\,d\omega^x(\xi)\right| dy\,dz. \end{aligned}$$

Denote

$$A_{y,z}(x) = \log\frac{|x-z|}{|x-y|} - \int (1-\varphi_B(\xi))\,\log\frac{|\xi-z|}{|\xi-y|}\,d\omega^x(\xi),$$

so that

$$I_1 \leqslant \sup_{y,z\in\bar{B}} |A_{y,z}(x)|.$$

To estimate $A_{y,z}(x)$ (for $y,z\in\bar{B}$) notice that both $A_{y,z}$ and f are harmonic in $\Omega\backslash 2\bar{B}$. Further, since

$$\frac{|x-z|}{|x-y|} \approx \frac{|\xi-z|}{|\xi-y|} \approx 1 \qquad \text{for all } x\in\Omega\backslash 2\bar{B},\ \xi\in\partial\Omega\backslash\tfrac{3}{2}\bar{B}, \text{ and } y,z\in\bar{B},$$

we infer that

$$|A_{y,z}(x)| \lesssim 1 \quad \text{for all } x\in\Omega\backslash 2\bar{B} \text{ and } y,z\in\bar{B}.$$

Further, using (5.6) it is immediate to check that

$$\lim_{\Omega\ni x\to\zeta} A_{y,z}(x) = 0 \qquad \text{for all } \zeta\in\partial\Omega\backslash 2\bar{B} \text{ and } y,z\in\bar{B}.$$

On the other hand,

$$f(x) \geqslant 1 \qquad \text{for all } x \in \Omega \cap a\bar{B}$$

and

$$f(x) \geqslant 0 \qquad \text{for all } x \in \Omega.$$

Then, by the maximum principle, it follows that

$$A_{y,z}(x) \leqslant C\, f(x) \qquad \text{for all } x \in \Omega\backslash 2\bar{B} \text{ and all } y, z \in \bar{B}.$$

Consequently,

$$I_1 = I_1(x) \leqslant \sup_{y,z\in\bar{B}} |A_{y,z}(x)| \lesssim f(x).$$

Together with the estimate we obtained for I_2, this proves the lemma. □

Lemma 7.23. *Let $\Omega \subset \mathbb{R}^2$ be an open Wiener regular set with compact boundary. Let $\bar{B}$ be a closed ball centered at $\partial\Omega$. Then,*

$$G^x(y) \lesssim \omega^x(8\bar{B}) \left(\log \frac{\mathrm{Cap}_L(\bar{B})}{\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega)}\right)^2 \qquad \text{for all } x \in \Omega\backslash 2\bar{B} \text{ and } y \in \tfrac{1}{5}\bar{B} \cap \Omega. \tag{7.15}$$

Proof. We can assume that Ω is bounded by proving first the estimate above for $\Omega_t = \Omega \cap B_t(0)$ and then letting $t \to \infty$.

To prove the lemma we will estimate $⨍_{\frac{1}{4}\bar{B}} G^x(z)\, dm(z)$ in terms of $⨍_{\bar{B}} \big|G^x(z) - m_{\bar{B}}G^x\big|\, dm(z)$ and then we will apply Lemmas 7.22 and 7.20.

Let $\bar{B} = \bar{B}_r(\xi)$, with $\xi \in \partial\Omega$. For $\frac{9}{10}r < s \leqslant r$, consider the open set $\Omega_s = B_s(\xi) \cap \Omega$. Then, for all $x \in \Omega\backslash 2\bar{B}$ and $y \in \frac{1}{4}\bar{B} \cap \Omega$, we have

$$G^x(y) = \int_{\partial\Omega_s} G^x(z)\, d\omega^y_{\Omega_s}(z) = \int_{\partial B_s(\xi)} G^x(z)\, d\omega^y_{\Omega_s}(z),$$

where ω_{Ω_s} is the harmonic measure for Ω_s and we took into account that $G^x(z)$ vanishes when $z \in \partial\Omega$. Notice that Ω_s may not be connected, in this case the harmonic measure is defined to be zero outside the boundary of the component containing the pole.

Remark that, for all $y \in \frac{1}{4}\bar{B} \cap \Omega$ there exists some function $\rho^y_s : \partial B_s(\xi) \to [0, \infty)$ such that

$$\omega^y_{\Omega_s}|_{\partial B_s(\xi)} = \rho^y_s \frac{\mathcal{H}^1|_{\partial B_s(\xi)}}{2\pi s},$$

with $\|\rho^y_s\|_\infty \lesssim 1$. This follows easily from the fact that, by the maximum principle,

$$\omega^y_{\Omega_s}(E) \leqslant \omega^y_{B_s(\xi)}(E) \qquad \text{for all } E \subset \partial B_s(\xi)$$

and the explicit formula for $\omega^y_{B_s(\xi)}$, see Example 5.27. Writing

$$\rho^y(z) = \rho^y_{|z-\xi|}(z),$$

by Fubini we have

$$G^x(y) = \frac{1}{0.1r}\int_{0.9r}^{r}\int_{\partial B_s(\xi)} G^x(z)\,d\omega^y_{\Omega_s}(z)\,ds \tag{7.16}$$
$$= \frac{10}{r}\int_{0.9r}^{r}\int_{\partial B_s(\xi)} G^x(z)\,\rho^y(z)d\mathcal{H}^1(z)\,\frac{ds}{2\pi s} = \int_{A_{0.9r,r}(\xi)} G^x(z)\,d\mu^y(z),$$

where μ^y is the measure

$$d\mu^y(z) = \frac{10}{2\pi\, r\,|z-\xi|}\,\rho^y(z)\,dm|_{A_{0.9r,r}(\xi)}(z).$$

Averaging (7.16) over $y \in \frac{1}{4}\bar{B}$ and applying Fubini, we get

$$m_{\frac{1}{4}\bar{B}}G^x = ⨍_{\frac{1}{4}\bar{B}}\int_{A_{0.9r,r}(\xi)} G^x(z)\,d\mu^y(z)\,dy = \int_{A_{0.9r,r}(\xi)} G^x(z)\,d\mu(z), \tag{7.17}$$

where

$$d\mu(z) = \rho(z)\,dm|_{A_{0.9r,r}(\xi)}(z), \qquad \rho(z) = \frac{10}{2\pi\, r\,|z-\xi|}\,⨍_{\frac{1}{4}\bar{B}}\rho^y(z)\,dy$$

understanding that $\rho^y(z) \equiv 0$ when $y \notin \Omega$. Notice that $\|\rho\|_\infty \lesssim r^{-2}$, since $\|\rho^y\|_\infty \lesssim 1$ for all $y \in \frac{1}{4}\bar{B}$.

Observe now that, by Lemma 7.20 and the subsequent remark, we have

$$\omega^y_{\Omega_s}(B_{0.9s}(\xi)) \gtrsim \frac{1}{\log\dfrac{s}{\mathrm{Cap}_L(B_{0.29s}(\xi)\backslash\Omega)}} \qquad \text{for all } y \in B_{0.29s}(\xi) \cap \Omega_s.$$

Since $\frac{1}{4}\bar{B} \subset B_{0.29s}(\xi)$ for $\frac{9}{10}r < s \leqslant r$, we infer that

$$\omega^y_{\Omega_s}(B_{0.9s}(\xi)) \gtrsim \frac{1}{\log\dfrac{s}{\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega)}} \approx \frac{1}{\log\dfrac{r}{\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega)}} \qquad \text{for all } y \in \tfrac{1}{4}\bar{B} \cap \Omega_s.$$

Thus,

$$\omega^y_{\Omega_s}(\partial B_s(\xi)) \leqslant 1 - \varepsilon_0,$$

where

$$\varepsilon_0 = \frac{c}{\log\dfrac{r}{\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega)}},$$

for some $c > 0$. Thus,

$$\|\mu\| = \mu(A_{0.9r,r}(\xi)) = ⨍_{\frac{1}{4}\bar{B}}\frac{1}{0.1r}\int_{0.9r}^{r}\omega^y_{\Omega_s}(\partial B_s(\xi))\,ds\,dy \leqslant 1 - \varepsilon_0.$$

Next we consider the measure

$$\nu = \frac{1}{2}\left(\mu + \frac{m|_{\frac{1}{4}\bar{B}}}{m(\frac{1}{4}\bar{B})}\right),$$

so that
$$\frac{1}{2} \leqslant \nu(\bar{B}) = \frac{1}{2}\big(\mu(\bar{B}) + 1\big) \leqslant 1 - \frac{\varepsilon_0}{2}.$$
From (7.17) and this estimate we infer that
$$\begin{aligned} m_{\frac{1}{4}\bar{B}} G^x &= \frac{1}{2}\int_{A_{0.9r,r}(\xi)} G^x(z)\, d\mu(z) + \frac{1}{2}\, m_{\frac{1}{4}\bar{B}} G^x \\ &= \nu(\bar{B}) \fint_{\bar{B}} G^x(z)\, d\nu(z) \leqslant \Big(1 - \frac{\varepsilon_0}{2}\Big) \fint_{\bar{B}} G^x(z)\, d\nu(z). \end{aligned}$$
Therefore,
$$\begin{aligned} \frac{\varepsilon_0}{2} \fint_{\bar{B}} G^x(z)\, d\nu(z) &\leqslant \fint_{\bar{B}} G^x(z)\, d\nu(z) - m_{\frac{1}{4}\bar{B}} G^x \\ &\leqslant \left| \fint_{\bar{B}} G^x(z)\, d\nu(z) - m_{\bar{B}} G^x \right| + \left| m_{\bar{B}} G^x - m_{\frac{1}{4}\bar{B}} G^x \right| \\ &\leqslant \fint_{\bar{B}} \left| G^x(z) - m_{\bar{B}} G^x \right| d\nu(z) + \fint_{\frac{1}{4}\bar{B}} \left| G^x(z) - m_{\bar{B}} G^x \right| dm(z). \end{aligned} \tag{7.18}$$
Recall now that $\nu(\bar{B}) \approx 1$ and that
$$\nu = \frac{1}{2}\Big(\rho\, \chi_{A_{0.9r,r}(\xi)} + \frac{1}{m(\frac{1}{4}\bar{B})}\, \chi_{\frac{1}{4}\bar{B}}\Big)\, m|_{\bar{B}} =: \widetilde{\rho}\, m|_{\bar{B}},$$
it is clear that $\|\widetilde{\rho}\|_{L^\infty(\bar{B})} \lesssim r^{-2}$. Hence,
$$\begin{aligned} \fint_{\bar{B}} \left| G^x(z) - m_{\bar{B}} G^x \right| d\nu(z) &\lesssim \frac{1}{r^2} \int_{\bar{B}} \left| G^x(z) - m_{\bar{B}} G^x \right| dm(z) \\ &\lesssim \fint_{\bar{B}} \left| G^x(z) - m_{\bar{B}} G^x \right| dm(z). \end{aligned}$$
By the definition of ν, (7.18), and the preceding estimate, we obtain
$$\frac{\varepsilon_0}{4} \fint_{\frac{1}{4}\bar{B}} G^x(z)\, dm(z) \leqslant \frac{\varepsilon_0}{2} \fint_{\bar{B}} G^x(z)\, d\nu(z) \lesssim \fint_{\bar{B}} \left| G^x(z) - m_{\bar{B}} G^x \right| dm(z),$$

From the preceding estimate, taking into account that G^x is subharmonic in $\mathbb{R}^2 \backslash \{x\}$ and using Lemmas 7.22 and 7.20, for all $y \in \frac{1}{5}\bar{B}$ we get
$$\begin{aligned} G^x(y) &\lesssim \fint_{\frac{1}{4}\bar{B}} G^x(z)\, dm(z) \lesssim \varepsilon_0^{-1} \fint_{\bar{B}} \left| G^x(z) - m_{\bar{B}} G^x \right| dm(z) \\ &\lesssim \frac{\omega^x(8\bar{B})}{\inf_{z \in 2\bar{B} \cap \Omega} \omega^z(8\bar{B})} \log \frac{r}{\mathrm{Cap}_L(\frac{1}{4}\bar{B} \backslash \Omega)} \lesssim \omega^x(8\bar{B}) \log \frac{8r}{\mathrm{Cap}_L(2\bar{B} \backslash \Omega)} \log \frac{r}{\mathrm{Cap}_L(\frac{1}{4}\bar{B} \backslash \Omega)} \\ &\lesssim \omega^x(8\bar{B}) \left(\log \frac{r}{\mathrm{Cap}_L(\frac{1}{4}\bar{B} \backslash \Omega)} \right)^2. \end{aligned}$$

□

Notice that, in the case when Ω is an NTA domain, we have $\omega^x(8\bar{B}) \approx \omega^x(\bar{B})$ and $\mathrm{Cap}_L(\frac{1}{4}\bar{B}\backslash\Omega) \approx \mathrm{Cap}_L(\bar{B}) = r(\bar{B})$, so that we recover the estimate

$$\omega^x(\bar{B}) \gtrsim G^x(y),$$

for $y \in \frac{1}{5}\bar{B}$, as in the case $d \geqslant 3$.

7.5 Capacity density condition

7.5.1 The CDC and Wiener regularity

Let $\Omega \subsetneq \mathbb{R}^d$ be an open set in $\mathbb{R}^d$ and let $\xi \in \partial\Omega$ and $r_0 > 0$. We say that Ω satisfies the (ξ, r_0)-local capacity density condition if there exists some constant $c > 0$ such that, for any $r \in (0, r_0)$,

$$\mathrm{Cap}(\bar{B}_r(\xi)\backslash\Omega) \geqslant c\, r^{d-2} \qquad \text{in the case } d \geqslant 3,$$

and

$$\mathrm{Cap}_L(\bar{B}_r(\xi)\backslash\Omega) \geqslant c\, r \qquad \text{in the case } d = 2.$$

We say that Ω satisfies the capacity density condition (CDC) if it satisfies the (ξ, r_0)-local capacity density condition for all $\xi \in \partial\Omega$ and all $r_0 \in (0, \mathrm{diam}(\partial\Omega))$ and moreover Ω^c contains more than one point. For example, a Jordan domain in the plane satisfies the CDC, or more generally, any planar bounded domain whose boundary consists of finitely many curves (we do not allow degenerate curves consisting of a single point).

The CDC can be understood as a strong form of Wiener regularity. In fact, we have:

Proposition 7.24. *Let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary and let $\xi \in \partial\Omega$ and $r_0 > 0$. If the (ξ, r_0)-local capacity density holds for Ω, then ξ is a regular point for the Dirichlet problem.*

As a corollary, if Ω satisfies the CDC, then it is Wiener regular.

Proof. This is an easy consequence of the Wiener criterion, more precisely of the implication (b) $\Rightarrow$ (a) in Theorem 6.23. Indeed, we just have to check that the (ξ, r_0)-local capacity density condition implies that

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{A}(\xi, 2^{-k-1}, 2^{-k})\backslash\Omega)}{\mathrm{Cap}(\bar{B}(\xi, 2^{-k}))} = \infty.$$

As shown in Remark 6.24, in the case $d \geqslant 3$ this is equivalent to the fact that

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cap}(\bar{B}_{2^{-k}}(\xi)\backslash\Omega)}{\mathrm{Cap}(\bar{B}_{2^{-k}}(\xi))} = \infty.$$

Now we just have to observe that (ξ, r_0)-local capacity density condition is equivalent to the fact that $\mathrm{Cap}(\bar{B}_r(\xi)\backslash\Omega) \geqslant c\,\mathrm{Cap}(\bar{B}_r(\xi))$ for $0 < r < r_0$, which clearly implies the above estimate.

The case $d = 2$ is a little trickier. Notice first that, for $r \in (0,1)$ the estimate $\mathrm{Cap}_L(\bar B_r(\xi)\backslash\Omega) \geqslant c\,r$ implies that

$$\frac{\mathrm{Cap}_W(\bar B_r(\xi)\backslash\Omega)}{\mathrm{Cap}_W(\bar B_r(\xi))} = \frac{\log\frac{1}{\mathrm{Cap}_L(\bar B_r(\xi))}}{\log\frac{1}{\mathrm{Cap}_L(\bar B_r(\xi)\backslash\Omega)}} \geqslant \frac{\log\frac{1}{r}}{\log\frac{1}{c\,r}} = \frac{\log\frac{1}{r}}{\log\frac{1}{r}+C} \geqslant \frac{1}{2},$$

assuming r small enough in the last inequality. Observe now that $\mathrm{Cap}_W(\bar B_{r^4}(\xi)) = \frac{1}{4}\mathrm{Cap}_W(\bar B_r(\xi))$. Then, by the subadditivity of Cap_W we deduce

$$\frac{1}{2} \leqslant \frac{\mathrm{Cap}_W((\bar B_r(\xi)\backslash\Omega)\backslash B_{r^4}(\xi)) + \mathrm{Cap}_W(\bar B_{r^4}(\xi))}{\mathrm{Cap}_W(\bar B_r(\xi))} = \frac{\mathrm{Cap}_W(\bar A_{r^4,r}(\xi)\backslash\Omega)}{\mathrm{Cap}_W(\bar B_r(\xi))} + \frac{1}{4}.$$

Hence

$$\frac{\mathrm{Cap}_W(\bar A_{r^4,r}(\xi)\backslash\Omega)}{\mathrm{Cap}_W(\bar B_r(\xi))} \geqslant \frac{1}{4}. \tag{7.19}$$

Now we can estimate the Wiener's series from below as follows, considering j_0 large enough,

$$\sum_{j\geqslant j_0}\sum_{4^j\leqslant k\leqslant 4^{j+1}-1}\frac{\mathrm{Cap}_W(\bar A(\xi,2^{-k-1},2^{-k})\backslash\Omega)}{\mathrm{Cap}_W(\bar B(\xi,2^{-k}))}$$
$$\geqslant \sum_{j\geqslant j_0}\sum_{4^j\leqslant k\leqslant 4^{j+1}-1}\frac{\mathrm{Cap}_W(\bar A(\xi,2^{-k-1},2^{-k})\backslash\Omega)}{\mathrm{Cap}_W(\bar B(\xi,2^{-4^j}))} \geqslant \sum_{j\geqslant j_0}\frac{\mathrm{Cap}_W(\bar A(\xi,2^{-4^{j+1}},2^{-4^j})\backslash\Omega)}{\mathrm{Cap}_W(\bar B(\xi,2^{-4^j}))}.$$

By (7.19), each of the summands on the right hand side is at least $1/4$ and so the sum is infinite. □

Remark 7.25. By Lemmas 7.16, 7.20, 7.19, and 7.23, if Ω satisfies the CDC, then it holds

$$\omega^x(\bar B) \gtrsim 1 \quad \text{for all } x \in \tfrac{1}{4}\bar B\cap\Omega, \text{ if } r(\bar B) < \mathrm{diam}(\partial\Omega) \tag{7.20}$$

and

$$G^x(y) \lesssim \frac{\omega^x(8\bar B)}{r(\bar B)^{d-2}} \quad \text{for all } x\in\Omega\backslash 2\bar B \text{ and } y \in \tfrac{1}{5}\bar B\cap\Omega, \text{ if } r(\bar B) < \mathrm{diam}(\partial\Omega), \tag{7.21}$$

with constants depending on the CDC.

Remark 7.26. It is immediate to check that if Ω and Ω' are open sets in $\mathbb{R}^d$ satisfying the CDC, then $\Omega\cap\Omega'$ also satisfies the CDC.

Exercise 7.5.1. Let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary and let $\xi \in \partial\Omega$. Prove that if there exist $c > 0$ and a sequence of radii $r_k \to 0$ such that

$$\mathrm{Cap}(\bar B_{r_k}(\xi)\backslash\Omega) \geqslant c\,r_k^{d-2} \qquad \text{in the case } d \geqslant 3,$$

and

$$\mathrm{Cap}_L(\bar B_{r_k}(\xi)\backslash\Omega) \geqslant c\,r_k \qquad \text{in the case } d = 2,$$

then ξ is a regular point for the Dirichlet problem.

7.5.2 Hölder continuity at the boundary

Lemma 7.27. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $\xi \in \partial\Omega$, and let $r > 0$. Suppose that $\Omega \cap B_r(\xi)$ is Wiener regular, and Ω satisfies the (ξ, r)-local capacity density condition with constant c. Let u be a nonnegative function which is continuous in $\overline{B_r(\xi) \cap \Omega}$ and harmonic in $B_r(\xi) \cap \Omega$, and vanishes on $B_r(\xi) \cap \partial\Omega$. Then there is $\alpha > 0$ depending on c (but not on r) such that*

$$u(x) \lesssim \left(\frac{|x-\xi|}{r}\right)^{\alpha} \sup_{B_r(\xi)\cap\Omega} u \quad \text{for all } x \in \Omega \cap B_r(\xi). \tag{7.22}$$

Proof. For very $k \geq 0$, let $B_k = B_{6^{-k}r}(\xi)$ and $\Omega_k = \Omega \cap B_k$. Since u vanishes identically on $\partial\Omega \cap B_k$, for all $x \in \partial B_{k+1} \cap \Omega$ we have

$$u(x) = \int_{\partial\Omega_k} u(y)\, d\omega^x_{\Omega_k}(y) = \int_{\partial B_k \cap \Omega} u(y)\, d\omega^x_{\Omega_k}(y) \leq \omega^x_{\Omega_k}(\partial B_k \cap \Omega) \sup_{\partial B_k \cap \Omega} u.$$

By the (ξ, r_0)-local capacity density condition (which also holds for Ω_k) and Lemmas 7.16 and 7.20,

$$\omega^x_{\Omega_k}(\partial B_k \cap \Omega) = 1 - \omega^x_{\Omega_k}(\partial\Omega \cap \bar{B}_k) \leq 1 - c_0$$

for some $c_0 \in (0,1)$. Thus,

$$\sup_{\partial B_{k+1}\cap\Omega} u \leq (1-c_0) \sup_{\partial B_k \cap \Omega} u.$$

By the maximum principle and iterating, we deduce that

$$\sup_{B_k \cap \Omega} u = \sup_{\partial B_k \cap \Omega} u \leq (1-c_0)^k \sup_{\partial B_0 \cap \Omega} u.$$

This readily proves the lemma. □

As an easy corollary we get a result about Hölder regularity:

Lemma 7.28. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let B be a ball with radius r_0 centered in $\partial\Omega$. Suppose that Ω satisfies the (ξ, r_0)-local CDC for every $\xi \in \partial\Omega \cap 2B$. Let u be a nonnegative function which is continuous in $\overline{2B \cap \Omega}$ and harmonic in $2B \cap \Omega$, and vanishes continuously on $2B \cap \partial\Omega$. Then there is $\alpha > 0$ such that*

$$|u(x) - u(y)| \lesssim \left(\frac{|x-y|}{r_0}\right)^{\alpha} \sup_{2B\cap\Omega} u \quad \text{for all } x, y \in B \cap \Omega. \tag{7.23}$$

Proof. Remark that every $\xi \in \partial(2B) \cap \Omega$ satisfies the local CDC with respect to $2B \cap \Omega$, so that in particular, by replacing Ω by $\Omega \cap 2B$ if necessary, we can assume that Ω is a bounded CDC open set, that is, the (ξ, r_0)-local CDC holds for all $\xi \in \partial\Omega$.

To prove the lemma, clearly we may assume that $|x - y| \leq r/4$. Denote as usual $\mathrm{d}_\Omega(z) := \mathrm{dist}(z, \partial\Omega)$, and suppose first that

$$|x-y| \leq \frac{1}{2} \max(\mathrm{d}_\Omega(x), \mathrm{d}_\Omega(y)) =: \frac{1}{2} \mathrm{d}_\Omega(x,y).$$

Assume that $d_\Omega(y) \leqslant d_\Omega(x) = d_\Omega(x,y)$, say, and consider the ball $B' = B(x, d_\Omega(x,y))$. Notice that $B' \subset \Omega \cap 2B$ and $x, y \in \frac{1}{2}B'$. So by standard arguments it follows that, for any $\alpha \in (0,1]$,

$$\begin{aligned}|u(x) - u(y)| &\leqslant \|\nabla u\|_{\infty,\frac{1}{2}B'} |x-y| \lesssim \|u\|_{\infty,B'} \frac{|x-y|}{r(B')} \leqslant \|u\|_{\infty,2B} \frac{|x-y|}{d_\Omega(x,y)} \\ &\leqslant \|u\|_{\infty,2B} \left(\frac{|x-y|}{d_\Omega(x,y)}\right)^\alpha.\end{aligned} \tag{7.24}$$

Notice also that the same estimate holds trivially in case that $|x-y| > \frac{1}{2} d_\Omega(x,y)$.

On the other hand, by Lemma 7.27, there exists some $\alpha \in (0,1)$ such that

$$u(x) \lesssim \left(\frac{d_\Omega(x)}{r_0}\right)^\alpha \|u\|_{\infty,2B},$$

whenever $d_\Omega(x) < r_0/2$. The same inequality holds trivially if $d_\Omega(x) \geqslant r_0/2$. Replacing x by y, we obtain the analogous estimate for y. Thus,

$$\begin{aligned}|u(x) - u(y)| \leqslant u(x) + u(y) &\lesssim \left(\frac{d_\Omega(x)}{r_0}\right)^\alpha \|u\|_{\infty,2B} + \left(\frac{d_\Omega(y)}{r_0}\right)^\alpha \|u\|_{\infty,2B} \\ &\lesssim \left(\frac{d_\Omega(x,y)}{r_0}\right)^\alpha \|u\|_{\infty,2B}.\end{aligned} \tag{7.25}$$

Taking the geometric mean of (7.24) and (7.25), the lemma follows (with $\alpha/2$ instead of α). □

As another immediate consequence of Lemma 7.27 we get the following:

Lemma 7.29. *Let $\Omega \subset \mathbb{R}^d$ be a Wiener regular open set with compact boundary, let $\xi \in \partial\Omega$, and let $r_0 > 0$. Suppose that Ω satisfies the (ξ, r_0)-local capacity density condition. Then there is $\alpha > 0$ such that, for all $r \in (0, r_0)$,*

$$\omega^x(B(\xi,r)^c) \lesssim \left(\frac{|x-\xi|}{r}\right)^\alpha \qquad \text{for } x \in \Omega \cap B_r(\xi). \tag{7.26}$$

7.5.3 Improving property of the CDC

As shown in Lemma 6.20, if a set $E \subset \mathbb{R}^d$ satisfies $\mathrm{Cap}(E) > 0$, then $\mathcal{H}^{d-2}_\infty(E) > 0$. Further, this estimate is sharp in the sense that one cannot infer that $\mathcal{H}^s_\infty(E) > 0$ for any $s > d-2$. In fact, it is not difficult to construct a compact set $E \subset \mathbb{R}^d$ such that $\mathrm{Cap}(E) > 0$ with $\dim_H(E) = d-2$, see Exercise 7.5.2 below. Similarly, if $\Omega \subset \mathbb{R}^d$ satisfies the CDC, then it easily follows from Lemma 6.20 that

$$\mathcal{H}^{d-2}_\infty(\Omega^c \cap \bar{B}_r(\xi)) \gtrsim r^{d-2} \qquad \text{for all } \xi \in \partial\Omega,\ r > 0.$$

From the previous discussion, it would appear that the exponent $d-2$ in this estimate might be sharp. Surprisingly, this can be improved, as the following theorem shows.

Theorem 7.30. *Let $r_0 > 0$ and let $\Omega \subset \mathbb{R}^d$ be an open set satisfying the (ξ, r_0)-local capacity density condition for every $\xi \in \partial\Omega$. Then there exists some $s > d-2$ and some $c > 0$ such that*

$$\mathcal{H}^s_\infty(\Omega^c \cap \bar{B}_r(\xi)) \geqslant c\, r^s \quad \textit{for all } \xi \in \partial\Omega,\ 0 < r \leqslant r_0.$$

The constant $c > 0$ and the precise $s > d-2$ depend only on d and on the constant involved in the local CDC.

Proof. Suppose first that $d \geqslant 3$. Denote $E = \Omega^c$. Observe first that the fact that Ω satisfies the (ξ, r_0)-local CDC for every $\xi \in \partial\Omega$ is equivalent to saying that

$$\mathrm{Cap}(E \cap \bar{B}_r(x)) \gtrsim r^{d-2} \quad \text{for all } x \in E,\ 0 < r \leqslant r_0.$$

Fix now a point $\xi \in \partial\Omega$ and $0 < R \leqslant r_0$, and let us see that $\mathcal{H}^s_\infty(E \cap \bar{B}_R(\xi)) \gtrsim R^s$ for some $s > d-2$, with both s and the implicit constant depending only on the local CDC. To this end, define $E_1 = E \cap \bar{B}_{R/4}(\xi)$. Note that $\mathbb{R}^d \backslash E_1$ may not satisfy the CDC. To deal with this issue, we consider the sets E_m defined inductively, for $m \geqslant 2$, by

$$E_m = E \cap \bigcup_{x \in E_{m-1}} \bar{B}_{2^{-m}R}(x).$$

It is immediate to check that the closure F of $\bigcup_{m \geqslant 1} E_m$ is contained in $B_R(\xi) \cap E$ and satisfies

$$\mathrm{Cap}(F \cap \bar{B}_r(x)) \gtrsim r^{d-2} \quad \text{for all } x \in F,\ 0 < r \leqslant R.$$

Equivalently, the open set $\mathbb{R}^d \backslash F$ satisfies the CDC.

Let μ_F be the equilibrium measure of F, and denote $\eta_s = R^s \mu_F$. We intend to show that there exists some $s > d-2$ such that

$$\eta_s(B_r(x)) \lesssim r^s \quad \text{for all } x \in F,\ 0 < r \leqslant R. \tag{7.27}$$

By Frostman's lemma, clearly this implies that

$$\mathcal{H}^s_\infty(E \cap \bar{B}_R(\xi)) \geqslant \mathcal{H}^s_\infty(F) \gtrsim \|\eta_s\| = R^s,$$

as wished. To prove (7.27), let $\eta = \eta_{d-2} = R^{d-2}\, \mu_F$, and notice that the CDC satisfied by F^c ensures that F^c is Wiener regular, so that by Lemma 6.26,

$$U_\eta(x) = R^{d-2} \frac{1}{\mathrm{Cap}(F)} \quad \text{for all } x \in F.$$

So the function

$$f(x) = R^{d-2} \frac{1}{\mathrm{Cap}(F)} - U_\eta(x)$$

is continuous in $\mathbb{R}^d$, harmonic in F^c, it vanishes in F, and it is non-negative in F^c, by the properties of the equilibrium potential. Further $\|f\|_\infty \leqslant R^{d-2} \frac{1}{\mathrm{Cap}(F)} \lesssim 1$. So by Lemma 7.28, f is Hölder continuous and, for some $\alpha > 0$ depending on the CDC it holds

$$|U_\eta(x) - U_\eta(y)| = |f(x) - f(y)| \lesssim \left(\frac{|x-y|}{R}\right)^\alpha \quad \text{for all } x, y \in \bar{B}_{2R}(\xi). \tag{7.28}$$

Fix $x \in F$ and $0 < r \leqslant R$, and let φ_r be a bump function such that $\chi_{B_r(x)} \leqslant \varphi_r \leqslant \chi_{B_{2r}(x)}$, with $\|\nabla\varphi_r\|_\infty \lesssim 1/r$. Since $-\Delta U_\eta = \eta$ in the sense of distributions, we have

$$\eta(B_r(x)) \leqslant \int \varphi_r \, d\eta = -\int U_\eta \, \Delta\varphi_r \, dy = -\int (U_\eta(y) - U_\eta(x)) \, \Delta\varphi_r(y) \, dy,$$

where, in the last identity, we used the fact that $\int \Delta\varphi_r \, dy = 0$. Plugging the estimate (7.28), we deduce

$$\eta(B_r(x)) \lesssim \frac{1}{r^2} \int_{B_{2r}(x)} |U_\eta(y) - U_\eta(x)| \, dy \lesssim r^{d-2} \left(\frac{r}{R}\right)^\alpha,$$

or equivalently,

$$\eta_{d-2+\alpha}(B_r(x)) \lesssim r^{d-2+\alpha}.$$

So (7.27) holds with $s = d - 2 + \alpha$.

In the case $d = 2$, by a suitable dilation, we may assume that $R = 1/4$, say. Then the arguments above work in a similar fashion, so that at the end we deduce that $\eta_\alpha(B_r(x)) \lesssim r^\alpha$. □

Corollary 7.31. *Let $r_0 > 0$ and let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary. Then, Ω satisfies the (ξ, r_0)-local capacity density condition for every $\xi \in \partial\Omega$ if and only if there exists some $s > d - 2$ and some $c > 0$ such that*

$$\mathcal{H}^s_\infty(\Omega^c \cap \bar{B}_r(\xi)) \geqslant c\, r^s \qquad \textit{for all } \xi \in \partial\Omega,\ 0 < r \leqslant r_0.$$

Proof. The fact that local CDC condition implies the s-lower content regularity above is shown in Theorem 7.30. The converse statement is an immediate consequence of the lower bound of $\mathrm{Cap}(\Omega^c \cap \bar{B}_r(\xi))$ in terms of $\mathcal{H}^s_\infty(\Omega^c \cap \bar{B}_r(\xi))$, for $s > d-2$, deduced from Lemma 6.20. □

Exercise 7.5.2. Construct a compact set $E \subset \mathbb{R}^d$ such that $\mathrm{Cap}(E) > 0$ with $\dim_H(E) = d - 2$, see [Tol14, Section 4.7], for a possible construction scheme.

7.6 Harmonic measure and Green's function with pole at infinity

In this section we will study the connection between harmonic measure with pole at infinity and Green's function with pole at infinity for unbounded open sets with compact boundary. We will study first the case of the plane, which is simpler, and later the higher dimensional case.

7.6.1 The case of the plane

Recall that for an unbounded open set with compact boundary the notion of harmonic measure with pole at ∞ was introduced in Definition 5.53. From that definition, it follows that for any function $f \in C(\partial\Omega)$,

$$\int_{\partial\Omega} f(\xi) d\omega^{\infty}(\xi) = \lim_{z\to\infty} \int_{\partial\Omega} f(\xi) d\omega^{z}(\xi). \tag{7.29}$$

Analogously, for any Borel set $E \subset \partial\Omega$, we have $\omega^z(E) \to \omega^{\infty}(E)$ as $z \to \infty$, see Exercise 5.5.1.

In the context above, denote by $G : \Omega \times \Omega \to \mathbb{R}$ the Green function for Ω. For any fixed point $y \in \Omega$, the function $G(y, \cdot)$ is harmonic at ∞ (i.e., it has a removable singularity at ∞), by Corollary 5.52. Thus we can define

$$G^{\infty}(y) = G(y, \infty) = \lim_{z\to\infty} G(y, z). \tag{7.30}$$

Theorem 7.32. *Let $\Omega \subset \mathbb{R}^2$ be a Wiener regular unbounded open set with compact boundary. Let $\{p_k\}_k \subset \Omega$ be a sequence of points such that $p_k \to \infty$. Then the functions G^{p_k} converge uniformly in bounded subsets of Ω to G^{∞}, the measures $\omega^{p_k}|_{\partial\Omega}$ converge weakly to ω^{∞}, and the following holds:*

(a) G^{∞} is harmonic and positive in Ω.

(b) ω^{∞} is mutually absolutely continuous with ω^p, for every p belonging to the unbounded component of Ω.

(c) For every $\varphi \in C_c^{\infty}(\mathbb{R}^2)$,

$$\int_{\Omega} G^{\infty}\, \Delta\varphi\, dm = \int \varphi\, d\omega^{\infty}.$$

(d) ω^{∞} coincides with the equilibrium measure of $\partial\Omega$ (and so it is a probability measure) and moreover, for every $z \in \Omega$,

$$G^{\infty}(z) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} - \mathcal{E} * \omega^{\infty}(z).$$

Proof. Statement (a) is immediate due to (7.30).

The weak convergence of ω^{p_k} to ω^{∞} is equivalent to (7.29). It is clear that this implies that ω^{∞} is a probability measure (this can also be derived directly from the definition of ω^{∞} and the Riesz representation theorem). Further, we already discussed the mutual absolute continuity of ω^{∞} and ω^p after Definition 5.53.

From the pointwise convergence given by (7.30) and an easy application of the Arzelà-Ascoli theorem, it follows that the functions G^{p_k} converge uniformly in compact subsets of Ω to G^{∞} as $p_k \to \infty$. To prove the uniform convergence in bounded subsets of Ω, let $r > 0$

be an arbitrary radius such that $\partial\Omega \subset S_r(0)$. Since the functions G^{p_k} vanish continuously on $\partial\Omega$, by the maximum principle the sequence $\{G^{p_k}\}_{k\geqslant 1}$ is a uniform Cauchy sequence in $\Omega \cap B_r(0)$, and so the convergence is uniform in $\Omega \cap B_r(0)$ and, therefore, in bounded subsets of Ω. In particular, G^∞ extends continuously to Ω^c as $G^\infty|_{\Omega^c} \equiv 0$.

The statement (c) of the theorem is a consequence of the fact that, for $\varphi \in C_c^\infty(\mathbb{R}^2)$ and ξ away from the support of φ,

$$\int_\Omega G^\xi(z)\,\Delta\varphi(z)\,dm(z) = \int \varphi\,d\omega^\xi.$$

Then we let $\xi \to \infty$ and use the uniform convergence of G^ξ to G^∞ in bounded sets and the weak convergence of ω^ξ to ω^∞, and (c) follows.

To prove (d), recall that

$$\begin{aligned} G(z,\xi) &= \frac{1}{2\pi}\int_{\partial\Omega} \log\frac{|\xi - x|}{|\xi - z|}\,d\mu(x) - \frac{1}{2\pi}\int_{\partial\Omega}\int_{\partial\Omega} \log\frac{|y-x|}{|y-z|}\,d\mu(x)\,d\omega^\xi(y) \\ &= \frac{1}{2\pi}\int_{\partial\Omega} \log\frac{|\xi - x|}{|\xi - z|}\,d\mu(x) - \int_{\partial\Omega} (\mathcal{E}^z(y) - U_\mu(y))\,d\omega^\xi(y) \end{aligned}$$

where μ is the equilibrium measure of $\partial\Omega$. Letting $\xi \to \infty$, since the potential is continuous and the harmonic measures ω^ξ converge weakly to ω^∞, we obtain

$$G^\infty(z) = 0 - \int_{\partial\Omega} (\mathcal{E}^z(y) - U_\mu(y))\,d\omega^\infty(y) = \int U_\mu(y)\,d\omega^\infty(y) - \int \mathcal{E}^z(y)\,d\omega^\infty(y).$$

For the first summand we take into account that

$$U_\mu(y) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} \quad \text{for all}\, y \in \partial\Omega,$$

since Ω is Wiener regular, and so

$$G^\infty(z) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} - U_{\omega^\infty}(z) \quad \text{for every } z \in \Omega.$$

Thus, U_{ω^∞} is continuous up to $\partial\Omega$, with

$$\frac{1}{\mathrm{Cap}_W(\partial\Omega)} = U_{\omega^\infty}(z) \quad \text{for every } z \in \partial\Omega.$$

By the uniqueness of the equilibrium measure μ (see Theorem 6.11), we infer that $\omega^\infty = \mu$. □

7.6.2 The higher dimensional case

For $d \geqslant 3$, let $\Omega \subset \mathbb{R}^d$ be an unbounded Wiener regular open set with compact boundary. In this case we cannot define the harmonic measure with pole at infinity directly as the weak limit of the measures ω^p with $p \to \infty$ because this limit is always zero. Instead we can define harmonic measure and the Green function with pole at infinity by a limiting process involving renormalization. The construction is summarized in the following theorem:

Theorem 7.33. *For $d \geqslant 3$, let $\Omega \subset \mathbb{R}^d$ be an unbounded Wiener regular open set with compact boundary. Let $\{p_k\}_k \subset \Omega$ be a sequence of points such that $p_k \to \infty$. Then the functions $\mathcal{E}(p_k)^{-1} G^{p_k}$ converge uniformly in bounded subsets of Ω to some function $G^\infty : \Omega \to \mathbb{R}$, the measures $\mathcal{E}(p_k)^{-1}\omega^{p_k}$ converge weakly to some measure ω^∞ supported in $\partial\Omega$, and the following holds:*

(a) G^∞ is harmonic and positive in Ω.

(b) ω^∞ is mutually absolutely continuous with ω^p, for every p in the unbounded component of Ω.

(c) For every $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_\Omega G^\infty \, \Delta\varphi \, dm = \int \varphi \, d\omega^\infty.$$

(d) ω^∞ is the equilibrium measure of $\partial\Omega$ times $\mathrm{Cap}(\partial\Omega)$ (and, so $\|\omega^\infty\| = \mathrm{Cap}(\partial\Omega)$) and moreover, for every $x \in \Omega$,

$$G^\infty(x) = 1 - \mathcal{E} * \omega^\infty(x) = 1 - \omega^x(\partial\Omega).$$

In particular, the limiting function G^∞ and the limiting measure ω^∞ do not depend on the chosen sequence $\{p_k\}_k$.

Proof. Let μ be the equilibrium measure of $\partial\Omega$. Observe first that, for all $p \in \Omega$,

$$\omega^p(\partial\Omega) = \mathrm{Cap}(\partial\Omega)\, U_\mu(p), \tag{7.31}$$

since the right hand side is a function that is harmonic in Ω and continuous in $\overline{\Omega}$, it equals 1 in $\partial\Omega$, and vanishes at ∞, see Proposition 5.39.

Consider now an arbitrary sequence $\{p_k\}_k \subset \Omega$ such that $p_k \to \infty$. We write

$$\mathcal{E}(p_k)^{-1}\omega^{p_k} = \mathrm{Cap}(\partial\Omega)\, \frac{U_\mu(p_k)}{\mathcal{E}(p_k)}\, \frac{1}{\omega^{p_k}(\partial\Omega)}\, \omega^{p_k}. \tag{7.32}$$

It is immediate to check that

$$\lim_{p_k \to \infty} \frac{U_\mu(p_k)}{\mathcal{E}(p_k)} = 1.$$

Thus there exists a subsequence $\{p_{k_j}\}_j$ such that $\mathcal{E}(p_{k_j})^{-1}\omega^{p_{k_j}}$ converges weakly $*$ to some measure $\widetilde{\omega}$ supported on $\partial\Omega$, with total mass $\mathrm{Cap}(\partial\Omega)$.

Notice also that the Green function satisfies

$$\mathcal{E}(p_k)^{-1} G(x, p_k) \leqslant \mathcal{E}(p_k)^{-1}\mathcal{E}(x - p_k) \to 1 \quad \text{as } k \to \infty, \quad \text{for all } x \in \Omega.$$

Thus there exists another subsequence $\{p_{k_h}\}_h$ such that the functions $\mathcal{E}(p_{k_h})^{-1} G^{p_{k_h}}$ converge locally uniformly in compact subsets of Ω to some harmonic function $\widetilde{g} : \Omega \to \mathbb{R}$ such that $\|\widetilde{g}\|_\infty \leqslant 1$. Without loss of generality, we may assume that the subsequences $\{p_{k_j}\}_j$ and $\{p_{k_h}\}_h$ coincide. Using that the functions $\mathcal{E}(p_{k_h})^{-1} G^{p_{k_h}}$ vanish continuously in

$\partial\Omega$, and using the maximum principle, as in the proof of Theorem 7.32, it follows that they converge uniformly on bounded subsets of Ω.

Given $\varphi \in C_c^\infty(\mathbb{R}^d)$, we have

$$\mathcal{E}(p_{k_j})^{-1} \int_\Omega G(x, p_{k_j})\, \Delta\varphi(x)\, dx = -\mathcal{E}(p_{k_j})^{-1}\varphi(p_{k_j}) + \mathcal{E}(p_{k_j})^{-1} \int \varphi\, d\omega^{p_{k_j}}.$$

By the uniform convergence of $\mathcal{E}(p_{k_j})^{-1}G(\cdot, p_{k_j})$ to $\widetilde{g}$ in bounded subsets of Ω, the left hand side converges to $\int_\Omega \widetilde{g}\, \Delta\varphi\, dx$ as $j \to \infty$, and by the weak $*$ convergence of $\mathcal{E}(p_{k_j})^{-1}\omega^{p_{k_j}}$ and the fact that $\varphi(p_{k_j}) = 0$ for j big enough, it is clear that the right hand side converges to $\int \varphi\, d\widetilde{\omega}$. So we deduce that

$$\int_\Omega \widetilde{g}\, \Delta\varphi\, dx = \int \varphi\, d\widetilde{\omega}.$$

From this fact, it is clear that $\widetilde{g}$ does not vanish identically on Ω. Taking into account that $\widetilde{g}$ is non-negative by construction and harmonic in Ω, it follows that $\widetilde{g}$ is (strictly) positive in Ω.

Next we will show that $\widetilde{\omega}$ coincides with the measure $\mathrm{Cap}(\partial\Omega)\,\mu$. To this end, recall that for any $x \in \Omega$,

$$G^{p_{k_j}}(x) = \mathcal{E}(x - p_{k_j}) - \int \mathcal{E}(x - z)\, d\omega^{p_{k_j}}(z).$$

Hence,

$$\mathcal{E}(p_{k_j})^{-1}G^{p_{k_j}}(x) = \mathcal{E}(p_{k_j})^{-1}\mathcal{E}(x - p_{k_j}) - \mathcal{E}(p_{k_j})^{-1} \int \mathcal{E}(x - z)\, d\omega^{p_{k_j}}(z).$$

The left side converges to $\widetilde{g}(x)$ as $j \to \infty$, while the first term on the right hand side tends to 1 and the last one to $\int \mathcal{E}(x - z)\, d\widetilde{\omega}(z)$. So we deduce that

$$\widetilde{g}(x) = 1 - \int \mathcal{E}(x - z)\, d\widetilde{\omega}(z) = 1 - U_{\widetilde{\omega}}(x). \tag{7.33}$$

Since $\widetilde{g}(x)$ is positive in Ω, we deduce that $U_{\widetilde{\omega}}(x) < 1$ for all $x \in \Omega$, and thus $U_{\widetilde{\omega}}(x) \leqslant 1$ for all $x \in \partial\Omega$. Since $\|\widetilde{\omega}\| = \mathrm{Cap}(\partial\Omega)$, by the uniqueness of the equilibrium measure μ of $\partial\Omega$, it follows that $\widetilde{\omega} = \mathrm{Cap}(\partial\Omega)\,\mu$, as claimed.

In particular, the identity $\widetilde{\omega} = \mathrm{Cap}(\partial\Omega)\,\mu$ ensures that the measure $\widetilde{\omega}$ does not depend on the chosen subsequence $\{p_{k_j}\}_j$, which in turn implies that the initial sequence of measures $\mathcal{E}(p_k)^{-1}\omega^{p_k}$ converges to $\widetilde{\omega}$. From the relationship between $\widetilde{g}$ and $\widetilde{\omega}$ in (7.33), we deduce that $\widetilde{g}$ does not depend on the subsequence $\{p_{k_j}\}_j$ either, and analogously this implies the local uniform convergence in bounded subsets of Ω of the functions $\mathcal{E}(p_k)^{-1}G^{p_k}$.

The preceding arguments show that setting $\omega^\infty = \widetilde{\omega}$ and $G^\infty = \widetilde{g}$, the properties (a), (c) and (d) hold. In particular, notice that the identities stated in (d) follow from (7.33) and (7.31). So it just remains to prove (b).

Consider a ball $B \subset \mathbb{R}^d$ centered at the origin such that $\partial\Omega \subset \frac{1}{2}B$. It suffices to show that ω^∞ is absolutely continuous with respect to ω^p with $p \in \partial B$. To this end, observe first that, by a Harnack chain argument,

$$\omega^p(E) \approx \omega^{p'}(E) \quad \text{for all } p, p' \in \partial B \text{ and all Borel set } E \subset \partial\Omega,$$

with the implicit constant independent of $p, p' \in E$. Consider the function

$$f_E(x) = \frac{r(B)^{n-1}}{|x|^{n-1}}\, \omega^p(E).$$

Observe that $f_E(p) = \omega^p(E) \approx \omega^q(E)$ for all $q \in \partial B$. Also,

$$\lim_{q\to\infty} f_E(q) = 0 = \lim_{q\to\infty} \omega^q(E).$$

So by the maximum principle we deduce that $f_E(x) \approx \omega^x(E)$ uniformly for all $x \in B^c$ and $E \subset \partial\Omega$. So we get

$$\frac{\omega^x(E)}{\omega^p(E)} \approx \frac{f_E(x)}{f_E(p)} = \frac{r(B)^{n-1}}{|x|^{n-1}} = \frac{f_{\partial\Omega}(x)}{f_{\partial\Omega}(p)} \approx \frac{\omega^x(\partial\Omega)}{\omega^p(\partial\Omega)}.$$

Thus,

$$\frac{\omega^p(E)}{\omega^p(\partial\Omega)} \approx \frac{\omega^x(E)}{\omega^x(\partial\Omega)} \qquad \text{for all } x \in B^c,$$

and then

$$\frac{\omega^p(E)}{\omega^p(\partial\Omega)} \approx \limsup_{y\to\infty} \frac{\omega^y(E)}{\omega^y(\partial\Omega)} \approx \liminf_{y\to\infty} \frac{\omega^y(E)}{\omega^y(\partial\Omega)}.$$

By the identity (7.32) and for k large enough, it follows that for $p \in \partial B$,

$$\frac{\mathcal{E}(p_k)^{-1}\omega^{p_k}(E)}{\mathrm{Cap}(\partial\Omega)} = \frac{U_\mu(p_k)}{\mathcal{E}(p_k)}\,\frac{\omega^{p_k}(E)}{\omega^{p_k}(\partial\Omega)} \approx \frac{U_\mu(p_k)}{\mathcal{E}(p_k)}\,\frac{\omega^p(E)}{\omega^p(\partial\Omega)}.$$

Letting $k \to \infty$, by Theorem 4.63 for every open set E we derive

$$\mathrm{Cap}(\partial\Omega)^{-1}\omega^\infty(E) \lesssim \frac{\omega^p(E)}{\omega^p(\partial\Omega)}$$

and for every compact set E we get

$$\mathrm{Cap}(\partial\Omega)^{-1}\omega^\infty(E) \gtrsim \frac{\omega^p(E)}{\omega^p(\partial\Omega)}.$$

By the regularity of Radon measures we infer

$$\mathrm{Cap}(\partial\Omega)^{-1}\omega^\infty(E) \approx \frac{\omega^p(E)}{\omega^p(\partial\Omega)}$$

for every Borel set $E \subset \partial\Omega$, which proves (c). □

Remark 7.34. Notice that the estimate in Lemma 7.18 also holds for the harmonic measure and the Green function with pole at ∞. To check this, just multiply the inequality (7.11) by $\mathcal{E}(x)^{-1}$ and take the limit as $x \to \infty$ and apply Theorem 4.63.

7.6.3 Immediate consequences

Using the same proofs as in Section 7.1, Theorems 7.32 and 7.33 immediately imply the following facts.

Proposition 7.35. *Let $\Omega \subset \mathbb{R}^d$ be an unbounded domain with compact boundary $\partial\Omega = E \cup \gamma$ where E is either compact or empty, γ is a C^1 curve and $E \cap \gamma = \varnothing$. If $G^\infty \in C^1(\Omega \cup \gamma)$, then*

$$\omega^\infty|_\gamma = -(\partial_\nu G^\infty)\,\sigma,$$

where ν is the unit outer normal to $\gamma \subset \partial\Omega$ and σ is the surface measure on γ.

Lemma 7.36. *Let $\Omega \subset \mathbb{R}^d$ be an unbounded Wiener regular open set with compact boundary. Let B be a ball centered in $\partial\Omega$. Then,*

$$\omega^\infty(B) \lesssim r(B)^{d-2} \fint_{2B} G^\infty(y)\,dy.$$

8 Harmonic measure in uniform domains satisfying the CDC and in NTA domains

This chapter deals with properties of harmonic measure on uniform domains satisfying the CDC and in NTA domains. Most of the material is based on [JK82]. In this chapter we assume that the domain Ω has compact boundary. We will use the following notation, in the spirit of Definition 4.17.

Definition 8.1. Let $\Omega \subset \mathbb{R}^d$. For every $\xi \in \partial\Omega$ and $r > 0$ we write the boundary ball

$$\Delta_{r,\xi} := \Delta_r(\xi) := B_r(\xi) \cap \partial\Omega.$$

We also use the classical notation for rescaled balls in the setting of boundary balls:

$$t\Delta_{r,\xi} := \Delta_{tr,\xi}.$$

8.1 CDC, uniform, and NTA domains

Definition 8.2. A CDC domain is a domain satisfying the CDC condition.

Recall that CDC domains are Wiener regular.

Definition 8.3. A domain $\Omega \subset \mathbb{R}^d$ satisfies the exterior corkscrew condition if there exist $r_0 > 0$ and $A > 0$ such that for every $\xi \in \partial\Omega$ and $r < r_0$ there exists a point $X_r^{\mathrm{ex}}(\xi) = X_{r,\xi}^{\mathrm{ex}} = X_{\Delta_{r,\xi}}^{\mathrm{ex}} \in \overline{\Omega}^c$ satisfying $|X_r^{\mathrm{ex}}(\xi) - \xi| < r$ and $\mathrm{d}_\Omega(X_r^{\mathrm{ex}}(\xi)) := \mathrm{dist}(X_r^{\mathrm{ex}}(\xi), \partial\Omega) > A^{-1}r$. We call $X_r^{\mathrm{ex}}(\xi)$ an *exterior corkscrew point* of ξ at scale r, and $B_{\Delta_{r,\xi}}^{\mathrm{ex}} := B_{r,\xi}^{\mathrm{ex}} := B_{\frac{r}{2A}}(X_{r,\xi}^{\mathrm{ex}})$ is called exterior corkscrew ball. Note that $B_{r,\xi}^{\mathrm{ex}} \subset 2B_{r,\xi}^{\mathrm{ex}} \subset \overline{\Omega}^c$.

It is immediate to check that, for any bounded domain, the exterior corkscrew condition implies the CDC condition, and thus the Wiener regularity of Ω.

Next we recall one of the Hölder regularity properties already shown for CDC domains.

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^d$ be a CDC domain with compact boundary, let $u \in C(B_r(\xi) \cap \Omega)$ be non-negative harmonic, vanishing continuously on $\Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $r < \mathrm{diam}(\partial\Omega)$. Then there are constants C_0 and α depending on d and the CDC character so that*

$$u(x) \leqslant C_0 \left(\frac{|x-\xi|}{r}\right)^\alpha \sup_{B_r(\xi)\cap\Omega} u \quad \textit{for every } x \in B_r(\xi) \cap \Omega.$$

Definition 8.5. A uniform domain $\Omega \subset \mathbb{R}^d$ is a domain satisfying

- Interior corkscrew condition: There exist $r_0 > 0$ and $A > 0$ such that for every $\xi \in \partial\Omega$ and $r < r_0$ there exists a point $X_r(\xi) = X_{r,\xi} = X_{\Delta_{r,\xi}} \in \Omega$ satisfying $|X_r(\xi) - \xi| < r$ and $d_\Omega(X_r(\xi)) > A^{-1}r$. We call $X_r(\xi)$ a *(interior) corkscrew point* of ξ at scale r, and $B^{\text{in}}_{\Delta_{r,\xi}} := B^{\text{in}}_{r,\xi} := B_{\frac{r}{2A}}(X_r(\xi))$ is called interior corkscrew ball. Note that $B^{\text{in}}_{r,\xi} \subset 2B^{\text{in}}_{r,\xi} \subset \overline{\Omega}$.
- Harnack chain condition: for $\varepsilon > 0$ and $x_1, x_2 \in \Omega$ with $d_\Omega(x_j) > \varepsilon$ and $|x_1 - x_2| = r < r_0$, there exists N depending only on $\frac{r}{\varepsilon}$ and a collection of balls $\{B_j\}_{j=0}^N$ with $x_1 \in B_0$, $x_2 \in B_N$ such that $2B_j \subset \Omega$ for every $0 \leqslant j \leqslant N$ and $B_j \cap B_{j-1} \neq \varnothing$ for every $1 \leqslant j \leqslant N$. This collection of balls is called a *Harnack chain* joining x_1 and x_2.

Lemma 8.6. *A domain $\Omega \subset \mathbb{R}^d$ is uniform if and only if for every $x_0, x_1 \in \Omega$ with $|x_0 - x_1| < r_0$ there exists a* non-tangential path*, i.e. a continuous map $\gamma : [0,1] \to \Omega$ such that*

1. *$\gamma(j) = x_j$ for $j \in \{0,1\}$,*
2. *the length of the curve $\ell(\gamma) \leqslant \widetilde{A}\,|x_0 - x_1|$ and*
3. *for $t \in (0,1)$ we have $d_\Omega(\gamma(t)) \geqslant \operatorname{dist}(\gamma(t), \{x_0, x_1\})/\widetilde{A}$.*

Proof. We can show first the 'if' part. Let $\xi \in \partial\Omega$, $r < \min\{r_0, \operatorname{diam}(\Omega)\}$. Consider $x_0 \in B_{\frac{r}{4}}(\xi) \cap \Omega$ and $x_1 \in \partial B_r(\xi) \cap \Omega$ (which exists by connectedness) and consider the path γ connecting x_0 and x_1. Then the point $X_r(\xi) := y \in \gamma(0,1) \cap \partial B_{\frac{r}{2}}(\xi)$ is a corkscrew point, so Ω satisfies de corkscrew condition.

Let us prove that the Harnack chain condition is also satisfied. To this end just consider $\varepsilon > 0$ and $x_1, x_2 \in \Omega$ with $d_\Omega(x_j) > \varepsilon$ and $|x_1 - x_2| = r < r_0$. We may assume that $r > d_\Omega(x_j)/4$, for $j = 1, 2$, because otherwise it suffices to consider a the ball $B_{d_\Omega(x_j)/2}(x_j)$. Take the collection of balls $\{B_{\frac{1}{10}d_\Omega(y)}(y)\}_{y \in \gamma([0,1])}$. By the $5r$-covering theorem there exists a subcollection of disjoint balls B_j, $j \in J$, such that $5B_j$ cover $\gamma([0,1])$. The radii of the balls are bounded below by a constant times $\operatorname{dist}(\gamma([0,1]), \partial\Omega) > C_{\widetilde{A}}^{-1}\varepsilon$ by the third condition.

We claim that, for every $k > 0$, the number of balls with $2^k C_{\widetilde{A}}^{-1}\varepsilon \leqslant r(B_j) < 2^{k+1} C_{\widetilde{A}}^{-1}\varepsilon$ is bounded by a constant C_1 depending on d and $\widetilde{A}$. It is enough to consider the balls whose center is closer to the endpoint x_1.

Note that the centers $x(B_j)$ of the balls B_j satisfy

$$\operatorname{dist}(x(B_j), x_1) \leqslant \widetilde{A} d_\Omega(x(B_j)) = 10\widetilde{A}\, r(B_j).$$

For any $t > 0$, the collection of balls B_j such that $t \leqslant r(B_j) < 2t$ is disjoint by assumption, each one has measure bounded below by a dimensional constant times t^d, and all of them are contained in the ball centered at x_1 with radius $10\widetilde{A}r(B_j) + r(B_j) \leqslant (20\widetilde{A} + 2)t$, whose measure is bounded above by a dimensional constant times $C_{\widetilde{A}}^d t^d$. Thus the number of balls is bounded above by $C_{\widetilde{A}}^d$ as claimed.

Also we can bound above the radii of the balls B_j, $j \in J$ as follows: by the assumption that $r > d_\Omega(x_1)/4$ and the second condition,

$$r(B_j) = \frac{1}{10} d_\Omega(x(B_j)) \leqslant \frac{1}{10}\left(d_\Omega(x_1) + |x_1 - x(B_j)|\right| \leqslant \frac{1}{10}\left(4r + \ell(\gamma)\right) \leqslant (1 + \widetilde{A})r.$$

Thus, the number of balls is bounded by

$$N \leqslant C_1\left(\log_2((1+\widetilde{A})r) - \log_2(C_{\widetilde{A}}^{-1}\varepsilon)\right) = C_1 \log_2 \frac{(1+\widetilde{A})r}{C_{\widetilde{A}}^{-1}\varepsilon}.$$

To show the converse, assume that Ω is uniform and let $x_0, x_1 \in \Omega$ with $d_\Omega(\{x_0, x_1\}) \leqslant |x_0 - x_1| < r_0$ (otherwise the straight segment joining x_0 and x_1 would be a non-tangential path trivially). Let $\xi_j \in \partial\Omega$ be points minimizing $\operatorname{dist}(x_j, \xi)$, and for every $0 \leqslant k \leqslant k_j := \lfloor \log_2(\frac{|x_0 - x_1|}{d_\Omega(x_j)}) \rfloor$ consider the corkscrew point $y_k^j := X_{2^k d_\Omega(x_j)}(\xi_j)$, and let also $y_{-1}^j = x_j$. The number of balls in a Harnack chain between two consecutive points y_k^j and y_{k+1}^j is uniformly bounded. The same can be said about the Harnack chain joining $y_{k_0}^0$ and $y_{k_1}^1$. Joining the centers of the balls in these Harnack chains between consecutive points we find a path satisfying the three conditions above. Indeed 1 holds trivially, 2 is a consequence of the fact that the number of balls of each scale is uniformly bounded and, therefore, the length of the curve can be controlled by a geometric sum whose bigger term is comparable to $|x_0 - x_1|$. The third condition follows from the fact that for every ball B from the Harnack chains d_Ω is comparable with $r(B)$ and the distance from the ball to the closest end-point is bounded again by a geometric series whose bigger term is comparable to $r(B)$. □

Put in plain words, the definition we give here of uniform domains in terms of corkscrew points and Harnack chains coincides with the definition in terms of “cigar” (i.e. non-tangential) paths from the Sobolev extension domains in [Jon81]. Also from the previous proof we can infer that the definition coincides with the one in [GO79], where the distance $\operatorname{dist}(\gamma(t), \{x_0, x_1\})$ in the third condition is replaced by the arc-length distance to the endpoints.

Roughly speaking, the domain cannot have outer cusps, thin tubes or slits. In two dimensions inner cusps are also banned.

The Harnack chain condition, using Lemma 2.15, gives us that, whenever u is a positive harmonic function on Ω,

$$C^{-N(\Lambda)} u(y) \leqslant u(x) \leqslant C^{N(\Lambda)} u(y) \text{ whenever } \frac{|x-y|}{d_\Omega(\{x,y\})} \leqslant \Lambda.$$

By the previous proof, uniformity tells us that for $k \geqslant 1$, by picking non-tangential paths we can assume

$$N(2^k) \leqslant C_1 \log_2\left(C_A 2^k\right) \leqslant C_1(k + \log_2(C_A)),$$

that is whenever $|x - y| \leqslant \min\{2^k d_\Omega(\{x,y\}), r_0\}$ with $k \geqslant 2$ we have

$$C_A^{-k} u(y) \leqslant u(x) \leqslant C_A^k u(y). \tag{8.1}$$

Note that the value of C_A may have increased in our reasoning, but it depends only on the constant A and the dimension d.

Definition 8.7. We say that a domain is UCDC (uniform domains satisfying the capacity-density condition) if it is both CDC with constant A and uniform with constants r_0 and A. More precisely, we assume that there exists a radius $0 < r_0 \leqslant \operatorname{diam}(\partial\Omega)$ and a constant A such that

1. The interior corkscrew and the Harnack chain conditions in Definition 8.5 are satisfied with constants r_0 and A.

2. Every pair of points $x_0, x_1 \in \Omega$ with $|x_0 - x_1| < r_0$ can be joined with a non-tangential path as in Lemma 8.6, with constant $\widetilde{A} = A$.

3. The domain satisfies the CDC with constant A^{-1}, that is, for any $r \in (0, \operatorname{diam}(\partial\Omega))$,
$$\operatorname{Cap}(\bar{B}_r(\xi)\backslash\Omega) \geqslant A^{-1}\, r^{d-2} \qquad \text{in the case } d \geqslant 3,$$
and
$$\operatorname{Cap}_L(\bar{B}_r(\xi)\backslash\Omega) \geqslant A^{-1}\, r \qquad \text{in the case } d = 2.$$

4. Further we assume that there exists a constant C_A such that (8.1) is satisfied, and we also assume that
$$N(t) \leqslant C_A(1 + \log_2(At + 1)).$$

5. If the boundary of the domain is bounded, we assume without loss of generality that $r_0 = \operatorname{diam}(\partial\Omega)$. Indeed, just by taking worse constants depending on the ratio $\frac{r_0}{\operatorname{diam}(\partial\Omega)}$ we can check that both corkscrew conditions and the Harnack chain condition are satisfied as well for $r_0 \leqslant r \leqslant \operatorname{diam}(\partial\Omega)$.

Note that, if Ω were unbounded with compact boundary, we could pick $r_0 = \infty$ regarding the uniformity constants, but the CDC would not hold for big balls, and estimate (7.20) would cease to be true in higher dimensions, so we will keep $r_0 = \operatorname{diam}(\partial\Omega)$ in this case to clarify ideas.

From this point onwards, we will write C_A and c_A for constants which depend only on the uniformity constants and the CDC as well. Note that in the preceding definition, we write A for the maximum constant between A and $\widetilde{A}$. In particular, for UCDC domains, by (7.20) and the Harnack chain property (8.1) we have:

Lemma 8.8. *Let $\Omega \subset \mathbb{R}^d$ be a UCDC domain with compact boundary and let $\xi \in \partial\Omega$ and $r \leqslant \operatorname{diam}(\partial\Omega)$. Then for $x \in B_{4r}(\xi) \cap \Omega$ with $\operatorname{dist}(x, \partial\Omega\backslash\Delta_{r,\xi}) \geqslant \frac{r}{A}$, we get*
$$\omega^x(\Delta_{r,\xi}) \geqslant c_A.$$

Proof. Let's write $\Delta = \Delta_{r,\xi}$. First assume that $x = X_\Delta$. Then using a Harnack chain, and (7.20) we get
$$\omega^{X_\Delta}(\Delta) \overset{(8.1)}{\approx_A} \omega^{X_{\Delta/4}}(\Delta) \overset{(7.20)}{\gtrsim_A} 1.$$

If $d_\Omega(x) \geqslant \frac{r}{2A}$, then using a Harnack chain again we obtain

$$\omega^x(\Delta) \overset{(8.1)}{\approx_A} \omega^{X_\Delta}(\Delta) \gtrsim_A 1.$$

If, instead, $d_\Omega(x) = \rho < \frac{r}{2A}$, then let $\zeta \in \partial\Omega$ such that $d_\Omega(x) = |x - \zeta|$. By our assumption, $\zeta \in \Delta$, and $\Delta_{\rho,\zeta} \subset \Delta$. Thus,

$$\omega^x(\Delta) \geqslant \omega^x(\Delta_{\rho,\zeta}) \overset{(8.1)}{\approx_A} \omega^{X_{\rho,\zeta}}(\Delta_{\rho,\zeta}) \gtrsim_A 1.$$

□

Definition 8.9. A non-tangentially accessible domain (NTA domain for short) is a uniform domain satisfying also the exterior corkscrew condition.

It is clear that any NTA domain is UCDC. The notion of NTA domain was introduced by Jerison and Kenig in [JK82]. In this work they studied the behavior of harmonic measure in this type of domains. Roughly speaking, NTA domains cannot have outer cusps, inner cusps, thin tubes, slits or isolated points in the boundary.

8.2 Green's function for UCDC domains

Next we show that the supremum of a nonnegative harmonic function in a ball coincides modulo constant with the value at the corkscrew point:

Lemma 8.10. *Let Ω be a UCDC domain. Let $u \geqslant 0$ harmonic in Ω, vanishing continuously on $\Delta_{2r,\xi}$ with $\xi \in \partial\Omega$ and $r < r_0$, then we have*

$$\sup_{\Omega \cap B_r(\xi)} u \leqslant C_A u(X_{r,\xi}).$$

Proof. Via Harnack inequality (8.1), we can control

$$\sup_{\Omega \cap B_r(\xi)} u \leqslant C_A \sup_{\Omega \cap B_r(\xi): d_\Omega(x) < r/8} u(x),$$

and for $\zeta \in \Delta_{r,\xi}$ we have

$$u(X_{r/8,\zeta}) \approx_A u(X_{r,\xi}).$$

Thus, to simplify notation, we can assume that $8r < r_0$, u vanishes on 8Δ with $\Delta := \Delta_{r,\xi}$, and let us assume that $u(X_{2\Delta}) = 1$. We will prove that

$$\sup_{\Omega \cap B_{2r}(\xi)} u \lesssim 1.$$

Theorem 8.4 implies the existence of a constant $A_1 > 1$ s.t. for every $\zeta \in 3\Delta$ and every $s < r$

$$\sup_{B(\zeta, A_1^{-1}s) \cap \Omega} u \leqslant \frac{1}{2} \sup_{B(\zeta,s) \cap \Omega} u. \tag{8.2}$$

The second observation is about the quantitative behavior of Harnack chains described in (8.1): if $x \in B_r(\zeta) \cap \Omega$ with $\zeta \in 3\Delta$, $n \in \mathbb{N}$, and $\mathrm{d}_\Omega(x) \geqslant A_1^{-n} r$, then

$$|X_{2r,\xi} - x| < 6r \leqslant 6A_1^n \mathrm{d}_\Omega(x) \implies C_A^{-k} u(x) \leqslant u(X_{2r,\xi}) = 1,$$

where $k = 1 + \lfloor \log_2(6A_1^n) \rfloor \approx n$. Thus, we can pick $A_2 := C_A^{k/n} > 1$ above, and we deduce that whenever $x \in B_{2r}(\zeta) \cap \Omega$, we have

$$u(x) > A_2^n \implies \mathrm{d}_\Omega(x) < A_1^{-n} r. \tag{8.3}$$

Now we argue by contradiction: consider N so that $2^N > A_2$ and let $n = N+3$. Assume that there exists $y_0 \in \Omega \cap B_{2r}(\xi)$ with $u(y_0) > A_2^n$. Then, by (8.3) we can find $\xi_0 \in \partial\Omega$ satisfying that

$$|y_0 - \xi_0| < A_1^{-n} r.$$

Note also that

$$|\xi - \xi_0| \leqslant |\xi - y_0| + |y_0 - \xi_0| \leqslant 2r + A_1^{-n} r < 3r.$$

for A_1 large enough, and by (8.2) we have

$$\sup_{B(\xi_0, A_1^{-n+N} r)} u > 2^N \sup_{B(\xi_0, A_1^{-n} r)} u > A_2 \cdot A_2^n = A_2^{n+1}.$$

We have proven the existence of $y_1 \in B(\xi_0, A_1^{-n+N} r)$ with $u(y_1) > A_2^{n+1}$. Since $N - n < 0$, we can apply (8.3) to find $\xi_1 \in \partial\Omega$ so that

$$|y_1 - \xi_1| < A_1^{-n-1} r.$$

Note also that

$$|\xi - \xi_1| \leqslant |\xi - \xi_0| + |\xi_0 - y_1| + |y_1 - \xi_1| \leqslant (2 + A_1^{-n} + A_1^{-n+N} + A_1^{-n-1}) r < 3r,$$

for A_1 large enough, and by (8.2) we have

$$\sup_{B(\xi_1, A_1^{-n-1+N} r)} u > 2^N \sup_{B(\xi_1, A_1^{-n-1} r)} u > A_2 \cdot u(y_1) > A_2^{n+2}.$$

Iterating the construction, we find $y_k \in B(\xi_{k-1}, A_1^{-n+N-k+1} r)$ with $u(y_k) > A_2^{n+k}$. We can apply (8.3) to find $\xi_k \in \partial\Omega$ so that

$$|y_k - \xi_k| < A_1^{-n-k} r.$$

Note also that

$$|\xi - \xi_k| \leqslant |\xi - \xi_{k-1}| + |\xi_{k-1} - y_k| + |y_k - \xi_k| \leqslant \left(2 + A_1^{-n} + \sum_{j=1}^{k} \left(A_1^{-n+N-j+1} + A_1^{-n-j}\right)\right) r < 3r,$$

for A_1 large enough, and by (8.2) we have

$$\sup_{B(\xi_k, A_1^{-n-k+N} r)} u > 2^N \sup_{B(\xi_k, A_1^{-n-k} r)} u > A_2 \cdot u(y_k) > A_2^{n+k+1},$$

so the induction can be carried on.

