Weighted estimates for Beltrami equations

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Abstract

We obtain a priori estimates in $L^p(\omega)$ for the generalized Beltrami equation, provided that the coefficients are compactly supported VMO functions with the expected ellipticity condition, and the weight ω lies in the Muckenhoupt class A_p. As an application, we obtain improved regularity for the jacobian of certain quasiconformal mappings.

1 Introduction

In this paper, we consider the inhomogeneous, Beltrami equation

$$\bar{\partial} f(z) - \mu(z)\, \partial f(z) - \nu(z)\, \overline{\partial f(z)} = g(z), \qquad a.e. z \in \mathbb{C} \tag{1}$$

where μ, ν are $L^\infty(\mathbb{C};\mathbb{C})$ functions such that $\||\mu| + |\nu|\|_\infty \leq k < 1$, and g is a measurable, $\mathbb{C}$-valued function. The derivatives $\partial f, \overline{\partial} f$ are understood in the distributional sense. In the work [2], the L^p theory of such equation was developed. More precisely, it was shown that if $1+k < p < 1+\frac{1}{k}$ and $g \in L^p(\mathbb{C})$ then (1) has a solution f, unique modulo constants, whose differential Df belongs to $L^p(\mathbb{C})$, and furthermore, the estimate

$$\|Df\|_{L^p(\mathbb{C})} \leq C\, \|g\|_{L^p(\mathbb{C})} \tag{2}$$

holds for some constant $C = C(k,p) > 0$. For other values of p, (1) the claim may fail in general. However, in the previous work [7], Iwaniec proved that if $\mu \in VMO(\mathbb{C})$ then for any $1 < p < \infty$ and any $g \in L^p(\mathbb{C})$ one can find exactly one solution f to the $\mathbb{C}$-linear equation

$$\overline{\partial} f(z) - \mu(z)\, \partial f(z) = g(z)$$

with $Df \in L^p(\mathbb{C})$. In particular, (2) holds whenever $p \in (1,\infty)$. Recently, Koski [9] has extended this result to the generalized equation (1). For results in other spaces of functions, see [4].

In this paper, we deal with weighted spaces, and so we assume $g \in L^p(\omega)$, $1 < p < \infty$. Here ω is a measurable function, and $\omega > 0$ at almost every point. By checking the particular case $\mu = 0$, one sees that, for a weighted version of the estimate (2) to hold, the Muckenhoupt condition $\omega \in A_p$ is necessary. It turns out that, for compactly supported $\mu \in VMO$, this condition is also sufficient.

Theorem 1. *Let $1 < p < \infty$. Let μ be a compactly supported function in $VMO(\mathbb{C})$, such that $\|\mu\|_\infty < 1$, and let $\omega \in A_p$. Then, the equation*

$$\overline{\partial} f(z) - \mu(z)\, \partial f(z) = g(z)$$

has, for $g \in L^p(\omega)$, a solution f with $Df \in L^p(\omega)$, which is unique up to an additive constant. Moreover, one has

$$\|Df\|_{L^p(\omega)} \leq C\, \|g\|_{L^p(\omega)}$$

for some $C > 0$ depending on μ, p and $[\omega]_{A_p}$.

The proof copies the scheme of [7]. In particular, our main tool is the following compactness Theorem, which extends a classical result of Uchiyama [15] about commutators of Calderón-Zygmund singular integral operators and VMO functions.

Theorem 2. *Let T be a Calderón-Zygmund singular integral operator. Let $\omega \in A_p$ with $1 < p < \infty$, and let $b \in VMO(\mathbb{R}^n)$. The commutator $[b, T]\colon L^p(\omega) \to L^p(\omega)$ is compact.*

Theorem 2 is obtained from a sufficient condition for compactness in $L^p(\omega)$. When $\omega = 1$, this sufficient condition reduces to the classical Frechet-Kolmogorov compactness criterion. Theorem 1 is then obtained from Theorem 2 by letting T be the Beurling-Ahlfors singular integral operator.

A counterpart to Theorem 1 for the *generalized Beltrami equation*,

$$\overline{\partial} f(z) - \mu(z)\, \partial f(z) - \nu(z)\, \overline{\partial f(z)} = g(z), \tag{3}$$

can also be obtained under the ellipticity condition $\||\mu| + |\nu|\|_\infty \leq k < 1$ and the VMO smoothness of the coefficients (see Theorem 8 below). Theorem 2 is again the main ingredient. However, for (3) the argument in Theorem 1 needs to be modified, because the involved operators are not $\mathbb{C}$-linear, but only $\mathbb{R}$-linear. In other words, $\mathbb{C}$-linearity is not essential. See also [9].

It turns out that any linear, elliptic, divergence type equation can be reduced to equation (3) (see e.g.[1, Theorem 16.1.6]). Therefore the following result is no surprise.

Corollary 3. *Let $K \geq 1$. Let $A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ be a matrix-valued function, satisfying the ellipticity condition*

$$\frac{1}{K} \leq v^t\, A(z)\, v \leq K, \qquad \text{whenever } v \in \mathbb{R}^2,\ |v| = 1$$

at almost every point $z \in \mathbb{R}^2$, and such that $A - \mathbf{Id}$ has compactly supported VMO entries. Let $p \in (1, \infty)$ be fixed, and $\omega \in A_p$. For any $g \in L^p(\omega)$, the equation

$$\operatorname{div}(A(z)\, \nabla u) = \operatorname{div}(g)$$

has a solution u with $\nabla u \in L^p(\omega)$, unique up to an additive constant, and such that

$$\|\nabla u\|_{L^p(\omega)} \le C\,\|g\|_{L^p(\omega)}$$

for some constant $C = C(A, \omega, p)$.

Other applications of Theorem 1 are found in connection to planar K-quasiconformal mappings. Remember that a $W^{1,2}_{loc}$ homeomorphism $\phi : \Omega \to \Omega'$ between domains $\Omega, \Omega' \subset \mathbb{C}$ is called K-quasiconformal if

$$|\overline{\partial}\phi(z)| \le \frac{K-1}{K+1}\,|\partial\phi(z)| \qquad\qquad \text{for } a.e. z \in \Omega.$$

In general, jacobians of K-quasiconformal maps are Muckenhoupt weights belonging to the class A_p for any $p > K$ (see [1, Theorem 14.3.2], or also [2]), and this is sharp. As a consequence of Theorem 1, we obtain the following improvement.

Corollary 4. *Let $\mu \in VMO$ be compactly supported, such that $\|\mu\|_\infty < 1$, and let ϕ be the principal solution of*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) = 0.$$

Then the jacobian determinant $J(\cdot, \phi^{-1})$ belongs to A_p for every $1 < p < \infty$.

We actually prove that composition with the inverse mapping ϕ^{-1} preserves the Muckenhoupt class A_2. Then, the above corollary follows by the results of Johnson and Neugebauer in [8], where the composition problem in all Muckenhoupt classes is completely solved.

The paper is structured as follows. In Section 2 we prove Theorem 2. In Section 3 we prove Theorem 1 and its counterpart for the generalized Beltrami equation. In Section 4 we study some applications. By C we denote a positive constant that may change at each occurrence. $B(x, r)$ denotes the open ball with center x and radius r, and $2B$ means the open ball concentric with B and having double radius.

2 Compactness

By singular integral operator T, we mean a linear operator on $L^p(\mathbb{R}^n)$ that can be written as

$$Tf(x) = \int_{\mathbb{R}^n} f(y)\,K(x, y)\,\mathrm{d}y.$$

Here $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\} \to \mathbb{C}$ obeys the bounds

1. $|K(x, y)| \le \frac{C}{|x-y|^n}$,
2. $|K(x, y) - K(x, y')| \le C\frac{|y-y'|}{|x-y|^{n+1}}$ whenever $|x - y| \ge 2|y - y'|$,
3. $|K(x, y) - K(x', y)| \le C\frac{|x-x'|}{|x-y|^{n+1}}$ whenever $|x - y| \ge 2|x - x'|$.

Given a singular integral operator T, we define the *truncated singular integral* as

$$T_\epsilon f(x) = \int_{|x-y|\geq\epsilon} K(x,y)f(y)\mathrm{d}y$$

and the *maximal singular integral* by the relationship

$$T_* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)| \,.$$

As usually, we denote $⨍_E f(x)\mathrm{d}x = \frac{1}{|E|}\int_E f(x)\,\mathrm{d}x$. A weight is a function $\omega \in L^1_{loc}(\mathbb{R}^n)$ such that $\omega(x) > 0$ almost everywhere. A weight ω is said to belong to the Muckenhoupt class A_p, $1 < p < \infty$, if

$$[\omega]_{A_p} := \sup \left(⨍_Q \omega(x)\mathrm{d}x\right)\left(⨍_Q \omega(x)^{-\frac{p'}{p}}\mathrm{d}x\right)^{\frac{p}{p'}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and where $\frac{1}{p} + \frac{1}{p'} = 1$. By $L^p(\omega)$ we denote the set of measurable functions f that satisfy

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x\right)^{\frac{1}{p}} < \infty. \tag{4}$$

The quantity $\|f\|_{L^p(\omega)}$ defines a complete norm in $L^p(\omega)$. It is well know that if T is a Calderón-Zygmund operator, then T and also T_* are bounded in $L^p(\omega)$ whenever $\omega \in A_p$ (see for instance [5, Cap. IV, Theorems 3.1 and 3.6]). Also the Hardy-Littlewood maximal operator M is bounded in L^p. For more about A_p classes and weighted spaces $L^p(\omega)$, we refer the reader to [5].

We first show the following sufficient condition for compactness in $L^p(\omega)$, $\omega \in A_p$. Remember that an metric space X is *totally bounded* if for every $\epsilon > 0$ there exists a finite number of open balls of radius ϵ whose union is the space X. In addition, a metric space is compact if and only if is complete and totally bounded.

