AUTOMORPHISMS OF FUSION SYSTEMS OF FINITE SIMPLE GROUPS OF LIE TYPE

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Abstract. For a finite group G of Lie type and a prime p, we compare the automorphism groups of the fusion and linking systems of G at p with the automorphism group of G itself. When p is the defining characteristic of G, they are all isomorphic, with a very short list of exceptions. When p is different from the defining characteristic, the situation is much more complex, but can always be reduced to a case where the natural map from $\mathrm{Out}(G)$ to outer automorphisms of the fusion or linking system is split surjective. This work is motivated in part by questions involving extending the local structure of a group by a group of automorphisms, and in part by wanting to describe self homotopy equivalences of $BG^\wedge_p$ in terms of $\mathrm{Out}(G)$.

When p is a prime, G is a finite group, and $S \in \mathrm{Syl}_p(G)$, the *fusion system* of G at S is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and whose morphisms are those homomorphisms between subgroups induced by conjugation in G. In this paper, we are interested in comparing automorphisms of G, when G is a simple group of Lie type, with those of the fusion system of G at a Sylow p-subgroup of G (for different primes p).

Rather than work with automorphisms of $\mathcal{F}_S(G)$ itself, it turns out to be more natural in many situations to study the group $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$ of outer automorphisms of the *centric linking system* of G. We refer to Section 1 for the definition of $\mathcal{L}_S^c(G)$, and to Definition 1.2 for precise definitions of $\mathrm{Out}(S, \mathcal{F}_S(G))$ and $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$. These are defined in such a way that there are natural homomorphisms

$$\mathrm{Out}(G) \xrightarrow{\ \kappa_G\ } \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \xrightarrow{\ \mu_G\ } \mathrm{Out}(S, \mathcal{F}_S(G)) \qquad \text{and} \qquad \bar{\kappa}_G = \mu_G \circ \kappa_G\,.$$

For example, if S controls fusion in G (i.e., if S has a normal complement), then $\mathrm{Out}(S, \mathcal{F}_S(G)) = \mathrm{Out}(S)$, and $\bar{\kappa}_G$ is induced by projection to S. The fusion system $\mathcal{F}_S(G)$ is *tamely realized by* G if κ_G is split surjective, and is *tame* if it is tamely realized by some finite group G^* where $S \in \mathrm{Syl}_p(G^*)$ and $\mathcal{F}_S(G) = \mathcal{F}_S(G^*)$. Tameness plays an important role in Aschbacher's program for shortening parts of the proof of the classification of finite simple groups by classifying simple fusion systems over finite 2-groups. We say more about this later in the introduction, just before the statement of Theorem C.

By [BLO1, Theorem B], $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \cong \mathrm{Out}(BG^\wedge_p)$: the group of homotopy classes of self homotopy equivalences of the p-completed classifying space of G. Thus one of the

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motivations for this paper is to compute $\mathrm{Out}(BG^\wedge_p)$ when G is a finite simple group of Lie type (in characteristic p or in characteristic different from p), and compare it with $\mathrm{Out}(G)$.

Following the notation used in [GLS3], for each prime p, we let $\mathfrak{Lie}(p)$ denote the class of finite groups of Lie type in characteristic p, and let $\mathfrak{Lie}$ denote the union of the classes $\mathfrak{Lie}(p)$ for all primes p. (See Definition 2.1 for the precise definition.) We say that $G \in \mathfrak{Lie}(p)$ is of *adjoint type* if $Z(G) = 1$, and is of *universal type* if it has no nontrivial central extensions which are in $\mathfrak{Lie}(p)$. For example, for $n \geq 2$ and q a power of p, $PSL_n(q)$ is of adjoint type and $SL_n(q)$ of universal type.

Our results can be most simply stated in the "equi-characteristic case": when working with p-fusion of $G \in \mathfrak{Lie}(p)$.

Theorem A. *Let p be a prime. Assume that $G \in \mathfrak{Lie}(p)$ and is of universal or adjoint type, and also that $(G, p) \not\cong (\mathrm{Sz}(2), 2)$. Fix $S \in \mathrm{Syl}_p(G)$. Then the composite homomorphism*

$$\bar{\kappa}_G \colon \mathrm{Out}(G) \xrightarrow{\ \kappa_G\ } \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G)) \xrightarrow{\ \mu_G\ } \mathrm{Out}(S, \mathcal{F}_S(G))$$

is an isomorphism, and κ_G and μ_G are isomorphisms except when $G \cong PSL_3(2)$.

Proof. Assume G is of adjoint type. When $G \not\cong GL_3(2)$, μ_G is an isomorphism by [O1, Proposition 4.3][1] or [O2, Theorems C & 6.2]. The injectivity of $\bar{\kappa}_G = \mu_G \circ \kappa_G$ (in all cases) is shown in Lemma 4.3. The surjectivity of κ_G is shown in Proposition 4.5 when G has Lie rank at least three, and in Proposition 4.8 when G has Lie rank 1 and $G \not\cong \mathrm{Sz}(2)$. When G has Lie rank 2, κ_G is onto (when $G \not\cong SL_3(2)$) by Proposition 4.12, 4.14, 4.15, 4.16, or 4.17. (See Notation 4.1(H) for the definition of Lie rank used here.)

If G is of universal type, then by Proposition 3.8, $G/Z(G) \in \mathfrak{Lie}(p)$ is of adjoint type where $Z(G)$ has order prime to p. Also, $\mathrm{Out}(G) \cong \mathrm{Out}(G/Z(G))$ by [GLS3, Theorem 2.5.14(d)]. Hence $\mathcal{F}_S(G) \cong \mathcal{F}_S(G/Z(G))$ and $\mathcal{L}^c_S(G) \cong \mathcal{L}^c_S(G/Z(G))$; and κ_G and/or $\bar{\kappa}_G$ is an isomorphism if $\kappa_{G/Z(G)}$ and/or $\bar{\kappa}_{G/Z(G)}$, respectively, is an isomorphism. □

When $G = PSL_3(2)$ and $p = 2$, $\mathrm{Out}(G) \cong \mathrm{Out}(S, \mathcal{F}_S(G)) \cong C_2$, while $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G)) \cong C_2^2$. When $G = \mathrm{Sz}(2) \cong C_5 \rtimes C_4$ and $p = 2$, $\mathrm{Out}(G) = 1$, while $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G)) \cong \mathrm{Aut}(C_4) \cong C_2$. Thus these groups are exceptions to Theorem A.

To simplify the statement of the next theorem, for finite groups G and H, we write $G \sim_p H$ to mean that there are Sylow subgroups $S \in \mathrm{Syl}_p(G)$ and $T \in \mathrm{Syl}_p(H)$, together with an isomorphism $\varphi\colon S \xrightarrow{\ \cong\ } T$ which induces an isomorphism of categories $\mathcal{F}_S(G) \cong \mathcal{F}_T(H)$ (i.e., φ is fusion preserving in the sense of Definition 1.2).

Theorem B. *Fix a pair of distinct primes p and q_0, and a group $G \in \mathfrak{Lie}(q_0)$ of universal or adjoint type. Assume that the Sylow p-subgroups of G are nonabelian. Then there is a prime $q_0^* \neq p$, and a group $G^* \in \mathfrak{Lie}(q_0^*)$ of universal or adjoint type, respectively, such that $G^* \sim_p G$ and κ_{G^*} is split surjective. If, furthermore, p is odd or G^* has universal type, then μ_{G^*} is an isomorphism, and hence $\bar{\kappa}_{G^*}$ is also split surjective.*

Proof. **Case 1:** Assume p is odd and G is of universal type. Since μ_G is an isomorphism by [O1, Theorem C], κ_G or κ_{G^*} is (split) surjective if and only if $\bar{\kappa}_G$ or $\bar{\kappa}_{G^*}$ is.

[1]Steve Smith recently pointed out to the third author an error in the proof of this proposition. One can get around this problem either via a more direct case-by-case argument (see the remark in the middle of page 345 in [O1]), or by applying [O4, Theorem C]. The proof of the latter result uses the classification of finite simple groups, but as described by Glauberman and Lynd [GLn, § 3], the proof in [O4] (for odd p) can be modified to use an earlier result of Glauberman [Gl2, Theorem A1.4], and through that avoiding the classification.

By Proposition 6.8, we can choose a prime q_0^* and a group $G^* \in \mathfrak{Lie}(q_0^*)$ such that either

(1.a) $G^* \cong \mathbb{G}(q^*)$ or ${}^2\mathbb{G}(q^*)$, for some $\mathbb{G}$ with Weyl group W and q^* a power of q_0^*, and has a σ-setup which satisfies the conditions in Hypotheses 5.1 and 5.10, and

- (1.a.1) $-\mathrm{Id} \notin W$ and G^* is a Chevalley group, or
- (1.a.2) $-\mathrm{Id} \in W$ and q^* has even order in $\mathbb{F}_p^\times$, or
- (1.a.3) $p \equiv 3 \pmod{p}$ and $p|(q^*-1)$; or

(1.b) $p=3$, $q_0^*=2$, $G \cong {}^3D_4(q)$ or ${}^2F_4(q)$ for q some power of q_0, and $G^* \cong {}^3D_4(q^*)$ or ${}^2F_4(q^*)$ for q^* some power of 2.

Also (by the same proposition), if $p=3$ and $G^* = F_4(q^*)$, then we can assume $q_0^*=2$.

In case (1.b), $\bar{\kappa}_{G^*}$ is split surjective by Proposition 6.9. In case (1.a), it is surjective by Proposition 5.14. In case (1.a.1), $\bar{\kappa}_{G^*}$ is split by Proposition 5.15(b,c). In case (1.a.3), $\bar{\kappa}_{G^*}$ is split by Proposition 5.15(b). In case (1.a.2), if G^* is a Chevalley group, then $\bar{\kappa}_{G^*}$ is split by Proposition 5.15(c).

This leaves only case (1.a.2) when G^* is a twisted group. The only irreducible root systems which have nontrivial graph automorphisms and for which $-\mathrm{Id} \in W$ are those of type D_n for even n. Hence $G^* = \mathrm{Spin}_{2n}^-(q^*)$ for some even $n \geq 4$. By the last statement in Proposition 6.8, G^* is one of the groups listed in Proposition 1.10, and so $q^n \equiv -1 \pmod{p}$. Hence $\bar{\kappa}_{G^*}$ is split surjective by Example 6.6(a), and we are done also in this case.