Note that y_k is a Cauchy sequence converging to a point in 3Δ. Therefore, we reach a contradiction with the continuity of u. $\square$

Lemma 8.11. *Let Ω be a UCDC domain with compact boundary, let $G := G_\Omega$ be its Green function and let $x \in \Omega \backslash B(\xi, 2r)$, with $\xi \in \partial\Omega$ and $r \leqslant \operatorname{diam}(\partial\Omega)$. Then the boundary ball $\Delta := \Delta_{r,\xi}$ satisfies*

$$\omega^x(\Delta) \leqslant C_A r^{d-2} G^x(X_\Delta)$$

Proof. Let $\phi \in C^\infty$ bump function so that $\chi_{B_r(\xi)} \leqslant \phi \leqslant \chi_{B_{5r/4}(\xi)}$ (so $\phi(x) = 0$) and $|D^2\phi| \lesssim r^{-2}$. Then

$$\int_\Omega G^x(y)\Delta\phi(y)\, dm(y) \overset{\text{L.7.13}}{=} \int \phi(\xi)\, d\omega^x(\xi) \geqslant \omega^x(\Delta).$$

Consider the domain $\widetilde{\Omega} := \Omega \backslash B_{\frac{1}{4}\mathrm{d}_\Omega(x)}(x)$, which is a UCDC domain with perhaps worse constant than the original one (but depending on it). Note that for $y \in \partial B_{\frac{1}{4}\mathrm{d}_\Omega(x)}(x)$ we have

$$|\xi - y| \geqslant |x - \xi| - \mathrm{d}_\Omega(x)/4 \geqslant \frac{3}{4}|x - \xi| \geqslant \frac{3}{2}r.$$

Thus, G^x is a function vanishing on the boundary ball $\Delta_{\frac{3}{2}r,\xi}$ with respect to the domain $\widetilde{\Omega}$. We can cover $\{y \in B_{\frac{5}{4}r,}(\xi) : \mathrm{d}_\Omega(y) < r/16\}$ with balls $B_\zeta := B_{\frac{r}{16}}(\zeta)$ so that G^x is a harmonic nonnegative function vanishing on $2\Delta_\zeta = 2\Delta_{\frac{r}{16}}(\zeta)$, and then apply Lemma 8.10 to conclude that $G^x(y) \lesssim_A G^x(X_{\Delta_\zeta})$ on $B_{r/16}(\zeta) \cap \Omega$. Using (8.1) we obtain $G^x(X_{\Delta_\zeta}) \approx_A G^x(X_\Delta)$, and also $G^x(y) \approx_A G^x(X_\Delta)$ for $y \in B_{\frac{5}{4}r(\xi)}$ such that $\mathrm{d}_\Omega(y) \geqslant r/16$. All in all, we get

$$\omega^x(\Delta) \leqslant \int_\Omega G^x(y)\Delta\phi(y)\, dm(y) \leqslant \int_{B_{\frac{5}{4}r(\xi)}} G^x(y)|\Delta\phi(y)|\, dm(y) \overset{\text{L 8.10}}{\lesssim}_{A,d} r^{d-2} G^x(X_\Delta).$$

□

Note that by the results in Chapter 7 and the Harnack chain condition, we get the converse inequality:

Lemma 8.12. *Let Ω be a UCDC domain with compact boundary, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $r \leqslant \operatorname{diam}(\partial\Omega)$. If $x \in \Omega \backslash B^{\text{in}}_{\frac{1}{2}\Delta}$, then*

$$r^{d-2} G^x(X_{\frac{1}{2}\Delta}) \lesssim_A \omega^x(\Delta).$$

Proof. From Lemmas 7.19 and 7.23, together with the CDC (see 3. in Definition 8.7), we have

$$r^{d-2} G^x(y) \lesssim_A \omega^x(\Delta) \qquad \text{for all } x \in \Omega \backslash \tfrac{1}{4}\bar{B} \text{ and } y \in (40)^{-1}\bar{B} \cap \Omega. \tag{8.4}$$

If $x \notin B^{\text{in}}_{(40A)^{-1}\Delta}$, set $Y := X_{(40A)^{-1}\Delta}$. Otherwise set $Y := X_{(40A)^{-2}\Delta}$, so that in both cases we get $x \notin B_{\frac{1}{2}\mathrm{d}_\Omega(Y)}(Y)$ and so $Y, X_{\frac{1}{2}\Delta} \notin B_{\frac{1}{4}\mathrm{d}_\Omega(x)}(x)$, see Exercise 8.2.1 below. Note that independently of our choice for Y, Green's function G^x is non-negative and harmonic in the domain $\Omega \backslash B_{\frac{1}{4}\mathrm{d}_\Omega(x)}(x)$, which is a UCDC domain with perhaps worse constants than Ω. Thus, (8.1) applies in this setting, and we get

$$G^x(X_{\frac{1}{2}\Delta}) \approx_A G^x(Y).$$

If $x \in \Omega \backslash \frac{1}{4}\bar{B}$, by (8.4) with $y = Y$ the lemma follows. If, instead, $x \in \Omega \cap \frac{1}{4}\bar{B}$, then consider two situations. First, if $d_\Omega(x) > (40A)^{-3}r$, then using (8.1) in $\Omega \backslash B_{\frac{1}{2}d_\Omega(Y)}(Y)$ we get

$$r^{d-2}G^x(Y) \overset{(8.1)}{\approx}_A r^{d-2}G^{X_\Delta}(Y) \overset{(8.4)}{\lesssim_A} \omega^{X_\Delta}(A^{-1}\Delta) \overset{(8.1)}{\lesssim}_A \omega^x(\Delta).$$

Finally, whenever $x \in \Omega \cap \frac{1}{4}\bar{B}$ with $d_\Omega(x) \leqslant (40A)^{-3}r$, then

$$r^{d-2}G^x(Y) \overset{(7.5)}{=} r^{d-2}G^Y(x) \overset{(8.1)}{\approx}_A r^{d-2}G^{X_\Delta}(x) \overset{(8.4)}{\lesssim_A} \omega^{X_\Delta}(A^{-1}\Delta) \overset{(5.11)}{\leqslant} 1 \overset{(7.20)}{\lesssim_A} \omega^x(\Delta).$$

□

Combining Lemmas 8.11 and 8.12 we get the following remarkable fact.

Theorem 8.13. *Let Ω be a UCDC domain with compact boundary, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $r \leqslant \operatorname{diam}(\partial\Omega)$. For $x \in \Omega \backslash B(\xi, 2r)$*

$$\frac{\omega^x(\Delta)}{r^{d-2}G^x(X_\Delta)} \approx_A 1.$$

Exercise 8.2.1. Let Ω be an open set. Given $x, y \in \Omega$, show that

$$x \notin B_{\frac{1}{2}d_\Omega(y)}(y) \implies y \notin B_{\frac{1}{4}d_\Omega(x)}(x).$$

8.3 Fundamental properties of the harmonic measure in UCDC domains

8.3.1 The doubling condition

Lemma 8.14 (Doubling condition)**.** *Let Ω be a UCDC domain with compact boundary. If $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $x \in \Omega$, then*

$$\omega^x(2\Delta) \leqslant C\omega^x(\Delta),$$

with C depending on $\frac{d_\Omega(x)}{\operatorname{diam}(\partial\Omega)}$, d, A, but neither on x nor on Δ.

Proof. Let $r_1 := (4A)^{-1}\operatorname{diam}(\partial\Omega)$ and assume first that $2r_1 \leqslant d_\Omega(x) \leqslant \frac{1}{2}\operatorname{diam}(\partial\Omega)$.

The case $2r \geqslant r_1$ follows by Lemma 8.8 and the Harnack inequality. Indeed, we can find a finite family of points ξ_j so that $\Delta_{r_1/4}(\xi_j)$ cover the boundary, so there is a ξ_{j_0} so that $\xi \in \Delta_{r_1/4,\xi_{j_0}}$ and thus $\Delta_{r_1/4,\xi_{j_0}} \subset \Delta$. Therefore

$$\omega^x(\Delta) \geqslant \omega^x(\Delta_{r_1/4,\xi_{j_0}}) \overset{(8.1)}{\gtrsim_A} \omega^{X_{r_1/4,\xi_{j_0}}}(\Delta_{r_1/4,\xi_{j_0}}) \overset{\text{L. } 8.8}{\geqslant} c_A \geqslant c_A\omega^x(2\Delta),$$

the constants of the second estimate depending only on A and perhaps on the dimension.

If $2r < r_1$, then we can use Theorem 8.13 twice and the Harnack chain:

$$\omega^x(2\Delta) \overset{\text{T } 8.13}{\approx_A} cr^{d-2}G^x(X_{2\Delta}) \overset{(8.1)}{\approx_A} cr^{d-2}G^x(X_\Delta) \overset{\text{T } 8.13}{\approx_A} \omega^x(\Delta).$$

For the cases $\mathrm{d}_\Omega(x) < 2r_1$ and $2\mathrm{d}_\Omega(x) > \mathrm{diam}(\partial\Omega)$, consider a corkscrew point $x_0 \in B_{\mathrm{diam}(\partial\Omega)/2}(\xi)$ so that $\mathrm{d}_\Omega(x_0) \geqslant 2r_1$, whose existence is granted by the interior corkscrew condition. Since $\omega^x(\Delta)$ and $\omega^x(2\Delta)$ are harmonic functions, we get that

$$\omega^x(\Delta) \approx_{\frac{\mathrm{d}_\Omega(x)}{\mathrm{diam}(\partial\Omega)},A} \omega^{x_0}(\Delta) \gtrsim_A \omega^{x_0}(2\Delta) \approx_{\frac{\mathrm{d}_\Omega(x)}{\mathrm{diam}(\partial\Omega)},A} \omega^x(2\Delta).$$

□

Note that one cannot expect to avoid the dependence on x: if $x \to 2\Delta\backslash\overline{\Delta}$, then $\omega^x(\Delta) \to 0$ and $\omega^x(2\Delta) \to 1$. However, the doubling constant for a fixed $\Delta = \Delta_{r,\xi}$ is universal if we pick the pole in $\Omega\backslash(B_{4r}(\xi)\backslash B_{(1-A^{-1})r}(\xi))$:

Lemma 8.15. *In Lemma 8.14, if $x \in \Omega\backslash B_{4r}(\xi)$, then C does not depend on $\frac{\mathrm{d}_\Omega(x)}{\mathrm{diam}(\partial\Omega)}$. The same can be said whenever $\mathrm{dist}(x, \partial\Omega\backslash\Delta) \geqslant \frac{r}{A}$, and $r \leqslant \mathrm{diam}(\partial\Omega)$.*

Proof. The case $2r_1 \leqslant \mathrm{d}_\Omega(x) \leqslant \frac{1}{2}\mathrm{diam}(\partial\Omega)$, where $r_1 := (4A)^{-1}\mathrm{diam}(\Omega)$ is already settled in the proof of Lemma 8.14.

The case $2\mathrm{d}_\Omega(x) > \mathrm{diam}(\partial\Omega)$ can be settled by standard maximum principle arguments combined with Harnack. Indeed, the constant is universal for $x \in S_{2\mathrm{diam}(\partial\Omega)}(\zeta)$ for a fixed $\zeta \in \partial\Omega$, by the Harnack inequality, since both functions $\omega^x(\Delta)$ and $\omega^x(2\Delta)$ are harmonic and non-negative:

$$\frac{\omega^x(\Delta)}{\omega^x(2\Delta)} \overset{\text{Harnack}}{\approx}_{d,A} \frac{\omega^{X_{\mathrm{diam}(\partial\Omega),\zeta}}(\Delta)}{\omega^{X_{\mathrm{diam}(\partial\Omega),\zeta}}(2\Delta)} \approx_A 1,$$

since we have reduced to the previous case. In fact, both functions satisfy the Dirichlet problem on unbounded domains (5.7) (see Theorem 5.42, Lemma 5.45, Remark 5.46 and Example 5.49), so the maximum principle allows us to extend

$$\omega^x(\Delta) \approx_A \omega^x(2\Delta)$$

to the whole $B_{2\mathrm{diam}(\partial\Omega)}(\zeta)^c$. The general case $2\mathrm{d}_\Omega(x) > \mathrm{diam}(\partial\Omega)$ follows by Harnack again.

Thus, we can assume $\mathrm{d}_\Omega(x) < 2r_1$. If $x \in B_{4r}(\xi)^c \cap \Omega$, that is, if $\mathrm{dist}(x, \xi) > 4r$, since $\mathrm{d}_\Omega(x) < (2A)^{-1}\mathrm{diam}(\partial\Omega)$, we infer that $r < \frac{1}{2}\mathrm{diam}(\partial\Omega)$. Thus, we can use Theorem 8.13 twice and the Harnack chain:

$$\omega^x(2\Delta) \overset{\text{T }8.13}{\approx_A} r^{d-2}G^x(X_{2\Delta}) \overset{(8.1)}{\approx_A} r^{d-2}G^x(X_\Delta) \overset{\text{T }8.13}{\approx_A} \omega^x(\Delta).$$

Finally, if $\mathrm{d}_\Omega(x) < 2r_1$, $x \in B_{4r}(\xi) \cap \Omega$ with $\mathrm{dist}(x, \partial\Omega\backslash\Delta) \geqslant A^{-1}r$, we get

$$\omega^x(\Delta) \overset{\text{L }8.8}{\gtrsim}_A 1 \geqslant \omega^x(2\Delta),$$

and the lemma follows.

□

Exercise 8.3.1. In Lemma 8.15, it suffices to require $\mathrm{dist}(x, 2\Delta\backslash\Delta) \geqslant \frac{r}{A}$.

8.3.2 The boundary Harnack principle

Next we find localized UCDC domains, that is, given a UCDC domain Ω, we provide intermediate domains contained in Ω which have diameter comparable to a ball, and at the same time coincide with Ω in a comparable, smaller ball. This is obtained using a Whitney covering, i.e., a covering of Ω with disjoint dyadic cubes, which are half-open cubes with sides parallel to the axis, vertices in the grid $2^{-k}\mathbb{Z}^d$ for $k \in \mathbb{Z}$ and with side–length

$$\ell(Q) := 2^{-k}.$$

Then, we denote by $\mathcal{W} := \mathcal{W}(\Omega)$ the set of maximal dyadic cubes $Q \subset \Omega$ such that $4Q \cap \Omega^c = \varnothing$. These cubes have disjoint interiors and can be easily shown to satisfy the following properties:

(a) $\operatorname{dist}(Q, \Omega^c) \lesssim \ell(Q) \lesssim \operatorname{dist}(Q, \Omega^c)$, where $\ell(Q)$ denotes the side length of the cube.

(b) If $Q, R \in \mathcal{W}$ and $4Q \cap 4R \neq \varnothing$, then $\ell(Q) \approx_d \ell(R)$. In particular we may assume that $\ell(Q) \leqslant 4\ell(R)$ whenever $\bar{Q} \cap \bar{R} \neq \varnothing$.

(c) $\sum_{Q \in \mathcal{W}} \chi_{2Q} \lesssim_d \chi_\Omega$.

When dealing with these cubes, we will usually refer to the the long distance

$$\mathrm{D}(Q, R) := \operatorname{diam}(Q) + \operatorname{diam}(R) + \operatorname{dist}(Q, R).$$

Lemma 8.16. *Let Ω be a UCDC domain. There exists a dimensional constant C such that for every $\xi \in \partial\Omega$ and $r < \operatorname{diam}(\partial\Omega)$, there exists a UCDC domain $\widetilde{\Omega}_{r,\xi}$ such that*

$$\Omega \cap B_{A^{-1}r}(\xi) \subset \widetilde{\Omega}_{r,\xi} \subset \Omega \cap B_{Cr}(\xi).$$

The constants of the UCDC domain are independent of ξ and r. Moreover, for $\zeta \in \partial\widetilde{\Omega}_{r,\xi} \backslash B_{\frac{r}{2}}(\xi)$, we have that $\mathrm{d}_\Omega(\zeta) \geqslant c_A r$.

Proof. Consider a Whitney covering $\mathcal{W} := \mathcal{W}(\Omega)$. Now, let $\Delta := \Delta_{A^{-1}r,\xi}$. For every $\zeta \in \Delta$ and $\rho \leqslant A^{-1}r$, there exists $Q^{\text{in}}_{\rho,\zeta} \in \mathcal{W}$ so that $Q^{\text{in}}_{\rho,\zeta} \cap B^{\text{in}}_{\rho,\zeta} \neq \varnothing$, and condition (a) ensures that

$$A^{-1}\rho \lesssim \ell(Q) \lesssim \rho. \tag{8.5}$$

Denote

$$\mathcal{F}_1 := \{Q \in \mathcal{W} : Q = Q^{\text{in}}_{\rho,\zeta} \text{ for some } \zeta \in \Delta \text{ and } \rho \leqslant A^{-1}r\}.$$

We can identify $Q \in \mathcal{F}_1$ with a pair (r_Q, ζ_Q) so that $Q = Q^{\text{in}}_{r_Q,\zeta_Q}$. Then, for $Q, R \in \mathcal{F}_1$ there exists a Harnack chain of balls $\{B^{Q,R}_j\}_{j=1}^{N_{Q,R}}$ following a non-tangential path joining $B^{\text{in}}_{r_Q,\zeta_Q}$ with $B^{\text{in}}_{r_R,\zeta_R}$ as in Definition 8.7, that is, $B^{Q,R}_j \cap B^{Q,R}_{j+1} \neq \varnothing$ with $r(B^{Q,R}_j) = \operatorname{dist}(B^{Q,R}_j, \partial\Omega)$, and

$$\sum_{j=1}^{N_{Q,R}} r(B^{Q,R}_j) \leqslant CAD(Q, R) \leqslant CA(5A^{-1}r) = Cr.$$

Note that in particular

$$B_j^{Q,R} \subset \{x \in \Omega : \operatorname{dist}(x, \Delta) \leqslant Cr\},$$

with C independent of A. By the third condition of non-tangential paths (see Lemma 8.6) we can obtain also

$$\operatorname{dist}(B_j^{Q,R}, \Delta) \overset{(8.5)}{\leqslant} \min\{\operatorname{dist}(B_j^{Q,R}, Q) + CA\ell(Q), \operatorname{dist}(B_j^{Q,R}, R) + CA\ell(R)\} \overset{\text{L.8.6,3.}}{\leqslant} C_A r(B_j^{Q,R}),$$

which improves the previous estimate when $B_j^{Q,R}$ is small.

Next we define

$$\mathcal{F}_2 := \{Q \in \mathcal{W} : Q \cap B_j^{R,S} \neq \varnothing \text{ for some } R, S \in \mathcal{F}_1 \text{ and } j \leqslant N_{R,S}\}.$$

At this point the reader may note that every pair of cubes in $\mathcal{F}_1$ can be connected by a chain of cubes in $\mathcal{F}_2$, whatever that means. However, we still need to show the existence of Harnack chains joining cubes in $\mathcal{F}_2 \backslash \mathcal{F}_1$ (but this fact cannot be granted), and to prove the inclusions of the domains. We will enlarge the family again.

To do so, note that given $Q \in \mathcal{F}_2$, there exists a couple of cubes $R_Q, S_Q \in \mathcal{F}_1$ so that $Q \cap B_j^{R_Q,S_Q} \neq \varnothing$ for some $j \leqslant N_{R_Q,S_Q}$. In particular,

$$\operatorname{dist}(Q, \Delta) \leqslant \operatorname{dist}(B_j^{R_Q,S_Q}, \Delta) + 2r(B_j^{R_Q,S_Q}) \leqslant \min\{Cr, C_A\ell(Q)\}.$$

Next we define

$$\mathcal{F}_3 := \{Q \in \mathcal{W} : \operatorname{dist}(Q, \Delta) \leqslant \min\{Cr, C_A\ell(Q)\}\}.$$

We get that $\mathcal{F}_2 \subset \mathcal{F}_3$ as discussed above, so the cubes in $\mathcal{F}_1$ can still be connected through $\mathcal{F}_3$. Now, note that if $\operatorname{dist}(Q, \xi) \leqslant r$, then

$$\operatorname{dist}(Q, \Delta) \leqslant \operatorname{dist}(Q, \xi) \leqslant r \leqslant Cr,$$

and

$$\operatorname{dist}(Q, \Delta) \geqslant \mathrm{d}_\Omega(Q) \overset{Q \in \mathcal{W}}{\geqslant} \ell(Q).$$

That is, enlarging the constants defining $\mathcal{F}_3$ if necessary, we get the inclusions

$$\Omega \cap B_{A^{-1}r}(\xi) \subset \left(\bigcup_{Q \in \mathcal{F}_3} \bar{Q}\right)^{\circ} \subset \Omega \cap B_{Cr}(\xi).$$

However, we cannot grant the existence of non-tangential paths yet.

Now, for $Q \in \mathcal{F}_3$ we claim that there exists $\Psi(Q) \in \mathcal{F}_1$ so that

$$\ell(Q) \approx_A \ell(\Psi(Q)) \approx_A \mathrm{D}(Q, \Psi(Q)). \tag{8.6}$$

Indeed, since $\operatorname{dist}(Q, \Delta) \leqslant \min\{Cr, C_A\ell(Q)\}\}$, we can take $\zeta_Q \in C_A Q \cap \Delta$. Then pick $\rho = \min\{C_A\ell(Q), A^{-1}r\}$ and define $\Psi(Q) := Q^{\text{in}}_{\rho,\zeta_Q}$, which satisfies (8.6).

Estimate (8.6) means in particular that all the balls in the chain $\left\{B_j^{Q,\Psi(Q)}\right\}$ joining Q and $\Psi(Q)$ are roughly of the same size and their number is bounded by universal constants depending only on A and d. Therefore, we define

$$\mathcal{F}_4 := \{R \in \mathcal{W} : B_j^{Q,\Psi(Q)} \cap R \neq \varnothing \text{ for some } Q \in \mathcal{F}_3,\, j \leqslant N_{Q,\Psi(Q)}\},$$

and let

$$\widetilde{\Omega} := \bigcup_{Q \in \mathcal{F}_4} 1.1Q.$$

The non-tangential paths condition is satisfied by construction: Ψ can easily be extended to $\mathcal{F}_4$ so that (8.6) is satisfied. Now, for points in neighboring Whitney cubes the path can be constructed thanks to the dilation of Whitney cubes. For points in Whitney cubes Q_1, Q_2 further away, connect each cube Q_j to $\Psi(Q_j)$ and then connect $\Psi(Q_1)$ and $\Psi(Q_2)$ by a Harnack chain of balls $B_j^{\Psi(Q_1),\Psi(Q_2)}$. Then the number of balls depends only on $\frac{\mathrm{D}(Q_1,Q_2)}{\min\{\ell(Q_1),\ell(Q_2)\}}$. Creating a non-tangential path out of this construction is an exercise left to the reader.

The fact that the CDC condition holds for $\widetilde{\Omega}$ can be checked easily: for $\xi \in \partial\widetilde{\Omega} \cap \partial\Omega$, use the fact that Ω satisfies the CDC. Otherwise, $\xi \in \partial\widetilde{\Omega} \cap \partial(1.1Q)$ for $Q \in \mathcal{F}_4$. For scales smaller than $\ell(Q)$, the CDC holds trivially (using condition (b) of the Whitney covering), while for greater scales one can use the CDC of Ω. □

Theorem 8.17 (Uniform boundary Harnack principle)**.** *Let Ω be a UCDC domain with compact boundary, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $3CAr < \operatorname{diam}(\partial\Omega)$, where C is the constant from Lemma 8.16. Let $u, v \geqslant 0$ be harmonic in Ω vanishing continuously on $2C\Delta$, and $u(X_\Delta) = v(X_\Delta)$. Then $\frac{u}{v} \approx_A 1$ on $A^{-1}B_r(\xi) \cap \Omega$.*

Proof. Consider the intermediate domain $\widetilde{\Omega} := \widetilde{\Omega}_{2r,\xi}$ from Lemma 8.16. We write $\widetilde{\Delta}_{r,\xi} := \widetilde{\Omega} \cap B_r(\xi)$, $\widetilde{\omega}$ for the harmonic measure in $\widetilde{\Omega}$ and so on.

Denote

$$L_1 := \{\zeta \in \partial\widetilde{\Omega}\backslash\partial\Omega : \operatorname{dist}(\zeta, \partial\Omega) < \frac{1}{2}A^{-3}r\}$$

and

$$L_2 := \partial\widetilde{\Omega}\backslash(L_1 \cup \partial\Omega).$$

Take a minimal covering of L_1 with surface balls $\widetilde{\Delta}_j = \widetilde{\Delta}_j(\zeta_j, (2A)^{-3}r) \subset \partial\widetilde{\Omega}$ with $j \in \{1, \ldots, N\}$. Since the covering is minimal and the balls $\widetilde{\Delta}_j$ are contained in a ball of radius Cr, the number of balls N only depends on d and A.

On the other hand, the corkscrew condition grants the existence of

$$y_1 \in B_{3CAr}(\xi) \cap \Omega\backslash B_{3Cr}(\xi) \subset \Omega\backslash\widetilde{\Omega}, \text{ such that } \mathrm{d}_\Omega(y_1) > 3Cr \text{ and}$$

and also of

$$y_2 \in B_{A^{-1}r}(\xi) \cap \Omega\backslash U_{A^{-2}r}(\partial\Omega) \subset \widetilde{\Omega}.$$

By the non-tangential path condition, there exists a point

$$\zeta_0 \in \partial\widetilde{\Omega} \cap \Omega \subset \Omega\backslash B_{2A^{-1}r}(\xi),$$

such that $d_\Omega(\zeta_0) > A^{-3}r$. Then the surface ball in $\partial\widetilde{\Omega}$ defined as $\widetilde{\Delta}_0 = \widetilde{\Delta}_{\frac{1}{2}A^{-3}r,\zeta_0} \subset L_2$.

Now, by Theorem 8.13 and the Harnack chain condition, given $x \in A^{-1}B_r(\xi) \cap \Omega$ we get

$$\widetilde{\omega}^x(\widetilde{\Delta}_j) \approx_A \left(\frac{r}{(2A)^3}\right)^{d-2} G^x(X_{\widetilde{\Delta}_j}) \approx_A \left(\frac{r}{2A^3}\right)^{d-2} G^x(X_{\widetilde{\Delta}_0}) \approx_A \widetilde{\omega}^x(\widetilde{\Delta}_0),$$

and therefore

$$\widetilde{\omega}^x(L_1) \leqslant \sum_{j=1}^{N} \widetilde{\omega}^x(\widetilde{\Delta}_j) \approx_A N\,\widetilde{\omega}^x(\widetilde{\Delta}_0) \leqslant N\widetilde{\omega}^x(L_2), \tag{8.7}$$

the constants not depending on x.

Applying Lemma 8.10 and the Harnack chain condition in Ω, assuming C_A large enough, we obtain

$$\sup_{\widetilde{\Omega}} u \leqslant \sup_{B_{Cr}(\xi)\cap\Omega} u \lesssim_A u(X_{\Delta_{Cr,\xi}}) \lesssim_A u(X_\Delta). \tag{8.8}$$

On the other hand, by Harnack inequality again $\inf_{L_2} v \gtrsim_A v(X_\Delta) = u(X_\Delta)$. All in all we get, for $x \in A^{-1}B_r(\xi) \cap \Omega$,

$$u(x) \overset{\text{Max.P.}}{\leqslant} \widetilde{\omega}^x((\partial\Omega)^c) \sup_{\widetilde{\Omega}} u \overset{(8.8)}{\lesssim_A} \widetilde{\omega}^x((\partial\Omega)^c) u(X_\Delta) \overset{(8.7)}{\lesssim_A} \widetilde{\omega}^x(L_2) \inf_{L_2} v \overset{\text{Max.P.}}{\leqslant} v(x).$$

□

The following corollary is immediate.

Corollary 8.18. *Let Ω be a UCDC domain, and let V be an open set. For any compact set $K \subset V$, there exists a constant $C = C_{V,K,A}$ such that for all positive harmonic functions u, v in Ω that vanish continuously on $\partial\Omega \cap V$, then for every $x, y \in \Omega \cap K$*

$$C^{-1}\frac{u(x)}{v(x)} \leqslant \frac{u(y)}{v(y)} \leqslant C\frac{u(x)}{v(x)}.$$

We also have:

Corollary 8.19. *Let Ω be a UCDC domain, and let $\Delta := \Delta_{r,\xi}$ with $\xi \in \partial\Omega$ and $0 < r \leqslant \operatorname{diam}(\partial\Omega)$. Let $u, v \geqslant 0$ be harmonic in Ω vanishing continuously on 2Δ. Then there exist $\alpha = \alpha(A) > 0$ and $C_A > 0$ such that*

$$\left|\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)}\right| \leqslant C_A \frac{u(X_\Delta)}{v(X_\Delta)} \left(\frac{|x-y|}{r}\right)^\alpha \qquad \textit{for every } x, y \in B_r(\xi) \cap \Omega.$$

Proof. We fix $\eta \in \Delta_{r,\xi}$ and we take $0 < s \leqslant r/4$. Then we set

$$M(s) = \sup_{y\in B_{2s}(\eta)\cap\Omega} \frac{u(y)}{v(y)}, \qquad m(s) = \inf_{y\in B_{2s}(\eta)\cap\Omega} \frac{u(y)}{v(y)}.$$

Note that

$$M(s) - \frac{u}{v} = \frac{M(s)v - u}{v}, \qquad \frac{u}{v} - m(s) = \frac{u - m(s)v}{v}$$

are quotients of non-negative harmonic functions in $B_{2s}(\eta) \cap \Omega$ which vanish in $\Delta_{2s,\eta}$. Then, by Corollary 8.18, we deduce that for all $x, y \in B_s(\eta) \cap \Omega$,

$$M(s) - \frac{u(x)}{v(x)} \leqslant C_A \left(M(s) - \frac{u(y)}{v(y)} \right).$$

Taking the infimum for $x \in B_s(\eta) \cap \Omega$ and the supremum for $y \in B_s(\eta) \cap \Omega$, we get

$$M(s) - m(s/2) \leqslant C_A \big(M(s) - M(s/2)\big),$$

or equivalently,

$$M(s/2) \leqslant \frac{C_A - 1}{C_A} M(s) + \frac{1}{C_A} m(s/2). \tag{8.9}$$

Analogously,

$$\frac{u(x)}{v(x)} - m(s) \leqslant C_A \left(\frac{u(y)}{v(y)} - m(s) \right)$$

for all $x, y \in B_s(\eta) \cap \Omega$. Thus,

$$M(s/2) - m(s) \leqslant C_A\big(m(s/2) - m(s)\big),$$

or equivalently,

$$m(s/2) \geqslant \frac{1}{C_A} M(s/2) + \frac{C_A - 1}{C_A} m(s). \tag{8.10}$$

Subtracting (8.10) from (8.9), we get

$$M(s/2) - m(s/2) \leqslant \frac{C_A - 1}{C_A} \big(M(s) - m(s)\big) + \frac{1}{C_A} \big(m(s/2) - M(s/2)\big).$$

That is,

$$M(s/2) - m(s/2) \leqslant \theta \big(M(s) - m(s)\big),$$

with $\theta := \frac{C_A - 1}{C_A + 1} < 1$. It is a routine task to check that this implies that

$$M(s) - m(s) \leqslant C\big(M(r/4) - m(r/4)\big) \left(\frac{s}{r}\right)^{\alpha} \lesssim \frac{u(X_\Delta)}{v(X_\Delta)} \left(\frac{s}{r}\right)^{\alpha}$$

for suitable $C > 0$ and $\alpha > 0$. The corollary follows immediately from this estimate. $\square$

8.3.3 The change of pole formula

Lemma 8.20. *Let Ω be a UCDC domain. Let u be harmonic and positive in Ω, with $\xi \in \partial\Omega$. If u vanishes continuously on $\partial\Omega \backslash \bar{\Delta}$ where $\Delta := \Delta_{r,\xi}$ with $r < \operatorname{diam}(\partial\Omega)$, then for all $x \in \Omega \backslash B_{2r}(\xi)$,*

$$u(x) \approx_A u(X_\Delta)\omega^x(\Delta).$$

Proof. Consider the annulus $U_r := B_{2r+\rho/2}(\xi)\backslash B_{2r-\rho/2}(\xi)$, with

$$\rho := \min\{(6CA)^{-1}\mathrm{diam}(\partial\Omega), (4C)^{-1}r\},$$

where C is the constant from Theorem 8.17. Cover $U_r \cap \partial\Omega$ with balls $B_\rho(\xi_j)$ of radius ρ, so that every $x \in B_\rho(\xi_j) \cap \Omega$ satisfies that

$$\frac{u(x)}{\omega^x(\Delta)} \overset{\text{T 8.17}}{\approx_A} \frac{u(X_{\rho,\xi_j})}{\omega^{X_{\rho,\xi_j}}(\Delta)} \overset{(8.1)}{\approx_A} \frac{u(X_\Delta)}{\omega^{X_\Delta}(\Delta)} \overset{\text{L 8.8}}{\approx_A} u(X_\Delta).$$

The estimates extend to $x \in \partial B_{2r}(\xi) \cap \Omega$ by the Harnack inequality, and the lemma follows by the maximum principle. □

Lemma 8.21 (Change of pole formula)**.** *Let $2r < \mathrm{diam}(\partial\Omega)$, $\Delta' \subset \Delta := \Delta_{r,\xi_0}$ and $x \in \Omega\backslash B_{2r}(\xi_0)$. Then*

$$\omega^{X_\Delta}(\Delta') \approx_A \frac{\omega^x(\Delta')}{\omega^x(\Delta)}.$$

In particular, $\frac{d\omega^x}{d\omega^{X_\Delta}} \approx \omega^x(\Delta)$ ω-almost everywhere.

Proof. Apply Lemma 8.20 to $u(x) = \omega^x(\Delta')$. The density estimate follows from Theorem 4.12 and the definition of density. □

Next we revisit the change of pole formula under the localization procedure. Let ξ be a boundary point, $r < \frac{1}{2}\mathrm{d}_\Omega(x_0)$. Consider the intermediate domain $\widetilde{\Omega} = \widetilde{\Omega}_{r,\xi}$ as in Lemma 8.16, $x = X_{r,\xi}$ with respect to Ω, $y \in B(x, A^{-3/2}r)\backslash B(x, A^{-2}r)$, $\Delta = \Delta_{A^{-2}r}(\xi)$. Then by Theorem 8.13 and Lemma 8.8 we get

$$G_{\widetilde{\Omega}}(y, x) \approx r^{2-d},$$

and, by Theorem 8.13 again and Harnack,

$$G_\Omega(y, x_0) \approx r^{2-d}\omega(\Delta).$$

Compare both functions on y using Theorem 8.17 to get

Claim 8.22. *For $z \in B_{A^{-2}r}(\xi) \cap \Omega$*

$$G_{\widetilde{\Omega}}(z, x) \approx \frac{G_\Omega(z, x_0)}{\omega(\Delta)}.$$

By Claim 8.22 and Theorem 8.13 we get

Claim 8.23. *For every surface ball $\Delta' \subset \Delta$, we have*

$$\omega^x_{\widetilde{\Omega}}(\Delta') \approx \frac{\omega(\Delta')}{\omega(\Delta)}$$

Finally, from Claim 8.23 and Lemma 8.21 we obtain

Theorem 8.24. *Let Ω be a UCDC domain and $x_0 \in \Omega$ a fixed point. Let ξ be a boundary point, $r < \frac{1}{2}\mathrm{d}_\Omega(x_0)$, and $\widetilde{\Omega} = \widetilde{\Omega}_{r,\xi}$ as in Lemma 8.16. For every Borel set $E \subset \Delta := \Delta_{A^{-2}r,\xi}$, we have*

$$\omega^{X_{r,\xi}}_{\widetilde{\Omega}}(E) \approx \frac{\omega^{x_0}(E)}{\omega^{x_0}(\Delta)} \approx \omega^{X_{r,\xi}}(E).$$

8.4 Estimates for the Radon-Nikodym derivative

Fix a UCDC domain Ω and a pole $x_0 \in \Omega$ and denote $\omega := \omega^{x_0}$. Then the Radon-Nikodym derivative $K(x,\xi) = \frac{d\omega^x}{d\omega}(\xi)$ equals $\lim_{r\to 0} \frac{\omega^x(\Delta_{r,\xi})}{\omega(\Delta_{r,\xi})}$ for ω-a.e. ξ (see Section 4.3).

To simplify the section, we define also $K(x,\xi) \equiv 0$ whenever $\frac{d\omega^x}{d\omega}(\xi)$ is not well defined or $\frac{d\omega^x}{d\omega}(\xi) \neq \lim_{r\to 0} \frac{\omega^x(\Delta_{r,\xi})}{\omega(\Delta_{r,\xi})}$.

Lemma 8.25. *Let $x = X_{r,\xi_0}$, $\Delta_j = \Delta_{2^j r,\xi_0}$ with $r \leqslant 2^j r \leqslant 2\mathrm{diam}(\partial\Omega)$ and $R_j = \Delta_j \backslash \Delta_{j-1}$. Then*

$$\sup_{\xi \in R_j} K(x,\xi) \leqslant \frac{C_{x_0} C 2^{-\gamma j}}{\omega(\Delta_j)},$$

with $\gamma, C > 0$ depending only on A; and C_{x_0} depending only on $\frac{\mathrm{d}_\Omega(x_0)}{\mathrm{diam}(\partial\Omega)}$ and A.

Proof. First we claim that whenever $\Delta' \subset R_j$, we get

$$\omega^x(\Delta') \leqslant C_A \omega^{X_{\Delta_j}}(\Delta') 2^{-j\alpha}.$$

Indeed, if $j \geqslant 2$, then combining Theorem 8.4, Lemma 8.10, and Harnack's inequality we get

$$\omega^x(\Delta') \overset{\text{T.8.4}}{\lesssim_A} \left(\frac{|x-\xi_0|}{2^{j-2}r}\right)^\alpha \sup_{y \in 2^{j-2}B_r(\xi_0)\cap\Omega} \omega^y(\Delta') \overset{\text{T.8.10, (8.1)}}{\lesssim_A} \left(\frac{|x-\xi_0|}{2^j r}\right)^\alpha \omega^{X_{\Delta_j}}(\Delta').$$

If, instead, $j \in \{0,1\}$, then

$$\omega^x(\Delta') \overset{(8.1)}{\approx_A} \omega^{X_{\Delta_j}}(\Delta') \approx_A \omega^{X_{\Delta_j}}(\Delta') 2^{-j\alpha},$$

and the claim is established.

To complete the proof, suppose first that $\mathrm{d}_\Omega(x_0) \geqslant r_1 := A^{-1}\mathrm{diam}(\partial\Omega)$. In this case, whenever $2^j r \leqslant 2^{-1} r_1$, we have $x_0 \notin B_{2\cdot 2^j r}(\xi_0)$ and so the change of pole formula implies

$$\omega(\Delta') \approx_A \omega^{X_{\Delta_j}}(\Delta')\, \omega(\Delta_j).$$

If, instead, $2^{-1} r_1 < 2^j r \leqslant 2\mathrm{diam}(\partial\Omega)$, then x_0 and X_{Δ_j} can be joined by a Harnack chain with a number of balls controled only by the dimension and A, so

$$\omega(\Delta') \overset{(8.1)}{\approx_A} \omega^{X_{\Delta_j}}(\Delta') \overset{\text{L.8.8}}{\approx_A} \omega^{X_{\Delta_j}}(\Delta')\, \omega(\Delta_j).$$

In any case, for $\xi \in R_j$ we get

$$K(x,\xi) \leqslant \limsup_{r\to 0} \frac{\omega^x(\Delta_{r,\xi})}{\omega(\Delta_{r,\xi})} \lesssim_A \frac{2^{-j\alpha}}{\omega(\Delta_j)}.$$

If instead, $\mathrm{d}_\Omega(x_0) < r_1$, then pick $\widetilde{x} \in \Omega$ such that $\mathrm{d}_\Omega(\widetilde{x}) \geqslant r_1$, whose existence is granted by the interior corkscrew condition. The Harnack chain condition (8.1) implies the existence of C_{x_0} depending only on $\frac{\mathrm{d}_\Omega(x_0)}{\mathrm{diam}(\partial\Omega)}$ and A such that

$$\omega^{\widetilde{x}}(\Delta) \approx C_{x_0} \omega^{x_0}(\Delta)$$

for every Δ, so applying the previous case we get

$$K(x,\xi) \leqslant \limsup_{r\to 0} \frac{\omega^x(\Delta_{r,\xi})}{\omega(\Delta_{r,\xi})} \approx C_{x_0} \limsup_{r\to 0} \frac{\omega^x(\Delta_{r,\xi})}{\omega^{\tilde{x}}(\Delta_{r,\xi})} \lesssim_A \frac{2^{-j\alpha}}{\omega(\Delta_j)}.$$

□

Lemma 8.26. *Let $r < \operatorname{diam}(\partial\Omega)$. Then*

$$\sup_{\xi\in\partial\Omega\setminus\Delta_{r,\xi_0}} K(x,\xi) \xrightarrow{x\to\xi_0} 0.$$

Proof. Note that if $\xi_x \in \partial\Omega$ is the point where $\mathrm{d}_\Omega(x) = \xi_x$, and picking $r_x = \mathrm{d}_\Omega(x)$, then $\xi \in R_j$ with $j_{x,\xi} \approx \log_2 \frac{|x-\xi|}{\mathrm{d}_\Omega(x)} \gtrsim \log_2 \frac{r}{\mathrm{d}_\Omega(x)} \xrightarrow{x\to\xi_0} \infty$. Thus, the previous lemma reads as

$$K(x,\xi) \leqslant \frac{C_{x_0} C 2^{-\gamma j_{x,\xi}}}{\omega(\Delta_{j_{x,\xi}})} \xrightarrow{x\to\xi_0} 0.$$

because $\omega(\Delta_{j_{x,\xi}})$ behaves like a constant by the doubling property. □

8.5 Global boundary behavior of harmonic functions in CDC uniform domains

A *kernel function* in Ω at $\xi \in \partial\Omega$ is a positive harmonic function u in Ω that vanishes continuously on $\partial\Omega\setminus\{\xi\}$ and such that $u(x_0) = 1$. Note that $\limsup_{x\to\xi} u(x) = \infty$. Otherwise $\{\xi\}$ would have positive harmonic measure, and this cannot happen by Theorem 6.33.

Lemma 8.27. *Let Ω be a UCDC domain. There exists a kernel function u at every boundary point.*

Proof. Let $\xi \in \partial\Omega$, and denote

$$u_m(x) = \frac{\omega^x(\Delta_{2^{-m},\xi})}{\omega(\Delta_{2^{-m},\xi})},$$

so that $u_m(x_0) = 1$.

By Harnack's inequality and Lemma 2.14 there is a partial $u_{m_j} \xrightarrow{j\to\infty} u$ uniformly on compact subsets of Ω, with u positive and harmonic in Ω.

Fix $r < \operatorname{diam}(\partial\Omega)$ and let $\Delta := \Delta_{r,\xi}$. For j big enough, we get

$$u_{m_j}(x) \overset{\text{L } 8.20}{\approx}_A u_{m_j}(X_\Delta)\omega^x(\Delta) \overset{(8.1)}{\approx}_{r,x_0,A} u_{m_j}(x_0)\omega^x(\Delta) = \omega^x(\Delta)$$

for every $x \in \Omega\setminus B_{2r}$. Therefore,

$$u(x) \approx_{r,x_0,A} \omega^x(\Delta) \quad \text{for every } x \in \Omega\setminus B_{2r}$$

and therefore u vanishes in $\partial\Omega\setminus 2\Delta$. The lemma follows letting $r \to 0$. □

Lemma 8.28. *Let Ω be a UCDC domain. Assume that u is a kernel function for Ω at ξ. Then*

$$u(x) \approx_A \frac{\omega^x(\Delta)}{\omega(\Delta)} \quad \textit{for every } x \in \Omega.$$

Proof. Let $r > 0$ be small enough and $\Delta := \Delta_{r,\xi}$. By Lemma 8.20

$$1 = u(x_0) \approx_A u(X_\Delta)\omega(\Delta).$$

and

$$u(x) \approx_A u(X_\Delta)\omega^x(\Delta).$$

Therefore

$$u(x) \approx_A \frac{\omega^x(\Delta)}{\omega(\Delta)}$$

for all $x \in \Omega \backslash B_{2r}(\xi)$ for r small enough. □

Theorem 8.29. *Let Ω be a UCDC domain. For every boundary point the kernel function is unique.*

Proof. We follow the approach of [CFMS81, Theorem 3.1]. Assume that u_1, u_2 are kernel functions for Ω at $\xi \in \partial\Omega$. Then, for $x \in \Omega$ we have $\frac{u_1(x)}{u_2(x)} \leqslant C_0 \frac{u_1(x_0)}{u_2(x_0)}$ by Lemma 8.28. Therefore

$$u_1 \leqslant C_0 u_2. \tag{8.11}$$

holds for every pair of kernel functions u_1, u_2.

If $C_0 = 1$ the lemma follows, so we may assume that $C_0 > 1$. In that case,

$$\frac{C_0}{C_0 - 1} u_2 - \frac{1}{C_0 - 1} u_1 = u_2 + \frac{1}{C_0 - 1}(u_2 - u_1)$$

is a kernel function as well by the maximum principle. Therefore (8.11) holds for this function, namely

$$u_1 \leqslant C_0 \left(u_2 + \frac{1}{C_0 - 1}(u_2 - u_1) \right)$$

so

$$\frac{C_0}{C_0 - 1} \left(u_2 + \frac{1}{C_0 - 1}(u_2 - u_1) \right) - \frac{1}{C_0 - 1} u_1 = u_2 + \frac{2}{C_0 - 1}(u_2 - u_1) + \frac{1}{(C_0 - 1)^2}(u_2 - u_1)$$

is also a kernel function.

In general, if

$$u_2 + \left(\frac{k}{C_0 - 1} + t_k \right)(u_2 - u_1) \tag{8.12}$$

is a kernel function, then (8.11) holds for this function as well, namely

$$u_1 \leqslant C_0 \left(u_2 + \left(\frac{k}{C_0 - 1} + t_k \right)(u_2 - u_1) \right),$$

so

$$\frac{C_0}{C_0-1}\left(u_2+\left(\frac{k}{C_0-1}+t_k\right)(u_2-u_1)\right)-\frac{1}{C_0-1}u_1$$
$$=u_2+\frac{k+1}{C_0-1}(u_2-u_1)+\frac{k+t_k(C_0-1)}{(C_0-1)^2}(u_2-u_1)$$

is also a kernel function. By induction, a kernel function as in (8.12) can be obtained for every $k\in\mathbb{N}$ with $t_k>0$.

Now, applying (8.11) again, we get that for every k

$$u_2+\frac{k}{C_0-1}(u_2-u_1)\leqslant u_2+\left(\frac{k}{C_0-1}+t_k\right)(u_2-u_1)\leqslant C_0u_2.$$

This implies that $u_2\leqslant u_1$. But interchanging the roles of u_1 and u_2 we obtain the converse inequality and the lemma follows.

□

Definition 8.30. A *non-tangential region* at $\xi\in\partial\Omega$ is denoted by

$$\Gamma_\alpha(\xi):=\{x\in\Omega:\ |x-\xi|<(1+\alpha)\mathrm{d}_\Omega(x)\}.$$

The *non-tangential maximal function* is denoted

$$\mathcal{N}_\alpha u(\xi):=\sup_{\Gamma_\alpha(\xi)}|u|$$

for u defined in Ω. Finally, we say that u converges to f *non-tangentially at* ξ if for any α,

$$\lim_{\Gamma_\alpha(\xi)\ni x\to\xi}u(x)=f(\xi).$$

Usually the value of α is of little importance when dealing with harmonic functions because typically the boundedness of the operator $\mathcal{N}_\alpha$ does not depend on α. Therefore we usually denote $\mathcal{N}u$ for some value of α.

Definition 8.31. The centered Hardy-Littlewood maximal function with respect to ω is defined as

$$M_\omega f(\xi):=\sup_r\fint_{\Delta_{r,\xi}}|f|\,d\omega$$

for every $f\in L^1_{\mathrm{loc}}(\omega)$, and, more generally,

$$M_\omega\mu(\xi):=\sup_r\frac{\mu(\Delta_{r,\xi})}{\omega(\Delta_{r,\xi})}$$

for every $\mu\in\mathcal{M}(\partial\Omega):=\{\text{Finite Radon measures supported in }\partial\Omega\}$.

The maximal function satisfies a weak-$(1,1)$ estimate, i.e.

$$\omega\{M_\omega f > \lambda\} \leqslant \frac{C}{\lambda}\|f\|_{L^1(\omega)}, \tag{8.13}$$

and for every $1 < p \leqslant \infty$

$$\|M_\omega f\|_{L^p(\omega)} \leqslant C\|f\|_{L^p(\omega)}, \tag{8.14}$$

see [Mat95, Theorem 2.19], for instance. In fact the weak estimate also holds for Radon measures, by the same covering arguments used to prove the weak $(1,1)$ bounds:

Lemma 8.32. *For $\mu \in \mathcal{M}(\partial\Omega)$ we have*

$$\omega\{M_\omega \mu > \lambda\} \leqslant \frac{C}{\lambda}|\mu(\partial\Omega)|. \tag{8.15}$$

Theorem 8.33. *Let Ω be a UCDC domain. If μ is a finite Borel measure on $\partial\Omega$ with Lebesgue decomposition (see Theorem 4.15) $d\mu = f d\omega + d\nu$, where ν is mutually singular with ω, and $u_\mu(x) := \int K(x,\zeta)\, d\mu(\zeta)$, then*

$$\mathcal{N} u_\mu \leqslant C_\alpha M_\omega \nu,$$

and u converges to f non-tangentially at ω-a.e. boundary point.

Proof. Consider the operator $\widetilde{N}$ defined on $\mathcal{M}(\partial\Omega)$ by

$$\widetilde{N}\mu := \mathcal{N} u_\mu,$$

where α is fixed (and the constants may depend on its value). First we claim that

$$\widetilde{N}\mu \leqslant C M_\omega \mu. \tag{8.16}$$

Indeed, let us assume that $y \in \Gamma_\alpha(\xi)$, with $\operatorname{dist}(y,\xi) \leqslant r \ll r_0$, and let $\Delta := \Delta_{r,\xi}$. By the Harnack inequality we have that

$$u_\mu(X_\Delta) = \int K(X_\Delta,\zeta)\, d\mu(\zeta).$$

Decomposing as in Lemma 8.25 we get

$$u_\mu(y) = \sum_j \int_{R_j} K(y,\zeta)\, d\mu(\zeta) \overset{\text{L 8.25}}{\lesssim_A} \sum_j \frac{2^{-\gamma_A j}}{\omega(\Delta_j)} \int_{R_j} d\mu(\zeta) \leqslant M_\omega \mu(\xi) \sum_j 2^{-\gamma_A j} \lesssim_A M_\omega \mu(\xi).$$

Since $\widetilde{N}\mu(\xi) = \sup_{y \in \Gamma_\alpha(\xi)} |u_\mu(y)|$, estimate (8.16) follows.

Note that combining (8.15) with (8.16) we obtain the weak type estimate

$$\omega\{\widetilde{N}\mu > \lambda\} \leqslant \frac{C}{\lambda}|\mu(\partial\Omega)|. \tag{8.17}$$

It remains to compute the nontangential limit of u_μ, proving that it coincides with f at ω-a.e. boundary point. Let us write $\text{n.t.}\limsup_{y\to\xi} := \limsup_{\Gamma_\alpha(\xi)\ni y\to\xi}$. Given $\varepsilon, \lambda > 0$, we want to prove that

$$\boxed{\boxed{0_{\mu,\lambda}}} := \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_\mu(y) - f(\xi)| > \lambda\right\} < \varepsilon. \tag{8.18}$$

First we will compute the case $\nu = 0$. Whenever $f \in C(\partial\Omega)$, we have that

$$u_f(x) = \int f(\zeta) K(x,\zeta)\, d\omega(\zeta) \overset{\text{E.4.3.2}}{=} \int f(\zeta)\, d\omega^x(\zeta) = Hf(x),$$

so

$$u_f(x) \to f(\xi) \text{ as } x \to \xi \in \partial\Omega \tag{8.19}$$

by Wiener regularity.

For $f \in L^1(\partial\Omega)$, consider simple functions $\{f_n\}_n$ converging in $L^1(\omega)$ to f. Since ω is a Radon measure, we can find continuous functions $\{f_{n,j}\}_j$ converging to f_n in $L^1(\omega)$. By a diagonal argument, we find a sequence of continuous functions $\{g_n\}_n$ converging in $L^1(\omega)$ to f.

Using the triangle inequality, we can decompose the left-hand side of (8.18) as

$$\begin{aligned}\boxed{\boxed{0_{f\omega,\lambda}}} \leqslant\ & \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_f(y) - u_{g_n}(y)| > \frac{\lambda}{3}\right\} \\ & + \omega\left\{\text{n.t.}\limsup_{y\to\xi} |u_{g_n}(y) - g_n(\xi)| > \frac{\lambda}{3}\right\} \\ & + \omega\left\{|g_n(\xi) - f(\xi)| > \frac{\lambda}{3}\right\} = \boxed{\boxed{1}} + \boxed{\boxed{2}} + \boxed{\boxed{3}}.\end{aligned}$$

By (8.13),

$$\boxed{\boxed{3}} \leqslant \frac{C}{\lambda}\|f - g_n\|_{L^1(\omega)}.$$

The continuity of g_n implies that $u_{g_n} = H_{g_n}$. By (8.19) Since Ω is Wiener regular, we get that

$$\boxed{\boxed{2}} = 0.$$

Finally,

$$\boxed{\boxed{1}} \leqslant \omega\left\{\widetilde{N}(f - g_n)(\xi) > \frac{\lambda}{3}\right\} \overset{(8.17)}{\leqslant} \frac{C}{\lambda}\|f - g_n\|_{L^1(\omega)}.$$

Combining the three estimates, we obtain

$$\omega\left\{\left|\text{n.t.}\limsup_{y\to\xi} u_f(y) - f(\xi)\right| > \lambda\right\} \leqslant \frac{C}{\lambda}\|f - g_n\|_{L^1(\omega)} < \varepsilon$$

for n big enough (depending on λ and f), so (8.18) is settled whenever $\nu = 0$. In particular,

$$\boxed{\boxed{0_{f\omega,\lambda}}} = 0.$$

If $\nu \neq 0$, we write

$$\boxed{0_{\mu,\lambda}} \leqslant \omega\left\{\text{n.t.}\limsup_{y\to\xi}|u_{f\omega}(y)-f(\xi)|>\lambda/2\right\}+\omega\left\{\text{n.t.}\limsup_{y\to\xi}|u_\nu(y)-0|>\lambda/2\right\}$$
$$=\omega\left\{\text{n.t.}\limsup_{y\to\xi}|u_\nu(y)-0|>\lambda/2\right\}.$$

Let $E\subset\partial\Omega$ be an ω-measurable set given by the Radon-Nikodym decomposition, i.e. so that $\omega(E)=0=\nu(\partial\Omega\backslash E)$. Since ν,ω are Radon measures, we can find a compact set $K\subset E$ and an open set $U\supset E$ so that $\nu(E\backslash K)<\delta$ and $\omega(U)<\delta$.

Now,

$$\boxed{0_{\mu,\lambda}} \leqslant \omega\left\{\text{n.t.}\limsup_{y\to\xi}\left|u_{\nu|_{E\backslash K}}(y)\right|>\lambda/4\right\}+\omega\left\{\text{n.t.}\limsup_{y\to\xi}\left|u_{\nu|_K}(y)\right|>\lambda/4\right\}$$
$$=\boxed{4}+\boxed{5}.$$

The weak estimate (8.17) implies that

$$\boxed{4}\leqslant\omega\left\{\left|\tilde{N}\nu_{E\backslash K}(\xi)\right|>\lambda/4\right\}\leqslant\frac{C}{\lambda}\nu(E\backslash K)\leqslant\frac{C}{\lambda}\delta.$$

Note also that

$$\boxed{5}\leqslant\omega(U)+\omega\left\{\xi\in U^c:\text{n.t.}\limsup_{y\to\xi}\left|u_{\nu|_K}(y)\right|>\lambda/4\right\}.$$

Let $r:=\operatorname{dist}(K,U^c)>0$. Now, for every $\xi\in U^c$, $y\in\Gamma_\alpha(\xi)$ we have that

$$u_{\nu|_K}(y):=\int_K K(y,\zeta)\,d\nu(\zeta)\leqslant\nu(K)\sup_{\zeta\in\partial\Omega\backslash\Delta_{r,\xi}}K(y,\zeta)\xrightarrow[y\to\xi]{\text{L }8.26}0,$$

so

$$\omega\left\{\xi\in U^c:\limsup_{y\to\xi}\left|u_{\nu|_K}(y)\right|>\lambda/4\right\}=0.$$

Combining all the estimates, we get

$$\boxed{0_{\mu,\lambda}}\leqslant\frac{C}{\lambda}\delta+\delta<\varepsilon$$

as long as we take δ small enough. □

Remark 8.34. Note that $f\in L^1(\omega^x)$ if and only if $f\in L^1(\omega)$ by Exercise 4.3.2 and Lemma 8.25. Moreover, by the previous theorem we can say that

$$u_f(x):=u_{f\omega}(x)=\int f(\xi)\,\frac{d\omega^x}{d\omega}(\xi)\,d\omega(\xi)\overset{\text{E.}4.3.2}{=}\int f(\xi)\,d\omega^x(\xi),$$

is the harmonic extension of $f\in L^1(\omega)$ in the ω-a.e. non-tangential sense, i.e.,

$$\text{n.t.}-\lim_{x\to\xi}u_f(x)=f(\xi)\qquad\text{for }\omega\text{-a.e. }x\in\partial\Omega.$$

Note that u_f coincides with its Perron extension H_f when f is continuous.

9 Harmonic measure in the complex plane

9.1 Introduction

In this chapter we will study some fundamental results regarding harmonic measure in the complex plane. We refer the interested reader to the book [GM05]. We will use the symbol $\mathbb{D}$ to refer to the unit disc (or ball) $B_1(0)$ in the complex plane.

Let us begin by citing some key results which we are going to use during this chapter. First, we say that a homeomorphism $\varphi : \Omega \to \Omega'$ with $\Omega, \Omega' \subset \mathbb{C}$ is conformal whenever $\bar{\partial}\varphi = 0$. Planar simply connected domains, i.e., domains $\Omega \subset \hat{\mathbb{C}} := \mathbb{C} \cup \infty$ such that Ω^c is connected, are conformally equivalent to the disc, as a consequence of the Riemann mapping theorem, see [Con78, Chapter VII] for a proof.

Theorem 9.1 (Riemann mapping Theorem). *Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and let $x \in \Omega$, $0 \leqslant \alpha < 2\pi$. Then there is a unique conformal map $\varphi : \mathbb{D} \to \Omega$ such that $\varphi(0) = x$ and $\arg(\varphi'(0)) = \alpha$.*

The mappings φ defined by the previous theorem are usually referred to as *Riemann mappings*. Note that changing the point x and the angle α in the theorem corresponds to precomposing φ with a Möbius transform. In this sense, once we obtain a Riemann mapping of a given domain, we can easily compute every single Riemann mapping of the domain.

The regularity properties of the boundary of a domain are related to the boundary behavior of their conformal mappings, see [Pom92, Theorem 2.6] for a detailed account.

Definition 9.2. We say that a set Γ is a *curve* whenever there exists a continuous parameterization $\gamma : \partial\mathbb{D} \to \Gamma$ (possibly with infinitely many self-intersections).

We say that a set E is locally connected if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every two points $x, y \in E$ with $|x - y| < \delta$ there exists a connected subset $F \subset E$ containing both points and such that $\operatorname{diam} F < \varepsilon$.

The following result is a combination of the continuity theorem and the prime ends theorem (see [Pom92, Theorem 2.1, Corollary 2.19])

Theorem 9.3 (Continuity theorem). *Let Ω be a simply connected domain and let $\varphi : \mathbb{D} \to \Omega$ be a Riemann mapping. Then the following are equivalent:*

- *The boundary $\partial\Omega$ is a curve.*
- *The boundary $\partial\Omega$ is locally connected.*
- *The function φ has a continuous extension to $\bar{\mathbb{D}}$.*

In particular, $\varphi : \partial\mathbb{D} \to \partial\Omega$ is a continuous parameterization of the curve and only a countable number of points in $\partial\Omega$ have more than two preimages.

Carathéodory Theorem (see [Pom92, Theorem 2.6]) is a natural counterpart to the continuity theorem above regarding homeomorphic mappings. We say that a set $\Gamma \subset \mathbb{C}$ is a Jordan curve if there exists a continuous, injective parameterization $\gamma : \partial\mathbb{D} \to \Gamma$, and we say that a domain Ω is a *Jordan domain* whenever $\partial\Omega$ is a Jordan curve (and therefore, $\partial\Omega$ is bounded). Note that given a Jordan curve Γ, there are two Jordan domains which have Γ as boundary, one of them is bounded, and the other one unbounded.

Theorem 9.4 (Carathéodory Theorem)**.** *Let $\varphi : \mathbb{D} \to \Omega$ be a Riemann mapping of a simply connected domain $\Omega \subset \mathbb{C}$. Then φ has a continuous and injective extension to $\overline{\mathbb{D}}$ if and only if Ω is a Jordan domain.*

As a matter of fact, the previous result can be easily extended to the case of unbounded Jordan domains, but we will omit these technicalities.

Another way to measure the regularity of the Riemann mapping is to find out to which function spaces it belongs. The smoother the domain is, the more regular the Riemann mapping will be. Next we define the Hardy spaces of analytic functions H^p, although in this chapter we will only consider H^1.

Definition 9.5. If $0 < p < \infty$, we say that an analytic function $f : \mathbb{D} \to \mathbb{C}$ is in the *Hardy space* H^p whenever

$$\|f\|_{H^p} := \sup_{0<r<1} \left(\int_{\partial\mathbb{D}} |f(r\zeta)|^p \, |d\zeta| \right)^{\frac{1}{p}} < \infty. \tag{9.1}$$

If $p = \infty$, then $f \in H^\infty$ whenever

$$\|f\|_{H^\infty} := \sup_{\mathbb{D}} |f| < \infty.$$

One can show that the term in the supremum in (9.1) is increasing in r and, therefore, it can be replaced by $\lim_{r \nearrow 1}$.

In virtue of Theorem 3.11, if $1 < p < \infty$, we get that whenever f extends non-tangentially to the boundary as an L^p function, then

$$\lim_{r\to 1} \int_{\partial\mathbb{D}} |f(r\zeta) - f(\zeta)|^p \, |d\zeta| = 0. \tag{9.2}$$

In fact, for every finite p and every $f \in H^p$ one can define the non-tangential limit $f(\zeta)$ almost everywhere, and identity (9.2) happens to be true, see [GM05, Appendix A]. Since we are only interested in H^1 we will refer only to this case:

Theorem 9.6. *Let $f \in H^1$. Then f has non-tangential limit $f(\zeta)$ $\mathcal{H}^1$-almost everywhere in $\partial\mathbb{D}$ and (9.2) is satisfied with $p = 1$. In particular,*

$$\|f\|_{H^1} = \|f\|_{L^1(\mathcal{H}^1|_{\partial\mathbb{D}})}.$$

If, moreover, $f \not\equiv 0$, then $f(\zeta) \neq 0$ $\mathcal{H}^1$-almost everywhere in $\partial\mathbb{D}$.

For the notion of *non-tangential limit*, see Definition 8.30.

9.2 Harmonic measure and conformal mappings

One of the basic facts that makes the study of harmonic measure in the plane different from higher dimensions is the availability of many conformal mappings in the plane and the good behavior of harmonic measure under those mappings. We will take advantage of this fact expressing the harmonic measure of a simply connected domain as the image measure of the arc-length by a Riemann mapping.

Recall that given a continuous map $\varphi : G \to G'$ and a Borel measure μ on G, then the image measure $\varphi_{\#}\mu$ is a measure on G' defined by

$$\varphi_{\#}\mu(A) = \mu(\varphi^{-1}(A))$$

for any Borel set $A \subset G'$. Then, for any Borel function $f : G' \to \mathbb{R}$, it holds

$$\int f \circ \varphi \, d\mu = \int f \, d\varphi_{\#}\mu.$$

See Chapter 4.

Proposition 9.7. *Let $\Omega, \Omega' \subset \mathbb{C}$ be bounded Wiener regular domains, and let $\varphi : \overline{\Omega} \to \overline{\Omega'}$ be a continuous surjective map such that $\varphi(\partial\Omega) = \partial\Omega'$. Suppose also that φ is holomorphic in Ω, and let $x \in \Omega$ and $x' = \varphi(x)$. Denote by ω_Ω and $\omega_{\Omega'}$ the respective harmonic measures for Ω and Ω'. Then,*

$$\omega_{\Omega'}^{x'} = \varphi_{\#}\omega_\Omega^x.$$

In particular, for any Borel set $A \subset \partial\Omega'$, we have $\omega_{\Omega'}^{x'}(A) = \omega_\Omega^x(\varphi^{-1}(A))$.

Proof. Let $f : \partial\Omega' \to \mathbb{R}$ be an arbitrary continuous function and let $u_{\Omega',f}$ be its harmonic extension to Ω'. Then $u_{\Omega',f} \circ \varphi$ is continuous in $\overline{\Omega}$, harmonic in Ω, and it coincides with the harmonic extension of $f \circ \varphi : \Omega \to \mathbb{R}$, i.e., $u_{\Omega',f} \circ \varphi = u_{\Omega, f\circ\varphi}$. Therefore,

$$\int f \, d\omega_{\Omega'}^{x'} = u_{\Omega',f}(x') = u_{\Omega',f}(\varphi(x)) = u_{\Omega,f\circ\varphi}(x) = \int f \circ \varphi \, d\omega_\Omega^x = \int f \, d\varphi_{\#}\omega_\Omega^x.$$

Since this holds for any continuous function f on $\partial\Omega'$, the proposition follows. □

Corollary 9.8. *Let $\Omega \subset \mathbb{C}$ be bounded and simply connected. Let $\varphi : \mathbb{D} \to \Omega$ be a conformal mapping which extends to a continuous map $\bar{\mathbb{D}} \to \overline{\Omega}$. Then*

$$\omega_\Omega^{\varphi(0)} = \frac{1}{2\pi} \varphi_{\#}\mathcal{H}^1|_{\partial\mathbb{D}}.$$

That is, for any Borel set $F \subset \partial\Omega$, and $E = \varphi^{-1}(F)$, we have

$$\omega_\Omega^{\varphi(0)}(F) = \frac{\mathcal{H}^1(E)}{2\pi}.$$

Proof. By topological arguments, $\varphi(\partial\mathbb{D}) = \partial\Omega$. By Proposition 9.7, we deduce that

$$\omega_\Omega^{\varphi(0)} = \varphi_\#\omega_{\mathbb{D}}^0 = \frac{1}{2\pi}\,\varphi_\#\mathcal{H}^1|_{\partial\mathbb{D}}.$$

□

Remark that, by the continuity theorem, if Ω is a simply connected domain with locally connected boundary (and in particular if it is a bounded Jordan domain), then the conformal mapping $\varphi : \mathbb{D} \to \Omega$ extends continuously to $\partial\mathbb{D}$, and thus the preceding corollary applies. Notice also that whenever we know how to find a conformal map $\varphi : \mathbb{D} \to \Omega$, we know how to find the harmonic measure ω_Ω.

9.3 The Riesz brothers theorem

In this section we will prove the following result:

Theorem 9.9 (F. and M. Riesz Theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain whose boundary has finite length, and let $\varphi : \mathbb{D} \to \Omega$ be a Riemann mapping for Ω. Then, $\varphi' \in H^1(\mathbb{D})$ and*

$$\omega(A) = 0 \iff \mathcal{H}^1(A) = 0.$$

The reader can find an elegant proof of this result in [GM05, Chapter VI], which covers the case of Jordan domains. In these notes we use the same approach, adding some technicalities to include every simply connected domain whose boundary has finite length. Notice that the result does not depend on the precise pole for harmonic measure, since harmonic measures for different poles (and the same domain) are mutually absolutely continuous, see Lemma 5.29.