Theorem 5. *Let $p \in (1,\infty)$, $\omega \in A_p$, and let $\mathfrak{F} \subset L^p(\omega)$. Then $\mathfrak{F}$ is totally bounded if it satisfies the next three conditions:*

1. *$\mathfrak{F}$ is uniformly bounded, i.e.* $\sup_{f\in\mathfrak{F}} \|f\|_{L^p(\omega)} < \infty$.
2. *$\mathfrak{F}$ is uniformly equicontinuous, i.e.* $\sup_{f\in\mathfrak{F}} \|f(\cdot + h) - f(\cdot)\|_{L^p(\omega)} \xrightarrow{h\to 0} 0$.
3. *$\mathfrak{F}$ uniformly vanishes at infinity, i.e.* $\sup_{f\in\mathfrak{F}} \|f - \chi_{Q(0,R)} f\|_{L^p(\omega)} \xrightarrow{R\to\infty} 0$, *where $Q(0,R)$ is the cube with center at the origin and sidelength R.*

Let us emphasize that Theorem 5 is a strong sufficient condition for compactness in $L^p(\omega)$, because for a general weight $\omega \in A_p$ the space $L^p(\omega)$ is not invariant under translations. Theorem 5 is proved by adapting the arguments in [6]. In particular, the following result (which can be found in [6, Lemma 1]) is essential.

Lemma 6. *Let X be a metric space. Suppose that for every $\epsilon > 0$ one can find a number $\delta > 0$, a metric space W and an application $\Phi\colon X \to W$ such that $\Phi(X)$ is totally bounded, and the implication*

$$d(\Phi(x), \Phi(y)) < \delta \qquad \Longrightarrow \qquad d(x, y) < \epsilon$$

holds for any $x, y \in X$. Then X is totally bounded.

Proof of Theorem 5. Suppose that the family $\mathfrak{F}$ satisfies the three conditions of the Theorem 5. Given $\rho \geq 0$, let Q be the largest open cube centered at 0 such that $2Q \subset B(0, \rho)$. For $R > 0$, let $Q_1, \ldots, Q_N$ be N copies of Q such that have not a overlap and such that

$$\overline{Q(0,R)} = \overline{\bigcup_i Q_i},$$

where $Q(0, R)$ is the open cube on the origin of side R. Let us define an application $f \mapsto \Phi f$ by setting

$$\Phi f(x) = \begin{cases} \fint_{Q_i} f(z)\mathrm{d}z, & x \in Q_i, i = 1, \ldots, N, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

For $f \in L^p(\omega)$ one has $f \in L^1_{loc}(\mathbb{R}^n)$, and thus Φf is well defined for $f \in \mathfrak{F}$. Moreover,

$$\begin{aligned} \int_{\mathbb{R}^n} |\Phi f(x)|^p \omega(x)\mathrm{d}x &= \sum_{i=1}^N \left| \fint_{Q_i} f(z)\mathrm{d}z \right|^p \int_{Q_i} \omega(x)\, \mathrm{d}(x) \\ &\leq \sum_{i=1}^N \left(\fint_{Q_i} |f(z)|^p \omega(z)\mathrm{d}z \right) \left(\fint_{Q_i} \omega^{\frac{-p'}{p}}(z)\mathrm{d}z \right)^{\frac{p}{p'}} \int_{Q_i} \omega(x)\mathrm{d}x \\ &\leq [\omega]_{A_p} \|f\|^p_{L^p(\omega)}. \end{aligned}$$

In particular, $\Phi : L^p(\omega) \to L^p(\omega)$ is a bounded operator. As $\mathfrak{F}$ is bounded, then $\Phi(\mathfrak{F})$ is a bounded subset of a finite dimensional Banach space, and hence $\Phi(\mathfrak{F})$ is totally bounded. By the vanishing condition at infinity, given $\epsilon > 0$ there exists $R_0 > 0$ such that

$$\sup_{f \in \mathfrak{F}} \|f - \chi_{Q(0,R)} f\|_{L^p(\omega)} < \frac{\epsilon}{4}, \qquad \text{if } R > R_0. \tag{6}$$

On the other hand, by Jensen's inequality,

$$\begin{aligned} \|f\chi_{Q(0,R)} - \Phi f\|^p_{L^p(\omega)} &= \sum_{i=1}^N \int_{Q_i} \left| f(x) - \fint_{Q_i} f(z)\mathrm{d}z \right|^p \omega(x)\mathrm{d}x \\ &\leq \sum_{i=1}^N \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(z)|^p \mathrm{d}z\, \omega(x)\mathrm{d}x \end{aligned}$$

Now, if $x, z \in Q_i$, then $z - x = h \in 2Q \subset B(0,\rho)$. Therefore, after a change of coordinates,

$$\begin{aligned}\|f\chi_{Q(0,R)} - \Phi f\|^p_{L^p(\omega)} &\leq \sum_{i=1}^N \frac{1}{|Q_i|}\int_{Q_i}\int_{2Q} |f(x) - f(x+h)|^p \mathrm{d}h\, \omega(x)\mathrm{d}x \\ &= \frac{1}{|Q|}\int_{2Q}\sum_{i=1}^N \int_{Q_i} |f(x) - f(x+h)|^p \omega(x)\mathrm{d}x\, \mathrm{d}h \\ &\leq \frac{1}{|Q|}\int_{2Q}\int_{\mathbb{R}^n} |f(x) - f(x+h)|^p \omega(x)\mathrm{d}x\, \mathrm{d}h \\ &\leq 2^n \sup_{|h|\leq\rho}\left(\sup_{f\in\mathfrak{F}} \|f(\cdot) - f(\cdot+h)\|^p_{L^p(\omega)}\right).\end{aligned}$$

Therefore, by the uniform equicontinuity, we can find $\rho > 0$ small enough, such that

$$\sup_{f\in\mathfrak{F}} \|f\chi_{Q(0,R)} - \Phi f\|_{L^p(\omega)} < \frac{\epsilon}{4}. \tag{7}$$

By (6) and (7) we have that

$$\sup_{f\in\mathfrak{F}} \|f - \Phi f\|_{L^p(\omega)} < \frac{\epsilon}{2},$$

whence

$$\|f\|_{L^p(\omega)} < \frac{\epsilon}{2} + \|\Phi f\|_{L^p(\omega)}, \qquad \text{whenever } f \in \mathfrak{F}. \tag{8}$$

Since Φ is linear, this means that

$$\|f - g\|_{L^p(\omega)} < \frac{\epsilon}{2} + \|\Phi f - \Phi g\|_{L^p(\omega)}, \qquad \text{whenever } f, g \in \mathfrak{F}.$$

Set $\delta = \epsilon/2$. The above inequality says that if $f, g \in \mathfrak{F}$ are such that $d(\Phi f, \Phi g) < \delta$, then $d(f, g) < \epsilon$. By the previous Lemma, it follows that $\mathfrak{F}$ is totally bounded. □

In order to prove Theorem 2, we will first reduce ourselves to smooth symbols b. Let us recall that commutators $C_b = [b, T]$ with $b \in BMO(\mathbb{R}^n)$ are continuous in $L^p(\omega)$ (*e. g.* Theorem 2.3 in [13]). Moreover, in [11, Theorem 1] the following estimate is shown,

$$\|C_b f\|_{L^p(\omega)} \leq C\,\|b\|_*\,\|M^2 f\|_{L^p(\omega)}, \tag{9}$$

where C may depend on ω, but not on b. Now, by the boudedness of the Hardy-Littlewood operator M on $L^p(\omega)$, we obtain

$$\|C_b f\|_{L^p(\omega)} \leq C\,\|b\|_*\,\|f\|_{L^p(\omega)}.$$

Since by assumption $b \in VMO(\mathbb{R}^n)$, we can approximate the function b by functions $b_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ in the BMO norm, and thus

$$\|C_b f - C_{b_j} f\|_{L^p(\omega)} = \|C_{b-b_j} f\|_{L^p(\omega)} \leq C\,\|b - b_j\|_*\,\|f\|_{L^p(\omega)}.$$

In particular, the commutators with smooth symbol C_{b_j} converge to C_b in the operator norm of $L^p(\omega)$. Therefore it suffices to prove compactness for the commutator with smooth symbol.

Another reduction in the proof of Theorem 2 will be made by slightly modifying the singular integral operator T. This technique comes from Krantz and Li [10]. More precisely, for every $\eta > 0$ small enough, let us take a continuous function K^η defined on $\mathbb{R}^n \times \mathbb{R}^n$, taking values in $\mathbb{R}$ or $\mathbb{C}$, and such that:

1. $K^\eta(x,y) = K(x,y)$ if $|x-y| \geq \eta$
2. $|K^\eta(x,y)| \leq \frac{C_0}{|x-y|^n}$ for $\frac{\eta}{2} < |x-y| < \eta$
3. $K^\eta(x,y) = 0$ si $|x-y| \leq \frac{\eta}{2}$

where C_0 is independent of η. Due to the growth properties of K, is not restrictive to suppose that the condition 2 holds for all $x, y \in \mathbb{R}^n$. Now, let

$$T^\eta f(x) = \int_{\mathbb{R}^n} K^\eta(x,y) f(y) \mathrm{d}y,$$

and let us also denote

$$C_b^\eta f(x) = [b, T^\eta] f(x) = \int_{\mathbb{R}^n} (b(x) - b(y)) K^\eta(x,y) f(y) \mathrm{d}y.$$

We now prove that the commutators C_b^η approximate C_b in the operator norm.

Lemma 7. *Let $b \in \mathcal{C}_c^1(\mathbb{R}^n)$. There exists a constant $C = C(n, C_0)$ such that*

$$|C_b f(x) - C_b^\eta f(x)| \leq C\,\eta\,\|\nabla b\|_\infty\, Mf(x) \qquad \textit{almost everywhere,}$$

for every $\eta > 0$. As a consequence,

$$\lim_{\eta \to 0} \|C_b^\eta - C_b\|_{L^p(\omega) \to L^p(\omega)} = 0$$

whenever $\omega \in A_p$ and $1 < p < \infty$.

