VARIATION AND OSCILLATION FOR SINGULAR INTEGRALS WITH ODD KERNEL ON LIPSCHITZ GRAPHS

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Abstract. We prove that, for $\rho > 2$, the ρ-variation and oscillation for the Cauchy transform on Lipschitz graphs with slope smaller than 1 is bounded in L^p for $1 < p < \infty$. The analogous result holds for the n-dimensional Riesz transform on n-dimensional Lipschitz graphs with slope smaller than 1, as well as for other singular integral operators with odd kernel. The restriction on the slope of the Lipschitz graph can be removed by using smooth truncations of singular integrals. In particular, our results strengthen the classical theorem on the L^2 boundedness of the Cauchy transform on Lipschitz graphs by Coifman, McIntosh, and Meyer.

1. Introduction

The ρ-variation and oscillation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis (see [Lé], [Bo], [JKRW], [CJRW1], and [JSW], for example). The purpose of this paper is to establish some new results concerning the ρ-variation and oscillation for families of singular integral operators defined on Lipschitz graphs. In particular, our results include the L^p boundedness of the ρ-variation and the oscillation for the Cauchy transform and the n-dimensional Riesz transform on Lipschitz graphs (with slope smaller than 1 if we consider truncations given by characteristic functions of balls), for $1 < p < \infty$ and $\rho > 2$.

Given a Borel measure μ in $\mathbb{R}^d$, one defines the n-dimensional Riesz transform of a function $f \in L^1(\mu)$ by $R^\mu f(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$R^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}}\, f(y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

When $d = 2$ (i.e., μ is a Borel measure in $\mathbb{C}$), one defines the Cauchy transform of $f \in L^1(\mu)$ by $C^\mu f(x) = \lim_{\epsilon \searrow 0} C^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$C^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, d\mu(y), \qquad x \in \mathbb{C}.$$

To avoid the problem of existence of the preceding limits, it is useful to consider the maximal operators $R^\mu_* f(x) = \sup_{\epsilon>0} |R^\mu_\epsilon f(x)|$ and $C^\mu_* f(x) = \sup_{\epsilon>0} |C^\mu_\epsilon f(x)|$.

The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón-Zygmund kernel. The kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that we consider

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in this paper satisfy

$$|K(x)| \le \frac{C}{|x|^n}, \quad |\partial_{x^i} K(x)| \le \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x^i}\partial_{x^j} K(x)| \le \frac{C}{|x|^{n+2}}, \tag{1.1}$$

for all $1 \le i, j \le d$ and $x = (x^1, \ldots, x^d) \in \mathbb{R}^d \setminus \{0\}$, where $0 < n < d$ is some integer and $C > 0$ is some constant; and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. K is odd). Notice that the n-dimensional Riesz transform corresponds to the vector kernel $(x^1, \ldots, x^d)/|x|^{n+1}$, and the Cauchy transform to $(x^1, -x^2)/|x|^2$ (so, we may consider K to be any scalar component of these vector kernels).

Given an odd kernel K satisfying (1.1) and a finite Borel measure μ in $\mathbb{R}^d$, for each $\epsilon > 0$, we consider the ϵ-truncated operator

$$T_\epsilon \mu(x) = \int_{|x-y|>\epsilon} K(x-y)\, d\mu(y), \qquad x \in \mathbb{R}^d,$$

and then we set $T\mu(x) = \lim_{\epsilon \searrow 0} T_\epsilon \mu(x)$ whenever the limit makes sense, and $T_* \mu(x) = \sup_{\epsilon>0} |T_\epsilon \mu(x)|$. Finally, given $f \in L^1(\mu)$, we define $T^\mu_\epsilon f(x) := T_\epsilon(f\mu)(x)$, $T^\mu f(x) := T(f\mu)(x)$ and $T^\mu_* f(x) := T_*(f\mu)(x)$. Thus, for a suitable choice of K, the operator T^μ coincides with the Cauchy or Riesz transforms.

Besides the operator T_ϵ defined above, one can consider other ϵ-truncated variants that we proceed to define. First we need some additional notation. Given $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$, we use the notation $\widetilde{x} := (x^1, \ldots, x^n) \in \mathbb{R}^n$. Let $\varphi_{\mathbb{R}} : [0, \infty) \to [0, \infty)$ be a non decreasing $\mathcal{C}^2$ function such that $\chi_{[3\sqrt{n},\infty)} \le \varphi_{\mathbb{R}} \le \chi_{[2.1\sqrt{n},\infty)}$ (the numbers $3\sqrt{n}$ and $2.1\sqrt{n}$ are chosen just for definiteness and they are not important). Given $\epsilon > 0$ and $x \in \mathbb{R}^d$, we denote

$$\varphi_\epsilon(x) := \varphi_{\mathbb{R}}(|x|/\epsilon) \quad \text{and} \quad \widetilde{\varphi}_\epsilon(x) := \varphi_{\mathbb{R}}(|\widetilde{x}|/\epsilon),$$
$$\chi_\epsilon(x) := \chi_{(1,\infty)}(|x|/\epsilon) \quad \text{and} \quad \widetilde{\chi}_\epsilon(x) := \chi_{(1,\infty)}(|\widetilde{x}|/\epsilon).$$

Definition 1.1 (family of truncations)**.** *We consider the following families of functions*

$$\varphi := \{\varphi_\epsilon\}_{\epsilon>0}, \quad \widetilde{\varphi} := \{\widetilde{\varphi}_\epsilon\}_{\epsilon>0}, \quad \chi := \{\chi_\epsilon\}_{\epsilon>0}, \quad \widetilde{\chi} := \{\widetilde{\chi}_\epsilon\}_{\epsilon>0}.$$

We say that a family of functions $\omega := \{\omega_\epsilon\}_{\epsilon>0}$ is a family of truncations *if $\omega \in \{\varphi, \widetilde{\varphi}, \chi, \widetilde{\chi}\}$.*

Let $\omega := \{\omega_\epsilon\}_{\epsilon>0}$ be a family of truncations. Given a kernel K as above, $x \in \mathbb{R}^d$, $0 < \epsilon$, and a finite Borel measure μ, we consider

$$(K\omega_\epsilon * \mu)(x) := \int \omega_\epsilon(x-y) K(x-y)\, d\mu(y).$$

We also denote $(K\omega * \mu)(x) := \{(K\omega_\epsilon * \mu)(x)\}_{\epsilon>0}$. Finally, given $f \in L^1(\mu)$, we define $T^\mu_{\omega_\epsilon} f(x) := (K\omega_\epsilon * (f\mu))(x)$, $T^\mu_\omega f(x) := \lim_{\epsilon\to 0} T^\mu_{\omega_\epsilon} f(x)$ (whenever the limit makes sense), $T^\mu_{\omega_*} f(x) := \sup_{\epsilon>0} |T^\mu_{\omega_\epsilon} f(x)|$, and $\mathcal{T}^\mu_\omega f(x) := \{T^\mu_{\omega_\epsilon} f(x)\}_{\epsilon>0}$. For the particular case of $\omega = \chi$, notice that $T^\mu_{\chi_\epsilon} f = T^\mu_\epsilon f$, thus we obtain the truncated Cauchy and Riesz transforms taking a suitable kernel K.

Definition 1.2 (ρ-variation and oscillation)**.** *Let $\mathcal{I}$ be a subset of $\mathbb{R}$ (in this paper, we will always have $\mathcal{I} = (0,\infty)$ or $\mathcal{I} = \mathbb{Z}$), and let $\mathcal{F} := \{F_\epsilon\}_{\epsilon\in\mathcal{I}}$ be a family of functions defined on $\mathbb{R}^d$. Given $\rho > 0$, the* ρ-variation *of $\mathcal{F}$ at $x \in \mathbb{R}^d$ is defined by*

$$\mathcal{V}_\rho(\mathcal{F})(x) := \sup_{\{\epsilon_m\}} \left(\sum_{m\in\mathbb{Z}} |F_{\epsilon_{m+1}}(x) - F_{\epsilon_m}(x)|^\rho \right)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m\in\mathbb{Z}} \subset \mathcal{I}$. *Fix a decreasing sequence* $\{r_m\}_{m\in\mathbb{Z}} \subset \mathcal{I}$. *The* oscillation *of* $\mathcal{F}$ *at* $x \in \mathbb{R}^d$ *is defined by*

$$\mathcal{O}(\mathcal{F})(x) := \sup_{\{\epsilon_m\},\{\delta_m\}} \left(\sum_{m\in\mathbb{Z}} |F_{\epsilon_m}(x) - F_{\delta_m}(x)|^2 \right)^{1/2},$$

where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m\in\mathbb{Z}} \subset \mathcal{I}$ *and* $\{\delta_m\}_{m\in\mathbb{Z}} \subset \mathcal{I}$ *such that* $r_{m+1} \leq \epsilon_m \leq \delta_m \leq r_m$ *for all* $m \in \mathbb{Z}$.

In this paper we are interested in studying the ρ-variation and oscillation of the families $\mathcal{T}^\mu_\omega f$, for the truncations ω introduced above. That is, we will deal with

$$(\mathcal{V}_\rho \circ \mathcal{T}^\mu_\omega) f(x) := \mathcal{V}_\rho(\mathcal{T}^\mu_\omega f)(x) = \mathcal{V}_\rho(K\omega * (f\mu))(x) \quad \text{and}$$
$$(\mathcal{O} \circ \mathcal{T}^\mu_\omega) f(x) := \mathcal{O}(\mathcal{T}^\mu_\omega f)(x) = \mathcal{O}(K\omega * (f\mu))(x),$$

for a Borel measure μ and $f \in L^1(\mu)$. Although it is not clear from the definitions, these operators are μ-measurable (see [CJRW1], [JSW]).

Given $E \subset \mathbb{R}^d$, we denote by $\mathcal{H}^n_E$ the n-dimensional Hausdorff measure restricted to E.

Let $\Gamma := \{x \in \mathbb{R}^d : x = (\widetilde{x}, A(\widetilde{x}))\}$ be the graph of a Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ with Lipschitz constant $\mathrm{Lip}(A)$. Let $H^1(\mathcal{H}^n_\Gamma)$ and $BMO(\mathcal{H}^n_\Gamma)$ be the (atomic) Hardy space and the space of functions with bounded mean oscillation, respectively, with respect to the measure $\mathcal{H}^n_\Gamma$. The following is our main result.

Main Theorem 1.1. *Let* $\rho > 2$, *let* K *be a kernel satisfying* (1.1), *let* ω *be a family of truncations, and set* $\mu := \mathcal{H}^n_\Gamma$. *If* $\omega \in \{\varphi, \widetilde{\varphi}, \widetilde{\chi}\}$, *the operators* $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\omega$ *and* $\mathcal{O} \circ \mathcal{T}^\mu_\omega$ *are bounded*

- *in* $L^p(\mu)$ *for* $1 < p < \infty$,
- *from* $L^1(\mu)$ *to* $L^{1,\infty}(\mu)$, *and*
- *from* $L^\infty(\mu)$ *to* $BMO(\mu)$.

The same holds if $\omega = \chi$ *and* $\mathrm{Lip}(A) < 1$. *Furthermore, if* $\omega \in \{\varphi, \widetilde{\varphi}\}$, *the operators* $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\omega$ *and* $\mathcal{O} \circ \mathcal{T}^\mu_\omega$ *are also bounded from* $H^1(\mu)$ *to* $L^1(\mu)$. *In all the cases above, the norm of* $\mathcal{O} \circ \mathcal{T}^\mu_\omega$ *is bounded independently of the sequence that defines* $\mathcal{O}$.

As remarked above, the theorem applies to the particular cases of the Cauchy transform (with $d = 2$, $n = 1$) and the n-dimensional Riesz transforms on n-dimensional Lipschitz graphs in $\mathbb{R}^d$.

We think that the Main Theorem 1.1 also holds without the assumption $\mathrm{Lip}(A) < 1$ in the case $\omega = \chi$. However, we have not been able to prove this (see Remark 8.3).

Let us recall that the $L^2(\mathcal{H}^1_\Gamma)$ boundedness of the Cauchy transform on Lipschitz graphs $\Gamma \subset \mathbb{C}$ with slope small enough was proved by A. P. Calderón in his celebrated paper [Ca]. The L^2 boundedness on Lipschitz graphs in full generality was proved later on by R. Coifman, A. McIntosh, and Y. Meyer [CMM].

Consider the Cauchy kernel $K(z) = 1/z$ ($z \in \mathbb{C}$), and set $\mu := \mathcal{H}^1_\Gamma$, so $C^\mu_\epsilon = T^\mu_{\chi_\epsilon}$. By standard Calderón-Zygmund theory (namely, Cotlar's inequality), the $L^2(\mu)$ boundedness of the Cauchy transform C^μ is equivalent to the $L^2(\mu)$ boundedness of the maximal operator C^μ_*. Let M^μ denote the Hardy-Littlewood maximal operator with respect to the measure μ. It is easy to check that, for $f \in L^1(\mu)$ with compact support, there exists some constant $C_0 > 0$ such that

$$C^\mu_\epsilon f(x) \leq T^\mu_{\widetilde{\chi}_\epsilon} f(x) + C_0 M^\mu f(x) \leq (\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\widetilde{\chi}}) f(x) + C_0 M^\mu f(x)$$

for all $\epsilon > 0$, thus $(\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\widetilde{\chi}}) + C_0 M^\mu$ controls the maximal operator C^μ_* and, in this sense, Theorem 1.1 (together with the known $L^p(\mu)$ boundedness of M^μ) strengthens the results of [Ca] and [CMM]. Analogous conclusions hold for the n-dimensional Riesz transform and the maximal operator R_*^μ.

Concerning the background on the ρ-variation and oscillation, a fundamental result is Lépingle's inequality [Lé], from which the L^p boundedness of the ρ-variation and oscillation for martingales follows, for $\rho > 2$ and $1 < p < \infty$ (see Theorem 2.4 below for more details). From this result on martingales, one deduces that the ρ-variation and oscillation are also bounded in L^p for the averaging operators (also called differentiation operators, see [JKRW]):

$$D_\epsilon f(x) = \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} f(y)\,dy, \qquad x \in \mathbb{R}. \tag{1.2}$$

As far as we know, the first work dealing with the ρ-variation and oscillation for singular integral operators is the one of J. Campbell, R. L. Jones, K. Reinhold and M. Wierdl [CJRW1], where the L^p and weak L^1 boundedness of the ρ-variation (for $\rho > 2$) and oscillation for the Hilbert transform was proved. Recall that, for $f \in L^p(\mathbb{R})$ and $x \in \mathbb{R}$,

$$H_\epsilon f(x) = \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{1}{x-y}\, f(y)\,dy,$$

and then the Hilbert transform of f is defined by $Hf(x) = \lim_{\epsilon\to 0} H_\epsilon f(x)$, whenever the limit exists. Later on, there appeared other papers showing the L^p boundedness of the ρ-variation and oscillation for singular integrals in $\mathbb{R}^d$ ([CJRW2]), with weights ([GT]), or for other operators such as the spherical averaging operator or singular integral operators on parabolas ([JSW]). Finally, we remark that, very recently, the case of the Carleson operator has been considered too ([LT], [OSTTW]).

Notice that the Hilbert transform is one of the simplest examples where Theorem 1.1 applies (one sets $\Gamma = \mathbb{R}$, i.e. $A \equiv 0$), and so one obtains a new proof of the L^p boundedness of the ρ-variation and oscillation for the Hilbert transform. In the original proof in [CJRW1], a key ingredient was the following classical identity, which follows via the Fourier transform:

$$Q_\epsilon = P_\epsilon * H, \tag{1.3}$$

where P_ϵ is the Poisson kernel and Q_ϵ is the conjugated Poisson kernel. Using this identity and the close relationship between the operators Q_ϵ and H_ϵ, Campbell *et al.* derived the L^p boundedness of the ρ-variation and oscillation for the Hilbert transform from the one of the family $\{D_\epsilon(Hf)\}_{\epsilon>0}$, where D_ϵ is the averaging operator in (1.2) (notice that P_ϵ can be written as a convex combination of operators D_δ, $\delta > 0$).

In most of the previous results concerning ρ-variation and oscillation of families of operators from harmonic analysis, the Fourier transform is a fundamental tool. However, this is not useful in order to prove Theorem 1.1, since the graph Γ is not invariant under translations in general. Moreover, even for the Cauchy transform, there is no formula like (1.3), which relates the truncations of a singular integral operator with an averaging operator applied to a singular integral operator, when Γ is a general Lipschitz graph.

The main ingredients of our proof of Theorem 1.1 are the known results on the ρ-variation and oscillation for martingales (Lépingle's inequality [Lé]) and a multiscale analysis which stems from the geometric proof of the L^2 boundedness of the Cauchy transform on Lipschitz graphs by P. W. Jones [Jn1] and his celebrated work [Jn2] on quantitative rectifiability in the plane, using the so called β coefficients. Some of the techniques in these papers were further developed in higher dimensions by David and Semmes [DS1] for Ahlfors-David regular sets. More recently, in [To] some coefficients denoted by α, in the spirit of the Jones' β's, were introduced, and they were shown to be useful for the study of the L^p-boundedness of Calderón-Zygmund operators on Lipschitz graphs and on uniformly rectifiable sets (see the definition below Theorem 1.3). In our paper, the α and β coefficients play a fundamental role.

Let us remark that Lépingle's inequality, which asserts the L^p boundedness of the ρ-variation of martingales, fails if one assumes $\rho \leq 2$ (see [Qi] and [JW], for example). Moreover, this fact can be brought to the ρ-variation of averaging operators and singular integral operators, thus it is essential to assume $\rho > 2$ in Theorem 1.1. Analogous conclusions hold if one replaces the ℓ^2-norm by and ℓ^ρ-norm with $\rho < 2$ in the definition of $\mathcal{O}$. See [CJRW1], or [AJS] for the case of martingales.

Concerning the applications of Theorem 1.1, it is easily seen that the L^p boundedness of $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\omega$ yields a new proof of the existence of the principal values $T^\mu_\omega f(x) = \lim_{\epsilon\to 0} T^\mu_{\omega_\epsilon} f(x)$ for all $f \in L^p(\mu)$ and almost all $x \in \Gamma$, without using a dense class of functions in $L^p(\mu)$ (as in the classical proof). Moreover, from Theorem 1.1 one also gets some information on the speed of convergence. In fact, a classical result derived from variational inequalities is the boundedness of the λ*-jump operator* $N_\lambda \circ \mathcal{T}^\mu_\omega$ and the (a, b)*-upcrossings operator* $N^b_a \circ \mathcal{T}^\mu_\omega$. Given $\lambda > 0$, $f \in L^1_{loc}(\mu)$ and $x \in \mathbb{R}^d$, one defines $(N_\lambda \circ \mathcal{T}^\mu_\omega)f(x)$ as the supremmum of all integers N for which there exists $0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \cdots \leq \epsilon_N < \delta_N$ so that

$$|T^\mu_{\omega_{\epsilon_i}} f(x) - T^\mu_{\omega_{\delta_i}} f(x)| > \lambda$$

for each $i = 1, \ldots, N$. Similarly, given $a < b$, one defines $(N^b_a \circ \mathcal{T}^\mu_\omega)f(x)$ to be the supremmum of all integers N for which there exists $0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \cdots \leq \epsilon_N < \delta_N$ so that $T^\mu_{\omega_{\epsilon_i}} f(x) < a$ and $T^\mu_{\omega_{\delta_i}} f(x) > b$ for each $i = 1, \ldots, N$. Using Theorem 1.1 one obtains (by the same arguments as in [CJRW1, Theorem 1.3 and Corollary 7.1]) the following:

Theorem 1.3. *Let $\rho > 2$, $\lambda > 0$, and let K, ω, and μ be as in Theorem 1.1. For $1 < p < \infty$, there exist constants C_1 and C_2 depending on ρ, n, d, K, and* $\mathrm{Lip}(A)$ *(and on p for the case of C_1) such that*

$$\|\big((N_\lambda \circ \mathcal{T}^\mu_\omega)f\big)^{1/\rho}\|_{L^p(\mu)} \leq \frac{C_1}{\lambda}\,\|f\|_{L^p(\mu)} \quad \textit{and}$$

$$\mu(\{x \in \Gamma : \, (N_\lambda \circ \mathcal{T}^\mu_\omega)f(x) > m\}) \leq \frac{C_2}{\lambda m^{1/\rho}}\,\|f\|_{L^1(\mu)}.$$

Trivially, $(N^b_a \circ \mathcal{T}^\mu_\omega)f \leq (N_{b-a} \circ \mathcal{T}^\mu_\omega)f$, thus Theorem 1.3 also holds replacing λ by $b - a$ and N_λ by N^b_a. In [JSW] it is shown that the results of Theorem 1.3 still hold when $\rho = 2$ for the particular case of the Hilbert transform. In our paper we do not pursue this endpoint result.

On the other hand, $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\omega$ is related to an important open problem posed by G. David and S. Semmes. We need some definitions to state it.

Recall that μ is said to be n-dimensional Ahlfors-David regular, or simply AD regular, if there exists some constant C such that $C^{-1}r^n \leq \mu(B(x, r)) \leq Cr^n$ for all $x \in \operatorname{supp}\mu$ and $0 < r \leq \operatorname{diam}(\operatorname{supp}\mu)$. It is not difficult to see that such a measure μ must be of the form $\mu = h\,\mathcal{H}^n_{\operatorname{supp}\mu}$, where h is some positive function bounded above and away from zero. A Borel set $E \subset \mathbb{R}^d$ is called AD regular if the measure $\mathcal{H}^n_E$ is AD regular.

One says that μ is n-uniformly rectifiable, or simply uniformly rectifiable, if there exist $\theta, M > 0$ so that, for each $x \in \operatorname{supp}\mu$ and $R > 0$, there is a Lipschitz mapping g from the n-dimensional ball $B^n(0, R) \subset \mathbb{R}^n$ into $\mathbb{R}^d$ such that $\mathrm{Lip}(g) \leq M$ and

$$\mu\big(B(x, R) \cap g(B^n(0, R))\big) \geq \theta R^n,$$

where $\mathrm{Lip}(g)$ stands for the Lipschitz constant of g. In the language of [DS2], this means that $\operatorname{supp}\mu$ *has big pieces of Lipschitz images of* $\mathbb{R}^n$. A Borel set $E \subset \mathbb{R}^d$ is called n-uniformly rectifiable if $\mathcal{H}^n_E$ is n-uniformly rectifiable. Of course, the n-dimensional graph of a Lipschitz function is n-uniformly rectifiable.

G. David and S. Semmes asked the following question, which is still open (see, for example, [Pa, Chapter 7]):

Problem 1.4. *Is it true that an n-dimensional AD regular measure μ is n-uniformly rectifiable if and only if R^μ_* is bounded in $L^2(\mu)$?*

It is proved in [DS1] that if μ is uniformly rectifiable, then R^μ_* is bounded in $L^2(\mu)$. However, the converse implication has been proved only in the case $n = 1$ and $d = 2$, by P. Mattila, M. Melnikov and J. Verdera [MMV], using the notion of curvature of measures (which seems to be useful only in this case).

Let $K(x)$ denote the n-dimensional Riesz kernel $x/|x|^{n+1}$ ($x \in \mathbb{R}^d$), so $R^\mu_\epsilon = T^\mu_{\chi_\epsilon}$. Combining some techniques from [DS2] and [To], one can show that the L^2 boundedness of $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\chi$ implies that μ is uniformly rectifiable (see [MT] for more details). Thus, if one proved that $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\chi$ is bounded in $L^2(\mu)$ when μ is AD regular and uniformly rectifiable, then we would have: *An n-dimensional AD regular measure μ is n-uniformly rectifiable if and only if $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\chi$ is a bounded operator in $L^2(\mu)$.* This statement can be considered as a weak version of Problem 1.4.

This paper is organized as follows. In section 2 we state some notation, definitions and preliminary results. In section 3 we sketch the proof of our Main Theorem 1.1, and in the subsequent sections we give the detailed proof.

2. Preliminaries

As we said in the introduction, throughout all the paper, n and d are two fixed integers such that $0 < n < d$. Given a point $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$, we use the notation $\widetilde{x} := (x^1, \ldots, x^n) \in \mathbb{R}^n$. Given a function $f : \mathbb{R}^m \to \mathbb{R}$, we denote by ∇f its gradient (when it makes sense), and by $\nabla^2 f$ the matrix of second derivatives of f. If f depends on different points $x_1, x_2, \ldots \in \mathbb{R}^m$, then $\nabla_{x_i} f$ denotes the gradient of f with respect to the x_i variable, and analogously for $\nabla^2_{x_i} f$.

For two sets $F_1, F_2 \subset \mathbb{R}^d$, we denote by $\mathrm{dist}_{\mathcal{H}}(F_1, F_2)$ the Hausdorff distance between F_1 and F_2. We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$, and for the sake of simplicity, we set $\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{L}^n)}$ for $1 \le p \le \infty$, and $dy := d\mathcal{L}^n(y)$ for $y \in \mathbb{R}^n$.

In the paper, when we refer to the angle between two affine n-planes in $\mathbb{R}^d$, we mean the angle between the n-dimensional subspaces associated to the n-planes. As usual, the letter 'C' stands for some constant which may change its value at different occurrences, and which quite often only depends on n and d. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is some fixed constant C such that $A \le CB$ ($A \ge CB$), with C as above. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

2.1. More about the families of truncations ω.

Given $x \in \mathbb{R}^d$, $0 < \epsilon \le \delta$, and a finite Borel measure μ, we set $\omega^\delta_\epsilon(x) := \omega_\epsilon(x) - \omega_\delta(x)$ and we define

$$(K\omega^\delta_\epsilon * \mu)(x) := \int \omega^\delta_\epsilon(x - y) K(x - y)\, d\mu(y),$$

thus $(K\omega^\delta_\epsilon * \mu)(x) = (K\omega_\epsilon * \mu)(x) - (K\omega_\delta * \mu)(x)$.

For $m \in \mathbb{N}$, $x \in \mathbb{R}^m$, and $R \ge r > 0$, we denote by $B^m(x, r)$ the closed ball of $\mathbb{R}^m$ with center x and radius r, and by $A^m(x, r, R)$ the closed annulus of $\mathbb{R}^m$ centered at x with inner radius r and outer radius R. We also use the notation $B(x, r)$ and $A(x, r, R)$ when there is no possible confusion about m.

If we take $\omega = \varphi$, each function $\varphi_\epsilon^\delta = \varphi_\epsilon - \varphi_\delta$ is non negative, and

$$\mathrm{supp}\varphi_\epsilon^\delta \subset A^d(0, 2.1\epsilon\sqrt{n}, 3\delta\sqrt{n}).$$

Moreover, $\sum_{j\in\mathbb{Z}} \varphi_{2^{-j-1}}^{2^{-j}}(x) = 1$ for $x \neq 0$, and there are at most two terms that do not vanish in the previous sum for a given $x \in \mathbb{R}^d$. For the case of $\omega = \widetilde{\varphi}$, one also has $\mathrm{supp}\widetilde{\varphi}_\epsilon^\delta \subset A^n(0, 2.1\epsilon\sqrt{n}, 3\delta\sqrt{n}) \times \mathbb{R}^{d-n} \subset \mathbb{R}^d$ and $\sum_{j\in\mathbb{Z}} \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x) = 1$ for $\widetilde{x} \neq 0$.

2.2. **The α and β coefficients. Special dyadic lattice.** Given $m \in \mathbb{N}$, $\lambda > 0$, and a cube $Q \subset \mathbb{R}^m$ (i.e. $Q := [0, b)^m + a$ with $a \in \mathbb{R}^m$ and $b > 0$), $\ell(Q)$ denotes the side length of Q, z_Q denotes the center of Q and λQ denotes the cube with center z_Q and side length $\lambda\ell(Q)$. Throughout the paper, we will only use cubes with sides parallel to the axes.

Let μ be a locally finite Borel measure on $\mathbb{R}^d$. Given $1 \leq p < \infty$ and a cube $Q \subset \mathbb{R}^d$, one sets (see [DS2])

$$\beta_{p,\mu(Q)} = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{2Q} \left(\frac{\mathrm{dist}(y, L)}{\ell(Q)} \right)^p d\mu(y) \right\}^{1/p}, \tag{2.1}$$

where the infimum is taken over all n-planes L in $\mathbb{R}^d$. For $p = \infty$ one replaces the L^p norm by the supremum norm:

$$\beta_{\infty,\mu}(Q) = \inf_L \left\{ \sup_{y\in\mathrm{supp}\mu\cap 2Q} \frac{\mathrm{dist}(y, L)}{\ell(Q)} \right\}, \tag{2.2}$$

where the infimum is taken over all n-planes L in $\mathbb{R}^d$ again. These coefficients were introduced by P. W. Jones in [Jn1] for $p = \infty$ and by G. David and S. Semmes in [DS1] for $1 \leq p < \infty$.

Let $F \subset \mathbb{R}^d$ be the closure of an open set. Given two finite Borel measures σ, ν on $\mathbb{R}^d$, one sets

$$\mathrm{dist}_F(\sigma, \nu) := \sup\left\{ \left| \int f\,d\sigma - \int f\,d\nu \right| : \mathrm{Lip}(f) \leq 1,\, \mathrm{supp} f \subset F \right\}. \tag{2.3}$$

It is easy to check that this is a distance in the space of finite Borel measures σ such that $\mathrm{supp}\sigma \subset F$ and $\sigma(\partial F) = 0$. Moreover, it turns out that this distance is a variant of the well known Wasserstein distance W_1 from optimal transportation (see [Vi, Chapter 1]). See [Ma, Chapter 14] for other properties of dist_F.

Given a cube Q which intersects $\mathrm{supp}\mu$, consider the closed ball $B_Q := B(z_Q, 6\ell(Q))$. Then one defines (see [To])

$$\alpha_\mu^n(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c\geq 0, L} \mathrm{dist}_{B_Q}(\mu,\, c\mathcal{H}_L^n), \tag{2.4}$$

where the infimum is taken over all constants $c \geq 0$ and all n-planes L in $\mathbb{R}^d$. For convenience, if Q does not intersect $\mathrm{supp}\mu$, we set $\alpha_\mu^n(Q) = 0$. To simplify notation, sometimes we will write $\alpha_\mu(Q)$ or $\alpha(Q)$ instead of $\alpha_\mu^n(Q)$ (and analogously for the β's).

The following result characterizes uniform rectifiability in terms of the α and β coefficients.

Theorem 2.1. *Let μ be an n-dimensional AD regular measure on $\mathbb{R}^d$, and consider any $p \in [1, 2]$. Then, the following are equivalent:*

(a) *μ is n-uniformly rectifiable.*
(b) *For any cube $R \subset \mathbb{R}^d$,*

$$\sum_{Q\in\mathcal{D}_{\mathbb{R}^d}(R)} \beta_{p,\mu}(Q)^2 \ell(Q)^n \leq C\ell(R)^n \tag{2.5}$$

with C independent of R, where $\mathcal{D}_{\mathbb{R}^d}(R)$ stands for the collection of cubes of $\mathbb{R}^d$ contained in R which are obtained by splitting R dyadically.

(c) *There exists $C>0$ such that, for any cube $R\subset\mathbb{R}^d$,*

$$\sum_{Q\in\mathcal{D}_{\mathbb{R}^d}(R)}\alpha_\mu(Q)^2\ell(Q)^n\leq C\ell(R)^n. \tag{2.6}$$

The equivalence (a)$\Longleftrightarrow$(b) in Theorem 2.1 was proved by G. David and S. Semmes in [DS1], and the equivalence (a)$\Longleftrightarrow$(c) was proved by X. Tolsa in [To].

In this paper we will use a slightly different definition of the α and β coefficients adapted to the n-uniformly rectifiable measure $\mu=f\mathcal{H}^n_\Gamma$, where $\Gamma:=\{x\in\mathbb{R}^d\,:\,x=(\widetilde{x},A(\widetilde{x}))\}$ is the n-dimensional graph of a given Lipschitz function $A:\mathbb{R}^n\to\mathbb{R}^{d-n}$ and $f\in L^\infty(\mathcal{H}^n_\Gamma)$ satisfies $f(x)\approx 1$ for almost all $x\in\Gamma$. To this end, we need to introduce a special dyadic lattice of sets related to Γ. Given a cube $\widetilde{Q}\subset\mathbb{R}^n$ (i.e. $\widetilde{Q}:=[0,b)^n+a$ with $a\in\mathbb{R}^n$ and $b>0$), we define $Q:=\widetilde{Q}\times\mathbb{R}^{d-n}$. This type of set will be called *v-cube* ("vertical" cube). We denote by $\ell(Q)$ and $\widetilde{z}_Q$ the side length and center of $\widetilde{Q}$, respectively, and given $\lambda>0$ we set $\lambda Q:=\lambda\widetilde{Q}\times\mathbb{R}^{d-n}$. Let $\widetilde{\mathcal{D}}$ denote the standard dyadic lattice of $\mathbb{R}^n$, and set $\mathcal{D}:=\{Q:\widetilde{Q}\in\widetilde{\mathcal{D}}\}$. It is easy to check that the v-cubes of $\mathcal{D}$ intersected with Γ provide a dyadic lattice associated to the graph Γ in the sense of [Da, Appendix 1]. Finally, for $m\in\mathbb{Z}$, set $\mathcal{D}_m:=\{Q\in\mathcal{D}:\,\ell(Q)=2^{-m}\}$.

Fix a constant $C_\Gamma>10\sqrt{n}(1+\mathrm{Lip}(A))$ (the precise value of C_Γ will not be relevant in the proofs given in the paper). Given $1\leq p\leq\infty$ and a v-cube $Q\subset\mathbb{R}^d$, we define the coefficient $\beta_{p,\mu}(Q)$ as in (2.1) and (2.2) but replacing $2Q$ by $C_\Gamma Q$. We also define $\alpha_\mu(Q)$ as in (2.4) but taking $B_Q:=B(\widetilde{z}_Q,C_\Gamma\ell(Q))\times\mathbb{R}^{d-n}\subset\mathbb{R}^d$. This new definition of the α and β coefficients (adapted to the graph Γ) is the one that we will use in the whole paper.

Remark 2.2. It is an exercise to check that, with this new definition of the α's and β's, inequalities (2.5) and (2.6) of Theorem 2.1 still hold. Moreover, the following is an easy consequence of (2.5) and (2.6): *Let Γ be an n-dimensional Lipschitz graph, $f\in L^\infty(\mathcal{H}^n_\Gamma)$ such that $f(x)\approx 1$ for almost all $x\in\Gamma$, and $\mu=f\mathcal{H}^n_\Gamma$. Let $1\leq p\leq 2$. Given $C_1,C_2,C_3\geq 1$, there exists a constant $C_4>0$ such that, for any $R\in\mathcal{D}$,*

$$\sum_{Q\in\mathcal{D}:\,Q\subset C_1R}\big(\,\beta_{p,\mu}(C_2Q)^2+\alpha_\mu(C_3Q)^2\,\big)\,\mu(Q)\leq C_4\mu(R),$$

and the dependence of C_4 with respect to Γ is only on $\mathrm{Lip}(A)$.

Remark 2.3. It is shown in [To, Lemma 3.2], that $\beta_{1,\mu}(Q)\lesssim\alpha_\mu(Q)$ for all $Q\in\mathcal{D}$. Given $Q\in\mathcal{D}$, let L_Q be a minimizing n-plane for $\alpha_\mu(Q)$. In general, $\beta_{\infty,\mu}(Q)$ can not be controlled by $\beta_{1,\mu}(Q)$, so given $x\in\mathrm{supp}\mu\cap C_\Gamma Q$, we can not control $\mathrm{dist}(x,L_Q)$ by means of $\alpha_\mu(Q)$. But it is shown in [To, Lemma 5.2] that

$$\mathrm{dist}(x,L_Q)\lesssim\sum_{R\in\mathcal{D}:\,x\in R\subset Q}\alpha_\mu(R)\ell(R),$$

and in particular, if $P\in\mathcal{D}$ is such that $P\subset Q$ and $x\in\mathrm{supp}\mu\cap C_\Gamma P$, and L_P denotes a minimizing n-plane for $\alpha_\mu(P)$, one has (see [To, Remark 5.3])

$$\mathrm{dist}(x,L_Q)\lesssim\mathrm{dist}(x,L_P)+\sum_{R\in\mathcal{D}:\,P\subset R\subset Q}\alpha_\mu(R)\ell(R). \tag{2.7}$$

2.3. **Martingales.** First of all, let us recall a particular case of Lépingle's inequality (see [JSW], or [Lé] and [JKRW, Theorem 6.4] for martingales in a probability space):

Theorem 2.4. *Let (X, Σ, λ) be a σ-finite measure space and $\rho > 2$. Then, there exist constants $C_1, C_2 > 0$ such that, for every martingale $\mathcal{G} := \{G_m\}_{m\in\mathbb{Z}} \in L^2(\lambda)$,*

$$\|\mathcal{V}_\rho(\mathcal{G})\|_{L^2(\lambda)} \leq C_1\|\mathcal{G}\|_{L^2(\lambda)} \quad \text{and} \quad \|\mathcal{O}(\mathcal{G})\|_{L^2(\lambda)} \leq C_2\|\mathcal{G}\|_{L^2(\lambda)},$$

where $\|\mathcal{G}\|_{L^2(\lambda)} := \sup_{m\in\mathbb{Z}} \|G_m\|_{L^2(\lambda)}$. The constants C_1 and C_2 do not depend on the measure λ, and C_2 neither depends on the fixed sequence that defines $\mathcal{O}$.