We begin by proving the Riesz brothers theorem for simply connected domains with locally connected boundary. Later on, in Theorem 9.14 we will prove that having finite length implies being locally connected for the boundary of a simply connected domain.

Theorem 9.10. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with locally connected boundary and let $\varphi : \mathbb{D} \to \Omega$ be conformal. Then $\partial\Omega$ has finite length if and only if $\varphi' \in H^1(\mathbb{D})$. If $\varphi' \in H^1(\mathbb{D})$, then*

$$\mathcal{H}^1(\partial\Omega) \leqslant \|\varphi'\|_{H^1(\mathbb{D})} \leqslant 2\mathcal{H}^1(\partial\Omega). \tag{9.3}$$

More precisely, for every Borel set $E \subset \partial\Omega$ we get

$$\mathcal{H}^1(E) \leqslant \varphi_\#\nu(E) \leqslant 2\mathcal{H}^1(E), \tag{9.4}$$

where ν is the Radon measure defined by

$$\nu(A) := \frac{1}{2\pi}\int_A |\varphi'|\,d\mathcal{H}^1, \tag{9.5}$$

for any Borel set $A \subset \partial\mathbb{D}$.

Proof. By the continuity theorem $\varphi : \partial\mathbb{D} \to \partial\Omega$ is a continuous parameterization of the curve and only a countable number of points in $\partial\Omega$ have more that two preimages.

Assume that $\varphi' \in H^1$. Then given a partition $0 = \theta_0 < \theta_1 < \cdots < \theta_n = 2\pi$, writing $\zeta_j := e^{i\theta_j}$, by the fundamental theorem of calculus we get

$$\sum_{j=1}^n |\varphi(\zeta_j) - \varphi(\zeta_{j-1})| \overset{\text{T.9.3}}{=} \lim_{r\to 1} \sum_{j=1}^n |\varphi(r\zeta_j) - \varphi(r\zeta_{j-1})| \overset{\text{FTC}}{=} \lim_{r\to 1} \sum_{j=1}^n \left| \int_{r\zeta_{j-1}}^{r\zeta_j} \varphi'(z)\, dz \right| \overset{\text{D.9.5}}{\leqslant} \|\varphi'\|_{H^1}.$$

Thus, the length of the parameterization (i.e., counting multiplicities) defined by

$$\ell(\varphi) := \sup_{\{\zeta_j\}_{j=1}^n} \sum_{j=1}^n |\varphi(\zeta_j) - \varphi(\zeta_{j-1})| \tag{9.6}$$

where the supremum is taken with respect to all the possible partitions, is bounded by $\ell(\varphi) \leqslant \|\varphi'\|_{H^1}$. Thus,

$$\mathcal{H}^1(\partial\Omega) \leqslant \ell(\varphi) \leqslant \|\varphi'\|_{H^1}, \tag{9.7}$$

and the boundary has finite length, see Exercise 9.3.1 for the details.

Conversely, let us assume that $\partial\Omega$ has finite length. First we claim that

$$\ell(\varphi) \leqslant 2\mathcal{H}^1(\partial\Omega).$$

Indeed, consider a partition $0 = \theta_0 < \theta_1 < \cdots < \theta_n = 2\pi$, and take $\zeta_j := e^{i\theta_j}$ and $F_j := \varphi \circ e^{i\cdot}([\theta_j, \theta_{j-1}))$, which is a Borel set. Writing $F := \{\zeta \in \partial\Omega : \#\varphi^{-1}(\zeta) \leqslant 2\}$, since $\partial\Omega \backslash F$ is countable (see Theorem 9.3 above), all these sets are Borel, so $\mathcal{H}^1(F_j) = \mathcal{H}^1(F \cap F_j)$. Since $\sum_{j=1}^n \chi_{F_j \cap F} \leqslant 2$, we get

$$\sum_{j=1}^n |\varphi(\zeta_j) - \varphi(\zeta_{j-1})| \leqslant \sum_{j=1}^n \mathcal{H}^1(F_j) = \sum_{j=1}^n \mathcal{H}^1(F_j \cap F) = \sum_{j=1}^n \int_F \chi_{F_j \cap F} d\mathcal{H}^1 \leqslant 2\mathcal{H}^1(F),$$

and the claim follows taking supremum on all the possible partitions, because $\mathcal{H}^1(F) = \mathcal{H}^1(\partial\Omega)$.

Given $r \in (0,1)$, let us choose a partition $0 = \theta_0 < \theta_1 < \cdots < \theta_n = 2\pi$, and let $\zeta_j := e^{i\theta_j}$, satisfying that

$$\sum_{j=1}^n |\varphi(r\zeta_j) - \varphi(r\zeta_{j-1})| \geqslant \ell(\varphi_r) - \varepsilon,$$

with $\ell(\varphi_r)$ defined as in (9.6), where $\varphi_r(\zeta) := \varphi(r\zeta)$ is a (rectifiable Jordan) curve. The function

$$\Psi(z) := \sum_{j=1}^n |\varphi(z\zeta_j) - \varphi(z\zeta_{j-1})|$$

is subharmonic on $\mathbb{D}$ and by the continuity theorem Ψ is continuous on $\bar{\mathbb{D}}$, so the maximum principle applies:

$$\sup_{\mathbb{D}} \Psi \overset{\text{L.5.4}}{=} \sup_{\partial\mathbb{D}} \Psi \leqslant \ell(\varphi) \leqslant 2\mathcal{H}^1(\partial\Omega).$$

Now,

$$\int_{\partial\mathbb{D}} \left|\varphi'(rz)\right| |dz| = \ell(\varphi_r) \leqslant \Psi(r) + \varepsilon \leqslant 2\mathcal{H}^1(\partial\Omega) + \varepsilon.$$

Thus, $\varphi' \in H^1$, and we get the estimate

$$\left\|\varphi'\right\|_{H^1} \leqslant 2\mathcal{H}^1(\partial\Omega).$$

Applying estimate (9.7), we obtain (9.3).

Next we turn our attention to the proof of (9.4). Let us assume that $\varphi' \in H^1(\mathbb{D})$. We can extend it non-tangentially to the boundary as an L^1 function via Theorem 9.6, so ν is a well defined Radon measure. First we show that

$$\mathcal{H}^1(\varphi(U)) \leqslant \nu(U) \leqslant 2\mathcal{H}^1(\varphi(U)) \tag{9.8}$$

for every relative open set $U \subset \partial\mathbb{D}$. It suffices to show this identity assuming that $U = J$ is an open arc $J = e^{i\cdot}(I)$, where I is an open interval $I = (a, b)$. Let $a = \theta_0 < \theta_1 < \cdots < \theta_n = b$ be a partition of I, and let $\zeta_j := e^{i\theta_j}$. Then, arguing as before we get

$$\mathcal{H}^1(\varphi(J)) \leqslant \ell(\varphi|_J) \leqslant \lim_{r\to 1^-} \int_J |\varphi'(r\zeta)||d\zeta| \overset{\text{T.9.6}}{=} \nu(J).$$

On the other hand, assuming the partition satisfies

$$\sum_{j=1}^{n} |\varphi(r\zeta_j) - \varphi(r\zeta_{j-1})| \geqslant \ell(\varphi_r|_J) - \varepsilon,$$

and defining Ψ as before, we obtain

$$\sup_{\mathbb{D}} \Psi = \sup_{\partial\mathbb{D}} \Psi \leqslant \ell(\varphi|_J) \leqslant 2\mathcal{H}^1(\varphi(J)),$$

so

$$\int_J |\varphi'(r\zeta)||d\zeta| = \ell(\varphi_r|_J) \leqslant \Psi(r) + \varepsilon \leqslant 2\mathcal{H}^1(\varphi(J)) + \varepsilon.$$

But, again by Theorem 9.6, we get

$$\nu(J) = \sup_{0<r<1} \int_J |\varphi'(r\zeta)||d\zeta| \leqslant 2\mathcal{H}^1(\varphi(J)) + \varepsilon,$$

and (9.8) follows.

Now, given a Borel set $E \subset \partial\Omega$, by the Borel regularity of ν we have

$$\varphi_{\#}\nu(E) = \inf_{U\supset\varphi^{-1}(E)} \nu(U) \overset{(9.8)}{\geqslant} \inf_{U\supset\varphi^{-1}(E)} \mathcal{H}^1(\varphi(U)) \geqslant \mathcal{H}^1(E),$$

establishing the left-hand side of (9.4). Regarding the right-hand side, we have

$$2\mathcal{H}^1(E) = \inf_{V\supset E} 2\mathcal{H}^1(V) = \inf_{V\supset E} 2\mathcal{H}^1(\varphi(\varphi^{-1}(V))).$$

Since φ is continuous by Theorem 9.3, we infer that $\varphi^{-1}(V)$ is an open set whenever V is open, so we get

$$2\mathcal{H}^1(E) \overset{(9.8)}{\geqslant} \inf_{V \supset E} \nu(\varphi^{-1}(V)) \geqslant \nu(\varphi^{-1}(E)).$$

□

Corollary 9.11. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with locally connected boundary. Then the harmonic measure and the arc-length measure in $\partial\Omega$ are mutually absolutely continuous.*

Proof. By Corollary 9.8, the harmonic measure is the pushforward of the arc-length measure, so

$$\frac{1}{2\pi}\mathcal{H}^1(\varphi^{-1}(E)) = \omega(E)$$

for every Borel set $E \subset \partial\Omega$.

The preceding theorem implies the comparability of the length in $\partial\Omega$ with respect to the pushforward of ν for Borel sets $E \subset \partial\Omega$:

$$\mathcal{H}^1(E) \leqslant \nu(\varphi^{-1}(E)) \leqslant 2\mathcal{H}^1(E),$$

where ν stands for the measure defined in (9.5). In particular, the length in $\partial\Omega$ and the pushforward of ν are mutually absolutely continuous, i.e.,

$$\mathcal{H}^1(E) = 0 \iff \nu(\varphi^{-1}(E)) = 0.$$

On the other hand, the arc-length in $\partial\mathbb{D}$ and ν are mutually absolutely continuous as well, i.e.,

$$\nu(A) = 0 \iff \mathcal{H}^1(A) = 0.$$

Indeed, by the inner regularity of Radon measures (see Proposition 4.5), we may assume that $A = K \subset \partial\mathbb{D}$ is compact. If $\mathcal{H}^1(K) = 0$, then $\nu(K) = 0$ by definition. On the other hand, if $\nu(K) = 0$, then since $\varphi' \neq 0$ a.e. (see Theorem 9.6), we get $\mathcal{H}^1(K) = 0$ as well.

Their image measures are also mutually absolutely continuous, i.e. for every Borel set $E \subset \partial\Omega$ we get

$$\nu(\varphi^{-1}(E)) = 0 \iff \omega(E) = \mathcal{H}^1(\varphi^{-1}(E)) = 0.$$

All in all, the harmonic measure and the arc-length measure in $\partial\Omega$ are mutually absolutely continuous as claimed. □

We need the following auxiliary result:

Theorem 9.12. *Let $E \subset \mathbb{R}^d$ be a compact connected set such that $\mathcal{H}^1(E) < \infty$. Then E is arc-connected.*

Proof. See Lemma 3.12 from [Fal85]. □

Theorem 9.13. *Let $E \subset \mathbb{R}^d$ be a compact connected set such that $\mathcal{H}^1(E) < \infty$. Then E is locally connected.*

Proof. We assume that E is not a single point. It suffices to check that for every $\xi \in E$ and $0 < r \leqslant \mathrm{diam}(E)/3$, there exists a connected set $F \subset E \cap \bar{B}_r(\xi)$ which is a neighborhood of ξ in the topology of E. To this end, denote by $\{\Gamma_i\}_{i \in I}$ the family of connected components of $E \cap \bar{B}_r(\xi)$. We claim that each component Γ_i, $i \in I$, intersects $\partial B_r(\xi)$. Indeed, by Theorem 9.12 E is arc-connected and, since $E \not\subset \bar{B}_r(\xi)$, there is an arc contained in E that joins Γ_i to some point $\xi' \in E \backslash \bar{B}_r(\xi)$. From this fact and the maximality of Γ_i, our claim follows easily.

Let $\{\Gamma_i\}_{i \in I_0}$, with $I_0 \subset I$, be the subfamily of connected components of $E \cap \bar{B}_r(\xi)$ which intersect $\bar{B}_{r/2}(\xi)$. Since each Γ_i, $i \in I_0$, intersects both $\partial B_r(\xi)$ and $\bar{B}_{r/2}(\xi)$, it holds that

$$\mathcal{H}^1(\Gamma_i) \geqslant r/2 \quad \text{for each } i \in I_0,$$

see [Fal85, Lemma 3.4]. Then, from the fact that $\mathcal{H}^1(E) < \infty$ and the disjointness of the components Γ_i, it turns out that I_0 is a finite set. That is, there are finitely many components Γ_i, $i \in I_0$.

Let $F = \Gamma_{k_0}$ be the component Γ_i, $i \in I_0$, that contains ξ. To see that F is a neighborhood of ξ in E, let

$$\delta = \min_{i \in I_0 \backslash \{k_0\}} \mathrm{dist}(\xi, \Gamma_i).$$

Notice that $\delta > 0$ because I_0 is finite. Next, let $\delta' = \frac{1}{2} \min(\delta, r/4)$. Then we have $E \cap B(\xi, \delta') \subset F$ and thus F is a neighborhood of ξ in E. $\square$

Corollary 9.14. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain whose boundary has finite length. Then its boundary is locally connected.*

Proof. Since Ω is simply connected its boundary is connected, and then we can apply Theorem 9.13. $\square$

Exercise 9.3.1. Show the first estimate in (9.7), that is, $\mathcal{H}^1(\partial\Omega) \leqslant \ell(\varphi)$.

9.4 The dimension of harmonic measure in the plane

The dimension of a Borel measure μ in $\mathbb{R}^d$ is defined as follows:

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) : E \subset \mathbb{R}^d \text{ Borel }, \mu(E^c) = 0\}.$$

This does not have to be confused with the dimension of suppμ. For example, let $\mathbb{Q} = \{q_k\}_{k \geqslant 1}$ be the set of all rational numbers, ordered in some way. Then consider the following measure in $\mathbb{R}$:

$$\mu = \sum_{k \geqslant 1} 2^{-k} \delta_{q_k},$$

where δ_{q_k} is the Dirac delta on q_k. It is immediate to check that $\dim_{\mathcal{H}} \mu = 0$, while $\mathrm{supp}\mu = \mathbb{R}$ and so $\dim_{\mathcal{H}}(\mathrm{supp}\mu) = 1$.

For simply connected domains Makarov [Mak85] proved in 1985 the following:

Theorem 9.15. *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then* $\dim_{\mathcal{H}} \omega = 1$. *Further,* $\omega(E) = 0$ *for any set* $E \subset \partial\Omega$ *with Hausdorff dimension* $\dim_{\mathcal{H}}(E) < 1$.

Remark that the dimension of harmonic measure is independent of the chosen pole in the domain. For arbitrary planar domains, Jones and Wolff proved the following result in 1988 [JW88]:

Theorem 9.16. *For any open set $\Omega \subset \mathbb{C}$, the associated harmonic measure satisfies*

$$\dim_{\mathcal{H}}(\omega) \leqslant 1.$$

Observe that the boundary of a planar domain may have Hausdorff dimension larger than 1. This is the case, for example, of the Jordan domain enclosed by the von Koch snowflake. It is well known that this curve has dimension $\log 4/\log 3$. Further, it is easy to check that, because of connectedness, the (closed) support of harmonic measure coincides with the full boundary for any domain Ω. In spite of this fact, the dimension of harmonic measure is always at most 1. So there is a set $E \subset \partial\Omega$ with $\dim_{\mathcal{H}} E \leqslant 1$ with full harmonic measure. By Corollary 5.36, such set E must be dense in $\partial\Omega$ whenever Ω is Wiener regular.

The Jones-Woff theorem was sharpened by Wolff [Wol93] a few years later:

Theorem 9.17. *For any open set $\Omega \subset \mathbb{C}$, there exists a set $E \subset \partial\Omega$ with σ-finite length and full harmonic measure.*

The rest of this chapter is devoted to the proof of the Jones-Wolff Theorem 9.16. We will not prove the other theorems by Makarov and Wolff mentioned above.

9.5 Preliminary reductions for the proof of the Jones–Wolff Theorem

We will prove Theorem 9.16 assuming $\partial\Omega$ to be bounded, since we have defined harmonic measure in this case. The case where $\partial\Omega$ is unbounded easily follows from the bounded case (once harmonic measure is properly defined). We will show first below that we may assume that Ω is Wiener regular.

Lemma 9.18. *To prove Theorem 9.16, it suffices to prove it when Ω is Wiener regular.*

Proof. We may assume that $\mathrm{Cap}_L(\partial\Omega) > 0$ because otherwise $\dim_{\mathcal{H}}(\omega) \leqslant \dim_{\mathcal{H}}(\partial\Omega) = 0$. For each $\varepsilon = 1/k$, let $\widetilde{\Omega}_k$ be the Wiener regular open set constructed in Proposition 6.37 (denoted by $\widetilde{\Omega}$ there). Also, denote by F_k the union of the closed balls $\bar{B}_i$, $i \in I$, in the construction of $\widetilde{\Omega}_k$. For a given $p \in \Omega$, let $k \geqslant k_0$ be small enough so that $p \in \widetilde{\Omega}_k$ and $d_\Omega(p) \approx d_{\widetilde{\Omega}_k}(p)$. Denote by ω and ω_k the respective harmonic measures for Ω and $\widetilde{\Omega}_k$. By Theorem 9.16 applied to $\widetilde{\Omega}_k$, there exists a subset $E_k \subset \partial\widetilde{\Omega}_k$ with full harmonic measure

ω_k^p and with Hausdorff dimension at most 1. Taking into account that $\partial\widetilde{\Omega}_k \subset F_k \cup \partial\Omega$, by Proposition 6.37(vi) we have[1]

$$\omega_k^p(E_k) \leqslant \omega_k^p(F_k) + \omega_k^p(E_k \cap \partial\Omega) \leqslant \frac{C}{k} + \omega_k^p(E_k \cap \partial\Omega), \tag{9.9}$$

with the constant C above possibly depending on $\mathrm{d}_\Omega(p)$.

Let

$$E = \bigcup_k (E_k \cap \partial\Omega).$$

Notice that $\dim_{\mathcal{H}}(E) = \sup_k \dim_{\mathcal{H}}(E_k) \leqslant 1$. By (9.9), we have

$$\omega_k^p(E) \geqslant \omega_k^p(E_k \cap \partial\Omega) \geqslant \omega_k^p(E_k) - \frac{C}{k} = \omega_k(\partial\widetilde{\Omega}_k) - \frac{C}{k} \geqslant \omega_k(\partial\Omega \cap \partial\widetilde{\Omega}_k) - \frac{C}{k}. \tag{9.10}$$

Now by Lemma 6.38, we know that

$$\omega^p(E) = \lim_{k\to\infty} \omega_k^p(E) \quad \text{ and } \quad \omega^p(\partial\Omega) = \lim_{k\to\infty} \omega_k^p(\partial\Omega \cap \partial\widetilde{\Omega}_k).$$

So letting $k \to \infty$ in (9.10), we deduce that E has full harmonic measure ω^p. Since E has Hausdorff dimension at most 1, we infer that $\dim_{\mathcal{H}} \omega^p \leqslant 1$. □

The next reduction is the following.

Lemma 9.19. *To prove Theorem 9.16 under the hypothesis of compact boundary, we may assume that Ω is an unbounded domain with compact boundary and that the pole for harmonic measure is ∞.*

Proof. We may assume that Ω is connected because the harmonic measure for Ω with pole at $p \in \Omega$ coincides with the harmonic measure for the component of Ω containing p, with pole at p.

Suppose now that $p \neq \infty$. Consider the map $\varphi(z) = 1/(z-p)$. This is a conformal mapping of the Riemann sphere, and by Proposition 9.7 (which also holds for unbounded domains with compact boundary), denoting $\Omega' = \varphi(\Omega)$, we have

$$\omega_{\Omega'}^{\infty} = \varphi_{\#}\omega_\Omega^p.$$

Hence, assuming that Theorem 9.16 holds for $\omega_{\Omega'}^{\infty}$, we infer that there exists some subset $E \subset \partial\Omega'$ with $\dim_{\mathcal{H}} E \leqslant 1$ and full measure $\omega_{\Omega'}^{\infty}$. Then $\varphi^{-1}(E)$ has full measure ω_Ω^p and, since $\varphi|_{\partial\Omega} : \partial\Omega \to \partial\Omega'$ is bilipschitz, we also have $\dim_{\mathcal{H}} \varphi^{-1}(E) \leqslant 1$. □

Recall that in Theorem 7.32 we showed the following properties for the harmonic measure and for the Green function with pole at ∞, for any unbounded Wiener regular domain Ω with compact boundary:

[1] Although $\omega_k^p(E_k) = \omega_k(\partial\widetilde{\Omega}_k) = 1$, we prefer not to use this fact, so that the proof of this lemma extends easily to unbounded domains in $\mathbb{R}^d$, $d \geqslant 3$. Recall that we cannot ensure that in these domains the harmonic measure of the boundary equals 1.

(i) For every $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\int_\Omega G^\infty(z)\,\Delta\varphi(z)\,dm(z) = \int \varphi\,d\omega^\infty.$$

(ii) ω^∞ coincides with the equilibrium measure of $\partial\Omega$ and moreover, for every $z \in \Omega$,

$$G^\infty(z) = \frac{1}{\mathrm{Cap}_W(\partial\Omega)} - \frac{1}{2\pi}\int_{\partial\Omega} \log\frac{1}{|\xi - z|}\,d\omega^\infty(\xi). \tag{9.11}$$

Recall also that, for any compact set $E \subset \mathbb{C}$,

$$\frac{1}{\mathrm{Cap}_W(E)} = \inf_{\mu\in M_1(E)} I(\mu) = \inf_{\mu\in M_1(E)} \int \mathcal{E} * \mu\,d\mu,$$

where the infimum is taken over all probability measure supported on E. The number

$$\gamma_E = \frac{2\pi}{\mathrm{Cap}_W(E)}$$

is called the Robin constant of E. So we have $\mathrm{Cap}_L(E) = e^{-\gamma_E}$.

Lemma 9.20. *To prove Theorem 9.16, it is enough to prove that for any $\varepsilon > 0$ the following holds:*

$$\text{For each } \eta_0 > 0 \text{ there is a set } A \subset \partial\Omega \text{ with } \mathcal{H}_\infty^{1+\varepsilon}(A) < \eta_0 \text{ and } \omega(\partial\Omega\backslash A) < \eta_0. \tag{9.12}$$

Proof. The statement (9.12) implies that for $\eta_0 > 0$ there is a set $A \subset \partial\Omega$ with $\mathcal{H}_\infty^{1+\varepsilon}(A) < \eta_0$ and $\omega(\partial\Omega\backslash A) = 0$, which in turn implies that there is $A \subset \partial\Omega$ with $\mathcal{H}_\infty^{1+\varepsilon}(A) = 0$ and $\omega(\partial\Omega\backslash A) = 0$. Now taking $\varepsilon_n \to 0$, one gets sets $A_n \subset \partial\Omega$ with $\mathcal{H}_\infty^{1+\varepsilon_n}(A_n) = 0$ and $\omega(\partial\Omega\backslash A_n) = 0$. Letting $E = \bigcap_n A_n$ we have $\mathcal{H}_\infty^{1+\varepsilon_n}(E) = 0$, for each n, which gives that the Hausdorff dimension of E is less than or equal to one, and $\omega(\partial\Omega\backslash E) = 0$. □

Sketch of the proof of Theorem 9.16

We will make a reduction to the case in which $K := \partial\Omega$ is a finite union of pieces of small diameter and rather well separated. Then we will construct an auxiliary compact K^*, which is a finite union of closed discs, using two special modification methods, called "the disc construction" and the "annulus construction". It is crucial to compare the harmonic measure associated with Ω and that associated with the new domain $\Omega^* = \mathbb{C}^*\backslash K^*$. This is simple for the annulus construction, but much more delicate for the disc construction; Lemma 9.21 below takes care of this issue. The gradient of the Green function $G = G^\infty$ of Ω^* with pole at ∞ can be estimated on some special curves surrounding K^* and contained in level sets of G. All these ingredients allow to estimate the harmonic measure of Ω in terms of the integral of the gradient of G on these curves. Lemma 9.24 is the main tool to end the proof estimating this integral in the appropriate way. An ingredient in the proof of Lemma 9.24 yields in the limiting case, assuming $\partial\Omega$ smooth, the formula

$$\int_{\partial\Omega} |\partial_\nu G|\,\log|\partial_\nu G|\,d\mathcal{H}^1 > -c_0,$$

where G is now the Green function of Ω with pole at ∞, ν is the outer unit normal to $\partial\Omega$ and $c_0 > 0$. If $\partial\Omega$ is analytic, by the reflection principle for harmonic functions (see Exercise 9.5.1 below) ∇G is harmonic in a neighborhood of $\partial\Omega$ and thus it is $C^1(\bar{\Omega})$. By Proposition 7.35, the harmonic measure is (in the smooth case)

$$d\omega^\infty = |\partial_\nu G|\, d\mathcal{H}^1|_{\partial\Omega} = -\partial_\nu G\, d\mathcal{H}^1|_{\partial\Omega}.$$

Let us do some heuristics here. Assume that at the point z the "dimension" of ω^∞ at z is $d(z)$, which means that $\omega(\bar{B}(z,r)) \sim r^{d(z)}$, or equivalently $\frac{\omega^\infty(\bar{B}(z,r))}{r} \sim r^{d(z)-1}$. By the previous identity and the Radon-Nikodym theorem, we have

$$|\partial_\nu G(z)| = \lim_{r\to 0} \frac{\omega^\infty(\bar{B}(z,r))}{2r},$$

so we get

$$\lim_{r\to 0} \int_{\partial\Omega} (d(z)-1)\log(2r)\, d\omega^\infty(z) \geqslant -c_0.$$

From this fact, we deduce that the integrand in the left hand side of the preceding identity does not tend to $-\infty$ in a set of positive measure as $r \to 0$, that is $d(z) \leqslant 1$ for ω^∞-a.e. $z \in \partial\Omega$, and so, ω^∞ lives in a set of dimension not greater than 1.

From now on, in the rest of this chapter, unless otherwise stated, we assume that Ω is a Wiener regular unbounded domain with compact boundary, and we denote by ω its harmonic measure with pole at ∞. We will also write $K = \partial\Omega$.

Exercise 9.5.1 (Reflection principle for planar harmonic functions)**.** Let $U \subset \mathbb{C}$ be a finitely connected domain bounded by disjoint analytic Jordan curves, and let $u : U \to \mathbb{R}$ be a harmonic function in U with $u \in C(\bar{U})$ and such that $u|_{\partial U} \equiv 0$. Show that $u \in C^1(\bar{U})$, that is, show that ∇u extends continuously to ∂U. (hint: solve first [Eva98, Problem 2.5.9] and then use that every analytic curve is locally the image of a segment by a conformal mapping.)

9.6 The disc and the annulus construction

Let us start with the disc construction.

Disc construction

Fix $\varepsilon > 0$. Let Q be a closed square with sides parallel to the axes and side length $\ell = \ell(Q)$ and set $E = Q \cap K$. Replace E by a closed disc $\bar{B}$ (B will stand for the corresponding open disc) with the same center as Q and radius $r(B)$ defined by

$$r(B) = \frac{1}{2}\frac{\mathrm{Cap}_L(E)^{1+\varepsilon}}{\ell^\varepsilon} = \frac{1}{2}\frac{e^{-\gamma_E(1+\varepsilon)}}{\ell^\varepsilon}. \tag{9.13}$$

By (6.4) this construction is *scale invariant*, i.e., if we dilate Q and E by a constant λ, then the ball defined by this method is dilated by the same factor λ as well. We get a new compact set $\widetilde{K} = (K \backslash E) \cup \bar{B}$, a new domain $\widetilde{\Omega} = \mathbb{C}^* \backslash \widetilde{K} = (\Omega \cup E) \backslash \bar{B}$ and a new harmonic measure $\widetilde{\omega} = \omega_{\widetilde{\Omega}}^{\infty}$.

Note that $\bar{B} \subset Q^{\circ}$. In fact, since the logarithmic capacity of a disc is the radius (see proposition 6.16), we have the estimate

$$\operatorname{Cap}_L(E) \leqslant \frac{\sqrt{2}}{2}\ell,$$

so that

$$r(B) \leqslant \frac{1}{2} \frac{\left(\sqrt{2}/2\right)^{1+\varepsilon} \cdot \ell^{1+\varepsilon}}{\ell^{\varepsilon}} = \frac{\ell}{2}\left(\sqrt{2}/2\right)^{1+\varepsilon} < \ell/2.$$

Annulus construction

Let Q be a closed square with sides parallel to the axis and take the square RQ, where R is a number larger than 1 that will be chosen later. The reader has to think that R is very large. Delete $K \cap (RQ \backslash Q)^0$ from K to obtain a new domain $\widetilde{\Omega} = \Omega \cup (RQ \backslash Q)^0$ and a new harmonic measure $\widetilde{\omega} = \omega_{\widetilde{\Omega}}$.

It is important to have some control on the harmonic measure of the new domain obtained after performing the disc or the annulus construction. For the annulus this is easy: any part of K which has not been removed has larger or equal harmonic measure. In other words, if $A \subset \partial\Omega$ satisfies $A \cap (RQ \backslash Q) = \varnothing$, then $\widetilde{\omega}(A) \geqslant \omega(A)$. This is a consequence of Lemma 5.32 because $A \subset \partial\Omega \cap \partial\widetilde{\Omega}$ and $\Omega \subset \widetilde{\Omega}$.

Estimating the harmonic measure after the disc construction is a difficult task. The result is the following.

Lemma 9.21. *Let Q be a square with sides parallel to the axis. Fix $\varepsilon > 0$ and perform the disc construction for this ε. Assume that $RQ \backslash Q \subset \Omega$. Then there exists a number $R_0(\varepsilon)$ such that for $R \geqslant R_0(\varepsilon)$ one has*

(a) $\widetilde{\omega}(\bar{B}) \geqslant c_0\, \omega(Q \cap K)$.

(b) $\widetilde{\omega}(A) \geqslant \omega(A)$, *if $A \subset \partial\Omega \backslash RQ$ is both relatively open and relatively closed.*

Above $\widetilde{\omega}$ and ω are harmonic measures with pole at ∞.

The proof of Lemma 9.21 will be presented in Section 9.11 and we will use it as a black box in the arguments below.

9.7 The Main Lemma and the domain modification

Let $\Omega = \mathbb{C}^* \backslash K$, $\operatorname{Cap}_L K > 0$ and assume that $K \subset \{|z| < 1/2\}$ (this assumption will be convenient later on, but it is not essential). Fix $\varepsilon > 0$ and let $R > 2 + R_0(\varepsilon)$, R integer, where $R_0(\varepsilon)$ is the constant given by Lemma 9.21. We let $M = M(\varepsilon, \eta)$ stand for a large constant that will be chosen later (see Section 9.10) and we let ρ be a small constant so that $M \leqslant \log 1/\rho$, and $\rho = \frac{1}{2^N}$, N a positive integer. Consider the grid $\mathcal{G}$ of dyadic squares of side length ρ and lower left corner at the points of the form $\{(m + ni)\rho;\, m, n \in \mathbb{Z}\}$. For each $1 \leqslant p, q \leqslant R$, let $\mathcal{G}_{pq}$ be the family of (closed) squares $Q \in \mathcal{G}$ with $(m, n) \equiv (p, q)$ (mod $R \times R$). Then $\mathcal{G} = \bigcup_{p,q=1}^{R} \mathcal{G}_{pq}$.

Write $K_{pq} = \bigcup_{Q \in \mathcal{G}_{pq}} K \cap Q$, $\Omega_{pq} = \mathbb{C}^* \backslash K_{pq}$, $\omega_{pq}(A) = \omega^{\infty}_{\Omega_{pq}}$. We will show the following:

Main Lemma 9.22. *For any $\varepsilon > 0$ and for any $\eta > 0$, one can choose $R(\varepsilon) > 0$ large enough and $\rho(\eta, \varepsilon)$ small enough so that for all $1 \leqslant p, q \leqslant R$ there is a Borel set $A_{pq} \subset K_{pq}$ satisfying*

$$\mathcal{H}^{1+\varepsilon}_{\infty}(A_{pq}) < \eta \quad \textit{and} \quad \omega_{pq}(K_{pq} \backslash A_{pq}) < \eta. \tag{9.14}$$

An important fact about the previous statement is that the constant $R = R(\varepsilon)$ does not depend on η, so that η can be chosen later depending on $R(\varepsilon)$.

Let us see how Lemma 9.20, and so the Jones-Wolff theorem, is derived from Main Lemma 9.22. Write $A = \bigcup_{1 \leqslant p,q \leqslant R} A_{pq}$. Then, we have

$$\mathcal{H}^{1+\varepsilon}_{\infty}(A_{pq}) \leqslant \sum_{1 \leqslant p,q \leqslant R} \mathcal{H}^{1+\varepsilon}_{\infty}(A_{pq}) \leqslant R^2 \, \eta,$$

and, by Lemma 5.32,

$$\omega(K \backslash A) \leqslant \sum_{1 \leqslant p,q \leqslant R} \omega(K_{pq} \backslash A) \leqslant \sum_{1 \leqslant p,q \leqslant R} \omega(K_{pq} \backslash A_{pq}) \overset{\text{L.5.32}}{\leqslant} \sum_{1 \leqslant p,q \leqslant R} \omega_{pq}(K_{pq} \backslash A_{pq}) \overset{(9.14)}{\leqslant} R^2 \eta.$$

Recalling that η can be taken arbitrarily small, for any given R, (9.12) follows.

Our next objective is to prove the Main Lemma 9.22. To this end, we need to perform a domain modification which we proceed to describe.

Domain modification.
From now on we fix p, q and let $\Omega = \Omega_{pq}$, $K = K_{pq}$, $\omega = \omega_{pq}$. We let $\{Q_j\}_j$ be the family of squares in $\mathcal{G}_{pq}$. We remark that, by the construction, for each square Q_j one has $RQ_j \backslash Q_j \subset \Omega$, so that we will be able to apply Lemma 9.21.

Fix $\varepsilon > 0$ and perform the disc construction for ε in every square Q_j, so that we get a finite family of closed discs $\{\bar{B}_j\}$, whose union is a compact set K_1, a new domain $\Omega_1 = \mathbb{C} \backslash K_1$ and a new harmonic measure $\omega_1 = \omega^{\infty}_{\Omega_1}$.

Next choose a dyadic square Q^1 of largest side $\ell(Q^1)$, not necessarily from $\mathcal{G}_{pq}$, such that

$$\ell(Q^1) \geqslant \rho \quad \text{and} \quad \omega_1(Q^1) \geqslant M \ell(Q^1).$$

If such Q^1 does not exist we stop the domain modification. If Q^1 exists we perform the annulus construction on Q^1 (with constant R) and after this we perform the disc construction on the square Q^1, replacing $K_1 \cap Q^1$ by a disc $\bar{B}^1$. So we obtain a new compact K_2, a new domain $\Omega_2 = \mathbb{C}^* \backslash K_2$ and a new harmonic measure $\omega_2 = \omega^{\infty}_{\Omega_2}$.

Now we continue and take Q^2 dyadic with largest side such that $Q^2 \not\subset Q^1$, $\ell(Q^2) \geqslant \rho$ and $\omega_2(Q^2) \geqslant M \ell(Q^2)$. If such Q^2 does not exist we stop. Otherwise we perform the annulus construction on Q^2 but with a special rule: If $\bar{B}^1 \cap (\partial(RQ^2 \backslash Q^2)) \neq \varnothing$, then we do not remove the set $\bar{B}^1 \cap (RQ^2 \backslash Q^2)$ from K_2. The reason for this rule is to get full balls in all cases.

After that we perform the disc construction on Q^2, replacing $K_2 \cap Q^2$ by the corresponding disc $\bar{B}^2$, getting a new compact K_3, a new domain Ω_3 and a new harmonic measure ω_3.

We continue this process so that if $K_1 \cap Q^1$, $K_2 \cap Q^2, \dots, K_{n-1} \cap Q^{n-1}$ have been substituted by $\bar{B}^1, \dots, \bar{B}^{n-1}$ we choose now (if there exists) a dyadic cube Q^n with largest side so that

$$Q^n \not\subset Q^j, \quad j = 1, \dots, n-1, \quad \ell(Q^n) \geqslant \rho, \quad \omega_n(Q^n) \geqslant M\ell(Q^n).$$

Then (if we do not stop) we perform the annulus construction with respect to Q^n but without removing $\bar{B}^j \cap (RQ^n \backslash Q^n)$, $j = 1, \dots, n-1$ in case that $\bar{B}^j \cap (\partial(RQ^n \backslash Q^n)) \neq \varnothing$ (this is the special rule). Finally we perform the disc construction on Q^n, getting $\bar{B}^n$, K_{n+1}, Ω_{n+1} and ω_{n+1}.

At each step there are only finitely many candidate dyadic squares, because $\rho \leqslant \ell(Q) \leqslant 1/M$. Since no Q^j can be repeated (because $Q^j \not\subset Q^\ell$, $\ell = 1, \dots, j-1$) the modification process stops after finitely many steps. Let $K^*, \Omega^* = \mathbb{C} \backslash K^*$, $\omega^* = \omega_{\Omega^*}^\infty$ be the final outcome so that K^* is the disjoint union of the non removed discs; more precisely,

$$K^* = \bigcup_{k \in S} \bar{B}^k \cup \bigcup_{j \in T} \bar{B}_j \quad \text{(some finite sets of indices } S \text{ and } T\text{)},$$

where the $\bar{B}_j$ are the original discs and the $\bar{B}^k$ are the new discs produced after performing the annulus and the disc constructions.

Now we want to prove by means of Lemma 9.21 the following estimates:

$$\omega^*(\bar{B}_j) \geqslant c_0\, \omega(Q_j), \qquad j \in T, \tag{9.15}$$

$$\omega^*(Q^j) \geqslant c_0\, M\ell(Q^j), \quad j \in S. \tag{9.16}$$

For (9.15) note first that we always have $RQ_j \backslash Q_j \subset \Omega$. Since Q_j has survived all steps we cannot have $RQ^k \supset Q_j$ at some step k. Since RQ^k is a union of dyadic squares, the other possibility is $RQ^k \cap Q_j = \varnothing$ for all k and we can apply both inequalities in Lemma 9.21.

For (9.16), when we select Q^j we have $\omega_j(Q^j) \geqslant M\ell(Q^j)$ and after performing the annulus and the disc constructions, we get $\omega_{j+1}(\bar{B}^j) \geqslant c_0\, \omega_j(Q^j) \geqslant c_0\, M\ell(Q^j)$. If $k > j$ there are three possibilities: i) $\bar{B}^j \subset RQ^k \backslash Q^k$, in which case $\bar{B}^j$ has disappeared and j would not be in S; ii) $\bar{B}^j \cap (RQ^k \backslash Q^k) = \varnothing$ in which case $\omega_{k+1}(\bar{B}^j) \geqslant \omega_k(\bar{B}^j)$ and iii) $\bar{B}^j \cap \partial(RQ^k \backslash Q^k) \neq \varnothing$.
In this last case we have $\ell(Q^k) \geqslant \ell(Q^j)$ since otherwise Q^k would have disappeared. But now since $R > 2 + R_0(\varepsilon)$ we get that $\bar{B}^j \cap (R_0(\varepsilon)Q^k \backslash Q^k) = \varnothing$ and so $\omega_{k+1}(\bar{B}^j) \geqslant \omega_k(\bar{B}^j)$ by Lemma 9.21 part b). At the end we obtain

$$\omega^*(Q^j) \geqslant \omega^*(\bar{B}^j) \geqslant \dots \geqslant \omega_k(\bar{B}^j) \geqslant \dots \geqslant \omega_{j+1}(\bar{B}^j) \geqslant c_0\, M\ell(Q^j).$$

We will also need the following estimate: If $z_0 \in Q_j$, $j \in T$ (or $z_0 \in Q^k$, $k \in S$) and $r \geqslant \ell(Q_j)$ ($r \geqslant \ell(Q^k)$), then

$$\omega^*\{|z - z_0| < r\} \leqslant CMr. \tag{9.17}$$

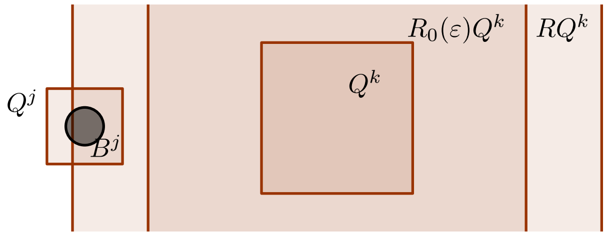
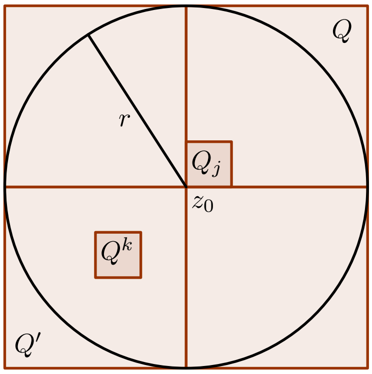


Figure 9.1: Disposition when special rule applies.

Let us discuss the case of Q_j, $z_0 \in Q_j$. We remark that if Q is a dyadic square with $Q \supset Q_j$, then one has $\omega^*(Q) \leqslant M\ell(Q)$ because otherwise the process would not have been stopped.



Take now a dyadic square $Q \supset Q_j$ with side length $2^m\ell(Q_j)$ such that $r \leqslant 2^m\ell(Q_j) \leqslant 2r$. We just said that $\omega^*(Q) \leqslant 2Mr$. Now the disc $\{|z-z_0| < r\}$ is contained in 4 dyadic squares of the same side length as Q. Take one of these squares Q' different from Q. If Q' does not contain any $Q_{j'}$ or Q^k then $\omega^*(Q') = 0$. Otherwise $\omega^*(Q') \leqslant 2Mr$.
The case $z_0 \in Q^k$ is dealt with similarly.

The next lemma shows that the union of the family of squares $\{Q_j\}_{j\in T}$ and a dilation of the family $\{Q_k\}_{k\in S}$ contains K.

Lemma 9.23. $K \subset \bigcup\limits_{k\in S} 2RQ^k \cup \bigcup\limits_{j\in T} Q_j$.

Proof. Recall that now $K = K_{pq} = \bigcup\limits_{Q\in\mathcal{G}_{pq}} K \cap Q$. So let $Q \in \mathcal{G}_{pq}$ and $E = K \cap Q$. If $Q = Q_j$ for some $j \in T$ then $E \subset Q_j$ and so $E \subset \bigcup\limits_{k} 2RQ^k \cup \bigcup\limits_{j\in T} Q_j$.

If $Q \neq Q_j$ for every $j \in T$ then there is a first index j_1 such that $Q \subset RQ^{j_1}\backslash Q^{j_1}$; if $j_1 \in S$ then $Q \subset RQ^{j_1}$, $j_1 \in S$, and we are done. If $j_1 \notin S$ there is a first index j_2 such that $Q^{j_1} \subset RQ^{j_2}\backslash Q^{j_2}$. In this case $\ell(Q^{j_2}) \geqslant 2\ell(Q^{j_1})$ because if we had $\ell(Q^{j_1}) \geqslant \ell(Q^{j_2})$ then $Q^{j_2} \subset RQ^{j_1}$ and $Q^{j_2} \subset RQ^{j_1}\backslash Q^{j_1}$, so that Q^{j_2} would have disappeared. If $j_2 \in S$ we have $Q \subset RQ^{j_2}$ and we are done. If $j_2 \notin S$ there is a first j_3 such that $Q^{j_2} \subset RQ^{j_3}\backslash Q^{j_3}$ and so on.

We get a sequence $j_1 < j_2 < \cdots < j_n$ with $j_1, \ldots, j_{n-1} \notin S$, $j_n \in S$ so that

$$Q \subset RQ^{j_1}, \qquad Q^{j_k} \subset RQ^{j_{k+1}} \quad \text{and} \quad \ell(Q^{j_{i+1}}) \geqslant 2\ell(Q^{j_i}).$$

Note that every pair of cubes Q_1 and Q_2 with $\ell(Q_2) \geqslant 2\ell(Q_1)$, satisfies that

$$Q \subset 2RQ_1 \text{ and } Q_1 \subset RQ_2 \quad \Longrightarrow \quad Q \subset 2RQ_2.$$

Then, using this argument inductively on $\{Q^{j_k}\}_{k=1}^n$, we get that $Q \subset 2RQ^{j_n}$. $\square$

9.8 Surrounding K^* by level curves of the Green function

To continue the proof of the Theorem, let Q be a square $Q = Q_j$, $j \in T$ or $Q = Q^k$, $k \in S$ and let $\bar{B}$ be the corresponding disc. Let $G(z) = G_{\Omega^*}^{\infty}(z)$ be the Green function of the domain Ω^* with pole at ∞. The goal of this section is to find a closed curve σ surrounding $\bar{B}$, contained in a level set of G, and such that

$$|\nabla G(z)| \leqslant CM^2 \log \frac{1}{\ell(Q)}, \quad z \in \sigma, \tag{9.18}$$

for a positive constant C.

By (9.11), the Green function G is the logarithmic potential of the equilibrium measure plus the Robin constant divided by 2π, that is,

$$\begin{aligned} 2\pi G(z) &= \int_{K^*} \log|z-w|\, d\omega^*(w) + \gamma_{K^*} \\ &= \int_{\bar{B}} \log|z-w|\, d\omega^*(w) + \int_{K^*\setminus\bar{B}} \log|z-w|\, d\omega^*(w) + \gamma_{K^*} =: u(z) + v(z) + \gamma_{K^*}. \end{aligned}$$

We have the estimate

$$|\nabla v(z)| \leqslant C \int_{K^*\setminus\bar{B}} \frac{d\omega^*(w)}{|z-w|} \leqslant CM \log \frac{1}{\ell(Q)}, \quad z \in Q. \tag{9.19}$$

To show this last inequality, fix $z \in Q$. We have

$$\begin{aligned} \int_{K^*\setminus\bar{B}} \frac{d\omega^*(w)}{|z-w|} &\lesssim \sum_{j=\log_2 \ell(Q)}^{\log_2 \operatorname{diam} K^*} \int_{B_{2^j}(z)\setminus B_{2^{j-1}}(z)} \frac{d\omega^*(w)}{|z-w|} \\ &\leqslant \sum_{j=\log_2 \ell(Q)}^{\log_2 \operatorname{diam} K^*} 2^{1-j}\omega^*(B_{2^j}(z)) \overset{(9.17)}{\lesssim} \sum_{j=\log_2 \ell(Q)}^{\log_2 \operatorname{diam} K^*} 2^{-j} CM 2^j \approx CM \log_2 \frac{1}{\ell(Q)}. \end{aligned}$$

Assume for simplicity that the center of the square Q, and so of the disc $\bar{B}$, is the origin. Next we will estimate the derivative $\frac{\partial u}{\partial r}(z)$ from below, namely

$$\frac{\partial u}{\partial r}(z) \geqslant c \frac{\omega^*(\bar{B})}{|z|}, \qquad \text{for } |z| \geqslant 2r(B), \tag{9.20}$$

with $0 < c < 1$ universal.

Write $z = re^{i\theta}$. Since

$$u(re^{i\theta}) = \frac{1}{2} \int_{\bar{B}} \log|re^{i\theta} - w|^2\, d\omega^*(w),$$

we have

$$\frac{\partial u}{\partial r}(z) = \frac{1}{2}\int_{\bar{B}} \frac{1}{|re^{i\theta}-w|^2}\frac{\partial}{\partial r}\Big((re^{i\theta}-w)(re^{-i\theta}-\bar{w})\Big)\, d\omega^*(w)$$

$$= \int_{\bar{B}} \mathrm{Re}\left(\frac{(z-w)\,\bar{z}}{|z-w|^2\,|z|}\right) d\omega^*(w).$$

Note that

$$\frac{\partial u}{\partial r}(z) = \mathrm{Re}\left(\frac{\bar{z}}{|z|}\int_{\bar{B}} \frac{z}{|z|^2}\, d\omega^*(w)\right) + \mathrm{Re}\left(\frac{\bar{z}}{|z|}\int_{\bar{B}}\left(\frac{(z-w)}{|z-w|^2} - \frac{z}{|z|^2}\right) d\omega^*(w)\right).$$

Trivially,

$$\mathrm{Re}\left(\frac{\bar{z}}{|z|}\int_{\bar{B}} \frac{z}{|z|^2}\, d\omega^*(w)\right) = \frac{\omega^*(\bar{B})}{|z|}.$$

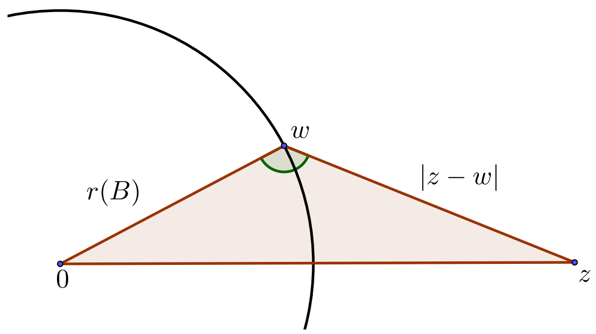


Figure 9.2: The scalar product $\langle w, z-w\rangle$ is negative when the angle $|\widehat{0wz}| < \frac{\pi}{2}$.

On the other hand,

$$\frac{(z-w)}{|z-w|^2} - \frac{z}{|z|^2} = \frac{1}{\overline{z-w}} - \frac{1}{\bar{z}} = \frac{\bar{w}}{\overline{z-w}\,\bar{z}},$$

so

$$\mathrm{Re}\left(\frac{\bar{z}}{|z|}\int_{\bar{B}}\left(\frac{(z-w)}{|z-w|^2} - \frac{z}{|z|^2}\right) d\omega^*(w)\right) = \frac{1}{|z|}\int_{\partial B} \frac{\mathrm{Re}\,(\bar{w}\,(z-w))}{|z-w|^2}\, d\omega^*(w).$$

Note that, whenever $|z| \geqslant 2r(B)$, then $\mathrm{Re}\,(\bar{w}\,(z-w)) = \langle w, z-w\rangle$ is positive on an open arc centered at $\frac{r(B)z}{\bar{z}}$, and subtaining an angle grater than $\frac{2\pi}{3}$. On the complementary arc

$$b = \{w \in \partial B : \langle w, z-w\rangle \leqslant 0\} \subset \{r(B)e^{i(t+\theta)} : \frac{\pi}{3} \leqslant t \leqslant \frac{5\pi}{3}\},$$

we get that $|z-w| > (1+\tau)r(B)$ for a universal $\tau > 0$ (in fact one can easily show that $\tau > \frac{1}{2}$), so

$$\left|\frac{w}{z-w}\right| \leqslant \frac{1}{1+\tau}.$$

All in all,

$$\frac{1}{|z|}\int_{\partial B}\frac{\operatorname{Re}\left(\bar{w}\,(z-w)\right)}{|z-w|^2}\,d\omega^*(w) \geqslant \frac{-1}{|z|}\int_b\left|\frac{w}{z-w}\right|\,d\omega^*(w) \geqslant -\frac{1}{1+\tau}\frac{\omega^*(\bar{B})}{|z|},$$

so

$$2\pi\,\frac{\partial u}{\partial r}(z) \geqslant \frac{\omega^*(\bar{B})}{|z|} - \frac{1}{1+\tau}\frac{\omega^*(\bar{B})}{|z|} = \frac{\tau}{1+\tau}\frac{\omega^*(\bar{B})}{|z|},$$

establishing (9.20).

We are now ready to estimate the gradient of the Green function G. Define

$$\alpha = \alpha(\bar{B}) = \max\left(\frac{\omega^*(\bar{B})}{M^2\log 1/\ell(Q)}, 2r(B)\right),$$

and distinguish two cases:

Case 1: $\alpha = 2r(B)$, that is, $\dfrac{\omega^*(\bar{B})}{r(B)} \leqslant 2M^2\log\frac{1}{\ell(Q)}$.

We let σ to be the circle ∂B, so we need to prove estimate (9.18) in this setting. First we claim that

$$\sup_{\partial B}|\nabla G| \leqslant C\inf_{\partial B}|\nabla G| \tag{9.21}$$

for some constant C, which we will prove below. In order to prove (12.30), assume that $z_0 = 0$ and take two points z and z' with $|z| = |z'| = 2r(B)$. Then we have

$$m^{-1}G(z') \leqslant G(z) \leqslant mG(z')$$

for some constant m by Harnack's inequality. Applying boundary Harnack's inequality to rotations of G in the domain $3B\backslash B$ (see for instance Corollary 8.18), we deduce that

$$m^{-1}G(z') \leqslant G(z) \leqslant mG(z') \quad \text{for } r(B) < |z| = |z'| < 2r(B).$$

Dividing by $|z| - r(b)$ and taking limit as $|z| \to r(B)$, we get

$$m^{-1}|\partial_\nu G|(z') \leqslant |\partial_\nu G|(z) \leqslant m|\partial_\nu G|(z'), \quad |z| = |z'| = r(B),$$

and (12.30) follows.

We have

$$\omega^*(\bar{B}) = \int_{\partial B}|\partial_\nu G|\,d\mathcal{H}^1 \geqslant 2\pi\inf_{\partial B}|\nabla G|\,r(B),$$

and for $z \in \partial B$, using (12.30) we get

$$|\nabla G(z)| \leqslant C\inf_{\partial B}|\nabla G(z)| \leqslant C\,\frac{\omega^*(\bar{B})}{r(B)}. \tag{9.22}$$

Since we are under the hypothesis $\dfrac{\omega^*(\bar{B})}{r(B)} \leqslant 2M^2\log\frac{1}{\ell(Q)}$, we get (9.18), i.e., $|\nabla G(z)| \leqslant CM^2\log\frac{1}{\ell(Q)}$.

Case 2: $\alpha > 2r(B)$, that is, $\dfrac{\omega^*(\bar{B})}{\alpha} = M^2 \log \frac{1}{\ell(Q)}$.

We note that
$$\alpha \leqslant \frac{\omega^*(Q)}{M^2 \log 2} \leqslant \frac{2M\ell(Q)}{M^2 \log 2} \leqslant \frac{4}{M}\,\ell(Q). \tag{9.23}$$
The inequality $\omega^*(Q) \leqslant M\ell(Q)$, for $Q = Q_j$, comes from the fact that Q_j has survived the process to get to ω^*. If $Q = Q^k$, take the dyadic square $\widetilde{Q}$ with side length $2\,\ell(Q^k)$ and containing Q^k. Since the process has stopped, $\omega^*(Q^k) \leqslant \omega^*(\widetilde{Q}) \leqslant M\ell(\widetilde{Q}) = 2M\ell(Q)$. Taking in (9.23) $M > 8$, we obtain $\alpha < \ell(Q)/2$ and so $\{|z - z_0| = \alpha\} \subset Q$.

Now we want to prove that
$$|\nabla G(z)| \leqslant 4\,M^2 \log \frac{1}{\ell(Q)}, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha, \tag{9.24}$$
where μ is such that $\mu > e^{2\pi C}$, a condition that will be used later, with C fixed in (9.28) below. Choosing $M > 8\mu$ we obtain $\alpha\mu < \ell(Q)/2$, by (9.23). Hence the annulus $\alpha \leqslant |z - z_0| \leqslant \mu\alpha$ is contained in $Q \backslash \bar{B}$, a fact that will be used in the sequel without further mention.

First, let us show
$$\frac{1}{2}\frac{\partial u}{\partial r}(z) \geqslant |\nabla v(z)|, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha. \tag{9.25}$$
By (9.20) we get
$$\frac{\partial u}{\partial r}(z) \geqslant c\frac{\omega^*(\bar{B})}{|z - z_0|} \geqslant c\frac{\omega^*(\bar{B})}{\mu\alpha}, \quad \alpha < |z - z_0| \leqslant \mu\alpha,$$
and since we are in case 2, that is, $\dfrac{\omega^*(\bar{B})}{\alpha} = M^2 \log \frac{1}{\ell(Q)}$, by taking the quotient M/μ big enough we obtain
$$\frac{1}{2}\frac{\partial u}{\partial r}(z) \geqslant \frac{c}{2\mu} M^2 \log \frac{1}{\ell(Q)} \overset{(9.19)}{\geqslant} \frac{cM|\nabla v|}{2\mu C} \geqslant |\nabla v(z)|, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha,$$
by (9.19), settling (9.25).

Finally, using (9.25) we get
$$2\pi|\nabla G(z)| \leqslant |\nabla u(z)| + |\nabla v(z)| \overset{(9.25)}{\leqslant} 2|\nabla u(z)| \leqslant C\int_{\partial B} \frac{d\omega^*(w)}{|z - w|}, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha,$$
and $|z - w| \geqslant |z - z_0| - |w - z_0| \geqslant \alpha - r(B) \geqslant \frac{\alpha}{2}$, which gives
$$|\nabla G(z)| \leqslant C\,\frac{\omega^*(\bar{B})}{\alpha} = C\,M^2 \log \frac{1}{\ell(Q)}, \quad \alpha \leqslant |z - z_0| \leqslant \mu\alpha,$$
establishing (9.24).

Assume $z_0 = 0$, let $c = \sup\{G(z) : |z| = \alpha\}$ and take as σ the connected component of $\{G = c\}$ that contains a point on $|z| = \alpha$. The curve σ encloses a domain that contains the disc $\{|z| < \alpha\}$.

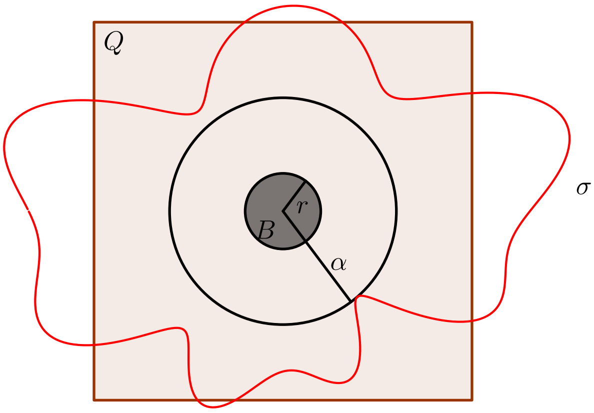


Figure 9.3: Disposition in case 2.

We claim that σ remains inside $\{|z| \leqslant \mu\alpha\}$, which, in view of (9.24), yields the required estimate (9.18).

We have

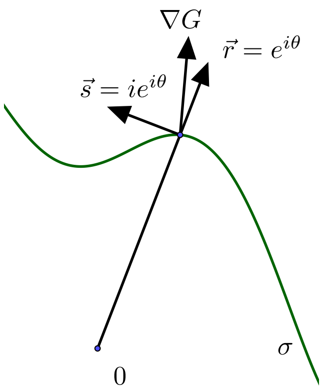
$$|\nabla u(z)| \leqslant \int_{\bar{B}} \frac{d\omega^*(w)}{|z-w|} \leqslant 2\frac{\omega^*(\bar{B})}{|z|}, \quad |z| > \alpha,$$

because

$$|z-w| \geqslant |z| - |w| \geqslant \frac{|z|}{2} + \frac{\alpha}{2} - r(B) > \frac{|z|}{2}.$$

By (9.20), for $|z| \geqslant \alpha \geqslant 2\,r(B)$ we get

$$\frac{\partial u}{\partial r}(z) \geqslant c\frac{\omega^*(\bar{B})}{|z|} \quad \text{and} \quad |\nabla u(z)| \leqslant 2c^{-1}\frac{\partial u}{\partial r}(z).$$



Therefore, combining the previous estimate with (9.25), for $\alpha \leqslant |z| \leqslant \mu\alpha$ we get

$$2\pi\,|\nabla G(z)| \leqslant |\nabla u(z)| + |\nabla v(z)| \leqslant C\frac{\partial u}{\partial r}(z). \quad (9.26)$$

Moreover, by(9.25) we have

$$\begin{aligned} 2\pi\,\frac{\partial G}{\partial r}(z) &= \frac{\partial u}{\partial r}(z) + \frac{\partial v}{\partial r}(z) \\ &\geqslant \frac{\partial u}{\partial r}(z) - |\nabla v(z)| \overset{(9.25)}{\geqslant} \frac{1}{2}\frac{\partial u}{\partial r}(z) \overset{(9.20)}{>} 0. \end{aligned} \quad (9.27)$$

The curve σ contains at least a point a on the circle $\{|z| = \alpha\}$. Consider the maximal subarc τ of σ containing a and contained in the disc $\{|z| \leqslant \mu\alpha\}$. By (9.27), each ray emanating from the origin intersects τ only once, and so τ can be parametrized by the polar angle θ in the form $r(\theta)e^{i\theta}$ with $\theta_1 \leqslant \theta \leqslant \theta_2$. Without loss of generality assume $\theta_1 < 0 < \theta_2$ and $r(0) = a$.

If $\tau = \sigma$ we are done. If not, $r(\theta_2) = \mu\alpha$ and we will reach a contradiction. If $\vec{r} = e^{i\theta}$ is the radial direction and $\vec{s} = ie^{i\theta}$ is the orthogonal direction to $\vec{r}$, then (12.37) yields

$$\left|\frac{\partial G}{\partial s}(z)\right| \leqslant |\nabla G(z)| \overset{(12.37)}{\leqslant} C\frac{\partial u}{\partial r}(z) \overset{(9.27)}{\leqslant} C\frac{\partial G}{\partial r}(z).$$

Since $G(r(\theta)e^{i\theta}) = c$, taking the derivative with respect to θ one gets

$$0 = \langle \nabla G(r(\theta)e^{i\theta}), r'(\theta)e^{i\theta} + ir(\theta)e^{i\theta}\rangle = r'(\theta)\frac{\partial G}{\partial r} + r(\theta)\frac{\partial G}{\partial s}$$

and so

$$\frac{|r'(\theta)|}{r(\theta)} \leqslant \left|\frac{\partial G}{\partial r}\right|^{-1}\left|\frac{\partial G}{\partial s}\right| \leqslant C. \tag{9.28}$$

Therefore

$$\log\frac{r(\theta_2)}{r(0)} = \int_0^{\theta_2}\frac{r'(\theta)}{r(\theta)}\,d\theta \leqslant 2\pi C$$

and, recalling the way μ has been chosen,

$$\mu\alpha = r(\theta_2) \leqslant e^{2\pi C}r(0) = e^{2\pi C}\alpha < \mu\alpha,$$

which is a contradiction. By (9.24) we obtain the desired inequality (9.18).

9.9 The estimate of the gradient of Green's function on the level curves

In the previous section we have exhibited for each disc $\bar{B} = \bar{B}_j$, $j \in T$ or $\bar{B} = \bar{B}^k$, $k \in S$, a simple curve σ contained in a level curve of G and surrounding $\bar{B}$, on which estimate (9.18) holds. Let now Γ be the curve formed by the set of σ's corresponding to each disc $\bar{B}_j$ or $\bar{B}^k$. Then Γ separates K^* from infinity.

In this section we prove the estimate

$$\int_\Gamma |\log|\nabla G|\,\partial_\nu G|\,d\mathcal{H}^1 \leqslant C\,\log\log(1/\rho). \tag{9.29}$$

At this point we write $\log^+ = \max\{\log, 0\}$ and $\log = \log^+ - \log^-$. Since we are assuming that $M \leqslant \log(1/\rho)$, we have, by (9.18),

$$\log^+|\nabla G(z)| \leqslant \log(CM^2\log 1/\ell(Q)) \leqslant C\,\log\log(1/\rho), \quad z \in \Gamma.$$

Note that

$$-\int_\Gamma \partial_\nu G\,d\mathcal{H}^1 = \sum_\sigma\int_\sigma \partial_\nu G\,d\mathcal{H}^1 = \sum_{\bar{B}}\omega^*(\bar{B})$$

which is clear for those terms for which $\sigma = \partial B$ and follows from the divergence theorem for the others, because σ surrounds ∂B, and Proposition 7.14 applies. In both cases we use that ∇G is continuous up to the analytic boundaries σ and ∂B by the reflection principle for harmonic functions.

Hence

$$\begin{aligned}\int_\Gamma |\partial_\nu G| \log^+ |\nabla G| \, d\mathcal{H}^1 &\leqslant C \, \log\log(1/\rho) \int_\Gamma |\partial_\nu G| \, d\mathcal{H}^1 \\ &= C \, \log\log(1/\rho) \sum_{\bar{B}} \omega^*(\bar{B}) = C \, \log\log(1/\rho).\end{aligned}$$

In order to estimate the integral on Γ of $\partial_\nu G \log^- |\nabla G|$ we need the following lemma.

Lemma 9.24. *Let $G(z) = G_\Omega^\infty(z)$ be the Green function of the domain Ω with pole at infinity and let $\Gamma = \bigcup_{j=1}^N \Gamma_j$ be the union of finitely many closed Jordan curves Γ_j enclosing disjoint (bounded) Jordan domains, so that $\Gamma \subset \{|z| < 1\}$, Γ separates $K = \mathbb{C}^* \backslash \Omega$ from infinity and there are constants c_j, $j = 1, \dots, N$ such that $\Gamma_j \subset \{G(z) = c_j\}$, $j = 1, \dots, N$. Then*

$$\int_\Gamma |\partial_\nu G| \, \log |\nabla G| \, d\mathcal{H}^1 > -\log 4\pi.$$

The proof of this lemma will be discussed in Section 9.11. By Lemma 9.24 we have

$$\int_\Gamma |\partial_\nu G| \, \log^- |\nabla G| \, d\mathcal{H}^1 \leqslant \int_\Gamma |\partial_\nu G| \, \log^+ |\nabla G| \, d\mathcal{H}^1 + \log 4\pi,$$

which completes the proof of (9.29).

9.10 End of the proof of the Main Lemma 9.22 and of the Jones-Wolff Theorem

Recall from (9.14) that for a fixed $\varepsilon > 0$ and for each $\eta > 0$ we have to find a set $A \subset K$ with $\mathcal{H}^{1+\varepsilon}_\infty(A) < \eta$ and $\omega(K \backslash A) < \eta$.

Decompose the set of indices T as $T = T_1 \cup T_2$ with

$$\begin{aligned} T_1 &= \{j \in T : \omega^*(\bar{B}_j) \geqslant \rho^{\varepsilon/2} \, r_j\}, \\ T_2 &= \{j \in T : \omega^*(\bar{B}_j) \leqslant \rho^{\varepsilon/2} \, r_j\}, \end{aligned}$$

where $r_j = r(B_j)$.

Set

$$A = \left[K \cap \left(\bigcup_{k \in S} 2RQ^k \right) \right] \cup \left[K \cap \left(\bigcup_{j \in T_1} Q_j \right) \right].$$

We know, by Lemma 9.23, that

$$K \backslash A = \bigcup_{j \in T_2} (K \cap Q_j).$$

Inequality (9.16) yields, using that $\sum_{k\in S}\omega^*(Q^k) \leqslant 1$,

$$\mathcal{H}_\infty^{1+\varepsilon}\left(K \cap \left(\bigcup_{k\in S} 2RQ^k\right)\right) \lesssim (2R)^{1+\varepsilon} \sum_{k\in S} \ell(Q^k)^{1+\varepsilon}$$

$$\leqslant \frac{R^{1+\varepsilon}}{(M\,c_0)^{1+\varepsilon}} \sum_{k\in S} \omega^*(Q^k)^{1+\varepsilon} \leqslant \left(\frac{R}{M\,c_0}\right)^{1+\varepsilon} \leqslant \frac{\eta}{2}$$

for M big enough. By Lemma 6.20 with $s = 1+\varepsilon$ and the definition of the radius of $\bar{B}_j$ in the disc construction (9.13) we obtain

$$\mathcal{H}_\infty^{1+\varepsilon}\left(\bigcup_{j\in T_1}(K\cap Q_j)\right) \leqslant \sum_{j\in T_1}\mathcal{H}_\infty^{1+\varepsilon}(K\cap Q_j) \leqslant C_\varepsilon \sum_{j\in T_1} \mathrm{Cap}_L(K\cap Q_j)^{1+\varepsilon}$$

$$= C_\varepsilon \sum_{j\in T_1} r_j\,\rho^\varepsilon = C_\varepsilon \sum_{j\in T_1} r_j\,\rho^{\varepsilon/2}\,\rho^{\varepsilon/2}$$

$$\leqslant C_\varepsilon \sum_{j\in T_1} \rho^{\varepsilon/2}\,\omega^*(\bar{B}_j) \leqslant C_\varepsilon \rho^{\varepsilon/2} \leqslant \frac{\eta}{2}$$

provided ρ is small enough.

We have got $\mathcal{H}_\infty^{1+\varepsilon}(A) < \eta$ and it remains to estimate $\omega(K\backslash A)$. By inequality (9.15)

$$\omega(K\backslash A) = \omega\left(\bigcup_{j\in T_2}(K\cap Q_j)\right) \overset{(9.15)}{\leqslant} \frac{1}{c_0}\sum_{j\in T_2}\omega^*(\bar{B}_j).$$

Now we remark that for $j \in T_2$ we are in the Case 1 of the Section 9.8, that is

$$\frac{\omega^*(\bar{B}_j)}{M^2\log(1/\rho)} \leqslant 2r_j.$$

Indeed, since $\omega^*(\bar{B}_j) \leqslant \rho^{\varepsilon/2} r_j$ it is enough to see that

$$\rho^{\varepsilon/2} \leqslant 2M^2\log(1/\rho),$$

which clearly holds for ρ sufficiently small.