Case 2: Now assume $p=2$ and G is of universal type. By Proposition 6.2, there is an odd prime q_0^*, a group $G^* \in \mathfrak{Lie}(q_0^*)$, and $S^* \in \mathrm{Syl}_p(G^*)$, such that $\mathcal{F}_S(G) \cong \mathcal{F}_{S^*}(G^*)$ and G^* has a σ-setup which satisfies Hypotheses 5.1 and 5.10. By the same proposition, if $G^* \cong G_2(q^*)$, then we can arrange that $q^*=5$ or $q_0^*=3$. If $G^* \cong G_2(5)$, then by Propositions 6.3 and A.12, $G^* \sim_2 G_2(3)$, $\bar{\kappa}_{G_2(3)}$ is split surjective, and $\mu_{G_2(3)}$ is injective.

In all remaining cases (i.e., $G^* \not\cong G_2(q^*)$ or $q_0^*=3$), $\bar{\kappa}_{G^*}$ is split surjective by Proposition 5.15(a). By Proposition A.3 or A.12, μ_{G^*} is injective, and hence κ_{G^*} is also split surjective.

Case 3: Now assume G is of adjoint type. Then $G \cong G_u/Z$ for some $G_u \in \mathfrak{Lie}(q_0)$ of universal type and $Z \leq Z(G_u)$. By Proposition 3.8, $Z = Z(G_u)$ and has order prime to q_0.

By Case 1 or 2, there is a prime $q_0^* \neq p$ and a group $G_u^* \in \mathfrak{Lie}(q_0^*)$ of universal type such that $G_u^* \sim_p G_u$ and $\kappa_{G_u^*}$ is split surjective. Also, G_u^* is p-perfect by definition of $\mathfrak{Lie}(q_0^*)$ (and since $q_0^* \neq p$), and $H^2(G_u^*;\mathbb{Z}/p)=0$ by Proposition 3.8. Set $G^* = G_u^*/Z(G_u^*)$. By Proposition 1.7, with $G_u^*/O_{p'}(G_u^*)$ in the role of G, κ_{G^*} is also split surjective.

It remains to check that $G \sim_p G^*$. Assume first that G_u and G_u^* have σ-setups which satisfy Hypotheses 5.1. Fix $S \in \mathrm{Syl}_p(G_u)$ and $S^* \in \mathrm{Syl}_p(G_u^*)$, and a fusion preserving isomorphism $\varphi\colon S \longrightarrow S^*$ (Definition 1.2(a)). By Corollary 5.9, $Z(\mathcal{F}_S(G_u)) = O_p(Z(G_u))$ and $Z(\mathcal{F}_{S^*}(G_u^*)) = O_p(Z(G_u^*))$. Since φ is fusion preserving, it sends $Z(\mathcal{F}_S(G_u))$ onto $Z(\mathcal{F}_{S^*}(G_u^*))$, and thus sends $O_p(Z(G_u))$ onto $O_p(Z(G_u^*))$. Hence φ induces a fusion preserving isomorphism between Sylow subgroups of $G = G_u/Z(G_u)$ and $G^* = G_u^*/Z(G_u^*)$.

The only cases we considered where G or G^* does not satisfy Hypotheses 5.1 were those in case (1.b) above. In those cases, $G \cong {}^2F_4(q)$ or ${}^3D_4(q)$ and $G^* \cong {}^2F_4(q^*)$ or ${}^3D_4(q^*)$ for some q and q^*, hence G and G^* are also of universal type ($d=1$ in the notation of [Ca, Lemma 14.1.2(iii)]), and so there is nothing more to prove. □

The last statement in Theorem B is *not* true in general when G^* is of adjoint type. For example, if $G^* \cong PSL_2(9)$, $p=2$, and $S^* \in \mathrm{Syl}_2(G^*)$, then $\mathrm{Out}(G^*) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_{S^*}^c(G^*)) \cong$

C_2^2, while $\mathrm{Out}(S^*, \mathcal{F}_{S^*}(G^*)) \cong C_2$. By comparison, if $\widetilde{G}^* \cong SL_2(9)$ is the universal group, then $\mathrm{Out}(\widetilde{S}^*, \mathcal{F}_{\widetilde{S}^*}(\widetilde{G}^*)) \cong C_2^2$, and $\kappa_{\widetilde{G}^*}$ and $\mu_{\widetilde{G}^*}$ are isomorphisms.

As noted briefly above, a fusion system $\mathcal{F}_S(G)$ is called *tame* if there is a finite group G^* such that $G^* \sim_p G$ and κ_{G^*} is split surjective. In this situation, we say that G^* *tamely realizes* the fusion system $\mathcal{F}_S(G)$. By [AOV, Theorem B], if $\mathcal{F}_S(G)$ is not tame, then some extension of it is an "exotic" fusion system; i.e., an abstract fusion system not induced by any finite group. (See Section 1 for more details.) The original goal of this paper was to determine whether all fusion systems of simple groups of Lie type (at all primes) are tame, and this follows as an immediate consequence of Theorems A and B. Hence this approach cannot be used to construct new, exotic fusion systems.

Determining which simple fusion systems over finite 2-groups are tame, and tamely realizable by finite simple groups, plays an important role in Aschbacher's program for classifying simple fusion systems over 2-groups (see [AKO, Part II] or [A2]). Given such a fusion system $\mathcal{F}$ over a 2-group S, and an involution $x \in S$, assume that the centralizer fusion system $C_{\mathcal{F}}(x)$ contains a normal quasisimple subsystem $\mathcal{E} \trianglelefteq C_{\mathcal{F}}(x)$. If $\mathcal{E}$ is tamely realized by a finite simple group K, then under certain additional assumptions, one can show that the entire centralizer $C_{\mathcal{F}}(x)$ is the fusion system of some finite extension of K. This is part of our motivation for looking at this question, and is also part of the reason why we try to give as much information as possible as to which groups tamely realize which fusion systems.

Theorem C. *For any prime p and any $G \in \mathfrak{Lie}$ of universal or adjoint type, the p-fusion system of G is tame. If the Sylow p-subgroups of G are nonabelian, or if p is the defining characteristic and $G \not\cong \mathrm{Sz}(2)$, then its fusion system is tamely realized by some other group in $\mathfrak{Lie}$.*

Proof. If $S \in \mathrm{Syl}_p(G)$ is abelian, then the p-fusion in G is controlled by $N_G(S)$, and $\mathcal{F}_S(G)$ is tame by Proposition 1.6. If $p = 2$ and $G \cong SL_3(2)$, then the fusion system of G is tamely realized by $PSL_2(9)$. In all other cases, the claims follow from Theorems A and B. □

We have stated the above three theorems only for groups of Lie type, but in fact, we proved at the same time the corresponding results for the Tits group:

Theorem D. *Set $G = {}^2F_4(2)'$ (the Tits group). Then for each prime p, the p-fusion system of G is tame. If $p = 2$ or $p = 3$, then κ_G is an isomorphism.*

Proof. The second statement is shown in Proposition 4.17 when $p = 2$, and in Proposition 6.9 when $p = 3$. When $p > 3$, the Sylow p-subgroups of G are abelian ($|G| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$), so G is tame by Proposition 1.6(b). □

As one example, if $p = 2$ and $G = PSL_2(17)$, then κ_G is not surjective, but $G^* = PSL_2(81)$ (of adjoint type) has the same 2-fusion system and κ_{G^*} is an isomorphism [BLO1, Proposition 7.9]. Also, $\bar{\kappa}_{G^*}$ is non-split surjective with kernel generated by the field automorphism of order two by [BLO1, Lemma 7.8]. However, if we consider the universal group $\widetilde{G}^* = SL_2(81)$, then $\bar{\kappa}_{\widetilde{G}^*}$ and $\kappa_{\widetilde{G}^*}$ are both isomorphisms by [BL, Proposition 5.5] (note that $\mathrm{Out}(S, \mathcal{F}) = \mathrm{Out}(S)$ in this situation).

As another, more complicated example, consider the case where $p = 41$ and $G = \mathrm{Spin}_{4k}^-(9)$. By [St1, (3.2)–(3.6)], $\mathrm{Outdiag}(G) \cong C_2$, and $\mathrm{Out}(G) \cong C_2 \times C_4$ is generated by a diagonal element of order 2 and a field automorphism of order 4 (whose square is a graph automorphism of order 2). Also, μ_G is an isomorphism by Proposition A.3, so κ_G is surjective, or split surjective, if and only if $\bar{\kappa}_G$ is. We refer to the proof of Lemma 6.5, and to Table 6.1 in that proof, for details of a σ-setup for G in which the normalizer of a maximal torus contains a Sylow p-subgroup S. In particular, S is nonabelian if $k \geq 41$. By Proposition 5.15(d) and Example 6.6(a,b), when $k \geq 41$, $\bar{\kappa}_G$ is surjective, $\bar{\kappa}_G$ is split (with $\mathrm{Ker}(\bar{\kappa}_G) = \mathrm{Outdiag}(G)$) when k is odd, and $\bar{\kappa}_G$ is not split ($\mathrm{Ker}(\bar{\kappa}_G) \cong C_2 \times C_2$) when k is even. By Proposition 1.9(c), when k is even, $G \sim_{41} G^*$ for $G^* = \mathrm{Spin}_{4k-1}(9)$, and κ_{G^*} is split surjective (with $\mathrm{Ker}(\kappa_{G^*}) = \mathrm{Outdiag}(G^*)$) by Proposition 5.15(c). Thus $\mathcal{F}_S(G)$ is tame in all cases: tamely realized by G itself when k is odd and by $\mathrm{Spin}_{4k-1}(9)$ when k is even. Note that when k is odd, since the graph automorphism does not act trivially on any Sylow p-subgroup, the p-fusion system of G (equivalently, of $SO^-_{4k}(9)$) is not isomorphic to that of the full orthogonal group $O^-_{4k}(9)$, so by [BMO, Proposition A.3(b)], it is not isomorphic to that of $\mathrm{Spin}_{4k+1}(9)$ either (nor to that of $\mathrm{Spin}_{4k-1}(9)$ since its Sylow p-subgroups are smaller).