To prove Theorem 1.1, we need to introduce a particular martingale, and to review some known results.

Lemma 2.5. *Fix a cube $\widetilde{P} \subset \mathbb{R}^n$ (not necessarily dyadic) and a Lipschitz graph $\Gamma := \{x \in \mathbb{R}^d : x = (\widetilde{x}, A(\widetilde{x}))\}$ such that* $\operatorname{supp} A \subset \widetilde{P}$. *Consider the measure $\mu := f\mathcal{H}^n_\Gamma$, where $f(x) = 1$ for all $\widetilde{x} \in \widetilde{P}^c$ and $C_0^{-1} \leq f(x) \leq C_0$ for all $\widetilde{x} \in \widetilde{P}$, for some fixed constant $C_0 > 0$. Also set $P := \widetilde{P} \times \mathbb{R}^{d-n}$. Then, the following hold:*

$$T_*\mu \in L^1_{loc}(\mu), \quad T_*(\chi_E\mu) \in L^1_{loc}(\mu) \text{ for every compact set } E \subset \mathbb{R}^d, \text{ and} \tag{2.8}$$

$$\|T\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}. \tag{2.9}$$

Remark 2.6. To avoid the problem of non-integrability near infinity, for this type of measures μ we redefine $T_\epsilon\mu(x) := \lim_{M\to\infty}(K\chi^M_\epsilon * \mu)(x)$, which exists because μ is flat outside a compact set and K is odd. All the results in this paper remain valid with this new definition and the adjustments that have to be done in the proofs are minimal.

Another way to avoid this problem consists in introducing kernels of the type $K\psi_M$, where K is as before and ψ_M is a smooth function such that $\chi_{B^d(0,M)} \leq \psi_M \leq \chi_{B^d(0,2M)}$ and $|\nabla\psi(x)| \lesssim M^{-1}$ for all $x \in \mathbb{R}^d$, and then obtaining estimates independent of ψ_M.

In this paper, we will deal with other integrals which concern the kernel K and the measure μ near infinity. The non-integrability problem can be avoided in the same manner.

Proof of Lemma 2.5. It is known that the operator T^μ_* is bounded in $L^2(\mu)$, because T^μ_* is the maximal operator associated to a Calderón-Zygmund singular integral and μ is an uniformly rectifiable measure (see [DS1]). Thus, $T_*(\chi_E\mu) = T^\mu_*(\chi_E) \in L^1_{loc}(\mu)$ for every compact set $E \subset \mathbb{R}^d$.

We are going to check that $\|T_*\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$. This will imply that $T_*\mu \in L^1_{loc}(\mu)$ and, since $T\mu$ exists (because μ is uniformly rectifiable) and $|T\mu| \leq T_*\mu$, we will also obtain $\|T\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$; so the lemma will be proved.

Using that T^μ_* is bounded in $L^2(\mu)$, we have

$$\begin{aligned}\|T_*\mu\|_{L^2(\mu)} &\leq \|T_*(\chi_{3P}\mu)\|_{L^2(\mu)} + \|T_*(\chi_{(3P)^c}\mu)\|_{L^2(\mu)} \\ &\lesssim \mu(P)^{1/2} + \|T_*(\chi_{(3P)^c}\mu)\|_{L^2(\mu)}.\end{aligned} \tag{2.10}$$

Set $L := \mathbb{R}^n \times \{0\}^{d-n} \subset \mathbb{R}^d$; obviously $\chi_{P^c}\mu = \mathcal{H}^n_{L\setminus P}$. Since L is an n-plane and K is odd, $T_*\mathcal{H}^n_L(x) = 0$ for all $x \in L$. Thus,

$$\|T_*\mathcal{H}^n_{L\setminus 3P}\|_{L^2(\mathcal{H}^n_L)} \leq \|T_*\mathcal{H}^n_L\|_{L^2(\mathcal{H}^n_L)} + \|T_*\mathcal{H}^n_{L\cap 3P}\|_{L^2(\mathcal{H}^n_L)} \lesssim \mu(P)^{1/2}. \tag{2.11}$$

Set $z_P := (\widetilde{z}_P, 0, \ldots, 0) \in L$ (recall that $\widetilde{z}_P$ denotes the center of $\widetilde{P}$). It is obvious that $\int \chi_\epsilon(z_P - y) K(z_P - y)\, d\mathcal{H}^n_{L\setminus 3P}(y) = 0$ for all $\epsilon > 0$. Thus, given $x \in \operatorname{supp}\mu \cap P$,

$$|(K\chi_\epsilon * \mathcal{H}^n_{L\setminus 3P})(x)| \le \int \chi_\epsilon(x-y)|K(x-y) - K(z_P - y)|\, d\mathcal{H}^n_{L\setminus 3P}(y) + \int |\chi_\epsilon(x-y) - \chi_\epsilon(z_P - y)||K(z_P - y)|\, d\mathcal{H}^n_{L\setminus 3P}(y).$$

Since Γ is a Lipschitz graph, $|x - z_P| \lesssim \ell(P)$. So, the first term on right hand side of the previous inequality is easily bounded by an absolute constant independent of ϵ, by standard arguments. For the second term, notice that $\operatorname{supp}(\chi_\epsilon(x - \cdot) - \chi_\epsilon(z_P - \cdot)) \cap (L \setminus 3P) = \emptyset$ for all $\epsilon < \ell(P)$, and $\mathcal{H}^n_L(\{y \in \mathbb{R}^n : \chi_\epsilon(x-y) - \chi_\epsilon(z_P - y) \neq 0\}) \lesssim \ell(P)\epsilon^{n-1}$ for all $\epsilon \ge \ell(P)$. Therefore, since $|z_P - y| \approx \epsilon$ for all $y \in \operatorname{supp}(\chi_\epsilon(x - \cdot) - \chi_\epsilon(z_P - \cdot)) \cap (L \setminus 3P)$, the second term can also be estimated by an absolute constant. Thus, we conclude $T_*\mathcal{H}^n_{L\setminus 3P}(x) = \sup_{\epsilon>0} |(K\chi_\epsilon * \mathcal{H}^n_{L\setminus 3P})(x)| \lesssim 1$ for all $x \in \operatorname{supp}\mu \cap P$.

Using the previous observations and (2.11), we have

$$\begin{aligned}\|T_*(\chi_{(3P)^c}\mu)\|^2_{L^2(\mu)} &= \|T_*\mathcal{H}^n_{L\setminus 3P}\|^2_{L^2(\chi_P\mu)} + \|T_*\mathcal{H}^n_{L\setminus 3P}\|^2_{L^2(\chi_{P^c}\mu)} \\ &\le \|T_*\mathcal{H}^n_{L\setminus 3P}\|^2_{L^2(\chi_P\mu)} + \|T_*\mathcal{H}^n_{L\setminus 3P}\|^2_{L^2(\mathcal{H}^n_L)} \lesssim \mu(P),\end{aligned}$$

which, combined with (2.10), gives $\|T_*\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$, as desired. □

We are ready to define the martingale. Let P and μ be as in Lemma 2.5. Given $m \in \mathbb{Z}$ and $a \in \mathbb{R}^n$, we set

$$\widetilde{D}^a_m := a + [0, 2^{-m})^n \subset \mathbb{R}^n \quad \text{and} \quad D^a_m := \widetilde{D}^a_m \times \mathbb{R}^{d-n} \subset \mathbb{R}^d.$$

Set $\mathcal{D}^a_m := \{D^{a+2^{-m}k}_m \subset \mathbb{R}^d : k \in \mathbb{Z}^n\}$ (notice that $\mathcal{D}^a_m$ coincides with $\mathcal{D}_m$ translated by a parameter $a \in \mathbb{R}^n$ and, for a fixed a, $\bigcup_{m\in\mathbb{Z}} \mathcal{D}^a_m$ is a translation of the standard dyadic lattice). Notice that $\mu(D^a_m) \approx 2^{-mn}$ for all $m \in \mathbb{Z}$, $a \in \mathbb{R}^n$. For $D \in \mathcal{D}^a_m$ and $x \in D$, we set

$$E_D\mu(x) := \frac{1}{\mu(D)} \int_D \int_{D^c} K(z-y)\, d\mu(y)\, d\mu(z)$$

(take into account Remark 2.6 for the meaning of $\int_{D^c} K(z-y)\, d\mu(y)$). Finally, for $x \in \mathbb{R}^d$, we define the martingale $E^a_m\mu(x) := \sum_{D\in\mathcal{D}^a_m} \chi_D(x) E_D\mu(x)$, $m \in \mathbb{Z}$.

Let us make some comments to understand better the nature of $E^a_m\mu$. First of all notice that, since $\mu(\partial D) = 0$, for any $D \in \mathcal{D}^a_m$ and μ-almost all $z \in D$ we have

$$\int_{D^c} K(z-y)\, d\mu(y) = \lim_{\epsilon\to 0} \int_{D^c} \chi_\epsilon(z-y) K(z-y)\, d\mu(y), \tag{2.12}$$

and for any $\epsilon > 0$, we have

$$\int_D \int_D \chi_\epsilon(z-y) K(z-y)\, d\mu(y)\, d\mu(z) = 0 \tag{2.13}$$

because of the antisymmetry of K. Therefore, by (2.12), (2.13), (2.8), and the dominated convergence theorem, $\int_D \left|\int_{D^c} K(z-y)\, d\mu(y)\right| d\mu(z) < \infty$ (in particular, we have seen that $E^a_m\mu$ is well defined) and $\int_D T(\chi_D\mu)\, d\mu = 0$. Using this and (2.12), we finally have that

$$E^a_m\mu(x) = \frac{1}{\mu(D)} \int_D T(\chi_{D^c}\mu)\, d\mu = \frac{1}{\mu(D)} \int_D T\mu\, d\mu \tag{2.14}$$

for $x \in D \in \mathcal{D}^a_m$, thus $E^a_m\mu(x)$ is the average of the function $T\mu$ on the v-cube $D \in \mathcal{D}^a_m$ which contains x. So, it is completely clear that, for a fixed $a \in \mathbb{R}^n$, $\{E^a_m\mu\}_{m\in\mathbb{Z}}$ is a martingale. In

[MV] it is shown that $\{E_m^a\mu\}_{m\in\mathbb{Z}}$ is well defined and it is a martingale without the assumption of the existence of $T\mu$ (i.e., for more general measures μ).

Now, we can use (2.14), the L^2 boundedness of the dyadic maximal operator and (2.9) to deduce that

$$\|E_m^a\mu\|_{L^2(\mu)} \lesssim \|T\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} \tag{2.15}$$

for all $a \in \mathbb{R}^n$ and $m \in \mathbb{Z}$, where the constants that appear in the previous inequalities only depend on C_0, n, d and $\mathrm{Lip}(A)$.

Set $E^a\mu := \{E_m^a\mu\}_{m\in\mathbb{Z}}$. Then, the martingale $E^a\mu$ belongs to $L^2(\mu)$ by (2.15); thus by Theorem 2.4, for all $a \in \mathbb{R}^n$,

$$\begin{aligned} \|\mathcal{V}_\rho(E^a\mu)\|_{L^2(\mu)} &\lesssim \|E^a\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} \qquad \text{for } \rho > 2,\\ \|\mathcal{O}(E^a\mu)\|_{L^2(\mu)} &\lesssim \|E^a\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}, \end{aligned} \tag{2.16}$$

where the constants in the previous inequalities only depend on C_0, n, d, and $\mathrm{Lip}(A)$ (and on ρ, in the case of $\mathcal{V}_\rho$).

Finally, for $x \in \mathbb{R}^d$, we define

$$E_m\mu(x) := 2^{mn} \int_{\{a\,:\,x\in D_m^a\}} E_m^a\mu(x)\,da$$

(notice that $\mathcal{L}^n(\{a : x \in D_m^a\}) = 2^{-mn}$). Thus, $E_m\mu$ is an average (of the m'th term) of some martingales depending on a parameter $a \in \mathbb{R}^n$.

Set $E\mu := \{E_m\mu\}_{m\in\mathbb{Z}}$. We want to obtain estimates like (2.16) for $\mathcal{V}_\rho(E\mu)$ and $\mathcal{O}(E\mu)$. We will only show the details for $\mathcal{V}_\rho(E\mu)$, because the case of $\mathcal{O}(E\mu)$ follows by similar arguments.

One can easily check that $E_m\mu(x) = 2^{Mn}\int_{[0,2^{-M}]^n} E_m^a\mu(x)\,da$ for all $m, M \in \mathbb{Z}$ with $M \le m$. Therefore, for all $M, r, s \in \mathbb{Z}$ with $M \le r \le s$, we have

$$E_r\mu(x) - E_s\mu(x) = 2^{Mn} \int_{[0,2^{-M}]^n} (E_r^a\mu(x) - E_s^a\mu(x))\,da. \tag{2.17}$$

Given $M \in \mathbb{Z}$, we consider the auxiliary transformation

$$\mathcal{V}_{\rho,M}(E\mu)(x) := \sup_{\{r_m\}} \Big(\sum_{m\in\mathbb{Z}} |E_{r_{m+1}}\mu(x) - E_{r_m}\mu(x)|^\rho \Big)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences of integers $\{r_m\}_{m\in\mathbb{Z}}$ such that $r_m \ge M$ for all $m \in \mathbb{Z}$. With this definition it is obvious that the sequence $\{\mathcal{V}_{\rho,M}(E\mu)(x)\}_{M\in\mathbb{Z}}$ is non increasing and $\mathcal{V}_\rho(E\mu)(x) = \lim_{M\to-\infty} \mathcal{V}_{\rho,M}(E\mu)(x)$ for all $x \in \mathbb{R}^d$. Minkowski's integral inequality and (2.17) yield the pointwise estimate

$$\begin{aligned} \mathcal{V}_{\rho,M}(E\mu)(x) &= \sup_{\{r_m\}\,:\,r_m\ge M} \Big(\sum_{m\in\mathbb{Z}} |E_{r_{m+1}}\mu(x) - E_{r_m}\mu(x)|^\rho \Big)^{1/\rho} \\ &\le 2^{Mn} \int_{[0,2^{-M}]^n} \sup_{\{r_m\}} \Big(\sum_{m\in\mathbb{Z}} |E_{r_{m+1}}^a\mu(x) - E_{r_m}^a\mu(x))|^\rho \Big)^{1/\rho} da \\ &= 2^{Mn} \int_{[0,2^{-M}]^n} \mathcal{V}_\rho(E^a\mu)(x)\,da. \end{aligned}$$

Therefore, by the previous estimate, Minkowski's integral inequality and (2.16),

$$\|\mathcal{V}_{\rho,M}(E\mu)\|_{L^2(\mu)} \leq 2^{Mn} \int_{[0,2^{-M}]^n} \|\mathcal{V}_\rho(E^a\mu)\|_{L^2(\mu)}\, da \leq C\mu(P)^{1/2},$$

where $C > 0$ only depends on C_0, n, d, $\mathrm{Lip}(A)$, and ρ. By the monotone convergence theorem, we conclude that $\|\mathcal{V}_\rho(E\mu)\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$. Thus we have proved the following theorem (which can be considered the starting point to prove Main Theorem 1.1):

Theorem 2.7. *Fix a cube $\widetilde{P} \subset \mathbb{R}^n$. Set $\Gamma := \{x \in \mathbb{R}^d : x = (\widetilde{x}, A(\widetilde{x}))\}$, where $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is a Lipschitz function supported in $\widetilde{P}$, and set $P := \widetilde{P} \times \mathbb{R}^{d-n}$. Set $\mu := f\mathcal{H}^n_\Gamma$, where $f(x) = 1$ for all $\widetilde{x} \in \widetilde{P}^c$ and $C_0^{-1} \leq f(x) \leq C_0$ for all $\widetilde{x} \in \widetilde{P}$, for some constant $C_0 > 0$.*

Let $\rho > 2$. Then, there exist constants $C_1, C_2 > 0$ such that $\|\mathcal{V}_\rho(E\mu)\|_{L^2(\mu)} \leq C_1\mu(P)^{1/2}$ and $\|\mathcal{O}(E\mu)\|_{L^2(\mu)} \leq C_2\mu(P)^{1/2}$, where C_1 and C_2 only depend on C_0, n, d, and $\mathrm{Lip}(A)$ *(and on ρ in the case of C_1).*

We need to introduce additional notation in order to express $E_m\mu$ in a more convenient way for our purposes. Let $\mu_1, \ldots, \mu_k$ be a finite collection of positive Borel measures such that $\mu_l(D^a_m) > 0$ for all $a \in \mathbb{R}^n$, $m \in \mathbb{Z}$ and $l = 1, \ldots, k$. Given $m \in \mathbb{Z}$ and $x_1, \ldots, x_i, y_1, \ldots, y_j \in \mathbb{R}^d$, we define

$$\Lambda^{\mu_1,\ldots,\mu_k}_m(x_1, \ldots, x_i\,;\, y_1, \ldots, y_j) := 2^{nm} \int_{\{a\,:\,x_1,\ldots,x_i \in D^a_m,\, y_1,\ldots,y_j \notin D^a_m\}} \frac{da}{\prod_{l=1}^k \mu_l(D^a_m)}.$$

Then, by Fubini's theorem,

$$\begin{aligned} E_m\mu(x) &= \int_{\{a\,:\,x\in D^a_m\}} \frac{2^{mn}}{\mu(D^a_m)} \int_{D^a_m} \int_{(D^a_m)^c} K(z-y)\, d\mu(y)\, d\mu(z)\, da \\ &= \iint \left(2^{mn} \int_{\{a\,:\,x,z\in D^a_m,\, y\notin D^a_m\}} \frac{da}{\mu(D^a_m)}\right) K(z-y)\, d\mu(z)\, d\mu(y) \qquad (2.18)\\ &= \iint \Lambda^\mu_m(x, z\,;\, y) K(z-y)\, d\mu(z)\, d\mu(y). \end{aligned}$$

3. Sketch of the proof of Main Theorem 1.1

The proof relies on two basic facts: the known L^2 boundedness of the ρ-variation and oscillation of martingales explained in the previous section and the good geometric properties of Lipschitz graphs from a measure-theoretic point of view.

As we said above, the starting point of the proof is Theorem 2.7, where the L^2 boundedness of the ρ-variation and oscillation (of a convex combination) of some particular martingales is stated. So, the first step consists in relating the results on martingales in Theorem 2.7 with the ρ-variation and oscillation of singular integrals on Lipschitz graphs, and this is the aim of the following two theorems:

Theorem 3.1. *Let Γ and μ be as in Theorem 2.7. For each $x \in \Gamma$, define*

$$W\mu(x)^2 := \sum_{m\in\mathbb{Z}} |(K\widetilde{\varphi}_{2^{-m}} * \mu)(x) - E_m\mu(x)|^2. \qquad (3.1)$$

Then, $\|W\mu\|^2_{L^2(\mu)} \leq C_1 \sum_{Q\in\mathcal{D}} \big(\alpha_\mu(C_2Q)^2 + \beta_{2,\mu}(Q)^2\big)\mu(Q)$, where $C_1, C_2 > 0$ depend only on C_0, n, d, K, and $\mathrm{Lip}(A)$.

Theorem 3.2. *Let Γ and μ be as in Theorem 2.7. For each $x \in \Gamma$, define*

$$S\mu(x)^2 := \sup_{\{\epsilon_m\}} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}:\, \epsilon_m,\epsilon_{m+1}\in I_j} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^2, \tag{3.2}$$

where $I_j = [2^{-j-1}, 2^{-j})$ and the supremum is taken over all decreasing sequences of positive numbers $\{\epsilon_m\}_{m\in\mathbb{Z}}$. Then, $\|S\mu\|_{L^2(\mu)}^2 \leq C \sum_{Q\in\mathcal{D}} \big(\alpha_\mu(Q)^2 + \beta_{2,\mu}(Q)^2\big)\mu(Q)$, where $C > 0$ only depends on C_0, n, d, K, and $\mathrm{Lip}(A)$.

Two fundamental tools to study $W\mu$ and $S\mu$ are the α and β coefficients, which will be used to measure the flatness of Γ at different scales, in order to estimate the terms which appear in the sums in (3.1) and (3.2). This will be done in sections 4 and 5. To use the α coefficients to relate the ρ-variation of martingales with the ρ-variation of singular integrals, it is a key fact that we are considering a family of smooth truncations like $\widetilde{\varphi}$, instead of χ, because the α's are defined in terms of Lipschitz functions. Moreover, for the moment, we are taking a truncation only on the first n-coordinates (i.e., $\widetilde{\varphi}$ instead of φ) because the average of martingales that we are using is taken over the parameter $a \in \mathbb{R}^n$, using the v-cubes D_M^a (see subsection 2.3).

Combining Theorem 3.1 and Theorem 3.2 with the L^2 estimates of the ρ-variation and oscillation on the average of martingales $E\mu$ in Theorem 2.7, we are able to obtain local L^2 estimates of $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$ when Γ is any Lipschitz graph. More precisely, we separate the sum in the definition of $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$ into two parts, which are classically called short and long variation (and analogously for $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$). The short variation corresponds to the sum $S\mu$ in Theorem 3.2 (here μ is a suitable modification of $\mathcal{H}_\Gamma^n$), where the indices run over $m \in \mathbb{Z}$ such that both ϵ_m and ϵ_{m+1} lie in the same dyadic interval, and can be handled using the α's and β's. The long variation corresponds to the sum over the indices $m \in \mathbb{Z}$ such that ϵ_m and ϵ_{m+1} lie in different dyadic intervals, so one may assume that the ϵ_m's are dyadic numbers. It is handled by comparing $K\widetilde{\varphi}_{2^{-m}} * \mu$ with $E_m\mu$, and then using Theorem 3.1 and the fact the ρ-variation and oscillation of $E\mu$ are bounded in $L^2(\mu)$, by Theorem 2.7. This will be done in section 6 (see Theorem 6.1).

Using the local L^2 estimates of Theorem 6.1, combined with rather standard techniques in Calderón-Zygmund theory, in section 7 we obtain the $H^1(\mathcal{H}_\Gamma^n) \to L^1(\mathcal{H}_\Gamma^n)$ and $L^\infty(\mathcal{H}_\Gamma^n) \to BMO(\mathcal{H}_\Gamma^n)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$. Then, by interpolation, we obtain the L^p boundedness of these operators in the whole range $1 < p < \infty$, and in particular the L^2 boundedness (see Theorem 7.1).

The next step is to replace the family of smooth truncations $\widetilde{\varphi}$ by the other families of truncations ω. We focus our interest on the case $\omega = \chi$, because we think that it is the most important one and the other (easier) cases follow using similar arguments. We obtain the L^2 boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}_\Gamma^n}$ by comparing these operators with $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$, and by estimating the difference in terms of the α and β coefficients, decomposing a function $f \in L^2(\mathcal{H}_\Gamma^n)$ using a suitable wavelet basis. It is in this step where we need the assumption $\mathrm{Lip}(A) < 1$ for $\omega = \chi$, if $\omega \in \{\varphi, \widetilde{\chi}\}$ then $\mathrm{Lip}(A) < \infty$ suffices. This is done in section 8 (see Theorem 8.1).

Finally, in section 9 (see Theorem 9.1) we show that the L^2 boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}_\Gamma^n}$ yields the $L^1(\mathcal{H}_\Gamma^n) \to L^{1,\infty}(\mathcal{H}_\Gamma^n)$ and $L^\infty(\mathcal{H}_\Gamma^n) \to BMO(\mathcal{H}_\Gamma^n)$ boundedness of these operators, and then we obtain the L^p boundedness in the whole range $1 < p < \infty$ by interpolation again. The same holds for the other families of truncations ω. This finishes the proof of the Main Theorem 1.1.

Let us stress that almost all the estimates in the proof of Main Theorem 1.1 (in particular, the constants involved in the relationships $\lesssim$, $\gtrsim$ and $\approx$) depend either on n, d, K or $\mathrm{Lip}(A)$, and possibly on other variables such as ρ or p.

4. Proof of Theorem 3.1

In order to study the difference $(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)-E_m\mu(x)$, we are going to split $E_m\mu(x)$ into two parts, the one we will compare with $(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)$ (which corresponds to integrate, in the definition of $E_m\mu(x)$, over the points $y\in\mathbb{R}^d$ such that $2^{-m}\lesssim|\widetilde{x}-\widetilde{y}|$), and the remaining part. Then, we will estimate each part of $(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)-E_m\mu(x)$ separately, using the cancelation properties of the kernel K and the uniform rectifiability of μ.

Recall from (2.18) that $E_m\mu(x)=\iint\Lambda_m^\mu(x,z\,;\,y)K(z-y)\,d\mu(z)\,d\mu(y)$. Given $\epsilon>0$, we set $\gamma_\epsilon:=1-\widetilde{\varphi}_\epsilon$. Then,

$$\begin{aligned}E_m\mu(x)&=\iint\widetilde{\varphi}_{2^{-m}}(x-y)\Lambda_m^\mu(x,z\,;\,y)K(z-y)\,d\mu(z)\,d\mu(y)\\&\quad+\iint\gamma_{2^{-m}}(x-y)\Lambda_m^\mu(x,z\,;\,y)K(z-y)\,d\mu(z)\,d\mu(y).\end{aligned}$$

The first term in the previous sum is the one that we will compare with $(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)$. For all $a\in\mathbb{R}^n$ such that $x\in D_m^a$, we have $\operatorname{supp}\widetilde{\varphi}_{2^{-m}}(x-\cdot)\cap D_m^a=\emptyset$, and thus $(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)=(K\widetilde{\varphi}_{2^{-m}}*(\chi_{(D_m^a)^c}\mu))(x)$. Hence, using Fubini's theorem and the definition of $\Lambda_m^\mu(x,z\,;\,y)$,

$$\begin{aligned}(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)&=2^{mn}\int_{\{a\,:\,x\in D_m^a\}}(K\widetilde{\varphi}_{2^{-m}}*(\chi_{(D_m^a)^c}\mu))(x)\,da\\&=2^{mn}\int_{\{a\,:\,x\in D_m^a\}}\mu(D_m^a)^{-1}\int_{D_m^a}(K\widetilde{\varphi}_{2^{-m}}*(\chi_{(D_m^a)^c}\mu))(x)\,d\mu(z)\,da\\&=\iint\widetilde{\varphi}_{2^{-m}}(x-y)\Lambda_m^\mu(x,z\,;\,y)K(x-y)\,d\mu(z)\,d\mu(y).\end{aligned}$$

We can decompose $(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)-E_m\mu(x)$ as

$$\begin{aligned}&(K\widetilde{\varphi}_{2^{-m}}*\mu)(x)-E_m\mu(x)\\&\quad=\iint\widetilde{\varphi}_{2^{-m}}(x-y)\Lambda_m^\mu(x,z\,;\,y)(K(x-y)-K(z-y))\,d\mu(z)\,d\mu(y)\\&\quad\quad-\iint\gamma_{2^{-m}}(x-y)\Lambda_m^\mu(x,z\,;\,y)K(z-y)\,d\mu(z)\,d\mu(y)\\&\quad=\sum_{j<m}F_j^m(x)-\sum_{j\in\mathbb{Z}}G_j^m(x),\end{aligned}\tag{4.1}$$

where

$$F_j^m(x):=\iint\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^\mu(x,z\,;\,y)(K(x-y)-K(z-y))\,d\mu(z)\,d\mu(y),\tag{4.2}$$

$$G_j^m(x):=\iint\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(z-y)\gamma_{2^{-m}}(x-y)\Lambda_m^\mu(x,z\,;\,y)K(z-y)\,d\mu(z)\,d\mu(y).\tag{4.3}$$

Fix a v-cube $D\in\mathcal{D}_m$, for some $m\in\mathbb{Z}$. In subsection 4.1 (see (4.18)) we will prove that

$$\sum_{j<m}|F_j^m(x)|\lesssim\frac{\operatorname{dist}(x,L_D)}{\ell(D)}+\sum_{Q\in\mathcal{D}\,:\,D\subset Q}\frac{\ell(D)}{\ell(Q)}\,\alpha(Q)\tag{4.4}$$

for all $x \in D \cap \Gamma$, where L_D denotes an n-plane that minimizes $\alpha(D)$, and in subsection 4.2 (see (4.37)) we will prove that there exists a constant $C_b > 1$ such that

$$\sum_{j\in\mathbb{Z}} |G_j^m(x)| \lesssim \alpha(C_bD) + \sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}}\,\alpha(Q) \tag{4.5}$$

for all $x \in D \cap \Gamma$. Assuming that these estimates hold, by (4.1),

$$\begin{aligned}
\|W\mu\|^2_{L^2(\mu)} &= \sum_{m\in\mathbb{Z}}\sum_{D\in\mathcal{D}_m}\int_D |(K\widetilde{\varphi}_{2^{-m}} * \mu)(x) - E_m\mu(x)|^2\,d\mu(x)\\
&\lesssim \sum_{D\in\mathcal{D}}\int_D \left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2 d\mu(x) + \sum_{D\in\mathcal{D}}\Bigg(\sum_{\substack{Q\in\mathcal{D}:\\ D\subset Q}} \frac{\ell(D)}{\ell(Q)}\,\alpha(Q)\Bigg)^2\mu(D)\\
&\quad + \sum_{D\in\mathcal{D}} \alpha(C_bD)^2\mu(D) + \sum_{D\in\mathcal{D}}\Bigg(\sum_{\substack{Q\in\mathcal{D}:\\ Q\subset C_bD}} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}}\,\alpha(Q)\Bigg)^2\mu(D)\\
&=: W_1\mu + W_2\mu + W_3\mu + W_4\mu.
\end{aligned} \tag{4.6}$$

If L_D^1 and L_D^2 denote a minimizing n-plane for $\beta_1(D)$ and $\beta_2(D)$, respectively, one can show that $\operatorname{dist}_{\mathcal{H}}(L_D \cap C_\Gamma D, L_D^1 \cap C_\Gamma D) \lesssim \alpha(D)\ell(D)$ and $\operatorname{dist}_{\mathcal{H}}(L_D^1 \cap C_\Gamma D, L_D^2 \cap C_\Gamma D) \lesssim \beta_2(D)\ell(D)$. This easily implies that, for $x \in D \cap \Gamma$, $\operatorname{dist}(x, L_D) \lesssim \operatorname{dist}(x, L_D^2) + \beta_2(D)\ell(D) + \alpha(D)\ell(D)$, so $W_1\mu \lesssim \sum_{D\in\mathcal{D}}(\alpha(D)^2 + \beta_2(D)^2)\mu(D)$.

By Cauchy-Schwarz inequality,

$$\begin{aligned}
W_2\mu &\le \sum_{D\in\mathcal{D}} \mu(D)\Bigg(\sum_{Q\in\mathcal{D}:\,D\subset Q} \frac{\ell(D)}{\ell(Q)}\,\alpha(Q)^2\Bigg)\Bigg(\sum_{Q\in\mathcal{D}:\,D\subset Q}\frac{\ell(D)}{\ell(Q)}\Bigg)\\
&\approx \sum_{D\in\mathcal{D}}\sum_{Q\in\mathcal{D}:\,D\subset Q} \frac{\ell(D)^{n+1}}{\ell(Q)}\,\alpha(Q)^2 \approx \sum_{Q\in\mathcal{D}} \alpha(Q)^2\mu(Q),
\end{aligned}$$

and also

$$\begin{aligned}
W_4\mu &\le \sum_{D\in\mathcal{D}} \mu(D)\Bigg(\sum_{Q\in\mathcal{D}:\,Q\subset C_bD} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}}\,\alpha(Q)^2\Bigg)\Bigg(\sum_{Q\in\mathcal{D}:\,Q\subset C_bD}\frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}}\Bigg)\\
&\approx \sum_{D\in\mathcal{D}}\sum_{Q\in\mathcal{D}:\,Q\subset C_bD} \ell(Q)^n\frac{\ell(Q)}{\ell(D)}\,\alpha(Q)^2 \lesssim \sum_{Q\in\mathcal{D}} \alpha(Q)^2\mu(Q).
\end{aligned}$$

Therefore, using (4.6) and that $\alpha(Q) \lesssim \alpha(C_bQ)$, we conclude that

$$\|W\mu\|^2_{L^2(\mu)} \lesssim \sum_{Q\in\mathcal{D}} (\alpha(C_bQ)^2 + \beta_2(Q)^2)\mu(Q),$$

and the theorem follows. It only remains to prove (4.4) and (4.5).

4.1. **Estimate of $\sum_{j<m} F_j^m(x)$ when $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$.** Assume that $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$ and $j < m$. Let L_D be an n-plane that minimizes $\alpha(D)$ and let $\sigma_D := c_D\mathcal{H}^n_{L_D}$ be a minimizing measure of $\alpha(D)$. Let L_D^x be the n-plane parallel to L_D that contains x and set $\sigma_D^x := c_D\mathcal{H}^n_{L_D^x}$.