For $z \in \partial\bar{B}_j$, $j \in T_2$, we know by (9.22) that

$$|\nabla G(z)| \leqslant C\,\frac{\omega^*(\bar{B}_j)}{r_j} \leqslant C\,\rho^{\varepsilon/2},$$

so that

$$\log|\nabla G(z)| \leqslant \log C + \frac{\varepsilon}{2}\log\rho \;\leqslant \frac{\varepsilon}{4}\log\rho,$$

for small enough ρ. Hence, for such small ρ,

$$|\log|\nabla G(z)|| \geqslant \frac{\varepsilon}{4}\log(1/\rho).$$

We then get

$$\begin{aligned}
\omega(K\backslash A) &\leqslant \frac{1}{c_0}\sum_{j\in T_2}\omega^*(\bar{B}_j) = \frac{1}{c_0}\sum_{j\in T_2}\int_{\partial\bar{B}_j}|\partial_\nu G|\,d\mathcal{H}^1\\
&\leqslant \frac{C}{c_0\,\varepsilon\log(1/\rho)}\sum_{j\in T_2}\int_{\partial\bar{B}_j}|\partial_\nu G|\,|\log|\nabla G||\,d\mathcal{H}^1\\
&\leqslant \frac{C}{c_0\,\varepsilon\log(1/\rho)}\int_{\Gamma}|\partial_\nu G|\,|\log|\nabla G||\,d\mathcal{H}^1\\
&\overset{(9.29)}{\leqslant} \frac{C}{\varepsilon\,c_0}\,\frac{\log\log(1/\rho)}{\log(1/\rho)},
\end{aligned}$$

due to (9.29). Thus $\omega(K\backslash A) < \eta$ if ρ is small enough. Therefore for fixed $\varepsilon > 0$ and given $\eta > 0$, we can choose M and ρ such that the set A satisfies the desired conclusion.

9.11 Proof of the lemmas

9.11.1 Proof of Lemma 9.21

Since the disc construction is scale invariant, changing scale we may assume that $\ell(Q) = 1$. Let ξ_0 stand for the center of Q.

Proof of a). Denote by μ the equilibrium measure for $(\Omega\cup E)^c$. Since $\Omega\cup E$ an unbounded domain, by (7.9) the Green function $G^\xi(z)$ of the domain $\Omega\cup E$ with pole at ξ can be written in the form

$$\begin{aligned}
G^\xi(z) &= \mathcal{E}^\xi(z) - U_\mu(z) - \int(\mathcal{E}^\xi(w) - U_\mu(w))\,d\omega^z_{\Omega\cup E}(w) \qquad (9.30)\\
&= \frac{1}{2\pi}\int\log\frac{|z-a|}{|z-\xi|}\,d\mu(a) + \frac{1}{2\pi}\iint\log\frac{|w-\xi|}{|w-a|}\,d\mu(a)\,d\omega^z_{\Omega\cup E}(w), \quad z\in\Omega\cup E.
\end{aligned}$$

Note that both measures μ and $\omega^z_{\Omega\cup E}$ are supported in $\partial\Omega\backslash RQ$. From (9.30) it is clear that the Green function can also be written in the form

$$G^\xi(z) = \frac{1}{2\pi}\log\frac{1}{|z-\xi|} + h(z,\xi), \quad z\in\Omega\cup E, \quad \xi\in\Omega\cup E, \qquad (9.31)$$

with

$$\begin{aligned}
h(z,\xi) &= -U_\mu(z) - \int(\mathcal{E}^\xi(w) - U_\mu(w))\,d\omega^z_{\Omega\cup E}(w) \qquad (9.32)\\
&= \frac{1}{2\pi}\iint\log\frac{|w-\xi|\,|z-a|}{|w-a|}\,d\omega^z_{\Omega\cup E}(w)\,d\mu(a), \quad z\in\Omega\cup E, \quad \xi\in\Omega\cup E.
\end{aligned}$$

Note that $h(z, \xi)$ is continuous by Lemma 6.26, and the change of integration order we have used above is well justified by Tonelli's theorem. Using the notation $\nabla = 2\bar{\partial}$, we obtain

$$|\nabla_\xi h(z, \xi)| \leqslant \frac{1}{2\pi} \left| \int_{\partial\Omega \backslash RQ} \frac{1}{\overline{w} - \overline{\xi}} \, d\omega^z_{\Omega \cup E}(w) \right| \leqslant O\left(\frac{1}{R}\right), \quad \xi \in Q, \quad z \in \Omega \cup E. \tag{9.33}$$

Next, for a given $z_0 \in \partial Q$, we wish to estimate $h(z_0, \xi_0)$ from below, where ξ_0 is the center of Q. To this end, note that, for all $a \in \operatorname{supp} \mu \subset \partial\Omega \backslash RQ$, $|z_0 - a| \geqslant \frac{1}{2}(R-1) \geqslant R/4 \geqslant \frac{1}{2}|\xi_0 - z_0|$ (because we assume $R \geqslant 2$), and thus, for all $w \in \partial\Omega \backslash RQ$,

$$|w - a| \leqslant |w - \xi_0| + |\xi_0 - z_0| + |z_0 - a| \leqslant |w - \xi_0| + 3|z_0 - a|.$$

Thus, using the two estimates $|z_0 - a| \geqslant R/4$ and $|w - \xi_0| \geqslant \frac{1}{2}R$, we derive

$$|w - a| \leqslant |w - \xi_0| \frac{|z_0 - a|}{R/4} + 3|z_0 - a| \frac{|w - \xi_0|}{R/2} = \frac{10|w - \xi_0| \, |z_0 - a|}{R}.$$

Hence,

$$\log \frac{|w - \xi_0| \, |z_0 - a|}{|w - a|} \geqslant \log \frac{R}{10}, \quad w \in \partial\Omega \backslash RQ, \quad a \in \partial\Omega \backslash RQ.$$

Plugging this into (9.32), we obtain

$$h(z_0, \xi_0) \geqslant \frac{1}{2\pi} \log \frac{R}{10}. \tag{9.34}$$

Let now μ_E and $\mu_{\bar{B}}$ be the equilibrium measures of E and $\bar{B}$ respectively and set

$$u(z) := \int_{\bar{B}} G^\xi(z) \, d\mu_{\bar{B}}(\xi), \quad v(z) := \int_E G^\xi(z) \, d\mu_E(\xi).$$

For every $z_0 \in \partial Q$ one has

$$\begin{aligned} u(\eta) &= \frac{\gamma_{\bar{B}}}{2\pi} + h(z_0, \xi_0) + O(1/R), \quad \eta \in \bar{B}, \\ v(\eta) &= \frac{\gamma_E}{2\pi} + h(z_0, \xi_0) + O(1/R), \quad \eta \in E, \end{aligned}$$

where the constant in $O(1/R)$ is independent of z_0. To see this just write

$$h(\eta, \xi) = (h(\eta, \xi) - h(\eta, \xi_0)) + (h(\xi_0, \eta) - h(\xi_0, z_0)) + h(z_0, \xi_0),$$

use (9.33), the symmetry of the Green's function and the fact that the equilibrium potential of a regular compact set is equal to the Robin constant on the set (see Lemma 6.26).

Now since $u = v = 0$ on $\partial\Omega \backslash RQ$ we get

$$\begin{aligned} u(z) &= \int_{\partial\widetilde{\Omega}} u(\xi) \, d\omega^z_{\widetilde{\Omega}}(\xi) = \int_{\partial B} u(\xi) \, d\omega^z_{\widetilde{\Omega}}(\xi), && z \in \widetilde{\Omega}, \\ v(z) &= \int_{\partial\Omega} v(\xi) \, d\omega^z_{\Omega}(\xi) = \int_{\partial E} v(\xi) \, d\omega^z_{\Omega}(\xi), && z \in \Omega. \end{aligned}$$

Hence, for $z \notin K \cup Q$,

$$u(z) = \left(\frac{\gamma_{\bar{B}}}{2\pi} + h(z_0, \xi_0) + O(1/R)\right) \omega^z_{\tilde{\Omega}}(\bar{B}),$$
$$v(z) = \left(\frac{\gamma_E}{2\pi} + h(z_0, \xi_0) + O(1/R)\right) \omega^z_{\tilde{\Omega}}(E).$$

Assume for the sake of simplicity that $\xi_0 = 0$. Then by plugging the identity (9.31) into the above definitions of u and v we obtain

$$u(z) = \frac{1}{2\pi} \log \frac{1}{|z|} + \int_{\bar{B}} h(z,\xi)\, d\mu_{\bar{B}}(\xi), \qquad z \notin \bar{B},$$

$$v(z) = \frac{1}{2\pi} \int_E \log \frac{1}{|z-\xi|}\, d\mu_E(\xi) + \int_E h(z,\xi)\, d\mu_E(\xi), \quad z \notin E.$$

Set

$$\varphi(z) := u(z) - v(z) = \int_E \left(\log \frac{1}{|z|} - \log \frac{1}{|z-\xi|}\right) d\mu_E(\xi)$$
$$+ \int_{\bar{B}} h(z,\xi)\, d\mu_{\bar{B}}(\xi) - \int_E h(z,\xi)\, d\mu_E(\xi)$$

Thus, for $z \in \Omega \backslash RQ$,

$$|\varphi(z)| \leqslant \left|\int_E \log \frac{|z-\xi|}{|z|}\, d\mu_E(\xi)\right| + \left|\int_{\bar{B}} (h(z,\xi) - h(z,0))\, d\mu_{\bar{B}}(\xi)\right|$$
$$+ \left|\int_E (h(z,\xi) - h(z,0))\, d\mu_E(\xi)\right| = O\left(\frac{1}{|z|}\right).$$

Therefore

$$u(z) = v(z) + O(1/|z|), \quad z \in \Omega \backslash RQ.$$

Recalling that $\operatorname{Cap}_L(\bar{B}) \overset{(9.13)}{=} \frac{1}{2} \operatorname{Cap}_L(E)^{1+\varepsilon}$ one gets

$$\omega^z_{\tilde{\Omega}}(\bar{B}) = \frac{u(z)}{(2\pi)^{-1}\gamma_{\bar{B}} + h(z_0, 0) + O(1/R)} = \frac{v(z) + O(1/|z|)}{(2\pi)^{-1}(\gamma_E(1+\varepsilon) + \log 2) + h(z_0,0) + O(1/R)}$$
$$= \frac{(\gamma_E + 2\pi h(z_0,0) + O(1/R))\, \omega^z_{\tilde{\Omega}}(E) + O(1/|z|)}{\gamma_E(1+\varepsilon) + \log 2 + 2\pi h(z_0,0) + O(1/R)}.$$

Clearly there exists R_0 such that for $R > R_0$ we have

$$\omega^z_{\tilde{\Omega}}(\bar{B}) \geqslant \frac{1}{2} \frac{\gamma_E + 2\pi h(z_0,0)}{\gamma_E(1+\varepsilon) + \log 2 + 2\pi h(z_0,0)}\, \omega^z_{\tilde{\Omega}}(E) + O\left(\frac{1}{|z|}\right),$$

since the denominator $\gamma_E(1+\varepsilon) + \log 2 + 2\pi h(z_0,0)$ is bounded below away from 0 by (9.34). Appealing again to (9.34) we obtain that, whenever $\varepsilon < \frac{1}{8}$ and $\log \frac{R}{10} \geqslant 4 \log 2$, then

$$\frac{\gamma_E + 2\pi h(z_0,0)}{\gamma_E(1+\varepsilon) + \log 2 + 2\pi h(z_0,0)} \geqslant \frac{1}{4}$$

and so
$$\omega^z_{\tilde{\Omega}}(\bar{B}) \geqslant \frac{1}{4}\omega^z_{\tilde{\Omega}}(E) + O\left(\frac{1}{|z|}\right).$$
Letting $z \to \infty$ completes the proof of a) in the lemma. □

Proof of b). Assume that $\xi_0 = 0$ and let $U = \{|z| < R/2\}$. The Green function of U is
$$G^w(\xi) = \log \left| \frac{1 - \frac{w}{R/2}\frac{\bar{\xi}}{R/2}}{\frac{w}{R/2} - \frac{\xi}{R/2}} \right|, \tag{9.35}$$
see Section 3.4 and Exercise 3.2.1. Let $G_{\bar{B}}$ be the Green function of $U \backslash \bar{B}$ and G_E the Green function of $U \backslash E$. We claim that
$$G^z_{\bar{B}}(\xi) - G^z(\xi) = \int_{\partial B} G^w(\xi)\, d\omega^z_{U\backslash \bar{B}}(w), \quad z, \xi \in U \backslash \bar{B}. \tag{9.36}$$
On one hand, both sides are harmonic and continuous up to the boundary. On the other hand, if z tends to a point in ∂U both sides converge to 0, while $z \to z_0 \in \partial B$ implies that both sides converge to $G^{z_0}(\xi)$. By the maximum principle both sides must coincide. Analogously one obtains
$$G^z_E(\xi) = G^z(\xi) - \int_{\partial E} G^w(\xi)\, d\omega^z_{U\backslash E}(w), \quad z, \xi \in U \backslash E. \tag{9.37}$$

Consider a relatively open subset A of $\partial\Omega \backslash RQ$. We want to prove
$$\omega^z(A) \leqslant \widetilde{\omega}^z(A), \quad |z| = \frac{R}{4}, \tag{9.38}$$
where $\omega^z(A) = \omega^z_{\tilde{\Omega}}(A)$ and $\widetilde{\omega}^z(A) = \omega^z_{\tilde{\Omega}}(A)$. Take a point z_0 with $|z_0| = \frac{R}{4}$ such that
$$\sup_{|z|=R/2} \frac{\omega^z(A)}{\widetilde{\omega}^z(A)} = \frac{\omega^{z_0}(A)}{\widetilde{\omega}^{z_0}(A)}.$$
Assume, to get a contradiction, that $\frac{\omega^{z_0}(A)}{\widetilde{\omega}^{z_0}(A)} = \lambda > 1$. Then, since $A \subset \partial\Omega$ is both relatively open and relatively closed, by Corollary 5.36 we get
$$\lim_{z \to \xi \in A} \lambda\widetilde{\omega}^z(A) - \omega^z(A) = \lambda - 1 > 0,$$
and
$$\lim_{z \to \xi \in A^c} \lambda\widetilde{\omega}^z(A) - \omega^z(A) = 0,$$
while we have
$$\lambda\widetilde{\omega}^z(A) - \omega^z(A) \geqslant 0, \quad |z| = \frac{R}{4}.$$
The maximum principle yields
$$\lambda\widetilde{\omega}^z(A) - \omega^z(A) > 0, \quad z \in \partial U.$$

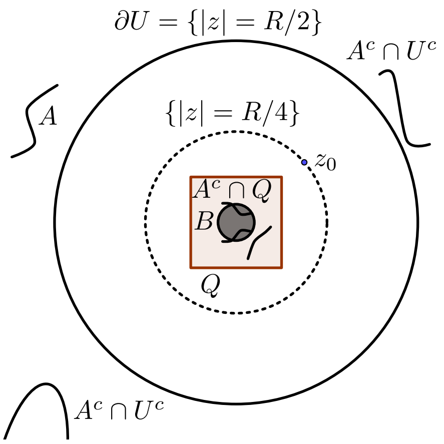


Figure 9.4: Disposition in the proof of *b)*.

Since $\omega^{\xi}(A)$ is a harmonic function on $U\backslash E$ vanishing on ∂E (by Corollary 5.36 again) and, similarly, $\widetilde{\omega}^{\xi}(A)$ is a harmonic function on $U\backslash\bar{B}$ vanishing on ∂B, we get,

$$\begin{aligned}0 &= \lambda\widetilde{\omega}^{z_0}(A) - \omega^{z_0}(A)\\ &= \frac{1}{2\pi}\int_{\partial U}\lambda\,\widetilde{\omega}^{\xi}(A)\,d\omega^{z_0}_{U\backslash\bar{B}}(\xi) - \frac{1}{2\pi}\int_{\partial U}\omega^{\xi}(A)\,d\omega^{z_0}_{\Omega\backslash E}(\xi).\end{aligned}$$

Now, since ∂U is analytic, ∇G is continuous in a neighborhood of ∂U and we can apply Proposition 7.14 to get $d\omega^{z_0}_{\Omega\backslash E}|_{\partial U} = \left|\partial_\nu G^{z_0}_E\right| d\mathcal{H}^1|_{\partial U}$. Thus,

$$0 \overset{\text{P.7.14}}{=} \frac{1}{2\pi}\int_{\partial U}\left|\partial_\nu G^{z_0}_{\bar{B}}(\xi)\right|\lambda\,\widetilde{\omega}^{\xi}(A)\,d\mathcal{H}^1(\xi) - \frac{1}{2\pi}\int_{\partial U}\left|\partial_\nu G^{z_0}_E(\xi)\right|\,\omega^{\xi}(A)\,d\mathcal{H}^1(\xi).$$

We will prove below the inequality

$$\left|\partial_\nu G^{z}_{\bar{B}}(\xi)\right| \geqslant \left|\partial_\nu G^{z}_E(\xi)\right|, \quad |z| = \frac{R}{4}, \quad \xi \in \partial U. \tag{9.39}$$

Using this fact, we obtain

$$0 \overset{(12.38)}{\geqslant} \frac{1}{2\pi}\int_{\partial U}\left|\partial_\nu G^{z_0}_{\bar{B}}(\xi)\right|\left(\lambda\widetilde{\omega}^{\xi}(A) - \omega^{\xi}(A)\right)\,d\mathcal{H}^1(\xi) > 0,$$

which is a contradiction. Then (9.38) holds.

By (9.38) and the maximum principle, $\omega^z(A) \leqslant \widetilde{\omega}^z(A)$ for $z \in \Omega$ and $|z| \geqslant \frac{R}{4}$, and letting $|z| \to \infty$, item b) of Lemma 9.21 follows.

It remains to prove (12.38), which follows from

$$G^{z}_{\bar{B}}(\xi) \geqslant G^{z}_E(\xi), \quad |z| = \frac{R}{4}, \quad \frac{3}{8}R \leqslant |\xi| < \frac{1}{2}R. \tag{9.40}$$

Since $G^z_{\bar{B}}(\xi) = G^z_E(\xi)$, $|\xi| = R/2$, then, by the maximum principle, it is enough to show (9.40) for $|\xi| = \frac{3}{8}R$.

We start by proving

$$\log\left(\frac{8}{3}\right) - \frac{C}{R} \leqslant G^w(\xi) \leqslant \log\left(\frac{8}{3}\right) + \frac{C}{R}, \quad |w| \leqslant 1, \quad |\xi| = \frac{3}{8}R, \tag{9.41}$$

where C is a positive constant and R is sufficiently large. We have

$$G^w(\xi) \overset{(9.35)}{=} \log\left(\frac{8}{3}\right) + \log\left|1 - \frac{w\bar{\xi}}{R^2}\right| - \log\left|1 - \frac{w}{\xi}\right|.$$

The absolute value of each of the last two terms is less than or equal to C/R for some constant C and (9.41) follows.

Inserting (9.41) into (9.36) and (9.37) we get

$$G^z_{\bar{B}}(\xi) \geqslant G^z(\xi) - \left(\log\left(\frac{8}{3}\right) + \frac{C}{R}\right)\omega^z_{U\backslash\bar{B}}(\bar{B}), \quad |z| = \frac{R}{4}, \quad |\xi| = \frac{3}{8}R,$$

$$G^z_E(\xi) \leqslant G^z(\xi) - \left(\log\left(\frac{8}{3}\right) - \frac{C}{R}\right)\omega^z_{U\backslash E}(E), \quad |z| = \frac{R}{4}, \quad |\xi| = \frac{3}{8}R.$$

Clearly (9.40) is a consequence of the two preceding inequalities and the following claim.

Claim 9.25. *For R large enough one has*

$$\left(\log\left(\frac{8}{3}\right) + \frac{C}{R}\right)\omega^z_{U\backslash\bar{B}}(\bar{B}) \leqslant \left(\log\left(\frac{8}{3}\right) - \frac{C}{R}\right)\omega^z_{U\backslash E}(E), \quad |z| = \frac{R}{4}.$$

Proof of the Claim. Recall that we are assuming $\ell(Q) = 1$, so that for all compact sets K, $\mathrm{Cap}_L(E) = \mathrm{Cap}_L(K \cap Q) \leqslant 1/\sqrt{2}$ and hence $\gamma_E \geqslant \log\sqrt{2} > 0$.

Moreover

$$\gamma_{\bar{B}} = \gamma_E(1+\varepsilon) + \log 2 > \gamma_E.$$

Let $r = r(B)$ be the radius of $\bar{B}$. The function

$$\log\left(\frac{R/2}{|z|}\right)\frac{1}{\log(R/2) - \log r}, \quad z \in U\backslash\bar{B},$$

is harmonic on $U\backslash\bar{B}$, vanishes on $|z| = R/2$ and is 1 on $|z| = r$. Thus it is precisely $\omega^z_{U\backslash\bar{B}}(\bar{B})$. Since $-\log r(B) = \gamma_{\bar{B}}$ we have

$$\omega^z_{U\backslash\bar{B}}(\bar{B}) = \log\left(\frac{R/2}{|z|}\right)\frac{1}{\log(R/2) + \gamma_{\bar{B}}}, \quad z \in U\backslash\bar{B}. \tag{9.42}$$

We turn now our attention to $\omega^z_{U\backslash E}(E)$. Consider the function

$$f(z) = \int_E \log\frac{R/2}{|z-w|}\,d\mu_E(w)\,\frac{1}{\log(R/2) + \gamma_E} \quad z \in U\backslash E.$$

Since $\int_E \log \frac{1}{|z-w|}\, d\mu_E(w) = \gamma_E$ for $z \in E$, (see Lemma 6.26), we infer that $f(z) = 1$ for $z \in E$.

If $w \in E$, $z \in \partial U$ one has $|z-w| = R/2 + O(1)$ and so

$$\log \frac{R/2}{|z-w|} = -\log\left(1 - \frac{R/2 - |z-w|}{R/2}\right) = -\log(1 + O(1/R)) = O(1/R).$$

We conclude that

$$|f(z)| \leqslant \frac{O(1/R)}{\log(R/2) + \gamma_E}, \quad z \in \partial U,$$

so that the function

$$\widetilde{f}(z) = f(z) - \frac{2C/R}{\log(R/2) + \gamma_E}$$

satisfies $\widetilde{f}(z) \leqslant 1$, $z \in E$, and $\widetilde{f}(z) \leqslant 0$, $z \in \partial U$, for an appropriate large constant C. From Corollary 5.36 and the maximum principle it follows that

$$\widetilde{f}(z) \leqslant \omega^z_{U\backslash E}(E), \quad z \in U\backslash E.$$

To estimate this harmonic measure we write

$$\omega^z_{U\backslash E}(E) \geqslant \frac{-2C}{R(\log R/2 + \gamma_E)} + \frac{1}{\log(R/2) + \gamma_E} \int_E \left(\log \frac{R/2}{|z-w|} - \log \frac{R/2}{|z|}\right) d\mu_E(w)$$
$$+ \frac{1}{\log(R/2) + \gamma_E} \log \frac{R/2}{|z|} = T_1 + T_2 + T_3.$$

By (9.42)

$$T_3 = \frac{1}{\log(R/2) + \gamma_{\bar{B}}} \log \frac{R/2}{|z|} + \left(\frac{1}{\log(R/2) + \gamma_E} - \frac{1}{\log(R/2) + \gamma_{\bar{B}}}\right) \log \frac{R/2}{|z|} = \omega^z_{U\backslash \bar{B}}(\bar{B}) + T_4.$$

For the term T_4 we have

$$T_4 = \frac{\gamma_{\bar{B}} - \gamma_E}{(\log(R/2) + \gamma_E)(\log(R/2) + \gamma_{\bar{B}})} \log \frac{R/2}{|z|} \geqslant \frac{\varepsilon\gamma_E + \log 2}{(\log(R/2) + 2\gamma_E + \log 2)^2},$$

provided $\varepsilon < 1$, because $\gamma_{\bar{B}} \leqslant 2\gamma_E + \log 2$.
For the term T_2 we have

$$|T_2| \leqslant \frac{1}{\log(R/2) + \gamma_E} \int_E \left|\log \frac{|z-w|}{|z|}\right| d\mu_E(\omega)$$

with

$$\log \frac{|z-w|}{|z|} = \log\left(1 + \frac{|z-w| - |z|}{|z|}\right) = \log(1 + O(1/R)) = O(1/R).$$

Hence

$$|T_2| \leqslant \frac{C}{R(\log(R/2) + \gamma_E)} \leqslant \frac{C}{R(\log R + \gamma_E)},$$

because $\gamma_E \geqslant \log\sqrt{2}$. Since $|T_1|$ obviously satisfies the same estimate, we conclude that

$$\omega^z_{U\setminus E}(E) \geqslant \omega^z_{U\setminus\bar B}(\bar B) + \frac{\varepsilon\gamma_E + \log 2}{(\log R + 2\gamma_E)^2} - \frac{C}{R(\log R + \gamma_E)}, \tag{9.43}$$

for some positive constant C.

Recall that the claim is

$$\left(\log\left(\frac{8}{3}\right) + \frac{C}{R}\right)\omega^z_{U\setminus\bar B}(\bar B) \leqslant \left(\log\left(\frac{8}{3}\right) - \frac{C}{R}\right)\omega^z_{U\setminus E}(E), \quad |z| = \frac{R}{4}.$$

From now to the end of the proof of the claim z denotes a point satisfying $|z| = \frac{R}{4}$. By (9.43), for $R \geqslant R_0(\varepsilon)$ we get

$$\omega^z_{U\setminus E}(E) \geqslant \omega^z_{U\setminus\bar B}(\bar B) + C\frac{\varepsilon\gamma_E}{(\log R + \gamma_E)^2},$$

as long as

$$\frac{\log R + \gamma_E}{R} \leqslant C\varepsilon\gamma_E, \tag{9.44}$$

which is clearly true for R large enough, because $\gamma_E \geqslant \log\sqrt{2}$. It is sufficient to show

$$\left(\log\left(\frac{8}{3}\right) + \frac{C}{R}\right)\omega^z_{U\setminus\bar B}(\bar B) \leqslant \left(\log\left(\frac{8}{3}\right) - \frac{C}{R}\right)\left(\omega^z_{U\setminus\bar B}(\bar B) + C\frac{\varepsilon\gamma_E}{(\log R + \gamma_E)^2}\right)$$

or

$$\frac{C\omega^z_{U\setminus\bar B}(\bar B)}{R} \leqslant -\frac{C}{R}\omega^z_{U\setminus\bar B}(\bar B) + \left(\log\left(\frac{4}{3}\right) - \frac{C}{R}\right)C\frac{\varepsilon\gamma_E}{(\log R + \gamma_E)^2},$$

which amounts to, for $R \geqslant R_0(\varepsilon)$,

$$\frac{\omega^z_{U\setminus\bar B}(\bar B)}{R} \leqslant C\,\frac{\varepsilon\gamma_E}{(\log R + \gamma_E)^2}.$$

By (9.42), for $|z| = R/4$, we have

$$\omega^z_{U\setminus\bar B}(\bar B) = \frac{\log 2}{\log(R/2) + \gamma_{\bar B}} = \frac{\log 2}{\log(R/2) + (1+\varepsilon)\gamma_E + \log 2} \leqslant \frac{2}{\log R + \gamma_E}.$$

Then, for $R \geqslant R_0(\varepsilon)$, we get

$$\frac{\omega^z_{U\setminus\bar B}(\bar B)}{R} \leqslant \frac{2}{R(\log R + \gamma_E)} \leqslant C\,\frac{\varepsilon\gamma_E}{(\log R + \gamma_E)^2},$$

where the last inequality is equivalent to (9.44) again, and the claim follows. □

9.11.2 Proof of Lemma 9.24

Recall that $G(z) = G_\Omega^\infty(z)$ stands for the Green function of the domain Ω with pole at infinity and $\Gamma = \bigcup_{j=1}^{N} \Gamma_j$ is a union of finitely many closed Jordan curves Γ_j enclosing disjoint (bounded) Jordan domains Ω_j, with $\Gamma \subset \mathbb{D}$, Γ separating $K = \mathbb{C}^* \backslash \Omega$ from infinity and there are constants c_j, $j = 1, \ldots, N$ such that $\Gamma_j \subset \{G(z) = c_j\}$, $j = 1, \ldots, N$.

We will use complex notation in order to keep ideas simple. Recall that a real-valued function f has Laplacian $\Delta f = 4\bar\partial\partial f = 4\partial\bar\partial f$, gradient $\nabla f = 2\bar\partial f = 2\overline{\partial f}$, and its normal (or any other directional) derivative is

$$\partial_\nu f = \langle \nabla f, \nu\rangle = \langle 2\bar\partial f, \nu\rangle = 2\,\mathrm{Re}\,(\overline{\bar\partial f}\nu) = 2\,\mathrm{Re}\,(\partial f \nu).$$

Moreover, whenever γ is a curve oriented counterclockwise, the tangent vector is $i\nu$, so $\nu d\mathcal{H}^1 = \frac{dz}{i}$. Green's formula in complex notation reads as

$$2\int_\Omega (\bar\partial g - \partial f)\, dm = \int_{\partial\Omega} \left(g\,\frac{dz}{i} + f\,\frac{d\bar z}{i}\right), \tag{9.45}$$

for $f, g \in W^{1,1}(\Omega) \cap C(\bar\Omega)$, see [AIM09, Theorem 2.9.1], for instance.

To study Lemma 9.24, we infer from the discussion above that

$$\int_\Gamma -\partial_\nu G \,\log|\nabla G|\, d\mathcal{H}^1 = -2\,\mathrm{Re} \int_\Gamma \partial G\,\log|\nabla G|\,\nu d\mathcal{H}^1 = 2\,\mathrm{Re}\int_\Gamma \partial G \log(2|\partial G|)\,\frac{dz}{i}.$$

Put in other words, we want to show that

$$2\,\mathrm{Re}\int_\Gamma \partial G\log(2|\partial G|)\,\frac{dz}{i} > -\log 4\pi. \tag{9.46}$$

Note that, replacing K by $\{g \leqslant \varepsilon\}$ for small $\varepsilon > 0$, we can assume Ω is a finitely connected domain with smooth boundary.

Now consider a disc B_R so that $\Gamma \subset B_R$, and write $U_R = B_R \backslash \bigcup_{j=1}^N \overline{\Omega_j}$. By Green's formula we get

$$\begin{aligned}\int_\Gamma \partial G \log(2|\partial G|)\,\frac{dz}{i} &\overset{(9.45)}{=} \int_\Gamma -G\bar\partial\log(2|\partial G|)\frac{d\bar z}{i} \\ &\quad + 2\int_{U_R} G\partial\bar\partial\log(2|\partial G|) - \bar\partial\partial G\log(2|\partial G|)\,dm \\ &\quad + \int_{\partial B_R}\left(\partial G\log(2|\partial G|)\,\frac{dz}{i} + G\bar\partial\log(2|\partial G|)\frac{d\bar z}{i}\right).\end{aligned} \tag{9.47}$$

First note that the first term in the right-hand side of (9.47) is

$$\int_\Gamma G\bar\partial\log(2|\partial G|)\frac{d\bar z}{i} = \sum_{i=1}^N c_i\int_{\Gamma_i}\bar\partial\log(|\partial G|)\frac{d\bar z}{i}.$$

Next we will use the following fact about level curves of harmonic functions:

Lemma 9.26. *Let γ be a smooth Jordan curve contained in a level set of a harmonic function f without critical points. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{\partial\partial f}{\partial f}dz = -1.$$

Proof. Indeed, $\Delta f = 4\bar\partial\partial f$, so ∂f is holomorphic by the Cauchy-Riemann equations. Then, writing $\tilde\gamma = \partial f \circ \gamma$, we get

$$\frac{1}{2\pi i}\int_\gamma \frac{\partial\partial f}{\partial f}dz = \frac{1}{2\pi i}\int_{\tilde\gamma}\frac{d\zeta}{\zeta} = \mathrm{Ind}(\tilde\gamma, 0) = \frac{\mathrm{Var}\arg\partial f}{2\pi} = -\frac{\mathrm{Var}\arg\nabla f}{2\pi}.$$

But the variation of the argument of the gradient along a smooth level curve is precisely 2π, so the lemma follows. □

Note that

$$\partial\log(|\partial G|) = \frac{1}{2}\partial\log(|\partial G|^2) = \frac{\partial\partial G\bar\partial G + \partial G\partial\bar\partial G}{2\partial G\bar\partial G} = \frac{\partial\partial G}{2\partial G}, \tag{9.48}$$

so

$$\int_{\Gamma_i}\bar\partial\log(|\partial G|)\frac{d\bar z}{i} = -\overline{\int_{\Gamma_i}\partial\log(|\partial G|)\frac{dz}{i}} \overset{\text{L.9.26}}{=} \pi.$$

Thus,

$$\int_\Gamma G\bar\partial\log(2|\partial G|)\frac{d\bar z}{i} = \sum_{i=1}^N c_i\int_{\Gamma_i}\bar\partial\log(|\partial G|)\frac{d\bar z}{i} = \pi\sum_{i=1}^N c_i. \tag{9.49}$$

Now we deal with the second integral in the right-hand side of (9.47). Here, since $\bar\partial\partial G = 0$, we only have to deal with $\int_{U_R} G\partial\bar\partial\log(2|\partial G|)\,dm$.

Lemma 9.27. *Let Ω be an open set, and let $f : \Omega \to \mathbb{R}$ a harmonic function. Then ∂f is a holomorphic function which has at most a countable number of zeroes $\{\xi_i\}_{i=1}^\infty \subset \Omega$ without accumulation points in Ω, with multiplicities $\{m_i\}_{i=1}^\infty$, and*

$$\bar\partial\partial\log|\partial f| = \frac{\pi}{2}\sum_{i=1}^\infty m_i\delta_{\xi_i},$$

i.e., for every $\varphi \in C_c(\Omega)$ we have

$$\langle\bar\partial\partial\log|\partial f|, \varphi\rangle = \frac{\pi}{2}\sum_{i=1}^\infty m_i\varphi(\xi_i). \tag{9.50}$$

Note that using real analysis notation, since $\Delta\log|\nabla f| = \Delta\log|\frac{1}{2}\nabla f| = 4\bar\partial\partial\log|\partial f|$, we are claiming that

$$\Delta\log|\nabla f| = 2\pi\sum_{i=1}^\infty m_i\delta_{\xi_i}.$$

Proof. Note that $\Delta f = 4\bar\partial\partial f$, so ∂f is holomorphic by the Cauchy-Riemann equations, and the first assertion is just a compendium of Cauchy local theory basic results. To see (9.50), consider a fixed critical point ξ_i. Note that for ε small enough, $B_\varepsilon(\xi_i)$ contains no other critical points, so we can apply the argument principle to obtain

$$\int_{\partial B_\varepsilon(\xi_i)} \frac{\partial\partial f}{\partial f}\, dz = 2\pi i m_i.$$

Now,

$$\begin{aligned}2\pi m_i\varphi(\xi_i) &= \int_{\partial B_\varepsilon(\xi_i)} \frac{\partial\partial f(z)}{\partial f(z)}\varphi(\xi_i)\,\frac{dz}{i}\\ &= \int_{\partial B_\varepsilon(\xi_i)} \frac{\partial\partial f(z)}{\partial f(z)}\varphi(z)\,\frac{dz}{i} + \mathcal{O}\left(\int_{\partial B_\varepsilon(\xi_i)} \left|\frac{\partial\partial f(z)}{\partial f(z)}\right| |\varphi(\xi_i)-\varphi(z)|\;|dz|\right).\end{aligned}$$

Since $|\varphi(\xi_i)-\varphi(z)| = \mathcal{O}(\varepsilon)$, and $\left|\frac{\partial\partial f(z)}{\partial f(z)}\right| = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ by l'Hôpital's rule, we conclude that

$$2\pi m_i\varphi(\xi_i) = \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(\xi_i)} \frac{\partial\partial f}{\partial f}\varphi\,\frac{dz}{i}.$$

As in (9.48) we have

$$\frac{\partial\partial f}{\partial f} = 2\partial\log|\partial f|, \tag{9.51}$$

so we obtain

$$2\pi m_i\varphi(\xi_i) = 2\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(\xi_i)} \varphi\partial\log|\partial f|\,\frac{dz}{i}.$$

On the other hand we have

$$\log|\partial f| = \mathcal{O}(\log(C\varepsilon^m)),$$

so

$$\lim_{\varepsilon\to 0}\left|\int_{\partial B_\varepsilon(\xi_i)} \bar\partial\varphi\log|\partial f|\frac{d\bar z}{i}\right| \leqslant \lim_{\varepsilon\to 0} m\,\mathcal{O}(\varepsilon\log(\varepsilon)) = 0.$$

All in all, we get

$$2\pi m_i\varphi(\xi_i) = 2\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(\xi_i)}\left(\varphi\partial\log|\partial f|\,\frac{dz}{i} + \bar\partial\varphi\log|\partial f|\frac{d\bar z}{i}\right).$$

Assume that there is an open set $U\subset\Omega$ so that $\operatorname{supp}\varphi\subset U$ and $\{\xi_i\}_{i=1}^\infty\cap U = \{\xi_i\}_{i=1}^N$. We get

$$\sum_{i=1}^N 2\pi m_i\varphi(\xi_i) \overset{(9.45)}{=} -4\lim_{\varepsilon\to 0}\int_{U\setminus\bigcup_{i=1}^N B_\varepsilon(\xi_i)} (\varphi\bar\partial\partial\log|\partial f| - \partial\bar\partial\varphi\log|\partial f|)\,dm.$$

In abscence of critical points, by (9.51) we get

$$\bar{\partial}\partial \log|\partial f| = \frac{\bar{\partial}\partial\partial f \partial f - \partial\partial f \bar{\partial}\partial f}{2(\partial f)^2} = 0,$$

so

$$\sum_{i=1}^{N} 2\pi m_i \varphi(\xi_i) \overset{(9.45)}{=} 4\int_\Omega (\partial\bar{\partial}\varphi \log|\partial f|)\, dm = 4\langle \bar{\partial}\partial \log|\partial f|, \varphi\rangle.$$

□

By the preceding lemma, in U_R there is a finite number of critical points, say $\{\xi_i\}_{i=1}^L$, and the second integral on the right-hand side of (9.47) is

$$2\int_{U_R} G\partial\bar{\partial}\log(2|\partial G|) - \bar{\partial}\partial G \log(2|\partial G|)\, dm = 2\int_{U_R} G\bar{\partial}\partial \log(2|\partial G|)\, dm = \pi \sum_{i=1}^{L} m_i G(\xi_i). \tag{9.52}$$

Next we turn our attention to the last integral in (9.47). We will let R tend to infinity, so we need to understand the asymptotic values of the relevant functions inside the integral. Recall that we can write the Green function G as

$$2\pi\, G(z) = \gamma_K + \int_K \log|z-w|\, d\mu_K(w) = \log|z| + \gamma_K + h_0(z), \tag{9.53}$$

where

$$h_0(z) = \int_K \log\frac{|z-w|}{|z|}\, d\mu_K(w).$$

is harmonic and satisfies $h_0(\infty) = 0$. In fact for $z \notin 2\mathbb{D}$, we get the bound

$$h_0(z) = \int_K \mathcal{O}\left(\frac{1}{|z|}\right) d\mu_K(w) = \mathcal{O}\left(\frac{1}{|z|}\right).$$

Differentiating, we obtain

$$\partial h_0(z) = \frac{1}{2z}\int_K \frac{w}{z-w}\, d\mu_K(w) = \mathcal{O}\left(\frac{1}{|z|^2}\right).$$

and

$$\partial^2 h_0(z) = \frac{-1}{2z^2}\int_K \frac{w}{z-w}\, d\mu_K(w) - \frac{1}{z}\int_K \frac{w}{(z-w)^2}\, d\mu_K(w) = \mathcal{O}\left(\frac{1}{|z|^3}\right).$$

Also

$$\partial G(z) = \frac{1}{4\pi z} + \frac{1}{2\pi}\partial h_0(z) = \frac{1}{4\pi z} + \mathcal{O}\left(\frac{1}{|z|^2}\right),$$

and its logarithm

$$\log(2|\partial G|) = \log\left|\frac{1}{2\pi\bar{z}} + \frac{1}{\pi}\overline{\partial h_0(z)}\right| = \log\left|\frac{1}{2\pi z}\right| + \mathcal{O}\left(\frac{1}{|z|}\right),$$

and finally

$$\bar\partial \log(2|\partial G|) = (\pi\bar z + \mathcal{O}(1))\left(\frac{-1}{2\pi\bar z^2} + \mathcal{O}\left(\frac{1}{|z|^3}\right)\right) = \frac{-1}{2\bar z} + \mathcal{O}\left(\frac{1}{|z|^2}\right).$$

With all this estimates at hand, we get

$$\begin{aligned}\boxed{\boxed{I_{\partial B_R}}} &:= \int_{\partial B_R}\left(\partial G \log(2|\partial G|)\,\frac{dz}{i} + G\bar\partial\log(2|\partial G|)\frac{d\bar z}{i}\right)\\ &= \int_{\partial B_R}\left(\frac{1}{4\pi z} + \mathcal{O}\left(\frac{1}{R^2}\right)\right)\left(\log\left|\frac{1}{2\pi z}\right| + \mathcal{O}\left(\frac{1}{R}\right)\right)\frac{dz}{i}\\ &\quad + \int_{\partial B_R}\frac{1}{2\pi}\left(\log|z| + \gamma_K + \mathcal{O}\left(\frac{1}{R}\right)\right)\left(\frac{-1}{2\bar z} + \mathcal{O}\left(\frac{1}{R^2}\right)\right)\frac{d\bar z}{i}\\ &= \int_{\partial B_R}\left(\frac{1}{4\pi z}\log\left|\frac{1}{2\pi z}\right| + \mathcal{O}\left(\frac{1}{R^{\frac{3}{2}}}\right)\right)\frac{dz}{i} + \int_{\partial B_R}\left(\frac{-1}{4\pi\bar z}\log|z| - \frac{\gamma_K}{4\pi\bar z} + \mathcal{O}\left(\frac{1}{R^{\frac{3}{2}}}\right)\right)\frac{d\bar z}{i}.\end{aligned}$$

In ∂B_R we have $\frac{|dz|}{R} = \frac{dz}{iz} = \frac{d\bar z}{-i\bar z}$, so

$$\boxed{\boxed{I_{\partial B_R}}} = \int_{\partial B_R}\left(\frac{1}{4\pi}\log\left|\frac{1}{2\pi}\right| + \frac{\gamma_K}{4\pi} + \mathcal{O}\left(\frac{1}{R^{\frac{1}{2}}}\right)\right)\frac{|dz|}{R} \xrightarrow{R\to\infty} \frac{\gamma_K}{2} - \frac{\log(2\pi)}{2}. \tag{9.54}$$

All in all, combining (9.47) with (9.49), (9.52) and (9.54) we have obtained

$$\int_\Gamma \partial G\log(2|\partial G|)\,\frac{dz}{i} \overset{(9.45)}{=} -\pi\sum_{j=1}^N c_j + \pi\sum_{i=1}^L m_i G(\xi_i) + \frac{\gamma_K}{2} - \frac{\log(2\pi)}{2}.$$

Note that

$$c_1 = G(\zeta) \overset{(9.53)}{=} \frac{\gamma_K}{2\pi} + \frac{1}{2\pi}\int_K \log|z-w|\,d\mu_K(w) \leqslant \frac{\gamma_K}{2\pi} + \frac{\log 2}{2\pi}.$$

Thus, estimate (9.46) follows immediately from the next claim.

Claim 9.28. *Let $\Gamma = \bigcup_{j=1}^N \Gamma_j$ be the union of finitely many closed Jordan curves Γ_j enclosing disjoint (bounded) Jordan domains Ω_j, with $K \subset \bigcup_j \Omega_j$, and there are constants c_j, $j = 1,\dots,N$ such that $\Gamma_j \subset \{G(z) = c_j\}$, $j = 1,\dots,N$. Then*

$$\sum_{j=1}^N c_j \leqslant \sum_{i=1}^L m_i G(\xi_i) + c_{j_0}, \tag{9.55}$$

for every $j_0 \leqslant N$.

In order to show (9.55), we will simply associate to each critical point ξ_i of multiplicity m_i a total amount of m_i curves Γ_j, such that $c_j \leqslant G(\xi_i)$, and we will do this by applying the argument principle conveniently.

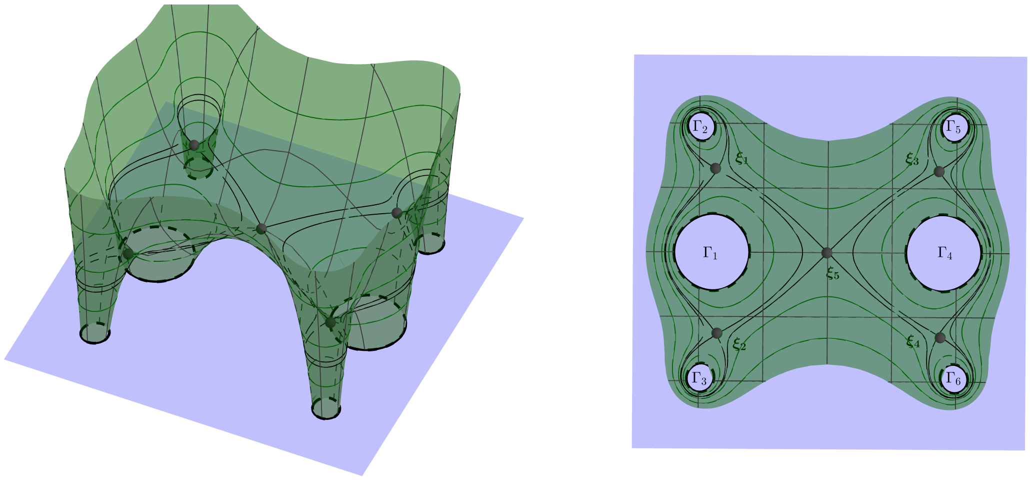


Figure 9.5: Green function with $K = \bigcup \Gamma_i$. In this case, there are five simple critical points, four of them sharing a common level set, but paired two by two in connected components of their level curve.

Recall that we have N Jordan curves Γ_j which bound Jordan domains Ω_j and such that $G|_{\Gamma_j} \equiv c_j$. Let us define $c_0 := \min_{1\leqslant j\leqslant N} c_j$. By Lemma 9.27 the derivative ∂G is holomorphic, and it has a finite number of critical points in $\{G > c_0\}$. Thus, we can find $\delta_0 > 0$ small enough, so that for each ξ_i there is no critical point ξ with $G(\xi_i) < G(\xi) \leqslant G(\xi_i) + \delta_0$. In particular, the level set $\{z : G(z) = G(\xi_i) + \delta_0\}$ is a finite union of smooth Jordan curves, and there exists a component Γ^i of this level set enclosing a Jordan domain Ω^i so that $\xi_i \in \Omega^i$. Note that we cannot grant that $\Gamma^i \subset \mathbb{D}$.

Note that several ξ_i may give rise to the same domain Ω^i (and the same level Jordan curve Γ^i), but for this to happen it must be that $G(\xi_{i_1}) = G(\xi_{i_2})$. For this reason, we may change our enumeration, so that we have a finite family of Jordan domains Ω^i bounded by Jordan curves Γ^i with $i \in \{1, \cdots, M\}$ (here $M \leqslant L$) satisfying that $G|_{\Gamma^i} = c^i$, and critical points $\{\xi_{i,k}\}_{k=1}^{N_i}$ with multiplicity $\{m_{i,k}\}_{k=1}^{N_i}$ so that $\xi_{i,k} \in \Omega^i$ and $G(\xi_{i,k}) = c^i - \delta_0$.

With this enumeration, we have a family of Jordan domains

$$\{\Omega_j\}_{j=1}^N \cup \{\Omega^i\}_{i=1}^M,$$

whose boundaries $\{\Gamma_j\}_{j=1}^N \cup \{\Gamma^i\}_{i=1}^M$, are smooth Jordan curves included in level sets of the Green function of levels

$$\{c_j\}_{j=1}^N \cup \{c^i\}_{i=1}^M.$$

Moreover, the domains are either disjoint or one is included in the other one.

Next we partition into a disjoint family of domains: let

$$\widetilde{\Omega}^i := \Omega^i \backslash \bigcup_{j=1}^N \overline{\Omega_j} \backslash \bigcup_{\ell=1}^M \overline{\Omega^\ell}.$$

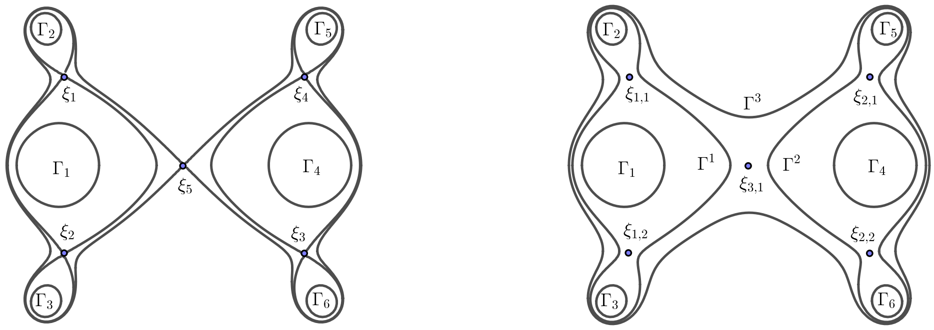


Figure 9.6: We pick level sets surrounding critical points, and then we rename the elements to create the tree structure.

Each of this new domains is bounded by a finite number of Jordan level curves, it contains the critical points $\{\xi_{i,k}\}_{k=1}^{N_i}$ and no other critical point. For this reason and by the maximum principle, one can infer that every Γ_j of level c_j such that $\Gamma_j \subset \partial\widetilde{\Omega}^i$, satisfies that $c_j \leqslant G(\xi_{i,k})$, and the same can be said about $\Gamma^j \subset \partial\widetilde{\Omega}^i$ whenever $j \neq i$. By Lemma 9.26 and the argument principle, we get

$$\#\{\text{components of } \partial\widetilde{\Omega}^i \text{ different from } \Gamma^i\} - 1 = \frac{1}{2\pi i}\int_{\partial\widetilde{\Omega}^i} \frac{\partial^2 G}{\partial G} dz = \sum_{k=1}^{N_i} m_{i,k}.$$

Thus, if we write $m_i := \sum_{k=1}^{N_i} m_{i,k}$, then

$$\#\{\text{components of } \partial\widetilde{\Omega}^i \text{ different from } \Gamma^i\} = m_i + 1.$$

Next we can create a graph of inclusion: The graph has nodes Γ^i and leaves Γ_j. a node or a leave is said to be a direct descendant of a node Γ^i if it is a component of $\partial\widetilde{\Omega}^i$ different from Γ^i. Each node Γ^i has exactly $m_i + 1$ descendants, as we have discussed. Moreover each descendant has level

$$c_j < G(\xi_{i,1}) < c^i \quad \text{ or } c^\ell < G(\xi_{i,1}) < c^i. \tag{9.56}$$

Since the graph has no loops, it is a tree. Here we begin an inductive pruning process.

Assume first that there are no critical points. Then it must be $N = 1$, and (9.55) holds trivially.

Otherwise, since the number of nodes and leaves is finite, we can find a node Γ^1 of multiplicity m_1 which only has leaves as direct descendants. Then, we can create a new tree by cutting away all the leaves which are descendant to Γ^1, say $\{\Gamma_j\}_{j=1}^{m_1+1}$, and convert Γ^1 to a leave of the new tree corresponding to the family of curves $\widetilde{\Gamma} = \{\widetilde{\Gamma}_j\}_{j=1}^{N-m_1}$, that is, a smaller number of level Jordan curves satisfying the hypothesis of Claim 9.28:

$$\widetilde{\Gamma}_1 = \Gamma^1 \quad \text{ and } \quad \widetilde{\Gamma}_j = \Gamma_{j+m_1} \text{ for } 2 \leqslant j \leqslant N - m_1,$$

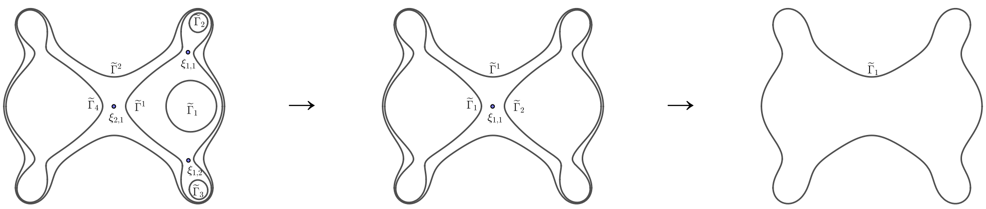


Figure 9.7: Inductive pruning process. In the first step we prune Γ^1 to get a reduced tree with four leaves and two nodes of multiplicities $\widetilde{m}_1 = 2$ and $\widetilde{m}_2 = 1$, then pruning Γ^2 we get two leaves and one simple node and finally pruning Γ^3 we are left with just one leave and no nodes, that is, without critical points.

with levels

$$\widetilde{c}_1 = c^1 \quad \text{ and } \quad \widetilde{c}_j = c_{j+m_1} \text{ for } 2 \leqslant j \leqslant N - m_1,$$

which satisfies (9.55) by induction hypothesis. The nodes of the new tree will be $\{\Gamma^i\}_{i=2}^L$, with multiplicities $\{m_i\}_{i=2}^L$ and (9.56) will be satisfied as well. Then in case $j_0 \leqslant m_1 + 1$ we can assume that $j_0 = 1$ and then

$$\sum_{j=1}^N c_j \overset{(9.56)}{\leqslant} c_{j_0} + m_1 G(\xi_1) + \sum_{j=m_1+2}^N c_j \leqslant m_1 G(\xi_1) + \sum_{j=1}^{N-m_1} \widetilde{c}_j + (c_{j_0} - \widetilde{c}_1).$$

Applying the induction hypothesis (9.55) with $j_0 = 1$, we get

$$\sum_{j=1}^N c_j \leqslant m_1 G(\xi_1) + \sum_{i=2}^L m_i G(\xi_i) + \widetilde{c}_1 + (c_{j_0} - \widetilde{c}_1).$$

If, instead, the singular index $j_0 > m_1 + 1$, then we just bound $c_1 \leqslant c^1 = \widetilde{c}_1$ and apply the induction hypothesis (9.55) with $j_0 = j_0 - m_1$

$$\sum_{j=1}^N c_j \overset{(9.56)}{\leqslant} m_1 G(\xi_1) + \sum_{j=1}^{N-m_1} \widetilde{c}_j \leqslant m_1 G(\xi_1) + \sum_{i=2}^L m_i G(\xi_i) + c_{j_0}.$$

□

10 Ahlfors regular domains

10.1 Some types of domains

In this chapter we will study the connection between harmonic measure and surface measure for some types of domains $\Omega \subset \mathbb{R}^d$ with finite surface $\mathcal{H}^{d-1}|_{\partial\Omega}$.

For $m > 0$, we say that a measure μ on $\mathbb{R}^d$ is m-Ahlfors regular if there exists some constant $C > 0$ such that

$$C^{-1}r^m \leqslant \mu(B_r(x)) \leqslant C\, r^m \quad \text{ for all } x \in \operatorname{supp}\mu \text{ and } 0 < r \leqslant \operatorname{diam}(\operatorname{supp}\mu).$$

In the case $m = d-1$, quite often we will just say that μ is Ahlfors regular. A set $E \subset \mathbb{R}^d$ is a called m-Ahlfors regular if the measure $\mathcal{H}^m|_E$ is m-Ahlfors regular.

For an easy notation, in this chapter we will set $d = n+1$ and we will work in $\mathbb{R}^{n+1}$. A domain $\Omega \subset \mathbb{R}^{n+1}$ whose boundary is n-Ahlfors regular is called an Ahlfors regular domain. Below we will consider $C^{1+\gamma}$ domains (with $\gamma \in (0,1)$), Lipschitz domains, and chord-arc domains. For simplicity we will assume all to be bounded. All of them are Ahlfors regular domains. In fact, it holds

$$C^{1,\gamma} \text{ domains } \subset \text{ Lipschitz domains } \subset \text{ chord-arc domains } \subset \text{ Ahlfors regular domains.}$$

Next we will define $C^{1,\gamma}$, Lipschitz, and chord-arc domains. First, a chord-arc domain is an NTA Ahlfors regular domain. To introduce Lipschitz domains takes some more work. We say that $Z \subset \mathbb{R}^{n+1}$ is a (τ,ℓ)-cylinder if there is an orthonormal coordinate system $x = (\bar{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$Z = \{(\bar{x}, x_{n+1}) : |\bar{x}| \leqslant \tau, |x_{n+1}| \leqslant 10\ell\tau\}.$$

Also, for all $s > 0$, we denote

$$sZ = \{(\bar{x}, x_{n+1}) : |\bar{x}| \leqslant s\tau, |x_{n+1}| \leqslant 10s\ell\tau\}.$$

We say that Ω is a Lipschitz domain with Lipschitz character (ℓ, N, C_0) is there is $r_0 > 0$ and at most N (τ,ℓ)-cylinders Z_j, $j = 1,\dots,N$, with $C_0^{-1}r_0 \leqslant \tau \leqslant C_0 r_0$ such that

(i) $8Z_j \cap \partial\Omega$ is the graph of a Lipschitz function A_j with $\|\nabla A_j\|_\infty \leqslant \ell$, $A_j(0) = 0$, in the coordinate system associated with Z_j,

(ii) $\partial\Omega = \bigcup_j (Z_j \cap \partial\Omega)$,

(iii) and
$$8Z_j \cap \Omega = \{(\bar{x}, x_{n+1}) \in 8Z_j : x_{n+1} > A_j(\bar{x})\}, \tag{10.1}$$
in the coordinate system associated with Z_j.

We also say that Ω is a *Lipschitz domain* with Lipschitz constant ℓ.

On the other hand we say that $\Omega \subset \mathbb{R}^{n+1}$ is a special Lipschitz domain if there is a coordinate system $x = (\bar{x}, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ and a Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Omega = \{(\bar{x}, x_{n+1}) : x_{n+1} > A(\bar{x})\}.$$

For $0 < \gamma \leqslant 1$, $\Omega \subset \mathbb{R}^{n+1}$ is a $C^{1,\gamma}$ domain if it is a Lipschitz domain such the Lipschitz functions A_j in (i), (ii), (iii) above are of class $C^{1,\gamma}$, and their derivatives are γ-Hölder uniformly on j. That is, there exists some constant C such that, for all j and all $\bar{x}, \bar{y} \in 8Z_j \cap \mathbb{R}^n$ (in the local coordinate system for Z_j),

$$|\nabla A_j(\bar{x}) - \nabla A_j(\bar{y})| \leqslant C\, |\bar{x} - \bar{y}|^\gamma.$$

Now we will prove a lemma which can be considered as a variant of Liouville's theorem for harmonic functions in a half-space. This lemma will play an important role in the study of harmonic measure both in $C^{1,\gamma}$ and Lipschitz domains.

Lemma 10.1. *Let u be a positive harmonic function in the upper half space $H = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ and continuous in $\overline{H}$ which vanishes in ∂H. Then there exists some constant $\lambda > 0$ such that*

$$u(x) = \lambda\, x_{n+1} \qquad \text{for all } x \in H.$$

Proof. Let $x_0 = e_{n+1}$. We choose $\lambda = u(x_0)$ and we let $v(x) = \lambda\, x_{n+1}$ for $x \in \overline{H}$. Since both u and v are positive and harmonic in H and vanish continuously in ∂H, by the boundary Harnack principle applied to $H \cap B_r(0)$ (see Theorem 8.17) with arbitrarily large $r > 0$, we have that $u(x) \approx v(x)$ for all $x \in H$. Thus, u grows at most linearly at ∞.

Since u vanishes in ∂H, it can be extended by reflection to the lower half space. Next we use the fact that that any harmonic function in $\mathbb{R}^{n+1}$ satisfying $|u(x)| \leqslant C(1 + |x|)$ in $\mathbb{R}^{n+1}$ is a polynomial of degree at most 1, by Proposition 2.13. From this fact one easily gets that $u = \lambda\, x_{n+1}$. $\square$

10.2 $C^{1,\gamma}$ domains

Our first result is the following.

Theorem 10.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded $C^{1,\gamma}$ domain, with $0 < \gamma < 1$. For all $x_0 \in \Omega$, the harmonic measure ω^{x_0} for Ω and the surface measure $\sigma = \mathcal{H}^n|_{\partial\Omega}$ are mutually absolutely continuous, and moreover the density $\frac{d\omega^{x_0}}{d\sigma}$ is bounded and bounded away from 0. That is, there exists some constant $C > 0$ such that*

$$C^{-1} \leqslant \frac{d\omega^{x_0}}{d\sigma}(\xi) \leqslant C \qquad \text{for } \sigma\text{-a.e. } \xi \in \partial\Omega. \tag{10.2}$$

Further, the Green function for Ω satisfies

$$|\nabla G^{x_0}(x)| \leqslant C \qquad \text{for all } x \in \Omega \backslash B(x_0, \tfrac{1}{2}\mathrm{d}_\Omega(x_0)). \tag{10.3}$$

The constant C in both inequalities only depends on γ, the $C^{1,\gamma}$ character of Ω, $\operatorname{diam}(\Omega)$, *and* $\mathrm{d}_\Omega(x_0)$.

Before going into the proof of the theorem, we will introduce Jones' β coefficients used to measure the flatness of sets. Given a set $E \subset \mathbb{R}^{n+1}$, a ball $B := B_r(x) \subset \mathbb{R}^{n+1}$, and a hyperplane $L \subset \mathbb{R}^{n+1}$, we let

$$\beta_{\infty,E}(B, L) = \beta_{\infty,E}(x, r, L) = \sup_{y \in E \cap B_r(x)} \frac{\operatorname{dist}(y, L)}{r}. \tag{10.4}$$

We recall also the notion of Hausdorff distance: Given two sets $E, F \subset \mathbb{R}^{n+1}$, we set

$$\operatorname{dist}_H(E, F) = \max\Big(\sup_{x \in E} \operatorname{dist}(x, F), \sup_{y \in F} \operatorname{dist}(y, E)\Big).$$

This is the so-called Hausdorff distance between E and F.

We will use the following auxiliary fact:

Lemma 10.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded $C^{1,\gamma}$ domain with $0 < \gamma < 1$. Then there exist constants $\gamma' > 0$ and $r_0 > 0$ such that, for all $\xi \in \partial\Omega$, if L_ξ denotes the tangent hyperplane to $\partial\Omega$ at ξ, we have*

$$\beta_{\infty,\partial\Omega}(\xi, r, L_\xi) \leqslant \left(\frac{r}{r_0}\right)^{\gamma'} \quad \text{for } 0 < r \leqslant r_0. \tag{10.5}$$

the constants γ', r_0 depend on γ and the $C^{1,\gamma}$ character of Ω.

The proof is standard and we leave this for the reader.

Proof of Theorem 10.2. Denote $\omega = \omega^{x_0}$ and $G = G^{x_0}$. We will show that there exists some $d_0 > 0$ depending on γ, the $C^{1,\gamma}$ character of Ω, $\operatorname{diam}(\Omega)$, and $d_\Omega(x_0)$ such that

$$G(x) \approx d_\Omega(x) \quad \text{for all } x \in \Omega \text{ with } d_\Omega(x) \leqslant d_0, \tag{10.6}$$

with the implicit constant depending also on γ, the $C^{1,\gamma}$ character of Ω, $\operatorname{diam}(\Omega)$, and $d_\Omega(x_0)$. Since Ω is an NTA domain, this condition implies that, for any surface ball $\Delta \subset \partial\Omega$ with radius $r(\Delta) \leqslant d_0$,

$$\omega(\Delta) \approx G(X_\Delta)\, r(\Delta)^{n-1} \approx r(\Delta)^{n-1}\, d_\Omega(X_\Delta) \approx r(\Delta)^n, \tag{10.7}$$

where X_Δ is a corkscrew point for Δ such that $r(\Delta) \lesssim d_\Omega(X_\Delta) \leqslant r(\Delta)$. So any small enough surface ball Δ satisfies $\omega(\Delta) \approx \sigma(\Delta)$. By the Radon-Nykodim-Lebesgue differentiation theorem, this implies (10.2). Further, from (10.6) and interior Caccioppoli estimates, (10.3) follows easily too.

Let r_0 be as in (10.5). Below we will choose $d_0 \leqslant \min(r_0/10, d_\Omega(x_0)/2)$. To prove (10.6) for a given $x \in \Omega$ with $d_\Omega(x) \leqslant d_0$, let $\xi \in \partial\Omega$ be such $|x - \xi| = d_\Omega(x)$ and, for each $k \geqslant 1$ such that $2^k d_\Omega(x) \leqslant d_0$, denote $B_k = B(\xi, 2^k d_\Omega(x))$ and $\Delta_k = B_k \cap \partial\Omega$. Also, let $y_k \in \Omega$ be a corkscrew point for Δ_k such that $y_k \in B_k$ and $d_\Omega(y_k) \approx 2^k d_\Omega(x)$. Without loss of generality, suppose that $\xi = 0$ and that the tangent hyperplane L_ξ to $\partial\Omega$ in ξ is horizontal. It is immediate to check that, for each k,

$$d_\Omega(y_k) \approx r(B_k) \approx y_{k,n+1},$$

where $y_{k,n+1}$ is the vertical component of y_k.

We wish to estimate

$$\left|\frac{G(y_k)}{y_{k,n+1}} - \frac{G(y_{k-1})}{y_{k-1,n+1}}\right|$$

for $1 \leqslant k \leqslant k_0$, where k_0 will be fixed in a moment. To this end, consider the ball $\widetilde{B}_k$ concentric with B_k and radius

$$r(\widetilde{B}_k) = \left(r(B_k)^{1+\frac{\gamma'}{2}}\, r_0^{\frac{\gamma'}{2}}\right)^{\frac{1}{1+\gamma'}} = 2^{k\frac{1+\frac{\gamma'}{2}}{1+\gamma'}}\left(\mathrm{d}_\Omega(x)^{1+\frac{\gamma'}{2}}\, r_0^{\frac{\gamma'}{2}}\right)^{\frac{1}{1+\gamma'}}, \tag{10.8}$$

with γ' as in (10.5). Notice that $B_k \subset \widetilde{B}_k$. Let k_0 be the maximal integer such that $r(\widetilde{B}_{k_0}) \leqslant d_0$, so that $d_0/2 < r(\widetilde{B}_{k_0}) \leqslant d_0$. Denote $\widetilde{\beta}_k = \beta_{\infty,\partial\Omega}(\xi, r(\widetilde{B}_k), L_\xi)$ and (for fixed k) let $h : \Omega \cap \widetilde{B}_k \to \mathbb{R}$ be the solution of the Dirichlet problem in $\Omega \cap \widetilde{B}_k$ with boundary data

$$h(\zeta) = \begin{cases} 0 & \text{if } \zeta_{n+1} \leqslant \widetilde{\beta}_k\, r(\widetilde{B}_k), \\ \zeta_{n+1} - \widetilde{\beta}_k\, r(\widetilde{B}_k) & \text{if } \zeta_{n+1} > \widetilde{\beta}_k\, r(\widetilde{B}_k), \end{cases}$$

for $\zeta \in \partial(\Omega \cap \widetilde{B}_k)$, where ζ_{n+1} is the $(n+1)$ component of ζ. Remark that the boundary data is continuous and $\Omega \cap \widetilde{B}_k$ is Wiener regular. Notice also that h vanishes in $\partial\Omega \cap \widetilde{B}_k$ and that

$$|h(z) - z_{n+1}| \leqslant \widetilde{\beta}_k\, r(\widetilde{B}_k) \quad \text{for all } z \in \Omega \cap \widetilde{B}_k, \tag{10.9}$$

by the maximum principle, since this inequality holds in the boundary of $\Omega \cap \widetilde{B}_k$ and the function $f(z) := h(z) - z_{n+1}$ is harmonic in that domain. Next, observe that

$$\widetilde{\beta}_k\, r(\widetilde{B}_k) \leqslant r_0^{-\gamma'}\, r(\widetilde{B}_k)^{\gamma'+1} = r(B_k)\left(\frac{r(B_k)}{r_0}\right)^{\frac{\gamma'}{2}} \leqslant r(B_k)\left(\frac{d_0}{r_0}\right)^{\frac{\gamma'}{2}} \ll r(B_k) \tag{10.10}$$

if we assume $d_0 \ll r_0$. Since $y_{k,n+1} \approx y_{k-1,n+1} \approx r(B_k)$, from (10.9) we infer that

$$h(y_k) \approx y_{k,n+1} \approx y_{k-1,n+1} \approx h(y_{k-1}). \tag{10.11}$$

We write

$$\begin{aligned}\left|\frac{G(y_k)}{y_{k,n+1}} - \frac{G(y_{k-1})}{y_{k-1,n+1}}\right| &\leqslant \left|\frac{G(y_k)}{h(y_k)} - \frac{G(y_{k-1})}{h(y_{k-1})}\right| + G(y_k)\left|\frac{1}{h(y_k)} - \frac{1}{y_{k,n+1}}\right| \\ &\quad + G(y_{k-1})\left|\frac{1}{h(y_{k-1})} - \frac{1}{y_{k-1,n+1}}\right|.\end{aligned} \tag{10.12}$$

By (10.9), (10.11), a Harnack chain estimate, and (10.10), the second and third term on the right hand side of (10.12) satisfy

$$\begin{aligned}G(y_k)&\left|\frac{1}{h(y_k)} - \frac{1}{y_{k,n+1}}\right| + G(y_{k-1})\left|\frac{1}{h(y_{k-1})} - \frac{1}{y_{k-1,n+1}}\right| \\ &\lesssim G(y_k)\frac{\widetilde{\beta}_k\, r(\widetilde{B}_k)}{y_{k,n+1}^2} + G(y_{k-1})\frac{\widetilde{\beta}_k\, r(\widetilde{B}_k)}{y_{k,n+1}^2} \\ &\approx G(y_k)\frac{\widetilde{\beta}_k\, r(\widetilde{B}_k)}{y_{k,n+1}^2} \leqslant G(y_k)\frac{r(B_k)}{y_{k,n+1}^2}\left(\frac{r(B_k)}{r_0}\right)^{\frac{\gamma'}{2}} \approx \frac{G(y_k)}{y_{k,n+1}}\left(\frac{r(B_k)}{r_0}\right)^{\frac{\gamma'}{2}}\end{aligned}$$

Finally, to deal with the first term on the right hand side of (10.12), we use Corollary 8.19:

$$\left|\frac{G(y_k)}{h(y_k)} - \frac{G(y_{k-1})}{h(y_{k-1})}\right| \lesssim \frac{G(y_k)}{h(y_k)}\left(\frac{|y_k - y_{k-1}|}{r(\widetilde{B}_k)}\right)^{\alpha} \lesssim \frac{G(y_k)}{y_{k,n+1}}\left(\frac{r(B_k)}{r(\widetilde{B}_k)}\right)^{\alpha},$$

where $\alpha \in (0,1)$ is some constant depending on the NTA character of Ω. Recalling the choice of $r(\widetilde{B}_k)$ in (10.8), we get

$$\left|\frac{G(y_k)}{h(y_k)} - \frac{G(y_{k-1})}{h(y_{k-1})}\right| \lesssim \frac{G(y_k)}{y_{k,n+1}}\left(\frac{r(B_k)}{r_0}\right)^{\frac{\alpha\gamma'}{2\gamma'+2}}$$

Putting altogether, since $\gamma'' := \frac{\alpha\gamma'}{2\gamma'+2} < \frac{\gamma'}{2}$, we derive

$$\left|\frac{G(y_k)}{y_{k,n+1}} - \frac{G(y_{k-1})}{y_{k-1,n+1}}\right| \lesssim \frac{G(y_k)}{y_{k,n+1}}\left(\frac{r(B_k)}{r_0}\right)^{\gamma''},$$

or equivalently,

$$\frac{G(y_k)}{y_{k,n+1}}\left(1 - \left(\frac{r(B_k)}{r_0}\right)^{\gamma''}\right) \leqslant \frac{G(y_{k-1})}{y_{k-1,n+1}} \leqslant \frac{G(y_k)}{y_{k,n+1}}\left(1 + \left(\frac{r(B_k)}{r_0}\right)^{\gamma''}\right).$$

Since

$$\sum_{k=1}^{k_0}\left(\frac{r(B_k)}{r_0}\right)^{\gamma''} < \infty,$$

we deduce that

$$\frac{G(x)}{x_{n+1}} \approx \frac{G(y_1)}{y_{1,n+1}} \approx \frac{G(y_{k_0})}{y_{k_0,n+1}} \approx \frac{G(y_{k_0})}{\mathrm{d}_\Omega(y_{k_0,n+1})} \approx \frac{\omega(B(\xi,d_0))}{d_0^n},$$

arguing as in (10.7) for the last estimate. As $\frac{\omega(B(\xi,d_0))}{d_0^n} \approx 1$ (with constants depending on $\mathrm{d}_\Omega(x_0)$, d_0, $\mathrm{diam}(\Omega)$, and the NTA character of Ω), the theorem follows. □

Remark 10.4. By inspection of the proof above, one can check that the following holds. If $\Omega \subset \mathbb{R}^{n+1}$ is an NTA domain, $x_0 \in \Omega$, $\xi \in \partial\Omega$, and there exists a hyperplane $L_\xi \ni \xi$ such that, for some $\gamma' > 0$ and $r_0 > 0$,

$$\beta_{\infty,\partial\Omega}(\xi, r, L_x) \leqslant \left(\frac{r}{r_0}\right)^{\gamma'} \quad \text{for } 0 < r \leqslant r_0,$$

then

$$0 < \liminf_{r\to 0}\frac{\omega^{x_0}(B(\xi,r))}{(2r)^n} \leqslant \limsup_{r\to 0}\frac{\omega^{x_0}(B(\xi,r))}{(2r)^n} < \infty.$$

The lim inf and lim sup above are called the lower and upper n-dimensional densities of ω^{x_0} at ξ, respectively.