Other examples are given in Examples 5.16 and 6.6. For more details, in the situation of Theorem B, about for which groups G the homomorphism $\bar{\kappa}_G$ is surjective or split surjective, see Propositions 5.14 and 5.15.

The following theorem was shown while proving Theorem B, and could be of independent interest. The case where p is odd was handled by Gorenstein and Lyons [GL, 10-2(1,2)].

Theorem E. *Assume $G \in \mathfrak{Lie}(q_0)$ is of universal type for some odd prime q_0. Fix $S \in \mathrm{Syl}_2(G)$. Then S contains a unique abelian subgroup of maximal order,* except *when $G \cong Sp_{2n}(q)$ for some $n \geq 1$ and some $q \equiv \pm 3 \pmod 8$.*

Proof. Assume S is nonabelian; otherwise there is nothing to prove. Since q_0 is odd, and since the Sylow 2-subgroups of ${}^2G_2(3^{2k+1})$ are abelian for all $k \geq 1$ [Ree, Theorem 8.5], G must be a Chevalley or Steinberg group. If $G \cong {}^3D_4(q)$, then (up to isomorphism) $S \in \mathrm{Syl}_2(G_2(q))$ by [BMO, Example 4.5]. So we can assume that $G \cong {}^r\mathbb{G}(q)$ for some odd prime power q, some $\mathbb{G}$, and $r = 1$ or 2.

If $q \equiv 3 \pmod 4$, then choose another prime power $q^* \equiv 1 \pmod 4$ such that $v_2(q^*-1) = v_2(q+1)$ (where $v_2(m) = k$ if $2^k | n$ and $2^{k+1} \nmid n$). Then $\overline{\langle q^* \rangle} = \overline{\langle -q \rangle}$ and $\overline{\langle -q^* \rangle} = \overline{\langle q \rangle}$ as closed subgroups of $(\mathbb{Z}_2)^\times$. By [BMO, Theorem A] (see also Theorem 1.8), there is a group $G^* \cong {}^t\mathbb{G}(q^*)$ (where $t \leq 2$) whose 2-fusion system is equivalent to that of G. We can thus assume that $q \equiv 1 \pmod 4$. So by Lemma 6.1, G has a σ-setup which satisfies Hypotheses 5.2. By Proposition 5.12(a), S contains a unique abelian subgroup of maximal order, unless $q \equiv 5 \pmod 8$ and $G \cong Sp_{2n}(q)$ for some $n \geq 1$. □

In fact, when $G \cong Sp_{2n}(q)$ for $q \equiv \pm 3 \pmod 8$, then $S \in \mathrm{Syl}_2(G)$ is isomorphic to $(Q_8)^n \rtimes P$ for $P \in \mathrm{Syl}_2(\Sigma_n)$, S contains 3^n abelian subgroups of maximal order 2^{2n}, and all of them are conjugate to each other in $N_G(S)$.

The main definitions and results about tame and reduced fusion systems are given in Section 1. We then set up our general notation for finite groups of Lie type in Sections 2 and 3, deal with the equicharacteristic case in Section 4, and with the cross characteristic case in Sections 5 and 6. The kernel of μ_G, and thus the relation between automorphism groups of the fusion and linking systems, is handled in an appendix.

Notation: In general, when $\mathcal{C}$ is a category and $x \in \mathrm{Ob}(\mathcal{C})$, we let $\mathrm{Aut}_{\mathcal{C}}(x)$ denote the group of automorphisms of x in $\mathcal{C}$. When $\mathcal{F}$ is a fusion system and $P \in \mathrm{Ob}(\mathcal{F})$, we set $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$.

For any group G and $g \in G$, $c_g \in \mathrm{Aut}(G)$ denotes the automorphism $c_g(h) = ghg^{-1}$. Thus for $H \leq G$, ${}^gH = c_g(H)$ and $H^g = c_g^{-1}(H)$. When G, H, K are all subgroups of a group Γ, we define

$$T_G(H,K) = \{g \in G \mid {}^gH \le K\}$$
$$\mathrm{Hom}_G(H,K) = \{c_g \in \mathrm{Hom}(H,K) \mid g \in T_G(H,K)\}\,.$$

We let $\mathrm{Aut}_G(H)$ be the group $\mathrm{Aut}_G(H) = \mathrm{Hom}_G(H,H)$. When $H \le G$ (so $\mathrm{Aut}_G(H) \ge \mathrm{Inn}(H)$), we also write $\mathrm{Out}_G(H) = \mathrm{Aut}_G(H)/\mathrm{Inn}(H)$.

1. Tame and reduced fusion systems

Throughout this section, p always denotes a fixed prime. Before defining tameness of fusion systems more precisely, we first recall the definitions of fusion and linking systems of finite groups, and of automorphism groups of fusion and linking systems.

Definition 1.1. *Fix a finite group G and a Sylow p-subgroup $S \le G$.*

(a) *The* fusion system *of G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and where $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for each $P,Q \le S$.*

(b) *A subgroup $P \le S$ is* p-centric *in* G *if $Z(P) \in \mathrm{Syl}_p(C_G(P))$; equivalently, if $C_G(P) = Z(P) \times C'_G(P)$ for a (unique) subgroup $C'_G(P)$ of order prime to p.*

(c) *The* centric linking system *of G is the category $\mathcal{L}^c_S(G)$ whose objects are the p-centric subgroups of G, and where $\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q) = T_G(P,Q)/C'_G(P)$ for each pair of objects P,Q. Let $\pi\colon \mathcal{L}^c_S(G) \longrightarrow \mathcal{F}_S(G)$ denote the natural functor: π is the inclusion on objects, and sends the class of $g \in T_G(P,Q)$ to $c_g \in \mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q)$.*

(d) *For $P,Q \le S$ p-centric in G and $g \in T_G(P,Q)$, we let $[\![g]\!]_{P,Q} \in \mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q)$ denote the class of g, and set $[\![g]\!]_P = [\![g]\!]_{P,P}$ if $g \in N_G(P)$. For each subgroup $H \le N_G(P)$, $[\![H]\!]_P$ denotes the image of H in $\mathrm{Aut}_{\mathcal{L}}(P) = N_G(P)/C'_G(P)$.*

The following definitions of automorphism groups are taken from [AOV, Definition 1.13 & Lemma 1.14], where they are formulated more generally for abstract fusion and linking systems.

Definition 1.2. *Let G be a finite group with $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{L} = \mathcal{L}^c_S(G)$.*

(a) *If H is another finite group with $T \in \mathrm{Syl}_p(H)$, then an isomorphism $\varphi\colon S \xrightarrow{\cong} T$ is called* fusion preserving *(with respect to G and H) if for each $P,Q \le S$,*

$$\mathrm{Hom}_H(\varphi(P),\varphi(Q)) = \varphi \circ \mathrm{Hom}_G(P,Q) \circ \varphi^{-1} .$$

(Composition is from right to left.) Equivalently, φ is fusion preserving if it induces an isomorphism of categories $\mathcal{F}_S(G) \xrightarrow{\cong} \mathcal{F}_T(H)$.

(b) *Let $\mathrm{Aut}(S,\mathcal{F}) \le \mathrm{Aut}(S)$ be the group of fusion preserving automorphisms of S. Set $\mathrm{Out}(S,\mathcal{F}) = \mathrm{Aut}(S,\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S)$.*

(c) *For each pair of objects $P \le Q$ in $\mathcal{L}$, set $\iota_{P,Q} = [\![1]\!]_{P,Q} \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, which we call the* inclusion *in $\mathcal{L}$ of P in Q. For each P, we call $[\![P]\!] = [\![P]\!]_P \le \mathrm{Aut}_{\mathcal{L}}(P)$ the* distinguished subgroup *of $\mathrm{Aut}_{\mathcal{L}}(P)$.*

(d) *Let $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ be the group of automorphisms α of the category $\mathcal{L}$ such that α sends inclusions to inclusions and distinguished subgroups to distinguished subgroups. For $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$, let $c_\gamma \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ be the automorphism which sends an object P to $\pi(\gamma)(P)$, and sends $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to $\gamma'\psi(\gamma'')^{-1}$ where γ' and γ'' are appropriate restrictions of γ. Set*

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) = \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})\big/\{c_\gamma \,|\, \gamma \in \mathrm{Aut}_{\mathcal{L}}(S)\} .$$

(e) *Let $\kappa_G\colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ be the homomorphism which sends the class $[\alpha]$, for $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(S) = S$, to the class of the induced automorphism of $\mathcal{L} = \mathcal{L}^c_S(G)$.*

(f) *Define* $\mu_G\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Out}(S,\mathcal{F})$ *by setting* $\mu_G([\beta]) = [\beta_S|_S]$ *for* $\beta \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}^c_S(G))$, *where* β_S *is the induced automorphism of* $\mathrm{Aut}_{\mathcal{L}}(S)$, *and* $\beta_S|_S \in \mathrm{Aut}(S)$ *is its restriction to* S *when we identify* S *with its image in* $\mathrm{Aut}_{\mathcal{L}}(S) = N_G(S)/C'_G(S)$.

(g) *Set* $\bar\kappa_G = \mu_G \circ \kappa_G\colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(S,\mathcal{F})$*: the homomorphism which sends the class of* $\alpha \in N_{\mathrm{Aut}(G)}(S)$ *to the class of* $\alpha|_S$.

By [AOV, Lemma 1.14], the above definition of $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is equivalent to that in [BLO2], and by [BLO2, Lemma 8.2], both are equivalent to that in [BLO1]. So by [BLO1, Theorem 4.5(a)], $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G)) \cong \mathrm{Out}(BG^\wedge_p)$: the group of homotopy classes of self homotopy equivalences of the space $BG^\wedge_p$.

We refer to [AOV, § 2.2] and [AOV, § 1.3] for more details about the definitions of κ_G and μ_G and the proofs that they are well defined. Note that μ is defined there for an arbitrary linking system, not necessarily one realized by a group.

We are now ready to define tameness. Again, we restrict attention to fusion systems of finite groups, and refer to [AOV, § 2.2] for the definition in the more abstract setting.