Proof. Let $f \in L^p(\omega)$. For every $x \in \mathbb{R}^n$ we have:

$$\begin{aligned} C_b f(x) - C_b^\eta f(x) &= \int_{|x-y|<\eta} (b(x)-b(y)) K(x,y) f(y) \mathrm{d}y - \int_{\frac{\eta}{2} \leq |x-y| < \eta} (b(x)-b(y)) K^\eta(x,y) f(y) \mathrm{d}y \\ &= I_1(x) + I_2(x). \end{aligned}$$

Using the smoothness of b and the size estimates of K^η, we have that

$$\begin{aligned} |I_1(x)| &\leq \int_{|x-y|<\eta} |b(y)-b(x)| |K(x,y)| |f(y)| \mathrm{d}y \leq C_0\, \|\nabla b\|_\infty \sum_{j=0}^\infty \int_{\frac{\eta}{2^{j+1}} < |x-y| < \frac{\eta}{2^j}} \frac{|f(y)|}{|x-y|^{n-1}} \mathrm{d}y \\ &\leq 2^n C_0\, \|\nabla b\|_\infty \sum_{j=0}^\infty \frac{\eta\, |B(0,1)|}{2^{j+1}} ⨍_{|x-y| < \frac{\eta}{2^j}} |f(y)| \mathrm{d}y \leq \eta\, 2^n\, C_0\, \|\nabla b\|_\infty\, |B(0,1)|\, Mf(x) \end{aligned}$$

for almost every x. For the other term, similarly

$$\begin{aligned} |I_2(x)| &\leq \eta\, \|\nabla b\|_\infty \int_{\frac{\eta}{2} < |x-y| < \eta} |K^\eta(x,y)|\, |f(y)|\, \mathrm{d}y \leq \eta\, C_0\, \|\nabla b\|_\infty \int_{\frac{\eta}{2} < |x-y| < \eta} \frac{|f(y)|}{|x-y|^n} \mathrm{d}y \\ &\leq \eta\, 2^n\, C_0\, \|\nabla b\|_\infty\, |B(0,1)| ⨍_{|x-y| < \eta} |f(y)|\, \mathrm{d}y \leq \eta\, 2^n\, C_0\, \|\nabla b\|_\infty\, |B(0,1)|\, Mf(x). \end{aligned}$$

Therefore, the pointwise estimate follows. Now, the boundedness of M in $L^p(\omega)$ for any A_p weight ω implies that

$$\begin{aligned}\|C_b f - C_b^\eta f\|_{L^p(\omega)} &\leq C\,\eta\,\|\nabla b\|_\infty\,\|Mf\|_{L^p(\omega)} \\ &\leq C\,\eta\,\|\nabla b\|_\infty\,\|f\|_{L^p(\omega)} \to 0, \qquad \text{as } \eta \to 0.\end{aligned}$$

This finishes the proof of Lemma 7. □

We are now ready to conclude the proof of Theorem 2. From now on, $\eta > 0$ and $b \in \mathcal{C}_c^1(\mathbb{R}^n)$ are fixed, and we have to prove that the commutator $C_b^\eta = [b, T^\eta]$ is compact. Thus, the constants that will appear may depend on b and η.

We denote $\mathfrak{F} = \{C_b^\eta f; f \in L^p(\omega), \|f\|_{L^p(\omega)} \leq 1\}$. Then $\mathfrak{F}$ is uniformly bounded, because C_b^η is a bounded operator on $L^p(\omega)$. To prove the uniform equicontinuity of $\mathfrak{F}$, we must see that

$$\lim_{h\to 0} \sup_{f\in\mathfrak{F}} \|C_b^\eta f(\cdot) - C_b^\eta f(\cdot + h)\|_{L^p(\omega)} = 0.$$

To do this, let us write

$$\begin{aligned}&C_b^\eta f(x) - C_b^\eta f(x+h) \\ &\quad = (b(x)-b(x+h))\int_{\mathbb{R}^n} K^\eta(x,y) f(y)\mathrm{d}y + \int_{\mathbb{R}^n} (b(x+h)-b(y))(K^\eta(x,y)-K^\eta(x+h,y))f(y)\mathrm{d}y \\ &\quad = \int_{\mathbb{R}^n} I_1(x,y,h)\mathrm{d}y + \int_{\mathbb{R}^n} I_2(x,y,h)\mathrm{d}y.\end{aligned}$$

For $I_1(x,y,h)$, using the regularity of the function b and the definition of the operator T_*,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} I_1(x,y,h)\mathrm{d}y\right| &\leq \|\nabla b\|_\infty |h| \left|\int_{|x-y|>\frac{\eta}{2}} (K^\eta(x,y)-K(x,y))f(y)\mathrm{d}y + \int_{|x-y|>\frac{\eta}{2}} K(x,y)f(y)\mathrm{d}y\right| \\ &\leq \|\nabla b\|_\infty |h| \left(\int_{|x-y|>\frac{\eta}{2}} |K^\eta(x,y) - K(x,y)|\,|f(y)|\mathrm{d}y + T_* f(x)\right) \\ &\leq \|\nabla b\|_\infty |h| \,(C\,Mf(x) + T_* f(x))\end{aligned}$$

for some constant $C > 0$ that may depend on η, but not on h. Therefore, by ,

$$\int \left|\int_{\mathbb{R}^n} I_1(x,y,h)\mathrm{d}y\right|^p \omega(x)\mathrm{d}x \leq C\,|h|\,\|f\|_{L^p(\omega)}, \tag{10}$$

for C independent of f and h. Here we used the boundedness of M and T_* on $L^p(\omega)$ (see [5, Chap. IV, Th. 3.6]). We will divide the integral of $I_2(x,y,h)$ into three regions:

$$\begin{aligned}A &= \left\{y \in \mathbb{R}^n : |x-y| > \frac{\eta}{2}, \quad |x+h-y| > \frac{\eta}{2}\right\}, \\ B &= \left\{y \in \mathbb{R}^n : |x-y| > \frac{\eta}{2}, \quad |x+h-y| < \frac{\eta}{2}\right\}, \\ C &= \left\{y \in \mathbb{R}^n : |x-y| < \frac{\eta}{2}, \quad |x+h-y| > \frac{\eta}{2}\right\}.\end{aligned}$$

Note that $I_2(x,y,h) = 0$ for $y \in \mathbb{R}^n \setminus A \cup B \cup C$. Now, for the integral over A, we use the smoothness of b and K^η,

$$\left|\int_A I_2(x,y,h)\mathrm{d}y\right| \leq C\|\nabla b\|_\infty |h| \int_{|x-y|>\frac{\eta}{4}} \frac{|f(y)|}{|x-y|^{n+1}}\mathrm{d}y$$

$$\leq C\|\nabla b\|_\infty \frac{|h|}{\eta} \sum_{j=0}^{\infty} 2^{-j} \fint_{|x-y|<\frac{2^j\eta}{4}} |f(y)|\mathrm{d}y \leq C\|\nabla b\|_\infty \frac{|h|}{\eta} \quad Mf(x),$$

thus

$$\int_{\mathbb{R}^n} \left|\int_A I_2(x,y,h)\mathrm{d}y\right|^p \omega(x)\mathrm{d}x \leq C|h|\,\|f\|_{L^p(\omega)}.$$

for some constant C that may depend on η, but not on h. In particular, the term on the right hand side goes to 0 as $|h| \to 0$.

The integrals of $I_2(x,y,h)$ over B and C are symmetric, so we only give the details once. For the integral over the set B, let us assume that $|h|$ is very small. We can first choose $R_0 > \eta/2 + |h|$ such that b vanishes outside the ball $B_0 = B(0,R_0)$. It then follows that $b(\cdot + h)$ has support in $2B_0$. Then, since $B \subset B(x, |h| + \eta/2)$, we have for $|x| < 3R_0$ that $B \subset 4B_0$ and therefore

$$\left|\int_B I_2(x,y,h)\mathrm{d}y\right| \leq C_0 \|\nabla b\|_\infty \int_{B\cap 4B_0} \frac{|x+h-y|\,|f(y)|}{|x-y|^n}\mathrm{d}y \leq C_0 \|\nabla b\|_\infty \int_{B\cap 4B_0} \frac{|f(y)|}{|x-y|^{n-1}}\mathrm{d}y$$

$$\leq C_0 \|\nabla b\|_\infty (2/\eta)^{n-1} \int_{B\cap 4B_0} |f(y)|\omega(y)^{\frac{1}{p}}\omega(y)^{-\frac{1}{p}}\mathrm{d}y$$

$$\leq C_0 \|\nabla b\|_\infty (2/\eta)^{n-1}\|f\|_{L^p(\omega)} \left(\int_{B\cap 4B_0} \omega(y)^{-\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{1}{p'}}$$

whence

$$\int_{3B_0} \left|\int_B I_2(x,y,h)\mathrm{d}y\right|^p \omega(x)\mathrm{d}x \leq C\,\|f\|_{L^p(\omega)}^p \left(\int_{3B_0} \omega(x)\mathrm{d}x\right) \left(\int_{B\cap 4B_0} \omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{p}{p'}}$$

for some constant C that might depend on η, but not on h. If, instead, we have $|x| \geq 3R_0$, then $b(x+h) = 0$ (because $|h| < R_0$ so that $|x+h| > 2R_0$). Note also that for $y \in B$ one has $|x| \leq C|x-y|$ where C depends only on η. Therefore

$$\left|\int_B I_2(x,y,h)\mathrm{d}y\right| \leq C\|b\|_\infty \int_{B\cap 4B_0} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y \leq \frac{C\|b\|_\infty}{|x|^n} \int_{B\cap 4B_0} |f(y)|\mathrm{d}y$$

$$\leq \frac{C\|b\|_\infty}{|x|^n}\,\|f\|_{L^p(\omega)} \left(\int_{B\cap 4B_0} \omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{1}{p'}}.$$

This implies that

$$\int_{\mathbb{R}^n\setminus 3B_0} \left|\int_B I_2(x,y,h)\mathrm{d}y\right|^p \omega(x)\mathrm{d}x \leq C\|b\|_\infty^p\,\|f\|_{L^p(\omega)}^p \left(\int_{\mathbb{R}^n\setminus 3B_0} \frac{\omega(x)}{|x|^{np}}\mathrm{d}x\right) \left(\int_{B\cap 4B_0} \omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{p}{p'}}$$

Summarizing,

$$\int_{\mathbb{R}^n} \left|\int_B I_2(x,y,h)\mathrm{d}y\right|^p \omega(x)\,\mathrm{d}x \leq C\,\|f\|_{L^p(\omega)} \left(\int_{B\cap 4B_0} \omega(y)^{-\frac{p'}{p}}\mathrm{d}y\right)^{\frac{p}{p'}} \left(\int_{3B_0} \omega(x)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus 3B_0} \frac{\omega(x)}{|x|^{np}}\mathrm{d}x\right) \tag{11}$$