Our next goal is to prove that, for a $C^{1,\gamma}$ domain $\Omega \subset \mathbb{R}^{n+1}$, the density function $\frac{d\omega^{x_0}}{d\sigma}$ is γ-Hölder continuous:

Theorem 10.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded $C^{1,\gamma}$ domain, with $0 < \gamma < 1$. For all $x_0 \in \Omega$, the density $\frac{d\omega^{x_0}}{d\sigma}$ belongs to C^γ. Further, $G^{x_0} \in C^{1,\gamma}(\Omega \backslash B(x_0, \frac{1}{2}\mathrm{d}_\Omega(x_0)))$.*

Remark that if $G^{x_0} \in C^{1,\gamma}(\Omega \backslash B(x_0, \frac{1}{2}\mathrm{d}_\Omega(x_0)))$, then the derivatives of G^{x_0} extend continuously to $\partial\Omega$, and thus $G^{x_0} \in C^1(\overline{\Omega} \backslash \bar{B}(x_0, \frac{1}{2}\mathrm{d}_\Omega(x_0)))$. Then, as shown in Proposition 7.7,

$$\omega^{x_0} = -(\partial_\nu G^{x_0})\,\sigma,$$

where ν is the unit outer normal to $\partial\Omega$. Therefore, for σ-a.e. $\xi \in \partial\Omega$,

$$\frac{d\omega^{x_0}}{d\sigma}(\xi) = -(\partial_\nu G^{x_0})(\xi) = -\langle \nabla G^{x_0}(\xi), \nu(\xi)\rangle.$$

Using that both ∇G^{x_0} and ν are γ-Hölder continuous and bounded in $\partial\Omega$, it follows immediately that $\frac{d\omega^{x_0}}{d\sigma}(\xi)$ is Hölder continuous. Hence, to prove Theorem 10.5 it suffices to show that $G^{x_0} \in C^{1,\gamma}(\Omega \backslash B(x_0, \frac{1}{2}\mathrm{d}_\Omega(x_0)))$. To do so, we will use PDE techniques.

For a function $f : E \to \mathbb{R}$ (or a vector field $f : E \to \mathbb{R}^d$) and $\gamma > 0$, we consider the seminorm

$$\|f\|_{\dot{C}^\gamma(E)} = \sup_{x,y \in E, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}.$$

We will prove the following result:

Theorem 10.6. *For $\gamma \in (0,1)$, let $\Omega \subset \mathbb{R}^{n+1}$ be a $C^{1,\gamma}$ bounded domain and let $u : \overline{\Omega} \to \mathbb{R}$ be harmonic in Ω and continuous in $\overline{\Omega}$. Let $f \in C^{1,\gamma}(\overline{\Omega})$ be such that $u = f$ on $\partial\Omega$ and suppose that $u \in C^{1,\gamma}(\overline{\Omega})$. Then*

$$\|\nabla u\|_{\dot{C}^\gamma(\overline{\Omega})} \lesssim \|\nabla u\|_{\infty,\overline{\Omega}} + \|\nabla f\|_{\dot{C}^\gamma(\partial\Omega)} + \|\nabla f\|_{\infty,\partial\Omega},$$

with the implicit constant depending on γ and the $C^{1,\gamma}$ character of Ω.

Remark 10.7. The a priori assumption that $u \in C^{1,\gamma}(\overline{\Omega})$ can be removed in Theorem 10.6, by an approximation argument and using suitable interpolation inequalities between different norms. See for example [GT01, Chapter 6]. However, to study harmonic measure in $C^{1,\gamma}$ domains we will only apply Theorem 10.6 in the particular case when u is the Green function for Ω. This will allow to use somewhat simpler arguments in the proof of Theorem 10.5.

The main step to prove Theorem 10.6 is the following.

Lemma 10.8. *For $\gamma \in (0,1)$, let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded $C^{1,\gamma}$ domain with $\mathrm{diam}(\Omega) > 1$. Let B_1 be a ball of radius 1 centered in $\partial\Omega$ and let $B_{1/2}$ be a concentric ball with radius $1/2$. For all $\delta > 0$, there exists some positive constant $C(\delta)$ such that for all functions $u, f \in C^{1,\gamma}(\overline{\Omega \cap B_1})$, with u harmonic in $\Omega \cap B_1$ and $u = f$ in $\partial\Omega \cap B_1$, it holds*

$$\|\nabla u\|_{\dot{C}^\gamma(B_{1/2}\cap\Omega)} \leqslant \delta\,\|\nabla u\|_{\dot{C}^\gamma(B_1\cap\Omega)} + C(\delta)\big(\|\nabla u\|_{\infty,B_1\cap\Omega} + \|\nabla f\|_{\dot{C}^\gamma(B_1\cap\partial\Omega)} + \|\nabla f\|_{\infty,B_1\cap\partial\Omega}\big).$$

Proof. Assume that the lemma does not hold. Then there exists some $\delta > 0$ such that for every $k > 1$ there are $C^{1,\alpha}$ domains $\Omega_k \subset \mathbb{R}^{n+1}$ (with a uniform $C^{1,\alpha}$ character) with $\operatorname{diam}(\Omega_k) > 1$ and functions $u_k, f_k \in C^{1,\gamma}(\overline{\Omega \cap B_1})$, with u_k harmonic in $\Omega \cap B_1$ and $u_k = f_k$ in $\partial\Omega_k \cap B_1$, so that

$$\begin{aligned}\|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2}\cap\Omega_k)} &> \delta\,\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)} \\ &\quad + k\left(\|\nabla u_k\|_{\infty,B_1\cap\Omega_k} + \|\nabla f_k\|_{\dot{C}^\gamma(B_1\cap\partial\Omega_k)} + \|\nabla f_k\|_{\infty,B_1\cap\partial\Omega_k}\right).\end{aligned} \tag{10.13}$$

Claim 10.9. *There are points $x_k, y_k \in B_{3/4} \cap \Omega_k$ such that*

$$|x_k - y_k| \approx d_{\Omega_k}(x_k) \approx d_{\Omega_k}(y_k) \tag{10.14}$$

and

$$\frac{|\nabla u_k(x_k) - \nabla u_k(y_k)|}{|x_k - y_k|^\gamma} \gtrsim \|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2}\cap\Omega_k)}. \tag{10.15}$$

Further, $|x_k - y_k| \lesssim k^{-1/\gamma}$.

Assume for the moment the claim to hold and denote $\rho_k = |x_k - y_k|$. Notice that $\rho_k \to 0$ as $k \to \infty$. Consider the domain $\widetilde{\Omega}_k = \rho_k^{-1}(\Omega_k - x_k)$. Clearly, $0 \in \widetilde{\Omega}_k$ for each k. Further, the fact that $d_{\Omega_k}(x_k) \approx \rho_k$ (by (10.14)), implies that $d_{\widetilde{\Omega}_k}(0) \approx 1$. Equivalently, there is some fixed constant $R > 0$ such that $\partial\widetilde{\Omega}_k \cap B_R(0) \neq \varnothing$ for all $k \geqslant 1$. Then, up to a subsequence we can assume that $\overline{\widetilde{\Omega}_k}$ and $\partial\widetilde{\Omega}_k$ converge locally in Hausdorff distance, respectively, to a domain H and a closed set $F = \partial H$. By Lemma 10.3, it follows that F is hyperplane and so H is a half-space containing the origin.

Consider the polynomial

$$p_k(z) = u_k(x_k) + \rho_k \nabla u_k(x_k) \cdot z,$$

and, for $\widetilde{\Omega}_k = \rho_k^{-1}(\Omega_k - x_k)$, let $\widetilde{u}_k : \widetilde{\Omega}_k \to \mathbb{R}$ be defined by

$$\widetilde{u}_k(z) = \frac{u_k(x_k + \rho_k z) - p_k(z)}{\rho_k^{1+\gamma}\,\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)}}.$$

Observe that

$$u_k(0) = \nabla u_k(0) = 0,$$

and for all $z \in \widetilde{\Omega}_k$ and k big enough so that $x_k + \rho_k z \in B_1$,

$$|\nabla\widetilde{u}_k(z)| = \frac{|\nabla u_k(x_k + \rho_k z) - \nabla u_k(x_k)|}{\rho_k^\gamma\,\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)}} \leqslant \frac{\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)}|\rho_k z|^\gamma}{\rho_k^\gamma\,\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)}} = |z|^\gamma. \tag{10.16}$$

Also, for all $x, y \in \widetilde{\Omega}_k$ and k big enough so that both $x_k + \rho_k y$ and $x_k + \rho_k y$ are in B_1,

$$\begin{aligned}|\nabla\widetilde{u}_k(x) - \nabla\widetilde{u}_k(y)| &= \frac{|\nabla u_k(x_k + \rho_k x) - \nabla u_k(x_k + \rho_k y)|}{\rho_k^\gamma\,\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)}} \\ &\leqslant \frac{\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)}|\rho_k x - \rho_k y|^\gamma}{\rho_k^\gamma\,\|\nabla u_k\|_{\dot{C}^\gamma(B_1\cap\Omega_k)}} = |x - y|^\gamma.\end{aligned} \tag{10.17}$$

From the above conditions it follows that $\widetilde{u}_k$ is a locally bounded equicontinuous family of harmonic functions, and by the Ascoli-Arzelà theorem, up to a subsequence, the functions u_k converge to another harmonic function $\widetilde{u} : H \to \mathbb{R}$ locally in $C^{1,\gamma}$ norm in compact subsets of H. From the above estimates, we infer that

$$\widetilde{u}(0) = \nabla\widetilde{u}(0) = 0, \tag{10.18}$$

for all $z \in H$,

$$|\nabla\widetilde{u}(z)| \leqslant |z|^\gamma, \tag{10.19}$$

and for all $x, y \in H$,

$$|\nabla\widetilde{u}(x) - \nabla\widetilde{u}(y)| \leqslant |x - y|^\gamma. \tag{10.20}$$

On the other hand, by (10.15) and (10.13), the point $\xi_k = \rho_k^{-1}(y_k - x_k)$ satisfies

$$|\nabla\widetilde{u}_k(\xi_k)| = \frac{|\nabla u_k(y_k) - \nabla u_k(x_k)|}{\rho_k^\gamma \, \|\nabla u_k\|_{\dot{C}^\gamma(B_1 \cap \Omega_k)}} \gtrsim \frac{\|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2} \cap \Omega_k)}}{\|\nabla u_k\|_{\dot{C}^\gamma(B_1 \cap \Omega_k)}} \geqslant \delta.$$

Notice also that $|\xi_k| = 1$ and $\mathrm{d}_{\widetilde{\Omega}_k}(\xi_k) \approx 1$, by (10.14). Hence, up to a subsequence, ξ_k converges to some point $\xi \in H$ such that

$$|\xi| = 1, \qquad |\nabla\widetilde{u}(\xi)| \gtrsim \delta. \tag{10.21}$$

From the conditions (10.18), (10.19), and (10.20), it follows that $\widetilde{u}$ and $\nabla\widetilde{u}$ can be extended continuously to the whole $\overline{H}$. We intend to show that $\nabla\widetilde{u}$ is constant in ∂H, which will lead to a contradiction. To this end, let $\widetilde{f}_k : \overline{\widetilde{\Omega}_k} \to \mathbb{R}$ be defined by

$$\widetilde{f}_k(z) = \frac{f_k(x_k + \rho_k z) - p_k(z)}{\rho_k^{1+\gamma} \, \|\nabla u_k\|_{\dot{C}^\gamma(B_1 \cap \Omega_k)}},$$

so that $\widetilde{u}_k|_{\partial\widetilde{\Omega}_k} = \widetilde{f}_k|_{\partial\widetilde{\Omega}_k}$. Denote by ∇_{T_k} and ∇_T the respective tangential gradients in $\partial\widetilde{\Omega}_k$ and ∂H. That is, $\nabla_{T_k} g(z) = \nabla g(z) - \nu_k(z)\,(\nu_k(z) \cdot \nabla g(z))$ for any function g and $z \in \partial\widetilde{\Omega}_k$, where ν_k is the outer unit normal of $\widetilde{\Omega}_k$. We define $\nabla_T g(z)$ for $z \in \partial H$ analogously. From the definitions of $\widetilde{u}_k$ and $\widetilde{f}_k$, we deduce that $\nabla_{T_k}\widetilde{u}_k = \nabla_{T_k}\widetilde{f}_k$.

Remark that since ∂H is a hyperplane, the outer unit normal is constant. Then, for any $z \in \overline{\widetilde{\Omega}_k}$, it makes sense to consider the "tangential gradient" $\nabla_T\widetilde{u}_k(z) = \nabla\widetilde{u}_k(z) - \nu(z)\,(\nu(z) \cdot \nabla\widetilde{u}_k(z))$, where ν is the outer unit normal of H. Next we intend to estimate $|\nabla_T\widetilde{u}_k(x) - \nabla_T\widetilde{u}_k(y)|$ for $x, y \in \partial H$:

$$\begin{aligned} &|\nabla_T\widetilde{u}_k(x) - \nabla_T\widetilde{u}_k(y)| \\ &\qquad \leqslant |\nabla_T\widetilde{u}_k(x) - \nabla_{T_k}\widetilde{u}_k(x)| + |\nabla_{T_k}\widetilde{f}_k(x) - \nabla_{T_k}\widetilde{f}_k(y)| + |\nabla_{T_k}\widetilde{u}_k(y) - \nabla_T\widetilde{u}_k(y)|. \end{aligned} \tag{10.22}$$

We estimate the first and third terms on the right hand side using (10.16):

$$\begin{aligned} |\nabla_T\widetilde{u}_k(x) - \nabla_{T_k}\widetilde{u}_k(x)| + |\nabla_{T_k}\widetilde{u}_k(y) - \nabla_T\widetilde{u}_k(y)| &\leqslant (|\nu_k(x) - \nu| + |\nu_k(y) - \nu|)\,\|\nabla\widetilde{u}_k\|_{\infty,\overline{\widetilde{\Omega}_k}} \\ &\leqslant |\nu_k(x) - \nu| + |\nu_k(y) - \nu|. \end{aligned}$$

Regarding the middle term on the right hand side of (10.22), by the definition of $\widetilde{f}_k$ we have

$$
\begin{aligned}
|\nabla_{T_k}\widetilde{f}_k(x) - \nabla_{T_k}\widetilde{f}_k(y)| \\
&\leqslant \frac{|\rho_k \nabla_{T_k} f_k(x_k+\rho_k x) - \rho_k \nabla_{T_k} f_k(x_k+\rho_k y)| + |\nabla_{T_k} p_k(x) - \nabla_{T_k} p_k(y)|}{\rho_k^{1+\gamma}\,\|\nabla u_k\|_{\dot C^\gamma(B_1\cap\Omega_k)}} \\
&= \frac{|\nabla_{T_k} f_k(x_k+\rho_k x) - \nabla_{T_k} f_k(x_k+\rho_k y)|}{\rho_k^{\gamma}\,\|\nabla u_k\|_{\dot C^\gamma(B_1\cap\Omega_k)}} + \frac{|\nabla_{T_k(x)} u_k(x_k) - \nabla_{T_k(y)} u_k(x_k)|}{\rho_k^{\gamma}\,\|\nabla u_k\|_{\dot C^\gamma(B_1\cap\Omega_k)}} \\
&=: S_1 + S_2.
\end{aligned}
$$

To we deal with S_1 we write, using (10.13), for k big enough,

$$
\begin{aligned}
S_1 &\leqslant \frac{|\nabla f_k(x_k+\rho_k x) - \nabla f_k(x_k+\rho_k y)| + |\nu_k(x)-\nu_k(y)|\,|\nabla f_k(y)|}{\rho_k^{\gamma}\,\|\nabla u_k\|_{\dot C^\gamma(B_1\cap\Omega_k)}} \\
&\leqslant \frac{\|\nabla f_k\|_{\dot C^\gamma(B_1\cap\partial\Omega_k)}|x-y|^\gamma}{\|\nabla u_k\|_{\dot C^\gamma(B_1\cap\Omega_k)}} + \frac{|\nu_k(x)-\nu_k(y)|\,\|\nabla f_k\|_{\infty,B_1\cap\partial\Omega_k}}{\rho_k^{\gamma}\,\|\nabla u_k\|_{\dot C^\gamma(B_1\cap\Omega_k)}} \\
&\leqslant \frac{|x-y|^\gamma}{k} + \frac{|\nu_k(x)-\nu_k(y)|}{k\,\rho_k^\gamma}.
\end{aligned}
$$

Denoting by ν_{Ω_k} the outer unit normal to Ω_k and using that Ω_k is $C^{1,\gamma}$ with a uniform character, we deduce that

$$
|\nu_k(x)-\nu_k(y)| = |\nu_{\Omega_k}(x_k+\rho_k x) - \nu_{\Omega_k}(x_k+\rho_k y)| \lesssim \rho_k^\gamma\,|x-y|^\gamma. \tag{10.23}
$$

So we get

$$
S_1 \lesssim \frac{|x-y|^\gamma}{k}.
$$

To estimate S_2 we use again (10.23) and (10.13) assuming k big enough:

$$
S_2 \leqslant \frac{|\nu_k(x)-\nu_k(y)|\,\|\nabla u_k\|_{\infty,B_1\cap\Omega_k}}{\rho_k^{\gamma}\,\|\nabla u_k\|_{\dot C^\gamma(B_1\cap\Omega_k)}} \lesssim \frac{|x-y|^\gamma}{k}.
$$

Putting altogether, we obtain

$$
|\nabla_T\widetilde{u}_k(x) - \nabla_T\widetilde{u}_k(y)| \leqslant |\nu_k(x)-\nu| + |\nu_k(y)-\nu| + C\,\frac{|x-y|^\gamma}{k}.
$$

Suppose now that $x,y\in B_M(0)$, for some fixed $M>10$. Using the fact that the domains Ω_k are $C^{1,\gamma}$, it is easy to check that then, up to a subsequence,

$$
|\nu_k(x)-\nu| + |\nu_k(y)-\nu| \leqslant \varepsilon_k,
$$

where $\varepsilon_k\to 0$ as $k\to\infty$. Hence,

$$
|\nabla_T\widetilde{u}_k(x) - \nabla_T\widetilde{u}_k(y)| \lesssim \varepsilon_k + \frac{M^\gamma}{k} \quad \text{for all } x,y\in\partial\widetilde{\Omega}_k\cap B_M(0). \tag{10.24}
$$

For a fixed $M > 10$ and any small $\tau > 0$, let us consider the neighborhood $V_\tau = U_\tau(\partial H) \cap B_M(0)$ and let us estimate $|\nabla_T \widetilde{u}_k(x) - \nabla_T \widetilde{u}_k(y)|$ for $x, y \in V_\tau \cap \widetilde{\Omega}_k$. Assume k to be large enough so that $\partial\widetilde{\Omega}_k \cap B(0, M) \subset V_\tau$ and $\partial H \cap B_M(0) \subset U_\tau(\partial\widetilde{\Omega}_k)$. Then, there exist $x', y' \in \partial\widetilde{\Omega}_k$ such that

$$|x - x'| \leqslant 2\tau \quad \text{and} \quad |y - y'| \leqslant 2\tau.$$

We split

$$\begin{aligned} &|\nabla_T \widetilde{u}_k(x) - \nabla_T \widetilde{u}_k(y)| \\ &\qquad \leqslant |\nabla_T \widetilde{u}_k(x) - \nabla_T \widetilde{u}_k(x')| + |\nabla_T \widetilde{u}_k(x') - \nabla_T \widetilde{u}_k(y')| + |\nabla_T \widetilde{u}_k(y') - \nabla_T \widetilde{u}_k(y)|. \end{aligned}$$

By (10.24), the middle term on the right hand side is bounded above by $C\varepsilon_k + C\frac{M^\gamma}{k}$. On the other hand, we can bound the first term using (10.17):

$$|\nabla_T \widetilde{u}_k(x) - \nabla_T \widetilde{u}_k(x')| \leqslant |\nabla \widetilde{u}_k(x) - \nabla_T \widetilde{u}_k(x')| \leqslant |x - x'|^\gamma \lesssim \tau^\gamma.$$

The third term is estimated in the same way. So we have

$$|\nabla_T \widetilde{u}_k(x) - \nabla_T \widetilde{u}_k(y)| \lesssim \tau^\gamma + \varepsilon_k + \frac{M^\gamma}{k}.$$

Letting $k \to \infty$, we deduce that

$$|\nabla_T \widetilde{u}(x) - \nabla_T \widetilde{u}(y)| \lesssim \tau^\gamma \quad \text{for all } x, y \in H \cap V_\tau..$$

By continuity, the same estimate holds for all $x, y \in \partial H \cap B_M(0)$. Since τ can be taken arbitrarily small and M arbitrarily large, we deduce that $\nabla_T \widetilde{u}$ is constant in ∂H, as wished.

Since $\nabla_T \widetilde{u}$ is constant in the hyperplane ∂H, there exists a first degree polynomial $p(z)$ such that $\widetilde{u} - p$ vanishes identically on ∂H. Now we can argue as in the proof of Lemma 10.1: we can extend the function $w := \widetilde{u} - p$ by reflection to the whole $\mathbb{R}^{n+1}$. The extension, which we still denote by w, is harmonic and by (10.19), $|\nabla w(z)| \leqslant C(1 + |z|)^\gamma$ in $\mathbb{R}^{n+1}$. By the mean value theorem, it follows that $|w(z)| \leqslant C(1 + |z|)^{1+\gamma}$, and then by Proposition 2.13 we deduce that w is a polynomial of degree at most 1. This implies that the gradient of $\widetilde{u} - p$, and so the one of $\widetilde{u}$, is constant in H, which contradicts the fact that $\nabla \widetilde{u}(0) = 0$ and $|\nabla \widetilde{u}(\xi)| \gtrsim \delta$, by (10.18) and (10.21). □

To conclude the proof of Lemma 10.8 it remains to prove Claim 10.9.

Proof of Claim 10.9. Let $a_k, b_k \in B_{1/2} \cap \Omega_k$ be such that

$$\frac{|\nabla u_k(a_k) - \nabla u_k(b_k)|}{|a_k - b_k|^\gamma} \geqslant \frac{1}{2} \|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2} \cap \Omega_k)}.$$

Denote $\ell_k = |a_k - b_k|$. From (10.13) it easily follows that $\ell_k \to 0$ as $k \to \infty$. Indeed, this implies

$$\|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2} \cap \Omega_k)} \leqslant 2\, \frac{|u_k(a_k) - u_k(b_k)|}{|a_k - b_k|^\gamma} \leqslant \frac{4\, \|\nabla u_k\|_{\infty, B_1 \cap \Omega_k}}{\ell_k^\gamma} \leqslant \frac{4\, \|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2} \cap \Omega_k)}}{k\, \ell_k^\gamma}, \tag{10.25}$$

and so $\ell_k \lesssim k^{-1/\gamma} \to 0$ as $k \to \infty$.

Observe now that if B is some ball such that $2B \subset \Omega_k$ with center x_B, from the harmonicity of u_k and the subharmonicity of $|\nabla u_k - \nabla u_k(x_B)|$ in $2B$, we deduce that, for all $x, y \in B$,

$$\begin{aligned}\frac{|\nabla u_k(x) - \nabla u_k(y)|}{|x-y|^\gamma} &\lesssim \frac{|\nabla u_k(x) - \nabla u_k(y)|\, r(B)^{1-\gamma}}{|x-y|} \leqslant \|\nabla^2 u_k\|_{\infty,B}\, r(B)^{1-\gamma} \qquad (10.26)\\ &\lesssim \|\nabla u_k - \nabla u_k(x_B)\|_{\infty,1.1B}\, r(B)^{-\gamma} \lesssim \max_{z\in\partial 1.2B} \frac{|\nabla u_k(z) - \nabla u_k(x_B)|}{r(B)^\gamma}.\end{aligned}$$

Suppose first that $|a_k - b_k| \geqslant \frac{1}{10}\,\mathrm{d}_{\Omega_k}(a_k)$. Consider a non-tangential path Γ joining a_k and b_k. We cover Γ by a family of ball $B_j = B_{r_j}(z_j)$, $j \in J$, so that the balls $\frac{1}{5}B_j$ are pairwise disjoint, with $r_j = \frac{1}{10}\mathrm{d}_{\Omega_k}(z_j)$. Notice that, for every $j \in J$,

$$r_j = \frac{1}{10}\mathrm{d}_{\Omega_k}(z_j) \leqslant \frac{1}{10}(\mathrm{d}_{\Omega_k}(a_k) + |a_k - z_j|) \leqslant \frac{1}{10}(\mathrm{d}_{\Omega_k}(a_k) + \mathcal{H}^1(\Gamma)) \leqslant 2\mathcal{H}^1(\Gamma). \qquad (10.27)$$

By the triangle inequality we have

$$|\nabla u_k(a_k) - \nabla u_k(b_k)| \leqslant \sum_{j\in J} \sup_{x,y\in B_j} |\nabla u_k(x) - \nabla u_k(y)|.$$

We claim that there exists some j such that

$$\sup_{x,y\in B_j} |\nabla u_k(x) - \nabla u_k(y)| \gtrsim \|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2}\cap\Omega_k)}\, r_j^\gamma. \qquad (10.28)$$

Indeed, suppose that for each j the supremum above is bounded by $\lambda\,\|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2}\cap\Omega_k)}\, r_j^\gamma$, for some small $\lambda > 0$ to be fixed below. For $i \geqslant 0$, let $\{B_j\}_{j\in J_i}$ be the family of the balls B_j such that $2^{-i}\mathcal{H}^1(\Gamma) < r(B_j) \leqslant 2^{-i+1}\,\mathcal{H}^1(\Gamma)$. It is easy to check that $\# J_i \lesssim 1$, with the implicit constant depending on the NTA character of Ω_k (see, for example, the proof of Lemma 8.6). Then we have

$$\begin{aligned}\|\nabla u_k\|^{-1}_{\dot{C}^\gamma(B_{1/2}\cap\Omega_k)} |\nabla u_k(a_k) - \nabla u_k(b_k)| &\leqslant \lambda \sum_{j\in J} r_j^\gamma \leqslant \lambda \sum_{i\geqslant 1} \# J_i\, (2^{-i+1}\mathcal{H}^1(\Gamma))^\gamma\\ &\lesssim \lambda \sum_{i\geqslant 0}\sum_{j\in J_i} (2^{-i+1}\mathcal{H}^1(\Gamma))^\gamma \approx \lambda\, \mathcal{H}^1(\Gamma))^\gamma \approx \lambda\, |a_k - b_k|^\gamma,\end{aligned}$$

which leads to a contradiction if λ is small enough.

Let $j \in J$ be such that (10.28) holds. From (10.26) we deduce that

$$\max_{z\in\partial 1.5B_j} \frac{|\nabla u_k(z) - \nabla u_k(z_j)|}{r(B_j)^\gamma} \gtrsim \|\nabla u_k\|_{\dot{C}^\gamma(B_{1/2}\cap\Omega_k)}.$$

We choose $x_k = z_j$ and we let y_k be the point in $\partial 1.5B_j$ which attains the maximum above. Since

$$\max(|x_k - a_k|, |y_k - a_k|) \lesssim \mathcal{H}^1(\Gamma) \lesssim |a_k - b_k| = \ell_k \to 0 \quad \text{ as } k \to \infty,$$

it follows that $a_k, b_k \in B_{3/4}$ for k large enough. Also, by (10.27), $|x_k - y_k| = r(B_j) \leqslant 2\mathcal{H}^1(\Gamma) \lesssim \ell_k \lesssim k^{-1/\gamma}$. It is easy to check that x_k and y_k satisfy the other properties required in the claim, too.

Consider now the case when $|a_k - b_k| < \frac{1}{10}\,\mathrm{d}_{\Omega_k}(a_k) =: d_a$. From (10.26) applied to the ball $B_{d_a/10}(a_k)$ we obtain

$$\frac{|\nabla u_k(a_k) - \nabla u_k(b_k)|}{|a_k - b_k|^\gamma} \lesssim \max_{z \in \partial B_{d_a/8}(a_k)} \frac{|\nabla u_k(z) - \nabla u_k(a_k)|}{d_a^\gamma}.$$

We take $x_k = a_k$ and we choose $y_k \in \partial B_{d_a/8}(a_k)$ to be a point where the maximum on the right hand side is attained, so that (10.15) holds. Notice that, since $d_a \leqslant 1/2$ and $a_k \in B_{1/2}$, we have $y_k \in B_{3/4}$. The same argument as in (10.25) shows that $|x_k - y_k| \lesssim k^{-1/\gamma}$. The property (10.14) is also easily checked. □

Proof of Theorem 10.6. Let τ be the constant in the definition of Lipschitz and $C^{1,\gamma}$ domains. Consider two points $x, y \in \overline{\Omega}$ and suppose first that $|x - y| \geqslant \tau/10$. Then we write

$$|\nabla u(x) - \nabla u(y)| \leqslant 2\,\|\nabla u\|_{\infty,\overline{\Omega}} \lesssim \|\nabla u\|_{\infty,\overline{\Omega}}\,\frac{|x-y|^\gamma}{\tau^\gamma}.$$

Suppose now that $|x - y| < \tau/10$ and $\mathrm{d}_\Omega(x) \geqslant \tau/5$. In this case, $y \in B_x := B_{\tau/10}(x)$ and $2B_x \subset \Omega$. So by interior Caccioppoli estimates, since ∇u is harmonic in $2B_x$,

$$|\nabla u(x) - \nabla u(y)| \leqslant \|\nabla^2 u\|_{\infty,B_x}\,|x - y| \lesssim \|\nabla u\|_{\infty,2B_x}\,\frac{|x-y|}{\tau} \leqslant \|\nabla u\|_{\infty,\overline{\Omega}}\,\frac{|x-y|^\gamma}{\tau^\gamma}.$$

In the case $|x-y| < \tau/10$ and $\mathrm{d}_\Omega(x) < \tau/5$, we apply Lemma 10.8 to the ball $B = B_\tau(\xi_x)$, where $\xi_x \in \partial\Omega$ satisfies $|x - \xi_x| = \mathrm{d}_\Omega(x)$. Since $x, y \in \frac{1}{2}B$, we derive

$$\begin{aligned}|\nabla u(x) - \nabla u(y)| &\leqslant \|\nabla u\|_{\dot C^\gamma(\frac{1}{2}B\cap\Omega)}\,|x-y|^\gamma\\ &\leqslant \Big[\delta\,\|\nabla u\|_{\dot C^\gamma(\overline{\Omega})} + C(\delta)\big(\|\nabla u\|_{\infty,\overline{\Omega}} + \|\nabla f\|_{\dot C^\gamma(\partial\Omega)} + \|\nabla f\|_{\infty,\partial\Omega}\big)\Big]\,|x-y|^\gamma.\end{aligned}$$

Gathering the estimates for the different cases, we infer that

$$\|\nabla u\|_{\dot C^\gamma(\overline{\Omega})} \lesssim \delta\,\|\nabla u\|_{\dot C^\gamma(\overline{\Omega})} + C(\delta,\tau)\big(\|\nabla u\|_{\infty,\overline{\Omega}} + \|\nabla f\|_{\dot C^\gamma(\partial\Omega)} + \|\nabla f\|_{\infty,\partial\Omega}\big).$$

Thus, choosing δ small enough and using the fact that $\|\nabla u\|_{\dot C^\gamma(\overline{\Omega})} < \infty$ by assumption, we get

$$\|\nabla u\|_{\dot C^\gamma(\overline{\Omega})} \lesssim C(\delta,\tau)\big(\|\nabla u\|_{\infty,\overline{\Omega}} + \|\nabla f\|_{\dot C^\gamma(\partial\Omega)} + \|\nabla f\|_{\infty,\partial\Omega}\big).$$

□

Proof of Theorem 10.5. We let $B_0 = B_{\frac{1}{2}\mathrm{d}_\Omega(x_0)}$. As explained above, it suffices to show that $G^{x_0} \in C^{1,\gamma}(\Omega \backslash \overline{B_0})$. To this end we will apply Theorem 10.6 and a suitable approximation argument.

We consider a sequence of domains Ω_j, $j \geqslant 1$, satisfying the following:

- $\Omega_j \subset \Omega_{j+1} \subset \Omega$ for every j, and $\Omega = \bigcup_j \Omega_j$.
- Each Ω_j is a C^∞ domain.
- The domains Ω_j have a uniform $C^{1,\gamma}$ character.

We leave for the reader to check that one can construct such sequence of domains Ω_j.

Denote by $G_j^{x_0}$ the Green function of Ω_j (assuming j large enough so that $x_0 \in \Omega_j$) and let $\widetilde{\Omega} = \Omega \backslash \overline{B_0}$ and $\widetilde{\Omega}_j = \Omega_j \backslash \overline{B_0}$. It is immediate to check that $\widetilde{\Omega}$ and $\widetilde{\Omega}_j$ are also $C^{1,\gamma}$ domains, uniformly on j for j big enough. Notice that $G_j^{x_0}$ vanishes identically on $\partial\Omega_j$ and satisfyes

$$|\nabla^i G_j^{x_0}| \lesssim_i \frac{1}{\mathrm{d}_\Omega(x_0)^{n-1+i}} \quad \text{in } \partial B_0, \text{ for all } i \geqslant 0.$$

In particular, $G_j^{x_0}$ is a harmonic function in $\widetilde{\Omega}_j$ with C^∞ boundary data. Then it follows that $G_j^{x_0} \in C^\infty(\overline{\Omega_j})$ (i.e., $G_j^{x_0} \in C^m(\overline{\Omega_j})$ for all $m \geqslant 1$). See for example [GM12, Theorem 4.14] or [Fol95, Chapter 7]. Then, by Theorem 10.6, choosing f_j to be a C^∞ function in $\mathbb{R}^{n+1}$ that vanishes in a neighborhood of $\partial\Omega_j$ and equals $G_j^{x_0}$ on ∂B_0, and applying also Theorem 10.2, we deduce that

$$\|\nabla G_j^{x_0}\|_{\dot{C}^\gamma(\widetilde{\Omega}_j)} \lesssim \|\nabla G_j^{x_0}\|_{\infty,\widetilde{\Omega}_j} + \|\nabla f_j\|_{\dot{C}^\gamma(\partial\widetilde{\Omega}_j)} + \|f_j\|_{\infty,\partial\widetilde{\Omega}_j} \lesssim \|\nabla G_j^{x_0}\|_{\infty,\widetilde{\Omega}_j} + C(\mathrm{d}_\Omega(x_0)) \leqslant M,$$

where M is some constant that depends on $\mathrm{d}_\Omega(x_0)$ and the $C^{1,\gamma}$ character of $\widetilde{\Omega}_j$, so that M is uniform on j. In other words,

$$|\nabla G_j^{x_0}(x) - \nabla G_j^{x_0}(y)| \leqslant M\,|x-y|^\gamma \quad \text{for all } j \text{ and } x, y \in \widetilde{\Omega}_j. \tag{10.29}$$

We assume that G^{x_0} and $G_j^{x_0}$ vanish identically in Ω^c and Ω_j^c, respectively. Notice that $G^{x_0} - G_j^{x_0}$ is harmonic in Ω_j. Further, by the Hölder continuity of G^{x_0} in a neighborhood of $\partial\Omega$, it holds that

$$|G^{x_0}(y) - G_j^{x_0}(y)| = G^{x_0}(y) \lesssim \mathrm{d}_\Omega(y)^\alpha \quad \text{for all } y \in \partial\Omega_j,$$

for some $\alpha > 0$ and some implicit constant depending on the CDC character of Ω_j. By the maximum principle, it follows that

$$\|G^{x_0} - G_j^{x_0}\|_{\infty,\Omega} \leqslant \|G^{x_0} - G_j^{x_0}\|_{\infty,\partial\Omega_j} \lesssim \mathrm{dist}_H(\partial\Omega, \partial\Omega_j)^\alpha \to 0,$$

as $j \to \infty$, where dist_H stands for the Hausdorff distance. Hence, $\nabla G_j^{x_0}$ converges locally uniformly in compact subsets of Ω to ∇G^{x_0}. So letting $j \to \infty$ in (10.29), we deduce that

$$|\nabla G^{x_0}(x) - \nabla G^{x_0}(y)| \leqslant M\,|x-y|^\gamma \quad \text{for all } x, y \in \widetilde{\Omega},$$

which proves that $G^{x_0} \in C^{1,\gamma}(\widetilde{\Omega})$ and completes the proof of the theorem. □

10.3 Dahlberg's theorem for Lipschitz domains

10.3.1 Introduction

Our objective in this section is to prove the following fundamental theorem of Dahlberg [Dah77]:

Theorem 10.10. *Let $\Omega \subset \mathbb{R}^{n+1}$ be either a bounded Lipschitz domain or a special Lipschitz domain and denote by σ the surface measure in $\partial\Omega$. Let B be a ball centered in $\partial\Omega$ and $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, 2B \cap \partial\Omega) \geqslant C_1^{-1} r(B)$. Then the following holds:*

(a) The harmonic measure ω^{x_0} and σ are mutually absolutely continuous.

(b) We have

$$\left(⨍_{B\cap\partial\Omega} \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \right)^{1/2} \leqslant C ⨍_{B\cap\partial\Omega} \frac{d\omega^{x_0}}{d\sigma}\, d\sigma = C\,\frac{\omega^{x_0}(B)}{\sigma(B)}, \tag{10.30}$$

where C depends only on n, the Lipschitz character of Ω, and C_1.

(c) $\omega^{x_0} \in A_\infty(\sigma)$, with the A_∞ constants depending only on n, the Lipschitz character of Ω, C_1, and $\operatorname{dist}(x_0, \partial\Omega)$.

Next we will describe the strategy for the proof of Dahlberg's theorem. First, notice that a Lipschitz domain is NTA, and thus its associated harmonic measure is doubling. Using this doubling property it is immediate to check that it suffices to prove the theorem for a ball B small enough such that $x_0 \notin 4B$ and $4B$ is contained in $2Z_j$, where Z_j is one of the cylinders in the definition of Lipschitz domain.

We will follow the notation in Definition 8.1. Namely, given B centered in $\partial\Omega$ we will write

$$\Delta_B := B \cap \partial\Omega$$

whenever $\partial\Omega$ is clear from the context.

Suppose that the boundary of Ω is smooth and that the Green function belongs to $C^2(\overline{\Omega})$, so that Green's formula can be applied to $G := G^{x_0}$ and to its partial derivatives (away from x_0). In this case ω^{x_0} and σ are mutually absolutely continuous and

$$\frac{d\omega^{x_0}}{d\sigma} = -\partial_\nu G,$$

where $\partial_\nu G$ is the normal derivative of G in $\partial\Omega$ (we assume that ν is the outer unit normal for Ω). Since G is constantly equal to 0 in $\partial\Omega$, the tangential derivative of G vanishes in $\partial\Omega$, and moreover

$$-\partial_\nu G = |\partial_\nu G| \approx \partial_{n+1} G \quad \text{in } 8Z_j \cap \partial\Omega,$$

in the coordinate system for Z_j. Therefore,

$$\int_{\Delta_B} \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \approx -\int_{\Delta_B} \partial_\nu G\, \partial_{n+1} G\, d\sigma.$$

Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a bump function which equals 1 in B and vanishes away from $2B$. Since G vanishes at the boundary and both G and $\partial_{n+1}G$ are harmonic in $2B$, by Green's formula,

$$\begin{aligned}\int_{\Delta_B}\left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma &\lesssim -\int_{\partial\Omega}\varphi\,\partial_\nu G\,\partial_{n+1}G\,d\sigma = -\int_{\partial\Omega}\partial_\nu(\varphi\,G)\,\partial_{n+1}G\,d\sigma\\ &= \int_\Omega\big(-\Delta(\varphi\,G)\,\partial_{n+1}G + \varphi\,G\,\Delta(\partial_{n+1}G)\big)\,dm = -\int_\Omega \Delta(\varphi\,G)\,\partial_{n+1}G\,dm\\ &= -\int_\Omega\big(\Delta\varphi\,G\,\partial_{n+1}G + 2\partial_{n+1}G\,\nabla\varphi\cdot\nabla G\big)\,dm.\end{aligned}$$

By the definition of φ, Theorem 8.13, and Caccioppoli's inequality, we obtain

$$\begin{aligned}&\int_\Omega\big|\Delta\varphi\,G\,\partial_{n+1}G + 2\partial_{n+1}G\,\nabla\varphi\cdot\nabla G\big|\,dm \qquad (10.31)\\ &\quad\lesssim \frac{1}{r(B)^2}\left(\int_{\Omega\cap 2B}G^2\,dm\right)^{1/2}\left(\int_{\Omega\cap 2B}|\partial_{n+1}G|^2\,dm\right)^{1/2} + \frac{1}{r(B)}\int_{\Omega\cap 2B}|\nabla G|^2\,dm\\ &\quad\lesssim \frac{1}{r(B)^3}\int_{\Omega\cap 3B}G^2\,dm \lesssim \frac{1}{r(B)^3}\left(\frac{\omega^{x_0}(B)}{r(B)^{n-1}}\right)^2 m(B) \approx \frac{\omega^{x_0}(B)^2}{\sigma(B)},\end{aligned}$$

which yields (10.30). The fact that ω^{x_0} is an $A_\infty(\sigma)$ weight follows then easily from this reverse Hölder property.

For arbitrary Lipschitz domains the argument above does not work because we cannot assume a priori that $\partial_\nu G$ and $\partial_{n+1}G$ are defined in $\partial\Omega$ and that the Green formula applied above holds. To prove Dahlberg's theorem with full rigor, first we will consider the case when the boundary $\partial\Omega$ is of class C^1 and we will prove a discrete version of (10.30) following an approach based on the arguments above. Later we will deduce the full result by an approximation argument

10.3.2 An auxiliary lemma

Lemma 10.11. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain, let B a ball centered in $\partial\Omega$, and let $H = \{y : y_{n+1} > 0\}$ and $L = \partial H$. For any $\varepsilon > 0$ there exists some $\delta = \delta(\varepsilon) > 0$ (depending on ε, the NTA character of Ω and the function β) such that the following holds. Suppose that $\Omega \cap \delta^{-1}B \subset H$ and that*

$$\mathrm{dist}_H(\partial\Omega\cap\delta^{-1}B, L\cap\delta^{-1}B) \leqslant \delta\,r(B). \qquad (10.32)$$

Let $u : \overline{\Omega\cap\delta^{-1}B} \to \mathbb{R}$ be a continuous function vanishing identically in $\partial\Omega\cap\delta^{-1}B$ and positive and harmonic in $\Omega\cap\delta^{-1}B$. Then there exists some constant $\lambda > 0$, depending on u, such that

$$|u(y) - \lambda\,y_{n+1}| \leqslant \varepsilon\,\|u\|_{\infty,B} \quad \textit{for all } y \in \overline{\Omega\cap B}, \qquad (10.33)$$

Further, if $y \in \Omega\cap B$ satisfies $\mathrm{dist}(y,\partial\Omega) \geqslant \frac14 r(B)$ and ε is small enough, then we have

$$|\nabla u(y)| \approx \partial_{n+1}u(y) \approx r(B)^{-1}\,u(y) \qquad (10.34)$$

and

$$r(B)\,|\nabla^2 u(y)| + |\nabla_L u(y)| \leqslant \varepsilon\,|\nabla u(y)| \ll |\nabla u(y)|, \tag{10.35}$$

where ∇_L denotes the tangential derivative in L.

Remark that, for δ small enough, the condition (10.32), the fact that $\Omega \cap \delta^{-1}B \subset H$, and the interior and exterior corkscrew properties of Ω, imply that the upper component of $\delta^{-1}B \setminus \overline{\mathcal{U}_{\delta r(B)}(L)}$ is contained in Ω, and the lower component in Ω^c.

Proof. Consider an arbitrary point $y_0 \in B \cap \Omega$ such that $\operatorname{dist}(y_0, \partial\Omega) \geqslant r(B)/4$. Then we will prove (10.33) with

$$\lambda = \frac{u(y_0)}{y_{0,n+1}}.$$

Denote $v(y) = \lambda\, y_{n+1}$. For the sake of contradiction, suppose that there exists some $\varepsilon > 0$ such that for any $\delta = 1/k$ there is an NTA domain Ω_k (with some bounded NTA character independent of k), a ball B_k centered in $\partial\Omega_k$ such that $\Omega_k \cap (kB_k) \subset H$ and $\operatorname{dist}_H(\partial\Omega_k \cap (kB_k), L \cap (kB_k)) \leqslant k^{-1}\, r(B_k)$, and a continuous function $u_k : \overline{\Omega_k \cap kB_k} \to \mathbb{R}$ vanishing identically in $\partial\Omega_k \cap kB_k \to \mathbb{R}$, positive and harmonic in $\Omega_k \cap kB_k$, such that

$$\|u_k - v_k\|_{\infty, B_k \cap \Omega_k} > \varepsilon\,\|u_k\|_{\infty, B_k}, \tag{10.36}$$

with $v_k(y) = \frac{u_k(y_0)}{y_{0,n+1}}\, y_{n+1}$. By translating and dilating B_k and Ω_k if necessary, we may assume that $B_k = B_1(0)$.

Since the domains Ω_k are NTA (with constants uniform in k), we infer that for any $1 \leqslant M \leqslant k/2$,

$$\|u_k\|_{\infty, MB} \overset{\text{L.8.10}}{\lesssim_M} u_k(y_0).$$

Hence, the sequence of functions $u_k(y_0)^{-1}\, u_k$ is uniformly locally bounded in compact subset of $\mathbb{R}^{n+1}$ (we assume these functions to be extended by zero in Ω_k^c). These functions are also uniformly Hölder continuous in $\frac{k}{2}B_k$ (by Lemma 7.28). Also, since

$$\operatorname{dist}_H(\partial\Omega \cap (kB_k), L \cap (kB_k)) \to 0,$$

by the Arzelà-Ascoli Theorem we infer that there is a subsequence $u_{k_j}(y_0)^{-1}\, u_{k_j}$ that convergences uniformly to some function $\tilde{u}$ which is positive and harmonic in H and vanishes continuously in $L = \partial H$. Clearly we have $\tilde{u}(y_0) = 1$ and so $\tilde{u}$ does not vanish identically in H. Thus, by Lemma 10.1 we know that $\tilde{u}(y) = \frac{1}{y_{0,n+1}}\, y_{n+1}$ in H.

On the other hand, notice also that $\frac{v_k(y)}{u_k(y_0)} = \frac{1}{y_{0,n+1}}\, y_{n+1}$ for all k, and thus by (10.36) we get the contradiction

$$0 = \|\tilde{u} - \tilde{u}\|_{\infty, B} = \lim_{j\to\infty} \frac{\|u_{k_j} - v_{k_j}\|_{\infty, B}}{u_{k_j}(y_0)} \gtrsim \limsup_{j\to\infty} \frac{\|u_{k_j} - v_{k_j}\|_{\infty, B}}{\|u_{k_j}\|_{\infty, B}} \geqslant \varepsilon,$$

which proves (10.33) with $\lambda = \frac{u(y_0)}{y_{0,n+1}}$.

Our next objective is to derive (10.34) and (10.35) from (10.33) with the preceding choice of λ, and with B replaced by $2B$ (it is clear that this estimate also holds in this case, by modifying suitably δ). By the mean value property and the usual interior Caccioppoli estimates for harmonic functions, we deduce that for all $y \in \Omega \cap B$ satisfying $\operatorname{dist}(y, \partial\Omega) \geqslant \frac{1}{4}\, r(B)$, we have

$$|\partial_{n+1} u(y) - \lambda| + |\nabla_L u(y)| \leqslant 2\, |\nabla u(y) - \lambda e_{n+1}| \lesssim \frac{1}{r(B)}\, \|u - v\|_{\infty, \Omega \cap 2B} \leqslant \frac{\varepsilon}{r(B)}\, \|u\|_{\infty, B} \tag{10.37}$$

and

$$|\nabla^2 u(y) - 0| \lesssim \frac{1}{r(B)^2}\, \|u - v\|_{\infty, \Omega \cap 2B} \leqslant \frac{\varepsilon}{r(B)^2}\, \|u\|_{\infty, B}. \tag{10.38}$$

Notice now that

$$\lambda = \frac{u(y_0)}{y_{0,n+1}} \approx \frac{u(y)}{r(B)} \approx \frac{1}{r(B)}\, \|u\|_{\infty, B},$$

and so from (10.37) we deduce that, for ε small enough,

$$|\partial_{n+1} u(y) - \lambda| \leqslant |\nabla u(y) - \lambda e_{n+1}| \leqslant \frac{\lambda}{2},$$

and so $\partial_{n+1} u(y) \approx |\nabla u(y)| \approx \lambda$, which yields (10.34). On the other hand, from (10.37) and (10.34) we derive

$$|\nabla_L u(y)| \overset{(10.37)}{\lesssim} \frac{\varepsilon}{r(B)}\, \|u\|_{\infty, B} \overset{\text{L.8.10}}{\approx} \varepsilon\, \frac{u(y)}{r(B)} \overset{(10.34)}{\approx} \varepsilon\, |\nabla u(y)|.$$

Finally, the estimate $r(B)\, |\nabla^2 u(y)| \lesssim \varepsilon\, |\nabla u(y)|$ in (10.35) follows from (10.38) in an analogous way. $\square$

10.3.3 A key lemma for the smooth case

As in Section 10.3.1, to prove Dahlberg's theorem, we will assume that the ball B is small enough, so that $x_0 \notin 4B$ and $4B$ is contained in $2Z$, where Z is one of the cylinders Z_j defined above. We denote by $\mathcal{D}(\partial\Omega, Z)$ the family of the following "dyadic cubes" of $\partial\Omega$ obtained as follows. Let $\mathcal{D}(\mathbb{R}^n)$ the usual dyadic lattice of $\mathbb{R}^n$. Let Π_Z be the orthogonal projection from $8Z$ to $\mathbb{R}^n \equiv \mathbb{R}^n \times \{0\}$, in the coordinate system associated with Z. Then we let

$$\mathcal{D}(\partial\Omega, Z) = \{\Pi_Z^{-1}(Q) \cap \partial\Omega : Q \in \mathcal{D}(\mathbb{R}^n), Q \subset 8Z \cap \mathbb{R}^n\}.$$

Here again we are identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$. Observe that the cubes from this family are contained in $\partial\Omega \cap 8Z$. We also denote $\ell(\Pi_Z^{-1}(Q) \cap \partial\Omega) := \ell(Q)$ and we call this the side length of $\Pi_Z^{-1}(Q) \cap \partial\Omega$. Its center is the point whose projection by Π_Z coincides with the center of Q. We let $\mathcal{D}_k(\partial\Omega, Z)$ be subfamily of the cubes from $\mathcal{D}(\partial\Omega, Z)$ with side length 2^{-k}, and given a cube $R \in \mathcal{D}(\partial\Omega, Z)$, we let $\mathcal{D}_k(\partial\Omega, Z, R)$ be the subfamily of the cubes from $\mathcal{D}(\partial\Omega, Z)$ which are contained in R and have side length $2^{-k}\ell(R)$.

Lemma 10.12. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a (bounded) Lipschitz domain. Let $Z \subset \mathbb{R}^{n+1}$ be one of the cylinders in the definition of the Lipschitz character of Ω. Let $R \in \mathcal{D}(\partial\Omega, Z)$ such that $4R \subset 4Z$ and $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, 4R) \geqslant 4\operatorname{diam}(R)$. Suppose that $\partial\Omega$ is C^1 in a neighborhood of $4R$. Then, for any $k \geqslant 1$, we have*

$$\sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 \sigma(Q) \leqslant C \left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2 \sigma(R), \tag{10.39}$$

with C depending only on the Lipschitz character of Ω.

Notice that (10.39) can be considered as a discrete version of (10.30).

Proof. Suppose that $\partial\Omega \cap Z$ coincides with the graph of the Lipschitz function $y_{n+1} = A(y)$ in Z. For $t > 0$, let $A_t(y) = A(y) + t$ and let $\Omega_t = \{y \in \Omega : y_{n+1} > A_t(y)\}$ (the definition of the function A away from $4Z$ does not matter).

For every $Q \in \mathcal{D}_k(\partial\Omega, Z, R)$ consider a C^∞ bump function φ_Q which equals 1 on $\frac{3}{2}Q$ and vanishes in $\mathbb{R}^{n+1} \backslash B_{\operatorname{diam}(Q)}(x_Q)$ and in $\Pi_Z^{-1}(2Q)$ (here x_Q is the center of Q), with $\ell(Q)|\nabla\varphi_Q| + \ell(Q)^2|\nabla^2\varphi_Q| \lesssim 1$. Since the function $G := G^{x_0}$ belongs to $W^{1,2}(\Omega\backslash\overline{B_r(x)})$ for any $r > 0$, we infer that

$$\omega^{x_0}(Q) \overset{\text{L.7.6}}{\leqslant} -\int_\Omega \nabla G\, \nabla\varphi_Q\, dm = -\lim_{t\to 0}\int_{\Omega_t} \nabla G\, \nabla\varphi_Q\, dm = -\lim_{t\to 0}\int_{\partial\Omega_t} \partial_{\nu_t} G\, \varphi_Q\, d\sigma_t,$$

where ν_t and σ_t denote the outer unit normal and the surface measure for $\partial\Omega_t$, respectively. Consequently, denoting $2Q_t = \Pi_Z^{-1}(2Q) \cap \partial\Omega_t$,

$$\begin{aligned} \sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 \sigma(Q) &\leqslant \limsup_{t\to 0} \sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \left(\int_{2Q_t} \partial_{\nu_t} G\, \varphi_Q\, d\sigma_t\right)^2 \sigma(Q)^{-1} \\ &\lesssim \limsup_{t\to 0} \sum_{Q \in \mathcal{D}_k(\partial\Omega, Z, R)} \int_{2Q_t} |\partial_{\nu_t} G|^2\, \varphi_Q^2\, d\sigma_t \\ &\lesssim \limsup_{t\to 0} \int_{2R_t} |\partial_{\nu_t} G|^2\, \varphi_R^2\, d\sigma_t. \end{aligned} \tag{10.40}$$

Since $A \in C^1(U)$ where $U \supset \Pi_Z(4R)$, by a compactness argument we get

$$\sup_{|x-y|\leqslant t} |\nabla A(\bar{x}) - \nabla A(\bar{y})| \leqslant \omega(t) \text{ for } x \in 3R$$

and t small enough, with $\lim_{t\to 0}\omega(t) = 0$. In particular, for every $\delta > 0$ there exists t_δ such that $\omega(4\delta^{-1}t_\delta) \leqslant \delta^2$. This implies that the tangent n-plane $L_y = \{x \in \mathbb{R}^{n+1} : x_{n+1} = A(y) + \nabla A(\bar{y}) \cdot (\bar{x} - \bar{y})\}$ satisfies

$$\beta_{\infty,\partial\Omega}(\delta^{-1}B_{4t_\delta}(y), L_y) \leqslant \sup_{|\bar{x}-\bar{y}|\leqslant\delta^{-1}t} \frac{|A(\bar{x}) - A(\bar{y}) - \nabla A(\bar{y}) \cdot (\bar{x} - \bar{y})|}{4\delta^{-1}t_\delta} \leqslant \omega(4\delta^{-1}t_\delta) \leqslant \delta^2.$$

by the mean value theorem applied to A.

Now, for every $\varepsilon > 0$ we can apply Lemma 10.11 to find $\delta = \delta(\varepsilon)$ so that fixing $t = t_{\delta(\varepsilon)}$ as in the previous paragraph, for every $y \in 2R_t$ we can infer (taking in the lemma $B = B_{4t}(y)$, and with L being a suitable n-plane T_y orthogonal to $\nu_t(y)$ so that ∂_{n+1} in Lemma 10.11 is precisely ∂_{ν_t} here) after applying perhaps Harnack's inequality, that for ε small enough and all $y \in 2R_t$,

$$|\nabla G(y)| \approx |\partial_{\nu_t} G(y)| = -\partial_{\nu_t} G(y) \approx t_{\delta(\varepsilon)}^{-1}\, G(y) \tag{10.41}$$

and

$$t_{\delta(\varepsilon)}\, |\nabla^2 G(y)| + |\nabla_{L_y} G(y)| \leqslant \varepsilon\, |\nabla G(y)| \ll |\nabla G(y)|, \tag{10.42}$$

where ∇_{L_y} denotes the tangential derivative in $\partial\Omega_t$. Let $\vec{v}_y$ be the orthogonal projection of the vertical unit vector e_{n+1} (in the local coordinates of Z) on the tangent n-plane L_y. Note that

$$\partial_{n+1} G(y) = e_{n+1} \cdot \nabla G(y) = \langle e_{n+1}, \nu_t(y)\rangle\, \partial_{\nu_t} G(y) + \langle e_{n+1}, |\vec{v}_y|^{-1} \vec{v}_y\rangle\, \partial_{|\vec{v}_y|^{-1}\vec{v}_y} G(y),$$

with the convention $\langle e_{n+1}, |\vec{v}_y|^{-1}\vec{v}_y\rangle\, \partial_{|\vec{v}_y|^{-1}\vec{v}_y} G(y) = 0$ whenever $\vec{v}_y = 0$. Since A is Lipschitz, the scalar product $\langle e_{n+1}, \nu_t(y)\rangle$ is bounded below, and taking into account (10.41) and (10.42), we derive

$$-\partial_{\nu_t} G(y) = |\partial_{\nu_t} G(y)| \approx \partial_{n+1} G(y) \quad \text{ for all } y \in 2R_t.$$

Thus, for ε small enough, and $t = t_{\delta(\varepsilon)}$ we have

$$\begin{aligned} I_\varepsilon := \int_{2R_{t_{\delta(\varepsilon)}}} |\partial_{\nu_{t_{\delta(\varepsilon)}}} G|^2\, \varphi_R^2\, d\sigma_{t_{\delta(\varepsilon)}} &\approx -\int_{2R_t} \partial_{\nu_t} G\, \partial_{n+1} G\, \varphi_R^2\, d\sigma_t \\ &= -\int_{2R_t} \partial_{\nu_t}(G\, \varphi_R^2)\, \partial_{n+1} G\, d\sigma_t + 2\int_{2R_t} G\, \varphi_R\, \partial_{\nu_t}\varphi_R\, \partial_{n+1} G\, d\sigma_t. \end{aligned} \tag{10.43}$$

We estimate the last integral on the right hand side above using Cauchy-Schwarz, the Hölder continuity of G in a neigborhood of $B_{\mathrm{diam}(R)}(x_R)$, (10.41), and the connection between ω^{x_0} and G:

$$\begin{aligned} \int_{2R_t} |G\, \varphi_R\, \partial_{\nu_t}\varphi_R\, \partial_{n+1} G|\, d\sigma_t &\lesssim \frac{1}{\ell(R)} \sup_{2R_t} G(y) \left(\int_{2R_t} |\varphi_R\, \partial_{n+1} G|^2\, d\sigma_t \right)^{1/2} \sigma(R)^{1/2} \\ &\overset{\text{L.7.28}}{\lesssim} \frac{1}{\ell(R)} \left(\frac{t_{\delta(\varepsilon)}}{\ell(R)} \right)^{\alpha} \sup_{y \in B_{3\mathrm{diam}(R)}(x_R)} G(y)\; I_\varepsilon^{1/2}\, \sigma(R)^{1/2} \\ &\overset{(7.13)/(7.15)}{\lesssim} \left(\frac{t_{\delta(\varepsilon)}}{\ell(R)} \right)^{\alpha} \frac{\omega^{x_0}(R)}{\sigma(R)^{1/2}}\, I_\varepsilon^{1/2}. \end{aligned}$$

To estimate the first integral on the right hand side of (10.43) we use Green's formula again and we take into account that $\partial_{n+1} G$ is harmonic away from x_0 in Ω:

$$\int_{2R_t} \partial_{\nu_t}(G\, \varphi_R^2)\, \partial_{n+1} G\, d\sigma_t = \int_{\Omega_t} \Delta(G\, \varphi_R^2)\, \partial_{n+1} G\, dm + \int_{2R_t} G\, \varphi_R^2\, \partial_{\nu_t}\partial_{n+1} G\, d\sigma_t \tag{10.44}$$

The first integral on the right hand side is estimated exactly as in (10.31). Indeed, denoting by B_R some ball centered in $\partial\Omega$ that contains suppφ_R and such that $\operatorname{diam}(B_R) \approx \ell(R)$, we get

$$\begin{aligned}
\int_{\Omega_t} |\Delta(G\,\varphi_R^2)\,\partial_{n+1}G|\,dm &\leqslant \int_{\Omega} \big|\Delta\varphi_R^2\,G\,\partial_{n+1}G + 2\partial_{n+1}G\,\nabla\varphi_R^2\cdot\nabla G\big|\,dm \\
&\lesssim \frac{1}{r(B_R)^2}\left(\int_{\Omega\cap B_R} G^2\,dm\right)^{1/2}\left(\int_{\Omega\cap B_R} |\partial_{n+1}G|^2\,dm\right)^{1/2} + \frac{1}{r(B_R)}\int_{\Omega\cap B_R} |\nabla G|^2\,dm \\
&\lesssim \frac{1}{r(B_R)^3}\int_{\Omega\cap 2B_R} G^2\,dm \lesssim \frac{1}{r(B_R)^3}\left(\frac{\omega^{x_0}(R)}{r(B_R)^{n-1}}\right)^2 m(B_R) \approx \frac{\omega^{x_0}(R)^2}{\sigma(R)}.
\end{aligned}$$

To deal with the last integral on the right hand side of (10.44) we apply (10.41) and (10.42):

$$\begin{aligned}
\int_{2R_t} |G\,\varphi_R^2\,\partial_{\nu_t}\partial_{n+1}G|\,d\sigma_t &\leqslant \int_{2R_t} G\,\varphi_R^2\,|\nabla^2 G|\,d\sigma_t \\
&\lesssim \int_{2R_t} (t|\partial_{\nu_t}G|)\,\varphi_R^2\,(\varepsilon t^{-1}|\partial_{\nu_t}G|)\,d\sigma_t \\
&= \varepsilon \int_{2R_t} |\partial_{\nu_t}G|^2\,\varphi_R^2\,d\sigma_t = \varepsilon I_\varepsilon.
\end{aligned}$$

Altogether, we obtain

$$I_\varepsilon \lesssim \left(\frac{t_{\delta(\varepsilon)}}{\ell(R)}\right)^{\alpha} \frac{\omega^{x_0}(R)}{\sigma(R)^{1/2}}\,I_\varepsilon^{1/2} + \frac{\omega^{x_0}(R)^2}{\sigma(R)} + \varepsilon I_\varepsilon.$$

For ε small enough, this yields

$$I_\varepsilon \lesssim \frac{\omega^{x_0}(R)^2}{\sigma(R)}.$$

Plugging this estimate into (10.40) for any sequence $t_{\delta(\varepsilon_j)}$ with $\varepsilon_j \to 0$, the lemma follows. □

10.3.4 Proof of Theorem 10.10

We assume that B is small enough so that $x_0 \notin 4B$ and $4B$ is contained in $2Z$, where Z is one of the cylinders in the definition of Lipschitz domain.

By reducing B and translating the dyadic lattice $\mathcal{D}(\partial\Omega, Z)$ if necessary, taking into account that ω^{x_0} is doubling, we may assume that $B \cap \partial\Omega$ is contained in some cube $R \in \mathcal{D}(Z, \partial\Omega)$ like the one in the statement of Lemma 10.12, so that moreover $\ell(R) \approx r(B)$. We claim that for any $k \geqslant 1$ we have

$$\sum_{Q\in\mathcal{D}_k(\partial\Omega,Z,R)} \left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 \sigma(Q) \leqslant C\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2 \sigma(R), \tag{10.45}$$

whith C depending only on the Lipschitz character of Ω.

To prove the claim we approximate Ω by a domain Ω_δ whose boundary is C^1 in $2Z$. To this end, we consider a smooth approximation of the identity $\{\phi_\delta\}_{\delta>0}$ in $\mathbb{R}^n$, we take a bump function $\eta : \mathbb{R}^n \to 0$ which equals 1 in a neighborhood of $3Z \cap \mathbb{R}^n$ and vanishes in $\mathbb{R}^n \backslash 3.1Z$, and for $z \in \mathbb{R}^n$ we denote

$$A_\delta(z) = A * \phi_{\eta(z)\delta}(z),$$

where $\delta \ll \ell(R)$ and we understand that $A * \phi_0(z) = A(z)$. It is easy to check that A_δ is Lipschitz (uniformly in δ), with $\|\nabla A_\delta\|_\infty \lesssim \|\nabla A\|_\infty$ (see Exercise 10.3.1 below), and that A_δ is C^∞ in a neighborhood of $3R$. We let Ω_δ be the domain whose boundary is the graph of A_δ in $4Z$ and coincides with $\partial\Omega$ in $\mathbb{R}^{n+1}\backslash 4Z$. We denote by $\omega_\delta^{x_0}$ the harmonic measure in Ω_δ with pole x_0, and we let $Q_\delta = \Pi_Z^{-1}(Q) \cap \partial\Omega_\delta$ for $Q \in \mathcal{D}(Z, \partial\Omega)$, so that $Q_\delta \in \mathcal{D}(Z, \partial\Omega_\delta)$.

For every $\varepsilon > 0$ and every $\delta < \delta_0(\varepsilon)$ small enough (possibly depending on k) we have

$$\omega^{x_0}(\tfrac{1}{2}Q) \leqslant \omega_\delta^{x_0}(Q_\delta) + \varepsilon\sigma(Q) \quad \text{for every } Q \in \mathcal{D}_k(\partial\Omega, Z, R). \tag{10.46}$$

Indeed, $\omega_\delta^{(\cdot)}(Q_\delta)$ is a function harmonic in Ω_δ, which extends continuously to 1 in $\frac{1}{2}Q_\delta$, with a Hölder modulus of continuity uniform in δ. This can be derived by applying Lemma 7.28 to the function $1 - \omega_\delta^{(\cdot)}(Q_\delta)$. Then, writing

$$v_\delta(x) := \begin{cases} \omega_\delta^x(Q_\delta), & \text{for } x \in \Omega \\ 1, & \text{for } x \in \Omega^c, \end{cases}$$

from Lemma 2.14 it easily follows that for ever sequence $\delta_j \to 0$ there exists a subsequence $\{j_k\}_k$ and a function v harmonic in Ω and Hölder continuous in a suitable ball B with $\frac{1}{2}Q \subset B$ (and $\frac{1}{2}Q_\delta \subset B$ for every δ) with $v|_{B\backslash\Omega} \equiv 1$, such that $v_{\delta_j} \to v$ uniformly in compact subsets of Ω and $v_{\delta_j} \to v$ in $C^\alpha(B)$. Note that v extends continuously to 1 in a neighborhood of $\frac{1}{2}Q$ and thus

$$\lim_{k\to\infty} \omega_{\delta_{j_k}}^{x_0}(Q_{\delta_{j_k}}) = v(x_0) \geqslant \omega^{x_0}(\tfrac{1}{2}Q)$$

by the maximum principle. Therefore,

$$\liminf_{\delta\to 0} \omega_\delta^{x_0}(Q_\delta) \geqslant \omega^{x_0}(\tfrac{1}{2}Q) \quad \text{for all } Q \in \mathcal{D}_k(\partial\Omega, Z, R),$$

which proves (10.46) because the number of cubes is finite. By a similar argument, we infer that for δ small enough we have

$$\omega^{x_0}(R) \geqslant \omega_\delta^{x_0}(\tfrac{1}{2}R_\delta) - \varepsilon\sigma(R). \tag{10.47}$$

From (10.46), Lemma 10.12, (10.47), and the doubling properties of ω and ω_δ, we get

$$\begin{aligned}\sum_{Q\in\mathcal{D}_k(\partial\Omega,Z,R)}\left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2\sigma(Q) &\lesssim \sum_{Q_\delta\in\mathcal{D}_k(\partial\Omega_\delta,Z,R)}\left[\left(\frac{\omega_\delta^{x_0}(Q_\delta)}{\sigma_\delta(Q_\delta)}\right)^2\sigma_\delta(Q_\delta)+\varepsilon^2\sigma(Q_\delta)\right]\\ &\lesssim \left(\frac{\omega_\delta^{x_0}(R_\delta)}{\sigma_\delta(R_\delta)}\right)^2\sigma_\delta(R_\delta)+\varepsilon^2\sigma(R_\delta)\\ &\lesssim \left[\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2+\varepsilon^2\right]\sigma(R).\end{aligned}$$

Now the claim (10.45) follows immediately by letting $\varepsilon \to 0$.