Definition 1.3. *For a finite group* G *and* $S \in \mathrm{Syl}_p(G)$*, the fusion system* $\mathcal{F}_S(G)$ *is* tame *if there is a finite group* G^* *which satisfies:*

- *there is a fusion preserving isomorphism* $S \xrightarrow{\ \cong\ } S^*$ *for some* $S^* \in \mathrm{Syl}_p(G^*)$*; and*
- *the homomorphism* $\kappa_{G^*}\colon \mathrm{Out}(G^*) \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}^c_S(G^*)) \cong \mathrm{Out}(BG^{*\wedge}_p)$ *is split surjective.*

In this situation, we say that G^* tamely realizes *the fusion system* $\mathcal{F}_S(G)$.

The above definition is complicated by the fact that two finite groups can have isomorphic fusion systems but different outer automorphism groups. For example, set $G = PSL_2(9) \cong A_6$ and $H = PSL_2(7) \cong GL_3(2)$. The Sylow subgroups of both groups are dihedral of order 8, and it is not hard to see that any isomorphism between Sylow subgroups is fusion preserving. But $\mathrm{Out}(G) \cong C_2^2$ while $\mathrm{Out}(H) \cong C_2$ (see Theorem 3.4 below). Also, κ_G is an isomorphism, while κ_H fails to be onto (see [BLO1, Proposition 7.9]). In conclusion, the 2-fusion system of both groups is tame, even though κ_H is not split surjective.

This definition of tameness was motivated in part in [AOV] by an attempt to construct new, “exotic” fusion systems (abstract fusion systems not realized by any finite group) as extensions of a known fusion system by an automorphism. Very roughly, if $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}^c_S(G))$ is not in the image of κ_G, and not in the image of κ_{G^*} for any other finite group G^* which has the same fusion and linking systems, then one can construct and extension of $\mathcal{F}_S(G)$ by α which is not isomorphic to the fusion system of any finite group. This shows why we are interested in the surjectivity of κ_G; to see the importance of its being split, we refer to the proof of [AOV, Theorem B].

It is usually simpler to work with automorphisms of a p-group which preserve fusion than with automorphisms of a linking system. So in most cases, we prove tameness for the fusion system of a group G by first showing that $\bar\kappa_G = \mu_G \circ \kappa_G$ is split surjective, and then showing that μ_G is injective. The following elementary lemma will be useful.

Lemma 1.4. *Fix a finite group* G *and* $S \in \mathrm{Syl}_p(G)$*, and set* $\mathcal{F} = \mathcal{F}_S(G)$*. Then*

(a) $\bar\kappa_G$ *is surjective if and only if each* $\varphi \in \mathrm{Aut}(S,\mathcal{F})$ *extends to some* $\bar\varphi \in \mathrm{Aut}(G)$*, and*

(b) $\mathrm{Ker}(\bar\kappa_G) \cong C_{\mathrm{Aut}(G)}(S)/\mathrm{Aut}_{C_G(S)}(G)$.

Proof. This follows from the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Aut}_{N_G(S)}(G) & \longrightarrow & N_{\mathrm{Aut}(G)}(S) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 0 \\ & & \big\downarrow\!\!\!\downarrow & & \big\downarrow{\scriptstyle \mathrm{restr}} & & \big\downarrow{\scriptstyle \bar{\kappa}_G} & & \\ 0 & \longrightarrow & \mathrm{Aut}_{N_G(S)}(S) & \longrightarrow & \mathrm{Aut}(S,\mathcal{F}) & \longrightarrow & \mathrm{Out}(S,\mathcal{F}) & \longrightarrow & 0 \end{array}$$

with exact rows. □

The next lemma can be useful when κ_G or $\bar{\kappa}_G$ is surjective but not split.

Lemma 1.5. *Fix a prime p, a finite group G, and $S \in \mathrm{Syl}_p(G)$.*

(a) *Assume $\widehat{G} \geq G$ is such that $G \trianglelefteq \widehat{G}$, $p \nmid |\widehat{G}/G|$, and $\mathrm{Out}_{\widehat{G}}(G) \leq \mathrm{Ker}(\bar{\kappa}_G)$. Then $\mathcal{F}_S(\widehat{G}) = \mathcal{F}_S(G)$ and $\mathcal{L}_S^c(\widehat{G}) \cong \mathcal{L}_S^c(G)$.*

(b) *If κ_G is surjective and $\mathrm{Ker}(\kappa_G)$ has order prime to p, then there is $\widehat{G} \geq G$ as in (a) such that $\kappa_{\widehat{G}}$ is split surjective. In particular, $\mathcal{F}_S(G)$ is tame, and is tamely realized by $\widehat{G}$.*

Proof. **(a)** Since $\mathrm{Out}_{\widehat{G}}(G) \leq \mathrm{Ker}(\bar{\kappa}_G)$, each coset of G in $\widehat{G}$ contains an element which centralizes S. (Recall that $\bar{\kappa}_G$ is induced by the restriction homomorphism from $N_{\mathrm{Aut}(G)}(S)$ to $\mathrm{Aut}(S,\mathcal{F})$.) Thus $\mathcal{F}_S(\widehat{G}) = \mathcal{F}_S(G)$ and $\mathcal{L}_S^c(\widehat{G}) = \mathcal{L}_S^c(G)$.

(b) Since G and $G/O_{p'}(Z(G))$ have isomorphic fusion systems at p, we can assume that $Z(G)$ is a p-group. Set $K = \mathrm{Ker}(\kappa_G) \leq \mathrm{Out}(G)$. Since $H^i(K; Z(G)) = 0$ for $i = 2, 3$, by the obstruction theory for group extensions [McL, Theorems IV.8.7–8], there is an extension $\widehat{G}$ of G by K such that $G \trianglelefteq \widehat{G}$, $\widehat{G}/G \cong K$, and $\mathrm{Out}_{\widehat{G}}(G) = K$. Since $K = \mathrm{Ker}(\kappa_G) \leq \mathrm{Ker}(\bar{\kappa}_G)$, $\mathcal{F}_S(\widehat{G}) = \mathcal{F}_S(G)$, and $\mathcal{L}_S^c(\widehat{G}) = \mathcal{L}_S^c(G)$ by (a).

By [OV, Lemma 1.2], and since $K \trianglelefteq \mathrm{Out}(G)$ and $H^i(K; Z(G)) = 0$ for $i = 1, 2$, each automorphism of G extends to an automorphism of $\widehat{G}$ which is unique modulo inner automorphisms. Thus $\mathrm{Out}(\widehat{G})$ contains a subgroup isomorphic to $\mathrm{Out}(G)/K$, and $\kappa_{\widehat{G}}$ sends this subgroup isomorphically onto $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(\widehat{G}))$. So $\kappa_{\widehat{G}}$ is split surjective, and $\mathcal{F}_S(G)$ is tame. □

The next proposition is really a result about constrained fusion systems (cf. [AKO, Definition I.4.8]): it says that every constrained fusion system is tame. Since we are dealing here only with fusion systems of finite groups, we state it instead in terms of p-constrained groups.

Proposition 1.6. *Fix a finite group G and a Sylow subgroup $S \in \mathrm{Syl}_p(G)$.*

(a) *If $C_G(O_p(G)) \leq O_p(G)$, then κ_G and μ_G are both isomorphisms:*

$$\mathrm{Out}(G) \xrightarrow[\cong]{\kappa_G} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \xrightarrow[\cong]{\mu_G} \mathrm{Out}(S, \mathcal{F}_S(G))\,.$$

(b) *If S is abelian, or more generally if $N_G(S)$ controls p-fusion in G, then $\mathcal{F}_S(G)$ is tame, and is tamely realized by $N_G(S)/O_{p'}(C_G(S))$.*

Proof. **(a)** Set $Q = O_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$, and $\mathcal{L} = \mathcal{L}_S^c(G)$. Then $\mathrm{Aut}_{\mathcal{L}}(Q) = G$, so $(\alpha \mapsto \alpha_Q)$ defines a homomorphism $\Phi\colon \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}) \longrightarrow \mathrm{Aut}(G, S)$. For each $\alpha \in \mathrm{Ker}(\Phi)$, $\alpha_Q = \mathrm{Id}_G$ and hence $\alpha = \mathrm{Id}_{\mathcal{L}}$. (Here, it is important that α sends inclusions to inclusions.) Thus Φ is an isomorphism. Also, $\alpha = c_\gamma$ for some $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$ if and only if $\alpha_Q = c_g$ for some $g \in N_G(S)$, so Φ factors through an isomorphism from $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ to $\mathrm{Aut}(G,S)/\mathrm{Aut}_G(S) \cong \mathrm{Out}(G)$, and this is an inverse to κ_G. Thus κ_G is an isomorphism.

In the terminology in [AKO, § I.4], G is a model for $\mathcal{F} = \mathcal{F}_S(G)$. By the uniqueness of models (cf. [AKO, Theorem III.5.10(c)]), each $\beta \in \mathrm{Aut}(S,\mathcal{F})$ extends to some $\chi \in \mathrm{Aut}(G)$, and χ is unique modulo $\mathrm{Aut}_{Z(S)}(G)$. Hence $\bar{\kappa}_G$ is an isomorphism, and so is μ_G.

(b) If $N_G(S)$ controls p-fusion in G, then $N_G(S) \sim_p G$. Also, $N_G(S) \sim_p G^*$ where $G^* = N_G(S)/O_{p'}(C_G(S))$, G^* satisfies the hypotheses of (a), and hence tamely realizes $\mathcal{F}_S(G)$. In particular, this holds whenever S is abelian by Burnside's theorem. □

When working with groups of Lie type when p is not the defining characteristic, it is easier to work with the universal groups rather than those in adjoint form (μ_G is better behaved in such cases). The next proposition is needed to show that tameness for fusion systems of groups of universal type implies the corresponding result for groups of adjoint type.