After proving that

$$\int_{|x|>3R_0} \frac{\omega(x)}{|x|^{np}} \mathrm{d}x < \infty$$

the left hand side of (11) will converge to 0 as $|h| \to 0$ since $|B| \to 0$ as $|h| \to 0$. To prove the above claim, let us choose $q < p$ such that $\omega \in A_q$ [5, Theorem 2.6, Ch. IV]. For such q, we have

$$\int_{|x|>R} \frac{\omega(x)}{|x|^{np}} \mathrm{d}x = \sum_{j=1}^{\infty} \int_{2^{j-1}<\frac{|x|}{R}<2^j} \frac{\omega(x)}{|x|^{np}} \mathrm{d}x \le \sum_{j=1}^{\infty} (2^{j-1}R)^{-np} \omega(B(0, 2^j R))$$

By [5, Lemma 2.2], we have

$$\int_{|x|>R} \frac{\omega(x)}{|x|^{np}} \mathrm{d}x \le \sum_{j=1}^{\infty} (2^{j-1}R)^{-np} (2^j R)^{nq} \omega(B(0,1)) = \frac{C}{R^{n(p-q)}} < \infty \tag{12}$$

as desired. The equicontinuity of $\mathfrak{F}$ follows.
Finally, we show the decay at infinity of the elements of $\mathfrak{F}$. Let x be such that $|x| > R > R_0$. Then, $x \notin \operatorname{supp} b$, and

$$\begin{aligned}
|C_b^\eta f(x)| &= \left| \int_{\mathbb{R}^n} (b(x) - b(y)) K^\eta(x,y) f(y) \mathrm{d}y \right| \\
&\le C_0 \|b\|_\infty \int_{\operatorname{supp} b} \frac{|f(y)|}{|x-y|^n} \mathrm{d}y \\
&\le \frac{C\|b\|_\infty}{|x|^n} \int_{\operatorname{supp} b} |f(y)| \, \mathrm{d}y \\
&\le \frac{C\|b\|_\infty}{|x|^n} \|f\|_{L^p(\omega)} \left(\int_{\operatorname{supp} b} \omega(y)^{-\frac{p'}{p}} dy \right)^{\frac{1}{p'}}
\end{aligned}$$

whence

$$\left(\int_{|x|>R} |C_b^\eta f(x)|^p \omega(x) \mathrm{d}x \right)^{\frac{1}{p}} \le C\|b\|_\infty \|f\|_{L^p(\omega)} \left(\int_{|x|>R} \frac{\omega(x)}{|x|^{np}} \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

The right hand side above converges to 0 as $R \to \infty$, due to (12). By Theorem 5, $\mathfrak{F}$ is totally bounded. Theorem 2 follows.

3 A priori estimates for Beltrami equations

We first prove Theorem 1. To do this, let us remember that the Beurling-Ahlfors singular integral operator is defined by the following principal value

$$\mathcal{B}f(z) = -\frac{1}{\pi} \, P.V. \int \frac{f(w)}{(z-w)^2} \, \mathrm{d}w.$$

This operator can be seen as the formal ∂ derivative of the Cauchy transform,

$$\mathcal{C}f(z) = \frac{1}{\pi} \int \frac{f(w)}{z-w} \, \mathrm{d}w.$$

At the frequency side, $\mathcal{B}$ corresponds to the Fourier multiplier $m(\xi) = \frac{\bar{\xi}}{\xi}$, so that $\mathcal{B}$ is an isometry in $L^2(\mathbb{C})$. Moreover, this Fourier representation also explains the important relation

$$\mathcal{B}(\overline{\partial} f) = \partial f$$

for smooth enough functions f. By $\mathcal{B}^*$ we mean the singular integral operator obtained by simply conjugating the kernel of $\mathcal{B}$, that is,

$$\mathcal{B}^*(f)(z) = -\frac{1}{\pi}\, P.V. \int \frac{f(w)}{(\bar{z}-\bar{w})^2}\, \mathrm{d}w.$$

Note that $\mathcal{B}^*$ has Fourier multiplier $m^*(\xi) = \frac{\xi}{\bar{\xi}}$. Thus,

$$\mathcal{B}\mathcal{B}^* = \mathcal{B}^*\mathcal{B} = \mathbf{Id}.$$

In other words, $\mathcal{B}^*$ is the L^2-inverse of $\mathcal{B}$. It also appears as the $\mathbb{C}$-linear adjoint of $\mathcal{B}$,

$$\int_{\mathbb{C}} \mathcal{B}f(z)\, \overline{g(z)}\, \mathrm{d}z = \int_{\mathbb{C}} f(z)\, \overline{\mathcal{B}^* g(z)}\, \mathrm{d}z.$$

The complex conjugate operator $\overline{\mathcal{B}}$ is the composition of $\mathcal{B}$ with the complex conjugation operator $\mathbf{C}f = \overline{f}$, that is,

$$\overline{\mathcal{B}}(f) = \mathbf{C}\mathcal{B}(f) = \overline{\mathcal{B}(f)}.$$

It then follows that

$$\overline{\mathcal{B}} = \mathbf{C}\mathcal{B} = \mathcal{B}^*\mathbf{C}.$$

Note that $\mathcal{B}$ and $\mathcal{B}^*$ are $\mathbb{C}$-linear operators, while $\overline{\mathcal{B}}$ is only $\mathbb{R}$-linear. See [1, Chapter 4] for more about the Beurling-Ahlfors transform.

Proof of Theorem 1. We follow Iwaniec's idea [7, pag. 42–43]. For every $N = 1, 2, ...$, let

$$P_N = \mathbf{Id} + \mu\mathcal{B} + \cdots + (\mu\mathcal{B})^N.$$

Then

$$(\mathbf{Id} - \mu\mathcal{B})P_{N-1} = P_{N-1}(\mathbf{Id} - \mu\mathcal{B}) = \mathbf{Id} - \mu^N\mathcal{B}^N + K_N \tag{13}$$

where $K_N = \mu^N\mathcal{B}^N - (\mu\mathcal{B})^N$. Each K_N consists of a finite sum of operators that contain the commutator $[\mu, \mathcal{B}]$ as a factor. Thus, by Theorem 2, each K_N is compact in $L^p(\omega)$. On the other hand, the iterates of the Beurling transform $\mathcal{B}^N$ have the kernel

$$b_N(z) = \frac{(-1)^N N}{\pi} \frac{\bar{z}^{N-1}}{z^{N+1}}.$$

Therefore,

$$\|\mathcal{B}^N\|_{L^p(\omega)} \le CN^2,$$

where the constant C depends on $[\omega]_{A_p}$, but not on N. As a consequence,

$$\|\mu^N\mathcal{B}^N f\|_{L^p(\omega)} \le CN^2 \|\mu\|_\infty^N \|f\|_{L^p(\omega)},$$

and therefore, for large enough N, the operator $\mathbf{Id} - \mu^N \mathcal{B}^N$ is invertible. This, together with (13), says that $\mathbf{Id} - \mu\mathcal{B}$ is an Fredholm operator. Now apply the index theory to $\mathbf{Id} - \mu\mathcal{B}$. The continuous deformation $\mathbf{Id} - t\mu\mathcal{B}$, $0 \le t \le 1$, is a homotopy from the identity operator to $\mathbf{Id} - \mu\mathcal{B}$. By the homotopical invariance of Index ,

$$\text{Index}\,(\mathbf{Id} - \mu\mathcal{B}) = \text{Index}\,(\mathbf{Id}) = 0.$$

Since injective operators with 0 index are onto, for the invertibility of $\mathbf{Id} - \mu\mathcal{B}$ it just remains to show that it is injective. So let $f \in L^p(\omega)$ be such that $f = \mu\mathcal{B}f$. Then f has compact support. Now, since belonging to A_p is an open-ended condition (see e.g. [5, Theorem IV.2.6]), there exists $\delta > 0$ such that $p - \delta > 1$ and $\omega \in A_{p-\delta}$. Then $\omega^{-\frac{1}{p-\delta}} \in L^1_{loc}(\mathbb{C})$. Taking $\epsilon = \frac{\delta}{p-\delta}$, we obtain

$$\begin{aligned} \int_{\mathbb{C}} |f(x)|^{1+\epsilon} \mathrm{d}x &\le \left(\int_{\operatorname{supp} f} |f(x)|^p \omega(x) \mathrm{d}x \right)^{\frac{1+\epsilon}{p}} \left(\int_{\operatorname{supp} f} \omega(x)^{-\frac{1+\epsilon}{p-(1+\epsilon)}} \mathrm{d}x \right)^{\frac{p-(1+\epsilon)}{p}} \\ &\le \|f\|^{1+\epsilon}_{L^p(\omega)} \left(\int_{\operatorname{supp} f} \omega(x)^{-\frac{1+\epsilon}{p-(1+\epsilon)}} \mathrm{d}x \right)^{\frac{p-(1+\epsilon)}{p}} < \infty, \end{aligned}$$

therefore $f \in L^{1+\epsilon}(\mathbb{C})$. But $\mathbf{Id} - \mu\mathcal{B}$ is injective on $L^p(\mathbb{C})$, $1 < p < \infty$ when $\mu \in VMO(\mathbb{C})$, by Iwaniec's Theorem. Hence, $f \equiv 0$.

Finally, since $\mathbf{Id} - \mu\mathcal{B} : L^p(\omega) \to L^p(\omega)$ is linear, bounded, and invertible, it then follows that it has a bounded inverse, so the inequality

$$\|g\|_{L^p(\omega)} \le C \, \|(\mathbf{Id} - \mu\mathcal{B})g\|_{L^p(\omega)}$$

holds for every $g \in L^p(\omega)$. Here the constant $C > 0$ depends only on the $L^p(\omega)$ norm of $\mathbf{Id} - \mu\mathcal{B}$, and therefore on p, k and $[\omega]_{A_p}$, but not on g. As a consequence, given $g \in L^p(\omega)$, and setting

$$f := \mathcal{C}(\mathbf{Id} - \mu\mathcal{B})^{-1} g,$$

we immediately see that f satisfies $\overline{\partial} f - \mu \partial f = g$. Moreover, since $\omega \in A_p$,

$$\begin{aligned} \|Df\|_{L^p(\omega)} &\le \|\partial f\|_{L^p(\omega)} + \|\overline{\partial} f\|_{L^p(\omega)} \\ &= \|\mathcal{B}(\mathbf{Id} - \mu\mathcal{B})^{-1} g\|_{L^p(\omega)} + \|(\mathbf{Id} - \mu\mathcal{B})^{-1} g\|_{L^p(\omega)} \le C\|g\|_{L^p(\omega)}, \end{aligned}$$

where still C depends only on p, k and $[\omega]_{A_p}$.