The theorem follows easily from (10.45). First we show that $\omega^{x_0} \in A_\infty(\sigma)$, with the A_∞ constants depending on the Lipschitz character of Ω and $\operatorname{dist}(x_0, \partial\Omega)$. To this end, it suffices to prove that there are $\delta_0, \varepsilon_0 \in (0,1)$ such that for any compact set $E \subset R$,

$$\sigma(E) \leqslant \delta_0\,\sigma(R) \quad\Rightarrow\quad \omega^{x_0}(E) \leqslant \varepsilon_0\,\omega^{x_0}(R). \tag{10.48}$$

Indeed, from the regularity of σ, we infer that for any $\delta_0 \in (0,1)$ there exists some k large enough and some family $I_k \subset \mathcal{D}_k(\partial\Omega, Z, R)$ such that the set $\widetilde{E} = \bigcup_{Q\in I_k} Q$ satisfies

$$E \subset \widetilde{E}, \qquad \sigma(\widetilde{E}) \leqslant \sigma(E) + \delta_0\,\sigma(R) \leqslant 2\delta_0\,\sigma(R).$$

By Cauchy-Schwarz and (10.45), we get

$$\begin{aligned}\omega^{x_0}(E) \leqslant \omega^{x_0}(\widetilde{E}) \leqslant \sum_{Q\in I_k}\frac{\omega^{x_0}(Q)}{\sigma(Q)}\sigma(Q) &\leqslant \left(\sum_{Q\in I_k}\left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2\sigma(Q)\right)^{1/2}\sigma(\widetilde{E})^{1/2}\\ &\leqslant C\left(\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2\sigma(R)\right)^{1/2}\delta_0^{1/2}\sigma(R)^{1/2} = C\delta_0^{1/2}\omega^{x_0}(R).\end{aligned}$$

So (10.48) holds if we choose δ_0 small enough. In particular, this implies that ω^{x_0} is absolutely continuous with respect to σ.

Finally we turn our attention to the estimate (10.30). Given any $\eta > 0$, by the Lebesgue differentiation theorem, for σ-a.e. $y \in R$ there exists some $k_y \geqslant 1$ such that

$$\left|\frac{d\omega^{x_0}}{d\sigma}(y) - \frac{\omega^{x_0}(Q)}{\sigma(Q)}\right| \leqslant \eta \quad \text{ if } x \in Q \in \mathcal{D}(\partial\Omega, Z) \text{ and } \ell(Q) \leqslant 2^{-k_y}\ell(R).$$

Denote $R(k_0) = \{y \in R : k_y \leqslant k_0\}$ for $k_0 \in \mathbb{N}$. Then, using again (10.45) we obtain

$$\begin{aligned}\int_{R(k_0)}\left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma &\leqslant 2\sum_{Q\in\mathcal{D}_{k_0}(\partial\Omega,Z,R)}\int_{R(k_0)\cap Q}\left(\frac{d\omega^{x_0}}{d\sigma} - \frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2 d\sigma\\ &\quad + 2\sum_{Q\in\mathcal{D}_{k_0}(\partial\Omega,Z,R)}\left(\frac{\omega^{x_0}(Q)}{\sigma(Q)}\right)^2\sigma(Q)\\ &\leqslant 2\eta^2\sigma(R) + C\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2\sigma(R).\end{aligned}$$

Since R coincides with $\bigcup_{k_0\geqslant 1} R(k_0)$ up to a set of zero σ measure, by the monotone converge theorem we derive

$$\int_B \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \leqslant \int_R \left(\frac{d\omega^{x_0}}{d\sigma}\right)^2 d\sigma \leqslant 2\eta^2\sigma(R) + C\left(\frac{\omega^{x_0}(R)}{\sigma(R)}\right)^2 \sigma(R).$$

Since η is arbitrarily small and $\omega^{x_0}(R) \approx \omega^{x_0}(B)$, clearly this yields (10.30). □

Exercise 10.3.1. Show that, in the proof above, A_δ is uniformly Lipschitz as claimed. Hint: show first that if $f_y(z) = \phi_{\eta(z)\delta}(z-y)$, then $|\nabla f_y(z)| \leqslant \frac{C}{(\eta(z)\delta)^{n+1}}$, and $|A_\delta(z) - A(z)| \leqslant \eta(z)\delta\|\nabla A\|_\infty$. Then treat separately the cases $0 < \eta(z') \leqslant \eta(z)$ with $B_{\eta(z')\delta}(z') \subset 3B_{\eta(z)\delta}(z)$, $0 < \eta(z') \leqslant \eta(z)$ with $B_{\eta(z')\delta}(z') \cap B_{\eta(z)\delta}(z) = \varnothing$, and $0 = \eta(z') < \eta(z)$, referring to the previous estimates.

10.4 Harmonic measure in chord-arc domains

Recall that a chord-arc domain in $\mathbb{R}^{n+1}$ is an NTA domain whose boundary is n-Ahlfors regular. A chord-arc domain in $\mathbb{R}^{n+1}$ is an NTA domain whose boundary is n-Ahlfors regular. Here we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the corkscrew condition if it satisfies the interior corkscrew condition from Definition 8.5 with $r_0 = \operatorname{diam}(\Omega)$, that is, for all $\xi \in \partial\Omega$ and $0 < r \leqslant \operatorname{diam}(\partial\Omega)$ there exists some ball $B \subset B_r(\xi) \cap \Omega$ with $r(B) \approx r$. We say that Ω is a two-sided corkscrew domain if both Ω and $\mathbb{R}^{n+1}\backslash\overline{\Omega}$ satisfy the corkscrew condition. It is clear that any chord-arc domain is also a two-sided corkscrew domain.

We will need the following geometric result, proved independently by David and Jerison [DJ90] and Semmes [Sem90]:

Theorem 10.13. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an Ahlfors regular and two-sided corkscrew domain. Then, for all $\xi \in \partial\Omega$ and all $r \in (0, \operatorname{diam}(\partial\Omega))$ there exists a Lipschitz domain $U_{\xi,r} \subset \Omega \cap B_r(\xi)$ such that*

$$\mathcal{H}^n(\Delta_{r,\xi} \cap \partial U_{\xi,r}) \gtrsim r^n,$$

where $\Delta_{r,\xi} = \partial\Omega \cap B_r(\xi)$. The Lipschitz character of the domains $U_{\xi,r}$ and the implicit constant above only depend on n and the parameters involved in the n-Ahlfors regularity of $\partial\Omega$ and the two-sided corkscrew condition for Ω.

Remark that, for the theorem above to hold, the two-sided corkscrew condition can be weakened, for example, by replacing the corkscrew balls by suitable discs of codimension one not intersecting $\partial\Omega$, see [Sem90]. An immediate corollary of the above result is that the boundary of an Ahlfors regular two-sided corkscrew domain is uniformly n-rectifiable (see [DS93] for the definition of uniform n-rectifiability). Another consequence is the following.

Theorem 10.14. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a chord-arc domain. The harmonic measure for Ω is an A_∞ weight with respect to the surface measure σ. In particular, there are constants $\delta, \varepsilon \in (0,1)$ such that for any ball B centered in $\partial\Omega$, any $x_0 \in \Omega\backslash 2B$, and any Borel set $E \subset \Delta_B = \partial\Omega \cap B$, the following holds:*

$$\sigma(E) > \delta\,\sigma(B) \quad \Rightarrow \quad \omega^{x_0}(E) \geqslant \varepsilon\,\omega^{x_0}(B).$$

Proof. By Theorem 10.13, for a ball B as above there is a Lipschitz domain $U \subset \Omega \cap \frac{1}{2}B$ such that

$$\sigma(\Delta_B \cap \partial U) \geqslant \eta\, \sigma(\Delta_B),$$

where $\eta > 0$ depends on the parameters of the chord-arc domain character of Ω. We claim that if δ is close enough to 1 and $\sigma(E) \geqslant \delta\, \sigma(\Delta_B)$ (for $E \subset B \cap \partial\Omega$), then

$$\mathcal{H}^n(E \cap \partial U) \gtrsim_{\delta,\eta} \mathcal{H}^n(\partial U). \tag{10.49}$$

Indeed,

$$\begin{aligned}\sigma(E \cap \partial U) &= \sigma(E) - \sigma(E \backslash \partial U) \geqslant \sigma(E) - \sigma(\Delta_B \backslash \partial U)\\ &= \sigma(E) - \sigma(\Delta_B) + \sigma(\Delta_B \cap \partial U) \geqslant \delta\, \sigma(\Delta_B) - (1-\eta)\, \sigma(\Delta_B)\\ &\approx (\delta + \eta - 1)\, r(B)^n \approx_{\delta,\eta} \mathcal{H}^n(\partial U).\end{aligned}$$

Consider a corkscrew point $x_B \in U$ such that $\operatorname{dist}(x_B, \partial U) \approx r(B)^n$. By Dahlberg's theorem, $\omega_U^{x_B}$ is an $A_\infty(\mathcal{H}^n|_{\partial U})$ weight, so (10.49) implies

$$\omega_U^{x_B}(E \cap \partial U) \gtrsim_{\delta,\eta} \omega_U^{x_B}(\partial U).$$

By the maximum principle for the harmonic measure of nested domains (see Lemma 5.32), we obtain

$$\omega_\Omega^{x_B}(E) \geqslant \omega_\Omega^{x_B}(E \cap \partial U) \overset{\text{L.5.32}}{\geqslant} \omega_U^{x_B}(E \cap \partial U) \gtrsim_{\delta,\eta} \omega_U^{x_B}(\partial U) = 1.$$

All in all,

$$\omega_\Omega^{x_B}(E) \gtrsim_{\delta,\eta} 1.$$

Then, by the change of pole formula for NTA domains we deduce

$$\omega_\Omega^{x_0}(E) \overset{\text{L.8.21}}{\gtrsim}_{\delta,\eta} \omega_\Omega^{x_0}(\Delta_B),$$

which proves the last claim in the theorem, see Remark 4.51.

□

10.5 L^p-solvability of the Dirichlet problem in terms of harmonic measure

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set and set $\sigma := \mathcal{H}^n|_{\partial\Omega}$ to be its surface measure. In Definition 8.30 we define the cone with vertex $\xi \in \partial\Omega$ and aperture $\alpha > 0$ by

$$\Gamma_\alpha(\xi) = \big\{y \in \Omega : |\xi - y| < (1+\alpha)\operatorname{dist}(y, \partial\Omega)\big\} \tag{10.50}$$

and the non-tangential maximal function operator of a measurable function $u : \Omega \to \mathbb{R}$ by

$$\mathcal{N}_\alpha(u)(\xi) := \sup_{y \in \Gamma_\alpha(\xi)} |u(y)|, \ \ \xi \in \partial\Omega. \tag{10.51}$$

Theorem 10.15. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with such that $\partial\Omega$ is n-Ahlfors regular. For $\alpha, \beta > 0$ and any function $u : \Omega \to \mathbb{R}$, we have*

$$\|\mathcal{N}_\alpha(u)\|_{L^p(\sigma)} \approx_{\alpha,\beta} \|\mathcal{N}_\beta(u)\|_{L^p(\sigma)}.$$

For the proof, see [HMT10, Proposition 2.2], for example.

Because of the preceding result, when estimating $\|\mathcal{N}_\alpha(u)\|_{L^p(\sigma)}$, quite often we will not just write $\mathcal{N}(u)$ in place of $\mathcal{N}_\alpha(u)$. For definiteness, we can think that $\alpha = 1$, although the relevant value of α will not be important for us.

For $1 \leqslant p \leqslant \infty$, we say that the Dirichlet problem is solvable in L^p for the Laplacian (writing (D_p) is solvable) if there exists some constant $C_p > 0$ such that, for any $f \in C_c(\partial\Omega)$, the solution $u : \Omega \to \mathbb{R}$ of the continuous Dirichlet problem for the Laplacian in Ω with boundary data f satisfies

$$\|\mathcal{N}(u)\|_{L^p(\sigma)} \leqslant C_p \, \|f\|_{L^p(\sigma)}.$$

By the maximum principle, it is clear that (D_∞) is solvable. Consequently, by interpolation, if (D_p) is solvable, then (D_q) is solvable for $q > p$.

The objective of this section is to characterize the solvability of (D_p) for $1 < p < \infty$ in terms of the analytic properties of harmonic measure. We need the following result.

Lemma 10.16. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with bounded n-Ahlfors regular boundary. Given $x \in \Omega$, denote by ω^x the harmonic measure for Ω with pole at x. Suppose that ω^x is absolutely continuous with respect to surface measure for every x. Let $p \in (1, \infty)$ and $\Lambda > 1$ and suppose that, for every ball B centered at $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\Omega)$ and all $x \in \Lambda B \cap \Omega$ such that $\operatorname{dist}(x, \partial\Omega) \geqslant \Lambda^{-1} r(B)$, it holds*

$$\left(\fint_{\Lambda B} \left(\frac{d\omega^x}{d\sigma} \right)^p d\sigma \right)^{1/p} \leqslant \kappa \, \sigma(B)^{-1}, \tag{10.52}$$

for some $\kappa > 0$. Then, if Λ is big enough, the Dirichlet problem is solvable in L^s, for $s > p'$. Further, for all $f \in L^{p'}(\sigma) \cap C(\partial\Omega)$, its harmonic extension u to Ω satisfies

$$\|\mathcal{N}(u)\|_{L^{p',\infty}(\sigma)} \lesssim \kappa \, \|f\|_{L^{p'}(\sigma)}. \tag{10.53}$$

Proof. Let $f \in C(\partial\Omega)$ and let u the solution of the Dirichlet problem in Ω with boundary data f. Suppose that $f \geqslant 0$. Consider a point $\xi \in \partial\Omega$ and a non-tangential cone $\Gamma(\xi) \subset \Omega$, with vertex ξ and with a fixed aperture. Fix a point $x \in \Gamma(\xi)$ and denote $\mathrm{d}_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. We intend to estimate $u(x)$, first assuming $\mathrm{d}_\Omega(x) \leqslant 2\operatorname{diam}(\partial\Omega)$.

To this end, we pick a smooth function φ which equals 1 in $B_1(0)$ and vanishes in $\mathbb{R}^{n+1} \backslash B_2(0)$. For some $M > 4$ to be chosen later, we denote

$$\varphi_M(y) = \varphi\Big(\frac{y}{M \mathrm{d}_\Omega(x)} \Big).$$

We set

$$f_0(y) = f(y)\, \varphi_M(y - \xi), \qquad f_1(y) = f(y) - f_0(y),$$

and we denote by u_0 and u_1 the corresponding solutions of the associated Dirichlet problems so that $u = u_0 + u_1$.

In the following computations, to shorten notation we denote $d_x = \mathrm{d}_\Omega(x)$. To estimate $u_0(x)$ we use (10.52) to show that

$$\begin{aligned}
u_0(x) &= \int f_0\, d\omega^x \leqslant \int_{B_{2Md_x}(\xi)} f\, \frac{d\omega^x}{d\sigma}\, d\sigma \\
&\leqslant \left(\int_{B_{2Md_x}(\xi)} |f|^{p'}\, d\sigma\right)^{1/p'} \left(\int_{B_{2Md_x}(\xi)} \left(\frac{d\omega^x}{d\sigma}\right)^p d\sigma\right)^{1/p} \\
&\leqslant \kappa\, C(M)\, \mathcal{M}_{\sigma,p'} f(\xi)\, \frac{\sigma(B_{2Md_x}(\xi))^{1/p'}}{\sigma(B_{d_x}(\xi))^{1/p'}} \lesssim \kappa\, C(M)\, \mathcal{M}_{\sigma,p'} f(\xi),
\end{aligned}$$

for $p' = p/(p-1)$, assuming $\Lambda \geqslant 2M$, where we wrote $\mathcal{M}_{\sigma,p'} f := \left(\mathcal{M}_\sigma(|f|^{p'})\right)^{1/p'}$.

Next we deal with $u_1(x)$, which we extende by 0 outside Ω. Note that $u_1 \neq 0$ implies that $B^c_{M\mathrm{d}_\Omega(x)} \cap \partial\Omega \neq \varnothing$, so $M\mathrm{d}_\Omega(x) \leqslant \mathrm{diam}(\partial\Omega)$. First we estimate $⨍_{B_{Md_x}(\xi)} u_1\, dm$ by the integral of its non-tangential maximal function. To do so, we use a classical trick of relating Whitney cubes $\mathcal{W} := \mathcal{W}(\Omega)$ in Ω (see Section 8.3.2) to a certain dyadic structure in $\partial\Omega$: denote by $I_B \subset \mathcal{W}$ the family of those cubes that intersect $B := B_{Md_x}(\xi)$. By the properties of $\mathcal{W}$, the cubes $P \in I_B$ are contained in $CB := B_{CMd_x}(\xi)$, for some C depending just on n and the parameters in the construction of $\mathcal{W}$. For every cube $P \in I_B$ we define $b(P)$ to be a Whitney cube of the same side-length intersecting $\partial\Omega$ such that for every $\xi \in b(P) \cap \partial\Omega$ we have $P \subset \Gamma(\xi)$. This is well defined as long as α is big enough, and the number of cubes $Q \in \mathcal{W}$ such that $b(Q) = b(P)$ is bounded by a dimensional constant (depending also on α). Again we have $b(P) \subset C'B$. Then, taking into account that $u_1 \leqslant u$, we have

$$\begin{aligned}
\int_{B_{Md_x}(\xi)} u_1\, dm &\leqslant \sum_{P \in I_B} \int_P u\, dm \leqslant \sum_{P \in I_B} \inf_{y \in b(P) \cap \partial\Omega} \mathcal{N}u(y)\, \ell(P)^{n+1} \\
&\lesssim \sum_{Q \in \mathcal{W}: Q \subset C'B} \inf_{Q \cap \partial\Omega} \mathcal{N}u \sum_{P \in I_B:\, Q = b(P)} \ell(P)^{n+1} \\
&\lesssim \sum_{Q \in \mathcal{W}: Q \subset C'B} \ell(Q) \int_{3Q \cap \partial\Omega} \mathcal{N}u\, d\sigma \lesssim M d_x \int_{C'B} \mathcal{N}u\, d\sigma.
\end{aligned} \tag{10.54}$$

So we deduce

$$⨍_{B_{Md_x}(\xi)} u_1\, dm \lesssim ⨍_{C'B} \mathcal{N}u\, d\sigma \lesssim \mathcal{M}_\sigma(\mathcal{N}u)(\xi).$$

Now, taking into account that f_1 vanishes in $B_{Md_x}(\xi)$, from the Hölder continuity of u_1 in $\Omega \cap B_{Md_x/2}(\xi)$ (see Lemma 7.27) and the fact that u_1 is subharmonic in $B_{Md_x(\xi)}$ (see Lemma 5.7), we infer that

$$u_1(x) \lesssim \frac{1}{M^\alpha} ⨍_{B_{Md_x}(\xi)} u_1\, dm \lesssim \frac{1}{M^\alpha}\, \mathcal{M}_\sigma(\mathcal{N}u)(\xi),$$

for some $\alpha > 0$ depending just on the Ahlfors regularity constant of $\partial\Omega$.

Altogether, for all $x \in \Gamma(\xi)$ with $d_\Omega(x) \leqslant 2\operatorname{diam}(\partial\Omega)$ we have

$$u(x) \leqslant \kappa\, C(M)\, \mathcal{M}_{\sigma,p'} f(\xi) + \frac{C}{M^\alpha}\, \mathcal{M}_\sigma(\mathcal{N}u)(\xi). \tag{10.55}$$

In case that Ω is unbounded, it turns out that the closure of $A := \{x \in \Omega : d_\Omega(x) \leqslant 2\operatorname{diam}(\partial\Omega) > 2\operatorname{diam}(\partial\Omega)\}$ is contained in the cone $\Gamma(\xi)$ if the aperture of $\Gamma(\xi)$ is assumed to be big enough. Thus, by the maximum principle, since (10.55) holds for $x \in \partial A$ and u vanishes at ∞, it follows that the same estimate is also valid for $x \in \Gamma(\xi) \cap A$. Hence (10.55) holds for all $x \in \Gamma(\xi)$ in any case. So we obtain

$$\mathcal{N}u(\xi) \leqslant \kappa\, C(M)\, \mathcal{M}_{\sigma,p'} f(\xi) + \frac{C}{M^\alpha}\, \mathcal{M}_\sigma(\mathcal{N}u)(\xi) \quad \text{ for all } \xi \in \partial\Omega. \tag{10.56}$$

Thus, for $s > p'$,

$$\begin{aligned}
\|\mathcal{N}u\|_{L^s(\sigma)} &\leqslant \kappa\, C(M)\, \|\mathcal{M}_{\sigma,p'} f\|_{L^s(\sigma)} + \frac{C}{M^\alpha}\, \|\mathcal{M}_\sigma(\mathcal{N}u)\|_{L^s(\sigma)} \\
&\leqslant \kappa\, C'(M)\, \|f\|_{L^s(\sigma)} + \frac{C'}{M^\alpha}\, \|\mathcal{N}u\|_{L^s(\sigma)}.
\end{aligned}$$

Since f is continuous and $\partial\Omega$ is bounded, $\|\mathcal{N}u\|_{L^s(\sigma)} < \infty$, and hence, choosing M (and thus Λ) big enough, we get

$$\|\mathcal{N}u\|_{L^s(\sigma)} \leqslant \kappa\, C'(M)\, \|f\|_{L^s(\sigma)}.$$

Regarding the last statement of the lemma, recall that $\mathcal{M}_{\sigma,p'}$ is bounded from $L^{p'}(\sigma)$ to $L^{p',\infty}(\sigma)$ and that $\mathcal{M}_\sigma$ is bounded in $L^{p',\infty}(\sigma)$. Then, from (10.56) we infer that

$$\begin{aligned}
\|\mathcal{N}u\|_{L^{p',\infty}(\sigma)} &\leqslant \kappa\, C(M)\, \|\mathcal{M}_{\sigma,p'} f\|_{L^{p',\infty}(\sigma)} + \frac{C}{M^\alpha}\, \|\mathcal{M}_\sigma(\mathcal{N}u)\|_{L^{p',\infty}(\sigma)} \\
&\lesssim \kappa\, C(M)\, \|f\|_{L^{p'}(\sigma)} + \frac{C}{M^\alpha}\, \|\mathcal{N}u\|_{L^{p',\infty}(\sigma)}.
\end{aligned}$$

Since $\|\mathcal{N}u\|_{L^{p',\infty}(\sigma)} < \infty$, the latter gives (10.53) for M and Λ big enough. □

Theorem 10.17. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with bounded n-Ahlfors regular boundary. Given $x \in \Omega$, denote by ω^x the harmonic measure for Ω with pole at x. For $p \in (1, \infty)$, the following are equivalent:*

(a) $(D_{p'})$ is solvable for Ω.

(b) The harmonic measure ω is absolutely continuous with respect to σ and for every ball B centered in $\partial\Omega$ and for all $x \in \Omega \cap 3B \backslash 2B$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$, it holds

$$\left(\fint_B \left(\frac{d\omega^x}{d\sigma} \right)^p d\sigma \right)^{1/p} \lesssim \sigma(B)^{-1}. \tag{10.57}$$

(c) *The harmonic measure ω is absolutely continuous with respect to σ and there is some $\Lambda > 1$ big enough such that, for every ball B centered in $\partial\Omega$ with* $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$ *and all $x \in \Lambda B \cap \Omega$ such that* $\operatorname{dist}(x, \partial\Omega) \geqslant \Lambda^{-1} r(B)$, *it holds*

$$\left(\fint_{\Lambda B} \left(\frac{d\omega^x}{d\sigma} \right)^p d\sigma \right)^{1/p} \lesssim_\Lambda \sigma(B)^{-1}.$$

By duality, (10.57) is equivalent to the following: for every ball B centered in $\partial\Omega$, for all $x \in \Omega \cap 3B \backslash 2B$, and all $f \in C_c(\partial\Omega \cap B)$,

$$\left| \int_B f \, d\omega^x \right| \lesssim \|f\|_{L^{p'}(\sigma)} \sigma(B)^{-1/p'}.$$

Denoting by u the harmonic extension of f to Ω, it can be rewritten as

$$|u(x)| \lesssim \|f\|_{L^{p'}(\sigma)} \sigma(B)^{-1/p'}. \tag{10.58}$$

Proof of Theorem 10.17. (a) $\Rightarrow$ (b). To prove (10.58), by standard arguments (as in (10.54), say) and the $L^{p'}$ solvability of the Dirichlet problem, it follows that

$$\fint_{4B} |u| \, dm \lesssim \fint_{CB \cap \partial\Omega} |\mathcal{N}(u)| \, d\sigma \leqslant \left(\fint_{CB \cap \partial\Omega} |\mathcal{N}(u)|^{p'} \, d\sigma \right)^{1/p'} \lesssim \|f\|_{L^{p'}(\sigma)} \sigma(B)^{-1/p'}.$$

By the subharmonicity of $|u|$ (extended by 0 in Ω^c, see Lemma 5.7) in $4B \backslash B$, we have

$$|u(x)| \lesssim \fint_{4B} |u| \, dm \quad \text{for all } x \in \Omega \cap 3B \backslash 2B.$$

Together with the previous estimate, this implies (b).

(a) $\Rightarrow$ (c). The arguments are almost the same as the ones in the proof of (a) $\Rightarrow$ (b), just replacing the condition $x \in \Omega \cap 3B \backslash 2B$ by $x \in \Omega \cap \Lambda B$, $\operatorname{dist}(x, \partial\Omega) \geqslant \Lambda^{-1} r(B)$. We leave the details for the reader.

(b) $\Rightarrow$ (a). First we will show that there exists some $\varepsilon > 0$ such that for any ball B centered in $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$ and for all $x \in \Omega \backslash 6B$,

$$\left(\fint_B \left(\frac{d\omega^x}{d\sigma} \right)^{p+\varepsilon} d\sigma \right)^{1/(p+\varepsilon)} \lesssim \sigma(B)^{-1}, \tag{10.59}$$

To this end, notice first that, for all $x \in \Omega \cap \partial(2B)$, by (7.20)

$$\omega^x(8B) \gtrsim 1.$$

Then, for any function $f \in C_c(B \cap \partial\Omega)$, the assumption in (b) and the preceding estimate give

$$|u(x)| \leqslant C \|f\|_{L^{p'}(\sigma)} \sigma(B)^{-1/p'} \leqslant C \|f\|_{L^{p'}(\sigma)} \frac{\omega^x(8B)}{\sigma(B)^{1/p'}} \quad \text{for all } x \in \Omega \cap \partial(2B),$$

where, as above, u is the harmonic extension of f to Ω. By the maximum principle we infer that the above inequality also holds for all $y \in \Omega \setminus 2B$. By duality it follows that

$$\left(\fint_B \left(\frac{d\omega^y}{d\sigma}\right)^p d\sigma\right)^{1/p} \lesssim \frac{\omega^y(8B)}{\sigma(B)} \quad \text{for all } y \in \Omega\setminus 2B.$$

So for any given ball B_0 centered in $\partial\Omega$ with $\operatorname{diam}(B_0) \leqslant 2\operatorname{diam}(\partial\Omega)$ and $y \in \Omega\setminus 6B_0$ and any ball B' centered at $1.1B_0 \cap \partial\Omega$ with $r(B') \leqslant 2r(B_0)$, we have

$$\left(\fint_{B'} \left(\frac{d\omega^y}{d\sigma}\right)^p d\sigma\right)^{1/p} \lesssim \frac{\omega^y(8B')}{\sigma(B')}.$$

By Gehring's lemma (see [GM12, Theorem 6.38], for example) adapted to n-Ahlfors regular sets, there exists some $\varepsilon > 0$ such that

$$\left(\fint_{B_0} \left(\frac{d\omega^y}{d\sigma}\right)^{p+\varepsilon} d\sigma\right)^{1/(p+\varepsilon)} \lesssim \frac{\omega^y(8B_0)}{\sigma(B_0)},$$

which yields (10.59).

Next we intend to apply Lemma 10.16 with $p+\varepsilon$ in place of p. To this end, given $\Lambda > 1$, a ball B centered in $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$, and $x \in \Lambda B$ with $\operatorname{dist}(x, \partial\Omega) \geqslant \Lambda^{-1} r(B)$, we cover $B \cap \partial\Omega$ with a family of balls B_i, $i \in I_B$, with $r(B_i) = (100\Lambda)^{-1} r(B)$, so that the balls B_i are centered at $B \cap \partial\Omega$, $x \notin 6B_i$ for any $i \in I_B$, and $\#I_B \leqslant C(\Lambda)$. Applying (10.59) to each of the balls B_i and summing over $i \in I_B$, we infer that

$$\left(\fint_{\Lambda B} \left(\frac{d\omega^x}{d\sigma}\right)^{p+\varepsilon} d\sigma\right)^{1/(p+\varepsilon)} \leqslant C(\Lambda)\, \sigma(B)^{-1}.$$

From Lemma 10.16 we deduce that (D_s) is solvable for $s > (p+\varepsilon)'$, and thus in particular for $s = p'$.

(c) $\Rightarrow$ (b). We will argue in the same way as in the proof of (a) $\Rightarrow$ (b), using the estimate (10.53) instead of the solvability of $(D_{p'})$. Again by duality, it suffices to show that for every ball B centered in $\partial\Omega$ with $\operatorname{diam}(B) \leqslant 2\operatorname{diam}(\partial\Omega)$, for all $x \in \Omega \cap 3B\setminus 2B$ and all $f \in C_c(\partial\Omega \cap B)$, the harmonic extension u of f to Ω satisfies

$$|u(x)| \lesssim \|f\|_{L^{p'}(\sigma)} \sigma(B)^{-1/p'}. \tag{10.60}$$

By standard arguments, the Kolmogorov inequality, and Lemma 10.16, we have

$$\fint_{4B} |u|\, dm \lesssim \fint_{CB} \mathcal{N}(u)\, d\sigma \overset{\text{L.4.10}}{\lesssim} \|\mathcal{N}(u)\|_{L^{p',\infty}(\sigma)}\, \sigma(B)^{-1/p'} \overset{(10.53)}{\lesssim} \|f\|_{L^{p'}(\sigma)}\, \sigma(B)^{-1/p'}.$$

Since f vanishes in $\partial\Omega\setminus B$, by the subharmonicity of $|u|$ (extended by 0 to Ω^c) in $4B\setminus B$ we have

$$|u(x)| \lesssim \fint_{4B} |u|\, dm \quad \text{for all } x \in \Omega \cap 3B\setminus 2B,$$

which, together with the previous estimate, implies (10.60). $\square$

Remark 10.18. The arguments in the above proof of (b) $\Rightarrow$ (a) show that solvability of $(D_{p'})$ for some $p' \in (1,\infty)$ implies solvability of $(D_{p'-\varepsilon})$ for some $\varepsilon > 0$.

Remark 10.19. The above theorem also holds if $\partial\Omega$ is unbounded. Indeed, the only place where the boundedness of $\partial\Omega$ is used is in Lemma 10.16, to ensure that $\|\mathcal{N}u\|_{L^s(\mu)} < \infty$ and $\|\mathcal{N}u\|_{L^{p',\infty}(\sigma)} < \infty$. A way of circumventing this technical problem is the following. For $r > 0$, consider the open set $\Omega_r := \Omega \cap B_r(0)$. It is easy to check that $\partial\Omega_r$ is n-Ahlfors regular and that an estimate such as (10.52) also holds for the harmonic measure ω_{Ω_r}, with bounds uniform on r, so that (D_s) is solvable for Ω_r, with $s > p'$, and (10.53) also holds. Given $f \in C(\partial\Omega)$ with compact support, let $r > 0$ be big enough so that $\operatorname{supp} f \subset B_r(0)$, and let $f_r : \partial\Omega_r \to \mathbb{R}$ be such that $f_r = f$ in $\partial\Omega \cap B_r(0)$ and $f_r = 0$ in $\partial\Omega_r \cap \Omega$. The we apply Lemma 10.16 to the solution u_r of the Dirichlet problem with data f_r in Ω_r. Letting $r \to \infty$, then one easily deduces that $\|\mathcal{N}u\|_{L^s(\sigma)} \lesssim \kappa \|f\|_{L^s(\sigma)}$, as well as the related estimate (10.53). We leave the details for the reader.

Theorem 10.20. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain. Then we have:*

(a) If Ω is a Lipschitz domain, then there exists some $\varepsilon_0 > 0$ depending just on the Lipschitz character of Ω such that (D_p) is solvable for $p \geqslant 2 - \varepsilon_0$.

(b) If Ω is chord-arc domain, then there exists some $p_0 > 1$ depending just on the chord-arc character of Ω such that (D_p) is solvable for $p \geqslant p_0$.

Proof. Suppose that Ω is a Lipschitz domain. Let $x_0 \in \Omega$ such that $\mathrm{d}_\Omega(x_0) := \operatorname{dist}(x_0, \partial\Omega) \approx \operatorname{diam}(\partial\Omega)$. By Dahlberg's theorem, the density function $\frac{d\omega^{x_0}}{d\sigma}$ satisfies the reverse Hölder inequality (10.30) with exponent 2. By Gehring's lemma (see Lemma 4.53) we deduce that an analogous reverse Hölder inequality holds for some exponent $q_0 > 2$. That is, for any ball B centered in $\partial\Omega$, with $\Delta_B = B \cap \Omega$, we get

$$\left(⨍_{\Delta_B} \left(\frac{d\omega^{x_0}}{d\sigma}\right)^{q_0} d\sigma \right)^{1/q_0} \leqslant C ⨍_{\Delta_B} \frac{d\omega^{x_0}}{d\sigma}\, d\sigma = C\, \frac{\omega^{x_0}(\Delta_B)}{\sigma(\Delta_B)}, \tag{10.61}$$

Note that the change of pole formula readily implies that for $x \in \Omega \cap 3B \backslash 2B$ with $\mathrm{d}_\Omega(x) \approx r(B)$, we obtain ω^x-a.e.

$$\frac{d\omega^x}{d\sigma} = \frac{d\omega^x}{d\omega^{x_0}} \frac{d\omega^{x_0}}{d\sigma} \overset{\text{L.8.21}}{\approx} \omega^{x_0}(\Delta_B)^{-1} \frac{d\omega^{x_0}}{d\sigma}.$$

In case $\mathrm{d}_\Omega(x) \leqslant A^{-1} r(B)$, then using Lemma 8.10 we get $\frac{d\omega^x}{d\sigma} \lesssim \omega^{x_0}(\Delta_B)^{-1} \frac{d\omega^{x_0}}{d\sigma}$. Consequently, the condition (b) in Theorem 10.17 is satisfied, with exponent q_0, which implies that $(D_{q_0'})$ is solvable, where q_0' is the conjugate exponent of q_0. By interpolation, (D_p) is solvable for $p \geqslant q_0'$, with $q_0' < 2$.

In case that Ω is assumed to be just a chord-arc domain, by Theorem 10.14 we know that $\frac{d\omega^{x_0}}{d\sigma}$ is an $A_\infty(\sigma)$ weight, and thus there exists some $q_0 > 1$ such that a reverse Hölder inequality such as (10.61) holds. As above, by the change of pole formula and by Theorem 10.17 we infer that $(D_{q_0'})$ is solvable, and by interpolation, (D_p) is solvable for $p \geqslant q_0'$, with $q_0' \in (1, \infty)$. $\square$

11 Rectifiability of harmonic measure

A set $E \subset \mathbb{R}^{n+1}$ is called n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $i = 1, 2, \ldots$, such that

$$\mathcal{H}^n\Big(E\setminus\bigcup_i f_i(\mathbb{R}^n)\Big) = 0. \tag{11.1}$$

A set $F \subset \mathbb{R}^{n+1}$ is called purely n-unrectifiable if $\mathcal{H}^n(F \cap E) = 0$ for every n-rectifiable set E. As for sets, one can define a notion of rectifiability also for measures: a measure μ is said to be n-rectifiable if it vanishes outside an n-rectifiable set $E \subset \mathbb{R}^{n+1}$ and, moreover, it is absolutely continuous with respect to $\mathcal{H}^n|_E$.

In this chapter we will investigate the connection between rectifiability and harmonic measure. First, under suitable assumptions on a domain $\Omega \subset \mathbb{R}^{n+1}$, we will show that harmonic measure for Ω and the Hausdorff measure $\mathcal{H}^n$ are mutually absolutely continuous on the set of cone points of $\partial\Omega$, which is a rectifiable set. Afterwards, in the converse direction, we will see that if the harmonic measure ω for Ω and that the Hausdorff measure $\mathcal{H}^n$ are mutually absolutely continuous on some subset $E \subset \partial\Omega$, then E is n-rectifiable, or equivalently, $\omega|_E$ is n-rectifiable.

11.1 Harmonic measure in the set of cone points

Definition 11.1. Let $\Omega \subset \mathbb{R}^{n+1}$ be open. A point $x \in \partial\Omega$ is called a cone point (for Ω) if there exist a unit vector $v \in \mathbb{R}^{n+1}$, $\alpha \in (0, 1)$, and $r > 0$ such that

$$C(x, v, \alpha, r) := \big\{y \in B_r(x) : (y - x) \cdot v > \alpha\, |y - x|\big\} \subset \Omega.$$

Remark that $C(x, v, \alpha, r)$ is a one sided truncated open cone with vertex x and axis parallel to v.

First we will show that the set of cone points for an open set Ω is n-rectifiable. We will prove this using the following basic lemma.

Lemma 11.2. *Let $0 < r \leqslant \infty$ and let $v \in \mathbb{R}^{n+1}$ be a unit vector and V the orthogonal vector space to v. Let $E \subset \mathbb{R}^{n+1}$ be such that* $\operatorname{diam}(E) < r$ *and*

$$E \cap C(x, v, \alpha, r) = \varnothing \quad \text{for all } x \in E. \tag{11.2}$$

Then E is contained in the graph of a Lipschitz function $A : V \to V^\perp$.

Proof. Denote by Π_V the orthogonal projection onto V. Consider $x, y \in E$. Since $y \notin C(x, v, \alpha, r)$, then $(y-x)\cdot v \leqslant \alpha\, |y-x|$; and since $x \notin C(y, v, \alpha, r)$, then $(x-y)\cdot v \leqslant \alpha\, |y-x|$. Therefore,

$$|(y - x) \cdot v| \leqslant \alpha\, |y - x|.$$

Consequently,

$$|\Pi_V(y) - \Pi_V(x)|^2 = |x-y|^2 - |(y-x)\cdot v|^2 \geqslant (1-\alpha^2)\,|x-y|^2.$$

So $\Pi_V|_E$ is one to one with Lipschitz inverse, with $\mathrm{Lip}((\Pi_V|_E)^{-1}) \leqslant (1-\alpha^2)^{-1/2}$. □

Proposition 11.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be open. Let $K \subset \partial\Omega$ the subset of all cone points. Then K can be covered by a countable collection of Lipschitz graphs and thus it is n-rectifiable. In particular, $\mathcal{H}^n|_K$ is σ-finite.*

Notice that the proposition ensures something stronger than the n-rectifiability of K: this is contained in a countable union of Lipschitz graphs, without leaving any subset of zero measure $\mathcal{H}^n$.

Proof. Let $\{v_i\}_{i\in I}$ be a countable and dense family of unit vectors in the sphere $\mathbb{S}^n$. For $i \in I$ and $m \geqslant 1$, let $K_{i,m}$ the subset of the cone points $x \in K$ such that

$$C(x, v_i, 1/m, 1/m) \subset \Omega.$$

It follows easily that

$$K = \bigcup_{i\in I}\bigcup_{m\geqslant 1} K_{i,m}.$$

For each i, m, consider a covering of $K_{i,m}$ with a finite or countable family of open balls B_j, $j \in J_{i,m}$, centered in $K_{i,m}$ with radii $1/(2m)$. For each i, m, j, the set $K_{i,m} \cap B_j$ satisfies the assumption (11.2), with $v = v_i$, $\alpha = r = 1/m$. So $K_{i,m} \cap B_j$ is contained in a Lipschitz graph. □

Our main objective in this section is to prove the following result.

Theorem 11.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, let $p \in \Omega$, and let $K \subset \partial\Omega$ be the subset of cone points for Ω. Then $\mathcal{H}^n|_K \ll \omega^p_\Omega|_K$, that is, $\mathcal{H}^n$ is absolutely continuous with respect to ω^p_Ω on K.*

Suppose moreover that the following holds: there exists some $c > 0$ such that

$$\mathcal{H}^n_\infty(B_r(\xi)\backslash\Omega) \geqslant c\, r^n \quad \text{for all } \xi \in \partial\Omega \text{ and } r > 0. \tag{11.3}$$

Then $\omega^p_\Omega|_K \ll \mathcal{H}^n|_K$, that is, ω^p_Ω is absolutely continuous with respect to $\mathcal{H}^n$ on K.

For simply connected domains in the plane, this result is a well-known theorem of McMillan [McM69]. The extension to higher dimensions announced above is due to Akman, Azzam, and Mourgoglou [AAM19].

If $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying (11.3) for some $c > 0$, we say that Ω has large n-dimensional complement, or just large complement, for short. Notice that if Ω has large complement, then it is Wiener regular and satisfies the CDC. The assumption of having large complement cannot be eliminated in the second statement in the theorem. For example, let $E \subset [0,1] \subset \mathbb{C}$ be the usual ternary Cantor set, and consider the planar domain $\Omega = B_2(0)\backslash E$. It is immediate to check that all the points from $\partial\Omega$ (and, in particular the ones from E) are cone points. However, $\omega(E) > 0$, while $\mathcal{H}^n(E) = 0$. On the other hand, remark that all planar simply connected domains have large complement, taking into account that $\partial\Omega$ is connected.

We will first prove the absolute continuity of surface measure with respect to harmonic measure on the set K of cone points for Ω. We will derive this from Dahlberg's theorem and the maximum principle.

Proof of $\mathcal{H}^n|_K \ll \omega^p|_K$ in Theorem 11.4. As in the proof of Proposition 11.3, let $\{v_i\}_{i\in I}$ be a countable and dense family of unit vectors in the sphere $\mathbb{S}^n$. For $i \in I$ and $m \geqslant 1$, let $K_{i,m}$ the subset of the cone points $x \in K$ such that

$$C(x, v_i, 1/m, 1/m) \subset \Omega,$$

so that

$$K = \bigcup_{i\in I}\bigcup_{m\geqslant 1} K_{i,m}.$$

For each i, m, consider a covering of $K_{i,m}$ with a finite family of open balls B_j, $j \in J_{i,m}$, centered in $K_{i,m}$ with radii $1/(5m)$. Observe that

$$\Omega_{i,m,j} := 2B_j \cap \bigcup_{x\in K_{i,m}\cap B_j} C(x, v_i, 1/m, 1/m) \subset \Omega.$$

Further, using Lemma 11.2, it is easy to check that each $\Omega_{i,m,j}$ is a Lipschitz domain, and that

$$K_{i,m} \subset \bigcup_{j\in J_{i,m}} \partial\Omega_{i,m,j}.$$

By Dahlberg's theorem and Lemma 5.32, it follows that

$$\mathcal{H}^n|_{\partial\Omega_{i,m,j}\cap K_{i,m}} \ll \omega_{\Omega_{i,m,j}}|_{K_{i,m}} \ll \omega_\Omega|_{K_{i,m}}.$$

Since this holds for all i, m, j, we deduce that $\mathcal{H}^n|_K \ll \omega_\Omega$. $\square$

The proof of the fact that $\omega^p|_K \ll \mathcal{H}^n|_K$ when Ω has large complement is more complicated. We will need the following auxiliary lemma.

Lemma 11.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a Wiener regular bounded domain, let $p \in \Omega$, and let $E \subset \partial\Omega$ be a Borel set. Then $\omega^p_\Omega(E) = 0$ if and only if*

$$\sup_{x\in\Omega} \omega^x_\Omega(E) < 1. \tag{11.4}$$

Proof. We write $\omega = \omega_\Omega$. We only have to show that the condition (11.4) implies that $\omega^p(E) = 0$, since the converse implication is trivial. To this end, assume first that E is closed and denote

$$\lambda = \sup_{x\in\Omega} \omega^x(E).$$

Since E is closed, by Corollary 5.36, for every $\xi \in \partial\Omega\backslash E$,

$$\lim_{x\to\xi} \omega^x(E) = 0.$$

For any $\varepsilon > 0$, let $f_\varepsilon \in C(\partial\Omega)$ be a function which equals λ on E and vanishes away from an ε-neighborhood of E. From the above conditions, it follows that the function defined by $u(x) = \omega^x(E)$ belongs to the Perron class $\mathcal{L}_{f_\varepsilon}$ for Ω, and thus

$$\omega^x(E) = u(x) \leqslant H_{f_\varepsilon}(x) = \int f_\varepsilon \, d\omega^x \quad \text{for all } x \in \Omega.$$

On the other hand, by the outer regularity of harmonic measure and the definition of f_ε, we have

$$\lim_{\varepsilon\to 0} \int f_\varepsilon \, d\omega^x = \lambda\, \omega^x(E).$$

Hence,

$$\omega^x(E) \leqslant \lambda\, \omega^x(E).$$

Since $\lambda < 1$, this implies that $\omega^x(E) = 0$.

In the case when E is an arbitrary Borel set, the condition (11.4) implies that for any closed subset $F \subset E$ it also holds $\sup_{x\in\Omega} \omega_\Omega^x(F) < 1$. Hence $\omega^p(F) = 0$ and thus, by the inner regularity of harmonic measure, we infer that $\omega^p(E) = 0$. □

The main tool to prove the second statement in Theorem 11.4 is the following:

Theorem 11.6. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with large complement and let $p \in \Omega$. Let Γ be a Lipschitz graph and denote by V_1 and V_2 the two connected components of $\mathbb{R}^{n+1}\backslash\Gamma$. Let $E \subset \partial\Omega \cap \Gamma$ be a Borel set with $\omega^p(E) > 0$. Then there are points $x_i \in V_i \cap \Omega$, for $i = 1, 2$, such that*

$$\omega^{x_1}_{\Omega\cap V_1}(E) + \omega^{x_2}_{\Omega\cap V_2}(E) > 0.$$

For $x = (\bar{x}, x_{n+1}) \in \mathbb{R}^{n+1}$ and $h, r > 0$ we will use the following notation for an open cylinder centered at x with height $2h$ and radius r:

$$\mathrm{Cy}(x, r, h) = \big\{x \in \mathbb{R}^{n+1} : |\bar{x}| < r,\ |x_{n+1}| < h\big\}.$$

For each $i = 1, 2$, we define

$$\widetilde{V}_i = \mathrm{Cy}(p,\, 4\,\mathrm{diam}(\partial\Omega),\, 4\,\mathrm{diam}(\partial\Omega)) \cap V_i,$$

in a coordinate system such that Γ is a Lipschitz graph with respect to a horizontal hyperplane. Notice that each set $\widetilde{V}_i$ is a bounded Lipschitz domain such that $\Omega \cap V_i \subset \widetilde{V}_i \subset V_i$.

Theorem 11.6 is proven in [AAM19] in the more general situation where Γ is the boundary of a two sided chord-arc domain (i.e., the sets V_1, V_2 above are assumed to be chord-arc domains). In turn, the results in [AAM19] are inspired by the ones of Wu [Wu86], where a similar result is proved for domains Ω satisfying an exterior corkscrew condition. The assumption that Γ is a Lipschitz graph in Theorem 11.6 simplifies some technical points in the arguments in [AAM19] and it suffices to complete the proof of Theorem 11.4.

To deduce the second statement in Theorem 11.4, recall that K can be covered by a countable collection of Lipschitz graphs Γ_j, $j \geqslant 1$. If $\omega^p(E) > 0$ for some Borel set $E \subset K$, then there exists some j such that $\omega^p(E \cap \Gamma_j) > 0$. Let V_1, V_2 the connected components of $\mathbb{R}^{n+1} \backslash \Gamma_j$, and let $\widetilde{V}_1, \widetilde{V}_2$ be associated bounded Lipschitz domains as above. By Theorem 11.6, $\omega^{x_i}_{\Omega\cap\widetilde{V}_i}(E \cap \Gamma_j) > 0$ either for $i = 1$ or 2. Since $\omega^{x_i}_{\widetilde{V}_i}(E \cap \Gamma_j) \geqslant \omega^{x_i}_{\Omega\cap\widetilde{V}_i}(E \cap \Gamma_j)$, either

$$\omega^{x_1}_{\widetilde{V}_1}(E \cap \Gamma_j) > 0 \quad \text{ or } \quad \omega^{x_2}_{\widetilde{V}_2}(E \cap \Gamma_j) > 0.$$

By Dahlberg's theorem for Lipschitz domains, this implies that $\mathcal{H}^n(E) \geqslant \mathcal{H}^n(E \cap \Gamma_j) > 0$. This shows that $\omega^p_\Omega|_K \ll \mathcal{H}^n|_K$.

To prove Theorem 11.6 we need the following auxiliary result.

Lemma 11.7. *Let Ω, Γ, $\widetilde{V}_1$, $\widetilde{V}_2$, x_1, and x_2 be as in Theorem 11.6. Let $x \in \Gamma \cap \Omega$ and denote $r = \delta_\Omega(x)$ and $U_i = \widetilde{V}_i \cap \Omega$. For $i = 1, 2$ and some $c > 0$, consider balls*

$$B_{cr}(y_i) \subset \widetilde{V}_i \cap B_r(x).$$

Let $E \subset \partial\Omega \cap \Gamma$ be a Borel set such that

$$\omega^{x_1}_{U_1}(E) = \omega^{x_2}_{U_2}(E) = 0$$

and suppose that

$$\omega^{y_i}_{U_i}(\partial U_i \backslash (\Gamma \cap \Omega)) \gtrsim 1 \qquad \text{either for } i = 1 \text{ or } i = 2. \tag{11.5}$$

Then there exists some $\gamma \in (0, 1)$ such that $\omega^x_\Omega(E) < \gamma$, with γ depending on the above implicit constant and c.

Proof. Let $B_i = B(y_i, \frac{1}{2}cr)$. By (11.5), there exists some $\eta \in (0, 1)$ such that

$$\min_{i=1,2} \omega^{y_i}_{U_i}(\Gamma \cap \Omega) \leqslant \eta.$$

Suppose the minimum is attained for $i = 1$, for example. Then, by the Markov property for harmonic measure in Theorem 5.54,

$$\omega^{y_1}_\Omega(E) = \omega^{y_1}_{U_1}(E) + \int_{\Gamma\cap\Omega} \omega^z_\Omega(E)\, d\omega^{y_1}_{U_1}(z) \leqslant 0 + \eta = \eta.$$

Therefore, $\omega^{y_1}(E^c) \geqslant 1 - \eta$, and then, by Harnack's inequality in the ball $B_{\delta_\Omega(x)}(x)$, $\omega^x(E^c) > t$ for some $t > 0$ depending on η. Hence,

$$\omega^x(E) < 1 - t.$$

□

Proof of Theorem 11.6. Let $E \subset \partial\Omega \cap \Gamma$ be a Borel set and let V_i, and $\widetilde{V}_i$ be as in the theorem. Suppose that

$$\omega^{x_1}_{\Omega\cap\widetilde{V}_1}(E) = \omega^{x_2}_{\Omega\cap\widetilde{V}_2}(E) = 0 \quad \text{for all } x_1 \in \Omega \cap \widetilde{V}_1,\, x_2 \in \Omega \cap \widetilde{V}_2.$$

We intend to show that this implies that $\omega^p_\Omega(E) = 0$, which will prove the theorem. To this end, we claim that it suffices to show that there exists some $\gamma > 0$ such that

$$\omega^x_\Omega(E) < \gamma \quad \text{for all } x \in \Gamma \cap \Omega. \tag{11.6}$$

Indeed, if this holds, then, by the Markov property in Theorem 5.54, for $i = 1, 2$ and for all $x \in \Omega \cap \widetilde{V}_i$ we have

$$\omega^x_\Omega(E) = \omega^x_{\Omega\cap\widetilde{V}_i}(E) + \int_{\Gamma\cap\Omega} \omega^y_\Omega(E)\, d\omega^x_{\Omega\cap\widetilde{V}_i}(y) \leqslant 0 + \gamma = \gamma.$$

Together with (11.6), this implies that $\omega^x_\Omega(E) \leqslant \gamma < 1$ for all $x \in \Omega$ and so, by Lemma 11.5, $\omega^p_\Omega(E) = 0$, which proves our claim.

Denote $U_i = \widetilde{V}_i \cap \Omega$. By Lemma 11.7, to prove (11.6) it suffices to show that

$$\omega^{y_i}_{U_i}(\partial U_i \backslash (\Gamma \cap \Omega)) \gtrsim 1 \quad \text{either for } i = 1 \text{ or } i = 2, \tag{11.7}$$

for $y_i \in U_i$ as in that lemma. We distinguish three cases. For a large $M_0 > 2$ and a small $\varepsilon_0 \in (0, 1/2)$ both to be fixed below, suppose first that there exists some $z_0 \in \partial\Omega \cap B_{M_0 r}(x) \cap \widetilde{V}_1$ such that $\operatorname{dist}(z_0, \Gamma) \geqslant \varepsilon_0 r$ (recall that $r = \delta_\Omega(x)$). Since $\widetilde{V}_1$ is an NTA domain, there exists a Harnack chain of balls $\{B^j\}_{1\leqslant j\leqslant N}$ such that $y_1 \in B^1$ and $z_0 \in B^N$, with $10B^j \subset \widetilde{V}_1$ for each j, and such that $N \leqslant C(\varepsilon_0, M_0)$. Notice that $B^N \cap \partial U_1 \neq \varnothing$ because $z_0 \in \partial U_1$. Let $j_0 \geqslant 1$ be the smallest integer such that $2B^{j_0} \cap \partial\Omega \neq \varnothing$ (notice that $j_0 \leqslant N$). Since U_1 satisfies the CDC, for all $y \in 2B^{j_0} \cap U_1$ we have

$$\omega^y_{U_1}(\partial U_1 \backslash (\Gamma \cap \Omega)) \geqslant \omega^y_{U_1}(\partial U_1 \cap 10B^{j_0} \backslash (\Gamma \cap \Omega)) = \omega^y_{U_1}(\partial U_1 \cap 10B^{j_0}) \gtrsim 1. \tag{11.8}$$

Then, by the Harnack inequality,

$$\begin{aligned} \inf_{y\in B^1} \omega^y_{U_1}(\partial U_1 \backslash (\Gamma \cap \Omega)) &\approx \inf_{y\in B^2} \omega^y_{U_1}(\partial U_1 \backslash (\Gamma \cap \Omega)) \approx \ldots \approx \inf_{y\in B^{j_0-1}} \omega^y_{U_1}(\partial U_1 \backslash (\Gamma \cap \Omega)) \\ &\gtrsim \inf_{y\in B^{j_0}} \omega^y_{U_1}(\partial U_1 \backslash (\Gamma \cap \Omega)) \gtrsim 1. \end{aligned} \tag{11.9}$$

So in this case (11.7) holds, with the implicit constant depending on N. Analogously, we also deduce that (11.7) is satisfied if there exists some $z_0 \in \partial\Omega \cap B_{M_0 r}(x) \cap \widetilde{V}_2$ such that $\operatorname{dist}(z_0, \Gamma) \geqslant \varepsilon_0 r$.

It remains to deal with the case when

$$\operatorname{dist}(z, \Gamma) \leqslant \varepsilon_0\, r \quad \text{for all } z \in \partial\Omega \cap B_{M_0,r}(x) \cap (\widetilde{V}_1 \cup \widetilde{V}_2). \tag{11.10}$$

Since $\partial B_r(x) \cap \partial\Omega \neq \varnothing$, from the fact that Ω has large complement it follows that $\mathcal{H}^n_\infty(B_{2r}(x)\backslash\Omega) \gtrsim r^n$. Thus, $\mathcal{H}^n_\infty\big((B_{2r}(x) \cap \overline{V_i})\backslash\Omega\big) \gtrsim r^n$ either for $i = 1$ or $i = 2$. Without loss of generality, we suppose that

$$\mathcal{H}^n_\infty\big((B_{2r}(x) \cap \overline{V_1})\backslash\Omega\big) \gtrsim r^n. \tag{11.11}$$

Of course, this is equivalent to saying that $\mathcal{H}^n_\infty\big((B_{2r}(x) \cap \overline{\widetilde{V}_1})\backslash\Omega\big) \gtrsim r^n$.

Claim 11.8. *There exists a Lipschitz domain $W \subset U_1$ satisfying the following:*

(a) $\operatorname{diam}(W) \leqslant Cr$, *$y_1 \in W$, and $\delta_W(y_1) \approx r$.*

(b) Either $\mathcal{H}^n(\partial W \cap \Gamma \cap \partial\Omega) \geqslant c\,r^n$ for some fixed $c > 0$, or there exists a Borel set $G \subset \partial W\backslash\partial U_1$ such that $\mathcal{H}^n(G) \gtrsim r^n$ and

$$\omega^z_{U_1}(\partial U_1\backslash(\Gamma \cap \Omega)) \gtrsim 1 \quad \text{for all } z \in G.$$

The Lipschitz character of W only depends on the Lipschitz constant of Γ.

The construction of the domain W requires a delicate stopping time argument and will be carried out later. First we will show how the theorem follows from the properties of W stated in the claim.

Suppose first that $\mathcal{H}^n(\partial W \cap \Gamma \cap \partial\Omega) \geqslant c\,r^n$ for some $c > 0$. In particular, this implies that $\mathcal{H}^n(\partial W \cap \Gamma \cap \partial\Omega) \approx \mathcal{H}^n(\partial W)$. Since $W \subset U_1$ and

$$\partial W \cap \Gamma \cap \partial\Omega \subset \partial U_1\backslash(\Gamma \cap \Omega),$$

we deduce that

$$\omega^{y_1}_{U_1}(\partial U_1\backslash(\Gamma \cap \Omega)) \geqslant \omega^{y_1}_{U_1}(\partial W \cap \Gamma \cap \partial\Omega) \geqslant \omega^{y_1}_{W}(\partial W \cap \Gamma \cap \partial\Omega). \tag{11.12}$$

From the fact that $\omega^{y_1}_W$ is an A_∞ weight with respect to $\mathcal{H}^n|_{\partial W}$ (by Dahlberg's theorem), using that $\mathcal{H}^n(\partial W \cap \Gamma \cap \partial\Omega) \approx \mathcal{H}^n(\partial W)$, we infer that

$$\omega^{y_1}_W(\partial W \cap \Gamma \cap \partial\Omega) \approx \omega^{y_1}_W(\partial W) \approx 1.$$

Together with (11.12), this gives (11.7), for $i = 1$.

Assume now that there exists a Borel set G as in the statement (b) in Claim 11.8. By the Markov property in Theorem 5.54, we have

$$\begin{aligned}\omega^{y_1}_{U_1}(\partial U_1\backslash(\Gamma \cap \Omega)) &= \omega^{y_1}_W(\partial U_1\backslash(\Gamma \cap \Omega)) + \int_{\partial W\backslash\partial U_1} \omega^z_{U_1}(\partial U_1\backslash(\Gamma \cap \Omega))\, d\omega^{y_1}_W(z)\\ &\geqslant \int_G \omega^z_{U_1}(\partial U_1\backslash(\Gamma \cap \Omega))\, d\omega^{y_1}_W(z) \gtrsim \omega^{y_1}_W(G).\end{aligned}$$

As above, using that $\omega^{y_1}_W$ is an A_∞ weight with respect to $\mathcal{H}^n|_{\partial W}$ and that $\mathcal{H}^n(G) \approx \mathcal{H}^n(\partial W)$, we get $\omega^{y_1}_W(G) \approx \omega^{y_1}_W(\partial W) \approx 1$. Thus,

$$\omega^{y_1}_{U_1}(\partial U_1\backslash(\Gamma \cap \Omega)) \gtrsim 1.$$

This concludes the proof of the theorem, modulo Claim 11.8. □

Proof of Claim 11.8. Without loss of generality, we assume that $\partial B(x,r)$ intersects $\partial\Omega$ at the origin and that Γ is a Lipschitz graph with respect to the horizontal hyperplane $\{(\bar y, y_{n+1}) : y_{n+1} = 0\}$. Abusing notation, we identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$. We let $A : \mathbb{R}^n \to \mathbb{R}$ be the Lipschitz function whose graph coincides with Γ. Recall that, by the choice of x and y_1,

$$c\,r \leqslant |x - y_1| \leqslant r \quad \text{ and } \quad c\,r \leqslant \operatorname{dist}(y_1, \Gamma) \leqslant 2r.$$

Denote $\lambda = \|\nabla A\|_\infty$ and $\Lambda = 10 + 10\lambda$. Notice that

$$\Gamma \cap \operatorname{Cy}(0, 10r, \infty) \subset \operatorname{Cy}(0, 10r, 10\lambda r).$$

We choose M_0 large enough so that

$$\operatorname{Cy}(0, 10r, \Lambda r) \subset B(x, M_0 r/2).$$

Also, we assume that y_1 is contained in the upper component of $\operatorname{Cy}(0, 10r, \Lambda r)\backslash\Gamma$.

Construction of W. To construct W, first we consider the function $h : \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(\bar y) = \sup\left\{(t - A(\bar y))^+ : (\bar y, t) \in \operatorname{Cy}(0, 10r, \Lambda r) \setminus \Omega\right\},$$

where $(s)^+ = \max(s, 0)$. In case the set on the right hand is empty, we set $h(\bar y) = 0$. Recall that $\operatorname{dist}(z, \Gamma) \leqslant \varepsilon_0\, r$ for all $z \in \partial\Omega \cap \operatorname{Cy}(0, 10r, \Lambda r)$, by (11.10). That is, $\partial\Omega \cap \operatorname{Cy}(0, 10r, \Lambda r)$ is contained in an $(\varepsilon_0\, r)$-neighborhood of Γ. By connectivity, since y_1 does not belong to this neighborhood and belongs to the upper component of $\operatorname{Cy}(0, 10r, \Lambda r)\backslash\Gamma$, and moreover $y_1 \in \Omega$, it follows that the upper component of $\operatorname{Cy}(0, 10r, \Lambda r) \setminus \mathcal{U}_{\varepsilon_0 r}(\Gamma)$ is contained in Ω. Consequently,

$$h(\bar y) \leqslant C\varepsilon_0 r \quad \text{ for all } \bar y \in \mathbb{R}^n. \tag{11.13}$$

Next we consider the following function $d : \mathbb{R}^n \to \mathbb{R}$:

$$d(\bar y) = \sup_{\bar z \in \mathbb{R}^n} \big(4\, h(\bar z) - \theta|\bar y - \bar z|\big),$$

for some large constant $\theta > 2$ to be fixed below. Notice that this is a θ-Lipschitz function, since the supremum of a family of θ-Lipschitz functions is θ-Lipschitz. Observe also that

$$d(\bar y) \geqslant 4\, h(\bar y) \geqslant 0 \quad \text{ for all } \bar y \in \mathbb{R}^n.$$

Also, by (11.13),

$$d(\bar y) \leqslant C\varepsilon_0 r \quad \text{ for all } \bar y \in \mathbb{R}^n. \tag{11.14}$$

We let $A_W : \mathbb{R}^n \to \mathbb{R}$ be the Lipschitz function

$$A_W(\bar y) = A(\bar y) + d(\bar y),$$

we denote by Γ_W its graph, and we define

$$W = \{(\bar y, y_{n+1}) \in \operatorname{Cy}(0, 5r, \Lambda r) : y_{n+1} > A_W(\bar y)\}.$$

Clearly, this is a Lipschitz domain with $\operatorname{diam}(W) \approx_\Lambda r$. Further, from (11.14) it follows that

$$\Gamma_W \cap \mathrm{Cy}(0, 10r, \Lambda r) \subset \mathcal{U}_{C\varepsilon_0 r}(\Gamma) \cap \mathrm{Cy}(0, 10r, \Lambda r).$$

This implies that $y_1 \in W$, since y_1 is in the upper component of $\mathrm{Cy}(0, 5r, \Lambda r)\backslash\Gamma$ and $\operatorname{dist}(y, \Gamma) \approx r$. Together with the fact that $|y_1 - x| \leqslant r$, this gives $\delta_W(y) \approx r$. Remark that $W \cap \Omega \neq \varnothing$ (because $y_1 \in W$), and $\partial\Omega \cap W = \varnothing$, by the definition of h and the fact that $d \leqslant h$. Hence, using also that W is connected,

$$W \subset \Omega \cap \widetilde{V}_1 = U_1.$$

Proof of (b). We introduce a lattice $\mathcal{D}(\Gamma)$ of "dyadic cubes" of Γ as follows. Let $\mathcal{D}(\mathbb{R}^n)$ be the usual dyadic lattice of $\mathbb{R}^n$. Let Π be the orthogonal projection from Γ to $\mathbb{R}^n \equiv \mathbb{R}^n \times \{0\}$. Then we set

$$\mathcal{D}(\Gamma) = \{\Pi^{-1}(Q) \cap \Gamma : Q \in \mathcal{D}(\mathbb{R}^n)\}.$$

Here again we are identifying $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\}$. We also denote $\ell(\Pi^{-1}(Q) \cap \Gamma) := \ell(Q)$ and we call this the side length of $\Pi^{-1}(Q) \cap \Gamma$. Its center is the point whose projection by Π coincides with the center of Q.

Also, for a given $a > 1$, if $P \in \mathcal{D}(\Gamma)$ is such that $P = \Pi(Q)$, for some $Q \in \mathcal{D}(\mathbb{R}^n)$, we set $aP = \Pi(aQ)$ (for definiteness, we assume aQ to be half open-closed, in the same way as Q). We denote by z_P the center of P and we let $B_P = B(z_P, \ell(P))$.

Now we consider the family $\mathcal{M}$ of the maximal cubes $Q \in \mathcal{D}(\Gamma)$ such that there exists some $y \in 3Q$ such that $h(\Pi(y)) > \ell(Q)$. From (11.13), it follows that $\ell(Q) \lesssim \varepsilon_0 r$ for all $Q \in \mathcal{M}$. By the definition of h and the family $\mathcal{M}$, one easily deduces that there exists some constant $C_1 > 2$, possibly depending on Λ, such that

$$V_1 \cap \mathrm{C_y}(0, 10r, \Lambda r) \setminus \Omega \subset \bigcup_{Q \in \mathcal{M}} C_1 B_Q.$$

So we can write

$$\overline{V_1} \cap \mathrm{C_y}(0, 5r, \Lambda r) \setminus \Omega \subset \bigcup_{Q \in \mathcal{M}} C_1 B_Q \cup \Big[\Big(\Gamma \backslash \bigcup_{Q \in \mathcal{M}} Q\Big) \cap \big(\mathrm{C_y}(0, 5r, \Lambda r) \setminus \Omega\big)\Big]. \tag{11.15}$$

Next we claim that

$$\Big[\Big(\Gamma \backslash \bigcup_{Q \in \mathcal{M}} Q\Big) \cap \big(\mathrm{C_y}(0, 5r, \Lambda r) \setminus \Omega\big)\Big] \subset \Gamma_W. \tag{11.16}$$

By the definition of Γ_W, this is equivalent to showing that any point y belonging to the set on the left hand side satisfies $d(\bar{y}) = 0$ (with $\bar{y} = \Pi(y)$). That is, $h(\bar{z}) - \theta\,|\bar{y} - \bar{z}| \leqslant 0$ for any $\bar{z} \in \mathbb{R}^n$. This is clear if $\bar{y} = \bar{z}$ (since $h(\bar{y}) = 0$). Otherwise, let $P \in \mathcal{D}(\mathbb{R}^n)$ be such that $y \in P$ and $\bar{z} \in 3\widehat{P}\backslash 3P$ (where $\widehat{P}$ stands for the parent of P), so that $|\bar{y} - \bar{z}| \approx \ell(P)$. If $|\bar{z}| \leqslant 10r$, then $h(\bar{z}) \lesssim \ell(P)$, since $\widehat{P} \notin \mathcal{M}$ and thus, if θ is chosen large enough,

$$4\,h(\bar{z}) - \theta|\bar{y} - \bar{z}| \leqslant C\ell(P) - c\,\theta\ell(P) \leqslant 0.$$

In case that $10r < |\bar z|$, then $|\bar y - \bar z| \geqslant 5r$ and from (11.13) it also follows that $h(\bar z) - \theta|\bar y - \bar z| \leqslant 0$. So (11.16) holds.

By (11.11) we know that $\mathcal{H}^n_\infty(\overline{V_1} \cap \mathrm{C_y}(0, 3r, \frac{1}{2}\Lambda r) \setminus \Omega) \gtrsim r^n$. Then, from (11.15) and (11.16) we infer that either

$$\mathcal{H}^n_\infty\Big(\bigcup_{Q\in\mathcal{M}} (C_1 B_Q \cap \mathrm{C_y}(0, 3r, \tfrac{1}{2}\Lambda r) \setminus \Omega)\Big) \gtrsim r^n, \tag{11.17}$$

or

$$\mathcal{H}^n_\infty\Big(\Gamma_{\mathcal{W}} \cap \mathrm{C_y}(0, 3r, \Lambda r) \setminus \Omega\Big) \geqslant \mathcal{H}^n_\infty\Big(\Big(\Gamma \setminus \bigcup_{Q\in\mathcal{M}} Q\Big) \cap \big(\mathrm{C_y}(0, 3r, \Lambda r) \setminus \Omega\big)\Big) \gtrsim r^n. \tag{11.18}$$

Since $W \subset \Omega$, it is clear that $\Gamma_{\mathcal{W}} \cap \mathrm{C_y}(0, 5r, \Lambda r) \setminus \Omega \subset \partial\Omega$. So in the last case we deduce that

$$\mathcal{H}^n(\partial W \cap \Gamma \cap \partial\Omega) \gtrsim r^n,$$

which gives (b), under the assumption (11.18).