Proposition 1.7. *Let G be a finite p-perfect group such that $O_{p'}(G) = 1$ and $H_2(G;\mathbb{Z}/p) = 0$ (i.e., such that each central extension of G by a finite p-group splits). Choose $S \in \mathrm{Syl}_p(G)$, and set $Z = Z(G) \leq S$. If $\mathcal{F}_S(G)$ is tamely realized by G, then $\mathcal{F}_{S/Z}(G/Z)$ is tamely realized by G/Z.*

Proof. Let $\mathcal{H}$ be the set of all $P \leq S$ such that $P \geq Z$ and P/Z is p-centric in G/Z, and let $\mathcal{L}_S^{\mathcal{H}}(G) \subseteq \mathcal{L}_S^c(G)$ be the full subcategory with object set $\mathcal{H}$. By [AOV, Lemma 2.17], $\mathcal{L}_S^{\mathcal{H}}(G)$ is a linking system associated to $\mathcal{F}_S(G)$ in the sense of [AOV, Definition 1.9]. Hence the homomorphism

$$R\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \xrightarrow{\ \cong\ } \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^{\mathcal{H}}(G))$$

induced by restriction is an isomorphism by [AOV, Lemma 1.17].

Set $\mathcal{F} = \mathcal{F}_S(G)$, $\mathcal{L} = \mathcal{L}_S^{\mathcal{H}}(G)$, $\bar{G} = G/Z$, $\bar{S} = S/Z$, $\bar{\mathcal{F}} = \mathcal{F}_{\bar{S}}(\bar{G})$, and $\bar{\mathcal{L}} = \mathcal{L}_{\bar{S}}^c(\bar{G})$ for short. Consider the following square:

$$\begin{array}{ccc} \mathrm{Out}(G) & \xrightarrow{\ \kappa_G\ } & \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \cong \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \\ \big\downarrow \mu & & 1{-}1 \big\uparrow \nu \\ \mathrm{Out}(\bar{G}) & \xrightarrow{\ \kappa_{\bar{G}}\ } & \mathrm{Out}_{\mathrm{typ}}(\bar{\mathcal{L}})\,. \end{array} \tag{1}$$

Here, μ sends the class of an automorphism of G to the class of the induced automorphism of $\bar{G} = G/Z(G)$.

Assume that ν has been defined so that (1) commutes and ν is injective. If κ_G is onto, then ν is onto and hence an isomorphism, so $\kappa_{\bar{G}}$ is also onto. Similarly, if κ_G is split surjective, then $\kappa_{\bar{G}}$ is also split surjective. Thus $\bar{\mathcal{F}}$ is tamely realized by $\bar{G}$ if $\mathcal{F}$ is tamely realized by G, which is what we needed to show.

It thus remains to construct the monomorphism ν, by sending the class of $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\bar{\mathcal{L}})$ to the class of a lifting of α to $\mathcal{L}$. So in the rest of the proof, we show the existence and uniqueness of such a lifting.

Let $\mathrm{pr}\colon \mathcal{L} \longrightarrow \bar{\mathcal{L}}$ denote the projection. Let $\mathrm{End}_{\mathrm{typ}}^I(\mathcal{L})$ be the monoid of functors from $\mathcal{L}$ to itself which send inclusions to inclusions and distinguished subgroups into distinguished subgroups. (Thus $\mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ is the group of elements of $\mathrm{End}_{\mathrm{typ}}^I(\mathcal{L})$ which are invertible.) We will prove the following two statements:

(2) For each $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\bar{\mathcal{L}})$, there is a functor $\widetilde{\alpha} \in \mathrm{End}_{\mathrm{typ}}^I(\mathcal{L})$ such that $\mathrm{pr} \circ \widetilde{\alpha} = \alpha \circ \mathrm{pr}$.

(3) If $\beta \in \mathrm{End}^I_{\mathrm{typ}}(\mathcal{L})$ is such that $\mathrm{pr} \circ \beta = \mathrm{pr}$, then $\beta = \mathrm{Id}_{\mathcal{L}}$.

Assume that (2) and (3) hold; we call $\widetilde{\alpha}$ a "lifting" of α in the situation of (2). For each $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\bar{\mathcal{L}})$, there are liftings $\widetilde{\alpha}$ of α and $\widetilde{\alpha}^*$ of α^{-1} in $\mathrm{End}^I_{\mathrm{typ}}(\mathcal{L})$, and these are inverses to each other by (3). Hence $\widetilde{\alpha} \in \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$, and is the unique such lifting of α by (3) again.

Define $\nu\colon \mathrm{Out}_{\mathrm{typ}}(\bar{\mathcal{L}}) \longrightarrow \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ by setting $\nu([\alpha]) = [\widetilde{\alpha}]$ when $\widetilde{\alpha}$ is the unique lifting of α. This is well defined as a homomorphism on $\mathrm{Aut}^I_{\mathrm{typ}}(\bar{\mathcal{L}})$ by the existence and uniqueness of the lifting; and it factors through $\mathrm{Out}_{\mathrm{typ}}(\bar{\mathcal{L}})$ since conjugation by $\bar{\gamma} \in \mathrm{Aut}_{\bar{\mathcal{L}}}(\bar{S})$ lifts to conjugation by $\gamma \in \mathrm{Aut}_{\mathcal{L}}(S)$ for any $\gamma \in \mathrm{pr}_S^{-1}(\bar{\gamma})$.

Thus ν is a well defined homomorphism, and is clearly injective. The square (1) commutes since for each $\beta \in \mathrm{Aut}(G)$ such that $\beta(S) = S$, $\kappa_G([\beta])$ and $\nu\kappa_{\bar{G}}\mu([\beta])$ are the classes of liftings of the same automorphism of $\bar{\mathcal{L}}$.

It remains to prove (2) and (3).

Proof of (2): For each $\alpha \in \mathrm{Aut}^I_{\mathrm{typ}}(\bar{\mathcal{L}})$, consider the pullback diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{L}} & \xrightarrow{\quad\rho_1\quad} & & & \mathcal{L} \\ \big\downarrow{\scriptstyle \rho_2} & & \overset{\widetilde{\alpha}}{\dashrightarrow} & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ \mathcal{L} & \xrightarrow{\mathrm{pr}} & \bar{\mathcal{L}} & \xrightarrow[\cong]{\alpha} & \bar{\mathcal{L}}\,. \end{array} \tag{4}$$

Each functor in (4) is bijective on objects, and the diagram restricts to a pullback square of morphism sets for each pair of objects in $\bar{\mathcal{L}}$ (and their inverse images in $\mathcal{L}$ and $\widetilde{\mathcal{L}}$).

Since the natural projection $G \longrightarrow \bar{G}$ is a central extension with kernel Z, the projection functor $\mathrm{pr}\colon \mathcal{L} \longrightarrow \bar{\mathcal{L}}$ is also a central extension of linking systems in the sense of [5a2, Definition 6.9] with kernel Z. Since ρ_2 is the pullback of a central extension, it is also a central extension of linking systems by [5a2, Proposition 6.10], applied with $\omega = \mathrm{pr}^*\alpha^*(\omega_0) \in Z^2(\mathcal{L}; Z)$, where ω_0 is a 2-cocycle on $\bar{\mathcal{L}}$ which determines the extension pr. By [BLO1, Proposition 1.1], $H^2(|\mathcal{L}|; \mathbb{F}_p) \cong H^2(G; \mathbb{F}_p)$, where the last group is zero by assumption. Hence $H^2(|\mathcal{L}|; Z) = 0$, so ω is a coboundary, and ρ_2 is the product extension by [5a2, Theorem 6.13]. In other words, $\widetilde{\mathcal{L}} \cong \mathcal{L}^c_Z(Z) \times \mathcal{L}$, where $\mathcal{L}^c_Z(Z)$ has one object and automorphism group Z, and there is a subcategory $\mathcal{L}_0 \subseteq \widetilde{\mathcal{L}}$ (with the same objects) which is sent isomorphically to $\mathcal{L}$ by ρ_2. Set $\widetilde{\alpha} = \rho_1 \circ (\rho_2|_{\mathcal{L}_0})^{-1}$.

We first check that $\widetilde{\alpha}$ sends distinguished subgroups to distinguished subgroups. Let $\mathrm{pr}_S\colon S \longrightarrow \bar{S} = S/Z$ be the projection. Fix an object P in $\mathcal{L}$, and set $Q = \widetilde{\alpha}(P)$. Then $Q/Z = \alpha(P/Z)$, and $\alpha_{P/Z}(\llbracket P/Z \rrbracket) = \llbracket Q/Z \rrbracket$, so $\widetilde{\alpha}_P(\llbracket P \rrbracket) \le \mathrm{pr}_S^{-1}(\llbracket Q/Z \rrbracket) = \llbracket Q \rrbracket$.

For each subgroup $P \in \mathrm{Ob}(\mathcal{L})$, there is a unique element $z_P \in Z$ such that $\widetilde{\alpha}(\iota_{P,S}) = \iota_{\widetilde{\alpha}(P),S} \circ \llbracket z_P \rrbracket_{\widetilde{\alpha}(P)}$. Note that $z_S = 1$. Define a new functor $\beta\colon \mathcal{L} \longrightarrow \mathcal{L}$ by setting $\beta(P) = \widetilde{\alpha}(P)$ on objects and for each $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, $\beta(\varphi) = \llbracket z_Q \rrbracket_{\widetilde{\alpha}(Q)} \circ \widetilde{\alpha}(\varphi) \circ \llbracket z_P \rrbracket^{-1}_{\widetilde{\alpha}(P)}$. Then β is still a lifting of α, and for each P:

$$\beta(\iota_{P,S}) = \llbracket z_S \rrbracket_S \circ \widetilde{\alpha}(\iota_{P,S}) \circ \llbracket z_P \rrbracket^{-1}_{\widetilde{\alpha}(P)} = \iota_{\widetilde{\alpha}(P),S} \circ \llbracket z_P \rrbracket_{\widetilde{\alpha}(P)} \circ \llbracket z_P \rrbracket^{-1}_{\widetilde{\alpha}(P)} = \iota_{\widetilde{\alpha}(P),S}\,.$$

For arbitrary $P \le Q$, since $\iota_{\widetilde{\alpha}(P),\widetilde{\alpha}(Q)}$ is the unique morphism whose composite with $\iota_{\widetilde{\alpha}(Q),S}$ is $\iota_{\widetilde{\alpha}(P),S}$ (see [BLO2, Lemma 1.10(a)]), β sends $\iota_{P,Q}$ to $\iota_{\widetilde{\alpha}(P),\widetilde{\alpha}(Q)}$.