Notice that, because of $x \in L_D^x$, the antisymmetry of $\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}} K$, and since $j < m$ (so, if $x \in D_m^a$ and $y \in \operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-\cdot)$, then $y \notin D_m^a$), we have

$$\begin{aligned} 0 &= \int \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)K(x-y)\,d\sigma_D^x(y) \\ &= \int_{\{a\,:\,x\in D_m^a\}} \frac{2^{mn}}{\sigma_D^x(D_m^a)} \int_{D_m^a}\int_{(D_m^a)^c} \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)K(x-y)\,d\sigma_D^x(y)\,d\sigma_D^x(z)\,da \\ &= \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\sigma_D^x}(x,z\,;\,y)K(x-y)\,d\sigma_D^x(z)\,d\sigma_D^x(y). \end{aligned} \tag{4.7}$$

Given $a \in \mathbb{R}^n$, let $b := a + \{2^{-m-1}\}^n \in \mathbb{R}^n$ be the center of $\widetilde{D}_m^a$. For $u \in \mathbb{R}^n$ we denote $\|u\|_\infty := \max_{i=1,\dots,n}|u^i|$. Then, given $t \in \mathbb{R}^d$, it is clear that $t \in \overline{D_m^a}$ if and only if $\|\tilde{t}-b\|_\infty \leq 2^{-m}$. Using that σ_D^x is a Hausdorff measure on an n-plane, that K is antisymmetric and that $\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}$ is symmetric, one can show that

$$0 = \int_{\|\widetilde{x}-b\|_\infty\leq 2^{-m}}\int_{\|\widetilde{z}-b\|_\infty\leq 2^{-m}}\int_{\|\widetilde{y}-b\|_\infty > 2^{-m}} \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)K(z-y)\,d\sigma_D^x(y)\,d\sigma_D^x(z)\,db.$$

By the change of variable $b = a + \{2^{-m-1}\}^n$, it is easy to see that this triple integral is equal to $\int_{\{a\,:\,x\in D_m^a\}}\int_{D_m^a}\int_{(D_m^a)^c} \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)K(z-y)\,d\sigma_D^x(y)\,d\sigma_D^x(z)\,da$. Thus, since $\sigma_D^x(D_m^a)$ does not depend on $a \in \mathbb{R}^n$ because σ_D^x is flat,

$$\begin{aligned} 0 &= \int_{\{a\,:\,x\in D_m^a\}} \frac{2^{mn}}{\sigma_D^x(D_m^a)} \int_{D_m^a}\int_{(D_m^a)^c} \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)K(z-y)\,d\sigma_D^x(y)\,d\sigma_D^x(z)\,da \\ &= \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\sigma_D^x}(x,z\,;\,y)K(z-y)\,d\sigma_D^x(z)\,d\sigma_D^x(y). \end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we conclude that

$$0 = \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\sigma_D^x}(x,z\,;\,y)(K(x-y)-K(z-y))\,d\sigma_D^x(z)\,d\sigma_D^x(y). \tag{4.9}$$

By definition, it is clear that $\Lambda_m^{\sigma_D^x}(x,z\,;\,y) = \Lambda_m^{\sigma_D}(x,z\,;\,y)$. Therefore, using (4.9), we can decompose

$$F_j^m(x) = F1_j^m(x) + F2_j^m(x) + F3_j^m(x) + F4_j^m(x), \tag{4.10}$$

where

$$F1_j^m(x) := \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\mu}(x,z\,;\,y) (K(x-y)-K(z-y))\,d(\mu-\sigma_D)(z)\,d\mu(y), \tag{4.11}$$

$$F2_j^m(x) := \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\mu}(x,z\,;\,y) (K(x-y)-K(z-y))\,d\sigma_D(z)\,d(\mu-\sigma_D)(y), \tag{4.12}$$

$$F3_j^m(x) := \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)(\Lambda_m^{\mu}(x,z\,;\,y)-\Lambda_m^{\sigma_D}(x,z\,;\,y)) (K(x-y)-K(z-y))\,d\sigma_D(z)\,d\sigma_D(y), \tag{4.13}$$

$$F4_j^m(x) := \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\sigma_D^x}(x,z\,;\,y) (K(x-y)-K(z-y))\,d(\sigma_D\times\sigma_D-\sigma_D^x\times\sigma_D^x)(z,y). \tag{4.14}$$

In the next subsections we will prove the following estimates:

$$|F1_j^m(x)| + |F3_j^m(x)| \lesssim 2^{j-m}\alpha(D), \tag{4.15}$$

$$|F2_j^m(x)| \lesssim 2^{j-m} \sum_{Q\in\mathcal{D}\,:\,D\subset Q,\,\ell(Q)\leq 2^{-j}} \alpha(Q), \tag{4.16}$$

$$|F4_j^m(x)| \lesssim 2^{j-m}\,\frac{\operatorname{dist}(x,L_D)}{\ell(D)}. \tag{4.17}$$

Then, using (4.10), we will finally get that, for all $D\in\mathcal{D}_m$ and $x\in D\cap\Gamma$,

$$\begin{aligned}\sum_{j<m}|F_j^m(x)| &\lesssim \frac{\operatorname{dist}(x,L_D)}{\ell(D)} + \sum_{j\leq m} 2^{j-m} \sum_{Q\in\mathcal{D}:\,D\subset Q,\,\ell(Q)\leq 2^{-j}} \alpha(Q)\\ &\lesssim \frac{\operatorname{dist}(x,L_D)}{\ell(D)} + \sum_{Q\in\mathcal{D}\,:\,D\subset Q}\frac{\ell(D)}{\ell(Q)}\,\alpha(Q),\end{aligned} \tag{4.18}$$

which gives (4.4).

4.1.1. **Estimate of** $F1_j^m(x)$. Notice that, if $|\widetilde{x}-\widetilde{z}| > 2^{-m}\sqrt{n}$, there is no $a\in\mathbb{R}^n$ such that $x,z\in D_m^a$, and this means that $\Lambda_m^\mu(x,z\,;\,y)=0$. Thus, we can assume that $|\widetilde{x}-\widetilde{z}|\leq 2^{-m}\sqrt{n}$. Therefore, if the constant C_Γ (see the definition of the α's in subsection 2.2) is big enough, $\operatorname{supp}\Lambda_m^\mu(x,\cdot\,;\,y)\subset B_D$.

For $y,z\in\Gamma$ such that $y\in\operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-\cdot)$, $j<m$ and $|\widetilde{x}-\widetilde{z}|\leq 2^{-m}\sqrt{n}$ (so, in particular, $|x-z|\lesssim|x-y|$), we have the following estimates:

$$|K(x-y)-K(z-y)| \lesssim |x-z||x-y|^{-n-1} \lesssim 2^{j(n+1)-m},$$
$$|\nabla_z(K(x-y)-K(z-y))| = |\nabla_z K(z-y)| \lesssim 2^{j(n+1)}.$$

Claim 4.1. *We have* $|\Lambda_m^\mu(x,z\,;\,y)|\lesssim 2^{mn}$ *and* $|\nabla_z\Lambda_m^\mu(x,z\,;\,y)|\lesssim 2^{m(n+1)}$ *for all* $x,y,z\in\mathbb{R}^d$.

Claim 4.1 and the subsequent ones 4.2,...,4.7 will be proved in subsection 4.3 below.

Putting all these estimates together we obtain that

$$\Big|\nabla_z\Big(\Lambda_m^\mu(x,z\,;\,y)(K(x-y)-K(z-y))\Big)\Big| \lesssim 2^{j(n+1)+mn},$$

and, since $\operatorname{supp}\Lambda_m^\mu(x,\cdot\,;\,y)\subset B_D$, recalling the definition of $\operatorname{dist}_{B_D}$ in (2.3),

$$\left|\int \Lambda_m^\mu(x,z\,;\,y)(K(x-y)-K(z-y))\,d(\mu-\sigma_D)(z)\right| \lesssim 2^{j(n+1)+mn}\operatorname{dist}_{B_D}(\mu,\sigma_D).$$

We can use this last estimate in (4.11) to obtain

$$\begin{aligned}|F1_j^m(x)| &\lesssim 2^{j(n+1)+mn}\operatorname{dist}_{B_D}(\mu,\sigma_D)\int\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)\,d\mu(y)\\ &\lesssim 2^{j+mn}\operatorname{dist}_{B_D}(\mu,\sigma_D)\approx 2^{j-m}\,\ell(D)^{-n-1}\operatorname{dist}_{B_D}(\mu,\sigma_D)\lesssim 2^{j-m}\alpha(D),\end{aligned}$$

which, together with the estimate of $|F3_j^m(x)|$ in subsection 4.1.3, gives (4.15).

4.1.2. **Estimate of** $F2_j^m(x)$. Arguing as in subsection 4.1.1, we can obtain the following estimates for x,y,z as above:

$$|\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)|\leq 1 \quad\text{and}\quad |\nabla_y\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-y)|\lesssim 2^j, \tag{4.19}$$

$$|K(x-y)-K(z-y)| \lesssim |x-z||x-y|^{-n-1}\lesssim 2^{j(n+1)-m}, \tag{4.20}$$

$$|\nabla_y(K(x-y)-K(z-y))| \lesssim |\nabla_y^2K(x-y)||x-z| \lesssim 2^{j(n+2)-m}. \tag{4.21}$$

(by $|\nabla_y^2 K(x-y)| \lesssim 2^{j(n+2)}$ we mean that all the components of the matrix $\nabla_y^2 K(x-y)$ are bounded in absolute value by $C2^{j(n+2)}$).

Claim 4.2. *For $j < m$, $y \in \operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(x-\cdot)$, and $|\widetilde{x}-\widetilde{z}| \leq 2^{-m}\sqrt{n}$, the following hold: $|\Lambda_m^\mu(x,z\,;\,y)| \lesssim 2^{mn}$ and $\nabla_y \Lambda_m^\mu(x,z\,;\,y) = 0$.*

Notice that the first estimate in Claim 4.2 is the same as the first one in Claim 4.1.

Let $D_j \in \mathcal{D}_j$ be the unique dyadic v-cube with $\ell(D_j) = 2^{-j}$ which contains D. Then, $\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(x-\cdot) \subset B_{D_j}$ for C_Γ big enough. Therefore, we can use the previous estimates to see that the gradient of the term inside the integral with respect to y in (4.12) is bounded by $2^{j(n+2)+m(n-1)}$ and is supported in B_{D_j}, and then by (2.3) we derive that

$$
\begin{aligned}
|F2_j^m(x)| \leq \int \bigg| \int \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(x-y)\Lambda_m^\mu(x,z\,;\,y) & \\
(K(x-y)-K(z-y))\, d(\mu-\sigma_D)(y) \bigg|\, d\sigma_D(z) & \\
\lesssim \int_{|\widetilde{x}-\widetilde{z}|\leq 2^{-m}\sqrt{n}} 2^{j(n+2)+m(n-1)} \operatorname{dist}_{B_{D_j}}(\mu,\sigma_D)\, d\sigma_D(z) & \\
\lesssim 2^{j(n+2)-m} \operatorname{dist}_{B_{D_j}}(\mu,\sigma_D). &
\end{aligned}
\tag{4.22}
$$

We shall estimate $\operatorname{dist}_{B_{D_j}}(\mu,\sigma_D)$ in terms of the α coeficients. Consider the unique sequence of dyadic v-cubes $D =: D_m \subset \ldots \subset D_{i+1} \subset D_i \subset \ldots \subset D_j$ such that each D_i belongs to $\mathcal{D}_i$, for $i = j,\ldots,m$. Let L_{D_i} be an n-plane that gives the minimum in the definition of $\alpha(D_i)$ and let $\sigma_{D_i} := c_{D_i} d\mathcal{H}^n_{L_{D_i}}$ be a minimizing measure. We will prove that

$$
\operatorname{dist}_{B_{D_j}}(\mu,\sigma_D) \lesssim 2^{-j(n+1)} \sum_{i=j}^{m-1} \alpha(D_i). \tag{4.23}
$$

Combining (4.23) with (4.22), we will finally obtain that $|F2_j^m(x)| \lesssim 2^{j-m}\sum_{i=j}^{m-1}\alpha(D_i)$, which gives (4.16).

Let us prove (4.23). By the triangle inequality,

$$
\begin{aligned}
\operatorname{dist}_{B_{D_j}}(\mu,\sigma_D) \leq{} & \operatorname{dist}_{B_{D_j}}(\mu,\sigma_{D_j}) + \textstyle\sum_{i=j}^{m-1} \operatorname{dist}_{B_{D_j}}(\sigma_{D_i},\sigma_{D_{i+1}}) \\
\lesssim{} & 2^{-j(n+1)}\alpha(D_j) + \textstyle\sum_{i=j}^{m-1} \operatorname{dist}_{B_{D_j}}(\sigma_{D_i},\sigma_{D_{i+1}}),
\end{aligned}
$$

so we are reduced to prove that, for all $i = j,\ldots,m-1$,

$$
\operatorname{dist}_{B_{D_j}}(\sigma_{D_i},\sigma_{D_{i+1}}) \lesssim 2^{-j(n+1)}\alpha(D_i). \tag{4.24}
$$

By definition, $\operatorname{dist}_{B_{D_j}}(\sigma_{D_i},\sigma_{D_{i+1}}) = \sup\big|\int g\, d(c_{D_i}\mathcal{H}^n_{L_{D_i}} - c_{D_{i+1}}\mathcal{H}^n_{L_{D_{i+1}}})\big|$, where the supremum is taken over all Lipschitz functions g supported in B_{D_j} such that $\operatorname{Lip}(g) \leq 1$. Fix one of such Lipschitz functions g. Then,

$$
\begin{aligned}
\int g\, d(c_{D_i}\mathcal{H}^n_{L_{D_i}} - c_{D_{i+1}}\mathcal{H}^n_{L_{D_{i+1}}}) = (c_{D_i} - c_{D_{i+1}}) \int g\, d\mathcal{H}^n_{L_{D_i}} & \\
+ c_{D_{i+1}} \int g\, d(\mathcal{H}^n_{L_{D_i}} - \mathcal{H}^n_{L_{D_{i+1}}}). &
\end{aligned}
\tag{4.25}
$$

It is shown in [To, Lemma 3.4] that $|c_{D_i} - c_{D_{i+1}}| \lesssim \alpha(D_i)$, so the first term on the right hand side of (4.25) is bounded in absolute value by $C2^{-j(n+1)}\alpha(D_i)$.

In order to estimate the second term of the right hand side of (4.25), set $L_{D_{i+1}} = \{(\widetilde{t}, a(\widetilde{t})) \in \mathbb{R}^d \,:\, \widetilde{t} \in \mathbb{R}^n\}$ (where $a : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is an appropriate affine map), and let $p : L_{D_i} \to L_{D_{i+1}}$ be the projection defined by $p(t) := (\widetilde{t}, a(\widetilde{t}))$. Since Γ is a Lipschitz graph, a is well defined and p is a homeomorphism. Let $p_\sharp \mathcal{H}^n_{L_{D_i}}$ be the image measure of $\mathcal{H}^n_{L_{D_i}}$ by p. It is easy to check that $\mathcal{H}^n_{L_{D_{i+1}}} = \tau p_\sharp \mathcal{H}^n_{L_{D_i}}$, where τ is some positive constant such that $|\tau - 1| \lesssim \alpha(D_i)$ and $\tau \lesssim 1$. Therefore,

$$
\begin{aligned}
\left| \int g \, d(\mathcal{H}^n_{L_{D_i}} - \mathcal{H}^n_{L_{D_{i+1}}}) \right| &= \left| \int (g(t) - \tau g(p(t))\,) \, d\mathcal{H}^n_{L_{D_i}}(t) \right| \\
&\le \left| \int (1-\tau) g(t) \, d\mathcal{H}^n_{L_{D_i}}(t) \right| + \left| \int \tau (g(t) - g(p(t))\,) \, d\mathcal{H}^n_{L_{D_i}}(t) \right| \\
&\lesssim 2^{-j(n+1)} \alpha(D_i) + \int |(g(t) - g(p(t))| \, d\mathcal{H}^n_{L_{D_i}}(t).
\end{aligned}
\tag{4.26}
$$

Since g and $g \circ p$ are supported in B_{D_j} and g is 1-Lipschitz, by [To, Lemma 3.4],

$$
\begin{aligned}
\int |(g - g \circ p)| \, d\mathcal{H}^n_{L_{D_i}} &\lesssim \int_{B_{D_j}} \operatorname{dist}_{\mathcal{H}}(L_{D_i} \cap B_{D_j}, L_{D_{i+1}} \cap B_{D_j}) \, d\mathcal{H}^n_{L_{D_i}} \\
&\lesssim \; 2^{-jn} \operatorname{dist}_{\mathcal{H}}(L_{D_i} \cap B_{D_j}, L_{D_{i+1}} \cap B_{D_j}) \\
&\lesssim \; 2^{-jn} 2^{i-j} \operatorname{dist}_{\mathcal{H}}(L_{D_i} \cap B_{D_i}, L_{D_{i+1}} \cap B_{D_i}) \lesssim \; 2^{-j(n+1)} \alpha(D_i).
\end{aligned}
$$

This last estimate together with (4.26) and the fact that $|c_{D_{i+1}}| \lesssim 1$ implies that the second term on the right hand side of (4.25) is also bounded in absolute value by $C 2^{-j(n+1)} \alpha(D_i)$. Therefore, to obtain (4.24) we only have to take the supremum in (4.25) over all admissible functions g.

4.1.3. **Estimate of** $F3^m_j(x)$. Notice that, by Fubini's theorem,

$$
\begin{aligned}
\Lambda^{\mu}_m(x, z\,;\, y) - \Lambda^{\sigma_D}_m(x, z\,;\, y) &= 2^{mn} \int_{\{a \,:\, x,z \in D^a_m,\, y \notin D^a_m\}} \left(\frac{1}{\mu(D^a_m)} - \frac{1}{\sigma_D(D^a_m)} \right) da \\
&= 2^{mn} \int_{\{a \,:\, x,z \in D^a_m,\, y \notin D^a_m\}} \frac{\sigma_D(D^a_m) - \mu(D^a_m)}{\mu(D^a_m)\sigma_D(D^a_m)} \, da \\
&= 2^{mn} \int_{\{a \,:\, x,z \in D^a_m,\, y \notin D^a_m\}} \left(\int_{t \in D^a_m} d(\sigma_D - \mu)(t) \right) \mu(D^a_m)^{-1} \sigma_D(D^a_m)^{-1} \, da \\
&= \int \Lambda^{\mu,\sigma_D}_m(x, z, t\,;\, y) \, d(\sigma_D - \mu)(t).
\end{aligned}
$$

Since $\Lambda^{\mu,\sigma_D}_m(x, z, t\,;\, y) = 0$ if $|\widetilde{x} - \widetilde{t}| > 2^{-m}\sqrt{n}$, we may assume that $\operatorname{supp} \Lambda^{\mu,\sigma_D}_m(x, z, \cdot\,;\, y) \subset B_D$ (by taking C_Γ big enough).

Claim 4.3. *We have* $|\Lambda^{\mu,\sigma_D}_m(x, z, t\,;\, y)| \lesssim 2^{2mn}$ *and* $|\nabla_t \Lambda^{\mu,\sigma_D}_m(x, z, t\,;\, y)| \lesssim 2^{m(2n+1)}$ *for all* $x, y, z, t \in \mathbb{R}^d$.

Using Claim 4.3, we deduce that $|\Lambda_m^\mu(x,z\,;\,y)-\Lambda_m^{\sigma_D}(x,z\,;\,y)|\lesssim 2^{m(2n+1)}\operatorname{dist}_{B_D}(\mu,\sigma_D)$, and then,

$$\begin{aligned}|F3_j^m(x)| &\lesssim \int \widetilde\varphi_{2^{-j-1}}^{2^{-j}}(x-y)\int_{|\widetilde x-\widetilde z|\le 2^{-m}\sqrt{n}} 2^{m(2n+1)}\operatorname{dist}_{B_D}(\mu,\sigma_D)\\ &\qquad\qquad |x-z||x-y|^{-n-1}\,d\sigma_D(z)\,d\sigma_D(y)\\ &\lesssim 2^{2mn+j(n+1)}\operatorname{dist}_{B_D}(\mu,\sigma_D)\iint_{\substack{|\widetilde x-\widetilde y|\le 2^{-j}3\sqrt{n}\\ |\widetilde x-\widetilde z|\le 2^{-m}\sqrt{n}}} d\sigma_D(z)\,d\sigma_D(y)\\ &\lesssim 2^{mn+j}\operatorname{dist}_{B_D}(\mu,\sigma_D)\lesssim 2^{j-m}\alpha(D),\end{aligned}$$

which, together with the estimate of $|F1_j^m(x)|$ (see subsection 4.1.1), gives (4.15).

4.1.4. **Estimate of** $F4_j^m(x)$. Set $L_D=\{(\widetilde y,a(\widetilde y))\in\mathbb{R}^d\,:\,\widetilde y\in\mathbb{R}^n\}$, where $a:\mathbb{R}^n\to\mathbb{R}^{d-n}$ is an appropriate affine map, and let $p:L_D^x\to L_D$ be the projection defined by $p(y):=(\widetilde y,a(\widetilde y))$. Since Γ is a Lipschitz graph, a is well defined and p is a homeomorphism. If $p_\sharp\sigma_D^x$ is the image measure of σ_D^x under p, we obviously have $\sigma_D=p_\sharp\sigma_D^x$ because L_D and L_D^x differ from a translation. Therefore, since $\widetilde{p(y)}=\widetilde y$, (4.14) becomes

$$\begin{aligned}F4_j^m(x)=\iint &\big(K(x-p(y))-K(p(z)-p(y))-(K(x-y)-K(z-y))\big)\\ &\qquad \widetilde\varphi_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\sigma_D^x}(x,z\,;\,y)\,d\sigma_D^x(z)\,d\sigma_D^x(y).\end{aligned}$$

For $y,z\in L_D^x$ such that $\widetilde\varphi_{2^{-j-1}}^{2^{-j}}(x-y)\Lambda_m^{\sigma_D^x}(x,z\,;\,y)\neq 0$, we have $K(p(z)-p(y))-K(z-y)=0$, so we can estimate

$$\begin{aligned}|K(x-p(y))-K(p(z)-p(y))-(K(x-y)-K(z-y))| &= |K(x-p(y))-K(x-y)|\\ &\lesssim \frac{|y-p(y)|}{|x-y|^{n+1}}\lesssim 2^{j(n+1)}|y-p(y)|\approx 2^{j(n+1)}\operatorname{dist}(x,L_D).\end{aligned}$$

By the same arguments as in the proof of Claim 4.1, one can easily see that $|\Lambda_m^{\sigma_D^x}(x,z\,;\,y)|\lesssim 2^{mn}$. Therefore,

$$\begin{aligned}|F4_j^m(x)| &\lesssim 2^{j(n+1)}\operatorname{dist}(x,L_D)2^{mn}\iint_{\substack{|\widetilde x-\widetilde y|\le 2^{-j}3\sqrt{n}\\ |\widetilde x-\widetilde z|\le 2^{-m}\sqrt{n}}} d\sigma_D^x(z)\,d\sigma_D^x(y)\\ &\lesssim 2^j\operatorname{dist}(x,L_D)\approx 2^{j-m}\operatorname{dist}(x,L_D)/\ell(D),\end{aligned}$$

which gives (4.17).

4.2. **Estimate of** $\sum_{j\in\mathbb{Z}}G_j^m(x)$ **when** $x\in D\cap\Gamma$ **for some** $D\in\mathcal{D}_m$**.** Assume that $x\in D$ for some $D\in\mathcal{D}_m$. Recall from (4.3) that

$$G_j^m(x)=\iint \widetilde\varphi_{2^{-j-1}}^{2^{-j}}(z-y)\gamma_{2^{-m}}(x-y)\Lambda_m^\mu(x,z\,;\,y)K(z-y)\,d\mu(z)\,d\mu(y),$$

where $0\le\gamma_{2^{-m}}(x-y)\le 1$, $|\nabla_y\gamma_{2^{-m}}(x-y)|\lesssim 2^m$ for all $x,y\in\mathbb{R}^d$, and $\gamma_{2^{-m}}(x-y)=0$ whenever $|\widetilde x-\widetilde y|>2^{-m}3\sqrt{n}$. Notice that $\Lambda_m^\mu(x,z\,;\,y)=0$ if $|\widetilde x-\widetilde z|>2^{-m}\sqrt{n}$, thus we can assume that $|\widetilde x-\widetilde z|\le 2^{-m}\sqrt{n}$ and $|\widetilde z-\widetilde y|\le 2^{-m+2}\sqrt{n}$ in the integral that defines $G_j^m(x)$. Hence, if $j\le m-2$, $\widetilde\varphi_{2^{-j-1}}^{2^{-j}}(z-y)\Lambda_m^\mu(x,z\,;\,y)=0$ for all $z,y\in\mathbb{R}^d$, because $\widetilde\varphi_{2^{-j-1}}^{2^{-j}}(z-y)=0$

if $|\widetilde{z}-\widetilde{y}| \leq 2^{-j-1}2.1\sqrt{n}$, and $2^{-j-1}2.1\sqrt{n} \geq 2^{-m+2}\sqrt{n}$ when $j \leq m-2$. Therefore, $G_j^m(x) = 0$ for $j \leq m-2$, and then

$$\sum_{j\in\mathbb{Z}} G_j^m(x) = \sum_{j\geq m-1} G_j^m(x); \tag{4.27}$$

so, from now on, we assume that $j \geq m-1$.

Let L_D be an n-plane that minimizes $\alpha(D)$ and let $\sigma_D := c_D\mathcal{H}^n_{L_D}$ be a minimizing measure of $\alpha(D)$. As we did in the beginning of subsection 4.1, given $a \in \mathbb{R}^n$, let $b := a+\{2^{-m-1}\}^n \in \mathbb{R}^n$ be the center of $\widetilde{D}^a_m$. Recall that, for $t \in \mathbb{R}^d$, $t \in \overline{D^a_m}$ if and only if $\|\tilde{t}-b\|_\infty \leq 2^{-m}$. Using that σ_D is a Hausdorff measure on an n-plane, that K is antisymmetric and that $\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}$ and $\gamma_{2^{-m}}$ are symmetric, one can show that

$$\begin{aligned}0 = \int_{\|\widetilde{x}-b\|_\infty\leq 2^{-m}} \int_{\|\widetilde{z}-b\|_\infty\leq 2^{-m}} \int_{\|\widetilde{y}-b\|_\infty> 2^{-m}} &\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\\ &\gamma_{2^{-m}}(x-y)K(z-y)\,d\sigma_D(y)\,d\sigma_D(z)\,db.\end{aligned}$$

By the change of variable $b = a+\{2^{-m-1}\}^n$, it is easy to see that this triple integral is equal to $\int_{\{a\,:\,x\in D^a_m\}}\int_{D^a_m}\int_{(D^a_m)^c} \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\gamma_{2^{-m}}(x-y)K(z-y)\,d\sigma_D(y)\,d\sigma_D(z)\,da$. Thus, since $\sigma_D(D^a_m)$ does not depend on $a \in \mathbb{R}^n$ because σ_D is flat,

$$\begin{aligned}0 &= \int_{\{a\,:\,x\in D^a_m\}} \frac{2^{mn}}{\sigma_D(D^a_m)} \int_{D^a_m}\int_{(D^a_m)^c} \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\\ &\qquad\qquad \gamma_{2^{-m}}(x-y)K(z-y)\,d\sigma_D(y)\,d\sigma_D(z)\,da\\ &= \iint \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\gamma_{2^{-m}}(x-y)\Lambda^{\sigma_D}_m(x,z\,;\,y)K(z-y)\,d\sigma_D(z)\,d\sigma_D(y).\end{aligned} \tag{4.28}$$

Let $\{\eta_Q\}_{Q\in\mathcal{D}_j}$ be a partition of the unity with respect to the v-cubes $Q \in \mathcal{D}_j$, i.e. $\eta_Q : \mathbb{R}^d \to \mathbb{R}$ are $\mathcal{C}^\infty$ functions such that: $\chi_{0.9Q} \leq \eta_Q \leq \chi_{1.1Q}$, $|\nabla\eta_Q| \lesssim \ell(Q)^{-1} = 2^j$, $\sum_{Q\in\mathcal{D}_j}\eta_Q = 1$ and $\eta_Q(y) = \eta_Q(\widetilde{y},0)$ for all $y \in \mathbb{R}^d$. It is easy to check that, if $j \geq m-1$, $Q \in \mathcal{D}_j$, and $\operatorname{supp}\eta_Q \cap \operatorname{supp}\gamma_{2^{-m}}(x-\cdot) \neq \emptyset$, then $Q \subset C_eD$ for some absolute constant $C_e > 1$.

Given $Q \in \mathcal{D}_j$, let L_Q and $\sigma_Q := c_Q\mathcal{H}^n_{L_Q}$ be a minimizing n-plane and measure for $\alpha(Q)$, respectively, and consider the measure

$$\lambda := \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} \eta_Q\sigma_Q.$$

By (4.28) and the properties of the partition of the unity $\{\eta_Q\}_{Q\in\mathcal{D}_j}$, for $j \geq m-1$ we can decompose $G_j^m(x)$ as

$$G_j^m(x) = G1_j^m(x) + G2_j^m(x) + G3_j^m(x) + G4_j^m(x) + G5_j^m(x), \tag{4.29}$$

where

$$G1_j^m(x) := \sum_{Q\in\mathcal{D}_j\colon Q\subset C_eD} \iint \ldots\; d(\mu-\sigma_Q)(z)\,d\mu(y), \tag{4.30}$$

$$G2_j^m(x) := \sum_{Q\in\mathcal{D}_j\colon Q\subset C_eD} \iint \ldots\; d\sigma_Q(z)\,d(\mu-\sigma_Q)(y), \tag{4.31}$$

$$G3_j^m(x) := \sum_{Q\in\mathcal{D}_j\colon Q\subset C_eD} \iint \ldots\; d(\sigma_Q\times\sigma_Q-\sigma_D\times\sigma_D)(z,y), \tag{4.32}$$

where "$\dots$" stands for $\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\gamma_{2^{-m}}(x-y)\eta_Q(y)K(z-y)\Lambda^\mu_m(x,z\,;\,y)$, and

$$G4^m_j(x) := \iint \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\gamma_{2^{-m}}(x-y)K(z-y) \left(\Lambda^\mu_m(x,z\,;\,y) - \Lambda^\lambda_m(x,z\,;\,y)\right) d\sigma_D(z)\,d\sigma_D(y), \tag{4.33}$$

$$G5^m_j(x) := \iint \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\gamma_{2^{-m}}(x-y)K(z-y) \left(\Lambda^\lambda_m(x,z\,;\,y) - \Lambda^{\sigma_D}_m(x,z\,;\,y)\right) d\sigma_D(z)\,d\sigma_D(y). \tag{4.34}$$

In the next subsections we will prove the following estimates:

$$|G1^m_j(x)| + |G2^m_j(x)| + |G4^m_j(x)| \lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\alpha(Q), \tag{4.35}$$

$$|G3^m_j(x)| + |G5^m_j(x)| \lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\Big(\alpha(C_bD) + \sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD}\alpha(R)\Big), \tag{4.36}$$

where C_b is some absolute constant bigger than C_e. Then, using (4.27) and (4.29), we will finally obtain that, for all $D\in\mathcal{D}_m$ and $x\in D\cap\Gamma$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|G^m_j(x)| &\lesssim \sum_{j\geq m-1}\sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\Big(\alpha(C_bD) + \sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD}\alpha(R)\Big)\\ &\lesssim \sum_{Q\in\mathcal{D}\,:\,Q\subset C_eD}\frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}}\Big(\alpha(C_bD) + \sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD}\alpha(R)\Big)\\ &\lesssim \alpha(C_bD) + \sum_{R\in\mathcal{D}\,:\,R\subset C_bD}\frac{\ell(R)^{n+1}}{\ell(D)^{n+1}}\,\alpha(R),\end{aligned} \tag{4.37}$$

which gives (4.5).

4.2.1. **Estimate of** $G1^m_j(x)$. If $\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\neq 0$ then $2^{-j-1}2.1\sqrt{n}\leq|\widetilde{z}-\widetilde{y}|\leq 2^{-j}3\sqrt{n}$, so if we also have that $y\in\operatorname{supp}\eta_Q$, then $z\in 8\sqrt{n}Q$ because $Q\in\mathcal{D}_j$. Therefore, we can assume that $\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)\eta_Q(y)\subset B_Q$ if C_Γ is big enough.

Claim 4.4. *For $z\in\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$, the following hold: $|\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{m(n+1)-j}$, $|\nabla_z\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{m(n+1)}$, and $|\nabla_y\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{m(n+1)}$.*

We have that $|K(z-y)|\lesssim 2^{jn}$ and $|\nabla_zK(z-y)|\lesssim 2^{j(n+1)}$ for all $z\in\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$. Using (4.19) and the first two estimates in Claim 4.4, we get

$$\big|\nabla_z\big(\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)K(z-y)\Lambda^\mu_m(x,z\,;\,y)\big)\big|\lesssim 2^{m(n+1)+jn}.$$

Therefore, for $y\in\operatorname{supp}\eta_Q$,

$$\begin{aligned}\left|\int\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)K(z-y)\Lambda^\mu_m(x,z\,;\,y)\,d(\mu-\sigma_Q)(z)\right| &\lesssim 2^{m(n+1)+jn}\operatorname{dist}_{B_Q}(\mu,\sigma_Q)\\ &\lesssim 2^{m(n+1)-j}\alpha(Q),\end{aligned}$$

and then,

$$|G1^m_j(x)|\lesssim\sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD}\int_{\operatorname{supp}\eta_Q}2^{m(n+1)-j}\alpha(Q)\,d\mu(y)\lesssim\sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD}2^{(m-j)(n+1)}\alpha(Q).$$

4.2.2. **Estimate of $G2_j^m(x)$.** It can be estimated using the arguments of subsection 4.2.1, but now we also have to take into account that $|\nabla_y \gamma_{2^{-m}}(x-y)| \lesssim 2^m \lesssim 2^j$, because we are assuming $j \geq m-1$, and we have to use the last estimate in Claim 4.4.

4.2.3. **Estimate of $G3_j^m(x)$.** Given $x \in D \cap \Gamma$ and $Q \in \mathcal{D}_j$, denote

$$H_Q(y,z) := \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(z-y)\gamma_{2^{-m}}(x-y)\eta_Q(y)K(z-y)\Lambda_m^\mu(x,z\,;\,y).$$

Then, (4.32) becomes

$$\begin{aligned} G3_j^m(x) &= \sum_{Q\in\mathcal{D}_j:\, Q\subset C_eD} \iint H_Q(y,z)\, d(\sigma_Q \times \sigma_Q - \sigma_D \times \sigma_D)(z,y) \\ &= \sum_{Q\in\mathcal{D}_j:\, Q\subset C_eD} \iint H_Q(y,z)\, d(c_Q^2\, \mathcal{H}^n_{L_Q} \times \mathcal{H}^n_{L_Q} - c_D^2\, \mathcal{H}^n_{L_D} \times \mathcal{H}^n_{L_D})(z,y) \\ &= \sum_{Q\in\mathcal{D}_j:\, Q\subset C_eD} \iint H_Q(y,z)(c_Q^2 - c_D^2)\, d\mathcal{H}^n_{L_Q}(z)\, d\mathcal{H}^n_{L_Q}(y) \\ &\quad + \sum_{Q\in\mathcal{D}_j:\, Q\subset C_eD} c_D^2 \iint H_Q(y,z)\, d(\mathcal{H}^n_{L_Q} \times \mathcal{H}^n_{L_Q} - \mathcal{H}^n_{L_D} \times \mathcal{H}^n_{L_D})(z,y) \\ &=: G3A_j^m(x) + G3B_j^m(x). \end{aligned} \tag{4.38}$$

We are going to estimate the terms $G3A_j^m(x)$ and $G3B_j^m(x)$ separately. Recall that $\ell(D) = 2^{-m}$. Given a v-cube $Q \in \mathcal{D}_j$ such that $Q \subset C_eD$, let $Q =: Q_j \subset \ldots \subset Q_{i+1} \subset Q_i \subset \ldots \subset Q_{m-1}$ be the sequence of v-cubes such that Q_i belongs to $\mathcal{D}_i$ for $i = m-1, \ldots, j$. Evidently, $Q_{m-1} \subset C_bD$ for some constant C_b big enough, because $\ell(Q_{m-1}) = 2\ell(D)$ and $Q \subset Q_{m-1} \cap C_eD$. Let L_{Q_i} be an n-plane that minimizes $\alpha(Q_i)$ and let $\sigma_{Q_i} := c_{Q_i}\mathcal{H}^n_{L_{Q_i}}$ be a minimizing measure of $\alpha(Q_i)$. Also, let L_{C_bD} and $\sigma_{C_bD} := c_{C_bD}\mathcal{H}^n_{L_{C_bD}}$ be a minimizing n-plane and measure of $\alpha(C_bD)$, respectively.

In order to estimate $G3A_j^m(x)$, notice that, by [To, Lemma 3.4] and the triangle inequality, $|c_{Q_i}| \lesssim 1$ for all $i = m-1, \ldots, j$, and

$$\begin{aligned} |c_Q^2 - c_D^2| &= |c_Q + c_D|\,|c_Q - c_D| \lesssim |c_{Q_j} - c_D| \\ &\lesssim |c_{Q_{m-1}} - c_{C_bD}| + |c_{C_bD} - c_D| + \sum_{i=m-1}^{j-1} |c_{Q_{i+1}} - c_{Q_i}| \\ &\lesssim \alpha(C_bD) + \sum_{i=m-1}^{j-1} \alpha(Q_i) \lesssim \alpha(C_bD) + \sum_{R\in\mathcal{D}:\, Q\subset R\subset C_bD} \alpha(R) \end{aligned} \tag{4.39}$$

(in the case that $j = m-1$, there are no intermediate scales between j and $m-1$, so one just obtains $|c_Q^2 - c_D^2| \lesssim \alpha(C_bD)$).