To complete the proof of Claim 11.8 we will show that if (11.17) holds, then there exists a subset set $G \subset \partial W \setminus \partial U_1$ such that $\mathcal{H}^n(G) \gtrsim r^n$ and $\omega^z_{U_1}(\partial U_1 \setminus (\Gamma \cap \Omega)) \gtrsim 1$ for all $z \in G$. Observe first that if $Q \in \mathcal{M}$ is such that $C_1 B_Q \cap \mathrm{C_y}(0, 3r, \frac{1}{2}\Lambda r) \neq \varnothing$, then $4Q \cap \Gamma \subset \mathrm{C_y}(0, 5r, \frac{1}{2}\Lambda r)$ for ε_0 small enough, because $\ell(Q) \lesssim \varepsilon_0 r$. Then, by by (11.17),

$$r^n \lesssim \mathcal{H}^n_\infty\Big(\bigcup_{Q\in\mathcal{M}} (C_1 B_Q \cap \mathrm{C_y}(0, 3r, \tfrac{1}{2}\Lambda r) \setminus \Omega)\Big) \lesssim \sum_{\substack{Q\in\mathcal{M}:\\ 4Q\cap\Gamma\subset \mathrm{C_y}(0,5r,\frac{1}{2}\Lambda r)}} \ell(Q)^n. \tag{11.19}$$

By the definition of $\mathcal{M}$, for each $Q \in \mathcal{M}$, there exists some $y_Q \in 3Q$ such that

$$d(\bar y_Q) \geqslant 4\, h(\bar y_Q) \geqslant 4\, \ell(Q).$$

Let us check that the converse estimate $d(\bar y_Q) \lesssim \ell(Q)$ holds. Indeed, given $\bar z \in \mathbb{R}^n$, let $P \in \mathcal{D}(\Gamma)$ be the minimal cube such that $P \supset \widehat{Q}$ (where $\widehat{Q}$ is the parent of Q) and $\bar z \in 3\Pi(P)$. If $P = \widehat{Q}$, then

$$4\, h(\bar z) - \theta\, |\bar y - \bar z| \leqslant 4\, h(\bar z) \leqslant 4\, \ell(\widehat{Q}) = 8\, \ell(Q)$$

by the maximality of $\mathcal{M}$. If $P \neq \widehat{Q}$, then $|\bar y_Q - \bar z| \approx \ell(P)$ and so, again by the definition of $\mathcal{M}$,

$$h(\bar z) - \theta\, |\bar y - \bar z| \leqslant \ell(P) - C\, \theta\, \ell(P) \leqslant 0,$$

assuming θ large enough. Taking the supremum over all $\bar z \in \mathbb{R}^n$, we deduce that $d(\bar y_Q) \leqslant 8\ell(Q)$.

Consider the points

$$y_{Q,\partial\Omega} := (\bar y_Q, A(\bar y_Q) + h(\bar y_Q)), \qquad y_{Q,\partial W} := (\bar y_Q, A(\bar y_Q) + d(\bar y_Q)).$$

Notice that

$$y_{Q,\partial\Omega} \in \partial\Omega \quad \text{ and } \quad y_{Q,\partial W} \in \partial W,$$

by the definitions of h and d. Further,

$$\mathrm{dist}(y_{Q,\partial\Omega},\Gamma) \approx |y_{Q,\partial\Omega} - y_Q| = h(\bar{y}_Q) \approx \ell(Q),$$

and

$$\mathrm{dist}(y_{Q,\partial W},\Gamma) \approx |y_{Q,\partial W} - y_Q| = d(\bar{y}_Q) \approx \ell(Q). \tag{11.20}$$

Moreover,

$$|y_{Q,\partial\Omega} - y_{Q,\partial W}| = d(\bar{y}_Q) - h(\bar{y}_Q) \leqslant 8\ell(Q). \tag{11.21}$$

This estimate implies that $\mathrm{dist}(y_{Q,\partial W},\partial\Omega) \lesssim \ell(Q)$. The converse estimate also holds. Indeed, by the definition of h and $\mathcal{M}$, for all $z \in \mathrm{C_y}(0,10r,\Lambda r) \cap \partial\Omega$ such that $\bar{z} \in \Pi(3\widehat{Q})$,

$$|y_{Q,\partial W} - z| \geqslant 4\,h(\bar{y}_Q) - h(\bar{z}) \geqslant 4\,\ell(Q) - \ell(\widehat{Q}) = 2\ell(Q).$$

Obviously, $|y_{Q,\partial W} - z| \gtrsim \ell(Q)$ if $z \notin \mathrm{C_y}(0,10r,\Lambda r)$, and so

$$\mathrm{dist}(y_{Q,\partial W},\partial\Omega) \gtrsim \ell(Q). \tag{11.22}$$

From (11.20), (11.21), and (11.22) we infer that there exists some constant $\eta \in (0,1/4)$ such that

$$|z - y_{Q,\partial\Omega}| \approx \mathrm{dist}(z,\partial\Omega) \approx \mathrm{dist}(z,\Gamma) \approx \ell(Q) \quad \text{for all } z \in B(y_{Q,\partial W},\eta\ell(Q)) \cap \partial W. \tag{11.23}$$

So for such points z, since $\widetilde{V}_1$ is an NTA domain, there exists a Harnack chain of balls $\{B^j\}_{1\leqslant j\leqslant N}$ such that $z \in B^1$ and $y_{Q,\partial\Omega}$ in B^N, with $10B^j \subset \widetilde{V}_1$ for each j, and such that $N \lesssim 1$. Notice that $B^N \cap \partial U_1 \neq \varnothing$ because $y_{Q,\partial\Omega} \in \partial U_1$. Taking the smallest integer $j_0 \geqslant 1$ such that $2B^{j_0} \cap \partial\Omega \neq \varnothing$ and arguing as in (11.8) and (11.9), we derive that

$$\omega^z_{U_1}(\partial U_1 \setminus (\Gamma \cap \Omega)) \gtrsim 1 \quad \text{for all } z \in B(y_{Q,\partial W},\eta\ell(Q)) \cap \partial W. \tag{11.24}$$

To define G, let $\mathcal{M}_0 \subset \mathcal{M}$ be a subfamily of cubes such that the cubes $4Q$, with $Q \in \mathcal{M}_0$, are pairwise disjoint, $4Q \cap \Gamma \subset \mathrm{C_y}(0,5r,\frac{1}{2}\Lambda r)$, and

$$\bigcup_{\substack{Q\in\mathcal{M}:\\ 4Q\cap\Gamma\subset \mathrm{C_y}(0,5r,\frac{1}{2}\Lambda r)}} 4Q \subset \bigcup_{Q\in\mathcal{M}_0} 20Q.$$

Then we set

$$G = \bigcup_{Q\in\mathcal{M}_0} B(y_{Q,\partial W},\eta\ell(Q)) \cap \partial W.$$

Notice that $G \subset \partial W \setminus \partial U_1$, by (11.23) and the above definition. Also, by (11.24), we have $\omega^z_{U_1}(\partial U_1 \setminus (\Gamma \cap \Omega)) \gtrsim 1$ for all $z \in G$, and by (11.19),

$$\mathcal{H}^n(G) \approx_\eta \sum_{Q\in\mathcal{M}_0} \mathcal{H}^n(Q) \gtrsim \sum_{\substack{Q\in\mathcal{M}:\\ 4Q\cap\Gamma\subset \mathrm{C_y}(0,5r,\frac{1}{2}\Lambda r)}} \mathcal{H}^n(Q) \gtrsim r^n.$$

So G satisfies all the required properties in (b). This completes the proof of Claim 11.8, and thus of Theorem 11.6. $\square$

11.2 Rectifiability of harmonic measure when it is absolutely continuous with respect to surface measure

In this section we will prove the following result.

Theorem 11.9. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open set and let $p \in \Omega$. Suppose that there exists a set $E \subset \partial\Omega$ such that $0 < \mathcal{H}^n(E) < \infty$ and that the harmonic measure $\omega_\Omega^p|_E$ is mutually absolutely continuous with respect to $\mathcal{H}^n|_E$. Then E is n-rectifiable.*

Of course, in the theorem above, saying that E is n-rectifiable is equivalent to saying that $\omega_\Omega^p|_E$ is n-rectifiable. Remark that the theorem also holds for unbounded open sets with compact boundary. In fact, the theorem for this type of domains can be easily be derived from the case when Ω is bounded. We leave the details for the reader.

The proof of Theorem 11.9 relies on the solution of David-Semmes problem from [NTV14b] and [NTV14c] about the connection between the L^2 boundedness of the Riesz transform and rectifiability. Given a measure μ in $\mathbb{R}^{n+1}$, its (n-dimensional) Riesz transform equals

$$\mathcal{R}\mu(x) = \int \frac{x-y}{|x-y|^{n+1}}\,d\mu(y),$$

whenever the integral makes sense (notice that this a vectorial integral). For $\varepsilon > 0$, we also consider the ε-truncared version, defined by

$$\mathcal{R}_\varepsilon\mu(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}}\,d\mu(y).$$

The maximal Riesz transform of μ is defined by

$$\mathcal{R}_*\mu(x) = \sup_{\varepsilon>0} |\mathcal{R}_\varepsilon\mu(x)|.$$

We also consider the maximal radial operator $\mathcal{M}_n$, defined by

$$\mathcal{M}_n\mu(x) = \sup_{r>0} \frac{\mu(B_r(x))}{r^n}.$$

For a given function $f \in L^1_{loc}(\mu)$, we denote

$$\mathcal{R}_\mu f(x) = \mathcal{R}(f\,\mu)(x), \quad \mathcal{R}_{\varepsilon,\mu} f(x) = \mathcal{R}_\varepsilon(f\,\mu)(x), \quad \mathcal{R}_{*,\mu} f(x) = \mathcal{R}_*(f\,\mu)(x).$$

We say that $\mathcal{R}_\mu$ is bounded in $L^2(\mu)$ if the operators $\mathcal{R}_{\varepsilon,\mu}$ are bounded in $L^2(\mu)$ uniformly on $\varepsilon > 0$.

The connection between the Riesz transform and harmonic measure stems from the fact that the Riesz kernel K equals the gradient of the fundamental solution $\mathcal{E}$ modulo a constant factor. That is,

$$K(x) = \frac{x}{|x|^{n+1}} = c_n\,\nabla\mathcal{E}(x).$$

Consequently, from the identity (7.2), we deduce

$$c_n \nabla_y G(x,y) = K(y-x) - \int_{\partial\Omega} K(y-z)\,d\omega^x(z) = K(y-x) - \mathcal{R}\omega^x(y) \quad \text{for } y \notin \operatorname{supp}\omega^x.$$

Next we show that it suffices to prove Theorem 11.9 for Wiener regular domains.

Lemma 11.10. *To prove Theorem 11.9 we can assume that Ω is Wiener regular.*

Proof. Let $E \subset \partial\Omega$ be as in Theorem 11.9. By the Borel regularity of $\mathcal{H}^n$ and ω, we can assume that E is in fact Borel. By an exhaustion argument, it suffices to show that there exists a subset $F \subset E$ with $\mathcal{H}^n(F) > 0$ which is n-rectifiable (see for example the argument below near (11.25)).

For any $\varepsilon > 0$, let $\widetilde{\Omega}_\varepsilon \subset \Omega$ be the Wiener regular open set constructed in Proposition 6.37. For E as above, let $E_\varepsilon = E \cap \partial\widetilde{\Omega}_\varepsilon$, so that by Lemma 6.38,

$$\lim_{\varepsilon\to 0} \omega^p_{\widetilde{\Omega}_\varepsilon}(E_\varepsilon) = \lim_{\varepsilon\to 0} \omega^p_{\widetilde{\Omega}_\varepsilon}(E) = \omega^p_\Omega(E).$$

Let $\varepsilon > 0$ be small enough so that $\omega^p_{\widetilde{\Omega}_\varepsilon}(E_\varepsilon) > 0$. By Lemma 5.32, we have

$$\omega^p_{\widetilde{\Omega}_\varepsilon}(A) \leqslant \omega^p_\Omega(A) \quad \text{for any Borel set } A \subset \partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon.$$

So $\omega^p_{\widetilde{\Omega}_\varepsilon}$ is absolutely continuous with respect to ω^p_Ω in $\partial\Omega \cap \partial\widetilde{\Omega}_\varepsilon$. Consequently, there exists a subset $F \subset E_\varepsilon$ where $\omega^p_{\widetilde{\Omega}_\varepsilon}$ and ω^p_Ω are mutually absolutely continuous and both $\omega^p_{\widetilde{\Omega}_\varepsilon}(F) > 0$, $\omega^p_\Omega(F) > 0$ (see exercise 4.3.1). Since F is a subset of E, $\omega^p_{\widetilde{\Omega}_\varepsilon}$ is also mutually absolutely continuous with $\mathcal{H}^n|_F$ and $\mathcal{H}^n(F) > 0$. By Theorem 11.9 applied to the Wiener regular domain $\widetilde{\Omega}_\varepsilon$, then we deduce that F is n-rectifiable, and so we are done. $\square$

To prove Theorem 11.9 we will use the following result.

Theorem 11.11. *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ and $E \subset \operatorname{supp}\mu$ such that $0 < \mathcal{H}^n(E) < \infty$ and $\mu|_E$ is mutually absolutely continuous with respect to $\mathcal{H}^n|_E$. If $\mathcal{R}_*\mu(x) < \infty$ for μ-a.e. $x \in E$, then $\mu|_E$ is n-rectifiable.*

This theorem follows from the following deep result from [NTV14c], which can be considered a non-quantitative version of the David-Semmes problem.

Theorem 11.12. *Let $E \subset \mathbb{R}^{n+1}$ be such that $0 < \mathcal{H}^n(E) < \infty$. Suppose that $\mathcal{R}_{\mathcal{H}^n|_E}$ is bounded in $L^2(\mathcal{H}^n|_E)$. Then E is n-rectifiable.*

The next result can be proved using a sophisticated Tb theorem of Nazarov, Treil, and Volberg [NTV14a], [Vol03] in combination with the methods in [Tol00]. For the detailed proof in the case of the Cauchy transform, see [Tol14, Theorem 8.13].

Theorem 11.13. *Let μ be a Radon measure with compact support in $\mathbb{R}^{n+1}$ and consider a μ-measurable set G with $\mu(G) > 0$ such that*

$$G \subset \{x \in \mathbb{R}^{n+1} : \mathcal{M}_n\mu(x) < \infty \text{ and } \mathcal{R}_*\mu(x) < \infty\}.$$

Then there exists a Borel subset $G_0 \subset G$ with $\mu(G_0) > 0$ such that $\sup_{x\in G_0} \mathcal{M}_n\mu|_{G_0}(x) < \infty$ and $\mathcal{R}_{\mu|_{G_0}}$ is bounded in $L^2(\mu|_{G_0})$.

We will prove neither Theorem 11.13 nor Theorem 11.12, since both results are out of the scope of these notes. Instead, we will outline how one can deduce Theorem 11.11 from Theorems 11.12 and 11.13.

Proof of Theorem 11.11 using Theorems 11.12 and 11.13. This follows by a standard exhaustion argument. Indeed, let μ and E satisfy the assumptions in Theorem 11.11. We can assume E to be bounded, so that $\mu(E) < \infty$. Let

$$\beta = \sup\{\mu(F) : \ F \subset E \text{ is Borel } n\text{-rectifiable}\}. \tag{11.25}$$

It is is immediate to check that the supremum is attained, that is, there exists a Borel n-rectifiable set $F \subset E$ such that $\mu(F) = \beta$.

We have to check that $\beta = \mu(E)$. Suppose that this is not the case, and let $G = E\backslash F$. By assumption, we have $\mathcal{R}_*\mu(x) < \infty$ for μ-a.e. $x \in G$. Also, for $x \in G$, we have

$$\limsup_{r\to 0} \frac{\mu(B_r(x))}{r^n} \leqslant \limsup_{r\to 0} \frac{\mu(B_r(x))}{\mathcal{H}^n(B_r(x) \cap E)} \ \limsup_{r\to 0} \frac{\mathcal{H}^n(B_r(x) \cap E)}{r^n}. \tag{11.26}$$

The first lim sup on the right hand side is finite $\mathcal{H}^n$-a.e. in G by Theorem 4.12 and thus μ-a.e. in G because of the absolute continuity of μ with respect to $\mathcal{H}^n$ in E, while the last one is also finite by the classical density bounds for Hausdorff measure (see for instance [Mat95, Theorem 6.2]). Hence the left hand side is also finite μ-a.e. in G, or equivalently,

$$\mathcal{M}_n\mu(x) < \infty \quad \text{ for } \mu\text{-a.e. } x \in G.$$

Then, by Theorem 11.13, there exists a Borel subset $G_0 \subset G$ with $\mu(G_0) > 0$ such that $\mathcal{R}_{\mu|_{G_0}}$ is bounded in $L^2(\mu|_{G_0})$. Denote by ρ the density of $\mu|_{G_0}$ with respect to $\mathcal{H}^n|_{G_0}$, so that $\mu|_{G_0} = \rho\,\mathcal{H}^n|_{G_0}$, and let $\tau > 0$ be such that the set

$$G_{0,\tau} = \{x \in G_0 : \rho(x) > \tau\}$$

has postive measure μ. It is immediate to check that $\mathcal{R}_{\mathcal{H}^n|_{G_{0,\tau}}}$ is bounded in $L^2(\mathcal{H}^n|_{G_{0,\tau}})$, and thus $G_{0,\tau}$ is n-rectifiable, by Theorem 11.12. As a consequence, the set $F' = F \cup G_{0,\tau}$ is n-rectifiable and $\mu(F') > \mu(F) = \beta$, which contradicts the definition of F and β. □

To prove Theorem 11.9, recall that Lemma 6.20 asserts the following: If $E \subset \mathbb{R}^{n+1}$ is compact and $n - 1 < s \leqslant n + 1$, in the case $n > 1$, we have

$$\mathrm{Cap}(E) \gtrsim_{s,n} \mathcal{H}^s_\infty(E)^{\frac{n-1}{s}}.$$

In the case $n = 1$,

$$\mathrm{Cap}_L(E) \gtrsim_s \mathcal{H}^s_\infty(E)^{\frac{1}{s}}.$$

Proof of Theorem 11.9. Let Ω, E, and p be as in Theorem 11.9, with Ω Wiener regular, and write ω instead of ω_Ω. By the regularity of $\mathcal{H}^n|_E$ and of ω, we can assume that E is compact. We will show that

$$\mathcal{R}_*\omega^p(x) < \infty \quad \text{for } \omega^p\text{-a.e. } x \in E,$$

which implies that $\omega^p|_E$ is n-rectifiable, by Theorem 11.11. For simplicity, in this proof we will work with closed balls $\bar{B}_s(\xi)$ (this is not essential, but it will ease some calculations because many lemmas in the preceding sections about the relationship between harmonic measure and the Green function are stated in terms of closed balls).

By the same argument as in (11.26), it follows that $\mathcal{M}_n\omega^p(x) < \infty$ for ω^p-a.e. $x \in E$. For $k \geqslant 1$, let

$$E_k = \{x \in E : \mathcal{M}_n\omega^p(x) \leqslant k\},$$

so that $E = \bigcup_{k\geqslant 1} E_k$, up to a set of ω^p-measure zero. For a fixed $k \geqslant 1$, let $x \in E_k$ be a density point of E_k, and let r_0 be small enough so that

$$\frac{\omega^p(\bar{B}_r(x) \cap E_k)}{\omega^p(\bar{B}_r(x))} \geqslant \frac{1}{2} \quad \text{for } 0 < r \leqslant r_0,$$

with $r_0 \leqslant |x-p|/100$. Observe that, since $\omega^p(\bar{B}_\rho(z) \cap E_k) \leqslant k\rho^n$ for all $z \in E_k$ and all $\rho > 0$, by Frostman's Lemma we have

$$\mathcal{H}^n_\infty(\bar{B}_r(x) \cap \partial\Omega) \geqslant \mathcal{H}^n_\infty(\bar{B}_r(x) \cap E_k) \geqslant c(k)\,\omega^p(\bar{B}_r(x) \cap E_k) \geqslant \frac{c(k)}{2}\,\omega^p(\bar{B}_r(x)), \tag{11.27}$$

for $0 < r \leqslant r_0$.

To show that $\mathcal{R}_*\omega^p(x) < \infty$ for $x \in E_k$ as above, clearly it suffices to show that

$$\sup_{0<r\leqslant r_0} |\mathcal{R}_r\omega^p(x)| < \infty. \tag{11.28}$$

To estimate $\mathcal{R}_r\omega^p(x)$ for $0 < r \leqslant r_0$, first we assume that

$$\omega^p(\bar{B}_{40r}(x)) \leqslant 50^n\omega^p(\bar{B}_r(x)). \tag{11.29}$$

We consider a radial C^∞ function $\varphi : \mathbb{R}^{n+1} \to [0,1]$ which vanishes in $\bar{B}_1(0)$ and equals 1 on $\mathbb{R}^{n+1}\backslash B_2(0)$, and for $r > 0$ and $z \in \mathbb{R}^{n+1}$ we denote $\varphi_r(z) = \varphi\left(\frac{z}{r}\right)$ and $\psi_r = 1 - \varphi_r$. We set

$$\widetilde{\mathcal{R}}_r\omega^p(x) = \int K(x-y)\,\varphi_r(x-y)\,d\omega^p(y).$$

Note that

$$\begin{aligned}|\mathcal{R}_r\omega^p(x)| &\leqslant \left|\int \varphi_r(x-y)K(x-y)\,d\omega^p(y)\right| + \int \left|\chi_{|x-y|>r} - \varphi_r(x-y)\right| \left|K(x-y)\right| d\omega^p(y) \\ &\leqslant |\widetilde{\mathcal{R}}_r\omega^p(x)| + C\,\mathcal{M}_n\omega^p(x).\end{aligned} \tag{11.30}$$

For a fixed $x \in E_k$ and $z \in \mathbb{R}^{n+1} \backslash \big[\mathrm{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\big]$, consider the function

$$u_r(z) = \mathcal{E}^p(z) - \int \mathcal{E}^y(z)\,\varphi_r(x-y)\,d\omega^p(y), \tag{11.31}$$

so that, by Lemma 7.4,

$$G^z(p) = u_r(z) - \int \mathcal{E}^z(y)\,\psi_r(x-y)\,d\omega^p(y) \quad \text{ for } m\text{-a.e. } z \in \mathbb{R}^{n+1}. \tag{11.32}$$

Differentiating (11.31) with respect to z, we obtain

$$\nabla u_r(z) = \nabla\mathcal{E}^p(z) - \int \nabla\mathcal{E}^y(z)\,\varphi_r(x-y)\,d\omega^p(y).$$

In the particular case $z = x$ we get

$$c_n\,\nabla u_r(x) = K(x-p) - \widetilde{\mathcal{R}}_r\omega^p(x),$$

and thus

$$|\widetilde{\mathcal{R}}_r\omega^p(x)| \lesssim \frac{1}{\mathrm{dist}(p,\partial\Omega)^n} + |\nabla u_r(x)|. \tag{11.33}$$

Since u_r is harmonic in $\mathbb{R}^{n+1} \backslash \big[\mathrm{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\big]$ (and so in $B_r(x)$), we have

$$|\nabla u_r(x)| \lesssim \frac{1}{r} \fint_{B_r(x)} |u_r(z) - \alpha|\,dz, \tag{11.34}$$

for any constant $\alpha \in \mathbb{R}$, possibly depending on x and r. From the identity (11.32) we deduce that

$$\begin{aligned}|\nabla u_r(x)| &\lesssim \frac{1}{r} \fint_{B_r(x)} G^z(p)\,dz + \frac{1}{r} \fint_{B_r(x)} \left| \int \big(\mathcal{E}^y(z) - \alpha'\big)\,\psi_r(x-y)\,d\omega^p(y) \right| dz \\ &=: I + II,\end{aligned}$$

for any constant $\alpha' \in \mathbb{R}$, possibly depending on x and r. To estimate the term II we use Fubini and the fact that $\mathrm{supp}\psi_r \subset B_{2r}(0)$:

$$II \lesssim \frac{1}{r^{n+2}} \int_{y \in B_{2r}(x)} \int_{z \in B_r(x)} |\mathcal{E}^y(z) - \alpha'|\,dz\,d\omega^p(y).$$

In the case $n \geqslant 2$ we choose $\alpha' = 0$, and we get

$$II \lesssim \frac{1}{r^{n+2}} \int_{y \in B_{2r}(x)} \int_{z \in B_r(x)} \frac{1}{|z-y|^{n-1}}\,dz\,d\omega^p(y) \lesssim \frac{\omega^p(B_{2r}(x))}{r^n} \lesssim \mathcal{M}_n\omega^p(x).$$

In the case $n = 1$ we take $\alpha' = \frac{1}{2\pi}\log\frac{1}{4r}$, and then we obtain

$$\begin{aligned}II &\lesssim \frac{1}{r^3} \int_{y \in B_{2r}(x)} \int_{z \in B_r(x)} \log\frac{4r}{|z-y|}\,dz\,d\omega^p(y) \\ &\leqslant \frac{1}{r^3} \int_{y \in B_{2r}(x)} \int_{z \in B_{3r}(y)} \log\frac{4r}{|z-y|}\,dz\,d\omega^p(y) \lesssim \frac{1}{r^3} \int_{y \in B_{2r}(x)} r^2\,d\omega^p(y) \lesssim \mathcal{M}_1\omega^p(x).\end{aligned}$$

Next we want to show that $I \lesssim_k 1$. Clearly it is enough to prove that

$$\frac{1}{r}\,|G^p(y)| \lesssim_k 1 \qquad \text{for all } y \in B_r(x) \cap \Omega \tag{11.35}$$

(now under the assumptions $x \in E_k$, $0 < r \leqslant r_0/2$, and (11.29)). To prove this, observe that, in the case $n \geqslant 2$, by Lemma 7.19,

$$G^p(y) \lesssim \frac{\omega^p(\bar{B}_{8r}(x))}{\operatorname{Cap}(\bar{B}_r(x)\backslash\Omega)} \qquad \text{for all } y \in \bar{B}_r(x) \cap \Omega.$$

Notice now that, by Lemma 6.20 and (11.27), we have

$$\operatorname{Cap}(\bar{B}_r(x)\backslash\Omega) \gtrsim \mathcal{H}^n_\infty(\bar{B}_r(x) \cap \partial\Omega)^{\frac{n-1}{n}} \gtrsim_k \omega^p(\bar{B}_r(x))^{\frac{n-1}{n}}.$$

Thus, by (11.29) and the fact that $\mathcal{M}_n\omega^p(x) \lesssim_k 1$,

$$\frac{1}{r}\,G^p(y) \lesssim_k \frac{\omega^p(\bar{B}_{8r}(x))}{r\,\omega^p(\bar{B}_r(x))^{\frac{n-1}{n}}} = \left(\frac{\omega^p(\bar{B}_{8r}(x))}{r^n}\right)^{\frac{1}{n}} \left(\frac{\omega^p(\bar{B}_{8r}(x))}{\omega^p(\bar{B}_r(x))}\right)^{\frac{n-1}{n}} \overset{(11.29)}{\lesssim_k} 1,$$

which proves (11.35). Almost the same arguments work in the case $n = 1$. Indeed, by Lemma 7.23,

$$\begin{aligned} G^p(y) &\lesssim \omega^p(\bar{B}_{40r}(x)) \left(\log \frac{r}{\operatorname{Cap}_L(\bar{B}_r(x)\backslash\Omega)}\right)^2 \\ &\lesssim \omega^p(\bar{B}_{40r}(x))\,\frac{r}{\operatorname{Cap}_L(\bar{B}_r(x)\backslash\Omega)} \qquad \text{for all } y \in \bar{B}_r(x) \cap \Omega. \end{aligned}$$

By Lemma 6.20 and (11.27), we have

$$\operatorname{Cap}_L(\bar{B}_r(x)\backslash\Omega) \gtrsim \mathcal{H}^1_\infty(\bar{B}_r(x) \cap \partial\Omega) \gtrsim_k \omega^p(\bar{B}_r(x)),$$

and thus, by (11.29),

$$\frac{1}{r}\,G^p(y) \lesssim_k \frac{\omega^p(\bar{B}_{40r}(x))}{\omega^p(\bar{B}_r(x))} \lesssim_k 1,$$

which proves again (11.35). So in any case we deduce that

$$|\mathcal{R}_r\omega^p(x)| \leqslant |\widetilde{\mathcal{R}}_r\omega^p(x)| + C\,\mathcal{M}_n\omega^p(x) \lesssim_k \frac{1}{\operatorname{dist}(p,\partial\Omega)^n} + 1 \tag{11.36}$$

for $x \in E_k$ and $0 < r \leqslant r_0/2$ satisfying (11.29).

In the case when (11.29) does not hold, we consider the smallest $r' > r$ of the form $r' = 40^j r$, $j > 0$, such that either $r' > r_0$ or (11.29) holds with r' replacing r. Let $j_0 \geqslant 1$ be such that $r' = 40^{j_0} r$ and write

$$|\mathcal{R}_r\omega^p(x)| \leqslant |\mathcal{R}_{r'}\omega^p(x)| + \int_{r<|x-y|\leqslant r'} |K(x-y)|\,d\mu(y) \leqslant |\mathcal{R}_{r'}\omega^p(x)| + C\sum_{j=1}^{j_0} \frac{\omega^p(\bar{B}_{40^j r}(x))}{(40^j r)^n}.$$

To estimate the last sum, notice that, for all $1 \leqslant j \leqslant j_0 - 1$,

$$\omega^p(\bar{B}_{40^j r}(x)) < 50^{-n} \omega^p(\bar{B}_{40^{j+1} r}(x)),$$

and thus, by iterating this estimate,

$$\sum_{j=1}^{j_0} \frac{\omega^p(\bar{B}_{40^j r}(x))}{(40^j r)^n} \leqslant \sum_{j=1}^{j_0} \frac{50^{-n(j_0-j)} \omega^p(\bar{B}_{40^{j_0} r}(x))}{40^{(j-j_0)n} \, (40^{j_0} r)^n} \lesssim \frac{\omega^p(\bar{B}_{r'}(x))}{(r')^n} \leqslant \mathcal{M}_n \omega^p(x).$$

On the other hand, in case that $r' < r_0$, then (11.36) holds (with r replaced by r'), and in case that $r' \geqslant r_0$, then we have $r' \approx r_0$ and we write

$$|\mathcal{R}_{r'} \omega^p(x)| \lesssim \frac{\omega^p(\partial\Omega)}{(r')^n} \lesssim \frac{1}{r_0^n}.$$

So in any case we deduce that

$$|\mathcal{R}_r \omega^p(x)| \lesssim_k \frac{1}{r_0^n} + \frac{1}{\operatorname{dist}(p, \partial\Omega)^n} + 1,$$

which yields (11.28). □

11.3 The maximal Riesz transform of harmonic measure under the CDC

In the previous section, to prove Theorem 11.9 we have estimated the maximal Riesz transform $\mathcal{R}_* \omega^p$ in terms of the maximal radial function $\mathcal{M}_n \omega^p$. For domains satisfying the CDC, a quite precise bound holds, as shown below.

Theorem 11.14. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with compact boundary satisfying the CDC and let $p \in \Omega$. Then, for every $x \in \partial\Omega$,*

$$\mathcal{R}_* \omega^p(x) \leqslant C \, \mathcal{M}_n \omega^p(x),$$

where the constant C depends only on n and the CDC.

We remark that in the case when Ω is unbounded with compact boundary, in the theorem we ask the CDC to hold with $r_0 = \operatorname{diam}(\partial\Omega)$ in the definition in Subsection 7.5.1. That is, for some $c > 0$, we require that for all $\xi \in \partial\Omega$ and all $r \in (0, \operatorname{diam}\partial\Omega)$,

$$\operatorname{Cap}(\bar{B}_r(\xi) \backslash \Omega) \geqslant c\, r^{d-2} \qquad \text{in the case } d \geqslant 3, \tag{11.37}$$

and

$$\operatorname{Cap}_L(\bar{B}_r(\xi) \backslash \Omega) \geqslant c\, r \qquad \text{in the case } d = 2. \tag{11.38}$$

Proof of Theorem 11.14. The arguments are quite similar (but somewhat simpler) to the ones used in the proof of Theorem 11.9. However, for the sake of completeness we will show the full details of the proof, repeating some of the estimates.

We have to show that, for all $x \in \partial\Omega$ and $r > 0$,

$$|\mathcal{R}_r\omega^p(x)| \leqslant C\,\mathcal{M}_n\omega^p(x),$$

where the constant C depends only on n and the CDC. We can assume that $r \leqslant \operatorname{diam}\partial\Omega$, because otherwise $\mathcal{R}_r\omega^p(x) = 0$. We will consider first the cases $n = 1$ with Ω bounded, and $n \geqslant 2$ with Ω bounded or unbounded with compact boundary. We will deal with the remaining case $n = 1$ with Ω unbounded with compact boundary at the end of the proof.

We consider a radial C^∞ function $\varphi : \mathbb{R}^{n+1} \to [0,1]$ which vanishes in $\bar{B}_1(0)$ and equals 1 on $\mathbb{R}^{n+1}\backslash\bar{B}_2(0)$, and for $r > 0$ and $z \in \mathbb{R}^{n+1}$ we denote $\varphi_r(z) = \varphi\left(\frac{z}{r}\right)$ and $\psi_r = 1 - \varphi_r$. We set

$$\widetilde{\mathcal{R}}_r\omega^p(x) = \int K(x-y)\,\varphi_r(x-y)\,d\omega^p(y).$$

Note that

$$\begin{aligned}|\mathcal{R}_r\omega^p(x)| &\leqslant \left|\int \varphi_r(x-y)K(x-y)\,d\omega^p(y)\right| + \int \left|\chi_{|x-y|>r} - \varphi_r(x-y)\right| \left|K(x-y)\right| d\omega^p(y) \\ &\leqslant |\widetilde{\mathcal{R}}_r\omega^p(x)| + C\,\mathcal{M}_n\omega^p(x).\end{aligned} \tag{11.39}$$

For a fixed $x \in \partial\Omega$ and $z \in \mathbb{R}^{n+1}\backslash\big[\operatorname{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\big]$, consider the function

$$u_r(z) = \mathcal{E}^p(z) - \int \mathcal{E}^y(z)\,\varphi_r(x-y)\,d\omega^p(y), \tag{11.40}$$

so that, by Lemma 7.4[1],

$$G^z(p) = u_r(z) - \int \mathcal{E}^y(z)\,\psi_r(x-y)\,d\omega^p(y) \qquad \text{for } m\text{-a.e. } z \in \mathbb{R}^{n+1}. \tag{11.41}$$

Differentiating (11.40) with respect to z, we obtain

$$\nabla u_r(z) = \nabla\mathcal{E}^p(z) - \int \nabla\mathcal{E}^y(z)\,\varphi_r(x-y)\,d\omega^p(y).$$

In the particular case $z = x$ we get

$$c_n\,\nabla u_r(x) = K(x-p) - \widetilde{\mathcal{R}}_r\omega^p(x),$$

and thus

$$|\widetilde{\mathcal{R}}_r\omega^p(x)| \lesssim \frac{1}{|x-p|^n} + |\nabla u_r(x)|. \tag{11.42}$$

[1] It is easy to check that the proof of this lemma extends easily to the case $n \geqslant 2$ with Ω unbounded with compact boundary

Since u_r is harmonic in $\mathbb{R}^{n+1}\setminus\big[\mathrm{supp}(\varphi_r(x-\cdot)\,\omega^p)\cup\{p\}\big]$ (and so in $B_r(x)$), we have

$$|\nabla u_r(x)| \lesssim \frac{1}{r} \fint_{B_r(x)} |u_r(z)-\alpha|\,dz, \tag{11.43}$$

for any constant $\alpha\in\mathbb{R}$, possibly depending on x and r. From the identity (11.41) we deduce that

$$\begin{aligned}|\nabla u_r(x)| &\lesssim \frac{1}{r}\fint_{B_r(x)} G^z(p)\,dz + \frac{1}{r}\fint_{B_r(x)} \left|\int \big(\mathcal{E}^y(z)-\alpha'\big)\,\psi_r(x-y)\,d\omega^p(y)\right| dz \\ &=: I + II,\end{aligned}$$

for any constant $\alpha'\in\mathbb{R}$, possibly depending on x and r. To estimate the term II we use Fubini and the fact that $\mathrm{supp}\psi_r\subset B_{2r}(0)$:

$$II \lesssim \frac{1}{r^{n+2}}\int_{y\in B_{2r}(x)}\int_{z\in B_r(x)} |\mathcal{E}^y(z)-\alpha'|\,dz\,d\omega^p(y).$$

In the case $n\geqslant 2$ we choose $\alpha'=0$, and we get

$$II \lesssim \frac{1}{r^{n+2}}\int_{y\in B_{2r}(x)}\int_{z\in B_r(x)} \frac{1}{|z-y|^{n-1}}\,dz\,d\omega^p(y) \lesssim \frac{\omega^p(B_{2r}(x))}{r^n} \lesssim \mathcal{M}_n\omega^p(x).$$

In the case $n=1$ we take $\alpha'=\frac{1}{2\pi}\log\frac{1}{4r}$, and then we obtain

$$\begin{aligned}II &\lesssim \frac{1}{r^3}\int_{y\in B_{2r}(x)}\int_{z\in B_r(x)} \log\frac{4r}{|z-y|}\,dz\,d\omega^p(y) \\ &\leqslant \frac{1}{r^3}\int_{y\in B_{2r}(x)}\int_{z\in B_{3r}(y)} \log\frac{4r}{|z-y|}\,dz\,d\omega^p(y) \lesssim \frac{1}{r^3}\int_{y\in B_{2r}(x)} r^2\,d\omega^p(y) \lesssim \mathcal{M}_1\omega^p(x).\end{aligned}$$

Next we want to show that $I\lesssim \mathcal{M}_n\omega^p(x)$. To this end, it is enough to prove that

$$\frac{1}{r}\,|G^p(y)| \lesssim \mathcal{M}_n\omega^p(x) \qquad \text{for all } y\in \bar{B}_r(x)\cap\Omega. \tag{11.44}$$

In the case $n\geqslant 2$, this is an immediate consequence of Lemma 7.19 and the CDC. Indeed,

$$\frac{1}{r}\,G^p(y) \lesssim \frac{\omega^p(\bar{B}_{8r}(x))}{r\ \mathrm{Cap}(\bar{B}_r(x)\setminus\Omega)} \lesssim \frac{\omega^p(\bar{B}_{8r}(x))}{r\,r^{n-1}} \lesssim \mathcal{M}_n\omega^p(x) \qquad \text{for all } y\in\bar{B}_r(x)\cap\Omega.$$

In the case $n=1$, we use Lemma 7.23 instead of Lemma 7.19, and we deduce

$$\begin{aligned}\frac{1}{r}\,G^p(y) &\lesssim \frac{\omega^p(\bar{B}_{40r}(x))}{r}\left(\log\frac{r}{\mathrm{Cap}_L(\bar{B}_r(x)\setminus\Omega)}\right)^2 \\ &\lesssim \frac{\omega^p(\bar{B}_{40r}(x))}{r} \lesssim \mathcal{M}_1\omega^p(x) \qquad \text{for all } y\in\bar{B}_r(x)\cap\Omega.\end{aligned}$$

So in any case (11.44) holds.

Combining (11.39), (11.42), and the estimates obtained for the terms I and II, we get

$$|\mathcal{R}_r\omega^p(x)| \lesssim |\widetilde{\mathcal{R}}_r\omega^p(x)| + \mathcal{M}_n\omega^p(x) \lesssim \mathcal{M}_n\omega^p(x) + \frac{1}{|x-p|^n}. \tag{11.45}$$

To complete the proof of the theorem, we will show that $\frac{1}{|x-p|^n} \lesssim \mathcal{M}_n\omega^p(x)$. Suppose first that $|p-x| \leqslant 2\,\mathrm{diam}(\partial\Omega)$. Then we use the fact that, by Lemmas 7.16 and 7.20 and the CDC, we have $\omega^p(\bar{B}_{4|p-x|}(x)) \gtrsim 1$, and thus

$$\frac{1}{|x-p|^n} \lesssim \frac{\omega^p(\bar{B}_{4|p-x|}(x))}{|x-p|^n} \lesssim \mathcal{M}_n\omega^p(x). \tag{11.46}$$

Consider now the case $|p-x| > 2\,\mathrm{diam}(\partial\Omega)$ (so Ω is unbounded in this case and $n \geqslant 2$). By Theorem 7.33 (d)

$$\omega^p(\partial\Omega) = \mathcal{E} * \omega^\infty(p) = \int \mathcal{E}^p(\xi)\,d\omega^\infty(\xi) \approx \frac{\|\omega^\infty\|}{\mathrm{dist}(p,\partial\Omega)^{n-1}} = \frac{\mathrm{Cap}(\partial\Omega)}{\mathrm{dist}(p,\partial\Omega)^{n-1}}.$$

Using the CDC, we have

$$\omega^p(\partial\Omega) \approx \frac{\mathrm{Cap}(\partial\Omega)}{\mathrm{dist}(p,\partial\Omega)^{n-1}} \approx \frac{\mathrm{diam}(\partial\Omega)^{n-1}}{|x-p|^{n-1}}$$

Thus,

$$\frac{1}{|x-p|^n} \leqslant \frac{1}{|x-p|^{n-1}\,\mathrm{diam}(\partial\Omega)} \lesssim \frac{\omega^p(\partial\Omega)}{\mathrm{diam}(\partial\Omega)^n} \lesssim \mathcal{M}_n\omega^p(x).$$

So (11.46) also holds, and the proof of the theorem is concluded in the cases $n=1$ with Ω bounded and $n \geqslant 2$ with Ω having compact boundary.

Suppose now that $n=1$ and Ω is unbounded with compact boundary. We will reduce this case to the case when Ω is bounded. To this end, consider $R>0$ large enough so that $\partial\Omega \subset B_{R/2}(0)$ and let $\Omega_R = \Omega \cap B_R(0)$. Arguing as above, we deduce that, for all $x \in \partial\Omega$ and all $0 < r \leqslant \mathrm{diam}(\partial\Omega)$, denoting by ω_R the harmonic measure for Ω_R,

$$|\widetilde{\mathcal{R}}_r\omega_R^p(x)| \lesssim \mathcal{M}_1\omega_R^p(x) + \frac{1}{|x-p|} \tag{11.47}$$

uniformly on R (notice that, to obtain this estimate, the CDC (11.38) for Ω_R is only required for $r \lesssim \mathrm{diam}(\partial\Omega)$, and this clearly holds).

Recall that for every $f \in C(\partial\Omega)$, and every $p \in \Omega$, we have

$$\int_{\partial\Omega} f\,d\omega^p = \lim_{R\to\infty}\int_{\partial\Omega} f\,d\omega^p_{\Omega_R},$$

by Remark 5.44. On the other hand, by Proposition 6.36 (b), $\omega^p(\partial\Omega)=1$. So we deduce that

$$\lim_{R\to\infty}\omega_R(\partial B_R(0)) = 0.$$

Consequently,

$$\lim_{R\to\infty} |\widetilde{\mathcal{R}}_r \omega_R^p(x)| = |\widetilde{\mathcal{R}}_r \omega^p(x)| \quad \text{and} \quad \limsup_{R\to\infty} \mathcal{M}_1\omega_R^p(x) \lesssim \mathcal{M}_1\omega^p(x).$$

So using (11.39) and letting $R \to \infty$ in (11.47), we derive

$$|\mathcal{R}_r\omega^p(x)| \lesssim |\widetilde{\mathcal{R}}_r\omega^p(x)| + \mathcal{M}_1\omega^p(x) \lesssim \mathcal{M}_1\omega^p(x) + \frac{1}{|x-p|}. \tag{11.48}$$

As above, in the case $|p-x| \leqslant 2\operatorname{diam}(\partial\Omega)$, by Lemma 7.20 and the CDC, we have $\omega^p(\bar{B}_{4|p-x|}(x)) \gtrsim 1$, and so (11.46) holds. For $|p-x| > 2\operatorname{diam}(\partial\Omega)$, we write

$$\frac{1}{|x-p|} \leqslant \frac{1}{\operatorname{diam}(\partial\Omega)} = \frac{\omega^p(\partial\Omega)}{\operatorname{diam}(\partial\Omega)} \lesssim \mathcal{M}_1\omega^p(x).$$

Thus, $|\mathcal{R}_r\omega^p(x)| \lesssim \mathcal{M}_1\omega^p(x)$ in any case. □

Recall that the upper n-dimensional density of a Borel measure μ at $x \in \mathbb{R}^{n+1}$ is defined by

$$\Theta^{n,*}(\mu, x) = \limsup_{t\to 0} \frac{\mu(B(x,r))}{(2r)^n}.$$

In the case when Ω is unbounded with compact boundary (satisfying the CDC, as above), we have the following result for harmonic measure with pole at ∞.

Theorem 11.15. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an unbounded open set with compact boundary satisfying the CDC. Then, for any $x \in \partial\Omega$,*

$$\mathcal{R}_*\omega^\infty(x) \leqslant C\, \mathcal{M}_n\omega^\infty(x) \tag{11.49}$$

and

$$\limsup_{\varepsilon\to 0} |\mathcal{R}_\varepsilon\omega^\infty(x)| \leqslant C\, \Theta^{n,*}(\omega^\infty, x). \tag{11.50}$$

where the constant C depends only on n and the CDC.

Proof. The first estimate follows from Theorem 11.14 by taking a sequence of poles $p_k \in \Omega$ tending to ∞, and and dividing by $\mathcal{E}(p_k)$ in the case $n \geqslant 2$.

To prove (11.50), by a quick inspection of the proof of Theorem 11.14, one can check that the following sharper version of (11.45) and (11.48) holds:

$$|\mathcal{R}_r\omega^p(x)| \lesssim \sup_{t>cr} \frac{\omega^p(B(x,t))}{t^n} + \frac{1}{|x-p|^n},$$

for some fixed constant $c > 0$ depending on n. By taking again a sequence of poles $p_k \in \Omega$ tending to ∞, and dividing by $\mathcal{E}(p_k)$ in the case $n \geqslant 2$, and then letting $k \to \infty$, we get

$$|\mathcal{R}_r\omega^\infty(x)| \lesssim \sup_{t>cr} \frac{\omega^\infty(B(x,t))}{t^n}.$$

Letting $r \to 0$, (11.50) follows. □

12 The dimension of harmonic measure

Recall that the dimension of a Borel measure μ in $\mathbb{R}^d$ is defined as follows:

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) : E \subset \mathbb{R}^d \text{ Borel }, \mu(E^c) = 0\}.$$

In Chapter 9 we showed that for planar domains, the dimension of harmonic measure is at most 1. In this chapter we will study the dimension of harmonic measure for domains in arbitrary dimensions. For $d \geqslant 3$, one might expect that the dimension of harmonic measure for domains in $\mathbb{R}^d$ is at most $d-1$, as in the complex plane. However, this is not the case. Indeed, Wolff in [Wol95] constructed a domain $\Omega \subset \mathbb{R}^3$ whose associated harmonic measure has dimension larger than 2. This example is easily extended to higher dimensions.

The main result that we will prove in this chapter is a theorem due to Bourgain [Bou87], which asserts that for any open set $\Omega \subset \mathbb{R}^d$, the dimension of ω_Ω is at most $d - \varepsilon(d)$, for some positive constant $\varepsilon(d)$. The sharp constant $\varepsilon(d)$ (which is smaller than 1, because of Wolff's example) is not known. We will also study the so-called dimension drop, i.e., the fact that $\dim_{\mathcal{H}}(\omega_\Omega) < \dim_{\mathcal{H}}(\partial\Omega)$, which occurs typically in fractional dimensions.

Before turning to Bourgain's theorem, we show a basic (but sharp) lower bound for the dimension of harmonic measure.

Proposition 12.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set with compact boundary which is not polar. Then, for any $x_0 \in \Omega$,*

$$\dim_{\mathcal{H}}(\omega_\Omega^{x_0}) \geqslant d-2.$$

The fact that $\partial\Omega$ is not polar ensures that ω^{x_0} is a non-trivial measure, by Proposition 6.36.

Proof. Remark first that the proposition is only meaningful for $d \geqslant 3$. We have to check that $\omega_\Omega^{x_0}(E) = 0$ for any Borel set $E \subset \partial\Omega$ such that $\dim_{\mathcal{H}}(E) < d-2$. To this end, notice that by Lemma 6.20 and the subsequent corollary, if $\dim_{\mathcal{H}}(E) < d-2$, then $\mathrm{Cap}(E) = 0$. In case that Ω is bounded, by Theorem 6.33 this implies that $\omega_\Omega^{x_0}(E) = 0$, as wished.

In case that Ω is unbounded with compact boundary, let $r > 0$ be such that $\{x_0\} \cup \partial\Omega \subset B_r(0)$ and denote $\Omega_r = \Omega \cap B_r(0)$. Then we have $\omega_{\Omega_r}^{x_0}(E) = 0$, and so by Lemma 5.45, we deduce

$$\omega_\Omega^{x_0}(E) = \lim_{r\to\infty} \omega_{\Omega_r}^{x_0}(E) = 0.$$

□

To check that the lower bound $d-2$ is sharp, one just has to consider a compact set $E \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(E) = d-2$ and with $\mathrm{Cap}(E) > 0$. Then, setting $\Omega = \mathbb{R}^d \setminus E$, it follows that $\partial\Omega$ is not polar and $\omega^{x_0}(\partial\Omega) > 0$ if x_0 belongs to the unbounded component of Ω, by Proposition 6.36. Obviously, we have $\dim_{\mathcal{H}} \omega^{x_0} \leqslant \dim_{\mathcal{H}} \partial\Omega = d-2$.

Exercise 12.0.1. For $d \geqslant 3$, construct a compact set $E \subset \mathbb{R}^d$ such that $\dim_{\mathcal{H}}(E) = d-2$ and $\mathrm{Cap}(E) > 0$.

12.1 Bourgain's theorem on the dimension of harmonic measure

In this section we will prove the following result:

Theorem 12.2. *For $d \geqslant 3$ there exists some constant $\varepsilon(d) > 0$ such that for every open set $\Omega \subset \mathbb{R}^d$ with compact boundary and every $x_0 \in \Omega$ we have*

$$\dim_H(\omega_\Omega^{x_0}) \leqslant d - \varepsilon(d).$$

Lemma 12.3. *To prove Theorem 12.2, we can assume that Ω is Wiener regular.*

The proof of this lemma is almost the same as the one of Lemma 9.18 and so we skip it.

From now on, in this section we assume that $\Omega \subset \mathbb{R}^d$ is an open Wiener regular set with compact boundary and we denote $E = \partial\Omega$.

Recall that the s-dimensional Hausdorff content of $F \subset \mathbb{R}^d$ equals

$$\mathcal{H}^s_\infty(F) = \inf\Big\{\sum_i \mathrm{diam}(A_i)^s : F \subset \bigcup_i A_i\Big\}.$$

Lemma 12.4. *Assume $d \geqslant 3$. Let $s \geqslant d-1$, let $Q \subset \mathbb{R}^d$ be an open cube, and let $Q_* = \frac{1}{8\,d^{1/2}}\bar{Q}$. Then, for any $\delta \in (0,1)$, one of the following alternatives holds:*

$$\omega_\Omega^x(E \cap Q) \geqslant c(d)\delta \qquad \text{for all } x \in Q_* \cap \Omega,$$

or

$$\mathcal{H}^s_\infty(E \cap Q_*) \leqslant \delta\, \ell(Q)^s,$$

with $c(d) > 0$.

Proof. Let $\bar{B}$ be a ball concentric with Q with radius

$$r(\bar{B}) = \frac{d^{1/2}}{2}\,\ell(Q_*) = \frac{1}{16}\,\ell(Q).$$

Notice that $Q_* \subset \bar{B} \subset 4\bar{B} \subset \frac{1}{2}\bar{Q} \subset Q$. Therefore, $4\bar{B} \cap \partial\Omega \subset E \cap Q$, and then , by Lemma 7.16 and Remark 7.17, for all $x \in \bar{B} \cap \Omega$ we have

$$\omega_\Omega^x(E \cap Q) \geqslant \omega_\Omega^x(4\bar{B}) \geqslant c(d)\,\frac{\mathrm{Cap}(\bar{B}\backslash\Omega)}{r(\bar{B})^{d-2}} = c(d)\,\frac{\mathrm{Cap}(\bar{B} \cap E)}{r(\bar{B})^{d-2}}.$$

Of course, this holds for all $x \in Q_* \cap \Omega$ because $Q_* \subset \bar{B}$.

Now, by Lemma 6.20, it holds that

$$\mathrm{Cap}(\bar{B} \cap E) \geqslant \mathrm{Cap}(E \cap Q_*) \gtrsim_d \mathcal{H}^s_\infty(E \cap Q_*)^{\frac{d-2}{s}}.$$

Actually, in Lemma 6.20 it is shown that this holds for $s \in (d-2, d]$ with the implicit constant depending both on d and s. However, it is immediate to check that when $s \geqslant d-1$, the proof in that lemma yields an estimate depending only on d. Gathering the estimates above, we obtain

$$\omega^x_\Omega(E \cap Q) \geqslant c'(d) \frac{\mathcal{H}^s_\infty(E \cap Q_*)^{\frac{d-2}{s}}}{r(\bar{B})^{d-2}} \approx_d \left(\frac{\mathcal{H}^s_\infty(E \cap Q_*)}{\ell(Q)^s}\right)^{\frac{d-2}{s}} \quad \text{for all } x \in Q_*.$$

If $\mathcal{H}^s_\infty(E \cap Q_*) > \delta\, \ell(Q)^s$, this implies that $\omega^x_\Omega(E \cap Q) \gtrsim_d \delta^{\frac{d-2}{s}} \geqslant \delta$, which proves the lemma. □

We introduce now two additional Hausdorff contents. For $s \in [d-1, d)$, $F \subset \mathbb{R}^d$, and $\varepsilon > 0$, we set

$$\mathcal{H}^s_\varepsilon(F) = \inf\Big\{\sum_i \mathrm{diam}(A_i)^s : F \subset \bigcup_i A_i,\ \mathrm{diam}(A_i) \leqslant \varepsilon\Big\}, \tag{12.1}$$

and

$$\mathcal{M}^s_\varepsilon(F) = \inf\Big\{\sum_i \ell(Q_i)^s : Q_i \in \mathcal{D},\ F \subset \bigcup_i Q_i,\ \ell(Q_i) \leqslant \varepsilon\Big\}, \tag{12.2}$$

where $\mathcal{D}$ stands for the family of the usual dyadic cubes in $\mathbb{R}^d$ and $\ell(Q_i)$ denotes the side length of Q_i. It is immediate to check that $\mathcal{H}^s_\varepsilon(F) \approx_d \mathcal{M}^s_\varepsilon(F)$.

Below we will the following notation. Given a cube $Q \in \mathcal{D}$ and $m \geqslant 0$, $\mathcal{D}_m(Q)$ is the family of the cubes $P \in \mathcal{D}$ such that $P \subset Q$ and $\ell(P) = 2^{-m}\ell(Q)$.

In the rest of the section we assume that we are under the assumptions of Theorem 12.2 (and that Ω is Wiener regular). Recall that $E = \partial\Omega$. Also, we denote $\omega = \omega^{x_0}_\Omega$. The proof of Theorem 12.2 is based on the following lemma.

Lemma 12.5. *There is some $s_0 < d$ and some $m_0 > 1$, both depending on d, such that for all $s \in [s_0, d)$ and every $Q_0 \in \mathcal{D}$ such that $x_0 \notin \overline{Q_0}$, one of the following alternatives holds:*

(a) $\mathcal{M}^s_{2^{-m_0}\ell(Q_0)}(E \cap Q_0) < \ell(Q_0)^s$, *or*

(b) $\sum_{P \in \mathcal{D}_{m_0}(Q_0)} \omega(P)^{1/2}\, \ell(P)^{d/2} \leqslant \frac{1}{4}\, \omega(Q_0)^{1/2}\, \ell(Q_0)^{d/2}$.

Remark that $\mathcal{M}^s_{\ell(Q_0)}(Q_0) = \ell(Q_0)^s$.

In the proof of the preceding lemma we will use Theorem 5.54. Given two Wiener regular open sets $V, \widetilde{V} \subset \mathbb{R}^d$ with compact boundary such that $\widetilde{V} \subset V$, that theorem asserts that, for every $x \in \widetilde{V}$ and every Borel set $A \subset \partial V$, it holds

$$\omega^x_V(A) = \omega^x_{\widetilde{V}}(A) + \int_{\partial\widetilde{V} \setminus \partial V} \omega^y_V(A)\, d\omega^x_{\widetilde{V}}(y). \tag{12.3}$$

Proof of Lemma 12.5. We will fix the constants s_0 and m_0 along the proof of the lemma. We denote $\ell_0 = \ell(Q_0)$.

Let k_d be the smallest natural number such that $k_d > 8\,d^{1/2}$. Let $Q \in \mathcal{D}_{m_0}(Q_0)$ and $Q^* = (2k_d + 1)Q$. By Lemma 12.4, choosing some positive absolute constant δ depending only on d, either

(i) $\mathcal{M}^s_{2^{-m_0}\ell_0}(E \cap Q) \leqslant \frac{1}{2}\,\ell(Q)^s$, or

(ii) $\omega^x_\Omega(E \cap Q^*) \geqslant c_1(d)$ for all $x \in \bar{Q} \cap \Omega$ and some fixed $c_1(d) > 0$.

We distinguish two cases:

Case 1. There exists some $Q \in \mathcal{D}_{m_0}(Q_0)$ satisfying (i). Since $Q_0 \backslash Q$ is covered by $2^{dm_0} - 1$ cubes from $\mathcal{D}_{m_0}(Q_0)$, we have

$$\begin{aligned}
\mathcal{M}^s_{2^{-m_0}\ell_0}(E \cap Q_0) &\leqslant \mathcal{M}^s_{2^{-m_0}\ell_0}(Q_0 \backslash Q) + \mathcal{M}^s_{2^{-m_0}\ell_0}(E \cap Q) \\
&\leqslant (2^{dm_0} - 1)\, 2^{-sm_0} \ell_0^s + \frac{1}{2}\,\ell(Q)^s \\
&= 2^{(d-s)m_0} \ell_0^s - \frac{1}{2}\, 2^{-m_0 s}\, \ell_0^s = (2^{(d-s)m_0} - 2^{-m_0 s - 1})\ell_0^s
\end{aligned}$$

Observe that, for any given m_0, by continuity, if $s < d$ is close enough to d,

$$2^{(d-s)m_0} - 2^{-m_0 s - 1} < 1,$$

and then the alternative (a) of the lemma holds. Below we will choose m_0 large enough independently of s.

Case 2. All the cubes $Q \in \mathcal{D}_{m_0}(Q_0)$ satisfy (ii). In this case we will prove that the alternative (b) of the lemma holds. To prove this we will show that the inner part of Q_0 has very small harmonic measure. To this end, denote $F_0 = Q_0$ and let I_0 be family of the cubes $Q \in \mathcal{D}_{m_0}(Q_0)$ whose boundaries intersect ∂F_0. Then we let

$$F_1 = F_0 \backslash \bigcup_{Q \in I_0} Q.$$

Inductively, let I_j be family of the cubes $Q \in \mathcal{D}_{m_0}(Q_0)$ whose boundaries intersect ∂F_j, for $j \geqslant 1$. Then we let

$$F_{j+1} = F_j \backslash \bigcup_{Q \in I_j} Q.$$

So F_{j+1} is the half open-closed cube obtained by eliminating the interior of the union of the "outer" cubes from $\mathcal{D}_{m_0}(Q_0)$ in F_j.

Observe that

$$G_j := \bigcup_{Q \in I_j} Q^* \subset F_{j-2k_d} \backslash F_{j+2k_d},$$

for $j \geqslant 2k_d$. Since $\omega^x_\Omega(E \cap Q^*) \geqslant c_1(d)$ for all $Q \in I_j$, with $j \geqslant 2k_d$, and all $x \in \bar{Q} \cap \Omega$, we deduce that

$$\omega^x_\Omega(E \cap (F_{j-2k_d} \backslash F_{j+2k_d})) \geqslant \omega^x_\Omega(E \cap G_j) \geqslant c_1(d) \qquad \text{for all } x \in \partial F_j \cap \Omega.$$

Consequently,

$$\begin{aligned}\omega_\Omega^x(E\cap F_{j+2k_d}) &= \omega_\Omega^x(E\cap F_{j-2k_d}) - \omega_\Omega^x(E\cap (F_{j-2k_d}\backslash F_{j+2k_d}))\\ &\leqslant \omega_\Omega^x(E\cap F_{j-2k_d}) - c_1(d)\\ &\leqslant (1-c_1(d))\,\omega_\Omega^x(E\cap F_{j-2k_d}) \quad \text{for all } x\in\partial F_j\cap\Omega. \end{aligned}\tag{12.4}$$

We claim that (12.4) also holds with x_0 in place of x. This would follow from the maximum principle if $\omega_\Omega^x(E\cap F_{j-2k_d})$ and $\omega_\Omega^x(E\cap F_{j+2k_d})$ were continuous functions of x in the closure of $\Omega\backslash\overline{F_j}$. Since this may fail, we need to be a bit more careful. Instead, we apply the Markov property (12.3) to the open sets Ω and $\Omega\backslash\overline{F_j}$. Then we deduce that, for every Borel set $A\subset\partial\Omega$,

$$\omega_\Omega^{x_0}(A) = \omega_{\Omega\backslash\overline{F_j}}^{x_0}(A) + \int_{\partial F_j\cap\Omega} \omega_\Omega^y(A)\, d\omega_{\Omega\backslash\overline{F_j}}^{x_0}(y).$$

In particular, choosing first $A = E\cap F_{j+2k_d}$ and later $A = E\cap F_{j-2k_d}$, from (12.4) we infer that

$$\begin{aligned}\omega_\Omega^{x_0}(E\cap F_{j+2k_d}) &= \omega_{\Omega\backslash\overline{F_j}}^{x_0}(E\cap F_{j+2k_d}) + \int_{\partial F_j\cap\Omega} \omega_\Omega^y(E\cap F_{j+2k_d})\, d\omega_{\Omega\backslash\overline{F_j}}^{x_0}(y)\\ &\leqslant 0 + (1-c_1(d))\int_{\partial F_j\cap\Omega} \omega_\Omega^y(E\cap F_{j-2k_d})\, d\omega_{\Omega\backslash\overline{F_j}}^{x_0}(y)\\ &\leqslant (1-c_1(d))\,\omega_\Omega^{x_0}(E\cap F_{j-2k_d}),\end{aligned}$$

which proves our claim.

Iterating, it follows that

$$\omega_\Omega^{x_0}(E\cap F_{4jk_d}) \leqslant (1-c_1(d)))^j\,\omega_\Omega^{x_0}(E\cap Q_0) \quad \text{for all } j\geqslant 0 \text{ such that } 4jk_d < 2^{m_0-1}.$$

Hence, for $n_0\geqslant 1$ being a multiple of $4k_d$ such that $n_0 < 2^{m_0-1}$,

$$\omega_\Omega^{x_0}(E\cap F_{n_0}) \leqslant (1-c_1(d)))^{\frac{n_0}{4k_d}}\,\omega_\Omega^{x_0}(E\cap Q_0). \tag{12.5}$$

Next we estimate the sum in (b). By Cauchy-Schwarz,

$$\begin{aligned}&\sum_{P\in\mathcal{D}_{m_0}(Q_0)} \omega(P)^{1/2}\,\ell(P)^{d/2}\\ &\quad= \sum_{P\in\mathcal{D}_{m_0}(Q_0):P\subset F_{n_0}} \omega(P)^{1/2}\,\mathcal{L}^d(P)^{1/2} + \sum_{P\in\mathcal{D}_{m_0}(Q_0):P\subset Q_0\backslash F_{n_0}} \omega(P)^{1/2}\,\mathcal{L}^d(P)^{1/2}\\ &\quad\leqslant \omega(F_{n_0})^{1/2}\,\mathcal{L}^d(F_{n_0})^{1/2} + \omega(Q_0\backslash F_{n_0})^{1/2}\,\mathcal{L}^d(Q_0\backslash F_{n_0})^{1/2}\\ &\quad\leqslant (1-c_1(d)))^{\frac{n_0}{8k_d}}\,\omega(Q_0)^{1/2}\,\mathcal{L}^d(Q_0)^{1/2} + \omega(Q_0)^{1/2}\,\mathcal{L}^d(Q_0\backslash F_{n_0})^{1/2}.\end{aligned}$$

Since $Q_0\backslash F_{n_0}$ is made up of at most $C(d)\,n_0\,2^{m_0(d-1)}$ cubes $Q\in\mathcal{D}_{m_0}(Q)$, we have

$$\mathcal{L}^d(Q_0\backslash F_{n_0}) \leqslant C(d)\,n_0\,2^{m_0(d-1)}\,2^{-m_0 d}\,\mathcal{L}^d(Q_0) = C(d)\,n_0\,2^{-m_0}\,\ell(Q_0)^d.$$

Therefore,

$$\sum_{P\in\mathcal{D}_{m_0}(Q_0)} \omega(P)^{1/2}\,\ell(P)^{d/2} \leqslant \big((1-c_1(d)))^{n_0/(8k_d)} + C(d)\,n_0\,2^{-m_0}\big)\,\omega(Q_0)^{1/2}\,\ell(Q_0)^{d/2}.$$

Choosing first n_0 large enough and later m_0 large enough too (depending on n_0), the statement (b) in the lemma follows. □

Proof of Theorem 12.2. As shown in Lemma 12.3, we can assume that Ω is Wiener regular. Denote $E = \partial\Omega$ and $\omega = \omega_\Omega^{x_0}$. We will show that for every dyadic cube $R_0 \in \mathcal{D}$ such that $x_0 \notin \overline{R_0}$, $\dim_{\mathcal{H}}(\omega|_{R_0}) \leqslant d - \varepsilon(d)$, with $\varepsilon(d) > 0$. This suffices to prove theorem. To this end, we will prove that there exists a some $t = t(d) \in (d-1,d)$ such that, for every $\tau > 0$, there exists a subset $E_\tau \subset E \cap R_0$ satisfying

$$\mathcal{H}^t_\infty(E_\tau) \leqslant \tau \quad \text{and} \quad \omega(E \cap R_0\backslash E_\tau) \leqslant \tau. \tag{12.6}$$

It is immediate to check that this implies that $\dim_{\mathcal{H}}(\omega|_{R_0}) \leqslant t$.

The tree $\mathcal{T}$ and the stopping cubes.

To prove the existence of the aforementioned set E_τ we will construct a suitable tree of dyadic cubes from $\mathcal{D}(R_0)$ which we proceed to define. First we need some terminology. We say that a cube $Q \in \mathcal{D}(R_0)$ is of type H (Hausdorff content estimate) if the alternative (a) in Lemma 12.5 holds for $Q_0 = Q$. Otherwise it satisfies (b) we say that this is of type W (harmonic measure estimate). We write $Q \in H$ and $Q \in W$ respectively. Now, for any $Q \in \mathcal{D}(R_0)$ of type H, we let $\mathrm{Next}(Q) \subset \mathcal{D}(Q)$ be a subfamily of cubes which cover $E \cap Q$, with $\ell(P) \leqslant 2^{-m_0}\ell(Q)$, and so that, for $s_0 \leqslant s < d$,

$$\sum_{P\in \mathrm{Next}(Q)} \ell(P)^s \leqslant \ell(Q)^s.$$

In case that $Q \in \mathcal{D}(R_0)$ is of type W, we let $\mathrm{Next}(Q)$ be the subfamily of the cubes from $\mathcal{D}_{m_0}(Q)$ that intersect E. Now we define inductive the following layers of cubes from $\mathcal{D}(R_0)$. We set $\mathcal{T}_0 = \{R_0\}$, and for $j \geqslant 1$, we set

$$\mathcal{T}_j = \bigcup_{Q\in\mathcal{T}_{j-1}} \mathrm{Next}(Q).$$

We also set $\mathcal{T} = \bigcup_{j\geqslant 0}\mathcal{T}_j$. One can think of $\mathcal{T}$ as a tree whose root is R_0 and whose branches join the every cube $Q \in \mathcal{T}$ with the descendants belonging to $\mathrm{Next}(Q)$. For $Q \in \mathcal{T}\backslash\{R_0\}$, we denote by $p(Q)$ the "parent" of Q in $\mathcal{T}$, that is, $p(Q)$ is the cube from $\mathcal{T}$ such that $Q \in \mathrm{Next}(p(Q))$. We also set $p(R_0) = R_0$.

We fix some small number $\delta \in (0,1)$. We let $\mathcal{T}^*$ be the subfamily of the cubes $Q \in \mathcal{T}$ such that $\ell(p(Q)) \geqslant \delta\,\ell(R_0)$ and we let Stop_δ be the family of minimal cubes from $\mathcal{T}^*$, so that any $Q \in \mathrm{Stop}_\delta$ belongs to $\mathcal{T}^*$ but no cube from $\mathrm{Next}(Q)$ belongs to $\mathcal{T}^*$ (which means that $\ell(Q) < \delta\,\ell(R_0)$). By construction, we have

$$E \cap R_0 \subset \bigcup_{Q\in\mathrm{Stop}_\delta} Q$$

and

$$\ell(Q) < \delta\,\ell(R_0) \quad \text{for all } Q \in \mathrm{Stop}_\delta.$$

Next we enumerate the cubes from $\mathcal{T}^* \cap W$ as follows. We denote by W_0 the family of the cubes from $\mathcal{T}^* \cap W$ which are maximal. Inductively, for $j \geqslant 1$, we let W_j be the family of the cubes from $\mathcal{T}^* \cap W$ which are contained in some cube from W_{j-1} and which are maximal. Of course, for j large, enough W_j will be empty.