For the uniqueness, let us choose two solutions f_1, f_2 to the inhomogeneous equation. The difference $F = f_1 - f_2$ defines a solution to the homogeneous equation $\overline{\partial} F - \mu \, \partial F = 0$. Moreover, one has that $DF \in L^p(\omega)$ and, arguing as before, one sees that $DF \in L^{1+\epsilon}(\mathbb{C})$. In particular, this says that $(I - \mu\mathcal{B})(\overline{\partial} F) = 0$. But for $\mu \in VMO(\mathbb{C})$, it follows from Iwaniec's Theorem that $\mathbf{Id} - \mu\mathcal{B}$ is injective in $L^p(\mathbb{C})$ for any $1 < p < \infty$, whence $\overline{\partial} F = 0$. Thus $DF = 0$ and so F is a constant. □

The $\mathbb{C}$-linear Beltrami equation is a particular case of the following one,

$$\overline{\partial} f(z) - \mu(z) \, \partial f(z) - \nu(z) \, \overline{\partial f(z)} = g(z),$$

which we will refer to as the *generalized Beltrami equation*. It is well known that, in the plane, any linear, elliptic system, with two unknowns and two first-order equations on the derivatives, reduces to the above equation (modulo complex conjugation), whence the interest in understanding it is very big. An especially interesting example is obtained by setting $\mu = 0$, when one obtains the so-called *conjugate Beltrami equation*,

$$\overline{\partial} f(z) - \nu(z)\, \overline{\partial f(z)} = g(z).$$

A direct adaptation of the above proof immediately drives the problem towards the commutator $[\nu, \overline{\mathcal{B}}]$. Unfortunately, as an operator from $L^p(\omega)$ onto itself, such commutator is not compact in general, even when $\omega = 1$. To show this, let us choose

$$\nu = i\, \nu_0\, \chi_{\mathbb{D}} + \nu_1 \chi_{\mathbb{C}\setminus\mathbb{D}}$$

where the constant $\nu_0 \in \mathbb{R}$ and the function ν_1 are chosen so that ν is continuous on $\mathbb{C}$, compactly supported in $2\mathbb{D}$, with $\|\nu\|_\infty < 1$. Let us also consider

$$E = \{f \in L^p;\, \|f\|_{L^p} \leq 1, \operatorname{supp}(f) \subset \mathbb{D}\},$$

which is a bounded subset of L^p. For every $f \in E$, one has

$$\begin{aligned} \nu\, \overline{\mathcal{B}(f)} - \overline{\mathcal{B}(\nu f)} &= \chi_{\mathbb{D}} i\nu_0 \overline{\mathcal{B}(f)} + \chi_{\mathbb{C}\setminus\mathbb{D}}\, \nu_1\, \overline{\mathcal{B}(f)} - \overline{\mathcal{B}(i\nu_0 f)} \\ &= \chi_{\mathbb{D}} i\nu_0 \overline{\mathcal{B}(f)} + \chi_{\mathbb{C}\setminus\mathbb{D}}\, \nu_1\, \overline{\mathcal{B}(f)} + i\nu_0 \overline{\mathcal{B}(f)} \\ &= \chi_{\mathbb{D}} 2i\nu_0 \overline{\mathcal{B}(f)} + \chi_{\mathbb{C}\setminus\mathbb{D}}\, (i\nu_0 + \nu_1)\, \overline{\mathcal{B}(f)}. \end{aligned}$$

In view of this relation, and since $\mathcal{B}$ is not compact, we have just cooked a concrete example of function $\nu \in VMO$ for wich the commutator $[\nu, \overline{\mathcal{B}}]$ is not compact. Nevertheless, it turns out that still a priori estimates hold, even for the generalized equation.

Theorem 8. *Let $1 < p < \infty$, $\omega \in A_p$, and let $\mu, \nu \in VMO(\mathbb{C})$ be compactly supported, such that $\||\mu| + |\nu|\|_\infty < 1$. Let $g \in L^p(\omega)$. Then the equation*

$$\overline{\partial} f(z) - \mu(z)\, \partial f(z) - \nu(z)\, \overline{\partial f(z)} = g(z)$$

has a solution f with $Df \in L^p(\omega)$ and

$$\|Df\|_{L^p(\omega)} \leq C\, \|g\|_{L^p(\omega)}.$$

This solution is unique, modulo an additive constant.

A previous proof for the above result has been shown in [9] for the constant weight $\omega = 1$. For the weighted counterpart, the arguments are based on a Neumann series argument similar to that in [9], with some minor modification. We write it here for completeness. The following Lemma will be needed.

Lemma 9. *Let $\mu, \nu \in L^\infty(\mathbb{C})$ be measurable, bounded with compact support, such that $\||\mu| + |\nu|\|_\infty < 1$. If $1 < p < \infty$ and $p' = \frac{p}{p-1}$, then the following statements are equivalent:*

1. *The operator* $\mathbf{Id} - \mu\,\mathcal{B} - \nu\,\overline{\mathcal{B}} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ *is bijective.*
2. *The operator* $\mathbf{Id} - \overline{\mu}\,\mathcal{B}^* - \nu\,\overline{\mathcal{B}^*} : L^{p'}(\mathbb{C}) \to L^{p'}(\mathbb{C})$ *is bijective.*

Proof. When $\nu = 0$, the above result is well known, and follows as an easy consequence of the fact that, with respect to the dual pairing

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\,\overline{g(z)}\,\mathrm{d}z, \tag{14}$$

the operator $\mathbf{Id} - \mu\mathcal{B} : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ has precisely $\mathbf{Id} - \mathcal{B}^*\overline{\mu} : L^{p'}(\mathbb{C}) \to L^{p'}(\mathbb{C})$ as its $\mathbb{C}$-linear adjoint. Unfortunately, when ν does not identically vanish, $\mathbb{R}$-linear operators do not have an adjoint with respect to this dual pairing. An alternative proof can be found in [9]. To this end, we think the space of $\mathbb{C}$-valued L^p functions $L^p(\mathbb{C})$ as an $\mathbb{R}$-linear space,

$$L^p(\mathbb{C}) = L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}),$$

by means of the obvious identification $u + iv = (u, v)$. According to this product structure, every bounded $\mathbb{R}$-linear operator $T : L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}) \to L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ has an obvious matrix representation

$$T(u + iv) = T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where every $T_{ij} : L^p_{\mathbb{R}}(\mathbb{C}) \to L^p_{\mathbb{R}}(\mathbb{C})$ is bounded. Similarly, bounded linear functionals $U : L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}) \to \mathbb{R}$ are represented by

$$U\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U_1 & U_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where every $U_j : L^p_{\mathbb{R}}(\mathbb{C}) \to \mathbb{R}$ is bounded. By the Riesz Representation Theorem, we get that $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ has precisely $L^{p'}_{\mathbb{R}}(\mathbb{C}) \oplus L^{p'}_{\mathbb{R}}(\mathbb{C})$ as its topological dual space. In fact, we have an $\mathbb{R}$-bilinear dual pairing,

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix}\rangle = \int u(z)\,u'(z)\,\mathrm{d}z + \int v(z)\,v'(z)\,\mathrm{d}z,$$

whenever $(u, v) \in L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ and $(u', v') \in L^{p'}_{\mathbb{R}}(\mathbb{C}) \oplus L^{p'}_{\mathbb{R}}(\mathbb{C})$, and which is nothing but the real part of (14). Under this new dual pairing, every $\mathbb{R}$-linear opeartor $T : L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C}) \to L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ can be associated another operator

$$T' : L^{p'}_{\mathbb{R}}(\mathbb{C}) \oplus L^{p'}_{\mathbb{R}}(\mathbb{C}) \to L^{p'}_{\mathbb{R}}(\mathbb{C}) \oplus L^{p'}_{\mathbb{R}}(\mathbb{C}),$$

called the $\mathbb{R}$-adjoint operator of T, defined by the common rule

$$\langle \begin{pmatrix} u \\ v \end{pmatrix}, T'\begin{pmatrix} u' \\ v' \end{pmatrix}\rangle = \langle T\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix}\rangle.$$

If T is a $\mathbb{C}$-linear operator, then T' is the same as the $\mathbb{C}$-adjoint T^* (i.e. the adjoint with respect to (14)) so in particular for the Beurling-Ahlfors transform $\mathcal{B}$ we have an $\mathbb{R}$-adjoint

$\mathcal{B}'$, and moreover $\mathcal{B}^* = \mathcal{B}'$. Similarly, the pointwise multiplication by μ and ν are also $\mathbb{C}$-linear operators. Thus their $\mathbb{R}$-adjoints μ', ν' agree with their respectives $\mathbb{C}$-adjoints μ^*, ν^*. But these are precisely the pointwise multiplication with the respective complex conjugates. Symbollically, $\mu' = \overline{\mu}$ and $\nu' = \overline{\nu}$. In contrast, general $\mathbb{R}$-linear operators need not have a $\mathbb{C}$-adjoint. For example, for the complex conjugation,

$$\mathbf{C} = \begin{pmatrix} \mathbf{Id} & 0 \\ 0 & -\mathbf{Id} \end{pmatrix}$$

one simply has $\mathbf{C}' = \mathbf{C}$. Putting all these things together, one easily sees that

$$\begin{aligned}
(\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}})' &= (\mathbf{Id} - \mu\mathcal{B} - \nu\mathbf{C}\mathcal{B})' \\
&= \mathbf{Id} - (\mu\mathcal{B})' - (\nu\mathbf{C}\mathcal{B})' \\
&= \mathbf{Id} - \mathcal{B}'\mu' - \mathcal{B}'\mathbf{C}'\nu' \\
&= \mathbf{Id} - \mathcal{B}^*\overline{\mu} - \mathcal{B}^*\mathbf{C}\overline{\nu} \\
&= \mathcal{B}^*\left(\mathbf{Id} - \overline{\mu}\mathcal{B}^* - \mathbf{C}\overline{\nu}\mathcal{B}^*\right)\mathcal{B} \\
&= \mathcal{B}^*\left(\mathbf{Id} - \overline{\mu}\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*\right)\mathcal{B}
\end{aligned}$$

where we used the fact that $\mathcal{B}^*\mathcal{B} = \mathcal{B}\mathcal{B}^* = \mathbf{Id}$. As a consequence, and using that both $\mathcal{B}$ and $\mathcal{B}^*$ are bijective in $L^p(\mathbb{C})$, we obtain that the bijectivity of the operator $\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}}$ in $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ is equivalent to that of $\mathbf{Id} - \overline{\mu}\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*$ in the dual space $L^{p'}_{\mathbb{R}}(\mathbb{C}) \oplus L^{p'}_{\mathbb{R}}(\mathbb{C})$. Similarly, one proves that

$$(\mathbf{Id} - \mu\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*)' = \mathcal{B}(\mathbf{Id} - \overline{\mu}\,\mathcal{B} - \nu\,\overline{\mathcal{B}})\mathcal{B}^*.$$

Hence, the bijectivity of $\mathbf{Id} - \mu\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*$ in $L^p_{\mathbb{R}}(\mathbb{C}) \oplus L^p_{\mathbb{R}}(\mathbb{C})$ is equivalent to the bijectivity of $\mathbf{Id} - \overline{\mu}\,\mathcal{B} - \nu\,\overline{\mathcal{B}}$ in $L^{p'}_{\mathbb{R}}(\mathbb{C}) \oplus L^{p'}_{\mathbb{R}}(\mathbb{C})$. □

Lemma 10. *If $1 < p < \infty$, $\omega \in A_p$, $\mu, \nu \in VMO$ have compact support, and $\||\mu| + |\nu|\|_\infty \leq k < 1$, then the operators*

$$\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}} \qquad \text{and} \qquad \mathbf{Id} - \mu\mathcal{B}^* - \nu\overline{\mathcal{B}^*}$$

are Fredholm operators in $L^p(\omega)$.