Claim 4.5. *For $z \in \operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, we have $|\Lambda_m^\mu(x,z\,;\,y)| \lesssim 2^{m(n+1)-j}$.*

Notice that this last estimate is the same as the first one in Claim 4.4. Using Claim 4.5 and that $|K(z-y)| \lesssim 2^{jn}$ for all $z \in \operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, we easily obtain $|H_Q(y,z)| \lesssim$

$2^{m(n+1)+j(n-1)}$. Therefore, using (4.39),

$$\begin{aligned}|G3A_j^m(x)| &\lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} |c_Q^2-c_D^2| \iint |H_Q(y,z)|\,d\mathcal{H}^n_{L_Q}(z)\,d\mathcal{H}^n_{L_Q}(y)\\ &\lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\Big(\alpha(C_bD)+\sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD}\alpha(R)\Big).\end{aligned} \tag{4.40}$$

To estimate $G3B_j^m(x)$ in (4.38), we set

$$G3B_j^m(x) = \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} c_D^2\, G3B(Q)_j^m(x), \tag{4.41}$$

where $G3B(Q)_j^m(x) := \iint H_Q\,d(\mathcal{H}^n_{L_Q}\times\mathcal{H}^n_{L_Q} - \mathcal{H}^n_{L_D}\times\mathcal{H}^n_{L_D})$. Given $Q\in\mathcal{D}_j$ such that $Q\subset C_eD$, we split $G3B(Q)_j^m(x)$ as follows:

$$\begin{aligned}G3B(Q)_j^m(x) = &\sum_{i=m-1}^{j-1}\iint H_Q\,d(\mathcal{H}^n_{L_{Q_{i+1}}}\times\mathcal{H}^n_{L_{Q_{i+1}}} - \mathcal{H}^n_{L_{Q_i}}\times\mathcal{H}^n_{L_{Q_i}})\\ &+\iint H_Q\,d(\mathcal{H}^n_{L_{Q_{m-1}}}\times\mathcal{H}^n_{L_{Q_{m-1}}} - \mathcal{H}^n_{L_{C_bD}}\times\mathcal{H}^n_{L_{C_bD}})\\ &+\iint H_Q\,d(\mathcal{H}^n_{L_{C_bD}}\times\mathcal{H}^n_{L_{C_bD}} - \mathcal{H}^n_{L_D}\times\mathcal{H}^n_{L_D})\end{aligned} \tag{4.42}$$

(as before, in the case $j=m-1$ the first term on the right hand side of (4.42) does not exist).

Fix $i\in\mathbb{Z}$ such that $m-1\le i<j$. Set $L_{Q_{i+1}} = \{(\widetilde{y},a(\widetilde{y}))\in\mathbb{R}^d : \widetilde{y}\in\mathbb{R}^n\}$, where $a:\mathbb{R}^n\to\mathbb{R}^{d-n}$ is an appropriate affine map, and let $p:L_{Q_i}\to L_{Q_{i+1}}$ be the map defined by $p(y):=(\widetilde{y},a(\widetilde{y}))$. Let $p_\sharp\mathcal{H}^n_{L_{Q_i}}$ be the image measure of $\mathcal{H}^n_{L_{Q_i}}$ by p. It is easy to check that $\mathcal{H}^n_{L_{Q_{i+1}}} = \tau p_\sharp\mathcal{H}^n_{L_{Q_i}}$, where τ is some positive constant such that $|\tau-1|\lesssim\alpha(Q_i)$ and $\tau\lesssim 1$. Therefore,

$$\begin{aligned}\iint H_Q(y,z)\,&d(\mathcal{H}^n_{L_{Q_{i+1}}}\times\mathcal{H}^n_{L_{Q_{i+1}}} - \mathcal{H}^n_{L_{Q_i}}\times\mathcal{H}^n_{L_{Q_i}})(z,y)\\ &=\iint \Big(\tau^2 H_Q(p(y),p(z)) - H_Q(y,z)\Big)\,d\mathcal{H}^n_{L_{Q_i}}(z)\,d\mathcal{H}^n_{L_{Q_i}}(y)\\ &=\iint \tau^2\Big(H_Q(p(y),p(z)) - H_Q(y,z)\Big)\,d\mathcal{H}^n_{L_{Q_i}}(z)\,d\mathcal{H}^n_{L_{Q_i}}(y)\\ &\quad+\iint(\tau^2-1)H_Q(y,z)\,d\mathcal{H}^n_{L_{Q_i}}(z)\,d\mathcal{H}^n_{L_{Q_i}}(y).\end{aligned} \tag{4.43}$$

Since $|\tau^2-1|\lesssim\alpha(Q_j)$ and we have seen that $|H_Q(y,z)|\lesssim 2^{m(n+1)+j(n-1)}$ after Claim 4.5, the second term on the right side of the last equality is bounded by $C2^{(m-j)(n+1)}\alpha(Q_i)$.

In order to estimate the first term on the right hand side of (4.43), notice that $\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y), \gamma_{2^{-m}}(x-y), \eta_Q(y)$ and $\Lambda^\mu_m(x,z\,;\,y)$ only depend on the first n coordinates of y and z (i.e., on $\widetilde{y}$ and $\widetilde{z}$), thus their values coincide on (y,z) and $(p(y),p(z))$. Then,

$$\begin{aligned}\iint \tau^2\Big(H_Q(p(y),p(z)) - H_Q(y,z)\Big)\, d\mathcal{H}^n_{L_{Q_i}}(z)\, d\mathcal{H}^n_{L_{Q_i}}(y)\\ = \tau^2 \iint \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\gamma_{2^{-m}}(x-y)\eta_Q(y)\Lambda^\mu_m(x,z\,;\,y)\\ \Big(K(p(z)-p(y)) - K(z-y)\Big)\, d\mathcal{H}^n_{L_{Q_i}}(z)\, d\mathcal{H}^n_{L_{Q_i}}(y).\end{aligned}$$

Let θ_i be the angle between L_{Q_i} and $L_{Q_{i+1}}$. One can easily see that, for $y,z\in L_{Q_i}$, $|(p(z)-p(y))-(z-y)|\lesssim \sin(\theta_i)|z-y|\lesssim \alpha(Q_i)|z-y|$. Thus, if also $z\in \operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$,

$$|K(p(z)-p(y)) - K(z-y)| \lesssim 2^{j(n+1)}|(p(z)-p(y))-(z-y)| \lesssim 2^{jn}\alpha(Q_i).$$

Together with Claim 4.5 and the fact that $\tau^2\lesssim 1$, this gives

$$\begin{aligned}\Big|\iint \tau^2\Big(H_Q(p(y),p(z)) - H_Q(y,z)\Big)\, d\mathcal{H}^n_{L_{Q_i}}(z)\, d\mathcal{H}^n_{L_{Q_i}}(y)\Big|\\ \lesssim \iint \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\eta_Q(y)2^{m(n+1)+j(n-1)}\alpha(Q_i)\, d\mathcal{H}^n_{L_{Q_i}}(z)\, d\mathcal{H}^n_{L_{Q_i}}(y)\\ \lesssim 2^{(m-j)(n+1)}\alpha(Q_i).\end{aligned}$$

From the last estimates and (4.43), we get

$$\Big|\iint H_Q\, d(\mathcal{H}^n_{L_{Q_{i+1}}}\times\mathcal{H}^n_{L_{Q_{i+1}}} - \mathcal{H}^n_{L_{Q_i}}\times\mathcal{H}^n_{L_{Q_i}})\Big| \lesssim 2^{(m-j)(n+1)}\alpha(Q_i)$$

for $i=m-1,\ldots,j-1$. With similar arguments, one also obtains

$$\begin{aligned}\Big|\iint H_Q\, d(\mathcal{H}^n_{L_{Q_{m-1}}}\times\mathcal{H}^n_{L_{Q_{m-1}}} - \mathcal{H}^n_{L_{C_bD}}\times\mathcal{H}^n_{L_{C_bD}})\Big| \lesssim 2^{(m-j)(n+1)}\alpha(C_bD),\\ \Big|\iint H_Q\, d(\mathcal{H}^n_{L_{C_bD}}\times\mathcal{H}^n_{L_{C_bD}} - \mathcal{H}^n_{L_D}\times\mathcal{H}^n_{L_D})\Big| \lesssim 2^{(m-j)(n+1)}\alpha(C_bD).\end{aligned}$$

These last three inequalities together with (4.42), (4.41) and the fact that $|c_D|\lesssim 1$ yield

$$\begin{aligned}|G3B^m_j(x)| &\lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\Big(\alpha(C_bD)+\sum_{i=m-1}^{j-1}\alpha(Q_i)\Big)\\ &\le \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\Big(\alpha(C_bD)+\sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD}\alpha(R)\Big).\end{aligned}\tag{4.44}$$

Finally, (4.40) and (4.44) applied to (4.38) give half of (4.36).

4.2.4. **Estimate of $G4^m_j(x)$.** By Fubini's theorem and the definitions of λ, Λ^μ_m and Λ^λ_m,

$$\begin{aligned}\Lambda^\mu_m(x,z\,;\,y) - \Lambda^\lambda_m(x,z\,;\,y) &= 2^{mn}\int_{\{a:\,x,z\in D^a_m,\,y\notin D^a_m\}} \frac{\lambda(D^a_m)-\mu(D^a_m)}{\mu(D^a_m)\lambda(D^a_m)}\,da\\ &= 2^{mn}\int_{\{a:\,x,z\in D^a_m,\,y\notin D^a_m\}}\Big(\int_{t\in D^a_m}\sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD}\eta_Q(t)\,d(\sigma_Q-\mu)(t)\Big)\frac{da}{\mu(D^a_m)\lambda(D^a_m)}\\ &= \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD}\int \eta_Q(t)\Lambda^{\mu,\lambda}_m(x,z,t\,;\,y)\,d(\sigma_Q-\mu)(t).\end{aligned}$$

We also used in the second equality that $1 = \sum_{Q\in\mathcal{D}_j}\eta_Q(t) = \sum_{Q\in\mathcal{D}_j:\,Q\subset C_eD}\eta_Q(t)$ for all $t\in D_m^a$ if C_e is big enough, and this is because $j\geq m-1$ and $|\widetilde{x}-\widetilde{t}|\lesssim 2^{-m}$ for all $t\in D_m^a$.

Claim 4.6. *For $x\in D$, $j\geq m-1$, $|x-y|\lesssim 2^{-m}$, and $z\in\operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(\cdot-y)$, the following hold: $|\Lambda_m^{\mu,\lambda}(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)-j}$ and $|\nabla_t\Lambda_m^{\mu,\lambda}(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)}$.*

Using Claim 4.6 and the properties of η_Q, we obtain

$$\begin{aligned}|\Lambda_m^{\mu}(x,z\,;\,y)-\Lambda_m^{\lambda}(x,z\,;\,y)| &\lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{m(2n+1)}\operatorname{dist}_{B_Q}(\mu,\sigma_Q)\\ &\lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{m(2n+1)-j(n+1)}\alpha(Q).\end{aligned}$$

Plugging this estimate into the definition of $G4_j^m(x)$ in (4.33), we get

$$\begin{aligned}|G4_j^m(x)| &\lesssim \iint \widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(z-y)\gamma_{2^{-m}}(x-y)|K(z-y)|\\ &\qquad\qquad \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{m(2n+1)-j(n+1)}\alpha(Q)\,d\sigma_D(z)\,d\sigma_D(y)\\ &\lesssim \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\alpha(Q),\end{aligned}$$

which, together with the estimates of $|G1_j^m(x)|$ and $|G2_j^m(x)|$ in subsections 4.2.1 and 4.2.2, gives (4.35).

4.2.5. **Estimate of $G5_j^m(x)$.** Arguing as in subsection 4.2.4, we have

$$\begin{aligned}\Lambda_m^{\lambda}(x,z\,;\,y)-\Lambda_m^{\sigma_D}(x,z\,;\,y) &= \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD}\int \eta_Q(t)\Lambda_m^{\lambda,\sigma_D}(x,z,t\,;\,y)\,d(\sigma_D-\sigma_Q)(t)\\ &= \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD}\int H_Q(t)\,d(\sigma_D-\sigma_Q)(t),\end{aligned}\tag{4.45}$$

where we have set $H_Q(t) := \eta_Q(t)\Lambda_m^{\lambda,\sigma_D}(x,z,t\,;\,y)$.

We are going to estimate the right hand side of (4.45) using the techniques of subsection 4.2.3. We have

$$\int H_Q\,d(\sigma_D-\sigma_Q) = (c_D-c_Q)\int H_Q\,d\mathcal{H}^n_{L_D} + c_Q\int H_Q\,d(\mathcal{H}^n_{L_D}-\mathcal{H}^n_{L_Q}).\tag{4.46}$$

We introduce the intermediate v-cubes between $Q\in\mathcal{D}_j$ and $D\in\mathcal{D}_m$ to obtain

$$|c_D-c_Q|\lesssim \alpha(C_bD)+\sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD}\alpha(R).\tag{4.47}$$

Claim 4.7. *For $x\in D$, $j\geq m-1$, $|x-y|\lesssim 2^{-m}$, and $z\in\operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(\cdot-y)$, the following holds: $|\Lambda_m^{\lambda,\sigma_D}(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)-j}$.*

Combining Claim 4.7 with (4.47), we derive that

$$|c_D-c_Q|\int |H_Q|\,d\mathcal{H}^n_{L_D}\lesssim 2^{m(2n+1)-j(n+1)}\Bigg(\alpha(C_bD)+\sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD}\alpha(R)\Bigg).\tag{4.48}$$

For the second term on the right side of (4.46), one can also use the arguments in subsection 4.2.3 (see (4.42) and following) to show that

$$\left|\int H_Q \, d(\mathcal{H}^n_{L_D} - \mathcal{H}^n_{L_Q})\right| \lesssim 2^{m(2n+1)-j(n+1)}\Big(\alpha(C_bD) + \sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD} \alpha(R)\Big) \tag{4.49}$$

(now it is easier because the function $H_Q(t)$ only depends on the first n coordinates of the points involved, i.e., it depends only on $\widetilde{x}$, $\widetilde{y}$, $\widetilde{z}$ and $\widetilde{t}$, so when we project vertically to deal with the image measure, the function H_Q is not affected). Therefore, by (4.48), (4.49), (4.46), and (4.45), we obtain

$$|\Lambda^\lambda_m(x,z\,;\,y) \,-\, \Lambda^{\sigma_D}_m(x,z\,;\,y)| \lesssim \sum_{\substack{Q\in\mathcal{D}_j:\\ Q\subset C_eD}} 2^{m(2n+1)-j(n+1)}\Big(\alpha(C_bD) + \sum_{\substack{R\in\mathcal{D}:\\ Q\subset R\subset C_bD}} \alpha(R)\Big).$$

From the definition of $G5^m_j(x)$ in (4.34), we conclude that

$$\begin{aligned}|G5^m_j(x)| \lesssim & \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{m(2n+1)-j(n+1)}\Big(\alpha(C_bD) + \sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD} \alpha(R)\Big) \\ & \qquad \iint \widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(z-y)\gamma_{2^{-m}}(x-y)|K(z-y)|\, d\sigma_D(z)\, d\sigma_D(y) \\ \lesssim & \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_eD} 2^{(m-j)(n+1)}\Big(\alpha(C_bD) + \sum_{R\in\mathcal{D}\,:\,Q\subset R\subset C_bD} \alpha(R)\Big),\end{aligned}$$

which, together with the estimate of $|G3^m_j(x)|$ in subsection 4.2.3, gives (4.36).

4.3. **Proof of Claims 4.1**, ..., **4.7.** We have to prove:

- **Claim 4.1:** We have $|\Lambda^\mu_m(x,z\,;\,y)| \lesssim 2^{mn}$ and $|\nabla_z\Lambda^\mu_m(x,z\,;\,y)| \lesssim 2^{m(n+1)}$ for all $x,y,z\in\mathbb{R}^d$.
- **Claim 4.2:** For $j<m$, $y\in \operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(x-\cdot)$, and $|\widetilde{x}-\widetilde{z}|\le 2^{-m}\sqrt{n}$, the following hold: $|\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{mn}$ and $\nabla_y\Lambda^\mu_m(x,z\,;\,y)=0$.
- **Claim 4.3:** We have $|\Lambda^{\mu;\sigma_D}_m(x,z,t\,;\,y)|\lesssim 2^{2mn}$ and $|\nabla_t\Lambda^{\mu;\sigma_D}_m(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)}$ for all $x,y,z,t\in\mathbb{R}^d$.
- **Claim 4.4:** For $z\in\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$, the following hold: $|\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{m(n+1)-j}$, $|\nabla_z\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{m(n+1)}$, and $|\nabla_y\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{m(n+1)}$.
- **Claim 4.5:** For $z\in\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$, $|\Lambda^\mu_m(x,z\,;\,y)|\lesssim 2^{m(n+1)-j}$.
- **Claim 4.6:** For $x\in D$, $j\ge m-1$, $|x-y|\lesssim 2^{-m}$, and $z\in\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$, the following hold: $|\Lambda^{\mu;\lambda}_m(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)-j}$ and $|\nabla_t\Lambda^{\mu;\lambda}_m(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)}$.
- **Claim 4.7:** For $x\in D$, $j\ge m-1$, $|x-y|\lesssim 2^{-m}$, and $z\in\operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$, the following holds: $|\Lambda^{\lambda;\sigma_D}_m(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)-j}$.

To prove the claims, we need to express the function Λ at the end of subsection 2.3 in a more convenient way. Notice that we can replace D^a_m by $\overline{D^a_m}$ in the definition of Λ because μ and the n-dimensional Hausdorff measure vanish on ∂D^a_m.

For $u\in\mathbb{R}^n$ and $r>0$, we denote $|u|_\infty := \max_{i=1,\dots,n}|u^i|$, $B_\infty(u,r) := \{v\in\mathbb{R}^n : |u-v|_\infty\le r\}$, and $B^m_\infty(u) := B_\infty(u,2^{-m-1})$. Given $a\in\mathbb{R}^n$, let $b := a+\{2^{-m-1}\}^n\in\mathbb{R}^n$ be the center of $\widetilde{D}^a_m$. Then, given $q\in\mathbb{R}^d$,

$$q\in\overline{D^a_m} \iff |\tilde{q}-b|_\infty\le 2^{-m} \iff b\in B^m_\infty(\widetilde{q}).$$

Let $\mu_1, \ldots, \mu_k$ be positive Borel measures such that $\mu_l(D_m^a) > 0$ and $\mu_l(\partial D_m^a) = 0$ for all $a \in \mathbb{R}^n$, $m \in \mathbb{Z}$ and $l = 1, \ldots, k$. Given $m \in \mathbb{Z}$ and $x_1, \ldots, x_i, y_1, \ldots, y_j \in \mathbb{R}^d$ we have, by the change of variable $b = a + \{2^{-m-1}\}^n \in \mathbb{R}^n$,

$$\begin{aligned}\Lambda_m^{\mu_1,\ldots,\mu_k}(x_1, \ldots, x_i\,;\, y_1, \ldots, y_j) &= \int_{\{a\in\mathbb{R}^n\,:\,x_1,\ldots,x_i\in\overline{D_m^a},\,y_1,\ldots,y_j\notin\overline{D_m^a}\}} \frac{2^{nm}\,da}{\prod_{l=1}^k \mu_l(D_m^a)}\\ &= 2^{nm}\int \frac{\chi_{B_\infty^m(\widetilde{x}_1)\cap\ldots\cap B_\infty^m(\widetilde{x}_i)\cap B_\infty^m(\widetilde{y}_1)^c\cap\ldots\cap B_\infty^m(\widetilde{y}_j)^c}(b)}{\prod_{l=1}^k \mu_l\big(D_m^{b-\{2^{-m-1}\}^n}\big)}\,db.\end{aligned} \tag{4.50}$$

Proof of Claim 4.1. By (4.50), we have

$$\Lambda_m^\mu(x, z\,;\, y) = 2^{nm}\int \mu\big(D_m^{b-\{2^{-m-1}\}^n}\big)^{-1}\chi_{B_\infty^m(\widetilde{x})\cap B_\infty^m(\widetilde{z})\cap B_\infty^m(\widetilde{y})^c}(b)\,db. \tag{4.51}$$

Since $\mu(D_m^b) \gtrsim 2^{-mn}$ for all $b \in \mathbb{R}^n$,

$$|\Lambda_m^\mu(x, z\,;\, y)| \lesssim 2^{2mn}\mathcal{L}^n(B_\infty^m(\widetilde{x}) \cap B_\infty^m(\widetilde{z}) \cap B_\infty^m(\widetilde{y})^c) \le 2^{2mn}\mathcal{L}^n(B_\infty^m(\widetilde{x})) \le 2^{mn}.$$

To deal with the second inequality in Claim 4.1, we will estimate

$$|\Lambda_m^\mu(x, z_1\,;\, y) - \Lambda_m^\mu(x, z_2\,;\, y)|/|z_1 - z_2|$$

for z_1 close enough to z_2. Recall that, given two sets $F_1, F_2 \subset \mathbb{R}^n$, $F_1\Delta F_2 := (F_1 \setminus F_2)\cup(F_2 \setminus F_1)$ denotes their symmetric difference. Using (4.51), we get

$$\begin{aligned}|\Lambda_m^\mu(x,&z_1\,;\, y) - \Lambda_m^\mu(x, z_2\,;\, y)|\\ &\lesssim 2^{2nm}\int |\chi_{B_\infty^m(\widetilde{x})\cap B_\infty^m(\widetilde{z}_1)\cap B_\infty^m(\widetilde{y})^c}(b) - \chi_{B_\infty^m(\widetilde{x})\cap B_\infty^m(\widetilde{z}_2)\cap B_\infty^m(\widetilde{y})^c}(b)|\,db\\ &= 2^{2nm}\mathcal{L}^n\Big(\big(B_\infty^m(\widetilde{x}) \cap B_\infty^m(\widetilde{z}_1) \cap B_\infty^m(\widetilde{y})^c\big)\Delta\big(B_\infty^m(\widetilde{x}) \cap B_\infty^m(\widetilde{z}_2) \cap B_\infty^m(\widetilde{y})^c\big)\Big)\\ &\le 2^{2nm}\mathcal{L}^n\big(B_\infty^m(\widetilde{z}_1)\Delta B_\infty^m(\widetilde{z}_2)\big) \lesssim 2^{2nm}|\widetilde{z}_1 - \widetilde{z}_2|2^{-m(n-1)} \le 2^{m(n+1)}|z_1 - z_2|,\end{aligned} \tag{4.52}$$

and the claim follows. □

Proof of Claim 4.2. The first estimate has been already proved in Claim 4.1. Let us deal with the second one. Notice that if $y \in \operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x - \cdot)$ then $|\widetilde{x} - \widetilde{y}| \ge 2^{-j-1}2.1\sqrt{n}$. Thus, if also $j < m$ and $|\widetilde{x} - \widetilde{z}| \le 2^{-m}\sqrt{n}$, then $|\widetilde{x} - \widetilde{y}| > 2^{-m}\sqrt{n}$ and $|\widetilde{z} - \widetilde{y}| > 2^{-m}\sqrt{n}$. Therefore, $B_\infty^m(\widetilde{x})\cap B_\infty^m(\widetilde{z})\cap B_\infty^m(\widetilde{y})^c = B_\infty^m(\widetilde{x})\cap B_\infty^m(\widetilde{z})$ for all $y \in \operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x-\cdot)$, if $|\widetilde{x}-\widetilde{z}| \le 2^{-m}\sqrt{n}$. This means, using (4.51), that $\Lambda_m^\mu(x, z\,;\, y)$ does not depend on y, so $\nabla_y\Lambda_m^\mu(x, z\,;\, y) = 0$ for all $y \in \operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(x - \cdot)$, and the claim is proved. □

Proof of Claim 4.3. This claim follows by arguments very similar to the ones in the proof of Claim 4.1. Just notice that $\mu(D_m^b)\sigma_D(D_m^b) \gtrsim 2^{-2mn}$ for all $b \in \mathbb{R}^n$. □

Proof of Claim 4.4. Using (4.51), we have that

$$|\Lambda_m^\mu(x, z\,;\, y)| \lesssim 2^{2mn}\mathcal{L}^n(B_\infty^m(\widetilde{x}) \cap B_\infty^m(\widetilde{z}) \cap B_\infty^m(\widetilde{y})^c) \le 2^{2mn}\mathcal{L}^n(B_\infty^m(\widetilde{z}) \cap B_\infty^m(\widetilde{y})^c).$$

Notice that $\mathcal{L}^n(B_\infty^m(\widetilde{z}) \cap B_\infty^m(\widetilde{y})^c) \lesssim 2^{-m(n-1)}|\widetilde{y} - \widetilde{z}|$. Since $z \in \operatorname{supp}\widetilde{\varphi}_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, $|\widetilde{y} - \widetilde{z}| \le 2^{-j}3\sqrt{n}$. Thus, $\mathcal{L}^n(B_\infty^m(\widetilde{z}) \cap B_\infty^m(\widetilde{y})^c) \lesssim 2^{-m(n-1)-j}$, and then $|\Lambda_m^\mu(x, z\,;\, y)| \lesssim 2^{m(n+1)-j}$.

In Claim 4.1 we already proved that $|\nabla_z\Lambda_m^\mu(x, z\,;\, y)| \lesssim 2^{m(n+1)}$. Finally, to prove that $|\nabla_y\Lambda_m^\mu(x, z\,;\, y)| \lesssim 2^{m(n+1)}$, one can repeat the computations done in (4.52) but applied to the y coordinate and use that $B_\infty^m(\widetilde{y}_1)^c\Delta B_\infty^m(\widetilde{y}_2)^c = B_\infty^m(\widetilde{y}_1)\Delta B_\infty^m(\widetilde{y}_2)$. □

Proof of Claim 4.5. This claim is included in the previous one. □

Proof of Claim 4.6. Recall that $\lambda = \sum_{Q\in\mathcal{D}_j\,:\,Q\subset C_e D}\eta_Q\sigma_Q$, where C_e is some fixed constant big enough (see the beginning of subsection 4.2). Using the properties of η_Q and that C_e is big enough, it is not difficult to show that $\lambda\big(D_m^{b-\{2^{-m-1}\}^n}\big)\gtrsim 2^{-mn}$ for all $b\in\mathbb{R}^n$ such that $b\in B^m_\infty(\widetilde{x})$ (recall that $x\in D$ and $j\geq m-1$). Therefore, by (4.50),

$$\begin{aligned}|\Lambda_m^{\mu,\lambda}(x,z,t\,;\,y)| &\lesssim 2^{3nm}\mathcal{L}^n\big(B^m_\infty(\widetilde{x})\cap B^m_\infty(\widetilde{z})\cap B^m_\infty(\widetilde{t})\cap B^m_\infty(\widetilde{y})^c\big)\\ &\leq 2^{3nm}\mathcal{L}^n\big(B^m_\infty(\widetilde{z})\cap B^m_\infty(\widetilde{y})^c\big).\end{aligned}$$

As in the proof of Claim 4.4, we have $\mathcal{L}^n\big(B^m_\infty(\widetilde{z})\cap B^m_\infty(\widetilde{y})^c\big)\lesssim 2^{-m(n-1)-j}$ for all $z\in \operatorname{supp}\widetilde{\varphi}^{2^{-j}}_{2^{-j-1}}(\cdot-y)$. Thus, $|\Lambda_m^{\mu,\lambda}(x,z,t\,;\,y)|\lesssim 2^{m(2n+1)-j}$, as wished.

For the second estimate in Claim 4.6, we argue as in (4.52). For t_1 and t_2 close enough,

$$\begin{aligned}&|\Lambda_m^{\mu,\lambda}(x,z,t_1\,;\,y)-\Lambda_m^{\mu,\lambda}(x,z,t_2\,;\,y)|\\ &\qquad\lesssim 2^{3nm}\int|\chi_{B^m_\infty(\widetilde{x})\cap B^m_\infty(\widetilde{z})\cap B^m_\infty(\widetilde{t_1})\cap B^m_\infty(\widetilde{y})^c}(b)-\chi_{B^m_\infty(\widetilde{x})\cap B^m_\infty(\widetilde{z})\cap B^m_\infty(\widetilde{t_2})\cap B^m_\infty(\widetilde{y})^c}(b)|\,db\\ &\qquad\leq 2^{3nm}\mathcal{L}^n\big(B^m_\infty(\widetilde{t_1})\Delta B^m_\infty(\widetilde{t_2})\big)\lesssim 2^{3nm}|\widetilde{t_1}-\widetilde{t_2}|2^{-m(n-1)}\leq 2^{m(2n+1)}|t_1-t_2|,\end{aligned}$$

and the claim follows by letting $t_1\to t_2$. □

Proof of Claim 4.7. This claim is proved as the first estimate in Claim 4.6, replacing μ by σ_D (we only used that $\mu(D^b_m)\gtrsim 2^{-mn}$ for all $b\in\mathbb{R}^n$, which also holds for σ_D). □

5. Proof of Theorem 3.2

Given $x\in\Gamma$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers such that

$$S\mu(x)^2\leq 2\sum_{j\in\mathbb{Z}}\;\sum_{m\in\mathbb{Z}\colon\epsilon_m,\epsilon_{m+1}\in I_j}|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*\mu)(x)|^2, \tag{5.1}$$

so $\{\epsilon_m\}_{m\in\mathbb{Z}}$ depends on x.

Fix $j\in\mathbb{Z}$ and assume that $x\in D$, for some $D\in\mathcal{D}_j$. Let L_D be an n-plane that minimizes $\alpha(D)$ and let $\sigma_D:=c_D\mathcal{H}^n_{L_D}$ be a minimizing measure for $\alpha(D)$. Let L^x_D be the n-plane parallel to L_D which contains x, and set $\sigma^x_D:=c_D\mathcal{H}^n_{L^x_D}$.

By the antisymmetry of the function $\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}K$, and since σ^x_D is a Hausdorff measure on the n-plane L^x_D and $x\in L^x_D$, we have

$$(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*\sigma^x_D)(x)=\int\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x-y)K(x-y)\,d\sigma^x_D(y)=0$$

for all $m\in\mathbb{Z}$. Therefore, we can decompose

$$(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*\mu)(x)=(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(\mu-\sigma_D))(x)+(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(\sigma_D-\sigma^x_D))(x). \tag{5.2}$$

For every $m\in\mathbb{Z}$ such that $\epsilon_m,\epsilon_{m+1}\in I_j$ we will prove the following inequalities:

$$|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(\mu-\sigma_D))(x)|\lesssim 2^j|\epsilon_m-\epsilon_{m+1}|\alpha(D), \tag{5.3}$$

$$|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(\sigma_D-\sigma^x_D))(x)|\lesssim 2^{2j}|\epsilon_m-\epsilon_{m+1}|\operatorname{dist}(x,L_D). \tag{5.4}$$

Assume for a moment that these estimates hold. Then, by (5.2),

$$|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*\mu)(x)|\lesssim 2^j|\epsilon_m-\epsilon_{m+1}|\left(\alpha(D)+2^j\operatorname{dist}(x,L_D)\right).$$

Then, using (5.1), we conclude that

$$\begin{aligned}\|S\mu\|_{L^2(\mu)}^2 &\leq 2\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D \sum_{m\in\mathbb{Z}:\,\epsilon_m,\epsilon_{m+1}\in I_j} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}*\mu)(x)|^2\,d\mu(x)\\ &\lesssim \sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\left(\alpha(D)+\frac{\operatorname{dist}(x,L_D)}{2^{-j}}\right)^2 \sum_{\substack{m\in\mathbb{Z}:\\ \epsilon_m,\epsilon_{m+1}\in I_j}}\left(\frac{|\epsilon_m-\epsilon_{m+1}|}{2^{-j}}\right)^2 d\mu(x)\\ &\lesssim \sum_{D\in\mathcal{D}}\alpha(D)^2\mu(D)+\sum_{D\in\mathcal{D}}\int_D\left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2 d\mu(x).\end{aligned}$$

The second term on the right hand side of the last inequality coincides with $W_1\mu$ (see (4.6)), thus it is bounded (modulo constants) by $\sum_{D\in\mathcal{D}}\big(\alpha(D)^2+\beta_2(D)^2\big)\mu(D)$, and Theorem 3.2 is proved.

It only remains to verify (5.3) and (5.4) for $x\in D\in\mathcal{D}_j$ and $m\in\mathbb{Z}$ such that $\epsilon_m,\epsilon_{m+1}\in I_j$. First of all, notice that $\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}$ satisfies

$$\begin{aligned}|\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)| &= \left|\varphi_{\mathbb{R}}\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_{m+1}}\right)-\varphi_{\mathbb{R}}\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_m}\right)\right| \leq \|\varphi_{\mathbb{R}}'\|_{L^\infty(\mathbb{R})}\left|\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_{m+1}}-\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_m}\right|\\ &= \|\varphi_{\mathbb{R}}'\|_\infty|\widetilde{x}-\widetilde{y}|\,\frac{\epsilon_m-\epsilon_{m+1}}{\epsilon_m\epsilon_{m+1}} \lesssim 2^j|\epsilon_m-\epsilon_{m+1}|\end{aligned}\tag{5.5}$$

for all $y\in\operatorname{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)$. For $i=1,\ldots,d$,

$$\partial_{y^i}\left(\widetilde{\varphi}_{\epsilon_m}(x-y)\right)=\varphi_{\mathbb{R}}'\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_m}\right)\frac{y^i-x^i}{\epsilon_m|\widetilde{x}-\widetilde{y}|}\,\chi_{[1,n]}(i).$$

Hence,

$$\begin{aligned}\left|\partial_{y^i}\left(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\right)\right| &\leq \left|\varphi_{\mathbb{R}}'\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_m}\right)\frac{1}{\epsilon_m}-\varphi_{\mathbb{R}}'\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_{m+1}}\right)\frac{1}{\epsilon_{m+1}}\right|\\ &\leq \left|\varphi_{\mathbb{R}}'\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_m}\right)\right|\left|\frac{1}{\epsilon_m}-\frac{1}{\epsilon_{m+1}}\right|+\left|\varphi_{\mathbb{R}}'\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_m}\right)-\varphi_{\mathbb{R}}'\left(\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_{m+1}}\right)\right|\frac{1}{\epsilon_{m+1}}\\ &\leq \left(\|\varphi_{\mathbb{R}}'\|_\infty+\|\varphi_{\mathbb{R}}''\|_\infty\frac{|\widetilde{x}-\widetilde{y}|}{\epsilon_{m+1}}\right)\frac{\epsilon_m-\epsilon_{m+1}}{\epsilon_m\epsilon_{m+1}}.\end{aligned}$$

Since $\epsilon_m,\epsilon_{m+1}\in I_j$, we deduce from the previous estimate that, for $x-y\in\operatorname{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}$,

$$|\nabla_y\left(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\right)|\lesssim\frac{\epsilon_m-\epsilon_{m+1}}{\epsilon_m\epsilon_{m+1}}\approx 2^{2j}|\epsilon_m-\epsilon_{m+1}|.\tag{5.6}$$

We are going to use (5.5) and (5.6) to prove (5.3) and (5.4). Let us start with (5.3). Since $\epsilon_m,\epsilon_{m+1}\in I_j$, we can assume that $\operatorname{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)\subset B_D$, by taking C_Γ big enough.

By (5.5) and (5.6), for all $y\in\operatorname{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)$,

$$\left|\nabla_y\left(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)K(x-y)\right)\right|\lesssim 2^{j(n+2)}|\epsilon_m-\epsilon_{m+1}|,$$

hence

$$|(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}*(\mu-\sigma_D))(x)|\lesssim 2^{j(n+2)}|\epsilon_m-\epsilon_{m+1}|\operatorname{dist}_{B_D}(\mu,\sigma_D)\lesssim 2^j|\epsilon_m-\epsilon_{m+1}|\alpha(D),$$

which gives (5.3).

In order to prove (5.4), set $L_D^x=\{(\widetilde{t},a(\widetilde{t}))\in\mathbb{R}^d\,:\,\widetilde{t}\in\mathbb{R}^n\}$, where $a:\mathbb{R}^n\to\mathbb{R}^{d-n}$ is an appropriate affine map, and let $p:L_D\to L_D^x$ be the map defined by $p(t):=(\widetilde{t},a(\widetilde{t}))$. Since

Γ is a Lipschitz graph, a is well defined and p is a homeomorphism. Let $p_\sharp\mathcal{H}^n_{L_D}$ be the image measure of $\mathcal{H}^n_{L_D}$ by p. It is easy to see that, $|y-p(y)|\approx \operatorname{dist}(x,L_D)$ for all $y\in L_D$. Notice also that the image measure $p_\sharp\mathcal{H}^n_{L_D}$ coincides with $\mathcal{H}^n_{L_D^x}$. Therefore, since $\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x-y)$ only depends on $\widetilde{x}-\widetilde{y}$,

$$
\begin{aligned}
(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(\sigma_D-\sigma_D^x))(x) &= c_D\int \widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x-y)K(x-y)\,d(\mathcal{H}^n_{L_D}-p_\sharp\mathcal{H}^n_{L_D})(y)\\
&= c_D\int \widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x-y)(K(x-y)-K(x-p(y)))\,d\mathcal{H}^n_{L_D}(y).
\end{aligned}
\tag{5.7}
$$

For $y\in\operatorname{supp}\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x-\cdot)\cap L_D$, we have

$$|K(x-y)-K(x-p(y))|\lesssim 2^{j(n+1)}|y-p(y)|\approx 2^{j(n+1)}\operatorname{dist}(x,L_D).$$

Plugging this estimate and (5.5) into (5.7), we conclude that

$$|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(\sigma_D-\sigma_D^x))(x)|\lesssim 2^{2j}|\epsilon_m-\epsilon_{m+1}|\operatorname{dist}(x,L_D),$$

which gives (5.4); and the theorem follows.