Thus, upon replacing $\widetilde{\alpha}$ by β, we can assume that $\widetilde{\alpha}$ sends inclusions to inclusions. This finishes the proof of (2).

Proof of (3): Assume that $\beta \in \mathrm{End}^I_{\mathrm{typ}}(\mathcal{L})$ is a lift of the identity on $\bar{\mathcal{L}}$. Let $\mathcal{B}(Z)$ be the category with one object $*$ and with morphism group Z. Define a functor $\chi\colon \mathcal{L} \longrightarrow \mathcal{B}(Z)$ by sending all objects in $\mathcal{L}$ to $*$, and by sending a morphism $[\![g]\!] \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to the unique element $z \in Z$ such that $\beta_{P,Q}([\![g]\!]) = [\![gz]\!] = [\![zg]\!]$. (Recall that $Z \le Z(G)$.)

Now,

$$H^1(|\mathcal{L}|;\mathbb{F}_p) \cong H^1(|\mathcal{L}^c_S(G)|;\mathbb{F}_p) \cong H^1(BG;\mathbb{F}_p) \cong H^1(G;\mathbb{F}_p) = 0\,,$$

where the first isomorphism holds by [5a1, Theorem B] and the second by [BLO1, Proposition 1.1]. Hence $\mathrm{Hom}(\pi_1(|\mathcal{L}|),\mathbb{F}_p) \cong \mathrm{Hom}(H_1(|\mathcal{L}|),\mathbb{F}_p) \cong H^1(|\mathcal{L}|;\mathbb{F}_p) = 0$, where the second isomorphism holds by the universal coefficient theorem (cf. [McL, Theorem III.4.1]), and so $\mathrm{Hom}(\pi_1(|\mathcal{L}|),Z) = 0$. In particular, the homomorphism $\widehat{\chi}\colon \pi_1(|\mathcal{L}|) \longrightarrow \pi_1(|\mathcal{B}(Z)|) \cong Z$ induced by χ is trivial.

Thus for each $\psi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, the loop in $|\mathcal{L}|$ formed by ψ and the inclusions $\iota_{P,S}$ and $\iota_{Q,S}$ is sent to $1 \in Z$. Since β sends inclusions to inclusions, this proves that $\chi_{P,Q}(\psi) = 1$, and hence that $\beta_{P,Q}(\psi) = \psi$. Thus $\beta = \mathrm{Id}_{\mathcal{L}}$. □

By Proposition 1.7, when proving tameness for fusion systems of simple groups of Lie type, it suffices to look at the universal groups (such as $SL_n(q)$, $SU_n(q)$) rather than the simple groups ($PSL_n(q)$, $PSU_n(q)$). However, it is important to note that the proposition is false if we replace automorphisms of the linking systems by those of the fusion system. For example, set $G = SL_2(3^4)$ and $\bar{G} = PSL_2(3^4)$. Then $S \cong Q_{32}$ and $\bar{S} \cong D_{16}$, $\mathrm{Out}(S,\mathcal{F}_S(G)) = \mathrm{Out}(S) \cong \mathrm{Out}(G) \cong C_4 \times C_2$ (and $\bar{\kappa}_G$ is an isomorphism), while $\mathrm{Out}(\bar{G}) \cong C_4 \times C_2$ and $\mathrm{Out}(\bar{S},\mathcal{F}_{\bar{S}}(\bar{G})) = \mathrm{Out}(\bar{S}) \cong C_2 \times C_2$.

We already gave one example of two groups which have the same fusion system but different outer automorphism groups. That is a special case of the main theorem in our earlier paper, where we construct many examples of different groups of Lie type with isomorphic fusion systems. Since this plays a crucial role in Section 6, where we handle the cross characteristic case, we restate the theorem here.

As in the introduction, we write $G \sim_p H$ to mean that there is a fusion preserving isomorphism from a Sylow p-subgroup of G to one of H.

Theorem 1.8 ([BMO, Theorem A])**.** *Fix a prime p, a connected reductive group scheme $\mathbb{G}$ over $\mathbb{Z}$, and a pair of prime powers q and q' both prime to p. Then the following hold.*

(a) $\mathbb{G}(q) \sim_p \mathbb{G}(q')$ *if* $\overline{\langle q\rangle} = \overline{\langle q'\rangle}$ *as subgroups of* $\mathbb{Z}_p^\times$.

(b) *If $\mathbb{G}$ is of type A_n, D_n, or E_6, and τ is a graph automorphism of $\mathbb{G}$, then* ${}^\tau\mathbb{G}(q) \sim_p {}^\tau\mathbb{G}(q')$ *if* $\overline{\langle q\rangle} = \overline{\langle q'\rangle}$ *as subgroups of* $\mathbb{Z}_p^\times$.

(c) *If the Weyl group of $\mathbb{G}$ contains an element which acts on the maximal torus by inverting all elements, then* $\mathbb{G}(q) \sim_p \mathbb{G}(q')$ *(or* ${}^\tau\mathbb{G}(q) \sim_p {}^\tau\mathbb{G}(q')$ *for τ as in (b)) if* $\overline{\langle -1,q\rangle} = \overline{\langle -1,q'\rangle}$ *as subgroups of* $\mathbb{Z}_p^\times$.

(d) *If $\mathbb{G}$ is of type A_n, D_n for n odd, or E_6, and τ is a graph automorphism of $\mathbb{G}$ of order two, then* ${}^\tau\mathbb{G}(q) \sim_p \mathbb{G}(q')$ *if* $\overline{\langle -q\rangle} = \overline{\langle q'\rangle}$ *as subgroups of* $\mathbb{Z}_p^\times$.

The next proposition is of similar type, but much more elementary.

Proposition 1.9. *Fix an odd prime p, a prime power q prime to p, $n \ge 2$, and $\varepsilon \in \{\pm1\}$. Then*

(a) $Sp_{2n}(q) \sim_p SL_{2n}(q)$ *if* $\mathrm{ord}_p(q)$ *is even;*

(b) *$Sp_{2n}(q) \sim_p \mathrm{Spin}_{2n+1}(q)$; and*

(c) *$\mathrm{Spin}^{\varepsilon}_{2n}(q) \sim_p \mathrm{Spin}_{2n-1}(q)$ if q is odd and $q^n \not\equiv \varepsilon$ (mod p).*

Proof. If we replace $\mathrm{Spin}^{\pm}_m(q)$ by $SO^{\pm}_m(q)$ in (b) and (c), then these three points are shown in [BMO, Proposition A.3] as points (d), (a), and (c), respectively. When q is a power of 2, (b) holds because the groups are isomorphic (see [Ta, Theorem 11.9]). So it remains to show that

$$\mathrm{Spin}^{\varepsilon}_m(q) \sim_p \Omega^{\varepsilon}_m(q) \sim_p SO^{\varepsilon}_m(q)$$

for all $m \geq 3$ (even or odd) and q odd. The first equivalence holds since p is odd and $\Omega^{\varepsilon}_m(q) \cong \mathrm{Spin}^{\varepsilon}_m(q)/K$ where $|K| = 2$. The second holds by Lemma 1.5(a), and since $\mathrm{Out}_{SO^{\varepsilon}_m(q)}(\Omega^{\varepsilon}_m(q))$ is generated by the class of a diagonal automorphism of order 2 (see, e.g., [GLS3, § 2.7]) and hence can be chosen to commute with a Sylow p-subgroup. This last statement is shown in Lemma 5.8 below, and holds since for appropriate choices of algebraic group $\overline{G}$ containing the given group G, and of maximal torus $\overline{T} \leq \overline{G}$, a Sylow p-subgroup of G is contained in $N_{\overline{G}}(\overline{T})$ (see [GLS3, Theorem 4.10.2]) and the diagonal automorphisms of G are induced by conjugation by elements in $N_{\overline{T}}(G)$ (see Proposition 3.5(c)). □

Theorem 1.8 and Proposition 1.9, together with some other, similar relations in [BMO], lead to the following proposition, which when p is odd provides a relatively short list of "p-local equivalence class representatives" for groups of Lie type in characteristic different from p.

Proposition 1.10. *Fix an odd prime p, and assume $G \in \mathfrak{Lie}(q_0)$ is of universal type for some prime $q_0 \neq p$. Assume also that the Sylow p-subgroups of G are nonabelian. Then there is a group $G^* \in \mathfrak{Lie}(q_0')$ of universal type for some $q_0' \neq p$, such that $G^* \sim_p G$ and G^* is one of the groups in the following list:*

(a) *$SL_n(q')$ for some $n \geq p$; or*

(b) *$\mathrm{Spin}^{\varepsilon}_{2n}(q')$, where $n \geq p$, $\varepsilon = \pm 1$, $(q')^n \equiv \varepsilon$ (mod p), and $\varepsilon = +1$ if n is odd; or*

(c) *${}^3D_4(q')$ or ${}^2F_4(q')$, where $p = 3$ and q' is a power of 2; or*

(d) *$\mathbb{G}(q')$, where $\mathbb{G} = G_2, F_4, E_6, E_7$, or E_8, $p \,\big|\, |W(\mathbb{G})|$, and $q' \equiv 1$ (mod p); or*

(e) *$E_8(q')$, where $p = 5$ and $q' \equiv \pm 2$ (mod 5).*

Furthermore, in all cases except (c), we can take q_0' to be any given prime whose class generates $(\mathbb{Z}/p^2)^{\times}$, and choose G^* so that $q' = (q_0')^b$ where $b|(p-1)p^k$ for some k.

Proof. Let q be such that $G \cong {}^{\tau}\mathbb{G}(q)$ for some τ and some $\mathbb{G}$. Thus q is a power of q_0. Fix a prime q_0' as specified above. By Lemma 1.11(a), there are positive integers b, c, and powers $q' = (q_0')^b$ and $q^{\vee} = (q_0')^c$ such that $\overline{\langle q\rangle} = \overline{\langle q'\rangle}$, $\overline{\langle -q\rangle} = \overline{\langle q^{\vee}\rangle}$, and $b, c|(p-1)p^{\ell}$ for some $\ell \geq 0$.