Proof. We will show the claim for the operator $\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}}$. For $\mathbf{Id} - \mu\mathcal{B}^* - \nu\overline{\mathcal{B}^*}$ the proof follows similarly. It will be more convenient for us to write $\overline{\mathcal{B}} = \mathbf{C}\mathcal{B}$. As in the proof of Theorem 1, we set

$$P_N = \sum_{j=0}^{N} (\mu\mathcal{B} + \nu\mathbf{C}\mathcal{B})^j.$$

Then

$$\begin{aligned}
(\mathbf{Id} - \mu\mathcal{B} - \nu\mathbf{C}\mathcal{B}) \circ P_{N-1} &= \mathbf{Id} - (\mu\mathcal{B} + \nu\mathbf{C}\mathcal{B})^N, \\
P_{N-1} \circ (\mathbf{Id} - \mu\mathcal{B} + \nu\mathbf{C}\mathcal{B}) &= \mathbf{Id} - (\mu\mathcal{B} + \nu\mathbf{C}\mathcal{B})^N.
\end{aligned}$$

We will show that

$$(\mu\mathcal{B} + \nu\mathbf{C}\mathcal{B})^N = R_N + K_N \tag{15}$$

where K_N is a compact operator, and R_N is a bounded, linear operator such that

$$\|R_N f\|_{L^p(\omega)} \leq C\, k^N\, N^3\, \|f\|_{L^p(\omega)}.$$

Then, the Fredholm property follows immediately. To prove (15), let us write, for any two operators T_1, T_2,

$$(T_1 + T_2)^N = \sum_{\sigma\in\{1,2\}^N} T_\sigma,$$

where $\sigma \in \{1,2\}^N$ means that $\sigma = (\sigma(1), \ldots, \sigma(N))$ and $\sigma(j) \in \{1,2\}$ for all $j = 1, \ldots, N$, and

$$T_\sigma = T_{\sigma(1)} T_{\sigma(2)} \ldots T_{\sigma(N)}.$$

By choosing $T_1 = \mu\mathcal{B}$ and $T_2 = \nu\mathbf{C}\mathcal{B}$, one sees that every $T_{\sigma(j)}$ can be written as

$$T_{\sigma(j)} = M_{\sigma(j)} C_{\sigma(j)} \mathcal{B}$$

being $M_1 = \mu$, $M_2 = \nu$, $C_1 = \mathbf{Id}$ and $C_2 = \mathbf{C}$. Thus

$$T_\sigma = M_{\sigma(1)} C_{\sigma(1)} \mathcal{B}\, M_{\sigma(2)} C_{\sigma(2)} \mathcal{B}\, \ldots M_{\sigma(N)} C_{\sigma(N)} \mathcal{B}.$$

Our main task consists of rewriting T_σ as

$$T_\sigma = M_{\sigma(1)} C_{\sigma(1)} M_{\sigma(2)} C_{\sigma(2)} \ldots M_{\sigma(N)} C_{\sigma(N)}\, B_\sigma + K_\sigma. \tag{16}$$

for some compact operator K_σ and some bounded operator $B_\sigma \in \{\mathcal{B}, \mathcal{B}^*\}^N$. If this is possible, then one gets that

$$\begin{aligned}(T_1 + T_2)^N &= \sum_{\sigma\in\{1,2\}^N} M_{\sigma(1)} C_{\sigma(1)} M_{\sigma(2)} C_{\sigma(2)} \ldots M_{\sigma(N)} C_{\sigma(N)}\, B_\sigma + \sum_{\sigma\in\{1,2\}^N} K_\sigma \\ &= R_N + K_N.\end{aligned}$$

It is clear that K_N is compact (it is a finite sum of compact operators). Moreover, from $B_\sigma \in \{\mathcal{B}, \mathcal{B}^*\}^N$, one has

$$|B_\sigma f(z)| \leq \sum_{j=1}^N |\mathcal{B}^n f(z)| + \sum_{j=1}^N |(\mathcal{B}^*)^n f(z)|.$$

Thus

$$\begin{aligned}|R_N f(z)| &\leq \sum_{\sigma\in\{1,2\}^N} |M_{\sigma(1)} C_{\sigma(1)} \ldots M_{\sigma(N)} C_{\sigma(N)}\, B_\sigma f(z)| \\ &\leq \sum_{\sigma\in\{1,2\}^N} |M_{\sigma(1)}(z)| \ldots |M_{\sigma(N)}(z)| \left(\sum_{n=1}^N |\mathcal{B}^n f(z)| + \sum_{j=1}^N |(\mathcal{B}^*)^n f(z)| \right) \\ &= \left(\sum_{n=1}^N |\mathcal{B}^n f(z)| + \sum_{j=1}^N |(\mathcal{B}^*)^n f(z)| \right) . (|M_1(z)| + |M_2(z)|)^N\end{aligned}$$

Now, since $\|\mathcal{B}^j f\|_{L^p(\omega)} \leq C_\omega\, j^2\, \|f\|_{L^p(\omega)}$ (and similarly for $(\mathcal{B}^*)^n$), one gets that

$$\begin{aligned}\|R_N f\|_{L^p(\omega)} &\leq \||M_1| + |M_2|\|_\infty^N\, C_\omega\, \left(\sum_{j=1}^N j^2\right) \|f\|_{L^p(\omega)}\\ &= C\, k^N\, N^3\, \|f\|_{L^p(\omega)}\end{aligned}$$

and so (15) follows from the representation (16). To prove that representation (16) can be found, we need the help of Theorem 2, according to which the differences $K_j = \mathcal{B}M_{\sigma(j)} - M_{\sigma(j)}\mathcal{B}$ are compact. Thus,

$$\begin{aligned}T_\sigma &= M_{\sigma(1)}C_{\sigma(1)}\mathcal{B}\, M_{\sigma(2)}C_{\sigma(2)}\mathcal{B}\, \ldots M_{\sigma(N)}C_{\sigma(N)}\mathcal{B}\\ &= M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}\mathcal{B}\, C_{\sigma(2)}M_{\sigma(3)}\, \ldots \mathcal{B}C_{\sigma(N)}\mathcal{B} + K_\sigma\end{aligned}$$

where all the factors containning K_j are includded in K_σ. In particular, K_σ is compact. Now, by reminding that

$$\mathbf{C}\,\mathcal{B} = \mathcal{B}^*\,\mathbf{C},$$

we have that $\mathcal{B}C_{\sigma(j+1)} = C_{\sigma(j+1)}B_j$ for some $B_j \in \{\mathcal{B}, \mathcal{B}^*\}$. Thus

$$T_\sigma = M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}C_{\sigma(2)}B_1\, M_{\sigma(3)}\, \ldots C_{\sigma(N)}B_{N-1}\mathcal{B} + K_\sigma$$

Now, one can start again. On one hand, the differences $B_j\, M_{\sigma(j+2)} - M_{\sigma(j+2)}B_j$ are again compact, because $B_j \in \{\mathcal{B}, \mathcal{B}^*\}$ and $M_{\sigma(j+2)} \in VMO$. Moreover, the composition $B_jC_{\sigma(j+2)}$ can be writen as $C_{\sigma(j+2)}\tilde{B}_j$, where $\tilde{B}_j$ need not be the same as B_j but still $\tilde{B}_j \in \{\mathcal{B}, \mathcal{B}^*\}$. So, with a little abbuse of notation, and after repeating this algorythm a total of $N-1$ times, one obtains (16). The claim follows. □

Proof of Theorem 8. The equation we want to solve can be rewritten, at least formally, in the following terms

$$(\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}})(\overline{\partial} f) = g,$$

so that we need to understand the $\mathbb{R}$-linear operator $T = \mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}}$. By Lemma 10, we know that T is a Fredholm operator in $L^p(\omega)$, $1 < p < \infty$. Now, we prove that it is also injective. Indeed, if

$$T(h) = 0$$

for some $h \in L^p(\omega)$ and $\omega \in A_p$, it then follows that

$$h = \mu\mathcal{B}(h) + \nu\overline{\mathcal{B}}(h)$$

so that h has compact support, and thus $h \in L^{1+\epsilon}(\mathbb{C})$ for some $\epsilon > 0$. We are then reduced to show that

$$T : L^{1+\epsilon}(\mathbb{C}) \to L^{1+\epsilon}(\mathbb{C}) \qquad \text{is injective.}$$