6. L^2 localization of $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}}$ and $\mathcal{O}\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}}$

From here till the end of the paper, $\Gamma:=\{x\in\mathbb{R}^d\,:\,x=(\widetilde{x},A(\widetilde{x}))\}$ will be the graph of a Lipschitz function $A:\mathbb{R}^n\to\mathbb{R}^{d-n}$, without any assumption on the support of A.

Theorem 6.1. *Let $\rho>2$. There exist $C_1,C_2>0$ such that, for every $f\in L^\infty(\mathcal{H}^n_\Gamma)$ supported in $\Gamma\cap D$ (where $D:=\widetilde{D}\times\mathbb{R}^{d-n}$ and $\widetilde{D}$ is a cube of $\mathbb{R}^n$),*

$$\int_D\big((\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})f\big)^2\,d\mathcal{H}^n_\Gamma \;\le\; C_1\|f\|^2_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(D)\quad\text{and}\tag{6.1}$$

$$\int_D\big((\mathcal{O}\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})f\big)^2\,d\mathcal{H}^n_\Gamma \;\le\; C_2\|f\|^2_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(D).\tag{6.2}$$

The constant C_2 does not depend on the fixed sequence that defines $\mathcal{O}$.

We will only give the proof of (6.1), because the proof of (6.2) follows by very similar arguments and computations.

We claim that it is enough to prove (6.1) for all functions f such that $f(x)\approx 1$ for all $x\in\Gamma\cap D$. Otherwise, we consider $g(x):=\|f\|^{-1}_{L^\infty(\mathcal{H}^n_\Gamma)}f(x)+2\chi_D(x)$, which clearly satisfies $g(x)\approx 1$ for all $x\in\Gamma\cap D$. Since $f=\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}(g-2\chi_D)$,

$$(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})f(x)\le\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\big((\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})g(x)+2(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})\chi_D(x)\big).$$

Applying (6.1) to the functions g and χ_D, we finally get

$$\int_D\big((\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})f\big)^2\,d\mathcal{H}^n_\Gamma\lesssim\|f\|^2_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(D).$$

Given f and D as in Theorem 6.1, from now on, we assume that $f\approx 1$ in $\Gamma\cap D$. Let $\widetilde{z}_D$ be the center of $\widetilde{D}$ and set $z_D:=(\widetilde{z}_D,A(\widetilde{z}_D))$. One can easily construct a Lipschitz function $A_D:\mathbb{R}^n\to\mathbb{R}^{d-n}$ such that $\operatorname{Lip}(A_D)\lesssim\operatorname{Lip}(A)$, $A_D(\widetilde{x})=A(\widetilde{z}_D)$ for all $\widetilde{x}\in(3\widetilde{D})^c$, and $A_D(\widetilde{x})=A(\widetilde{x})$ for all $\widetilde{x}\in\widetilde{D}$. Let Γ_D be the graph of A_D and define the measure $\mu:=\mathcal{H}^n_{\Gamma_D\setminus D}+f\mathcal{H}^n_{\Gamma_D\cap D}$. Notice that $\chi_{(3D)^c}\mu$ is supported in the n-plane $L:=\mathbb{R}^n\times\{A(\widetilde{z}_D)\}$ and $\chi_D\mu=f\mathcal{H}^n_\Gamma$.

Since $f \approx 1$ in $\Gamma \cap D$ and $\chi_D\mu = (1 - \chi_{(3D)^c} - \chi_{3D\setminus D})\mu$, we can decompose

$$\int_D \big((\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})f\big)^2 \, d\mathcal{H}^n_\Gamma \approx \int_D \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_D\mu))^2 \, d\mu$$
$$\lesssim \int_D \big(\mathcal{V}_\rho(K\widetilde{\varphi} * \mu) + \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{(3D)^c}\mu)) + \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{3D\setminus D}\mu))\big)^2 \, d\mu.$$

In the next subsections, we will see that $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * \mu)^2 \, d\mu$, $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{(3D)^c}\mu))^2 \, d\mu$, and $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{3D\setminus D}\mu))^2 \, d\mu$ are bounded by $C\mu(D)$, and (6.1) will follow.

6.1. **Proof of $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * \mu)^2 \, d\mu \lesssim \mu(D)$.** Fix $x \in \operatorname{supp}\mu$, and let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$(\mathcal{V}_\rho(K\widetilde{\varphi} * \mu)(x))^\rho \leq 2 \sum_{m\in\mathbb{Z}} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^\rho. \tag{6.3}$$

For $j \in \mathbb{Z}$ we set $I_j := [2^{-j-1}, 2^{-j})$. We decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$, where

$$\begin{aligned} &\mathcal{S} := \bigcup_{j\in\mathbb{Z}} \mathcal{S}_j, \quad \mathcal{S}_j := \{m \in \mathbb{Z} \,:\, \epsilon_m, \epsilon_{m+1} \in I_j\}, \\ &\mathcal{L} := \{m \in \mathbb{Z} \,:\, \epsilon_m \in I_i,\, \epsilon_{m+1} \in I_j \text{ for } i < j\}. \end{aligned} \tag{6.4}$$

Then, $\sum_{m\in\mathbb{Z}} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^\rho = \sum_{m\in\mathcal{S}} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^\rho + \sum_{m\in\mathcal{L}} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^\rho$.

Notice that, since the $\ell^\rho(\mathbb{Z})$-norm is smaller than the $\ell^2(\mathbb{Z})$-norm for $\rho > 2$,

$$\sum_{m\in\mathcal{S}} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^\rho \leq S\mu(x)^\rho, \tag{6.5}$$

where $S\mu(x)$ has been defined in Theorem 3.2.

Let us now estimate the sum over the indices $m \in \mathcal{L}$. For $m \in \mathbb{Z}$ we define $j(\epsilon_m)$ as the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Since $\{\epsilon_m\}_{m\in\mathbb{Z}}$ is decreasing, given $j \in \mathbb{Z}$, there is at most one index $m \in \mathcal{L}$ such that $\epsilon_m \in I_j$. Thus, if $k, m \in \mathcal{L}$ and $k < m$, one has $j(\epsilon_k) < j(\epsilon_m)$.

With this notation, we have

$$\begin{aligned} \sum_{m\in\mathcal{L}} |(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^\rho &= \sum_{m\in\mathcal{L}} |(K\widetilde{\varphi}_{\epsilon_{m+1}} * \mu)(x) - (K\widetilde{\varphi}_{\epsilon_m} * \mu)(x)|^\rho \\ &\lesssim \sum_{m\in\mathcal{L}} |(K\widetilde{\varphi}_{\epsilon_{m+1}} * \mu)(x) - (K\widetilde{\varphi}_{2^{-j(\epsilon_{m+1})-1}} * \mu)(x)|^\rho \\ &\quad + \sum_{m\in\mathcal{L}} |(K\widetilde{\varphi}_{2^{-j(\epsilon_{m+1})-1}} * \mu)(x) - E_{j(\epsilon_{m+1})+1}\mu(x)|^\rho \\ &\quad + \sum_{m\in\mathcal{L}} |E_{j(\epsilon_{m+1})+1}\mu(x) - E_{j(\epsilon_m)+1}\mu(x)|^\rho \\ &\quad + \sum_{m\in\mathcal{L}} |E_{j(\epsilon_m)+1}\mu(x) - (K\widetilde{\varphi}_{2^{-j(\epsilon_m)-1}} * \mu)(x)|^\rho \\ &\quad + \sum_{m\in\mathcal{L}} |(K\widetilde{\varphi}_{2^{-j(\epsilon_m)-1}} * \mu)(x) - (K\widetilde{\varphi}_{\epsilon_m} * \mu)(x)|^\rho \\ &\lesssim S\mu(x)^\rho + W\mu(x)^\rho + \mathcal{V}_\rho(E\mu)(x)^\rho, \end{aligned} \tag{6.6}$$

where $S\mu(x)$ and $W\mu(x)$ have been defined in Theorems 3.2 and 3.1, respectively, and $\mathcal{V}_\rho(E\mu)$ is the ρ-variation of the average of martingales $\{E_m\mu\}_{m\in\mathbb{Z}}$ from subsection 2.3. Therefore, by (6.5), (6.6), and (6.3), we deduce that

$$\mathcal{V}_\rho(K\widetilde{\varphi} * \mu)(x) \lesssim S\mu(x) + W\mu(x) + \mathcal{V}_\rho(E\mu)(x)$$

for all $x \in \operatorname{supp}\mu$, and so

$$\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * \mu)^2 \, d\mu \lesssim \|S\mu\|^2_{L^2(\mu)} + \|W\mu\|^2_{L^2(\mu)} + \|\mathcal{V}_\rho(E\mu)\|^2_{L^2(\mu)}. \tag{6.7}$$

Clearly, Theorem 2.7, Theorem 3.1, and Theorem 3.2 can be applied to the measure μ, because $\operatorname{supp}\mu$ is a translation of the graph of a Lipschitz function with compact support. These theorems in combination with (6.7) yield

$$\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * \mu)^2 \, d\mu \leq C_1\bigg(\mu(3D) + \sum_{Q\in\mathcal{D}} \big(\alpha_\mu(C_2Q)^2 + \beta_{2,\mu}(Q)^2\big)\mu(Q)\bigg), \tag{6.8}$$

where $C_1, C_2 > 0$ only depend on n, d, K, $\operatorname{Lip}(A)$, and ρ (the condition $\rho > 2$ is used to ensure the L^2 boundedness of $\mathcal{V}_\rho(E\mu)$). Obviously, $\mu(3D) \approx \mu(D)$ and, since $\chi_{(3D)^c}\mu$ coincides with the n-dimensional Hausdorff measure on an n-plane, using Remark 2.2 it is easy to check that $\sum_{Q\in\mathcal{D}} \big(\alpha_\mu(C_2Q)^2 + \beta_{2,\mu}(Q)^2\big)\mu(Q) \lesssim \mu(3D)$. Hence, we conclude that $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * \mu)^2 \, d\mu \lesssim \mu(D)$ by (6.8).

6.2. **Proof of $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{(3D)^c}\mu))^2 \, d\mu \lesssim \mu(D)$.** Fix $x \in \operatorname{supp}\mu \cap D$, and let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$\big(\mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{(3D)^c}\mu))(x)\big)^\rho \leq 2 \sum_{m\in\mathbb{Z}} |(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{(3D)^c}\mu))(x)|^\rho. \tag{6.9}$$

Recall that $\widetilde{z}_D$ is the center of $\widetilde{D}$, $z_D := (\widetilde{z}_D, A(\widetilde{z}_D))$ and $L := \mathbb{R}^n \times \{A(\widetilde{z}_D)\}$. Since $\chi_{(3D)^c}\mu = \mathcal{H}^n_{L\setminus 3D}$ and z_D is the center of $L \cap D$, $(K\widetilde{\varphi}^\delta_\epsilon * (\chi_{(3D)^c}\mu))(z_D) = 0$ for all $0 < \epsilon \leq \delta$. Thus, $|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{(3D)^c}\mu))(x)| = |(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{(3D)^c}\mu))(x) - (K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{(3D)^c}\mu))(z_D)| \leq \Theta 1_m + \Theta 2_m$, where

$$\begin{aligned} \Theta 1_m &:= \int_{(3D)^c} \widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x-y)|K(x-y) - K(z_D - y)| \, d\mu(y), \\ \Theta 2_m &:= \int_{(3D)^c} |\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x-y) - \widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(z_D - y)| K(z_D - y) \, d\mu(y). \end{aligned} \tag{6.10}$$

Since $x \in \operatorname{supp}\mu \cap D$ and A is a Lipschitz function, we have $|x - z_D| \lesssim \ell(D)$, and then $|K(x-y) - K(z_D - y)| \lesssim |x - z_D||z_D - y|^{-n-1} \lesssim \ell(D)|z_D - y|^{-n-1}$ for all $y \in (3D)^c$. Therefore, using that $\sum_{m\in\mathbb{Z}} \widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} \leq 1$ and that $\rho > 1$,

$$\bigg(\sum_{m\in\mathbb{Z}} \Theta 1_m^\rho\bigg)^{1/\rho} \leq \sum_{m\in\mathbb{Z}} \Theta 1_m \lesssim \int_{(3D)^c} \ell(D)|z_D - y|^{-n-1} \, d\mu(y) \lesssim 1. \tag{6.11}$$

To deal with $\Theta 2_m$, we decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$ as in (6.4). As before, given $m \in \mathbb{Z}$, let $j(\epsilon_m)$ be the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Observe that

$$\operatorname{supp} \widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}(x - \cdot) \subset A(\widetilde{x}, 2.1\sqrt{n}2^{-j(\epsilon_{m+1})-1}, 3\sqrt{n}2^{-j(\epsilon_m)}) \times \mathbb{R}^{d-n} =: A_m(x).$$

Notice also that the sets $A_m(x)$ have finite overlap for $m \in \mathcal{L}$, and the same is true for the sets $A'_j(x) := A(\widetilde{x}, 2.1\sqrt{n}2^{-j-1}, 3\sqrt{n}2^{-j}) \times \mathbb{R}^{d-n}$ for $j \in \mathbb{Z}$. The same observations hold if we replace x by z_D (and $\widetilde{x}$ by $\widetilde{z}_D$). Obviously, $A_m(x) \subset A'_j(x)$ (and $A_m(z_D) \subset A'_j(z_D)$) for $m \in \mathcal{S}_j$.

Assume that $m \in \mathcal{S}$. With the same computations as those carried out in (5.6), one can easily prove that, for all $z - y \in \operatorname{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}$,

$$|\nabla_z(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(z-y))| \lesssim \left(\|\varphi_{\mathbb{R}}'\|_{L^\infty(\mathbb{R})} + \|\varphi_{\mathbb{R}}''\|_{L^\infty(\mathbb{R})} \frac{|\widetilde{z}-\widetilde{y}|}{\epsilon_{m+1}}\right) \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}} \lesssim 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|\widetilde{z}-\widetilde{y}|},$$

because $|\widetilde{z}-\widetilde{y}| \approx \epsilon_m \approx \epsilon_{m+1} \approx 2^{j(\epsilon_m)}$ for all $z-y \in \operatorname{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}$ and $m \in \mathcal{S}$. In particular, if $z \in D$ and $y \in (3D)^c$, $|\nabla_z(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(z-y))| \lesssim 2^{j(\epsilon_m)}|\epsilon_m - \epsilon_{m+1}||\widetilde{z}_D - \widetilde{y}|^{-1}$. Hence,

$$\Theta 2_m \lesssim \int_{(A_m(x)\cup A_m(z_D))\setminus 3D} \ell(D) 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|\widetilde{z}_D - \widetilde{y}|^{n+1}}\, d\mu(y),$$

and then,

$$\begin{aligned}\Big(\sum_{m\in\mathcal{S}} \Theta 2_m^\rho\Big)^{1/\rho} &\lesssim \sum_{m\in\mathcal{S}} \int_{(A_m(x)\cup A_m(z_D))\setminus 3D} \ell(D) 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|\widetilde{z}_D - \widetilde{y}|^{n+1}}\, d\mu(y)\\ &\le \sum_{j\in\mathbb{Z}} \int_{(A_j'(x)\cup A_j'(z_D))\setminus 3D} \frac{\ell(D)}{|\widetilde{z}_D - \widetilde{y}|^{n+1}} \sum_{m\in\mathcal{S}_j} \frac{|\epsilon_m - \epsilon_{m+1}|}{2^{-j}}\, d\mu(y)\\ &\lesssim \int_{(3D)^c} \frac{\ell(D)}{|\widetilde{z}_D - \widetilde{y}|^{n+1}}\, d\mu(y) \lesssim 1.\end{aligned} \tag{6.12}$$

Assume now that $m \in \mathcal{L}$. It is easy to check that $|\nabla_z(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(z-y))| \lesssim |\widetilde{z}-\widetilde{y}|^{-1}$ for all $z, y \in \mathbb{R}^d$. So, if also $z \in D$ and $y \in (3D)^c$, $|\nabla_z(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(z-y))| \lesssim |\widetilde{z}_D - \widetilde{y}|^{-1}$. Therefore,

$$\begin{aligned}\Big(\sum_{m\in\mathcal{L}} \Theta 2_m^\rho\Big)^{1/\rho} &\lesssim \sum_{m\in\mathcal{L}} \int_{(A_m(x)\cup A_m(z_D))\setminus 3D} \frac{\ell(D)}{|z_D - y|^{n+1}}\, d\mu(y)\\ &\lesssim \int_{(3D)^c} \frac{\ell(D)}{|\widetilde{z}_D - \widetilde{y}|^{n+1}}\, d\mu(y) \lesssim 1.\end{aligned} \tag{6.13}$$

Finally combining (6.11), (6.12), and (6.13), with (6.9) and the fact that $(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * (\chi_{(3D)^c}\mu))(x)| \le \Theta 1_m + \Theta 2_m$, we conclude that

$$\mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{(3D)^c}\mu))(x) \lesssim \Big(\sum_{m\in\mathbb{Z}} \Theta 1_m^\rho\Big)^{1/\rho} + \Big(\sum_{m\in\mathcal{S}} \Theta 2_m^\rho\Big)^{1/\rho} + \Big(\sum_{m\in\mathcal{L}} \Theta 2_m^\rho\Big)^{1/\rho} \lesssim 1$$

for all $x \in \operatorname{supp}\mu \cap D$. Therefore, $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{(3D)^c}\mu))^2\, d\mu \lesssim \mu(D)$.

6.3. **Proof of $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{3D\setminus D}\mu))^2\, d\mu \lesssim \mu(D)$.** Fix $x \in \operatorname{supp}\mu \cap D$. Since $\rho > 1$,

$$\begin{aligned}\mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{3D\setminus D}\mu))(x) &= \sup_{\{\epsilon_m\}} \Big(\sum_{m\in\mathbb{Z}} \Big|\int_{3D\setminus D} \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y) K(x-y)\, d\mu(y)\Big|^\rho\Big)^{1/\rho}\\ &\le \sup_{\{\epsilon_m\}} \sum_{m\in\mathbb{Z}} \int_{3D\setminus D} \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)|K(x-y)|\, d\mu(y) \le \int_{3D\setminus D} |K(x-y)|\, d\mu(y).\end{aligned}$$

By a standard computation, one can show that

$$\int_D \Big(\int_{3D\setminus D} |K(x-y)|\, d\mu(y)\Big)^2 d\mu(x) \lesssim \mu(D),$$

hence we conclude that $\int_D \mathcal{V}_\rho(K\widetilde{\varphi} * (\chi_{3D\setminus D}\mu))^2 d\mu \lesssim \mu(D)$.

7. L^p AND ENDPOINT ESTIMATES FOR $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ AND $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$

We denote by $H^1(\mathcal{H}^n_\Gamma)$ and $BMO(\mathcal{H}^n_\Gamma)$ the (atomic) Hardy space and the space of functions with bounded mean oscillation, respectively, with respect to the measure $\mathcal{H}^n_\Gamma$. These spaces are defined as the classical $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$ (see [Du, Chapter 6], for example), but by replacing the true cubes of $\mathbb{R}^d$ by our special v-cubes.

Theorem 7.1. *Let $\rho > 2$. The operators $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ and $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ are bounded*

- *in $L^p(\mathcal{H}^n_\Gamma)$ for $1 < p < \infty$,*
- *from $H^1(\mathcal{H}^n_\Gamma)$ to $L^1(\mathcal{H}^n_\Gamma)$, and*
- *from $L^\infty(\mathcal{H}^n_\Gamma)$ to $BMO(\mathcal{H}^n_\Gamma)$,*

and the norm of $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ in the cases above is bounded independently of the sequence that defines $\mathcal{O}$.

We will only give the proof of Theorem 7.1 in the case of the ρ-variation, because the proof for the oscillation follows by analogous arguments.

7.1. The operator $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma} : H^1(\mathcal{H}^n_\Gamma) \to L^1(\mathcal{H}^n_\Gamma)$ is bounded.

Fix a cube $\widetilde{D} \subset \mathbb{R}^n$ and set $D := \widetilde{D} \times \mathbb{R}^{d-n}$. Let f be an atom, i.e., a function defined on Γ and such that

$$\operatorname{supp} f \subset D, \quad \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)} \le \frac{1}{\mathcal{H}^n_\Gamma(D)}, \quad \text{and} \quad \int f \, d\mathcal{H}^n_\Gamma = 0. \tag{7.1}$$

We have to prove that $\int (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f \, d\mathcal{H}^n_\Gamma \le C$, for some constant $C > 0$ which does not depend on f or D.

First of all, by Hölder's inequality, Theorem 6.1, and (7.1),

$$\begin{aligned}\int_{3D} (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f \, d\mathcal{H}^n_\Gamma &\le \mathcal{H}^n_\Gamma(3D)^{1/2} \left(\int_{3D} \left((\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f \right)^2 d\mathcal{H}^n_\Gamma \right)^{1/2} \\ &\lesssim \mathcal{H}^n_\Gamma(3D)^{1/2} \left(\|f\|^2_{L^\infty(\mathcal{H}^n_\Gamma)} \mathcal{H}^n_\Gamma(3D) \right)^{1/2} \lesssim 1.\end{aligned}$$

Thus, it remains to prove that $\int_{(3D)^c} (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f \, d\mathcal{H}^n_\Gamma \le C$.

Given $x \in \Gamma \setminus 3D$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$\left((\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f(x) \right)^\rho \le 2 \sum_{m \in J} |(K \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * (f \mathcal{H}^n_\Gamma))(x)|^\rho, \tag{7.2}$$

where $J := \{m \in \mathbb{Z} : \operatorname{supp} \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot) \cap \operatorname{supp} f \neq \emptyset\}$, thus $\{\epsilon_m\}_{m \in \mathbb{Z}}$ depends on x.

Set $z_D := (\widetilde{z}_D, A(\widetilde{z}_D)) \in D \cap \Gamma$, where $\widetilde{z}_D$ is the center of $\widetilde{D}$. By (7.1), we have $\int \widetilde{\varphi}_\epsilon^\delta(x - z_D) K(x - z_D) f(y) \, d\mathcal{H}^n_\Gamma(y) = 0$ for all $0 < \epsilon \le \delta$. Thus, given $m \in J$, we can decompose

$$\begin{aligned}(K \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * (f \mathcal{H}^n_\Gamma))(x) &= \int \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x - y) \left(K(x - y) - K(x - z_D) \right) f(y) \, d\mathcal{H}^n_\Gamma(y) \\ &\quad + \int \left(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x - y) - \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x - z_D) \right) K(x - z_D) f(y) \, d\mathcal{H}^n_\Gamma(y),\end{aligned}$$

and we obtain $|(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * (f\mathcal{H}^n_\Gamma))(x)| \leq \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}(\Theta 1_m + \Theta 2_m)$, where

$$\Theta 1_m := \int_D \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,|K(x-y) - K(x-z_D)|\; d\mathcal{H}^n_\Gamma(y),$$
$$\Theta 2_m := \int_D \big|\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y) - \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-z_D)\big|\big|K(x-z_D)\big|\, d\mathcal{H}^n_\Gamma(y).$$

The term $\Theta 1_m$ can be easily handled. For $x \in \Gamma \setminus 3D$, we have

$$\Theta 1_m \lesssim \ell(D)\,\mathrm{dist}(x,D)^{-n-1}\int_D \widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\, d\mathcal{H}^n_\Gamma(y). \tag{7.3}$$

Let us estimate $\Theta 2_m$. Decompose $J = \mathcal{S} \cup \mathcal{L}$, where $\mathcal{S}$ and $\mathcal{L}$ are as in (6.4) but replacing $m \in \mathbb{Z}$ by $m \in J$, and as before, let $j(\epsilon_m)$ be the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Using that $x \in \Gamma \setminus 3D$ and $\mathrm{supp} f \subset D$, one can easily check that $\mathcal{L}$ contains a finite number of elements, and this number only depends on n and d. Similarly, $\mathcal{S}_j = \emptyset$ for all $j \in \mathbb{Z}$ except on a finite number which only depends on n and d.

Assume that $m \in \mathcal{S}$. With the same computations as those in (5.6), one can prove that, for all $y \in \mathrm{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)$, $|\nabla_y(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y))| \lesssim 2^{j(\epsilon_m)}|\epsilon_m - \epsilon_{m+1}||\widetilde{x}-\widetilde{y}|^{-1}$, because $|\widetilde{x}-\widetilde{y}| \approx \epsilon_m \approx \epsilon_{m+1} \approx 2^{-j(\epsilon_m)}$ for all $y \in \mathrm{supp}\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)$. Thus,

$$\Theta 2_m \lesssim \ell(D)^{n+1}\,\mathrm{dist}(x,D)^{-n-1}2^{j(\epsilon_m)}|\epsilon_m - \epsilon_{m+1}|. \tag{7.4}$$

Assume now that $m \in \mathcal{L}$. It is easy to verify that $|\nabla_y(\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y))| \lesssim |\widetilde{x}-\widetilde{y}|^{-1}$, so $\Theta 2_m \lesssim \ell(D)^{n+1}\,\mathrm{dist}(x,D)^{-n-1}$.

Combining this last estimate with (7.3), (7.4), the fact that $|(K\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m} * (f\mathcal{H}^n_\Gamma))(x)| \leq \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}(\Theta 1_m + \Theta 2_m)$, the remark on $\mathcal{S}$ and $\mathcal{L}$ made just after (7.3), (7.2), and that $\rho > 1$, we have that, for all $x \in \Gamma \setminus 3D$,

$$\begin{aligned}
(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})f(x) &\lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\Big(\sum_{m\in J}\Theta 1_m + \sum_{m\in\mathcal{S}}\Theta 2_m + \sum_{m\in\mathcal{L}}\Theta 2_m\Big)\\
&\lesssim \frac{\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\ell(D)^{n+1}}{\mathrm{dist}(x,D)^{n+1}}\Big(\sum_{m\in J}\int_D \frac{\widetilde{\varphi}_{\epsilon_{m+1}}^{\epsilon_m}(x-y)}{\ell(D)^n}\,d\mathcal{H}^n_\Gamma(y) + \sum_{m\in\mathcal{S}}\frac{|\epsilon_m - \epsilon_{m+1}|}{2^{-j(\epsilon_m)}} + \sum_{m\in\mathcal{L}}1\Big)\\
&\lesssim \frac{\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\ell(D)^{n+1}}{\mathrm{dist}(x,D)^{n+1}}.
\end{aligned}$$

Then, using (7.1) and standard estimates, we conclude that

$$\int_{(3D)^c}(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})f(x)\,d\mathcal{H}^n_\Gamma(x) \lesssim \int_{(3D)^c}\frac{\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\ell(D)^{n+1}}{\mathrm{dist}(x,D)^{n+1}}\,d\mathcal{H}^n_\Gamma(x) \lesssim 1.$$

7.2. **The operator $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma} : L^\infty(\mathcal{H}^n_\Gamma) \to BMO(\mathcal{H}^n_\Gamma)$ is bounded.** We have to prove that there exists a constant $C > 0$ such that, for any $f \in L^\infty(\mathcal{H}^n_\Gamma)$ and any cube $\widetilde{D} \subset \mathbb{R}^n$, there exists some constant c depending on f and $\widetilde{D}$ such that

$$\int_D \big|(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})f - c\big|\,d\mathcal{H}^n_\Gamma \leq C\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(D).$$

Let f and D be as above, and set $f_1 := f\chi_{3D}$ and $f_2 := f - f_1$. First of all, by Hölder's inequality and Theorem 6.1, we have

$$\int_D (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_1 \, d\mathcal{H}^n_\Gamma \le \mathcal{H}^n_\Gamma(D)^{1/2} \left(\int_{3D} \left((\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_1 \right)^2 d\mathcal{H}^n_\Gamma \right)^{1/2}$$
$$\lesssim \mathcal{H}^n_\Gamma(D)^{1/2} \left(\|f_1\|^2_{L^\infty(\mathcal{H}^n_\Gamma)} \mathcal{H}^n_\Gamma(3D) \right)^{1/2} \lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)} \mathcal{H}^n_\Gamma(D).$$

Notice that $|(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})(f_1+f_2) - (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2| \le (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_1$, because $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ is sublinear and positive. Then, for any $c \in \mathbb{R}$,

$$|(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})(f_1+f_2) - c| \le |(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})(f_1+f_2) - (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2| + |(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2 - c|$$
$$\le (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_1 + |(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2 - c|,$$

hence we are reduced to prove that, for some constant $c \in \mathbb{R}$,

$$\int_D \left| (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2 - c \right| d\mathcal{H}^n_\Gamma \le C \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)} \mathcal{H}^n_\Gamma(D). \tag{7.5}$$

Set $z_D := (\widetilde{z}_D, A(\widetilde{z}_D))$, where z_D is the center of $\widetilde{D}$, and take $c := (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2(z_D)$. We may assume that $c < \infty$ (this is the case if, for example, f has compact support). By the triangle inequality,

$$\left| (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2(x) - c \right|^\rho \le \sup_{\{\epsilon_m \searrow 0\}} \sum_{m \in \mathbb{Z}} |(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (f_2 \mathcal{H}^n_\Gamma))(x) - (K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (f_2 \mathcal{H}^n_\Gamma))(z_D)|^\rho.$$

Given $x \in \Gamma \cap D$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$\left| (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2(x) - c \right|^\rho \le 2 \sum_{m \in \mathbb{Z}} |(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (f_2 \mathcal{H}^n_\Gamma))(x) - (K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (f_2 \mathcal{H}^n_\Gamma))(z_D)|^\rho.$$

Notice that $|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (f_2 \mathcal{H}^n_\Gamma))(x) - (K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}} * (f_2 \mathcal{H}^n_\Gamma))(z_D)| \le \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)} (\Theta 1_m + \Theta 2_m)$, where $\Theta 1_m$ and $\Theta 2_m$ are as in (6.10) but replacing μ by $\mathcal{H}^n_\Gamma$. It is straightforward to check that the arguments and computations given in subsection 6.2 to estimate the two terms in (6.10) (see (6.11), (6.12), and (6.13)) still hold if we replace μ by $\mathcal{H}^n_\Gamma$. Therefore, we have

$$\sum_{m \in \mathbb{Z} = \mathcal{S} \cup \mathcal{L}} (\Theta 1_m + \Theta 2_m)^\rho \lesssim 1,$$

which impies that $\left| (\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}) f_2(x) - c \right| \lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}$ and, by integrating in D, gives (7.5).

7.3. **The operator $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma} : L^p(\mathcal{H}^n_\Gamma) \to L^p(\mathcal{H}^n_\Gamma)$ is bounded for all $1 < p < \infty$.** Since $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ is sublinear, the L^p boundedness follows by applying the results of subsection 7.1 and subsection 7.2, and the interpolation theorem between the pairs $(H^1(\mathcal{H}^n_\Gamma), L^1(\mathcal{H}^n_\Gamma))$ and $(L^\infty(\mathcal{H}^n_\Gamma), BMO(\mathcal{H}^n_\Gamma))$ in [Ju, page 43].

Given a v-cube $Q \subset \mathbb{R}^d$, set $m_Q(f) := \mathcal{H}^n_\Gamma(Q)^{-1} \int_Q f \, d\mathcal{H}^n_\Gamma$, and let M be the Hardy-Littlewood maximal operator, i.e. for $x \in \Gamma$, $M(f)(x) := \sup m_Q(|f|)$, where the supremum is taken over all v-cubes $Q \subset \mathbb{R}^d$ containing $x \in \Gamma$. Let $M^\sharp$ be the sharp maximal operator defined by $M^\sharp(f)(x) := \sup m_Q(|f - m_Q(f)|)$, where the supremum is also taken over all v-cubes $Q \subset \mathbb{R}^d$ containing $x \in \Gamma$.

One comment about the interpolation theorem in [Ju, page 43] is in order. Given an operator F bounded form H^1 to L^1 and from L^∞ to BMO, in the proof of the interpolation theorem applied to F, one uses that $M^\sharp \circ F$ is sublinear (i.e. $(M^\sharp \circ F)(f+g) \le (M^\sharp \circ F)f + (M^\sharp \circ F)g$ for all functions f, g). This is the case when F is linear. However, $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ is not linear, and then it is not clear if $M^\sharp \circ \mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ is sublinear. Nevertheless, this problem can be fixed easily using that $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ is sublinear and positive (that is $(\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma})f(x) \ge 0$ for all f and x), as the following lemma shows.

Lemma 7.2. *Let $F : L^1_{loc}(\mathcal{H}^n_\Gamma) \to L^1_{loc}(\mathcal{H}^n_\Gamma)$ be a positive and sublinear operator. Then $(M^\sharp \circ F)(f+g) \lesssim (M \circ F)f + (M^\sharp \circ F)g$ for all functions f, g.*

Proof. If F is sublinear and positive, one has that $|F(f)(x) - F(g)(x)| \le F(f-g)(x)$ for all functions $f, g \in L^1_{loc}(\mathcal{H}^n_\Gamma)$. Then, for $x, y \in Q \cap \Gamma$,

$$\begin{aligned}|F(f+g)(y) - m_Q(Fg)| &\le |F(f+g)(y) - Fg(y)| + |Fg(y) - m_Q(Fg)| \\ &\le |Ff(y)| + |Fg(y) - m_Q(Fg)|.\end{aligned}$$

Hence, $m_Q|F(f+g) - m_Q(Fg)| \le m_Q|Ff| + m_Q|Fg - m_Q(Fg)| \le (M \circ F)f(x) + (M^\sharp \circ F)g(x)$ and, by taking the supremum over all possible v-cubes $Q \ni x$, we conclude $(M^\sharp \circ F)(f+g)(x) \lesssim (M \circ F)f(x) + (M^\sharp \circ F)g(x)$. □

By using Lemma 7.2 and the fact that $\|Mf\|_{L^p(\mathcal{H}^n_\Gamma)} \lesssim \|M^\sharp f\|_{L^p(\mathcal{H}^n_\Gamma)}$ for $f \in L^{p_0}(\mathcal{H}^n_\Gamma) \cap L^p(\mathcal{H}^n_\Gamma)$ and $1 \le p_0 \le p < \infty$, one can reprove Journé's interpolation theorem applied to $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}^n_\Gamma}$ with minor modifications in the original proof.

8. L^2 estimates for $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$

Recall that $\Gamma := \{x \in \mathbb{R}^d : x = (\widetilde{x}, A(\widetilde{x}))\}$. Let $A_\Gamma : \mathbb{R}^n \to \mathbb{R}^d$ be the parametrization of Γ, i.e. $A_\Gamma(y) := (y, A(y))$ for $y \in \mathbb{R}^n$. We may and will assume that the Lipschitz function A has compact support, and our estimates will not depend on the support of A. By a limiting argument, one easily obtains the same estimates for the case of a general Lipschitz graph.

Abusing notation, throughout this section we will identify the cubes $D \subset \mathbb{R}^n$ with the v-cubes $D \times \mathbb{R}^{d-n} \subset \mathbb{R}^d$, so we will use the same symbol D to denote both objects. In particular, $\mathcal{D}$ will denote the dyadic lattice of cubes in $\mathbb{R}^n$ and the dyadic lattice of v-cubes in $\mathbb{R}^d$. It will be clear from the context to which object we are referring to in each particular circumstance.

Recall that we have set $\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{L}^n)}$ for $1 \le p \le \infty$, and $dy := d\mathcal{L}^n(y)$ for $y \in \mathbb{R}^n$.

Theorem 8.1. *Let $\rho > 2$, and assume $\mathrm{Lip}(A) < 1$. The operators $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ are bounded in $L^2(\mathcal{H}^n_\Gamma)$, and the norm of $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ is bounded above independently of the sequence that defines $\mathcal{O}$.*

As in the previous sections, we will only prove Theorem 8.1 in the case of the ρ-variation, because the proof for the oscillation follows by very similar arguments.

We will use the following lemma, which is proved in subsection 8.6 (see Lemma 8.9).