(i) Assume $G \cong \mathrm{Sz}(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, or $G \cong {}^3D_4(q)$. Since $p \neq q_0$, and since $S \in \mathrm{Syl}_p(G)$ is nonabelian, p divides the order of the Weyl group W of $\mathbb{G}$ by [GL, 10-1]. The Weyl group of B_2 is a 2-group, and 2 and 3 are the only primes which divide the orders of the Weyl groups of G_2, F_4, and D_4. Hence $p = 3$, $G \not\cong {}^2G_2(q)$ since that is defined only in characteristic 3, and so $G \cong {}^2F_4(q)$ or ${}^3D_4(q)$. Set $G^* = {}^2F_4(q')$ or ${}^3D_4(q')$, respectively, where $q_0' = 2$. Then $G^* \sim_p G$, and we are in case (c).

(ii) If $G = SU_n(q)$ or ${}^2E_6(q)$, then by Theorem 1.8(d), $G \sim_p G^*$ where $G^* \cong SL_n(q^{\vee})$ or $E_6(q^{\vee})$, respectively. So we can replace G by a Chevalley group in these cases.

(iii) Assume $G = Sp_{2n}(q)$ for some n and q. If $\mathrm{ord}_p(q)$ is even, then by Proposition 1.9(a), $G \sim_p SL_{2n}(q)$. If $\mathrm{ord}_p(q)$ is odd, then $\mathrm{ord}_p(q^\vee)$ is even (recall that $q^\vee \equiv -q \pmod p$), and $G \sim_p Sp_{2n}(q^\vee)$ by Theorem 1.8(c). So G is always p-locally equivalent to a linear group in this case.

(iv) Assume $G = \mathrm{Spin}_{2n+1}(q)$ for some n and q. Then $G \sim_p Sp_{2n}(q)$ by Proposition 1.9(b). So G is p-locally equivalent to a linear group by (iii).

(v) If $G = SL_n(q)$, set $G^* = SL_n(q')$. Then $G^* \sim_p G$ by Theorem 1.8(a), $n \geq p$ since the Sylow p-subgroups of G are nonabelian, and we are in the situation of (a).

(vi) Assume $G = \mathrm{Spin}^\varepsilon_{2n}(q)$ for some n and q, and $\varepsilon = \pm 1$. If q is a power of 2, then by using point (a) or (b) of Theorem 1.8, we can arrange that q be odd. If $q^n \not\equiv \varepsilon \pmod p$, then $G \sim_p \mathrm{Spin}_{2n-1}(q)$ by Proposition 1.9(c), and this is p-equivalent to a linear group by (iv). So we are left with the case where $q^n \equiv \varepsilon \pmod p$. If n is odd and $\varepsilon = -1$, set $G^* = \mathrm{Spin}^+_{2n}(q^\vee) \sim_p G$ (Theorem 1.8(d)). Otherwise, set $G^* = \mathrm{Spin}^\varepsilon_{2n}(q') \sim_p G$ (Theorem 1.8(a,b)). In either case, we are in the situation of (b).

We are left with the cases where $G = \mathbb{G}(q)$ for some exceptional Lie group $\mathbb{G}$. By [GL, 10-1(2)] and since the Sylow p-subgroups of G are nonabelian, $p \bigm| |W(\mathbb{G})|$. If $\mathrm{ord}_p(q) = 1$, then $G^* = \mathbb{G}(q') \sim_p G$ by Theorem 1.8(a). If $\mathrm{ord}_p(q) = 2$ and $\mathbb{G} \neq E_6$, then $G^* = \mathbb{G}(q^\vee) \sim_p G$ by Theorem 1.8(c), where $q^\vee \equiv 1 \pmod p$. In either case, we are in the situation of (d).

If $\mathrm{ord}_p(q) = 2$ and $G = E_6(q)$, then $\overline{\langle q \rangle} = \overline{\langle -q^2 \rangle}$ as closed subgroups of $\mathbb{Z}_p^\times$ (note that $v_p(q^2-1) = v_p((-q^2)^2-1)$). So by Theorem 1.8(d) and Example 4.4 in [BMO], $G = E_6(q) \sim_p {}^2E_6(q^2) \sim_p F_4(q^2)$. So we can choose G^* satisfying (d) as in the last paragraph.

Assume $\mathrm{ord}_p(q) > 2$. By [GL, 10-1(3)], for $S \in \mathrm{Syl}_p(G)$ to be nonabelian, there must be some $n \geq 1$ such that $p \cdot \mathrm{ord}_p(q) \bigm| n$, and such that $q^n - 1$ appears as a factor in the formula for $|\mathbb{G}(q)|$ (see, e.g., [GL, Table 4-2] or [Ca, Theorem 9.4.10 & Proposition 10.2.5]). Since $\mathrm{ord}_p(q)|(p-1)$, this shows that the case $\mathrm{ord}_p(q) > 2$ appears only for the group $E_8(q)$, and only when $p = 5$ and $m = 4$. In particular, $q, q' \equiv \pm 2 \pmod 5$. Set $G^* = E_8(q')$; then $G^* \sim_p G$ by Theorem 1.8(a), and we are in the situation of (e). □

The following lemma was needed in the proof of Proposition 1.10 to reduce still further the prime powers under consideration.

Lemma 1.11. *Fix a prime p, and an integer $q > 1$ prime to p.*

(a) *If p is odd, then for any prime r_0 whose class generates $(\mathbb{Z}/p^2)^\times$, there is $b \geq 1$ such that $\overline{\langle q \rangle} = \overline{\langle (r_0)^b \rangle}$, and $b|(p-1)p^\ell$ for some ℓ.*

(b) *If $p = 2$, then either $\overline{\langle q \rangle} = \overline{\langle 3 \rangle}$, or $\overline{\langle q \rangle} = \overline{\langle 5 \rangle}$, or there are $\varepsilon = \pm 1$ and $k \geq 1$ such that $\varepsilon \equiv q \pmod 8$ and $\overline{\langle q \rangle} = \overline{\langle \varepsilon \cdot 3^{2^k} \rangle}$.*

Proof. Since $q \in \mathbb{Z}$ and $q > 1$, $\overline{\langle q \rangle}$ is infinite.

(a) If p is odd, then for each $n \geq 1$, $(\mathbb{Z}/p^n)^\times \cong (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p^{n-1})$ is cyclic and generated by the class of r_0. Hence $\mathbb{Z}_p^\times \cong (\mathbb{Z}/p)^\times \times (\mathbb{Z}_p, +)$, and $\overline{\langle r_0 \rangle} = \mathbb{Z}_p^\times$. Also, $\overline{\langle q \rangle} \geq 1 + p^\ell \mathbb{Z}_p$ for some $\ell \geq 1$, since each infinite, closed subgroup of $(\mathbb{Z}_p, +)$ contains $p^k\mathbb{Z}_p$ for some k.

Set $b = [\mathbb{Z}_p^\times : \overline{\langle q \rangle}] = [(\mathbb{Z}/p^\ell)^\times : \langle q + p^\ell \mathbb{Z} \rangle] \bigm| (p-1)p^{\ell-1}$. Then $\overline{\langle q \rangle} = \overline{\langle (r_0)^b \rangle}$.

(b) If $p = 2$, then $\mathbb{Z}_2^\times = \{\pm 1\} \times \overline{\langle 3 \rangle}$, where $\overline{\langle 3 \rangle} \cong (\mathbb{Z}_2, +)$. Hence the only infinite closed subgroups of $\overline{\langle 3 \rangle}$ are those of the form $\overline{\langle 3^{2^k} \rangle}$ for some $k \geq 0$. So $\overline{\langle q \rangle} = \overline{\langle \varepsilon \cdot 3^{2^k} \rangle}$ for some $k \geq 0$ and some $\varepsilon = \pm 1$, and the result follows since $\overline{\langle 5 \rangle} = \overline{\langle -3 \rangle}$. □

We also note, for use in Section 4, the following more technical result.

Lemma 1.12. *Let G be a finite group, fix $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Let $P \leq S$ be such that $C_G(P) \leq P$ and $N_S(P) \in \mathrm{Syl}_p(N_G(P))$. Then for each $\varphi \in \mathrm{Aut}(S, \mathcal{F})$ such that $\varphi(P) = P$, $\varphi|_{N_S(P)}$ extends to an automorphism $\bar{\varphi}$ of $N_G(P)$.*

Proof. Since $C_G(P) \leq P$ and $N_S(P) \in \mathrm{Syl}_p(N_G(P))$, $N_G(P)$ is a model for the fusion system $\mathcal{E} = \mathcal{F}_{N_S(P)}(N_G(P))$ in the sense of [AKO, Definition I.4.8]. By the strong uniqueness property for models [AKO, Theorem I.4.9(b)], and since $\varphi|_{N_S(P)}$ preserves fusion in $\mathcal{E}$, $\varphi|_{N_S(P)}$ extends to an automorphism of the model. □

The following elementary lemma will be useful in Sections 5 and 6; for example, when computing orders of Sylow subgroups of groups of Lie type.

Lemma 1.13. *Fix a prime p. Assume $q \equiv 1 \pmod p$, and $q \equiv 1 \pmod 4$ if $p = 2$. Then for each $n \geq 1$, $v_p(q^n - 1) = v_p(q - 1) + v_p(n)$.*

Proof. Set $r = v_p(q - 1)$, and let k be such that $q = 1 + p^r k$. Then $q^n = 1 + np^r k + \xi$, where $v_p(np^r k) = v_p(n) + r$, and where each term in ξ has strictly larger valuation. □

2. Background on finite groups of Lie type

In this section and the next, we fix the notation to be used for finite groups of Lie type, and list some of the (mostly standard) results which will be needed later. We begin by recalling the following concepts used in [GLS3]. We do not repeat the definitions of maximal tori and Borel subgroups in algebraic groups, but refer instead to [GLS3, §§ 1.4–1.6].