Let us first see how the proof finishes. Injectivity of T in $L^{1+\epsilon}(\mathbb{C})$ gives us that $h = 0$. Therefore, T is injective also in $L^p(\omega)$. Being as well Fredholm, it is also surjective, so by the open map Theorem it has a bounded inverse $T^{-1}: L^p(\omega) \to L^p(\omega)$. As a consequence, given any $g \in L^p(\omega)$, the function

$$f = \mathcal{C}T^{-1}(g)$$

is well defined, and has derivatives in $L^p(\omega)$ satisfying the estimate

$$\begin{aligned}
\|Df\|_{L^p(\omega)} &\leq \|\partial f\|_{L^p(\omega)} + \|\overline{\partial} f\|_{L^p(\omega)} \\
&= \|\mathcal{B}T^{-1}(g)\|_{L^p(\omega)} + \|T^{-1}(g)\|_{L^p(\omega)} \\
&\leq (C+1)\,\|T^{-1}(g)\|_{L^p(\omega)} \\
&\leq C\,\|g\|_{L^p(\omega)},
\end{aligned}$$

because $\omega \in A_p$. Moreover, we see that f solves the inhomogeneous equation

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)} = g(z).$$

Finally, if there were two such solutions f_1, f_2, then their difference $F = f_1 - f_2$ solves the homogeneous equation, and also $DF \in L^p(\omega)$. Thus

$$T(\overline{\partial} F) = 0.$$

By the injectivity of T we get that $\overline{\partial} F = 0$, and from $DF \in L^p(\omega)$ we get that $\partial F = 0$, whence F must be a constant.
We now prove the injectivity of T in $L^p(\mathbb{C})$, $1 < p < \infty$. First, if $p \geq 2$ and $h \in L^p(\mathbb{C})$ is such that $T(h) = 0$, then h has compact support, whence $h \in L^2(\mathbb{C})$. But $\mathcal{B}, \overline{\mathcal{B}}$ are isometries in $L^2(\mathbb{C})$, whence

$$\|h\|_2 \leq k\,\|\mathcal{B}h\|_2 = k\|f\|_2$$

and thus $h = 0$, as desired. For $p < 2$, we recall from Lemma 9 that the bijectivity of T in $L^p(\mathbb{C})$ is equivalent to that of $T' = \mathbf{Id} - \overline{\mu}\mathcal{B}^* - \nu\overline{\mathcal{B}^*}$ in the dual space $L^p(\mathbb{C})$. For this, note that the injectivity of T' in $L^{p'}(\mathbb{C})$ follows as above (since $p' \geq 2$). Note also that, by Lemma 10 we know that T' is a Fredholm operator in $L^{p'}(\mathbb{C})$, since $\overline{\mu}$ and ν are compactly supported VMO functions. The claim follows. □

4 Applications

We start this section by recalling that if $\mu, \nu \in L^\infty(\mathbb{C})$ are compactly supported with $\||\mu| + |\nu|\|_\infty \leq k < 1$ then the equation

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) - \nu(z)\,\overline{\partial\phi(z)} = 0$$

admits a unique homeomorphic $W^{1,2}_{loc}(\mathbb{C})$ solution $\phi : \mathbb{C} \to \mathbb{C}$ such that $|\phi(z) - z| \to 0$ as $|z| \to \infty$. We call it the *principal* solution, and it defines a global K-quasiconformal map, $K = \frac{1+k}{1-k}$.

Applications of Theorem 1 are based in the following change of variables lemma, which is already proved in [2, Lemma 14]. We rewrite it here for completeness.

Lemma 11. *Given a compactly supported function* $\mu \in L^\infty(\mathbb{C})$ *such that* $\|\mu\|_\infty \le k < 1$, *let* ϕ *denote the principal solution to the equation*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) = 0.$$

For a fixed weight ω, *let us define*

$$\eta(\zeta) = \omega(\phi^{-1}(\zeta))\,J(\zeta,\phi^{-1})^{1-\frac{p}{2}}.$$

The following statements are equivalent:

(a) *For every* $h \in L^p(\omega)$, *the inhomogeneous equation*

$$\overline{\partial}f(z) - \mu(z)\,\partial f(z) = h(z) \tag{17}$$

has a solution f *with* $Df \in L^p(\omega)$ *and*

$$\|Df\|_{L^p(\omega)} \le C_1\,\|h\|_{L^p(\omega)}. \tag{18}$$

(b) *For every* $\tilde{h} \in L^p(\eta)$, *the equation*

$$\overline{\partial}g(\zeta) = \tilde{h}(\zeta) \tag{19}$$

has a solution g *with* $Dg \in L^p(\eta)$ *and*

$$\|Dg\|_{L^p(\eta)} \le C_2\,\|\tilde{h}\|_{L^p(\eta)}. \tag{20}$$

Proof. Let us first assume that (b) holds. To get (a), we have to find a solution f of (17) such that $Df \in L^p(\omega)$ with the estimate (18). To this end, we make in (17) the change of coordinates $g = f \circ \phi^{-1}$. We obtain for g the following equation

$$\overline{\partial}g(\zeta) = \tilde{h}(\zeta), \tag{21}$$

where $\zeta = \phi(z)$ and

$$\tilde{h}(\zeta) = h(z)\,\frac{\partial\phi(z)}{J(z,\phi)} \quad .$$

In order to apply the assumption (b), we must check that $\tilde{h} \in L^p(\eta)$. However,

$$\begin{aligned}\|\tilde{h}\|^p_{L^p(\eta)} = \int |\tilde{h}(\zeta)|^p\,\eta(\zeta)\mathrm{d}\zeta &= \int |\tilde{h}(\phi(z))|^p\,\omega(z)\,J(z,\phi)^{\frac{p}{2}}\,\mathrm{d}z \\ &= \int |h(z)\,|^p\;\frac{\omega(z)}{(1-|\mu(z)|^2)^{\frac{p}{2}}}\,\mathrm{d}z \le \frac{1}{(1-k^2)^{\frac{p}{2}}}\,\|h\|^p_{L^p(\omega)}.\end{aligned}$$

Since $\tilde{h} \in L^p(\eta)$, (b) applies, and a solution g to (21) can be found with the estimate

$$\|Dg\|_{L^p(\eta)} \le C_2\,\|\tilde{h}\|_{L^p(\eta)} \le \frac{C_2}{(1-k^2)^{\frac{1}{2}}}\,\|h\|_{L^p(\omega)}.$$

With such a g, the function $f = g \circ \phi$ is well defined, and

$$\begin{aligned}\int |Df(z)|^p \,\omega(z)\,\mathrm{d}z &= \int |Dg(\phi(z))\,D\phi(z)|^p \,\omega(z)\,\mathrm{d}z \\ &= \int |Dg(\zeta)\,D\phi(\phi^{-1}(\zeta))|^p \,\omega(\phi^{-1}(z))\,J(\zeta,\phi^{-1})\mathrm{d}\zeta \\ &\le \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Dg(\zeta)|^p \,J(\phi^{-1}(\zeta),\phi)^{\frac{p}{2}}\,\omega(\phi^{-1}(z))\,J(\zeta,\phi^{-1})\mathrm{d}\zeta \\ &= \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Dg(\zeta)|^p \,\eta(\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

due to the $\frac{1+k}{1-k}$-quasiconformality of ϕ. Moreover, f satisfies the desired equation, and so 1 follows, with constant $C_1 = \frac{C_2}{1-k}$.

To show that (a) implies (b), for a given $\tilde{h} \in L^p(\eta)$ we have to find a solution of (19) satisfying the estimate (20). Since this is a $\overline{\partial}$-equation, this could be done by simply convolving $\tilde{h}$ with the Cauchy kernel $\frac{1}{\pi z}$. However, the desired estimate for the solution g cannot be obtained in this way, because at this point the weight η is not known to belong to A_p. So we will proceed in a different maner. Namely, we make the change of coordinates $f = g \circ \phi$. We obtain for f the equation

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) = h(z),$$

where $h(z) = \tilde{h}(\phi(z))\,\overline{\partial\phi(z)}\,(1-|\mu(z)|^2)$. Moreover,

$$\int |h(z)|^p \,\omega(z)\,\mathrm{d}z = \int |\tilde{h}(\zeta)|^p \,(1-|\mu(\phi^{-1}(\zeta))|^2)^{p/2}\,\eta(\zeta)\,\mathrm{d}\zeta \le \int |\tilde{h}(\zeta)|^p \,\eta(\zeta)\,\mathrm{d}\zeta.$$

Therefore (a) applies, and a solution f can be found with $Df \in L^p(\omega)$ and $\|Df\|_{L^p(\omega)} \le C_1 \,\|\tilde{h}\|_{L^p(\eta)}$. As before, once f is found, one simply constructs $g = f \circ \phi^{-1}$. By the chain rule,

$$\begin{aligned}\int |Dg(\zeta)|^p \eta(\zeta)\mathrm{d}\zeta &= \int |Dg(\phi^{-1}(z))|^p J(z,\phi^{-1})\eta(\phi^{-1}(z))\mathrm{d}z \\ &= \int |D(g\circ\phi^{-1})(z)\,(D\phi^{-1}(z))^{-1}|^p J(z,\phi^{-1})\eta(\phi^{-1}(z))\mathrm{d}z \\ &\le \int |Df(z)|^p \,|D\phi(\phi^{-1}(z))|^p J(z,\phi^{-1})\eta(\phi^{-1}(z))\mathrm{d}z \\ &\le \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Df(z))|^p \,J(\phi^{-1}(z),\phi)^{\frac{p}{2}} J(z,\phi^{-1})\eta(\phi^{-1}(z))\mathrm{d}z \\ &= \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Df(z))|^p \,\omega(z)\mathrm{d}z.\end{aligned}$$

Thus, $\|Dg\|_{L^p(\eta)} \le C_2 \,\|\tilde{h}\|_{L^p(\eta)}$ with $C_2 = \left(\frac{1+k}{1-k}\right)^{\frac{1}{2}} C_1$, and (b) follows. □

According to the previous Lemma, a priori estimates for $\overline{\partial} - \mu\,\partial$ in $L^p(\omega)$ are equivalent to a priori estimates for $\overline{\partial}$ in $L^p(\eta)$. However, by Theorem 1, if ω is taken in A_p, the first statement holds, at least, when μ is compactly supported and belongs to VMO. We then obtain the following consequence.

Corollary 12. *Let $\mu \in VMO$ be compactly supported, such that $\|\mu\|_\infty < 1$, and let ϕ be the principal solution of*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) = 0.$$

If $1 < p < \infty$ and $\omega \in A_p$, then the weight

$$\eta(z) = \omega(\phi^{-1}(z))\,J(z,\phi^{-1})^{1-p/2}$$

belongs to A_p. Moreover, its A_p constant $[\eta]_{A_p}$ can be bounded in terms of μ, p and $[\omega]_{A_p}$.