Lemma 8.2. *If $\mathrm{Lip}(A) < 1$, then*

$$\mathcal{H}^n_\Gamma(A^d(z,a,b)) \lesssim (b-a)b^{n-1} \quad \text{for all } 0 < a \le b \text{ and } z \in \Gamma. \tag{8.1}$$

Remark 8.3. Without the assumption $\mathrm{Lip}(A) < 1$, the lemma fails (see Remark 8.11 below). The estimate (8.1) is essential for some of the arguments below. This is the reason why we have been able to prove Theorem 8.1 only under the assumption $\mathrm{Lip}(A) < 1$.

This lemma is only required to study the ρ-variation and oscillation for singular integrals when the family of truncations is χ. If we considered the family $\widetilde{\chi}$ instead of χ, we would have to estimate $\mathcal{H}^n_\Gamma(A^n(z,a,b)\times\mathbb{R}^{d-n})$, which is easily seen to be bounded by $C(b-a)b^{n-1}$ in any case, i.e. without the extra assumption $\mathrm{Lip}(A)<1$. On the other hand, if we worked with φ, we would not need to estimate the size of any annulus in our computations. Instead, we would use the regularity of the functions φ_ϵ for $\epsilon>0$, as we did with $\widetilde{\varphi}_\epsilon$ in the preceding sections. For more details, see Remark 8.7.

8.1. **Beginning of the proof of Theorem 8.1.** Let $f\in L^2(\mathcal{H}^n_\Gamma)$. Given $x\in\Gamma$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$\big((\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_\chi)f(x)\big)^\rho\le 2\sum_{m\in\mathbb{Z}}|(K\chi^{\epsilon_m}_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))(x)|^\rho. \tag{8.2}$$

Given $j\in\mathbb{Z}$, we denote $I_j:=[2^{-j-1},2^{-j})$. Let $j(\epsilon_m)$ be the integer such that $\epsilon_m\in I_{j(\epsilon_m)}$. As before, we set $\mathcal{S}:=\bigcup_{j\in\mathbb{Z}}\mathcal{S}_j$, $\mathcal{S}_j:=\{m\in\mathbb{Z}\,:\,\epsilon_m,\epsilon_{m+1}\in I_j\}$, and $\mathcal{L}:=\{m\in\mathbb{Z}\,:\,\epsilon_m\in I_i,\ \epsilon_{m+1}\in I_j\text{ for }i<j\}$.

For $\epsilon>0$, we define $\kappa_\epsilon:=\chi_\epsilon-\widetilde{\varphi}_\epsilon$. Then, by (8.2) and the triangle inequality,

$$\begin{aligned}\big((\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_\chi)f(x)\big)^\rho\lesssim&\sum_{m\in\mathcal{S}}|(K\chi^{\epsilon_m}_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))(x)|^\rho+\sum_{m\in\mathcal{L}}|(K\kappa_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))(x)|^\rho\\&+\sum_{m\in\mathcal{L}}|(K\kappa_{\epsilon_m}*(f\mathcal{H}^n_\Gamma))(x)|^\rho+\sum_{m\in\mathcal{L}}|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))(x)|^\rho.\end{aligned}\tag{8.3}$$

Notice that $\sum_{m\in\mathcal{L}}|(K\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))(x)|^\rho\le\big((\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})f(x)\big)^\rho$.

We will prove the following estimate in subsections 8.3, 8.4 and 8.5:

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|(K\chi^{\epsilon_m}_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))|^2\,d\mathcal{H}^n_\Gamma\\&\quad+\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{L}:\,\epsilon_m\in I_j}|(K\kappa_{\epsilon_m}*(f\mathcal{H}^n_\Gamma))|^2\,d\mathcal{H}^n_\Gamma\\&\qquad+\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{L}:\,\epsilon_{m+1}\in I_j}|(K\kappa_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))|^2\,d\mathcal{H}^n_\Gamma\lesssim\|f\|^2_{L^2(\mathcal{H}^n_\Gamma)}.\end{aligned}\tag{8.4}$$

Using (8.4), (8.3), Theorem 7.1 for $p=2$, and that $\rho>2$, we finally get

$$\begin{aligned}\|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_\chi)f\|^2_{L^2(\mathcal{H}^n_\Gamma)}&\lesssim\|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}})f\|^2_{L^2(\mathcal{H}^n_\Gamma)}+\int\Big(\sum_{m\in\mathcal{S}}|(K\chi^{\epsilon_m}_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))|^\rho\Big)^{2/\rho}d\mathcal{H}^n_\Gamma\\&\quad+\int\Big(\sum_{m\in\mathcal{L}}|(K\kappa_{\epsilon_m}*(f\mathcal{H}^n_\Gamma))|^\rho\Big)^{2/\rho}d\mathcal{H}^n_\Gamma+\int\Big(\sum_{m\in\mathcal{L}}|(K\kappa_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))|^\rho\Big)^{2/\rho}d\mathcal{H}^n_\Gamma\\&\lesssim\|f\|^2_{L^2(\mathcal{H}^n_\Gamma)}+\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|(K\chi^{\epsilon_m}_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))|^2\,d\mathcal{H}^n_\Gamma\\&\quad+\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{L}\,:\,\epsilon_m\in I_j}|(K\kappa_{\epsilon_m}*(f\mathcal{H}^n_\Gamma))|^2\,d\mathcal{H}^n_\Gamma\\&\quad+\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{L}\,:\,\epsilon_{m+1}\in I_j}|(K\kappa_{\epsilon_{m+1}}*(f\mathcal{H}^n_\Gamma))|^2\,d\mathcal{H}^n_\Gamma\lesssim\|f\|^2_{L^2(\mathcal{H}^n_\Gamma)}.\end{aligned}$$

It only remains to prove (8.4).

8.2. **Estimate of $\sum_{m\in\mathcal{S}_j}|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(f\mathcal{H}^n_\Gamma))(x)|^2$ for $x\in\Gamma\cap D$, for $D\in\mathcal{D}_j$.** Using the parametrization A_Γ of Γ, we have

$$
\begin{aligned}
(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(f\mathcal{H}^n_\Gamma))(x) &= \int_{\mathbb{R}^n} K(x-A_\Gamma(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))f(A_\Gamma(y))|J(A_\Gamma)(y)|\,dy\\
&= \int_{\mathbb{R}^n} K(x-A_\Gamma(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))g(y)\,dy,
\end{aligned}
\tag{8.5}
$$

where $J(A_\Gamma)$ stands for the n-dimensional Jacobian of the map $A_\Gamma:=y\mapsto(y,A(y))$, and we have set $g(y):=f(A_\Gamma(y))|J(A_\Gamma)(y)|$ for $y\in\mathbb{R}^n$ (notice that $x\in\mathbb{R}^d$ but $y\in\mathbb{R}^n$, we have not used the notation $\widetilde{y}$ here to make it simpler). Since Γ is a Lipschitz graph $|J(A_\Gamma)|\approx 1$, so $g\in L^2(\mathcal{L}^n)$.

Definition 8.4. *Let $\{\psi_Q^k\}_{Q\in\mathcal{D},\,k=1,\dots,2^n-1}$ be an orthonormal basis of $\mathcal{C}^1$ wavelets on $\mathbb{R}^n$ in the following manner (see* [Da, Part I]*):*

(a) $\psi_Q^k:\mathbb{R}^n\to\mathbb{R}$ *is a $\mathcal{C}^1$ function for all $Q\in\mathcal{D}$ and $k=1,\dots,2^n-1$.*
(b) *There exists $C>1$ and $\psi_0:[0,C]^n\to\mathbb{R}$ with $\|\psi_0\|_2=1$, $\|\psi_0\|_\infty\lesssim 1$, and such that, for any $Q\in\mathcal{D}$ and $k=1,\dots,2^n-1$, there exists $l\in\mathbb{Z}^n$ such that $\psi_Q^k(y)=\psi_0(y/\ell(Q)-l)\ell(Q)^{-n/2}$ for all $y\in\mathbb{R}^n$.*
(c) $\|\psi_Q^k\|_2=1$, $\int\psi_Q^k\,d\mathcal{L}^n=0$ *and* $\int\psi_Q^k\psi_R^l\,d\mathcal{L}^n=0$, *for all $Q,R\in\mathcal{D}$ and $k,l=1,\dots,2^n-1$ such that $(Q,k)\neq(R,l)$.*
(d) $\operatorname{supp}\psi_Q^k\subset C_wQ$ *for all $Q\in\mathcal{D}$ and $k=1,\dots,2^n-1$, where $C_w>1$ is some fixed constant (which depends on n). In particular, the supports of the functions in $\{\psi_Q^k\}_{Q\in\mathcal{D},\,k=1,\dots,2^n-1}$ have finite overlap.*
(e) $\|\psi_Q^k\|_\infty\lesssim\ell(Q)^{-n/2}$ *and* $\|\nabla\psi_Q^k\|_\infty\lesssim\ell(Q)^{-n/2-1}$ *for all $Q\in\mathcal{D}$, $k=1,\dots,2^n-1$.*
(f) *If $h\in L^2(\mathcal{L}^n)$, then $h=\sum_{Q\in\mathcal{D},\,k=1,\dots,2^n-1}\Delta_Q^k h$, where $\Delta_Q^k h:=\big(\int h\psi_Q^k\,d\mathcal{L}^n\big)\psi_Q^k$.*

In order to reduce the notation, we may think that a cube of $\mathcal{D}$ is not only a subset of $\mathbb{R}^n$, but a couple (Q,k), where Q is a subset of $\mathbb{R}^n$ and $k=1,\dots,2^n-1$. In particular, there exist 2^n-1 cubes in $\mathcal{D}$ such that the subsets that they represent in $\mathbb{R}^n$ coincide. We make this abuse of notation to avoid using the superscript k in the previous definition. Then, we can rewrite the wavelet basis as $\{\psi_Q\}_{Q\in\mathcal{D}}$, with the evident adjustments of the properties (a),…,(f) in Definition 8.4.

Remark 8.5. Since Γ is Lipschitz graph, $|J(\mathcal{A}_\Gamma)(y)|\approx 1$ for all $y\in\mathbb{R}^n$. Then, using Definition 8.4(c) and Definition 8.4(f), one easily obtains

$$\|f\|^2_{L^2(\mathcal{H}^n_\Gamma)}\approx\|g\|_2^2=\sum_{Q\in\mathcal{D}}\|\Delta_Q g\|_2^2.$$

Given $x\in D\cap\Gamma$, if $m\in\mathcal{S}_j$ and $\operatorname{supp}\psi_Q\cap\operatorname{supp}\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(\cdot))\neq\emptyset$, then either $D\subset C_bQ$ or $Q\subset C_bD$, where C_b is some big fixed constant. Set $J:=\{Q\in\mathcal{D}:\,D\subset C_bQ$ and $Q\not\subset C_bD\}$

and $\Psi_D g := \sum_{Q\in J} \Delta_Q g$. Using (8.5) and Definition 8.4(f), we have

$$\begin{aligned}(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mathcal{H}^n_\Gamma))(x) &= \int_{\mathbb{R}^n} K(x-A_\Gamma(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))g(y)\,dy\\ &= \int_{\mathbb{R}^n} K(x-A_\Gamma(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))(\Psi_D g(y)-\Psi_D g(x))\,dy\\ &\quad+\Psi_D g(x)\int_{\mathbb{R}^n} K(x-A_\Gamma(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))\,dy\\ &\quad+\sum_{\substack{Q\in\mathcal{D}:\\ Q\subset C_bD}}\int_{\mathbb{R}^n} K(x-A_\Gamma(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))\Delta_Q g(y)\,dy\\ &=: U1_m(x)+U2_m(x)+U3_m(x).\end{aligned} \tag{8.6}$$

8.2.1. **Estimate of** $\sum_{m\in\mathcal{S}_j}|U1_m(x)|^2$. Notice that, by Definition 8.4(e), $\|\nabla(\Delta_Q g)\|_\infty \lesssim \|\Delta_Q g\|_2 \ell(Q)^{-n/2-1}$. Then,

$$\begin{aligned}|U1_m(x)| &\le \sum_{Q\in J}\int_{\mathbb{R}^n}|K(x-A_\Gamma(y))|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))|\Delta_Q g(y)-\Delta_Q g(x)|\,dy\\ &\le \sum_{Q\in J}\int_{\mathbb{R}^n}|K(x-A_\Gamma(y))|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))\|\nabla(\Delta_Q g)\|_\infty|x-y|\,dy\\ &\lesssim \sum_{Q\in J}\ell(D)^{-n+1}\|\Delta_Q g\|_2\ell(Q)^{-n/2-1}\int_{\mathbb{R}^n}\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))\,dy,\end{aligned}$$

thus, by Cauchy-Schwarz inequality and the fact that $\sum_{m\in\mathcal{S}_j}\int_{\mathbb{R}^n}\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))\,dy \lesssim \ell(D)^n$,

$$\sum_{m\in\mathcal{S}_j}|U1_m(x)|^2 \lesssim \Bigg(\sum_{\substack{Q\in\mathcal{D}:\\ D\subset C_bQ}}\ell(Q)^{-n/2}\frac{\ell(D)}{\ell(Q)}\|\Delta_Q g\|_2\Bigg)^2 \lesssim \sum_{\substack{Q\in\mathcal{D}:\\ D\subset C_bQ}}\frac{\ell(D)}{\ell(Q)^{n+1}}\|\Delta_Q g\|_2^2. \tag{8.7}$$

8.2.2. **Estimate of** $\sum_{m\in\mathcal{S}_j}|U3_m(x)|^2$. To estimate $U3_m(x)$, we denote

$$U3_m(x,Q) := \int \chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_\Gamma(y))K(x-A_\Gamma(y))\Delta_Q g(y)\,dy,$$

so $U3_m(x)=\sum_{Q\in\mathcal{D}:Q\subset C_bD}U3_m(x,Q)$. Notice that, if $C_wQ\cap\Gamma\cap A(x,\epsilon_{m+1},\epsilon_m)=\emptyset$, then $U3_m(x,Q)=0$. So, if we set

$$\begin{aligned}J_m^1 &:= \{Q\in\mathcal{D} : Q\subset C_bD,\, C_wQ\cap\Gamma\cap A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset,\, 100\sqrt{n}C_w\ell(Q)\ge\epsilon_m-\epsilon_{m+1}\},\\ J_m^2 &:= \{Q\in\mathcal{D} : Q\subset C_bD,\, C_wQ\cap\Gamma\cap A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset,\, 100\sqrt{n}C_w\ell(Q)<\epsilon_m-\epsilon_{m+1}\},\end{aligned}$$

we have

$$U3_m(x)=\sum_{Q\in J_m^1}U3_m(x,Q)+\sum_{Q\in J_m^2}U3_m(x,Q). \tag{8.8}$$

For $m \in \mathcal{S}_j$ and $Q \in J^1_m$, we have $\int \chi^{\epsilon_m}_{\epsilon_{m+1}}(x - A_\Gamma(y))|\Delta_Q g(y)|\,dy \lesssim |\epsilon_m - \epsilon_{m+1}|\ell(Q)^{-1}\|\Delta_Q g\|_1$. This follows from the smoothness of the wavelet ψ_Q, Definition 8.4(b), and Lemma 8.2. Then,

$$\begin{aligned}|U3_m(x,Q)| &\lesssim \int \chi^{\epsilon_m}_{\epsilon_{m+1}}(x - A_\Gamma(y))\ell(D)^{-n}|\Delta_Q g(y)|\,dy \lesssim \frac{|\epsilon_m - \epsilon_{m+1}|}{\ell(D)^n\ell(Q)}\|\Delta_Q g\|_1\\ &\lesssim \frac{\ell(Q)^{n/2-1}}{\ell(D)^n}\,|\epsilon_m - \epsilon_{m+1}|\|\Delta_Q g\|_2.\end{aligned}$$

Therefore, by Cauchy-Schwarz,

$$\begin{aligned}\sum_{m\in\mathcal{S}_j}\Big(\sum_{Q\in J^1_m}|U3_m(x,Q)|\Big)^2 &\lesssim \sum_{m\in\mathcal{S}_j}\Big(\sum_{Q\in J^1_m}\frac{\ell(Q)^{n/2-1}}{\ell(D)^n}\,|\epsilon_m-\epsilon_{m+1}|\|\Delta_Q g\|_2\Big)^2\\ &\le \frac{1}{\ell(D)^{2n}}\sum_{m\in\mathcal{S}_j}\Big(\sum_{Q\in J^1_m}\ell(Q)^{n-1}\Big)\Big(\sum_{Q\in J^1_m}\frac{|\epsilon_m-\epsilon_{m+1}|^2}{\ell(Q)}\,\|\Delta_Q g\|_2^2\Big).\end{aligned}\tag{8.9}$$

From the definition of J^1_m, it is not difficult to check that

$$\sum_{Q\in J^1_m}\ell(Q)^{n-1} \lesssim \ell(D)^{n-1}\log_2\big(\ell(D)/|\epsilon_m-\epsilon_{m+1}|\big).\tag{8.10}$$

To check this, recall that $\epsilon_m, \epsilon_{m+1} \in I_j$, $D \in \mathcal{D}_j$, and $Q \in J^1_m$. Then, split the sum according to the different scales of the v-cubes and use that, given $i \in \mathbb{Z}$ such that $\sqrt{n}C_w2^{-i} \ge |\epsilon_m - \epsilon_{m+1}|$, the number of v-cubes $Q \in \mathcal{D}$ such that $\ell(Q) = 2^{-i}$, $Q \subset C_bD$, and $C_wQ \cap \Gamma \cap A(x,\epsilon_{m+1},\epsilon_m) \neq \emptyset$ is bounded by $C\ell(D)^{n-1}2^{i(n-1)}$, since for all these v-cubes, $C_wQ\cap\Gamma \subset A(x,\epsilon_{m+1}-C2^{-i},\epsilon_m+C2^{-i})\cap\Gamma$, and by Lemma 8.2, $\mathcal{H}^n_\Gamma(A(x,\epsilon_{m+1}-C2^{-i},\epsilon_m+C2^{-i})) \lesssim \ell(D)^{n-1}2^{-i}$.

Then, by (8.9) and since $t^{1/2}\log_2(1/t) \lesssim 1$ for all $0 < t \lesssim 1$,

$$\begin{aligned}\sum_{m\in\mathcal{S}_j}\Big(\sum_{Q\in J^1_m}|U3_m(x,Q)|\Big)^2 &\lesssim \frac{1}{\ell(D)^{n+1}}\sum_{m\in\mathcal{S}_j}\log_2\Big(\frac{\ell(D)}{|\epsilon_m-\epsilon_{m+1}|}\Big)\sum_{Q\in J^1_m}\frac{|\epsilon_m-\epsilon_{m+1}|^2}{\ell(Q)}\,\|\Delta_Q g\|_2^2\\ &\lesssim \frac{1}{\ell(D)^n}\sum_{m\in\mathcal{S}_j}\sum_{Q\in J^1_m}\frac{|\epsilon_m-\epsilon_{m+1}|^{3/2}}{\ell(D)^{1/2}\ell(Q)}\,\|\Delta_Q g\|_2^2\\ &\lesssim \sum_{\substack{Q\in\mathcal{D}:\\ Q\subset C_bD}}\;\sum_{\substack{m\in\mathcal{S}_j\,:\,100\sqrt{n}C_w\ell(Q)\ge\epsilon_m-\epsilon_{m+1},\\ C_wQ\cap\Gamma\cap A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}}\frac{|\epsilon_m-\epsilon_{m+1}|}{\ell(Q)}\,\frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}}\,\|\Delta_Q g\|_2^2.\end{aligned}$$

Since $\sum_{\substack{m\in\mathcal{S}_j\,:\,100\sqrt{n}C_w\ell(Q)\ge\epsilon_m-\epsilon_{m+1},\\ C_wQ\cap\Gamma\cap A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}} |\epsilon_m-\epsilon_{m+1}|/\ell(Q) \lesssim 1$, we finally obtain

$$\sum_{m\in\mathcal{S}_j}\Big(\sum_{Q\in J^1_m}|U3_m(x,Q)|\Big)^2 \lesssim \sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD}\frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}}\,\|\Delta_Q g\|_2^2.\tag{8.11}$$

Assume now that $Q \in J_m^2$. Let z_Q be the center of $Q \subset \mathbb{R}^n$. Since $\int \Delta_Q g(y)\,dy = 0$ (see Definition 8.4(c)), we decompose

$$
\begin{aligned}
U3_m(x,Q) &= \int \chi_{\epsilon_{m+1}}^{\epsilon_m}(x - A_\Gamma(y))\big(K(x - A_\Gamma(y)) - K(x - A_\Gamma(z_Q))\big)\Delta_Q g(y)\,dy \\
&\quad + \int \big(\chi_{\epsilon_{m+1}}^{\epsilon_m}(x - A_\Gamma(y)) - \chi_{\epsilon_{m+1}}^{\epsilon_m}(x - A_\Gamma(z_Q))\big)K(x - A_\Gamma(z_Q))\Delta_Q g(y)\,dy \\
&=: U3_m^A(x,Q) + U3_m^B(x,Q).
\end{aligned}
\tag{8.12}
$$

The first term of the previous sum can be easily handled:

$$
\begin{aligned}
\sum_{m\in\mathcal{S}_j}\Big(\sum_{Q\in J_m^2}|U3_m^A(x,Q)|\Big)^2 &\lesssim \Big(\sum_{m\in\mathcal{S}_j}\sum_{Q\in J_m^2}\frac{\ell(Q)}{\ell(D)^{n+1}}\int \chi_{\epsilon_{m+1}}^{\epsilon_m}(x - A_\Gamma(y))|\Delta_Q g(y)|\,dy\Big)^2 \\
&\le \Big(\sum_{Q\in\mathcal{D}:\,Q\subset C_bD}\frac{\ell(Q)}{\ell(D)^{n+1}}\|\Delta_Q g\|_1\Big)^2 \lesssim \Big(\sum_{Q\in\mathcal{D}:\,Q\subset C_bD}\frac{\ell(Q)^{n/2+1}}{\ell(D)^{n+1}}\|\Delta_Q g\|_2\Big)^2.
\end{aligned}
$$

Then, using Cauchy-Schwarz and that $\sum_{Q\in\mathcal{D}:\,Q\subset C_bD}\ell(Q)^{n+1} \lesssim \ell(D)^{n+1}$, we conclude

$$
\begin{aligned}
\sum_{m\in\mathcal{S}_j}\Big(\sum_{Q\in J_m^2}|U3_m^A(x,Q)|\Big)^2 &\lesssim \Big(\sum_{\substack{Q\in\mathcal{D}:\\ Q\subset C_bD}}\ell(Q)^{n+1}\Big)\Big(\sum_{\substack{Q\in\mathcal{D}:\\ Q\subset C_bD}}\frac{\ell(Q)}{\ell(D)^{2n+2}}\|\Delta_Q g\|_2^2\Big) \\
&\lesssim \sum_{Q\in\mathcal{D}:\,Q\subset C_bD}\frac{\ell(Q)}{\ell(D)^{n+1}}\|\Delta_Q g\|_2^2 \lesssim \sum_{Q\in\mathcal{D}:\,Q\subset C_bD}\frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}}\|\Delta_Q g\|_2^2.
\end{aligned}
\tag{8.13}
$$

To deal with $U3_m^B(x,Q)$, notice that, if $C_wQ\cap\Gamma\cap\partial A(x,\epsilon_{m+1},\epsilon_m) = \emptyset$, then $\chi_{\epsilon_{m+1}}^{\epsilon_m}(x - A_\Gamma(y)) - \chi_{\epsilon_{m+1}}^{\epsilon_m}(x - A_\Gamma(z_Q)) = 0$ for all $y \in C_wQ \subset \mathbb{R}^n$. So, $\sum_{Q\in J_m^2}|U3_m^B(x,Q)| = \sum_{Q\in J_m^3}|U3_m^B(x,Q)|$, where

$$
J_m^3 := \{Q\in\mathcal{D} : Q\subset C_bD,\ C_wQ\cap\Gamma\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset,\ 100\sqrt{n}C_w\ell(Q) < \epsilon_m - \epsilon_{m+1}\}.
$$

We use the easy estimate $|U3_m^B(x,Q)| \lesssim \ell(D)^{-n}\|\Delta_Q g\|_1 \lesssim \ell(Q)^{n/2}\ell(D)^{-n}\|\Delta_Q g\|_2$ for all $Q\in J_m^3$, and then, by Cauchy-Schwarz inequality,

$$
\Big(\sum_{Q\in J_m^3}|U3_m^B(x,Q)|\Big)^2 \lesssim \ell(D)^{-2n}\Big(\sum_{Q\in J_m^3}\ell(Q)^{n-1/2}\Big)\Big(\sum_{Q\in J_m^3}\ell(Q)^{1/2}\|\Delta_Q g\|_2^2\Big).
\tag{8.14}
$$

It is not difficult to show that $\sum_{Q\in J_m^3}\ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}|\epsilon_m - \epsilon_{m+1}|^{1/2}$. To check this, split the sum according to the different scales of the v-cubes and use that, given $i\in\mathbb{Z}$ such that $100\sqrt{n}C_w 2^{-i} < \epsilon_m - \epsilon_{m+1}$, the number of v-cubes $Q\in\mathcal{D}$ such that $\ell(Q) = 2^{-i}$ and $C_wQ\cap\Gamma\cap\partial B(x,\epsilon_{m+1})\neq\emptyset$ or $C_wQ\cap\Gamma\cap\partial B(x,\epsilon_m)\neq\emptyset$ is bounded by $C\ell(D)^{n-1}2^{i(n-1)}$ due to Lemma 8.2, arguing as below (8.10). Further, for a fixed $Q\in\mathcal{D}$ such that $Q\subset C_bD$,

$$
\sum_{\substack{m\in\mathcal{S}_j:\,100\sqrt{n}C_w\ell(Q)<\epsilon_m-\epsilon_{m+1},\\ C_wQ\cap\Gamma\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}}\frac{|\epsilon_m - \epsilon_{m+1}|^{1/2}}{\ell(D)^{1/2}} \lesssim 1,
$$

because $|\epsilon_m - \epsilon_{m+1}| < \ell(D)$ and the sum contains finitely many terms, depending only on d, n and $\mathrm{Lip}(A)$. Applying these remarks in (8.14) and interchanging the order of summation, we obtain

$$
\begin{aligned}
\sum_{m\in\mathcal{S}_j}\bigg(\sum_{Q\in J_m^3}|U3_m^B(x,Q)|\bigg)^2 &\lesssim \sum_{m\in\mathcal{S}_j}\ell(D)^{-n-1}|\epsilon_m-\epsilon_{m+1}|^{1/2}\sum_{Q\in J_m^3}\ell(Q)^{1/2}\|\Delta_Q g\|_2^2\\
&=\sum_{m\in\mathcal{S}_j}\sum_{Q\in J_m^3}\frac{|\epsilon_m-\epsilon_{m+1}|^{1/2}}{\ell(D)^{1/2}}\frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}}\|\Delta_Q g\|_2^2\lesssim\sum_{Q\in\mathcal{D}\colon Q\subset C_bD}\frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}}\|\Delta_Q g\|_2^2.
\end{aligned}
\tag{8.15}
$$

Finally, combining (8.8), (8.11), (8.12), (8.13), and (8.15), we conclude

$$
\sum_{m\in\mathcal{S}_j}|U3_m(x)|^2\lesssim\sum_{Q\in\mathcal{D}\colon Q\subset C_bD}\frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}}\|\Delta_Q g\|_2^2.\tag{8.16}
$$

8.2.3. **Estimate of** $\sum_{m\in\mathcal{S}_j}|U2_m(x)|^2$. Let L_D be a minimizing n-plane for $\alpha_\mu(D)$ and let L_D^x be the n-plane parallel to L_D which contains x. Let $A_x:\mathbb{R}^n\to L_D^x$ be the parametrization of the n-plane L_D^x given by $A_x(\widetilde{z}):=(\widetilde{z},a_x(\widetilde{z}))$ where $a_x:\mathbb{R}^n\to\mathbb{R}^{d-n}$ is some affine map, and let $|J(A_x)|$ denote its n-dimensional Jacobian (notice that $|J(A_x)|$ is bounded by some constant depending only on $\operatorname{Lip}(A)$). Given $z\in\mathbb{R}^d$, let p_0^x denote the orthogonal projection onto L_D^x, and for $z\in\mathbb{R}^d\setminus(p_0^x)^{-1}(x)$, consider the angular projection given by

$$
p^x(z):=x+(p_0^x(z)-x)\frac{|z-x|}{|p_0^x(z)-x|}.\tag{8.17}
$$

If $z\in\mathbb{R}^d\setminus(p_0^x)^{-1}(x)$, then $|p_0^x(z)-x|\neq0$, so p^x is well defined and $|z-x|=|p^x(z)-x|$. Since Γ is a Lipschitz graph with slope strictly less than 1, we have $(p_0^x)^{-1}(x)\cap\Gamma=\{x\}$, because the slope of the n-plane L_D^x is also smaller than 1 and then the $(d-n)$-plane passing through x and orthogonal to L_D^x does not intersect the cone

$$
\{y\in\mathbb{R}^d\,:\,|(y-\widetilde{y})-(x-\widetilde{x})|<|\widetilde{y}-\widetilde{x}|\},
$$

so it cannot contain any other point of Γ different from x. Thus, we can extend p^x to the whole graph Γ by setting $p^x(x)=x$. Notice that $p^x|_\Gamma$ is a Lipschitz function (with Lipschitz constant depending on $\operatorname{Lip}(A)$).

For $y\in\Gamma$, set $d\mu(y):=|J(A_\Gamma)(\widetilde{y})|^{-1}\,d\mathcal{H}^n_\Gamma(y)$ and $\nu_x:=p^x_\sharp\mu$. Then, by the definition of $U2_m$ in (8.6),

$$
\begin{aligned}
U2_m(x)&=\Psi_Dg(x)\int_\Gamma K(x-y)\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d\mu(y)\\
&=\Psi_Dg(x)\int_\Gamma K(x-y)\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d(\mu-\nu_x)(y)\\
&\quad+\Psi_Dg(x)\int_\Gamma K(x-y)\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d\nu_x(y)=:U4_m(x)+U5_m(x).
\end{aligned}
\tag{8.18}
$$

Notice that $\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)=\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-p^x(y))$. This is the main reason why we use the angular projection p^x instead of p_0^x or a "vertical" one. Since $|y-p^x(y)|\lesssim\operatorname{dist}(y,L_D^x)\le\operatorname{dist}(y,L_D)+\operatorname{dist}(x,L_D)$ for all $y\in\Gamma$,

$$
\begin{aligned}
|U4_m(x)|&\le|\Psi_Dg(x)|\int_\Gamma|K(x-y)-K(x-p^x(y))|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d\mu(y)\\
&\lesssim|\Psi_Dg(x)|\ell(D)^{-n-1}\int_\Gamma|y-p^x(y)|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d\mu(y)\\
&\lesssim|\Psi_Dg(x)|\ell(D)^{-n-1}\int_\Gamma(\operatorname{dist}(y,L_D)+\operatorname{dist}(x,L_D))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d\mu(y).
\end{aligned}
\tag{8.19}
$$

If L_D^1 denotes a minimizing n-plane for $\beta_1(D)$, one can show that $\operatorname{dist}_{\mathcal{H}}(L_D\cap C_\Gamma D, L_D^1\cap C_\Gamma D)\lesssim \alpha(D)\ell(D)$, so $\operatorname{dist}(y,L_D)\lesssim \operatorname{dist}(y,L_D^1)+\alpha_\mu(D)\ell(D)$ for $y\in C_\Gamma D\cap\Gamma$. Therefore,

$$
\begin{aligned}
&\textstyle\sum_{m\in\mathcal{S}_j}|U4_m(x)|^2\\
&\quad\lesssim |\Psi_D g(x)|^2\bigg(\ell(D)^{-n-1}\sum_{m\in\mathcal{S}_j}\int_\Gamma(\operatorname{dist}(y,L_D)+\operatorname{dist}(x,L_D))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d\mu(y)\bigg)^2\\
&\quad\lesssim |\Psi_D g(x)|^2\bigg(\ell(D)^{-n-1}\int_{C_\Gamma D}(\operatorname{dist}(y,L_D)+\operatorname{dist}(x,L_D))\,d\mu(y)\bigg)^2\\
&\quad\lesssim |\Psi_D g(x)|^2\bigg(\beta_{1,\mu}(D)^2+\alpha_\mu(D)^2+\bigg(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\bigg)^2\bigg).
\end{aligned}
\tag{8.20}
$$

Let us consider $U5_m(x)$ now. We can assume that ν_x is absolutely continuous with respect to $\mathcal{H}^n_{L_D^x}$, because the set of points $x\in\Gamma$ for which this statement does not hold has countable many elements, thus it has μ-measure zero. Let h_x be the corresponding density, so $\nu_x=h_x\mathcal{H}^n_{L_D^x}$. Finally, set $u_x(y):=h_x(A_x(y))|J(A_x)|$ for $y\in\mathbb{R}^n$. Then, since $\int_{\mathbb{R}^n}K(x-A_x(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))\,dy=0$ because L_D^x is an n-plane containing x,

$$
\begin{aligned}
U5_m(x)&=\Psi_D g(x)\int_\Gamma K(x-y)\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\,d\nu_x(y)\\
&=\Psi_D g(x)\int_{\mathbb{R}^n}K(x-A_x(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))u_x(y)\,dy\\
&=\Psi_D g(x)\int_{\mathbb{R}^n}K(x-A_x(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))(u_x(y)-|J(A_x)|)\,dy.
\end{aligned}
\tag{8.21}
$$

Since A has compact support, we may assume that $h_x-1\in L^2(\mathcal{H}^n_{L_D^x})$. Therefore, $u_x(y)-|J(A_x)|\in L^2(\mathcal{L}^n)$. Consider the decomposition of this function with respect to the wavelet basis, i.e., $u_x-|J(A_x)|=\sum_{Q\in\mathcal{D}}\Delta_Q(u_x-|J(A_x)|)=\sum_{Q\in\mathcal{D}}\Delta_Q u_x$ (notice that, for any $Q\in\mathcal{D}$, $\int|J(A_x)|\psi_Q\,d\mathcal{L}^n=0$).

Set $J:=\{Q\in\mathcal{D}\,:\,\operatorname{supp}\psi_Q\cap\operatorname{supp}\chi_{2^{-j-1}}^{2^{-j}}(x-A_x(\cdot))\neq\emptyset\}$. Recall that $D\in\mathcal{D}_j$ and $m\in\mathcal{S}_j$. Since $x\in D$ and $\ell(D)=2^{-j}$, if $Q\in J$, then $D\subset C_bQ$ or $Q\subset C_bD$ for C_b big enough. In particular, if $100\sqrt{n}C_w\ell(Q)>\ell(D)$ then $D\subset C_bQ$, and if $100\sqrt{n}C_w\ell(Q)\le\ell(D)$ then $Q\subset C_bD$ and $\operatorname{dist}(x,C_wQ)\gtrsim\ell(D)$.