We split Stop_δ into two subfamilies: for some natural number n_0 to be fixed later, we let

$$\mathrm{Stop}_H = \{Q \in \mathrm{Stop}_\delta : Q \text{ is not contained in any cube from } W_{n_0}\}$$

and

$$\mathrm{Stop}_W = \mathrm{Stop}_\delta \backslash \mathrm{Stop}_H.$$

That is, Stop_H is the family of cubes from Stop_δ which is contained in less than n_0 cubes from $\mathcal{T}^* \cap W$, while the cubes from Stop_W are contained in more that n_0 cubes from $\mathcal{T}^* \cap W$.

Estimates to prove (12.6).

Recall that if $Q \in H$, then

$$\sum_{P \in \mathrm{Next}(Q)} \ell(P)^s \leqslant \ell(Q)^s$$

for $s_0 \leqslant s < d$. On the other hand, if $Q \in W$,

$$\sum_{P \in \mathrm{Next}(Q)} \omega(P)^{1/2}\,\ell(P)^{d/2} \leqslant \frac{1}{4}\,\omega(Q_0)^{1/2}\,\ell(Q_0)^{d/2}. \tag{12.7}$$

Further since all the cubes from $\mathrm{Next}(Q)$ have side lengths $2^{-m_0}\ell(Q)$, we have

$$\sum_{P \in \mathrm{Next}(Q)} \ell(P)^s \leqslant 2\,\ell(Q)^s$$

assuming s to be close enough to d (so s depends on d and m_0).

We claim that

$$\sum_{P \in W_j} \ell(P)^s \leqslant 2 \sum_{P \in W_{j-1}} \ell(P)^s. \tag{12.8}$$

Indeed, let $Q \in W_{j-1}$, with $Q \in \mathcal{T}_k$ for some $k \geqslant 0$. Denote by $\widetilde{\mathcal{T}}_i^j$ the cubes from $\mathcal{T}_i$ which are contained in some cube from W_{j-1} and that are not contained in any cube from W_j. Then, using that there are no cubes of type W between the layers W_{j-1} and W_j, we get

$$\begin{aligned}
\frac{1}{2}\,\ell(Q)^s &\geqslant \sum_{P \in \mathcal{T}_{k+1}: P \subset Q} \ell(P)^s = \sum_{P \in \mathcal{T}_{k+1} \cap W_j: P \subset Q} \ell(P)^s + \sum_{P \in \widetilde{\mathcal{T}}_{k+1}^j: P \subset Q} \ell(P)^s \\
&\geqslant \sum_{P \in \mathcal{T}_{k+1} \cap W_j: P \subset Q} \ell(P)^s + \sum_{P \in \widetilde{\mathcal{T}}_{k+1}^j: P \subset Q} \; \sum_{P' \in \mathcal{T}_{k+2}: P' \subset P} \ell(P')^s \\
&= \sum_{P \in \mathcal{T}_{k+1} \cap W_j: P \subset Q} \ell(P)^s + \sum_{P' \in \widetilde{\mathcal{T}}_{k+2}^j: P' \subset Q} \ell(P')^s.
\end{aligned}$$

Iterating, we obtain

$$\frac{1}{2}\,\ell(Q)^s \geqslant \sum_{i\geqslant k+1}\ \sum_{P\in\mathcal{T}_i\cap W_j:P\subset Q} \ell(P)^s = \sum_{P\in W_j:P\subset Q} \ell(P)^s.$$

Summing over all the cubes $Q \in W_{j-1}$, (12.8) follows.

Iterating the estimate (12.8), we deduce that

$$\sum_{P\in W_j} \ell(P)^s \leqslant 2^j\,\ell(R_0)^s.$$

Therefore,

$$\sum_{P\in \mathrm{Stop}_H} \ell(P)^s \leqslant \sum_{j=0}^{n_0-1}\ \sum_{P\in W_j} \ell(P)^s \leqslant \sum_{j=0}^{n_0-1} 2^j\,\ell(R_0)^s \leqslant 2^{n_0}\,\ell(R_0)^s.$$

Choosing $s' = (s+d)/2$, we get

$$\sum_{P\in \mathrm{Stop}_H} \ell(P)^{s'} \leqslant \sum_{P\in \mathrm{Stop}_H} \ell(P)^s\,(\delta\,\ell(R_0))^{s'-s} \leqslant 2^{n_0}\,\delta^{(d-s)/2}\,\ell(R_0)^{s'}.$$

Hence, choosing

$$n_0 = \lfloor \log_2 \delta^{(s-d)/2} \rfloor,$$

it follows that

$$\sum_{P\in \mathrm{Stop}_H} \ell(P)^{s'} \leqslant \ell(R_0)^{s'}. \tag{12.9}$$

It remains to deal with the family of cubes from Stop_W. By Hölder's inequality and (12.7), for each $j \geqslant 1$, we have

$$\begin{aligned}\sum_{P\in W_j} \omega(P)^{1/2}\,\ell(P)^{d/2} &= \sum_{R\in W_{j-1}}\ \sum_{Q\in \mathrm{Next}(R)}\ \sum_{P\in W_{j-1}:P\subset Q} \omega(P)^{1/2}\,\ell(P)^{d/2}\\ &\leqslant \sum_{R\in W_{j-1}}\ \sum_{Q\in \mathrm{Next}(R)} \omega(Q)^{1/2}\,\ell(Q)^{d/2}\\ &\leqslant \frac{1}{4} \sum_{R\in W_{j-1}} \omega(R)^{1/2}\,\ell(R)^{d/2}.\end{aligned}$$

Iterating and using Hölder again, we obtain

$$\sum_{P\in W_{n_0-1}} \omega(P)^{1/2}\,\ell(P)^{d/2} \leqslant \frac{1}{4^{n_0-1}} \sum_{R\in W_1} \omega(R)^{1/2}\,\ell(R)^{d/2} \leqslant \frac{1}{4^{n_0-1}}\,\omega(R_0)^{1/2}\,\ell(R_0)^{d/2}.$$

Assume that δ is a dyadic number. That is, $\delta = 2^{-h}$ for some natural number h. Denote by $\mathcal{S}_W$ the family of cubes $Q \in \mathcal{D}_h(R_0)$ that contain some cube from Stop_W. Then, by construction, the cubes from $\mathcal{S}_W$ are contained in cubes from W_{n_0-1},. So once more by Hölder, and using that $n_0 \geqslant \log_2 \delta^{(s-d)/2} - 1$, we get

$$\begin{aligned}\sum_{Q\in\mathcal{S}_W} \omega(Q)^{1/2}\,\ell(Q)^{d/2} &= \sum_{P\in W_{n_0-1}} \sum_{Q\in\mathcal{S}_W:Q\subset P} \omega(Q)^{1/2}\,\ell(Q)^{d/2}\\ &\leqslant \sum_{P\in W_{n_0-1}} \omega(P)^{1/2}\,\ell(P)^{d/2}\\ &\leqslant \frac{1}{4^{n_0-1}}\,\omega(R_0)^{1/2}\,\ell(R_0)^{d/2} \leqslant 16\,\delta^{d-s}\,\omega(R_0)^{1/2}\,\ell(R_0)^{d/2}. \qquad (12.10)\end{aligned}$$

Consider the families

$$\mathcal{S}_W^1 = \left\{Q\in\mathcal{S}_W : \omega(Q) \geqslant \left(\frac{\ell(Q)}{\ell(R_0)}\right)^s \omega(R_0)\right\}, \qquad \mathcal{S}_W^2 = \mathcal{S}_W\backslash\mathcal{S}_W^1.$$

We have

$$\ell(Q)^s \leqslant \frac{\omega(Q)}{\omega(R_0)}\,\ell(R_0)^s \quad \text{for each } Q\in\mathcal{S}_W^1,$$

and thus

$$\sum_{Q\in\mathcal{S}_W^1} \ell(Q)^s \leqslant \sum_{Q\in\mathcal{S}_W^1} \frac{\omega(Q)}{\omega(R_0)}\,\ell(R_0)^s \leqslant \ell(R_0)^s. \qquad (12.11)$$

On the other hand, the cubes $Q\in\mathcal{S}_W^2$ satisfy

$$\omega(Q) < \left(\frac{\ell(Q)}{\ell(R_0)}\right)^s \omega(R_0) = \delta^{s-d}\left(\frac{\ell(Q)}{\ell(R_0)}\right)^d \omega(R_0).$$

and so, by (12.10),

$$\sum_{Q\in\mathcal{S}_W^2} \omega(Q) \leqslant \delta^{(s-d)/2} \sum_{Q\in\mathrm{Stop}_W} \omega(Q)^{1/2}\left(\frac{\ell(Q)}{\ell(R_0)}\right)^{d/2} \omega(R_0)^{1/2} \leqslant 16\,\delta^{(d-s)/2}\,\omega(R_0). \qquad (12.12)$$

Let $t = (s'+d)/2 = (s+3d)/4$ and denote

$$E_\tau = \bigcup_{Q\in\mathrm{Stop}_H} Q \cup \bigcup_{Q\in\mathcal{S}_W^1} Q.$$

Since the cubes Q in the unions above satisfy $\ell(Q)\leqslant\delta\,\ell(R_0)$, by (12.9) and (12.11), we have

$$\begin{aligned}\mathcal{H}^t_\infty(E_\tau) &\lesssim (\delta\,\ell(R_0))^{t-s'}\,\mathcal{M}^{s'}_{\delta\ell(R_0)}\Big(\bigcup_{Q\in\mathrm{Stop}_H} Q\Big) + (\delta\,\ell(R_0))^{t-s}\,\mathcal{M}^{s}_{\delta\ell(R_0)}\Big(\bigcup_{Q\in\mathcal{S}_W^1} Q\Big)\\ &\lesssim \left(\delta^{t-s'} + \delta^{t-s}\right)\,\ell(R_0)^t \lesssim \delta^{(d-s)/4}\,\ell(R_0)^t.\end{aligned}$$

On the other hand, notice that

$$E\cap R_0 \subset E_\tau \cup \bigcup_{Q\in\mathcal{S}_W^2} Q.$$

Then, by (12.12),

$$\omega(E \cap R_0 \backslash E_\tau) \leqslant \omega\Big(\bigcup_{Q \in \mathcal{S}_W^2} Q\Big) \leqslant 16\,\delta^{(d-s)/2}\,\omega(R_0).$$

So (12.6) holds with $\tau = \min(C\delta^{(d-s)/4}, 16\,\delta^{(d-s)/2})$. □

12.2 Dimension drop

For a domain $\Omega \subset \mathbb{R}^d$, when the (Hausdorff) codimension of $\partial\Omega$ is not 1 or $\partial\Omega$ is of fractal type, many examples show that we may have $\dim \omega < \dim \partial\Omega$. This is the so-called "dimension drop" for harmonic measure, which seems to be a frequent phenomenon. This was first observed by Carleson [Car85] for some domains defined as complements of suitable Cantor type sets in the plane.

In this section we will show that if $\partial\Omega$ is s-Ahlfors regular for some $s \in (d-1, d)$, then the harmonic measure has a dimension drop. We will prove that the same holds in the planar case if $\partial\Omega$ is contained in a line and $s \in [1/2, 1)$. The first result is due to Azzam [Azz20] and the second one to Tolsa [Tol24].

Recall that, for $s > 0$, a measure μ on $\mathbb{R}^d$ is called s-Ahlfors regular if there exists some constant $C_0 > 0$ such that

$$C_0^{-1} r^s \leqslant \mu(B_r(x)) \leqslant C_0\, r^s \quad \text{ for all } x \in \operatorname{supp}\mu \text{ and } 0 < r \leqslant \operatorname{diam}(\operatorname{supp}\mu).$$

A set $E \subset \mathbb{R}^d$ is a called s-Ahlfors regular if the measure $\mathcal{H}^s|_E$ is s-Ahlfors regular. If we want to specify the constant C_0 involved in the Ahlfors regularity, we may say that that μ or E are (s, C_0)-Ahlfors regular.

12.2.1 A general result about dimension drop on Ahlfors regular sets

Our proof of the dimension drop for harmonic measure is based on the following result, which has an independent interest.

Theorem 12.6. *For $d \geqslant 1$, $s > 0$, $C_0 > 1$, there exists an $M = M(d, s, C_0) > 1$ (sufficiently big) such that the following holds. Let $E \subset \mathbb{R}^d$ be an (s, C_0)-Ahlfors regular set. Let ν be a measure supported on E and $c_1 \in (0,1)$ such that, for all $x \in E$, $0 < r \leqslant \operatorname{diam}(E)$, there exists a ball $B_\rho(y)$ with $y \in B_r(x) \cap E$, $c_1\, r \leqslant \rho \leqslant r$, satisfying either*

$$\frac{\nu(B_\rho(y))}{\rho^s} \geqslant M\,\frac{\nu(B_r(x))}{r^s} \qquad \text{or} \qquad \frac{\nu(B_\rho(y))}{\rho^s} \leqslant M^{-1}\,\frac{\nu(B_r(x))}{r^s}. \tag{12.13}$$

Then $\dim \nu < s - \varepsilon$, with ε depending on c_1, d, M.

The arguments to prove this theorem stem from the techniques developed by Bourgain in Theorem 12.2, and later used by Batakis [Bat96], and more recently by Azzam [Azz20]. First we will prove the following.

Lemma 12.7. *Under the assumptions of Proposition 12.6, let $\mu = \mathcal{H}^s|_E$ and let $\mathcal{D}_\mu$ be a dyadic lattice associated with μ as in Definition 4.23 and Theorem 4.26. Then there exist some $m_0 \geqslant 1$ depending on c_1 and some $\gamma \in (0,1)$ depending on c_1 and M such that*

$$\sum_{P\in\mathcal{D}_{\mu,m_0}(Q)} \nu(P)^{1/2}\mu(P)^{1/2} \leqslant \gamma\,\nu(Q)^{1/2}\mu(Q)^{1/2}. \tag{12.14}$$

In the lemma, $\mathcal{D}_{\mu,m_0}(Q)$ stands for the family of cubes $P \in \mathcal{D}_\mu$ contained in Q with side length $\ell(P) = \ell_0^{-m_0}$, that is, if $Q \in \mathcal{D}_{\mu,k}$, then $P \in \mathcal{D}_{m_0+k}$.[1]

Proof. By Theorem 4.26, there exists a dyadic lattice associated with μ, which we denote by $\mathcal{D}_\mu$. For $Q \in \mathcal{D}_\mu$, we denote

$$\theta_\nu(Q) = \frac{\nu(Q)}{\ell(Q)^s}.$$

We claim that the assumptions of the theorem imply that there exists some constant $a \in (0,1)$ such that for any $Q \in \mathcal{D}_\mu$ there exists another cube $P_0 \in \mathcal{D}_\mu$ contained Q satisfying:

(a) either $\theta_\nu(P_0) \geqslant C^{-1}M^{1/2}\theta_\nu(Q)$ or $\theta_\nu(P_0) \leqslant CM^{-1/2}\theta_\nu(Q)$ (for some constant C depending on s and the parameters involved in $\mathcal{D}_\mu$), and

(b) $\ell(P_0) \geqslant a\,\ell(Q)$.

Indeed, let z_Q be the center of Q and let B_Q be a ball centered at z_Q as in the Definition 4.23, so that $B_Q \cap \operatorname{supp}\mu \subset Q$, with radius $r(B_Q) \approx \ell(Q)$. By the assumptions of the theorem applied to $\frac{1}{2}B_Q$, there exists a ball $B_\rho(y)$ with $y \in \frac{1}{2}B_Q$, $c_1'\,r(B_Q) \leqslant \rho \leqslant \frac{1}{2}r(B_Q)$, satisfying either

$$\frac{\nu(B_\rho(y))}{\rho^s} \geqslant C^{-1}M\,\frac{\nu(\frac{1}{2}B_Q)}{\ell(Q)^s} \qquad \text{or} \qquad \frac{\nu(B_\rho(y))}{\rho^s} \leqslant CM^{-1}\,\frac{\nu(\frac{1}{2}B_Q)}{\ell(Q)^s}. \tag{12.15}$$

Observe that in any case $B_\rho(y) \cap \operatorname{supp}(\mu) \subset Q$. If the second option in (12.15) holds, then we take a cube $P_0 \in \mathcal{D}_\mu$ contained in $B_\rho(y)$ with $\ell(P_0) \approx \rho$, and then we have

$$\theta_\nu(P_0) \lesssim \frac{\nu(B_\rho(y))}{\rho^s} \lesssim M^{-1}\,\frac{\nu(\frac{1}{2}B_Q)}{\ell(Q)^s} \lesssim M^{-1}\theta_\nu(Q).$$

So P_0 satisfies both (a) and (b).

If the first option in (12.15) holds, then we can assume that $\frac{\nu(\frac{1}{2}B_Q)}{\ell(Q)^s} \geqslant M^{-1/2}\theta_\nu(Q)$, because otherwise we can take a cube $P_0 \in \mathcal{D}_\mu$ contained in $\frac{1}{2}B_Q$ with $\ell(P_0) \approx \ell(Q)$ and then, arguing as above we deduce that $\theta_\nu(P_0) \lesssim M^{-1/2}\theta_\nu(Q)$, and thus P_0 does the job.

[1]It is easy to check that we can take the constant $r_0 = 1/2$ in Definition 4.23 just eliminating or repeating intermediate generations if necessary, allowing for example cubes Q to have a unique child.

So suppose that $\frac{\nu(\frac{1}{2}B_Q)}{\ell(Q)^s} \geqslant M^{-1/2}\theta_\nu(Q)$ and that the first option in (12.15) holds. Then there exists $B_\rho(y)$ with $y \in \frac{1}{2}B_Q$, $c_1'\, r(B_Q) \leqslant \rho \leqslant \frac{1}{2}r(B_Q)$, such that

$$\frac{\nu(B_\rho(y))}{\rho^s} \gtrsim M\,\frac{\nu(\frac{1}{2}B_Q)}{\ell(Q)^s} \gtrsim M^{1/2}\theta_\nu(Q).$$

Since $B_\rho(y) \cap \operatorname{supp}(\mu)$ is contained in Q, it can be covered by a finite number of cubes $P \in \mathcal{D}_\mu$ contained in Q with $\ell(P) \approx \rho$. The cube $P = P_0$ with maximal ν measure satisfies $\nu(P_0) \gtrsim \nu(B_\rho(y))$ and so

$$\theta_\nu(P_0) \gtrsim \frac{\nu(B_\rho(y))}{\rho^s} \gtrsim M^{1/2}\theta_\nu(Q).$$

This completes the proof of the claim.

Assume that the constant $a > 0$ in (b) is of the form $a = \ell_0^{-m_0}$, for some natural number m_0 and let n_0, with $1 \leqslant n_0 \leqslant m_0$, be such that $P_0 \in \mathcal{D}_{\mu,n_0}(Q)$. Suppose that $\theta_\nu(P_0) \leqslant CM^{-1/2}\theta_\nu(Q)$. By Cauchy-Schwarz, we have

$$\sum_{P\in\mathcal{D}_{\mu,n_0}(Q):P\neq P_0} \nu(P)^{1/2}\mu(P)^{1/2} \leqslant \nu(Q\backslash P_0)^{1/2}\mu(Q\backslash P_0)^{1/2} \leqslant \nu(Q)^{1/2}\big(\mu(Q)-\mu(P_0)\big)^{1/2}.$$

On the other hand, we have

$$\begin{aligned}\nu(P_0)^{1/2}\mu(P_0)^{1/2} &= \frac{\nu(P_0)^{1/2}}{\mu(P_0)^{1/2}}\,\mu(P_0) \approx \theta_\nu(P_0)^{1/2}\mu(P_0)\\ &\lesssim M^{-1/4}\theta_\nu(Q)^{1/2}\,\mu(P_0) \approx M^{-1/4}\nu(Q)^{1/2}\mu(Q)^{1/2}\,\frac{\mu(P_0)}{\mu(Q)}.\end{aligned}$$

Gathering the two previous estimates and using the inequality $(1-x)^{1/2} \leqslant 1-\frac{1}{2}x$ for $0 \leqslant x \leqslant 1$, we obtain

$$\begin{aligned}\sum_{P\in\mathcal{D}_{\mu,n_0}(Q)} \nu(P)^{1/2}\mu(P)^{1/2} &\leqslant \nu(Q)^{1/2}\big(\mu(Q)-\mu(P_0)\big)^{1/2} + CM^{-1/4}\nu(Q)^{1/2}\mu(Q)^{1/2}\,\frac{\mu(P_0)}{\mu(Q)}\\ &= \nu(Q)^{1/2}\mu(Q)^{1/2}\left(\left(1-\frac{\mu(P_0)}{\mu(Q)}\right)^{1/2} + CM^{-1/4}\frac{\mu(P_0)}{\mu(Q)}\right)\\ &\leqslant \nu(Q)^{1/2}\mu(Q)^{1/2}\left(1-\frac{1}{4}\frac{\mu(P_0)}{\mu(Q)}\right),\end{aligned}$$

assuming $CM^{-1/4} \leqslant \frac{1}{4}$ for the last inequality. Taking into account the s-Ahlfors regularity of μ, we have

$$\frac{\mu(P_0)}{\mu(Q)} \approx \frac{\ell(P_0)^s}{\ell(Q)^s} \leqslant \ell_0^{-m_0 s}.$$

Hence, taking $\gamma = 1 - c\ell_0^{-m_0 s}$, we have

$$\sum_{P\in\mathcal{D}_{\mu,n_0}(Q)} \nu(P)^{1/2}\mu(P)^{1/2} \leqslant \gamma\,\nu(Q)^{1/2}\mu(Q)^{1/2}.$$

On the other hand, by Cauchy-Schwarz, splitting each $P \in \mathcal{D}_{\mu,n_0}(Q)$ into its descendants from $\mathcal{D}_{\mu,m_0}(Q)$, we get

$$\sum_{P\in\mathcal{D}_{\mu,n_0}(Q)} \nu(P)^{1/2}\mu(P)^{1/2} \leqslant \sum_{P\in\mathcal{D}_{\mu,m_0}(Q)} \nu(P)^{1/2}\mu(P)^{1/2}, \tag{12.16}$$

and thus the Proposition follows in this case.

Suppose now that $\theta_\nu(P_0) \geqslant CM^{-1/2}\theta_\nu(Q)$. The arguments are quite similar to the previous ones, interchanging the roles of μ and ν. Indeed, By Cauchy-Schwarz,

$$\sum_{P\in\mathcal{D}_{\mu,n_0}(Q):P\neq P_0} \nu(P)^{1/2}\mu(P)^{1/2} \leqslant \big(\nu(Q)-\nu(P_0)\big)^{1/2}\mu(Q)^{1/2}.$$

Also, we have

$$\begin{aligned}\nu(P_0)^{1/2}\mu(P_0)^{1/2} &= \nu(P_0)\,\frac{\mu(P_0)^{1/2}}{\nu(P_0)^{1/2}} \approx \nu(P_0)\,\theta_\nu(P_0)^{-1/2}\\ &\lesssim M^{-1/4}\nu(P_0)\,\theta_\nu(Q)^{-1/2} \approx M^{-1/4}\frac{\nu(P_0)}{\nu(Q)}\,\nu(Q)^{1/2}\mu(Q)^{1/2}.\end{aligned}$$

From two previous estimates and using again the inequality $(1-x)^{1/2} \leqslant 1-\frac{1}{2}x$ for $0 \leqslant x \leqslant 1$, we obtain

$$\begin{aligned}\sum_{P\in\mathcal{D}_{\mu,n_0}(Q)} \nu(P)^{1/2}\mu(P)^{1/2} &\leqslant \big(\nu(Q)-\nu(P_0)\big)^{1/2}\mu(Q)^{1/2} + CM^{-1/4}\frac{\nu(P_0)}{\nu(Q)}\,\nu(Q)^{1/2}\mu(Q)^{1/2}\\ &= \nu(Q)^{1/2}\mu(Q)^{1/2}\bigg(\bigg(1-\frac{\nu(P_0)}{\nu(Q)}\bigg)^{1/2} + CM^{-1/4}\frac{\nu(P_0)}{\nu(Q)}\bigg)\\ &\leqslant \nu(Q)^{1/2}\mu(Q)^{1/2}\bigg(1-\frac{1}{4}\frac{\nu(P_0)}{\nu(Q)}\bigg),\end{aligned}$$

since we are assuming that $CM^{-1/4} \leqslant \frac{1}{4}$. Observe now that

$$\frac{\nu(P_0)}{\nu(Q)} = \frac{\theta_\nu(P_0)}{\theta_\nu(Q)}\,\frac{\ell(P_0)^s}{\ell(Q)^s} \geqslant M^{1/2}\,\ell_0^{-m_0 s} \geqslant C\,\gamma,$$

where as before, $\gamma = 1-\ell_0^{-m_0 s}$. Therefore,

$$\sum_{P\in\mathcal{D}_{\mu,n_0}(Q)} \nu(P)^{1/2}\mu(P)^{1/2} \leqslant \gamma\,\nu(Q)^{1/2}\mu(Q)^{1/2}.$$

Finally, the same estimate as in (12.16) completes the proof of the lemma. □

Proof of Theorem 12.6. The arguments to prove the theorem are quite similar to the ones used in the proof of Bourgain's Theorem 12.2. Indeed, notice that the estimate (12.14) in Lemma 12.7 is similar to (b) in Lemma 12.5. We will use a construction analogous to the tree type construction in the proof of Theorem 12.2, but a bit simpler due to the fact that now we do not have to distinguish between two options such as (a) and (b) in Lemma 12.5. For the convenience of the reader, we will show the full details.

We introduce now a dyadic Hausdorff content for subsets of $E := \operatorname{supp}(\mu)$ analogous to the one in (12.2). For $F \subset E$ and $t, \varepsilon > 0$, we denote

$$\mathcal{M}^t_{\mu,\varepsilon}(F) = \inf\Big\{\sum_i \ell(Q_i)^t : Q_i \in \mathcal{D}_\mu,\, F \subset \bigcup_i Q_i,\, \ell(Q_i) \leqslant \varepsilon\Big\}. \tag{12.17}$$

By the properties of $\mathcal{D}_\mu$, it is immediate to check that that $\mathcal{H}^t_\varepsilon(F) \approx \mathcal{M}^t_{\mu,\varepsilon}(F)$ with the implicit constant depending on the parameters in the definition of $\mathcal{D}_\mu$.

We will show that for every cube $R_0 \in \mathcal{D}_\mu$, $\dim_{\mathcal{H}}(\nu|_{R_0}) \leqslant t$ for some $t < s$ depending on γ in Lemma 12.7, which suffices to prove theorem. To this end, we will prove that, for every $\tau > 0$, there exists a subset $E_\tau \subset E \cap R_0$ satisfying

$$\mathcal{H}^t_\infty(E_\tau) \leqslant \tau \quad \text{ and } \quad \nu(E \cap R_0 \backslash E_\tau) \leqslant \tau. \tag{12.18}$$

It is immediate to check that this implies that $\dim_{\mathcal{H}}(\nu|_{R_0}) \leqslant t$.

For m_0 as in Lemma 12.7, for every $k \geqslant 1$ we have

$$\begin{aligned}\sum_{Q \in \mathcal{D}_{\mu,km_0}(R_0)} \nu(Q)^{1/2}\mu(Q)^{1/2} &= \sum_{P \in \mathcal{D}_{\mu,(k-1)m_0}(R_0)} \sum_{Q \in \mathcal{D}_{\mu,m_0}(P)} \nu(Q)^{1/2}\mu(Q)^{1/Q} \\ &\leqslant \gamma \sum_{P \in \mathcal{D}_{\mu,(k-1)m_0}(R_0)} \nu(P)^{1/2}\mu(P)^{1/2}.\end{aligned}$$

Iterating, we deduce that

$$\sum_{Q \in \mathcal{D}_{\mu,km_0}(R_0)} \nu(Q)^{1/2}\mu(Q)^{1/2} \leqslant \gamma^k\, \nu(R_0)^{1/2}\mu(R_0)^{1/2} \quad \text{ for all } k \geqslant 1. \tag{12.19}$$

For any fixed $k \geqslant 1$, denote $\delta_k = \ell_0^{-km_0}$, so that $\ell(Q) = \delta_k\, \ell(R_0)$ for $Q \in \mathcal{D}_{\mu,km_0}(R_0)$. For some $t' \in (0, s)$ to be fixed below, consider the families

$$\mathcal{S}^1_k = \left\{Q \in \mathcal{D}_{\mu,km_0}(R_0) : \nu(Q) \geqslant \left(\frac{\ell(Q)}{\ell(R_0)}\right)^{t'} \nu(R_0)\right\}, \qquad \mathcal{S}^2_k = \mathcal{D}_{\mu,km_0}(R_0) \backslash \mathcal{S}^1_k.$$

We have

$$\ell(Q)^{t'} \leqslant \frac{\nu(Q)}{\nu(R_0)}\, \ell(R_0)^{t'} \quad \text{ for each } Q \in \mathcal{S}^1_k,$$

and thus

$$\sum_{Q \in \mathcal{S}^1_k} \ell(Q)^{t'} \leqslant \sum_{Q \in \mathcal{S}^1} \frac{\nu(Q)}{\nu(R_0)}\, \ell(R_0)^{t'} \leqslant \ell(R_0)^{t'}. \tag{12.20}$$

On the other hand, the cubes $Q \in \mathcal{S}^2_k$ satisfy

$$\nu(Q) < \left(\frac{\ell(Q)}{\ell(R_0)}\right)^{t'} \nu(R_0) = \delta_k^{t'-s} \left(\frac{\ell(Q)}{\ell(R_0)}\right)^{s} \nu(R_0).$$

and so, by (12.19),

$$\sum_{Q\in\mathcal{S}_k^2} \nu(Q) \leqslant \delta_k^{(t'-s)/2} \sum_{Q\in\mathcal{D}_{\mu,km_0}(R_0)} \nu(Q)^{1/2} \left(\frac{\ell(Q)}{\ell(R_0)}\right)^{s/2} \nu(R_0)^{1/2}$$

$$\approx \delta_k^{(t'-s)/2} \sum_{Q\in\mathcal{D}_{\mu,km_0}(R_0)} \nu(Q)^{1/2}\,\mu(Q)^{1/2}\left(\frac{\nu(R_0)}{\mu(R_0)}\right)^{1/2} \leqslant \delta_k^{(t'-s)/2}\,\gamma^k\,\nu(R_0).$$

Recalling that $\delta_k = \ell_0^{-km_0}$, for t_1 is close enough to s we have

$$\delta_k^{(t'-s)/2}\,\gamma^k = \ell_0^{-km_0(t'-s)/2}\,\gamma^k \leqslant \gamma^{k/2},$$

and thus

$$\sum_{Q\in\mathcal{S}_k^2} \nu(Q) \lesssim \gamma^{k/2}\,\nu(R_0). \tag{12.21}$$

Let $t = (t' + s)/2$ and denote

$$E_\tau = \bigcup_{Q\in\mathcal{S}_k^1} Q.$$

Since the cubes Q in the union above satisfy $\ell(Q) \leqslant \delta_k\,\ell(R_0)$, by (12.20), we have

$$\mathcal{H}^t_\infty(E_\tau) \lesssim (\delta_k\,\ell(R_0))^{t-t'}\,\mathcal{M}^{t'}_{\delta_k\ell(R_0)}\Big(\bigcup_{Q\in\mathcal{S}_k^1} Q\Big) \lesssim \delta_k^{t-t'}\,\ell(R_0)^t = \delta^{s-t}\,\ell(R_0)^t.$$

On the other hand, notice that

$$E \cap R_0 \subset E_\tau \cup \bigcup_{Q\in\mathcal{S}_k^2} Q.$$

Then, by (12.21), $\nu(E\cap R_0\backslash E_\tau) \lesssim \gamma^{k/2}\,\nu(R_0)$. Hence, for k large enough (12.18) follows. □

12.2.2 Dimension drop in the case of codimension smaller than one

Theorem 12.8. *Let $\Omega \subset \mathbb{R}^d$ be an open set with compact (s, C_0)-Ahlfors regular boundary, for some $s \in (d-1, d)$ and $C_0 \geqslant 1$. Then, for any $p \in \Omega$, $\dim_{\mathcal{H}} \omega_\Omega^{x_0} < s - \varepsilon$, for some $\varepsilon > 0$ depending on s and C_0.*

Remark that in the plane this theorem is a consequence of the Jones-Wolff Theorem 9.16 about the the dimension of harmonic measure in the plane. Indeed, this implies that $\dim_{\mathcal{H}} \omega_\Omega^{x_0} \leqslant 1 < s$.

Proof. Notice first that the s-Ahlfors regularity of $\partial\Omega$ for some $s > d-1$ implies that Ω satisfies the CDC.

By Theorem 12.6, it suffices to prove that for every $x \in \partial\Omega$ and $0 < r \leqslant \delta_\Omega(p)/4$, there exists a ball $B_\rho(y)$ with $y \in B_r(x) \cap E$, $c_1 r \leqslant \rho \leqslant r$, satisfying (12.13). For a fixed $p \in \Omega$ and a big constant M to be chosen below, we may assume that

$$\omega_\Omega^p(B_r(x)) \leqslant 4^s M\, \omega^p(B_{r/4}(x)),$$

because otherwise the second option in (12.13) holds. We claim that the above estimate implies that there exists some point $q_0 \in B_{r/2}(x) \cap \Omega$ such that

$$G(p,q_0) \gtrsim \frac{\omega^p(B_r(x))}{r^{d-2}} \quad \text{and} \quad \delta_\Omega(q_0) \gtrsim_M r. \tag{12.22}$$

To prove this, consider a C^∞ function φ such that $\chi_{B_{r/4}(x)} \leqslant \varphi \leqslant \chi_{B_{3r/8}(x)}$ and $\|\nabla\varphi\|_\infty \lesssim r^{-1}$. For some small $\lambda \in (0, 1/50)$ to be chosen, denote

$$U_\lambda = \left\{x \in \Omega \cap B_{3r/8}(x) : \delta_\Omega(x) \leqslant \lambda r\right\}$$

We write

$$\begin{aligned} \omega^p(B_{r/4}(x)) &\leqslant \int \varphi\, d\omega^p = -\int \nabla G^p(y)\, \nabla\varphi(y)\, dy \\ &\leqslant \int_{U_\lambda} |\nabla G^p(y)\, \nabla\varphi(y)|\, dy + \int_{\Omega\setminus U_\lambda} |\nabla G^p(y)\, \nabla\varphi(y)|\, dy =: I_1 + I_2. \end{aligned} \tag{12.23}$$

Next we intend to show that $I_1 \leqslant \frac{1}{2}\omega^p(B_{r/4}(x))$ if λ is taken small enough. To this end, by Vitali's covering theorem we can cover U_λ with a family of balls $\{B_i\}_{i\in I} := \{B(x_i, 6\lambda r)\}_{i\in I}$, with $x_i \in U_\lambda$, so that the balls $\frac{1}{5}B_i$, for $i \in I$, are disjoint. From the fact that the balls $\frac{1}{5}B_i$ are disjoint and they have the same radius, it is immediate to check that the larger balls $30B_i$ have finite superposition, that is,

$$\sum_{i\in I} \chi_{30B_i} \leqslant C.$$

We also assume λ small enough so that the balls $30B_i$ are contained in $B_r(x)$. Then, using Caccioppoli and Lemma 7.19, together with CDC for Ω, we deduce that

$$\begin{aligned} I_1 &\leqslant \sum_{i\in I} \int_{B_i} |\nabla G^p(y)\, \nabla\varphi(y)|\, dy \lesssim \frac{1}{r} \sum_{i\in I} \left(\int_{B_i} |\nabla G^p(y)|^2\, dy\right)^{1/2} m(B_i)^{1/2} \\ &\lesssim \frac{1}{r(\lambda r)} \sum_{i\in I} \left(\int_{B_i} |G^p(y)|^2\, dy\right)^{1/2} m(B_i)^{1/2} \lesssim \frac{(\lambda r)^d}{\lambda r^2} \sum_{i\in I} \sup_{y\in 2B_i} |G^p(y)| \\ &\lesssim \lambda^{d-1} r^{d-2} \sum_{i\in I} \frac{\omega^p(30B_i)}{(\lambda r)^{d-2}} \lesssim \lambda\, \omega^p\Big(\bigcup_{i\in I} 30B_i\Big) \leqslant \lambda\, \omega^p(B_r(x)) \lesssim M\lambda\, \omega^p(B_{r/4}(x)). \end{aligned}$$

Hence, taking $\lambda = c/M$ with c sufficiently small, we derive $I_1 \leqslant \omega^p(B_{r/4}(x))/2$, as wished.

From (12.23) and the last estimate obtained for I_1, we infer that

$$\omega^p(B_{r/4}(x)) \leqslant 2 \int_{\Omega\setminus U_\lambda} |\nabla G^p(y)\,\nabla\varphi(y)|\,dy \lesssim \frac{1}{r}\int_{B_{3r/8}(x)\setminus U_\lambda} |\nabla G^p(y)|\,dy$$
$$\lesssim r^{d-1} \sup_{y\in B_{3r/8}(x)\setminus U_\lambda} |\nabla G^p(y)| \lesssim_\lambda r^{d-2} \sup_{y\in B_{r/2}(x)\cap\Omega\setminus U_{\lambda/2}} G^p(y).$$

Thus, there exists some $q_0 \in B_{r/2}(x) \cap \Omega\setminus U_{\lambda/2}$ such that

$$G^p(q_0) \gtrsim_\lambda \frac{1}{r^{d-2}}\,\omega^p(B_{r/4}(x)) \approx_M \frac{1}{r^{d-2}}\,\omega^p(B_r(x)),$$

which proves our claim (12.22).

Consider the ball $B_0 = B_{\delta_\Omega(q_0)}(q_0)$, so that $B_0 \subset \Omega$ and $\partial B_0 \cap \partial\Omega \neq \varnothing$. Let $\xi \in \partial B_0 \cap \partial\Omega$ and take $q_1 = \frac{q_0+\xi}{2}$, so that $q_1 \in B_0$. Then we have $|\xi - q_1| = \delta_\Omega(q_1)$, and letting $B_1 = B_{|\xi-q_1|}(q_1)$, it holds

$$B_1 \subset B_0 \subset \Omega \quad \text{and} \quad \{\xi\} = \partial B_1 \cap \partial\Omega.$$

Notice also that $|\xi - q_0| \leqslant |x - q_0| < r/2$ and so $\xi \in B_r(x)$.

Assume that q_1 is the origin in $\mathbb{R}^d$ and that $\xi = (0,\dots,0,-|\xi|)$. Let Γ be the upper half of the sphere ∂B_1. Notice that $\operatorname{dist}(\Gamma,\partial B_0) \approx r(B_0) \approx r$. Then, by a Harnack chain argument, since G^p is harmonic in B_0, it follows that

$$G^p(y) \approx G^p(q_0) \quad \text{for all } y \in \Gamma. \tag{12.24}$$

Denote by r_1 the radius of B_1. That is, $r_1 = |\xi - q_1|$. Let us check that

$$G^p(z) \gtrsim \frac{\operatorname{dist}(z,\partial B_1)}{r}\,G^p(q_0) \quad \text{for all } z \in B_1 \cap B_{r_1/4}(\xi). \tag{12.25}$$

Indeed, by (12.24) and the maximum principle,

$$G^p(z) \gtrsim \omega^z_{B_1}(\Gamma)\,G^p(q_0) \quad \text{in } B_1. \tag{12.26}$$

By the explicit formula for $\omega^z_{B_1}(\Gamma)$ in Example 5.27, we have

$$\omega^z_{B_1}(\Gamma) = \int_{y\in\Gamma} \frac{r_1^2 - |z|^2}{\kappa_d\, r_1\, |y-z|^d}\,d\sigma(z) \gtrsim \frac{\operatorname{dist}(z,\partial B_1)}{r} \quad \text{for all } z \in B_1 \cap B_{r_1/4}(\xi),$$

which, together with (12.26), gives (12.25).

Now, by Lemma 7.19, (12.25) and (12.22), one easily deduces that the first option in (12.13) holds. Indeed, for $0 < \rho \leqslant r_1/4$, we have

$$\omega^p(B_\rho(\xi)) \gtrsim \rho^{d-2} \sup_{z\in B_{\rho/8}(\xi)} G^p(z) \gtrsim \frac{\rho^{d-1}}{r}\,G^p(q_0) \gtrsim_M \frac{\rho^{d-1}}{r^{d-1}}\,\omega^p(B_r(x)).$$

Equivalently,

$$\frac{\omega^p(B_\rho(\xi))}{\rho^s} \gtrsim_M \frac{\omega^p(B_r(x))}{r^s}\left(\frac{r}{\rho}\right)^{s-(d-1)},$$

and so, taking ρ small enough, the first estimate in (12.13) follows. □

12.2.3 Dimension drop for subsets of lines in the plane

Theorem 12.9. *Let $\Omega \subset \mathbb{R}^2$ be an open set with compact (s, C_0)-Ahlfors regular boundary contained in a line, for some $s \in [1/2, 1)$ and $C_0 \geqslant 1$. Then, for any $p \in \Omega$, $\dim_{\mathcal{H}} \omega_\Omega^p < s - \varepsilon$, for some $\varepsilon > 0$ depending on s and C_0.*

To prove this theorem, we will use the following result due to David, Feneuil, and Mayboroda [DFM21], which has its own interest.

Lemma 12.10. *Let $\Omega \subset \mathbb{R}^d$ be an open set with (s, C_0)-Ahlfors regular boundary, for some $s \in (0, d-1)$. Then Ω is a uniform domain.*

Proof. The fact that $s < d-1$ implies that $\mathbb{R}^d \backslash \partial\Omega$ is connected and so $\Omega = \mathbb{R}^d \backslash \partial\Omega$. Then, the corkscrew condition follows easily from the Ahlfors regularity of $\partial\Omega$.

To prove the Harnack chain condition, let $x_1, x_2 \in \Omega$ and $r, \Lambda > 0$ be such that $\operatorname{dist}(x_i, \partial\Omega) \geqslant r$ and $|x_1 - x_2\| \leqslant \Lambda r$. We claim that there are points $y_i \in B_{r/3}(x_i)$ such that the segment $S := [y_1, y_2]$ satisfies $\operatorname{dist}(S, \partial\Omega) \geqslant \delta\, r$, with $\delta = \delta(\Lambda, d, s) > 0$. That is, there is a thick tube contained in Ω that connects $B_{r/3}(x_1)$ and $B_{r/3}(x_2)$.

To prove our claim, denote by L a hyperplane through the origin orthogonal to S. Let $\delta \in (0,1)$ be a small constant to be chosen soon. We can find $N \geqslant C^{-1}\delta^{1-d}$ points $z_j \in L \cap B_{r/3}(0)$ such that $|z_i - z_j| \geqslant 4\delta r$ for $i \neq j$. For each j, let S_j be the translated segment $S_j = z_j + S$. Suppose that $\operatorname{dist}(S_j, \partial\Omega) \leqslant \delta r$. Then we can find points $w_j \in \partial\Omega$ such that $\operatorname{dist}(w_j, S_j) \leqslant \delta r$. The balls $B_j = B_{\delta r}(w_j)$ are disjoint because $\operatorname{dist}(S_i, S_j) \geqslant 4\delta r$. Then, from the (s, C_0)-Ahlfors regularity of $\partial\Omega$ and the fact that all the balls B_j are contained in $B_{2r+|x_1-x_2|}(w_k)$ (for any w_k), we deduce that

$$N C_0^{-1} (\delta r)^s \leqslant \sigma(B_j \cap \partial\Omega) = \sigma\Big(\bigcup_j B_j \cap \partial\Omega\Big) \leqslant \sigma(B_{2r+|x_1-x_2|}(w_k)) \leqslant C_0 (2+\Lambda)^s r^s.$$

Thus,

$$C^{-1} C_0^{-1} \delta^{1-d} \delta^s \leqslant C_0 (2+\Lambda)^s,$$

which gives a contradicition if δ is chosen small enough, depending on C_0, d, s. So there exists a segment S_j such that $\operatorname{dist}(S_j, \partial\Omega) \geqslant \delta r$, and taking $y_1 = x_1 + z_j$ and $y_2 = x_2 + z_j$ the claim follows.

To construct a Harnack chain between x_1 and x_2, we can choose $B_{r/3}(x_1)$ and $B_{r/3}(x_2)$ as the first and last balls of the chain, respectively. To choose the other balls of the chain, we consider a family of points $a_1, \dots, a_m \in S$ such that $|a_k - a_{k+1}| \leqslant \delta r/4$, $m \leqslant C\Lambda\delta^{-1}$, and we take balls $B_{\delta r/2}(a_k)$, for $1 \leqslant k \leqslant m$. It is immediate to check that this chain of balls satisfies the required properties in Definition 8.5. $\square$

Proof of Theorem 12.9. Notice that Ω is a uniform domain satisfying the CDC.

First we will show that, for $E = \partial\Omega$, $B_r(x)$, and x_0 as above, there exists $y \in B_r(x) \cap E$ such that

$$\frac{\omega^{p_0}(B_\rho(y))}{\omega^{p_0}(B_r(x))} \geqslant c(s, C_0) \left(\frac{\rho}{r}\right)^{\frac{1}{2}} \tag{12.27}$$

for all $\rho \in (0, c' r)$, for some $c' \in (0,1)$ depending only on s, C_0, and n. Clearly, this implies the first estimate in (12.13) for $s \in (\frac{1}{2}, 1)$ and ρ small enough. For the cases $s = \frac{1}{2}$ we will need more careful estimates. Notice also that, modulo a constant factor, the estimate (12.27) is independent of the pole p_0 as soon as p_0 is far enough from $B_r(x)$, since Ω is a uniform domain.

The arguments for the case $s \in [\frac{1}{2}, 1)$.
Without loss of generality, we assume that $E \subset \mathbb{R} \equiv \mathbb{R} \times \{0\}$. Let $x \in E$ and $0 < r \leqslant \operatorname{diam} E$. Taking into account that $s < 1$, by a pigeon-hole argument, there is an open interval $I = (a, b) \subset [x - r, x]$ which does not intersect E and satisfies $\ell := \mathcal{H}^1(I) \approx_s r$. By enlarging I if necessary, we can assume that $b \in E$. Notice that b is contained in $[x - (1-c)r, x]$ because $x \in E$, for some $c > 0$ depending on s.

We choose $y = b$. Again by the s-Ahlfors regularity of E and the pigeon-hole principle, there exist radii r_1, r_2 with $\ell/2 \leqslant r_1 < r_2 \leqslant \ell$, $r_2 - r_1 \approx_s \ell \approx r$ such that

$$A_{r_1,r_2}(y) \cap E = \varnothing.$$

Here $A_{r_1,r_2}(y)$ stands for the open annulus centered in x with inner radius r_1 and outer radius r_2. Observe that the left component of $A_{r_1,r_2}(y) \cap \mathbb{R}$ is contained in I.

Next we apply a "localization argument". We denote $E_1 = E \cap \bar{B}_{r_1}(y)$, $\Omega_1 = E_1^c$, $r' = (r_1 + r_2)/2$. It is immediate to check that E_1 is still s-Ahlfors regular and thus Ω_1 is a uniform domain too. We claim that for any subset $F \subset E_1$ and any $p \in \partial B_{r'}(y)$,

$$\omega_1^p(F) \approx_s \omega^p(F), \tag{12.28}$$

where ω_1 stands for the harmonic measure for Ω_1. To prove the claim, consider first $p \in \partial B_{r'}(y)$ such that

$$\omega_1^p(F) = \max_{q \in \partial B_{r'}(y)} \omega_1^q(F).$$

Using that $\omega_1^z(F)$ is harmonic in Ω and vanishes in $E_1 \backslash F$ and the maximum principle, we get

$$\begin{aligned} \omega_1^p(F) &= \int_E \omega_1^z(F)\, d\omega^p(z) = \omega^p(F) + \int_{E \backslash E_1} \omega_1^z(F)\, d\omega^p(z) \\ &\leqslant \omega^p(F) + \sup_{z \in E \backslash E_1} \omega_1^z(F)\, \omega^p(E \backslash E_1). \end{aligned}$$

Observe that, by Lemma 7.20, Lemma 6.20, the CDC, and a Harnack chain argument, $\omega^p(E_1) \geqslant \delta_0$, for some $\delta_0 > 0$ depending just on s. Hence, $\omega^p(E \backslash E_1) \leqslant 1 - \delta_0$. Also, since $\omega_1^z(F)$ is harmonic in $\mathbb{C}_\infty \backslash B_{r'}(y)$ and $E \backslash E_1 \subset \mathbb{C}_\infty \backslash B_{r'}(y)$, by the maximum principle we have

$$\sup_{z \in E \backslash E_1} \omega_1^z(F) \leqslant \max_{q \in \partial B_{r'}(y)} \omega_1^q(F) = \omega_1^p(F).$$

Therefore,

$$\omega_1^p(F) \leqslant \omega^p(F) + \omega_1^p(F)\,(1 - \delta_0),$$

or equivalently, $\omega_1^p(F) \leqslant \delta_0^{-1}\,\omega^p(F)$. By the definition of r' and Harnack's inequality, we infer

$$\omega_1^p(F) \lesssim \omega^p(F)$$

for *all* $p \in \partial B_{r'}(y)$. On the other hand, by the maximum principle, we have trivially that $\omega_1^p(F) \geqslant \omega^p(F)$, which concludes the proof of the claimed estimate (12.28).

Next we will perform another modification of the domain Ω_1. For a fixed $\rho \in (0, r_1/4)$, consider the intervals $J = [y, y + \rho/2]$, $J' = [y, y + \rho]$ and define $E_2 = E_1 \cup J$ and $\Omega_2 = E_2^c = \Omega_1 \backslash J$. By the CDC and the uniformity of Ω_1, we infer that, for all $q \in \partial B_{\rho/2}(y)$,

$$\omega_1^q(J' \cap E_1) \gtrsim 1 \geqslant \omega_2^q(J).$$

We also have $\omega_1^q(J' \cap E_1) \geqslant \omega_2^q(J) = 0$ for all $q \in J^c \cap E_1$. Then, by the maximum principle, since both $\omega_1^z(J' \cap E_1)$ and $\omega_2^z(J)$ are harmonic in $\Omega_1 \backslash \bar{B}_{\rho/2}(y) = \Omega_2 \backslash \bar{B}_{\rho/2}(y)$ we deduce that

$$\omega_1^q(J' \cap E_1) \gtrsim \omega_2^q(J)$$

for all $q \in \Omega_2 \backslash \bar{B}_{\rho/2}(y)$, and in particular for all $p \in \partial B_{r'}(y)$.

Finally we let $E_3 = [y, y + r_1]$ and $\Omega_3 = E_3^c$, so that $E_2 \subset E_3$. By the maximum principle, we have

$$\omega_2^p(J) \geqslant \omega_3^p(J)$$

for all $p \in \partial B_{r'}(y)$. Hence, gathering the above estimates, we infer that, for all $p \in \partial B_{r'}(y)$,

$$\omega^p(J' \cap E) \approx_s \omega_1^p(J' \cap E) \gtrsim \omega_2^p(J) \geqslant \omega_3^p(J). \tag{12.29}$$

Now it just remains to estimate $\omega_3^p(J)$. We can do this by means of a conformal transformation. Indeed, observe first that, by a Harnack chain argument and the maximum principle, $\omega_3^p(J) \approx \omega_3^\infty(J)$ for all $p \in \partial B_{r'}(y)$. Next, suppose for simplicity that $y = -r_1/2$, so that $E_3 = [-r_1/2, r_1/2]$. The map $f : \bar{B}_1(0) \to \overline{\Omega_3}$ defined by

$$f(z) = \left(z + \frac{1}{z}\right)\frac{r_1}{4} \tag{12.30}$$

is a conformal transformation from $B_1(0)$ to Ω_3 such that $f(0) = \infty$, with $f(\partial B_1(0)) = \partial\Omega_3 = E_3$. Thus,

$$\omega_3^\infty(J) = \frac{1}{2\pi}\,\mathcal{H}^1(f^{-1}(J)).$$

An easy computation shows that

$$f^{-1}(J) = \{e^{i\alpha} : \pi - \theta \leqslant \alpha \leqslant \pi + \theta\},$$

with

$$\theta = \arccos\left(1 - \frac{2\mathcal{H}^1(J)}{r_1}\right) = \arccos\left(1 - \frac{\rho}{r_1}\right) \approx \left(\frac{\rho}{r_1}\right)^{1/2}. \tag{12.31}$$

Thus,

$$\omega_3^\infty(J) = \frac{\theta}{\pi} \approx \left(\frac{\rho}{r_1}\right)^{1/2} \approx \left(\frac{\rho}{r}\right)^{1/2}. \tag{12.32}$$

Consequently, by the change of pole formula for uniform CDC domains and (12.29), we deduce that, for $p \in \partial Br'(y)$,

$$\frac{\omega^{p_0}(B_\rho(y))}{\omega^{p_0}(B_r(x))} \approx \omega^p(\bar{B}_\rho(y)) = \omega^p(J' \cap E) \gtrsim \omega_3^p(J) \approx \left(\frac{\rho}{r}\right)^{1/2}, \tag{12.33}$$

which completes the proof of (12.27).

The case $s = 1/2$.
In this case the inequality (12.27) does not suffice to prove (12.13) and we need a better estimate. We consider the preceding domains $\Omega_1, \Omega_2, \Omega_3$, so that, for all $p \in \partial B_{r'}(y)$, (12.29) holds. However, the estimate $\omega_2^p(J) \geqslant \omega_3^p(J)$ is too coarse for our purposes. Instead, we write

$$\omega_2^p(J) \approx \omega_2^\infty(J) = \int_{E_3} \omega_2^z(J)\, d\omega_3^\infty(z).$$

The density $\frac{d\omega_3^\infty}{d\mathcal{H}^1|_{E_3}}$ can be computed explicitly by means of the conformal transformation in (12.30). Using the identity $\omega_3^\infty(J) = \pi^{-1} \arccos\left(1 - \frac{2\mathcal{H}^1(J)}{r_1}\right)$ and differentiating, it follows that

$$\frac{d\omega_3^\infty}{d\mathcal{H}^1|_{E_3}}(t) = \frac{1}{\pi\sqrt{(\frac{r_1}{2} - t)(\frac{r_1}{2} + t)}}.$$

Thus,

$$\frac{d\omega_3^\infty}{d\mathcal{H}^1|_{E_3}}(t) \approx \frac{1}{\sqrt{r_1(t + \frac{r_1}{2})}} \qquad \text{for } t \in [-r_1/2, 0], \tag{12.34}$$

and so

$$\omega_2^p(J) \gtrsim \int_{\frac{-r_1}{2}}^0 \omega_2^t(J)\, \frac{dt}{\sqrt{r_1(t + \frac{r_1}{2})}} \tag{12.35}$$

(recall that we are identifying $\mathbb{R} \equiv \mathbb{R} \times \{0\}$).

To estimate the integral in (12.35) from below, consider the annuli $A_k = A_{2^k\rho, 2^{k+1}\rho}(y)$ for $k \geqslant 1$, and let $N = [\log_2 \frac{r_1}{\rho}]$. By the s-Ahlfors regularity of E and pigeonholing, for every $k \in [1, N]$ there exists an interval $I_k \subset A_k \cap E_3$ (recall E_3 is an interval) such that $\mathcal{H}^1(I_k) \approx 2^k\rho$ and $I_k \cap E = I_k \cap E_2 = \varnothing$. Let $\hat{I}_k$ be another interval concentric with I_k and half length. Then we write

$$\int_{\frac{-r_1}{2}}^0 \omega_2^t(J)\, \frac{dt}{\sqrt{r_1(t + \frac{r_1}{2})}} \geqslant \sum_{k=1}^N \int_{\hat{I}_k} \omega_2^t(J)\, \frac{dt}{\sqrt{r_1(t + \frac{r_1}{2})}}. \tag{12.36}$$

We claim that

$$\omega_2^t(J) \gtrsim \left(\frac{\rho}{|t - y|}\right)^{1/2} \qquad \text{for all } t \in \textstyle\bigcup_{k=1}^N \hat{I}_k. \tag{12.37}$$

Assuming this for the moment, we obtain

$$\begin{aligned}\omega_2^p(J) &\gtrsim \sum_{k=1}^N \int_{\hat I_k}\left(\frac{\rho}{|t-y|}\right)^{1/2}\frac{dt}{\sqrt{r_1(t+\frac{r_1}{2})}}\\ &\approx \sum_{k=1}^N \int_{\hat I_k}\left(\frac{\rho}{\mathcal{H}^1(\hat I_k)}\right)^{1/2}\frac{dt}{r_1^{1/2}\mathcal{H}^1(\hat I_k)^{1/2}} = N\left(\frac{\rho}{r_1}\right)^{1/2}\approx \log\frac{r}{\rho}\left(\frac{\rho}{r}\right)^{1/2}.\end{aligned}$$

By (12.29) and the change of pole formula, arguing as in the preceding subsection, we obtain

$$\frac{\omega^{p_0}(B_\rho(y))}{\omega^{p_0}(B_r(x))}\approx \omega^p(\bar B_\rho(y)) = \omega^p(J'\cap E)\gtrsim \omega_2^p(J)\approx \log\frac{r}{\rho}\left(\frac{\rho}{r}\right)^{1/2}, \tag{12.38}$$

which implies (12.13) for ρ small enough.

It remains to prove (12.37). To this end, for each $t\in\hat I_k$, let $t'\in\mathbb{R}$ be the point symmetric to t with respect to y. That is, $t'=-r_1-t$. Notice that t' is on the left side of the interval E_3 (recall that the leftmost point of E_3 is $y=-r_1/2$). By a Harnack chain argument and the maximum principle, we have

$$\omega_2^t(J)\approx\omega_2^{t'}(J)\geqslant\omega_3^{t'}(J).$$

Now we can compute explicitly $\omega_3^{t'}(J)$ by means of the conformal transformation in (12.30). Indeed, consider the change of variable $t'=y-\frac{r_1}{2}h$. Then, it follows easily that

$$f^{-1}(t')=\frac{-1}{1+h+\sqrt{h(2+h)}}=-(1+h)+\sqrt{h(2+h)}.$$

So $f^{-1}(t')$ is a point in the unit disk belonging to the segment $(-1,0)$ such that

$$|-1-f^{-1}(t')| = -h+\sqrt{h(2+h)}\approx h^{1/2}=\left(\frac{2|t'-y|}{r_1}\right)^{1/2}.$$

Recall that $f^{-1}(J)=[\pi-\theta,\pi+\theta]$, with $\theta\approx\left(\frac{\rho}{r_1}\right)^{1/2}$, by (12.31). Hence, $|-1-f^{-1}(t')|\gtrsim \mathcal{H}^1(f^{-1}(J))$. Taking into account that, for any point $q\in B_1(0)$ and $\eta:=10\operatorname{dist}(q,\partial B_1(0))$, $\omega^q_{B_1(0)}|_{B_\eta(q)}$ is comparable to $\eta^{-1}\mathcal{H}^1|_{\partial B_1(0)\cap B_\eta(q)}$, we deduce that

$$\omega_3^{t'}(J)=\omega_{B_1(0)}^{f^{-1}(t')}(f^{-1}(J))\approx\frac{\theta}{|-1-f^{-1}(t')|}\approx\frac{\left(\frac{\rho}{r_1}\right)^{1/2}}{\left(\frac{2|t'-y|}{r_1}\right)^{1/2}}\approx\frac{\rho^{1/2}}{|t'-y|^{1/2}}=\frac{\rho^{1/2}}{|t-y|^{1/2}},$$

which yields (12.37). □

Theorem 12.9 does not hold for $0<s<0.249$. Indeed, for such values of s, David, Jeznach, and Julia [DJJ23] have constructed an s-Ahlfors regular compact subset $E\subset\mathbb{R}\times\{0\}$ for which $\mathcal{H}^s|_E$ and harmonic measure for $\Omega=\mathbb{R}^2\backslash E$ are mutually absolutely continuous. An interesting open problem consists in finding the sharp threshold s_0 such that for all s-AD regular sets with $s \in (s_0, 1)$ contained in a line in the plane, the dimension drop for harmonic measure occurs. Clearly, by Theorem 12.9 and [DJJ23], we have $0.249 < s_0 < 1/2$. Also, for s-Ahlfors regular sets E in the plane not contained in a line, it is an open question if there exists some $s_0' < 1$ such that the dimension drop for $\omega_{\mathbb{R}^2 \setminus E}$ occurs whenever $s_0' < s < 1$.

Exercise 12.2.1. Let $E \subset \mathbb{R}^2$ be the 1/4 planar Cantor set, defined inductively by setting $Q_1^0 = [0,1]^2$, and then choosing the squares Q_k^{m+1} of the $(m+1)$-th generation by replacing each square Q_j^m from the m-th generation by four closed sub-squares in the corners with side length equal to $\frac{1}{4}\ell(Q_j^m)$, so that the the cubes Q_k^{m+1} have side length 4^{-m-1}. Then we set $E_m = \bigcup_{j=1}^{4^m} Q_j^m$ and $E = \bigcap_{m=0}^{\infty} E_m$. One can check that this set satisfies $0 < \mathcal{H}^1(E) < \infty$. See [Mat95, Section 4.12]. Also, from the fact that it has orthogonal projections of zero length both on the horizontal and vertical axis, it follows that E is purely 1-unrectifiable.

Prove that, for $\Omega = \mathbb{C} \setminus E$, $\dim_{\mathcal{H}} \omega_\Omega^\infty < 1$. To do so, you could first check that if Q is a given closed square and $\Omega_Q = \mathbb{R}^2 \setminus Q$, for a point ξ in a corner of Q, it holds that $\omega^\infty(B_r(\xi)) \approx r^{2/3}$, via a suitable conformal transformation. Then, try to argue as in the proof of Theorem 12.9.

Bibliography

[AAM19] Murat Akman, Jonas Azzam, and Mihalis Mourgoglou. Absolute continuity of harmonic measure for domains with lower regular boundaries. *Adv. Math.*, 345:1206–1252, 2019. 246, 248, 249

[AG01] David H. Armitage and Stephen J. Gardiner. *Classical potential theory*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001. 1

[AHM$^+$16] Jonas Azzam, Steve Hofmann, José María Martell, Svitlana Mayboroda, Mihalis Mourgoglou, Xavier Tolsa, and Alexander Volberg. Rectifiability of harmonic measure. *Geom. Funct. Anal.*, 26(3):703–728, 2016. 1

[AIM09] Kari Astala, Tadeusz Iwaniec, and Gaven Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press, 2009. 207

[Azz20] Jonas Azzam. Dimension drop for harmonic measure on Ahlfors regular boundaries. *Potential Anal.*, 53(3):1025–1041, 2020. 276

[Bat96] Athanassios Batakis. Harmonic measure of some Cantor type sets. *Ann. Acad. Sci. Fenn. Math.*, 21(2):255–270, 1996. 276

[Bou87] J. Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. *Invent. Math.*, 87(3):477–483, 1987. 267

[Car85] Lennart Carleson. On the support of harmonic measure for sets of Cantor type. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 10:113–123, 1985. 276

[Car98] Lennart Carleson. *Selected Problems on Exceptional Sets*. Van Nostrand Company, 1998. 1

[CFMS81] Luis A Caffarelli, Eugene B. Fabes, S. Mortola, and Sandro Salsa. Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana University Mathematics Journal*, 30(4):621–640, 1981. 170

[Chr90] Michael Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, 60/61(2):601–628, 1990. 33

[Con78] John B Conway. *Functions of One Complex Variable*, volume 11 of *Graduate Texts in Mathematics*. Springer, 2$^{\text{nd}}$ edition, 1978. 175

[CS05] Luis A Caffarelli and Sandro Salsa. *A geometric approach to free boundary problems*, volume 68. American Mathematical Society Providence, RI, 2005. 1

[CTV18] Julià Cufí, Xavier Tolsa, and Joan Verdera. About the Jones-Wolff theorem on the Hausdorff dimension of harmonic measure. 09 2018. 1

[Dah77] Björn E. J. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977. 228

[DFM21] G. David, J. Feneuil, and S. Mayboroda. Elliptic theory for sets with higher co-dimensional boundaries. *Mem. Amer. Math. Soc.*, 274(1346):vi+123, 2021. 284

[DJ90] Guy David and David Jerison. Lipschitz approximation to hypersurfaces, harmonic measure, and singular-integrals. *Indiana University Mathematics Journal*, 39(3):831–845, 1990. 237

[DJJ23] Guy David, Cole Jeznach, and Antoine Julia. Cantor sets with absolutely continuous harmonic measure. *J. Éc. polytech. Math.*, 10:1277–1298, 2023. 288, 289

[DS93] Guy David and Stephen Semmes. *Analysis of and on uniformly rectifiable sets.* American Mathematical Soc., 1993. 237

[EG15] Lawrence Craig Evans and Ronald F Gariepy. *Measure theory and fine properties of functions.* CRC press, 2015. 25, 26, 126, 127

[Eva98] Lawrance C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics.* Oxford University Press, 1998. 186

[Fal85] K. J. Falconer. *The geometry of fractal sets.* Cambridge University Press, 1985. 181, 182

[Fol95] Gerald B Folland. *Introduction to partial differential equations.* Princeton University Press, 2^{nd} edition, 1995. 1, 3, 10, 12, 23, 227

[GM05] John B Garnett and Donald E Marshall. *Harmonic measure*, volume 2. Cambridge University Press, 2005. 1, 175, 176, 178

[GM12] Mariano Giaquinta and Luca Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012. 227, 243

[GO79] Frederick W Gehring and Brad G Osgood. Uniform domains and the quasi-hyperbolic metric. *J. Anal. Math.*, 36(1):50–74, 1979. 155

[Gra08] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. New York: Springer, 2$^{\text{nd}}$ edition, 2008. 12, 27, 65

[Gra09] Loukas Grafakos. *Modern Fourier analysis*, volume 250. Springer, 2009. 35

[GT01] David Gilbarg and Neil S Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224. Springer Science & Business Media, 2001. 220

[HK12] Tuomas P Hytönen and Anna Kairema. Systems of dyadic cubes in a doubling metric space. *Colloq. Math.*, 126(1):1–33, 2012. 33

[HMT10] Steve Hofmann, Marius Mitrea, and Michael Taylor. Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains. *Int. Math. Res. Not. IMRN*, (14):2567–2865, 2010. 239

[JK82] David S Jerison and Carlos E Kenig. Boundary behavior of harmonic functions in non-tangentially accessible domains. *Advances in Mathematics*, 46(1):80–147, 1982. 1, 153, 157

[Jon81] Peter W. Jones. Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.*, 147(1):71–88, 1981. 155

[JW88] Peter W. Jones and Thomas H. Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Mathematica*, 161(1):131–144, 1988. 183

[Ken94] Carlos E Kenig. *Harmonic analysis techniques for second order elliptic boundary value problems*, volume 83. American Mathematical Soc., 1994. 1, 66

[Lan72] Naum Samoĭlovich Landkof. *Foundations of modern potential theory*, volume 180. Springer, 1972. 1, 87

[Mak85] N. G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc. (3)*, 51(2):369–384, 1985. 182

[Mat95] Pertti Mattila. *Geometry of sets and measures in euclidean spaces: fractals and rectifiability*, volume 44. Cambridge University Press, 1995. 25, 26, 27, 34, 56, 101, 172, 258, 289

[McM69] J. E. McMillan. Boundary behavior of a conformal mapping. *Acta Math.*, 123:43–67, 1969. 246

[NTV14a] F. Nazarov, S. Treil, and A. Volberg. The Tb-theorem on non-homogeneous spaces that proves a conjecture of Vitushkin. 2014. 257

[NTV14b] Fedor Nazarov, Xavier Tolsa, and Alexander Volberg. On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1. *Acta Math.*, 213(2):237–321, 2014. 256

[NTV14c] Fedor Nazarov, Xavier Tolsa, and Alexander Volberg. The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions. *Publ. Mat.*, 58(2):517–532, 2014. 256, 257

[Pom92] Christian Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der mathematischen Wissenschaften A Series of Comprehensive Studies in Mathematics*. Springer, 1992. 175, 176

[Ran95] Thomas Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995. 1, 87

[Rud87] Walter Rudin. *Real and complex analysis*. Tata McGraw-Hill Education, 1987. 28, 52, 53

[Rud91] Walter Rudin. *Functional analysis. International series in pure and applied mathematics*. McGraw-Hill, Inc., New York, 1991. 128

[Sch02] Martin Schechter. *Principles of functional analysis*, volume 36 of *Graduate Studies in Mathematics*. American Mathematical Society, 2$^{\text{nd}}$ edition, 2002. 13, 14

[Sem90] Stephen Semmes. Analysis vs. geometry on a class of rectifiable hypersurfaces in Rn. *Indiana University Mathematics Journal*, 39(4):1005–1035, 1990. 237

[Sto48] M. H. Stone. The generalized Weierstrass approximation theorem. *Mathematics Magazine*, 21(4):167–184, 1948. 69

[Tol00] Xavier Tolsa. Principal values for the Cauchy integral and rectifiability. *Proc. Amer. Math. Soc.*, 128(7):2111–2119, 2000. 257

[Tol14] Xavier Tolsa. *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory*. Springer, 2014. 146, 257

[Tol24] Xavier Tolsa. The dimension of harmonic measure on some AD-regular flat sets of fractional dimension. *Int. Math. Res. Not. IMRN*, (8):6579–6605, 2024. 276

[Tri83] Hans Triebel. *Theory of function spaces*. Birkhäuser, reprint (2010) edition, 1983. 19

[Vol03] Alexander Volberg. *Calderón-Zygmund capacities and operators on nonhomogeneous spaces*, volume 100 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2003. 257

[Wol93] Thomas H. Wolff. Plane harmonic measures live on sets of σ-finite length. *Ark. Mat.*, 31(1):137–172, 1993. 183

[Wol95] Thomas H. Wolff. Counterexamples with harmonic gradients in $\mathbb{R}^3$. In *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, volume 42 of *Princeton Math. Ser.*, pages 321–384. Princeton Univ. Press, Princeton, NJ, 1995. 267

[Wu86] Jang-Mei Wu. On singularity of harmonic measure in space. *Pacific J. Math.*, 121(2):485–496, 1986. 248