Proof. Under the above assumptions, by Theorem 1, we know that if $h \in L^p(\omega)$ then the equation $\overline{\partial}f - \mu\,\partial f = h$ can be found a solution f with $Df \in L^p(\omega)$ and such that $\|Df\|_{L^p(\omega)} \leq C_0\,\|h\|_{L^p(\omega)}$, for some constant $C_0 > 0$ depending on k, p and $[\omega]_{A_p}$. Equivalently, by Lemma 11, for every $\tilde{h} \in L^p(\eta)$ we can find a solution g of the inhomogeneous Cauchy-Riemann equation

$$\overline{\partial}g = \tilde{h},$$

with $Dg \in L^p(\eta)$ and in such a way that the estimate

$$\|Dg\|_{L^p(\eta)} \leq C\,\|\tilde{h}\|_{L^p(\eta)}$$

holds for some constant C depending on C_0, k and p. Now, let us choose $\varphi \in \mathcal{C}_0^\infty(\mathbb{C})$ and set $\tilde{h} = \overline{\partial}\varphi$. Then of course $g = \varphi$ and $\partial\varphi = \mathcal{B}(\overline{\partial}\varphi)$, and the above inequality says that

$$\||\partial\varphi| + |\overline{\partial}\varphi|\|_{L^p(\eta)} \leq C\,\|\overline{\partial}\varphi\|_{L^p(\eta)},$$

whence the estimate

$$\|\mathcal{B}(\psi)\|_{L^p(\eta)} \leq (C^p - 1)^{\frac{1}{p}}\,\|\psi\|_{L^p(\eta)} \tag{22}$$

holds for any $\psi \in \mathcal{D}^* = \{\psi \in \mathcal{C}_c^\infty(\mathbb{C}); \int\psi = 0\}$. It turns out that $\mathcal{D}^*$ is a dense subclass of $L^p(\eta)$, provided that $\eta \in L^1_{loc}$ is a positive function with infinite mass. But this is actually the case. Indeed, one has

$$\int_{D(0,R)} \eta(\zeta)\,\mathrm{d}\zeta = \int_{\phi^{-1}(D(0,R))} \omega(z)\,J(z,\phi)^{\frac{p}{2}}\,\mathrm{d}z.$$

Above, the integral on the right hand side certainly grows to infinite as $R \to \infty$. Otherwise, one would have that $J(\cdot,\phi)^{\frac{1}{2}} \in L^p(\omega)$. But ϕ is a principal quasiconformal map, hence $J(z,\phi) = 1 + O(1/|z|^2)$ as $|z| \to \infty$. Thus for large enough $N > M > 0$,

$$\int_{M<|z|<N} J(z,\phi)^{\frac{p}{2}}\,\omega(z)\,\mathrm{d}z \geq C\int_{M<|z|<N} \omega(z)\,\mathrm{d}z$$

and the last integral above blows up as $N \to \infty$, because ω is an A_p weight.
Therefore, the estimate (22) holds for all ψ in $L^p(\eta)$. By [14, Ch. V, Proposition 7], this implies that $\eta \in A_p$, and moreover, $[\eta]_{A_p}$ depends only on the constant $(C^p - 1)^{\frac{1}{p}}$, that is, on k, p and $[\omega]_{A_p}$. ◻

The above Corollary is especially interesting in two particular cases. First, for the constant weight $\omega = 1$ the above result says that

$$J(\cdot, \phi^{-1})^{1-p/2} \in A_p, \qquad\qquad 1 < p < \infty.$$

Without the VMO assumption, this is only true for the smaller range $1 + \|\mu\|_\infty < p < 1 + \frac{1}{\|\mu\|_\infty}$ (see e.g. [1, Theorem 13.4.2]). Secondly, by setting $p = 2$ in Corollary 12 we get the following.

Corollary 13. *Let $\mu \in VMO$ be compactly supported, and assume that $\|\mu\|_\infty < 1$. Let ϕ be the principal solution of*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) = 0.$$

Then, for every $\omega \in A_2$ one has $\omega \circ \phi^{-1} \in A_2$.

The above result drives us to the problem of finding what homeomorphisms ϕ preserve the A_p classes under composition with ϕ^{-1}. Note that preserving A_p forces also the preservation of the space BMO of functions with bounded mean oscillation, and thus such homeomorphisms ϕ must be quasiconformal [12]. However, at level of Muckenhoupt weights, the question is deeper. As an example, simply consider the weight

$$\omega(z) = \frac{1}{|z|^\alpha},$$

and its composition with the inverse of a radial stretching $\phi(z) = z|z|^{K-1}$. It is clear that the values of p for which A_p contains ω and $\omega \circ \phi^{-1}$ are *not* the same, whence preservation of A_p requires something else. This question was solved by Johnson and Neugebauer [8] as follows.

Theorem 14. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be K-quasiconformal. Then, the following statements are equivalent:*

1. *If $\omega \in A_2$ then $\omega \circ \phi^{-1} \in A_2$ quantitatively.*
2. *For a fixed $p \in (1, \infty)$, if $\omega \in A_p$ then $\omega \circ \phi^{-1} \in A_p$ quantitatively.*
3. *$J(\cdot, \phi^{-1}) \in A_p$ for every $p \in (1, \infty)$.*

It follows from Corollary 13 and Theorem 14 that, if $\mu \in VMO$ is compactly supported, $\|\mu\|_\infty \le k < 1$ and ϕ is the principal solution to the $\mathbb{C}$-linear equation $\overline{\partial}\phi = \mu\,\partial\phi$, then

$$J(\cdot, \phi^{-1}) \in \bigcap_{p>1} A_p.$$

Note that if the VMO assumption is removed, then we can only guarantee

$$J(\cdot, \phi^{-1}) \in \bigcap_{p>\frac{1+k}{1-k}} A_p.$$

It is not clear to the authors what is the role of $\mathbb{C}$-linearity in the above results concerning the regularity of the jacobian. In other words, there seems to be no reason for Theorem 13 to fail if one replaces the $\mathbb{C}$-linear equation by the generalized one, while mantainning the ellipticity, compact support and smoothness on the coefficients. In fact, there is a deep connection between this question and the problem of determining those weights $\omega > 0$ for which the estimate

$$\|Df\|_{L^2(\omega)} \leq C\,\|\overline{\partial} f - \mu\,\partial f - \nu\,\overline{\partial f}\|_{L^2(\omega)}$$

holds for any $f \in \mathcal{C}_0^\infty(\mathbb{C})$. The following result, which is a counterpart of Lemma 11, explains this connection.

Lemma 15. *To each pair $\mu, \nu \in L^\infty(\mathbb{C})$ of compactly supported functions with $\||\mu| + |\nu|\|_\infty \leq k < 1$, let us associate, on one hand, the principal solution ϕ to the equation*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) - \nu(z)\,\overline{\partial\phi(z)} = 0,$$

and on the other, the function λ defined by $\lambda \circ \phi = \frac{-2i\nu}{1-|\mu|^2+|\nu|^2}$. For a fixed weight ω, let us define

$$\eta(\zeta) = \omega(\phi^{-1}(\zeta))\,J(\zeta, \phi^{-1})^{1-\frac{p}{2}}.$$

The following statements are equivalent:

(a) *For every $h \in L^p(\omega)$, the equation*

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)} = h(z)$$

has a solution f with $Df \in L^p(\omega)$ and $\|Df\|_{L^p(\omega)} \leq C\,\|h\|_{L^p(\omega)}$.

(b) *For every $\tilde{h} \in L^p(\eta)$, the equation*

$$\overline{\partial} g(\zeta) - \lambda(\zeta)\,\mathrm{Im}(\partial g(\zeta)) = \tilde{h}(\zeta)$$

has a solution g with $Dg \in L^p(\eta)$ and $\|Dg\|_{L^p(\eta)} \leq C\,\|\tilde{h}\|_{L^p(\eta)}$.

Although the proof requires quite tedious calculations, it follows the scheme of Lemma 11, and thus we omit it. From this Lemma, we would be very interested in answering the following question.

Question 16. *Let $\omega \in L^1_{loc}(\mathbb{C})$ be such that $\omega(z) > 0$ almost everywhere, and let $\lambda \in L^\infty(\mathbb{C})$ be a compactly supported VMO function, such that $\|\lambda\|_\infty < 1$. If the estimate*

$$\|Df\|_{L^p(\omega)} \leq C\,\|\overline{\partial} f - \lambda\,\mathrm{Im}(\partial f)\|_{L^p(\omega)}$$

holds for every $f \in \mathcal{C}_0^\infty$, is it true that $\omega \in A_2$?

What we actually want is to find planar, elliptic, first order differential operators, different from the $\overline{\partial}$, that can be used to characterize the Muckenhoupt classes A_p. In this direction, an affirmative answer tho Question 16 would allow us to characterize A_2 weights as follows: given $\mu, \nu \in VMO$ uniformly elliptic and compactly supported, a positive a.e. function $\omega \in L^1_{loc}$ is an A_2 weight if and only if there is a constant $C > 0$ such that

$$\|Df\|_{L^2(\omega)} \leq C\,\|\overline{\partial} f - \mu\,\partial f - \nu\,\overline{\partial f}\|_{L^2(\omega)}, \qquad \text{for every } f \in \mathcal{C}_0^\infty(\mathbb{C}). \tag{23}$$

Note that if $\||\mu| + |\nu|\|_\infty < \epsilon$ is small enough, (23) says that

$$\|\partial f\|^2_{L^2(\omega)} + \|\overline{\partial} f\|_{L^2(\omega)} \leq C\, \|\overline{\partial} f\|_{L^2(\omega)} + C\,\epsilon\, \|\partial f\|_{L^2(\omega)},$$

so if $\epsilon < \frac{1}{C}$ one easily gets that

$$\|\partial f\|_{L^2(\omega)} \leq \frac{C-1}{1-C\epsilon} \|\overline{\partial} f\|_{L^2(\omega)}.$$

From the above estimate, weighted bounds for $\mathcal{B}$ easily follow, and so if $\||\mu| + |\nu|\|_\infty < \epsilon$ then such a characterization holds. Question 16 has an affirmative answer.

Acknowledgements. Sections 1 and 2 of the present paper were obtained by the second author in [3], jointly with J. Mateu and J. Orobitg. The authors were partially supported by projects 2009-SGR-420 (Generalitat de Catalunya), 2010-MTM-15657 (Spanish Ministry of Science) and NF-129254 (Programa Ramón y Cajal).

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