We define $J_1:=\{Q\in J\,:\,100\sqrt{n}C_w\ell(Q)\le\ell(D)\}\subset\{Q\in\mathcal{D}\,:\,Q\subset C_bD\}$ and $J_2:=J\setminus J_1\subset\{Q\in\mathcal{D}\,:\,D\subset C_bQ\}$, thus $\operatorname{dist}(x,C_wQ)\gtrsim\ell(D)$ for all $Q\in J_1$. Then, since $\operatorname{supp}\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(\cdot))\subset\operatorname{supp}\chi_{2^{-j-1}}^{2^{-j}}(x-A_x(\cdot))$ for all $m\in\mathcal{S}_j$,

$$
\begin{aligned}
U5_m(x)&=\Psi_D g(x)\int_{\mathbb{R}^n}K(x-A_x(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))\sum_{Q\in J_1}\Delta_Q u_x(y)\,dy\\
&\qquad+\Psi_D g(x)\int_{\mathbb{R}^n}K(x-A_x(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))\sum_{Q\in J_2}\Delta_Q u_x(y)\,dy\\
&=:U6_m(x)+U7_m(x).
\end{aligned}
\tag{8.22}
$$

The sum $\sum_{m\in\mathcal{S}_j}|U6_m(x)|^2$ can be estimated using almost the same arguments as the ones for $\sum_{m\in\mathcal{S}_j}|U3_m(x)|^2$ in subsection 8.2.2 (see (8.16)), and then one obtains

$$\sum_{m\in\mathcal{S}_j}|U6_m(x)|^2\lesssim|\Psi_D g(x)|^2\sum_{Q\in J_1}\frac{\ell(Q)^{1/2}}{\ell(D)^{n+1/2}}\,\|\Delta_Q u_x\|_2^2. \tag{8.23}$$

For the case of $U7_m(x)$, recall that $\int_{\mathbb{R}^n}K(x-A_x(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))\,dy=0$ because of the antisymmetry of K and the flatness of L_D^x. Therefore,

$$U7_m(x)=\Psi_D g(x)\sum_{Q\in J_2}\int_{\mathbb{R}^n}K(x-A_x(y))\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))\big(\Delta_Q u_x(y)-\Delta_Q u_x(x)\big)\,dy,$$

and then, since $|\Delta_Q u_x(y)-\Delta_Q u_x(x)|\lesssim\|\Delta_Q u_x\|_2\ell(Q)^{-n/2-1}|x-y|$ by Definition 8.4(e),

$$|U7_m(x)|\lesssim|\Psi_D g(x)|\sum_{Q\in J_2}\ell(D)^{-n+1}\|\Delta_Q u_x\|_2\ell(Q)^{-n/2-1}\int_{\mathbb{R}^n}\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-A_x(y))\,dy.$$

Finally, by Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{m\in\mathcal{S}_j}|U7_m(x)|^2&\lesssim|\Psi_D g(x)|^2\bigg(\sum_{Q\in J_2}\frac{\ell(D)}{\ell(Q)}\,\ell(Q)^{-n/2}\|\Delta_Q u_x\|_2\bigg)^2\\&\lesssim|\Psi_D g(x)|^2\sum_{Q\in J_2}\frac{\ell(D)}{\ell(Q)}\,\ell(Q)^{-n}\|\Delta_Q u_x\|_2^2.\end{aligned}\tag{8.24}$$

Lemma 8.6. *Given $Q\in\mathcal{D}$, one has $\|\Delta_Q u_x\|_2\lesssim\alpha_{\nu_x}(Q)\ell(Q)^{n/2}$. Moreover, there exist absolute constants $C_1,C_2>1$ such that, given $Q\in\mathcal{D}$,*

(a) *if $D\subset C_bQ$, then*

$$\alpha_{\nu_x}(Q)\lesssim\sum_{R\in\mathcal{D}:\,D\subset R\subset C_1Q}\alpha_\mu(C_1R)+\frac{\operatorname{dist}(x,L_D)}{\ell(D)},\quad \textit{and}$$

(b) *if $Q\subset C_bD$ and $\operatorname{dist}(x,C_wQ)\gtrsim\ell(D)$, there exists a v-cube $Q_0\equiv Q_0(x,Q)\in\mathcal{D}$ depending on x and Q such that $Q_0\subset C_2D$, $\ell(Q_0)\approx\ell(Q)$, $\Gamma\cap Q_0\cap(p^x)^{-1}(Q\cap L_D^x)\neq\emptyset$ and*

$$\alpha_{\nu_x}(Q)\lesssim\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)+\frac{\operatorname{dist}(x,L_D)}{\ell(D)}.$$

Proof. Given $Q\in\mathcal{D}$, define the function $\phi_Q:\mathbb{R}^d\to\mathbb{R}$ by $\phi_Q(y)=\psi_Q(\widetilde{y})$ (see Definition 8.4 for ψ_Q). Then $\operatorname{supp}\phi_Q\subset C_wQ$ and $\int\phi_Q\,d\mathcal{H}^n_L=0$ for all n-planes $L\subset\mathbb{R}^d$ which are not perpendicular to $\mathbb{R}^n\times\{0\}^{d-n}$. Notice also that $|\nabla\phi_Q|\lesssim\ell(Q)^{-n/2-1}$. Let λ_Q be a minimizing measure for $\alpha_{\nu_x}(Q)$. Then,

$$\begin{aligned}\|\Delta_Q u_x\|_2=\|\langle u_x,\psi_Q\rangle\psi_Q\|_2=|\langle u_x,\psi_Q\rangle|&=\Big|\int_{\mathbb{R}^n}\psi_Q(y)h_x(A_x(y))|J(A_x)|\,dy\Big|\\&=\Big|\int_{L_D^x}\phi_Q(y)\,d\nu_x(y)\Big|=\Big|\int\phi_Q(y)\,d(\nu_x-\lambda_Q)(y)\Big|\\&\lesssim\ell(Q)^{-n/2-1}\operatorname{dist}_{B_Q}(\nu_x,\lambda_Q)\lesssim\ell(Q)^{n/2}\alpha_{\nu_x}(Q),\end{aligned}$$

which proves the first statement of the lemma.

Assume now that $D \subset C_b Q$. Let L_Q be a minimizing n-plane for $\alpha_\mu(C_1 Q)$, where $C_1 > 1$ is some big constant to be fixed below, and let L_Q^x be the n-plane parallel to L_Q which contains x. Let $\sigma_Q := c_Q \mathcal{H}^n_{L_Q}$ be a minimizing measure for $\alpha_\mu(C_1 Q)$ and define $\sigma_Q^x := c_Q \mathcal{H}^n_{L_Q^x}$. Finally, set $\sigma := c_Q \mathcal{H}^n_{L_D^x}$.

Similarly to what we said below (8.17), one can verify that p^x is well defined on L_Q^x. Since σ is a flat measure,

$$\begin{aligned}\alpha_{\nu_x}(Q) &\le \ell(Q)^{-n-1}\text{dist}_{B_Q}(\nu_x, \sigma)\\ &\le \ell(Q)^{-n-1}\text{dist}_{B_Q}(\nu_x, p^x_\sharp \sigma_Q^x) + \ell(Q)^{-n-1}\text{dist}_{B_Q}(p^x_\sharp\sigma_Q^x, \sigma).\end{aligned} \tag{8.25}$$

To deal with the first term on the right hand side of (8.25), let h be a Lipschitz function such that supp$h \subset B_Q$ and $\text{Lip}(h) \le 1$. Then, the function $h \circ p^x$ restricted to $\Gamma \cup L_Q^x$ can be extended to a Lipschitz function supported in $B_{C_1 Q}$ (if C_1 is big enough) with $\text{Lip}(h \circ p^x)$ bounded by a constant which only depends on n, d, and $\text{Lip}(A)$. Therefore,

$$\begin{aligned}\Big|\int_{B_Q} h\, d(\nu_x - p^x_\sharp \sigma_Q^x)\Big| &= \Big|\int_{B_{C_1Q}} h\circ p^x\, d(\mu - \sigma_Q^x)\Big| \lesssim \text{dist}_{B_{C_1Q}}(\mu, \sigma_Q^x)\\ &\le \text{dist}_{B_{C_1Q}}(\mu, \sigma_Q) + \text{dist}_{B_{C_1Q}}(\sigma_Q, \sigma_Q^x)\\ &\lesssim \alpha_\mu(C_1Q)\ell(Q)^{n+1} + \text{dist}(x, L_Q)\ell(Q)^n.\end{aligned} \tag{8.26}$$

By Remark 2.3 (see (2.7)), since $x \in \Gamma \cap D$ and $D \subset C_1 Q$ (if $C_1 > C_b$),

$$\text{dist}(x, L_Q) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q} \alpha_\mu(R)\ell(R) + \text{dist}(x, L_D). \tag{8.27}$$

Taking the supremum over all possible Lipschitz functions h in (8.26) and using that $\ell(D) \le \ell(R) \le C_b \ell(Q)$ in the sum above, we get

$$\ell(Q)^{-n-1}\text{dist}_{B_Q}(\nu_x, p^x_\sharp \sigma_Q^x) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q} \alpha_\mu(C_1R) + \text{dist}(x, L_D)\ell(D)^{-1}. \tag{8.28}$$

To estimate the second term on the right hand side of (8.25), notice that $p^x_\sharp \sigma = \sigma$ because $p^x|_{L_D^x} = \text{Id}$. Hence, as in (8.26),

$$\begin{aligned}\text{dist}_{B_Q}(p^x_\sharp\sigma_Q^x, \sigma) &= \text{dist}_{B_Q}(p^x_\sharp\sigma_Q^x, p^x_\sharp\sigma) \lesssim \text{dist}_{B_{C_1Q}}(\sigma_Q^x, \sigma)\\ &\le \text{dist}_{B_{C_1Q}}(\sigma_Q^x, \sigma_Q) + \text{dist}_{B_{C_1Q}}(\sigma_Q, \sigma)\\ &\lesssim \text{dist}_{B_{C_1Q}}(\mathcal{H}^n_{L_Q^x}, \mathcal{H}^n_{L_Q}) + \text{dist}_{B_{C_1Q}}(\mathcal{H}^n_{L_Q}, \mathcal{H}^n_{L_D}) + \text{dist}_{B_{C_1Q}}(\mathcal{H}^n_{L_D}, \mathcal{H}^n_{L_D^x})\\ &\lesssim \text{dist}(x, L_Q)\ell(Q)^n + \text{dist}_{B_{C_1Q}}(\mathcal{H}^n_{L_Q}, \mathcal{H}^n_{L_D}) + \text{dist}(x, L_D)\ell(Q)^n.\end{aligned}$$

The term $\text{dist}_{B_{C_1Q}}(\mathcal{H}^n_{L_Q}, \mathcal{H}^n_{L_D})$ can be estimated using the intermediate v-cubes between D and C_1Q as we did in subsection 4.1.2 (see (4.24) for example), and we obtain

$$\text{dist}_{B_{C_1Q}}(\mathcal{H}^n_{L_Q}, \mathcal{H}^n_{L_D}) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q} \alpha_\mu(C_1R)\ell(Q)^{n+1}.$$

Thus, by (8.27) and since $\ell(D) \lesssim \ell(Q)$,

$$\text{dist}_{B_Q}(p^x_\sharp\sigma_Q^x, \sigma) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q} \alpha_\mu(C_1R)\ell(Q)^{n+1} + \text{dist}(x, L_D)\ell(D)^{-1}\ell(Q)^{n+1}.$$

Then, Lemma 8.6(a) follows by plugging this last inequality and (8.28) in (8.25).

Let us turn our attention to Lemma 8.6(b), so assume that $Q \subset C_b D$. Let C_2 be some constant bigger than C_b, and let $Q_0 \in \mathcal{D}$ be a minimal v-cube such that $C_2 Q_0$ contains $\Gamma \cap (p^x)^{-1}(Q \cap L_D^x)$. We can assume $Q_0 \subset C_2 D$ if C_2 is big enough. We may also suppose that $\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)$ is small enough, otherwise the estimate that we want to prove would be trivial; indeed, if $C_0\leq\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)$ for some absolute constant $C_0>0$, then $\alpha_{\nu_x}(Q)\lesssim 1\leq C_0^{-1}\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)$.

One can show that, if $\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)\leq C_0$ with C_0 small enough, then

$$\operatorname{diam}(\Gamma\cap(p^x)^{-1}(Q\cap L_D^x))\lesssim\ell(Q). \tag{8.29}$$

Indeed, since $\alpha_\mu(C_2D)$ is small by our assumption, then $\beta_{\infty,\mu}(C_2D)$ is also small. Take $z_1,z_2\in\Gamma\cap(p^x)^{-1}(Q\cap L_D^x)$ such that $|z_1-z_2|=\operatorname{diam}(\Gamma\cap(p^x)^{-1}(Q\cap L_D^x))$, and set $y_1:=p^x(z_1)$ and $y_2:=p^x(z_2)$. We claim that $|z_1-z_2|\lesssim|y_1-y_2|$. Otherwise, the angle between $L_{z_1,x}$ and L_{z_1,z_2} would be big, where $L_{u,v}$ denotes the line passing through the points $u,v\in\mathbb{R}^d$. Since $\beta_{\infty,\mu}(C_2D)$ is small by hypothesis, the angle between $L_{z_1,x}$ and L_D^x is also small. Therefore, the angle between L_{z_1,z_2} and L_D^x would be big. Since $\alpha_\mu(C_2Q_0)$ is small by hypothesis (and the same holds for $\beta_{\infty,\mu}(C_2Q_0)$), the angle between L_{z_1,z_2} and L_{Q_0} is also small, where L_{Q_0} is a minimizing n-plane for $\beta_{\infty,\mu}(C_2Q_0)$. Therefore, the angle between L_{Q_0} and L_D^x would be big, but this can not happen because that angle is bounded by $\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)$, which is small by hypothesis. Hence, $|z_1-z_2|\lesssim|y_1-y_2|$, and this easily implies (8.29). By hypothesis, $\Gamma\cap(p^x)^{-1}(Q\cap L_D^x)\subset C_2Q_0$ and, by (8.29), $\ell(Q_0)\approx\ell(Q)$ if C_0 is small enough.

Let L_{Q_0} and $\sigma_{Q_0}:=c_{Q_0}\mathcal{H}^n_{L_{Q_0}}$ be a minimizing n-plane and measure for $\alpha_\mu(C_2Q_0)$, respectively. Fix $z_{Q_0}\in L_{Q_0}\cap B_{C_2Q_0}$ and let L_r be an n-plane parallel to L_D^x which contains z_{Q_0}. Finally, define the measures $\sigma_r:=c_{Q_0}\mathcal{H}^n_{L_r}$ and $\sigma':=c_{Q_0}\mathcal{H}^n_{L_D^x}$.

Notice that p^x is well defined on $(L_{Q_0}\cup L_r)\cap B_{C_2Q_0}$ because $\operatorname{dist}(x,Q)\gtrsim\ell(D)$ (we may assume that $\ell(Q)$ is small enough). Since σ' is a flat measure, by the triangle inequality,

$$\begin{aligned}\alpha_{\nu_x}(Q)\ell(Q)^{n+1}&\leq\operatorname{dist}_{B_Q}(\nu_x,\sigma')\\&\leq\operatorname{dist}_{B_Q}(\nu_x,p^x_\sharp\sigma_{Q_0})+\operatorname{dist}_{B_Q}(p^x_\sharp\sigma_{Q_0},p^x_\sharp\sigma_r)+\operatorname{dist}_{B_Q}(p^x_\sharp\sigma_r,\sigma').\end{aligned}\tag{8.30}$$

Arguing as in (8.26), if C_2 is big enough, we have

$$\operatorname{dist}_{B_Q}(\nu_x,p^x_\sharp\sigma_{Q_0})=\operatorname{dist}_{B_Q}(p^x_\sharp\mu,p^x_\sharp\sigma_{Q_0})\lesssim\alpha_\mu(C_2Q_0)\ell(Q)^{n+1}, \tag{8.31}$$

and

$$\operatorname{dist}_{B_Q}(p^x_\sharp\sigma_{Q_0},p^x_\sharp\sigma_r)\lesssim\operatorname{dist}_{B_{C_2Q_0}}(\sigma_{Q_0},\sigma_r)\lesssim\operatorname{dist}_{\mathcal{H}}(L_{Q_0}\cap B_{C_2Q_0},L_r\cap B_{C_2Q_0})\ell(Q)^n.$$

Let γ be the angle between L_r and L_{Q_0} (which is the same as the one between L_D and L_{Q_0}). Since $z_{Q_0}\in L_{Q_0}\cap L_r\cap B_{C_2Q_0}$, we have $\operatorname{dist}_{\mathcal{H}}(L_{Q_0}\cap B_{C_2Q_0},L_r\cap B_{C_2Q_0})\lesssim\sin(\gamma)\ell(Q)$, and it is not difficult to show that $\sin(\gamma)\lesssim\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)$. Thus,

$$\operatorname{dist}_{B_Q}(p^x_\sharp\sigma_{Q_0},p^x_\sharp\sigma_r)\lesssim\sum_{R\in D:\,Q_0\subset R\subset C_2D}\alpha_\mu(C_2R)\ell(Q)^{n+1}. \tag{8.32}$$

Let us estimate the last term on the right hand side of (8.30). Since $c_{Q_0}\lesssim 1$, we have $\operatorname{dist}_{B_Q}(p^x_\sharp\sigma_r,\sigma')\lesssim\operatorname{dist}_{B_Q}(p^x_\sharp\mathcal{H}^n_{L_r},\mathcal{H}^n_{L_D^x})$. Let h be a 1-Lipschitz function supported in B_Q and such that

$$\operatorname{dist}_{B_Q}(p^x_\sharp\mathcal{H}^n_{L_r},\mathcal{H}^n_{L_D^x})\approx\left|\int h\,d(p^x_\sharp\mathcal{H}^n_{L_r}-\mathcal{H}^n_{L_D^x})\right|. \tag{8.33}$$

Set $d:=\operatorname{dist}(z_{Q_0},L_D^x)$. Without loss of generality, we may assume that $x=0$ and that $L_D^x=\mathbb{R}^n\times\{0\}^{d-n}$, so $L_r=z_{Q_0}+\mathbb{R}^n\times\{0\}^{d-n}$. Then, if we set $z'_{Q_0}:=(z^{n+1}_{Q_0},\dots,z^d_{Q_0})$, we have that $d = |z'_{Q_0}|$ and p^x restricted to L_r can be written in the following manner: $p^x : y = (y^1, \dots, y^n, z'_{Q_0}) \mapsto (F(y^1, \dots, y^n), 0)$, where $F : \mathbb{R}^n \setminus \{0\}^n \to \mathbb{R}^n$ is defined by

$$F(y) = y\frac{\sqrt{|y|^2 + d^2}}{|y|} = y\sqrt{1 + \frac{d^2}{|y|^2}}.$$

Therefore, $\int h\, d(p^x_\sharp \mathcal{H}^n_{L_r}) = \int h \circ p^x\, d\mathcal{H}^n_{L_r} = \int_{\mathbb{R}^n} (h \circ p^x)(y, z'_{Q_0})\, dy = \int_{\mathbb{R}^n} h(F(y), 0)\, dy$, and we also have $\int h\, d\mathcal{H}^n_{L^x_D} = \int_{\mathbb{R}^n} h((y, 0))\, dy = \int_{\mathbb{R}^n} h(F(y), 0) J(F)(y)\, dy$ by a change of variables, where $J(F)$ denotes the Jacobian of F (we may assume $\operatorname{dist}(0, \operatorname{supp} h(F(\cdot), 0)) \gtrsim \ell(D)$, because $\operatorname{dist}(x, Q) \gtrsim \ell(D)$ and we can assume that $\ell(Q)$ is small enough). Hence, by (8.33),

$$\operatorname{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \lesssim \int_{\mathbb{R}^n} |h(F(y), 0)||1 - J(F)(y)|\, dy.$$

Notice that, because of the assumptions on $\operatorname{supp} h(F(\cdot), 0)$ and since $z_{Q_0} \in B_{C_2 Q_0}$ and $Q_0 \subset C_2 D$, we have $d \lesssim |y|$ for all $y \in \operatorname{supp} h(F(\cdot), 0)$. If F_i denotes the i'th coordinate of F, it is straightforward to check that $\partial_{y^j} F_i(y) = -d^2 y^i y^j |y|^{-3}(|y|^2 + d^2)^{-1/2}$ if $i \neq j$ and $\partial_{y^i} F_i(y) = (1 + d^2/|y|^2)^{1/2} - d^2 (y^i)^2 |y|^{-3}(|y|^2 + d^2)^{-1/2}$. Thus, we easily obtain $|1 - J(F)(y)| \lesssim d/|y| \lesssim d/\ell(D)$ for all $y \in \operatorname{supp} h(F(\cdot), 0)$.

Since $\operatorname{diam}(\operatorname{supp} h(F(\cdot), 0)) \lesssim \ell(Q)$ and $h((F(\cdot), 0))$ is Lipschitz, using the previous comments we have $\operatorname{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \lesssim \ell(Q)^{n+1} d/\ell(D)$. Finally, by Remark 2.3 (see (2.7)) and since $z_{Q_0} \in L_{Q_0}$,

$$d \lesssim \operatorname{dist}(z_{Q_0}, L_D) + \operatorname{dist}(L_D, L^x_D) \lesssim \sum_{R \in \mathcal{D} \colon Q_0 \subset R \subset C_2 D} \alpha_\mu(C_2 R)\ell(R) + \operatorname{dist}(x, L_D),$$

and thus

$$\operatorname{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \lesssim \sum_{R \in \mathcal{D} \colon Q_0 \subset R \subset C_2 D} \alpha_\mu(C_2 R)\ell(Q)^{n+1} + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\ell(Q)^{n+1}. \tag{8.34}$$

Lemma 8.6(b) follows by applying (8.31), (8.32), and (8.34) to (8.30). □

8.3. **Estimate of** $\sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{S}_j} |K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f\mathcal{H}^n_\Gamma)|^2\, d\mathcal{H}^n_\Gamma$ **in (8.4).** From (8.6), (8.18), and (8.22), we have

$$\begin{aligned} &\sum_{m \in \mathcal{S}_j} |(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f\mathcal{H}^n_\Gamma))(x)|^2 \\ &\qquad \lesssim \sum_{m \in \mathcal{S}_j} (|U1_m(x)|^2 + |U3_m(x)|^2 + |U4_m(x)|^2 + |U6_m(x)|^2 + |U7_m(x)|^2), \end{aligned} \tag{8.35}$$

First of all, by (8.7) we have

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathcal{S}_j} |U1_m|^2\, d\mathcal{H}^n_\Gamma &\lesssim \sum_{D \in \mathcal{D}} \sum_{Q \in \mathcal{D} \,:\, D \subset C_b Q} \frac{\ell(D)^{n+1}}{\ell(Q)^{n+1}} \|\Delta_Q g\|_2^2 \\ &= \sum_{Q \in \mathcal{D}} \|\Delta_Q g\|_2^2 \sum_{D \in \mathcal{D} \,:\, D \subset C_b Q} \frac{\ell(D)^{n+1}}{\ell(Q)^{n+1}} \lesssim \sum_{Q \in \mathcal{D}} \|\Delta_Q g\|_2^2 = \|g\|_2^2, \end{aligned}$$

and by (8.16),

$$\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|U3_m|^2\,d\mathcal{H}^n_\Gamma \lesssim \sum_{D\in\mathcal{D}}\;\sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD}\frac{\ell(Q)^{1/2}}{\ell(D)^{1/2}}\,\|\Delta_Q g\|_2^2$$
$$=\sum_{Q\in\mathcal{D}}\|\Delta_Q g\|_2^2\sum_{D\in\mathcal{D}\,:\,Q\subset C_bD}\frac{\ell(Q)^{1/2}}{\ell(D)^{1/2}}\lesssim\sum_{Q\in\mathcal{D}}\|\Delta_Q g\|_2^2=\|g\|_2^2.$$

For the case of $U4_m(x)$, it is known that $|\Psi_D g(x)|\lesssim|g|_{C_aD}$ for $x\in D$ (see [Da, Part I]), where $C_a>0$ is some constant depending on C_b (see the definition of $\Psi_D g$ just after Remark 8.5) and $|g|_{C_aD}:=\mathcal{L}^n(C_aD)^{-1}\int_{C_aD}|g(y)|\,dy$. If L^1_D and L^2_D denote a minimizing n-plane for $\beta_{1,\mu}(D)$ and $\beta_{2,\mu}(D)$, respectively, one can show that $\operatorname{dist}_{\mathcal{H}}(L_D\cap C_\Gamma D, L^1_D\cap C_\Gamma D)\lesssim\alpha_\mu(D)\ell(D)$ and $\operatorname{dist}_{\mathcal{H}}(L^1_D\cap C_\Gamma D, L^2_D\cap C_\Gamma D)\lesssim\beta_{2,\mu}(D)\ell(D)$, so we have $\operatorname{dist}(x,L_D)\lesssim\operatorname{dist}(x,L^2_D)+\beta_{2,\mu}(D)\ell(D)+\alpha_\mu(D)\ell(D)$ for $x\in D\cap\Gamma$. Then, by (8.20),

$$\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|U4_m|^2\,d\mathcal{H}^n_\Gamma$$
$$\lesssim\sum_{D\in\mathcal{D}}|g|^2_{C_aD}(\beta_{1,\mu}(D)^2+\alpha_\mu(D)^2)\ell(D)^n+\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\int_D\left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2d\mathcal{H}^n_\Gamma(x)$$
$$\lesssim\sum_{D\in\mathcal{D}}|g|^2_{C_aD}(\beta_{1,\mu}(D)^2+\beta_{2,\mu}(D)^2+\alpha_\mu(D)^2)\ell(D)^n\lesssim\|g\|_2^2,$$

where we used in the last inequality that the α_μ, $\beta_{1,\mu}$ and $\beta_{2,\mu}$ coefficients satisfy a Carleson packing condition, and so we can apply the Carleson's embedding theorem.

For $U6_m(x)$, by (8.23) and Lemma 8.6(b), we have the estimate

$$\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|U6_m|^2\,d\mathcal{H}^n_\Gamma$$
$$\lesssim\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\int_D\sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2d\mathcal{H}^n_\Gamma(x)$$
$$+\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\int_D\sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}:\,Q_0(x,Q)\subset R\subset C_2D}\alpha_\mu(C_2R)\Bigg)^2d\mathcal{H}^n_\Gamma(x).$$

Since $\sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD}(\ell(Q)\ell(D)^{-1})^{n+1/2}\lesssim1$ and $\operatorname{dist}(x,L_D)\lesssim\operatorname{dist}(x,L^2_D)+\beta_{2,\mu}(D)\ell(D)+\alpha_\mu(D)\ell(D)$ for $x\in D\cap\Gamma$, the first term on the right hand side of the last inequality is bounded by $\sum_{D\in\mathcal{D}}|g|^2_{C_aD}(\beta_{2,\mu}(D)^2+\alpha_\mu(D)^2)\ell(D)^n$, and hence by $C\|g\|_2^2$, by Carleson's embedding theorem on Carleson measures. For the second term on the right side, since $\ell(Q)\approx\ell(Q_0(x,Q))$ (recall the definition of $Q_0\equiv Q_0(x,Q)$ in Lemma 8.6(b)), $Q_0(x,Q)\subset C_2D$, and every $Q_0\in\mathcal{D}$ intersects $\Gamma\cap(p^x)^{-1}(Q\cap L^x_D)$ for finitely many v-cubes $Q\in\mathcal{D}$ (with a bound for the number of such v-cubes Q independent of x and Q_0), we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD}&\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}\colon Q_0(x,Q)\subset R\subset C_2D}\alpha_\mu(C_2R)\Bigg)^2\\&=\sum_{S\in\mathcal{D}\,:\,S\subset C_2D}\;\sum_{\substack{Q\in\mathcal{D}\,:\\ Q\subset C_bD,\,Q_0(x,Q)=S}}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}\colon S\subset R\subset C_2D}\alpha_\mu(C_2R)\Bigg)^2\\&\lesssim\sum_{S\in\mathcal{D}\,:\,Q\subset C_2D}\left(\frac{\ell(S)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}\colon S\subset R\subset C_2D}\alpha_\mu(C_2R)\Bigg)^2.\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{S\in\mathcal{D}\,:\,S\subset C_2D}&\left(\frac{\ell(S)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}\colon S\subset R\subset C_2D}\alpha_\mu(C_2R)\Bigg)^2\\&\lesssim\sum_{S\in\mathcal{D}\,:\,S\subset C_2D}\left(\frac{\ell(S)}{\ell(D)}\right)^{n+1/2}\log_2\left(\frac{\ell(D)}{\ell(S)}\right)\sum_{R\in\mathcal{D}\colon S\subset R\subset C_2D}\alpha_\mu(C_2R)^2\\&\lesssim\sum_{S\in\mathcal{D}\,:\,S\subset C_2D}\left(\frac{\ell(S)}{\ell(D)}\right)^{n+1/4}\sum_{R\in\mathcal{D}\colon S\subset R\subset C_2D}\alpha_\mu(C_2R)^2\\&\leq\sum_{R\in\mathcal{D}\colon R\subset C_2D}\alpha_\mu(C_2R)^2\sum_{S\in\mathcal{D}\,:\,S\subset R}\left(\frac{\ell(S)}{\ell(D)}\right)^{n+1/4}\\&\lesssim\sum_{R\in\mathcal{D}\colon R\subset C_2D}\alpha_\mu(C_2R)^2\left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/4}=:\lambda_1(D).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\int_D\sum_{Q\in\mathcal{D}\,:\,Q\subset C_bD}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}\colon Q_0(x,Q)\subset R\subset C_2D}\alpha_\mu(C_2R)\Bigg)^2d\mathcal{H}^n_\Gamma(x)\\\lesssim\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\lambda_1(D)^2\ell(D)^n.\end{aligned}$$

Let us check that the coefficients $\lambda_1(D)$ satisfy a Carleson packing condition, so they originate a Carleson measure. For all $S\in\mathcal{D}$,

$$\begin{aligned}\sum_{D\in\mathcal{D}\colon D\subset S}\lambda_1(D)^2\ell(D)^n&=\sum_{D\in\mathcal{D}\colon D\subset S}\;\sum_{R\in\mathcal{D}\colon R\subset C_2D}\alpha_\mu(C_2R)^2\left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/4}\ell(D)^n\\&\leq\sum_{R\in\mathcal{D}\colon R\subset C_2S}\alpha_\mu(C_2R)^2\ell(R)^n\sum_{D\in\mathcal{D}\colon R\subset C_2D}\left(\frac{\ell(R)}{\ell(D)}\right)^{1/4}\\&\lesssim\sum_{R\in\mathcal{D}\colon R\subset C_2S}\alpha_\mu(C_2R)^2\ell(R)^n\lesssim\ell(S)^n.\end{aligned}$$

Then, $\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|U6_m|^2\,d\mathcal{H}^n_\Gamma\lesssim\|g\|_2^2$, by the Carleson embedding theorem.

For $U7_m(x)$, using (8.24) and Lemma 8.6(a), we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|U7_m|^2\,d\mathcal{H}^n_\Gamma\\ &\lesssim \sum_{D\in\mathcal{D}}|g|^2_{C_aD}\int_D\sum_{Q\in\mathcal{D}\,:\,D\subset C_bQ}\frac{\ell(D)}{\ell(Q)}\left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2 d\mathcal{H}^n_\Gamma(x)\\ &\quad+\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\ell(D)^n\sum_{Q\in\mathcal{D}\,:\,D\subset C_bQ}\frac{\ell(D)}{\ell(Q)}\bigg(\sum_{R\in\mathcal{D}:\,D\subset R\subset C_1Q}\alpha_\mu(C_1R)\bigg)^2.\end{aligned}$$

Since $\operatorname{dist}(x,L_D)\lesssim \operatorname{dist}(x,L^2_D)+\beta_{2,\mu}(D)\ell(D)+\alpha_\mu(D)\ell(D)$ for $x\in D\cap\Gamma$, by Cauchy-Schwarz and Carleson's embedding theorem for the $\beta_{2,\mu}$'s and α_μ's,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|U7_m|^2\,d\mathcal{H}^n_\Gamma &\lesssim \sum_{D\in\mathcal{D}}|g|^2_{C_aD}\int_D\left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2 d\mathcal{H}^n_\Gamma(x)\\ &\quad+\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\ell(D)^n\sum_{Q\in\mathcal{D}\,:\,D\subset C_bQ}\frac{\ell(D)}{\ell(Q)}\log_2\left(\frac{\ell(Q)}{\ell(D)}\right)\sum_{R\in\mathcal{D}:\,D\subset R\subset C_1Q}\alpha_\mu(C_1R)^2\\ &\lesssim \|g\|_2^2+\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\ell(D)^n\sum_{Q\in\mathcal{D}\,:\,D\subset C_bQ}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\sum_{R\in\mathcal{D}:\,D\subset R\subset C_1Q}\alpha_\mu(C_1R)^2\\ &\le \|g\|_2^2+\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\ell(D)^n\sum_{R\in\mathcal{D}\,:\,D\subset R}\alpha_\mu(C_1R)^2\sum_{Q\in\mathcal{D}:\,R\subset C_1Q}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\\ &\lesssim \|g\|_2^2+\sum_{D\in\mathcal{D}}|g|^2_{C_aD}\ell(D)^n\sum_{R\in\mathcal{D}\,:\,D\subset R}\alpha_\mu(C_1R)^2\left(\frac{\ell(D)}{\ell(R)}\right)^{1/2}.\end{aligned}$$

We are going to check that the coefficients $\lambda_2(D)^2:=\sum_{R\in\mathcal{D}\,:\,D\subset R}\alpha_\mu(C_1R)^2\frac{\ell(D)^{1/2}}{\ell(R)^{1/2}}$ satisfy a Carleson packing condition, so they provide a Carleson measure; and then we will conclude that $\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D\sum_{m\in\mathcal{S}_j}|U7_m|^2\,d\mathcal{H}^n_\Gamma\lesssim\|g\|_2^2$. For all $S\in\mathcal{D}$, we have

$$\begin{aligned}\sum_{D\in\mathcal{D}:\,D\subset S}\lambda_2(D)^2\ell(D)^n &= \sum_{D\in\mathcal{D}:\,D\subset S}\;\sum_{R\in\mathcal{D}\,:\,D\subset R}\alpha_\mu(C_1R)^2\left(\frac{\ell(D)}{\ell(R)}\right)^{1/2}\ell(D)^n\\ &= \sum_{D\in\mathcal{D}:\,D\subset S}\;\sum_{R\in\mathcal{D}\,:\,D\subset R\subset S}\alpha_\mu(C_1R)^2\left(\frac{\ell(D)}{\ell(R)}\right)^{n+1/2}\ell(R)^n\\ &\quad+\sum_{D\in\mathcal{D}:\,D\subset S}\;\sum_{R\in\mathcal{D}\,:\,S\subset R}\alpha_\mu(C_1R)^2\left(\frac{\ell(D)}{\ell(R)}\right)^{n+1/2}\ell(R)^n=:I+II.\end{aligned}$$

Concerning I, since the α_μ's satisfy a Carleson packing condition, we get

$$\begin{aligned}I&=\sum_{R\in\mathcal{D}:\,R\subset S}\alpha_\mu(C_1R)^2\ell(R)^n\sum_{D\in\mathcal{D}\,:\,D\subset R}\left(\frac{\ell(D)}{\ell(R)}\right)^{n+1/2}\\ &\lesssim\sum_{R\in\mathcal{D}:\,R\subset S}\alpha_\mu(C_1R)^2\ell(R)^n\lesssim\ell(S)^n,\end{aligned}$$

as wished. For the case of II we use the estimate $\alpha_\mu(C_1 R) \lesssim 1$ for all $R \in \mathcal{D}$, thus

$$\begin{aligned}
II &\lesssim \sum_{R\in\mathcal{D}\,:\,S\subset R}\ \sum_{D\in\mathcal{D}\colon D\subset S} \left(\frac{\ell(D)}{\ell(R)}\right)^{n+1/2} \ell(R)^n \\
&= \sum_{R\in\mathcal{D}\,:\,S\subset R} \left(\frac{\ell(S)}{\ell(R)}\right)^{n+1/2} \ell(R)^n \sum_{D\in\mathcal{D}\colon D\subset S} \left(\frac{\ell(D)}{\ell(S)}\right)^{n+1/2} \\
&\lesssim \ell(S)^n \sum_{R\in\mathcal{D}\,:\,S\subset R} \left(\frac{\ell(S)}{\ell(R)}\right)^{1/2} \lesssim \ell(S)^n.
\end{aligned}$$

Therefore, $\sum_{D\in\mathcal{D}\colon D\subset S} \lambda_2(D)^2 \ell(D)^n \lesssim \ell(S)^n$, as claimed.

Finally, plugging all these estimates in (8.35), we conclude that

$$\sum_{j\in\mathbb{Z}} \sum_{D\in\mathcal{D}_j} \int_D \sum_{m\in\mathcal{S}_j} |K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mathcal{H}^n_\Gamma)|^2 \, d\mathcal{H}^n_\Gamma \lesssim \|g\|_2^2 \approx \|f\|^2_{L^2(\mathcal{H}^n_\Gamma)},$$

as desired.

8.4. **Estimate of $\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D \sum_{m\in\mathcal{L}\,:\,\epsilon_m\in I_j} |K\kappa_{\epsilon_m} * (f\mathcal{H}^n_\Gamma)|^2\, d\mathcal{H}^n_\Gamma$ in (8.4).** Recall that, for $\epsilon > 0$, we have set $\kappa_\epsilon := \chi_\epsilon - \widetilde{\varphi}_\epsilon$ (see the line before (8.3)).

The arguments to estimate $\sum_{j\in\mathbb{Z}}\sum_{D\in\mathcal{D}_j}\int_D \sum_{m\in\mathcal{L}\,:\,\epsilon_m\in I_j} |K\kappa_{\epsilon_m} * (f\mathcal{H}^n_\Gamma)|^2\, d\mathcal{H}^n_\Gamma$ are very similar to the ones in the previous subsections. Basically, we have to replace the function $\chi^{\epsilon_m}_{\epsilon_{m+1}}$ by κ_{ϵ_m} in all the proofs in subsections 8.2 and 8.3, because, in most of the estimates, we only used the properties of the support and the symmetry of the function $\chi^{\epsilon_m}_{\epsilon_{m+1}}$, and κ_{ϵ_m} satisfies analogous properties (κ_ϵ is supported in the closure of $(B^n(0, 3\sqrt{n}\epsilon) \times \mathbb{R}^{d-n}) \setminus B^d(0,\epsilon) \subset \mathbb{R}^d$). Notice that the sum $\sum_{m\in\mathcal{L}\,:\,\epsilon_m\in I_j} |(K\kappa_{\epsilon_m} * (f\mathcal{H}^n_\Gamma))(x)|^2$ only has one term (or none) for each $x \in \Gamma$ and $j \in \mathbb{Z}$.

There are only two details that have to be pointed. The first one is in equation (8.12). Instead, now we have

$$\begin{aligned}
U3_m(x,Q) &= \int \kappa_{\epsilon_m}(x - A_\Gamma(y))\Big(K(x - A_\Gamma(y)) - K(x - A_\Gamma(z_Q))\Big)\Delta_Q g(y)\, dy \\
&\quad + \int \Big(\kappa_{\epsilon_m}(x - A_\Gamma(y)) - \kappa_{\epsilon_m}(x - A_\Gamma(z_Q))\Big) K(x - A_\Gamma(z_Q))\Delta_Q g(y)\, dy \\
&= \int \kappa_{\epsilon_m}(x - A_\Gamma(y))\Big(K(x - A_\Gamma(y)) - K(x - A_\Gamma(z_Q))\Big)\Delta_Q g(y)\, dy \\
&\quad + \int \Big(\chi_{\epsilon_m}(x - A_\Gamma(y)) - \chi_{\epsilon_m}(x - A_\Gamma(z_Q))\Big) K(x - A_\Gamma(z_Q))\Delta_Q g(y)\, dy \\
&\quad + \int \Big(\widetilde{\varphi}_{\epsilon_m}(x - A_\Gamma(z_Q)) - \widetilde{\varphi}_{\epsilon_m}(x - A_\Gamma(y))\Big) K(x - A_\Gamma(z_Q))\Delta_Q g(y)\, dy \\
&=: U3^A_m(x,Q) + U3^B_m(x,Q) + U3^C_m(x,Q).
\end{aligned}$$

The terms $U3^A_m(x,Q)$ and $U3^B_m(x,Q)$ can be handled as above, at the end of subsection 8.2.2, and the term $U3^C_m(x,Q)$ can be easily estimated using the smoothness of $\widetilde{\varphi}_{\epsilon_m}$. Indeed, notice that $|\nabla\widetilde{\varphi}_{\epsilon_m}| \lesssim \ell(D)^{-1}$ for $\epsilon_m \in I_j$ and $D \in \mathcal{D}_j$. Therefore, $|U3^C_m(x,Q)| \lesssim \ell(Q)\ell(D)^{-n-1}\|\Delta_Q g\|_1$, and then one can continue with the same arguments as when we estimated $U3^A_m(x,Q)$ (see the equation before (8.13)).

The other point that has to be mentioned is in (8.18). Instead, now we have

$$\begin{aligned} U2_m(x) &= \Psi_D g(x) \int_\Gamma K(x-y)\kappa_{\epsilon_m}(x-y)\, d\mathcal{H}^n_\Gamma(y) \\ &= \Psi_D g(x) \int_\Gamma K(x-y)\kappa_{\epsilon_m}(x-y)\, d(\mathcal{H}^n_\Gamma - \nu_x)(y) \\ &\quad + \Psi_D g(x) \int_\Gamma K(x-y)\kappa_{\epsilon_m}(x-y)\, d\nu_x(y) =: U4_m(x) + U5_m(x). \end{aligned}$$

The term $U5_m(x)$ can be handled as we did previously, in the case of the function $\chi^{\epsilon_m}_{\epsilon_{m+1}}$ (see (8.21) and the subsequent arguments). To deal with $U4_m(x)$, using that $\chi_{\epsilon_m}(x - p^x(y)) = \chi_{\epsilon_m}(x-y)$, we obtain

$$\begin{aligned} U4_m(x) &= \Psi_D g(x) \int_\Gamma K(x-y)\kappa_{\epsilon_m}(x-y)\, d(\mu - \nu_x)(y) \\ &= \Psi_D g(x) \int_\Gamma \Big(K(x-y)\kappa_{\epsilon_m}(x-y) - K(x-p^x(y))\kappa_{\epsilon_m}(x-p^x(y))\Big)\, d\mu(y) \\ &= \Psi_D g(x) \int_\Gamma \Big(K(x-y) - K(x-p^x(y))\Big)\chi_{\epsilon_m}(x-y)\, d\mu(y) \\ &\quad + \Psi_D g(x) \int_\Gamma \Big(K(x-p^x(y))\widetilde{\varphi}_{\epsilon_m}(x-p^x(y)) - K(x-y)\widetilde{\varphi}_{\epsilon_m}(x-y)\Big)\, d\mu(y) \\ &= \Psi_D g(x) \int_\Gamma \Big(K(x-y) - K(x-p^x(y))\Big)\kappa_{\epsilon_m}(x-y)\, d\mu(y) \\ &\quad + \Psi_D g(x) \int_\Gamma K(x-p^x(y))\Big(\widetilde{\varphi}_{\epsilon_m}(x-p^x(y)) - \widetilde{\varphi}_{\epsilon_m}(x-y)\Big)\, d\mu(y). \end{aligned}$$

The first term on the right hand side of the last equality can be handled as we did for the case of the function $\chi^{\epsilon_m}_{\epsilon_{m+1}}$ (see (8.19)). The second term can be easily estimated by $|\Psi_D g(x)|(\beta_{1,\mu}(D) + \operatorname{dist}(x, L_D)\ell(D)^{-1})$, using the smoothness of $\widetilde{\varphi}_{\epsilon_m}$ and that $|y - p^x(y)| \lesssim \operatorname{dist}(y, L^x_D) \le \operatorname{dist}(y, L_D) + \operatorname{dist}(x, L_D)$ for all $y \in \Gamma$. Therefore, (8.20) still holds replacing $\chi^{\epsilon_m}_{\epsilon_{m+1}}$ by κ_{ϵ_m}.

8.5. **Estimate of** $\sum_{j\in\mathbb{Z}} \sum_{D\in\mathcal{D}_j} \int_D \sum_{m\in\mathcal{L}\,:\,\epsilon_{m+1}\in I_j} |K\kappa_{\epsilon_{m+1}} * (f\mathcal{H}^n_\Gamma)|^2\, d\mathcal{H}^n_\Gamma$**.** One argues exactly as in subsection 8.4 and obtains the same estimates.

Remark 8.7. By easier arguments one can also show that the operators $\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\chi}}$ and $\mathcal{O} \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\chi}}$ are bounded in $L^2(\mathcal{H}^n_\Gamma)$. To estimate these operators, one does not need to introduce the angular projection p^x that we used in the previous subsections. It is enough to use vertical projections, as in the case of $\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_{\widetilde{\varphi}}$ in sections 4 and 5. These projections behave well with respect to the truncations $\widetilde{\chi}_\epsilon$. Furthermore, as we remarked at the beginning of section 8, the use of Lemma 8.2 is not necessary and the L^2 boundedness holds for any Lipschitz graph Γ (i.e. for any $\operatorname{Lip}(A) < \infty$).

The L^2 boundedness of the operators $\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_{\varphi}$ and $\mathcal{O} \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_{\varphi}$ is easier to obtain than in the case $\omega \in \{\chi, \widetilde{\chi}\}$ (using similar arguments), because now the difficult parts (which were the ones involving differences of characteristic functions) are estimated using the regularity of the functions φ_ϵ for $\epsilon > 0$, as in section 7 with the truncations $\widetilde{\varphi}_\epsilon$. Once we know the L^2 boundedness of $\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_{\varphi}$ and $\mathcal{O} \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_{\varphi}$, we can argue as in subsection 7.1 to prove that these operators are also bounded from $H^1(\mathcal{H}^n_\Gamma)$ to $L^1(\mathcal{H}^n_\Gamma)$, because the L^2 boundedness implies the local L^2 estimates of section 6.

8.6. **Proof of Lemma 8.2.** We need the followng auxiliary result:

Lemma 8.8. *Let $0<n<d$. For $x:=(x_1,\dots,x_d)\in\mathbb{R}^d$ we denote*

$$x_H:=(x_1,\dots,x_n,0,\dots,0)\in\mathbb{R}^d \quad\text{and}\quad x_V:=(0,\dots,0,x_{n+1},\dots,x_d)\in\mathbb{R}^d.$$

Given $x,y\in\mathbb{R}^d\setminus\{0\}$, if there exists $0<s<1$ such that $|x_V|\le s|x_H|$, $|y_V|\le s|y_H|$, and $|x_V-y_V|\le s|x_H-y_H|$, then there exists $C>0$ depending only on s such that

$$|x_V-y_V|\le C\Big||x||x_H|^{-1}x_H-|y||y_H|^{-1}y_H\Big|. \tag{8.36}$$

Proof. We set $\Phi(x,y):=\big||x||x_H|^{-1}x_H-|y||y_H|^{-1}y_H\big|$. Since Φ is symmetric in x and y, we can assume that $|x_H|\le|y_H|$. If $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$, using the polarization identity,

$$\begin{aligned}\Phi(x,y)^2&=|x|^2+|y|^2-2|x||x_H|^{-1}|y||y_H|^{-1}\langle x_H,y_H\rangle\\&=|x|^2+|y|^2+|x||x_H|^{-1}|y||y_H|^{-1}\big(|x_H-y_H|^2-|x_H|^2-|y_H|^2\big)\\&=|x|^2+|y|^2-2|x||y|\\&\quad+|x||x_H|^{-1}|y||y_H|^{-1}\big(|x_H-y_H|^2-|x_H|^2-|y_H|^2+2|x_H||y_H|\big)\\&=\big(|x|-|y|\big)^2+|x||x_H|^{-1}|y||y_H|^{-1}\big(|x_H-y_H|^2-(|x_H|-|y_H|)^2\big).\end{aligned}$$

Since $|x_H-y_H|^2-(|x_H|-|y_H|)^2\ge0$, $|x_H|\le|x|$, and $|y_H|\le|y|$, we have

$$\Phi(x,y)^2\ge\big(|x|-|y|\big)^2+|x_H-y_H|^2-(|x_H|-|y_H|)^2. \tag{8.37}$$

Assume that $2|x|\le|y|$. Then, using (8.37),

$$|x_V-y_V|\le|x|+|y|\le\frac{3}{2}|y|=3\Big(|y|-\frac{1}{2}\,|y|\Big)\le3(|y|-|x|)\le3\Phi(x,y),$$

and we obtain (8.36). By the same arguments, if $2|y|\le|x|$, then $|x_V-y_V|\le3\Phi(x,y)$ and (8.36) holds. Thus, from now on we assume $\frac{1}{2}|x|\le|y|\le2|x|$.

Let $0<\delta<1$ be a small number that will be fixed below. Assume that $(1-\delta)|x_H-y_H|\ge\big||y_H|-|x_H|\big|$. Then, by (8.37),

$$\begin{aligned}\Phi(x,y)^2&\ge|x_H-y_H|^2-(|x_H|-|y_H|)^2\ge|x_H-y_H|^2-(1-\delta)^2|x_H-y_H|^2\\&=\delta(2-\delta)|x_H-y_H|^2\ge\delta(2-\delta)s^{-2}|x_V-y_V|^2,\end{aligned}$$

and then (8.36) holds with $C=s/\sqrt{\delta(2-\delta)}$.

Therefore, we can suppose that $(1-\delta)|x_H-y_H|\le\big||y_H|-|x_H|\big|=|y_H|-|x_H|$, since we are also assuming $|x_H|\le|y_H|$. If we set $z=y-x$, we have $(1-\delta)|z_H|\le|x_H+z_H|-|x_H|$, so $(1-\delta)|z_H|+|x_H|\le|x_H+z_H|$. Hence,

$$\begin{aligned}(1-\delta)^2|z_H|^2+|x_H|^2+2(1-\delta)|z_H||x_H|&=\big((1-\delta)|z_H|+|x_H|\big)^2\\&\le|x_H+z_H|^2=|x_H|^2+|z_H|^2+2\langle x_H,z_H\rangle\end{aligned}$$

and we obtain

$$\langle x_H,z_H\rangle\ge-\frac{1}{2}\,\delta(2-\delta)|z_H|^2+(1-\delta)|z_H||x_H|. \tag{8.38}$$

Using (8.38), that $\langle x_V, z_V\rangle \geq -|x_V||z_V|$, and that $|x_V| \leq s|x_H|$ and $|z_V| \leq s|z_H|$, we get

$$\begin{aligned}\langle x, z\rangle &= \langle x_H + x_V, z_H + z_V\rangle = \langle x_H, z_H\rangle + \langle x_V, z_V\rangle\\ &\geq -\frac{1}{2}\,\delta(2-\delta)|z_H|^2 + (1-\delta)|z_H||x_H| - |x_V||z_V|\\ &\geq -\frac{1}{2}\,\delta(2-\delta)|z_H|^2 + (1-\delta-s^2)|z_H||x_H|.\end{aligned} \tag{8.39}$$

Notice that, if $\delta > 0$ is small enough depending on s, then $-\frac{1}{4}\,(1-s^2)(1+s^2)^{-1} < -\frac{3}{2}\,\delta(2-\delta) < 0$ and $1-\delta-s^2 > \frac{1}{2}\,(1-s^2)$. Let $\gamma(x,z)$ be the angle between x and z (by definition, $0 \leq \gamma(x,z) \leq \pi$). Using that $\langle x, z\rangle = |x||z|\cos(\gamma(x,z))$, that $|x| \leq \sqrt{1+s^2}|x_H|$ and $|z| \leq \sqrt{1+s^2}|z_H|$, and that $|z| \leq |x| + |y| \leq 3|x|$, we finally obtain from (8.39) that

$$\begin{aligned}\cos(\gamma(x,z)) &\geq -\frac{1}{2}\,\delta(2-\delta)|z_H|^2|x|^{-1}|z|^{-1} + (1-\delta-s^2)|z_H||x_H||x|^{-1}|z|^{-1}\\ &\geq -\frac{3}{2}\,\delta(2-\delta) + (1-\delta-s^2)(1+s^2)^{-1} \geq \frac{1}{4}\,(1-s^2)(1+s^2)^{-1} =: a.\end{aligned}$$

Notice that $a > 0$, because $0 < s < 1$ by hypothesis. Hence, since $\cos(\gamma(-x, y-x)) = \cos(\gamma(-x, z)) = -\cos(\gamma(x,z))$ (because $z = y - x$ and $\langle -x, z\rangle = -\langle x, z\rangle$), we have $c_0 := \cos(\gamma(-x, y-x)) \leq -a < 0$ (notice that $c_0 \leq 0$ implies that $|x| \leq |y|$). By the cosinus theorem, $|y|^2 = |x|^2 - |y-x|^2 - 2|x||y-x|c_0$. Since $c_0 < 0$, we solve the second degree equation in $|y-x|$ and obtain

$$\begin{aligned}|y-x| &= \sqrt{|y|^2 - |x|^2(1-c_0^2)} - |x||c_0| = \frac{|y|^2 - |x|^2(1-c_0^2) - |x|^2c_0^2}{\sqrt{|y|^2 - |x|^2(1-c_0^2)} + |x||c_0|}\\ &= \frac{(|y|-|x|)(|y|+|x|)}{\sqrt{|y|^2 - |x|^2(1-c_0^2)} + |x||c_0|} \leq \frac{(|y|-|x|)(|y|+|x|)}{|x||c_0|} \leq (|y|-|x|)\frac{3}{a},\end{aligned}$$

where we also used that $|y| \leq 2|x|$ in the last inequality. Therefore, by (8.37),

$$|x_V - y_V| \leq |x - y| \leq \frac{3}{a}\,(|y| - |x|) \leq \frac{3}{a}\,\Phi(x,y),$$

and (8.36) follows with $C = 3/a$, where $a > 0$ only depends on s. This completes the proof of the lemma. □

Let us recall Lemma 8.2 and let us prove it:

Lemma 8.9. *Let* $\Gamma := \{x \in \mathbb{R}^d : x = (\widetilde{x}, A(\widetilde{x}))\}$ *be the graph of a Lipschitz function* $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ *such that* $\operatorname{Lip}(A) < 1$. *Then,* $\mathcal{H}^n_\Gamma(A^d(z,a,b)) \lesssim (b-a)b^{n-1}$ *for all* $0 < a \leq b$ *and* $z \in \Gamma$.

Proof. We keep the notation introduced in Lemma 8.8. Fix $z \in \Gamma$. We can assume that $z = 0$, by taking a translation of Γ if it is necessary.

For $x \in \mathbb{R}^d$ with $x_H \neq 0$, consider the map $\Upsilon(x) = |x||x_H|^{-1}x_H + x_V$. It is not difficult to show that Υ is a bilipschitz mapping from (a neighborhood of) the cone

$$L := \{x \in \mathbb{R}^d \setminus \{0\} : |x_V| \leq \operatorname{Lip}(A)|x_H|\}$$

to itself, whose inverse equals $\Upsilon^{-1}(x) = |x_H||x|^{-1}x_H + x_V$. Moreover, $\operatorname{Lip}(\Upsilon)$ and $\operatorname{Lip}(\Upsilon^{-1})$ only depend on n, d, and $\operatorname{Lip}(A)$. Notice that, given $0 < a \leq b$, $\Upsilon(L \cap A^d(0,a,b)) \subset A^n(0,a,b) \times \mathbb{R}^{d-n}$. Therefore, $\mathcal{H}^n_\Gamma(A^d(0,a,b)) = \mathcal{H}^n(\Gamma \cap A^d(0,a,b)) \approx \mathcal{H}^n(\Upsilon(\Gamma \cap A^d(0,a,b)))$ for all $0 < a \leq b$, because $\Gamma \subset L \cup \{0\}$.

Consider the set $\Upsilon(\Gamma)$. Since Γ has slope smaller than 1, by Lemma 8.8 there exists a constant $C > 0$ depending only on n, d, and $\operatorname{Lip}(A)$ such that for any two points $x, y \in \Upsilon(\Gamma)$

one has $|x_V - y_V| \leq C|x_H - y_H|$. Then, it is known that $\Upsilon(\Gamma)$ is contained in the n-dimensional graph Γ' of some Lipschitz function (see for example the proof of [Ma, Lemma 15.13]). Therefore,

$$\mathcal{H}^n_\Gamma(A^d(0,a,b)) \approx \mathcal{H}^n(\Upsilon(\Gamma \cap A^d(0,a,b))) \leq \mathcal{H}^n(\Gamma' \cap (A^n(0,a,b) \times \mathbb{R}^{d-n})) \lesssim (b-a)b^{n-1},$$

and the lemma is proved. □

Remark 8.10. With a little more of efford, one can show that $\Upsilon(\Gamma)$ is a Lipschitz graph.

Remark 8.11. Lemma 8.2 is sharp in the sense that the estimate fails if $\mathrm{Lip}(A) \geq 1$ (notice that the constant C in Lemma 8.8 is bigger than $(1+\mathrm{Lip}(A)^2)/(1-\mathrm{Lip}(A)^2)$). Given $\epsilon > 0$, one can easily construct a Lipschitz graph Γ such that $1 < \mathrm{Lip}(A) < 1+\epsilon$ and such that, for some $z \in \Gamma$ and $r > 0$, Γ contains a set $P \subset \partial B(z,r)$ with $\mathcal{H}^n_\Gamma(P) > 0$. Then, if Lemma 8.2 were true for Γ, we would have $0 < \mathcal{H}^n_\Gamma(P) \leq \mathcal{H}^n_\Gamma(A(z, r-\delta, r+\delta)) \lesssim 2\delta(r+\delta)^{n-1}$, and we would have a contradiction by making $\delta \to 0$. By a similar argument, one can also show that the lemma fails in the limiting case $\mathrm{Lip}(A) = 1$.

9. L^p and endpoint estimates for $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$

Theorem 9.1. *Let $\rho > 2$ and assume* $\mathrm{Lip}(A) < 1$. *The operators $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ and $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ are bounded*

- *in $L^p(\mathcal{H}^n_\Gamma)$ for $1 < p < \infty$,*
- *from $L^1(\mathcal{H}^n_\Gamma)$ to $L^{1,\infty}(\mathcal{H}^n_\Gamma)$, and*
- *from $L^\infty(\mathcal{H}^n_\Gamma)$ to $BMO(\mathcal{H}^n_\Gamma)$,*

and the norm of $\mathcal{O} \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ in the cases above is bounded independently of the sequence that defines $\mathcal{O}$.

We will only give the proof of Theorem 9.1 in the case of the ρ-variation, because the proof for the oscillation follows by very close arguments.

9.1. **The operator $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma} : L^1(\mathcal{H}^n_\Gamma) \to L^{1,\infty}(\mathcal{H}^n_\Gamma)$ is bounded.** By Theorem 8.1, we know that $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma} : L^2(\mathcal{H}^n_\Gamma) \to L^2(\mathcal{H}^n_\Gamma)$ is bounded. In Theorem B of [CJRW2] it is proved that, if the variation for a singular integral in $\mathbb{R}^n$ with respect to the measure $\mathcal{L}^n$ is a bounded operator in L^2 and the kernel satisfies standard estimates, then the variation is also bounded from L^1 to $L^{1,\infty}$. Because of the AD regularity of the measure $\mathcal{H}^n_\Gamma$, it is not difficult to adapt Theorem B of [CJRW2] to our setting (i.e., when the space is not $\mathbb{R}^n$ but an n-dimensional Lipschitz graph) by using Lemma 8.2, and then the weak-L^1 estimate follows.

9.2. **The operator $\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma} : L^\infty(\mathcal{H}^n_\Gamma) \to BMO(\mathcal{H}^n_\Gamma)$ is bounded.** The arguments are very similar to the ones in subsection 7.2, and we will use analogous techniques and notation (replacing $\widetilde{\varphi}$ by χ). We set $f_1 := f\chi_{3D}$ and $f_2 := f - f_1$. The case of f_1 is handled as in subsection 7.2, but replacing Theorem 6.1 by Theorem 8.1. In the case of f_2, for $x \in \Gamma \cap D$, we decompose

$$\left|(\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}) f_2(x) - c\right|^\rho \lesssim \|f\|^\rho_{L^\infty(\mathcal{H}^n_\Gamma)} \sum_{m\in\mathbb{Z}} (\Theta 1_m + \Theta 2_m)^\rho,$$

where $c := (\mathcal{V}_\rho \circ \mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}) f_2(z_D)$, and

$$\Theta 1_m := \int_{(3D)^c} \chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)\, |K(x-y) - K(z_D - y)|\, d\mathcal{H}^n_\Gamma(y),$$
$$\Theta 2_m := \int_{(3D)^c} |\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) - \chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)||K(z_D-y)|\, d\mathcal{H}^n_\Gamma(y).$$

Arguing as in subsection 7.2, we have

$$\Big(\sum_{m\in\mathbb{Z}} \Theta 1_m^\rho\Big)^{1/\rho} \leq \sum_{m\in\mathbb{Z}} \Theta 1_m \lesssim \ell(D) \int_{(3D)^c} |z_D - y|^{-n-1}\, d\mathcal{H}^n_\Gamma(y) \lesssim 1.$$

The case of $\Theta 2_m$ is more delicate. Since Γ is a Lipschitz graph, there exists an integer $M > 10$ depending only on $\mathrm{Lip}(A)$ such that any $x \in \Gamma \cap D$ satisfies $|x - z_D| < 2^M \ell(D)$. Without loss of generality, we can assume that there exists $m_0 \in \mathbb{Z}$ such that $\epsilon_{m_0} = 2^{M+2}\ell(D)$, just by adding the term $2^{M+2}\ell(D)$ to the fixed sequence $\{\epsilon_m\}_{m\in\mathbb{Z}}$.

We set $J_0 := \{m \in \mathbb{Z} : \epsilon_m \leq 2^{M+2}\ell(D)\} = \{m \in \mathbb{Z} : m \geq m_0\}$ and, for $j > M+2$,

$$J_j^1 := \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} \leq \epsilon_m \leq 2^j \ell(D) \text{ and } \epsilon_m - \epsilon_{m+1} \geq 2^M \ell(D)\},$$
$$J_j^2 := \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} \leq \epsilon_m \leq 2^j \ell(D) \text{ and } \epsilon_m - \epsilon_{m+1} < 2^M \ell(D)\},$$
$$J_j^3 := \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} \leq 2^j \ell(D) < \epsilon_m\}.$$

Then $\mathbb{Z} = J_0 \cup \big(\bigcup_{j>M+2} (J_j^1 \cup J_j^2 \cup J_j^3)\big)$. For the case of $m \in J_0$, we have the easy estimate

$$\begin{aligned}\Big(\sum_{m\in J_0} \Theta 2_m^\rho\Big)^{1/\rho} &\lesssim \sum_{m\in J_0} \int_{(3D)^c} \big(\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) + \chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)\big)\ell(D)^{-n}\, d\mathcal{H}^n_\Gamma(y)\\ &\leq \int_{|x-y|\leq 2^{M+2}\ell(D)} \frac{d\mathcal{H}^n_\Gamma(y)}{\ell(D)^n} + \int_{|z_D-y|\leq 2^{M+2}\ell(D)} \frac{d\mathcal{H}^n_\Gamma(y)}{\ell(D)^n} \lesssim 1.\end{aligned}$$

Assume that $m \in J_j^1$. Notice that $\mathrm{supp}\big(\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot) - \chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-\cdot)\big) \subset A_m(x, z_D)$, where $A_m(x,z_D)$ denotes the symmetric difference between $A(x,\epsilon_{m+1},\epsilon_m)$ and $A(z_D,\epsilon_{m+1},\epsilon_m)$. Notice also that, since $m \in J_j^1$ and $x \in D \cap \Gamma$, the set $A_m(x,z_D)$ is contained in the union of annuli $A_1 := A(x, \epsilon_{m+1} - 2^M\ell(D), \epsilon_{m+1} + 2^M\ell(D))$ and $A_2 := A(x, \epsilon_m - 2^M\ell(D), \epsilon_m + 2^M\ell(D))$. For $z \in \Gamma$ and $0 < a \leq b$, we have $\mathcal{H}^n_\Gamma(A(z,a,b)) \lesssim (b-a)b^{n-1}$ by Lemma 8.2. Hence, since $m \in J_j^1$,

$$\begin{aligned}&\mathcal{H}^n_\Gamma(\{y \in \mathbb{R}^d : |\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) - \chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)| \neq 0\}) \leq \mathcal{H}^n_\Gamma(A_1 \cup A_2)\\ &\quad\lesssim 2^{M+1}\ell(D)\Big(\epsilon_m + 2^M\ell(D)\Big)^{n-1} + 2^{M+1}\ell(D)\Big(\epsilon_{m+1} + 2^M\ell(D)\Big)^{n-1}\\ &\quad\lesssim 2^{j(n-1)}\ell(D)^n.\end{aligned} \tag{9.1}$$

Using that $|K(z_D - y)| \lesssim (2^j\ell(D))^{-n}$ for all $y \in A_m(x,z_D) \cap (3D)^c$, we get $\Theta 2_m \lesssim (2^j\ell(D))^{-n} 2^{j(n-1)}\ell(D)^n = 2^{-j}$ and, since J_j^1 contains at most 2^{j-M-1} indices and $\rho > 2$, we have $\sum_{m\in J_j^1} \Theta 2_m^\rho \lesssim 2^{-j}$.

Assume now that $m \in J_j^2$. Then, using Lemma 8.2, we obtain

$$\begin{aligned}&\mathcal{H}^n_\Gamma(\{y \in \mathbb{R}^d : |\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) - \chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)| \neq 0\})\\ &\quad\leq \mathcal{H}^n_\Gamma(\{y \in \mathbb{R}^d : \chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) = 1\} + \mathcal{H}^n_\Gamma\{y \in \mathbb{R}^d : \chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y) = 1\})\\ &\quad\lesssim (\epsilon_m - \epsilon_{m+1})\epsilon_m^{n-1},\end{aligned}$$

and, as above, $|K(z_D-y)|\lesssim(2^j\ell(D))^{-n}$ for all $y\in A_m(x,z_D)\cap(3D)^c$. Since $m\in J_j^2$,

$$\begin{aligned}\Theta2_m^\rho&\lesssim(2^j\ell(D))^{-\rho n}\big((\epsilon_m-\epsilon_{m+1})\epsilon_m^{n-1}\big)^\rho\\&\lesssim(2^j\ell(D))^{-\rho n}(\epsilon_m-\epsilon_{m+1})(2^M\ell(D))^{\rho-1}(2^j\ell(D))^{(n-1)\rho}\lesssim2^{-j\rho}\ell(D)^{-1}(\epsilon_m-\epsilon_{m+1})\end{aligned}$$

and then, since $\rho>2$ and $j>M+2>12$,

$$\sum_{m\in J_j^2}\Theta2_m^\rho\lesssim2^{-j\rho}\sum_{m\in J_j^2}\frac{\epsilon_m-\epsilon_{m+1}}{\ell(D)}\le2^{-j\rho}2^{j-1}\approx2^{-j(\rho-1)}\le2^{-j}.$$

Finally, assume that $m\in J_j^3$. Obviously, the set J_j^3 contains at most one term. If $\epsilon_m-\epsilon_{m+1}<2^M\ell(D)$, arguing as in the case $m\in J_j^2$, we have

$$\begin{aligned}\mathcal{H}^n_\Gamma(\{y\in\mathbb{R}^d\,:\,|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)-\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)|\neq0\})&\lesssim(\epsilon_m-\epsilon_{m+1})\epsilon_m^{n-1}\\&\lesssim2^M\ell(D)(2^j\ell(D)+2^M\ell(D))^{n-1}\lesssim2^{j(n-1)}\ell(D)^n,\end{aligned}$$

and then $\Theta2_m\lesssim2^{j(n-1)}\ell(D)^n(2^{j-1}\ell(D))^{-n}\lesssim2^{-j}$. On the contrary, if $\epsilon_m-\epsilon_{m+1}\ge2^M\ell(D)$, arguing as in the case $m\in J_j^1$, we have $\operatorname{supp}\big(\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)-\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-\cdot)\big)\subset A_m(x,z_D)\subset A_1\cup A_2$. Similarly to (9.1), we have

$$\mathcal{H}^n_\Gamma(A_1)\lesssim2^{M+1}\ell(D)(\epsilon_{m+1}+2^M\ell(D))^{n-1}\lesssim\epsilon_{m+1}^{n-1}\ell(D)\le2^{j(n-1)}\ell(D)^n,$$

and $|K(z_D-y)|\lesssim(2^j\ell(D))^{-n}$ for all $y\in A_1\cap(3D)^c$. If we denote by $j(\epsilon_m)$ the positive integer such that $2^{j(\epsilon_m)-1}\ell(D)\le\epsilon_m\le2^{j(\epsilon_m)}\ell(D)$ (obviously, $j(\epsilon_m)>j$), we have $\mathcal{H}^n_\Gamma(A_2)\lesssim\epsilon_m^{n-1}\ell(D)\le2^{j(\epsilon_m)(n-1)}\ell(D)^n$, and $|K(z_D-y)|\lesssim(2^{j(\epsilon_m)}\ell(D))^{-n}$ for all $y\in A_2\cap(3D)^c$. Hence, $\Theta2_m\lesssim2^{j(n-1)}\ell(D)^n(2^j\ell(D))^{-n}+2^{j(\epsilon_m)(n-1)}\ell(D)^n(2^{j(\epsilon_m)}\ell(D))^{-n}\lesssim2^{-j}+2^{-j(\epsilon_m)}\lesssim2^{-j}$. Therefore, since J_j^3 contains at most one term, $\sum_{m\in J_j^3}\Theta2_m^\rho\lesssim2^{-j\rho}\le2^{-j}$.

We put all these estimates together and conclude that

$$\begin{aligned}\big|(\mathcal{V}_\rho\circ\mathcal{T}_\chi^{\mathcal{H}^n_\Gamma})f_2(x)-c\,\big|&\lesssim\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{m\in\mathbb{Z}}(\Theta1_m+\Theta2_m)^\rho\bigg)^{1/\rho}\\&\lesssim\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{m\in\mathbb{Z}}\Theta1_m^\rho\bigg)^{1/\rho}+\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{m\in J_0}\Theta2_m^\rho\bigg)^{1/\rho}\\&\quad+\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{j>M+2}\bigg(\sum_{m\in J_j^1}\Theta2_m^\rho+\sum_{m\in J_j^2}\Theta2_m^\rho+\sum_{m\in J_j^3}\Theta2_m^\rho\bigg)\bigg)^{1/\rho}\\&\lesssim\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(1+1+\bigg(\sum_{j>12}2^{-j}\bigg)^{1/\rho}\bigg)\lesssim\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)},\end{aligned}$$

and so the boundedness of $\mathcal{V}_\rho\circ\mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}:L^\infty(\mathcal{H}^n_\Gamma)\to BMO(\mathcal{H}^n_\Gamma)$ follows, as in subsection 7.2.

9.3. **The operator** $\mathcal{V}_\rho\circ\mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}:L^p(\mathcal{H}^n_\Gamma)\to L^p(\mathcal{H}^n_\Gamma)$ **is bounded for** $1<p<\infty$**.** We deduce the L^p boundedness of the positive sublinear operator $\mathcal{V}_\rho\circ\mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ by interpolation between the pairs $(L^1(\mathcal{H}^n_\Gamma),L^{1,\infty}(\mathcal{H}^n_\Gamma))$ and $(L^2(\mathcal{H}^n_\Gamma),L^2(\mathcal{H}^n_\Gamma))$ for $1<p<2$, and between $(L^2(\mathcal{H}^n_\Gamma),L^2(\mathcal{H}^n_\Gamma))$ and $(L^\infty(\mathcal{H}^n_\Gamma),BMO(\mathcal{H}^n_\Gamma))$ for $2<p<\infty$.

Let us remark that, in the latter case, the classical interpolation theorem (see [Du, Theorem 2.4], for instance) would require the operator $\mathcal{V}_\rho\circ\mathcal{T}_\chi^{\mathcal{H}^n_\Gamma}$ to be linear. Clearly, this fails in our case. However, an easy modification of the arguments in [Du] using Lemma 7.2 shows that the interpolation theorem between (L^2, L^2) and (L^∞, BMO) is also valid for positive sublinear operators.

Remark 9.2. From Remark 8.7, we know that the operators $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\chi}}^{\mathcal{H}_\Gamma^n}$ and $\mathcal{V}_\rho \circ \mathcal{T}_{\varphi}^{\mathcal{H}_\Gamma^n}$ are also bounded in $L^2(\mathcal{H}_\Gamma^n)$. The endpoint estimates and the interpolation theorem can also be obtained for these operators by very similar arguments: for the family of truncations $\widetilde{\chi}$ one argues as for χ (but now Lemma 8.2 is not necessary), and for φ one uses the regularity of the functions φ_ϵ, as we pointed out in Remark 8.7 (for example, the proof of the boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_{\varphi}^{\mathcal{H}_\Gamma^n} : L^\infty(\mathcal{H}_\Gamma^n) \to BMO(\mathcal{H}_\Gamma^n)$ is analogous to the one in subsection 7.2). The same holds for the operators $\mathcal{O} \circ \mathcal{T}_{\widetilde{\chi}}^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_{\varphi}^{\mathcal{H}_\Gamma^n}$.

For the case of $\mathcal{V}_\rho \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_{\widetilde{\varphi}}^{\mathcal{H}_\Gamma^n}$, the weak L^1 estimate can be obtained similarly to the case of the family of truncations φ, that is, by adapting Theorem B of [CJRW2] to our specific setting.

We want to emphasize that the assumption $\mathrm{Lip}(A) < 1$ is not necessary when we deal with any of the families of truncations $\widetilde{\chi}$, φ or $\widetilde{\varphi}$ (see the comment that follows Lemma 8.2). Therefore, Main Theorem 1.1 is finally proven.

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