Definition 2.1 ([GLS3, Definitions 1.7.1, 1.15.1, 2.2.1]). *Fix a prime q_0.*

(a) *A connected algebraic group $\overline{G}$ over $\overline{\mathbb{F}}_{q_0}$ is* simple *if $[\overline{G},\overline{G}] \neq 1$, and all proper closed normal subgroups of $\overline{G}$ are finite and central. If $\overline{G}$ is simple, then it is of* universal type *if it is simply connected, and of* adjoint type *if $Z(\overline{G}) = 1$.*

(b) *A* Steinberg endomorphism *of a connected simple algebraic group $\overline{G}$ is a surjective algebraic endomorphism $\sigma \in \mathrm{End}(\overline{G})$ whose fixed subgroup is finite.*

(c) *A* σ-setup *for a finite group G is a pair $(\overline{G}, \sigma)$, where $\overline{G}$ is a simple algebraic group over $\overline{\mathbb{F}}_{q_0}$, and where σ is a Steinberg endomorphism of $\overline{G}$ such that $G = O^{q_0'}(C_{\overline{G}}(\sigma))$.*

(d) *Let $\mathfrak{Lie}(q_0)$ denote the class of finite groups with σ-setup $(\overline{G}, \sigma)$ where $\overline{G}$ is simple and is defined in characteristic q_0, and let $\mathfrak{Lie}$ be the union of the classes $\mathfrak{Lie}(q_0)$ for all primes q_0. We say that G is of universal (adjoint) type if $\overline{G}$ is of universal (adjoint) type.*

If $\overline{G}$ is universal, then $C_{\overline{G}}(\sigma)$ is generated by elements of q_0-power order (see [St3, Theorem 12.4]), and hence $G = C_{\overline{G}}(\sigma)$ in (c) above. In general, $C_{\overline{G}}(\sigma) = G \cdot C_{\overline{T}}(\sigma)$ (cf. [GLS3, Theorem 2.2.6]).

A *root group* in a connected algebraic group $\overline{G}$ over $\overline{\mathbb{F}}_{q_0}$ with a given maximal torus $\overline{T}$ is a one-parameter closed subgroup (thus isomorphic to $\overline{\mathbb{F}}_{q_0}$) which is normalized by $\overline{T}$. The *roots* of $\overline{G}$ are the characters for the $\overline{T}$-actions on the root groups, and lie in the $\mathbb{Z}$-lattice $X(\overline{T}) = \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_{q_0}^\times)$ of characters of $\overline{T}$. (Note that this is the group of *algebraic* homomorphisms, and that $\mathrm{Hom}(\overline{\mathbb{F}}_{q_0}^\times, \overline{\mathbb{F}}_{q_0}^\times) \cong \mathbb{Z}$.) The roots are regarded as lying in the $\mathbb{R}$-vector space $V = \mathbb{R} \otimes_{\mathbb{Z}} \overline{T}^*$. We refer to [GLS3, § 1.9] for details about roots and root subgroups of algebraic groups, and to [Brb, Chapitre VI] for a detailed survey of root systems.

The following notation and hypotheses will be used throughout this paper, when working with a finite group of Lie type defined via a σ-setup.

Notation 2.2. *Let $(\overline{G}, \sigma)$ be a σ-setup for the finite group G, where $\overline{G}$ is a connected, simple algebraic group over $\overline{\mathbb{F}}_{q_0}$ for a prime q_0. When convenient, we also write $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_{q_0})$, where $\mathbb{G}$ is a group scheme over $\mathbb{Z}$.*

(A) ***The maximal torus and Weyl group of $\overline{G}$.*** *Fix a maximal torus $\overline{T}$ in $\overline{G}$ such that $\sigma(\overline{T}) = \overline{T}$. Let $W = N_{\overline{G}}(\overline{T})/\overline{T}$ be the Weyl group of $\overline{G}$ (and of $\mathbb{G}$).*

(B) ***The root system of $\overline{G}$.*** *Let Σ be the set of all roots of $\overline{G}$ with respect to $\overline{T}$, and let $\overline{X}_\alpha < \overline{G}$ denote the root group for the root $\alpha \in \Sigma$. Thus $\overline{X}_\alpha = \{x_\alpha(u) \,|\, u \in \overline{\mathbb{F}}_{q_0}\}$ with respect to some fixed Chevalley parametrization of $\overline{G}$. Set $V = \mathbb{R} \otimes_{\mathbb{Z}} \overline{T}^*$: a real vector space with inner product $(-,-)$ upon which the Weyl group W acts orthogonally. Let $\Pi \subseteq \Sigma$ be a fundamental system of roots, and let $\Sigma_+ \subseteq \Sigma$ be the set of positive roots with respect to Π. For each $\alpha \in \Sigma_+$, let $\mathrm{ht}(\alpha)$ denote the* height *of α: the number of summands in the decomposition of α as a sum of fundamental roots.*

For each $\alpha \in \Sigma$, *let* $w_\alpha \in W$ *be the reflection in the hyperplane* $\alpha^\perp \subseteq V$.

For $\alpha \in \Sigma$ *and* $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$, *let* $n_\alpha(\lambda) \in \langle \overline{X}_\alpha, \overline{X}_{-\alpha} \rangle$ *and* $h_\alpha(\lambda) \in \bar{T} \cap \langle \overline{X}_\alpha, \overline{X}_{-\alpha} \rangle$ *be as defined in* [Ca, § 6.4] *or* [GLS3, Theorem 1.12.1]*: the images of* $\left(\begin{smallmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, *respectively, under the homomorphism* $SL_2(\bar{\mathbb{F}}_{q_0}) \longrightarrow \overline{G}$ *which sends* $\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ *to* $x_\alpha(u)$ *and* $\left(\begin{smallmatrix} 1 & 0 \\ v & 1 \end{smallmatrix}\right)$ *to* $x_{-\alpha}(v)$. *Equivalently,* $n_\alpha(\lambda) = x_\alpha(\lambda)x_{-\alpha}(-\lambda^{-1})x_\alpha(\lambda)$ *and* $h_\alpha(\lambda) = n_\alpha(\lambda)n_\alpha(1)^{-1}$.

(C) ***The maximal torus, root system and Weyl group of G.*** *Set* $T = \bar{T} \cap G$.
Let $\tau \in \mathrm{Aut}(V)$ *and* $\rho \in \mathrm{Aut}(\Sigma)$ *be the orthogonal automorphism and permutation, respectively, such that for each* $\alpha \in \Sigma$, $\sigma(\overline{X}_\alpha) = \overline{X}_{\rho(\alpha)}$ *and* $\rho(\alpha)$ *is a positive multiple of* $\tau(\alpha)$. *Set* $W_0 = C_W(\tau)$.
If $\rho(\Pi) = \Pi$, *then set* $V_0 = C_V(\tau)$, *and let* $\mathrm{pr}_{V_0}^\perp$ *be the orthogonal projection of* V *onto* V_0. *Let* $\widehat{\Sigma}$ *be the set of equivalence classes in* Σ *determined by* τ, *where* $\alpha, \beta \in \Sigma$ *are equivalent if* $\mathrm{pr}_{V_0}^\perp(\alpha)$ *is a positive scalar multiple of* $\mathrm{pr}_{V_0}^\perp(\beta)$ *(see* [GLS3, Definition 2.3.1] *or* [Ca, § 13.2]*). Let* $\widehat{\Pi} \subseteq \widehat{\Sigma}_+$ *denote the images in* $\widehat{\Sigma}$ *of* $\Pi \subseteq \Sigma_+$.
For each $\widehat{\alpha} \in \widehat{\Sigma}$, *set* $\overline{X}_{\widehat{\alpha}} = \langle \overline{X}_\alpha \,|\, \alpha \in \widehat{\alpha} \rangle$ *and* $X_{\widehat{\alpha}} = C_{\overline{X}_{\widehat{\alpha}}}(\sigma)$. *When* $\alpha \in \Sigma$ *is of minimal height in its class* $\widehat{\alpha} \in \widehat{\Sigma}$, *and* $q' = |X_{\widehat{\alpha}}^{\mathrm{ab}}|$, *then for* $u \in \mathbb{F}_{q'}$, *let* $\widehat{x}_\alpha(u) \in X_{\widehat{\alpha}}$ *be an element whose image under projection to* X_α *is* $x_\alpha(u)$ *(uniquely determined modulo* $[X_{\widehat{\alpha}}, X_{\widehat{\alpha}}]$*).*
For $\alpha \in \Pi$ *and* $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$, *let* $\widehat{h}_\alpha(\lambda) \in T$ *be an element in* $G \cap \langle h_\beta(\bar{\mathbb{F}}_{q_0}^\times) \,|\, \beta \in \widehat{\alpha} \rangle$ *whose component in* $h_\alpha(\bar{\mathbb{F}}_{q_0}^\times)$ *is* $h_\alpha(\lambda)$ *(if there is such an element).*

To see that τ and ρ exist as defined in point (C), recall that the root groups $\overline{X}_\alpha$ for $\alpha \in \Sigma$ are the unique closed subgroups of $\overline{G}$ which are isomorphic to $(\bar{\mathbb{F}}_{q_0}, +)$ and normalized by $\bar{T}$ (see, e.g., [GLS3, Theorem 1.9.5(a,b)]). Since σ is algebraic (hence continuous) and bijective, σ^{-1} sends root subgroups to root subgroups, and σ permutes the root subgroups (hence the roots) since there are only finitely many of them. Using Chevalley's commutator formula, one sees that this permutation ρ of Σ preserves angles between roots, and hence (up to positive scalar multiple) extends to an orthogonal automorphism of V.

These definitions of $\widehat{x}_\alpha(u) \in X_{\widehat{\alpha}}$ and $\widehat{h}_\alpha(\lambda) \in T$ are slightly different from the definitions in [GLS3, § 2.4] of elements $x_{\widehat{\alpha}}(u)$ and $h_{\widehat{\alpha}}(\lambda)$. We choose this notation to emphasize that these elements depend on the choice of $\alpha \in \Sigma$, not only on its class $\widehat{\alpha} \in \widehat{\Sigma}$. This will be important in some of the relations we need to use in Section 4.

Lemma 2.3. *Under the assumptions of Notation 2.2, the action of* W *on* $\bar{T}$ *restricts to an action of* W_0 *on* T, *and the natural isomorphism* $N_{\overline{G}}(\bar{T})/\bar{T} \cong W$ *restricts to an isomorphism*

$$\big(N_G(T) \cap N_{\overline{G}}(\bar{T})\big)/T \cong C_W(\tau) = W_0\,.$$

Proof. For each $\alpha \in \Sigma$, $n_\alpha(1) = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ represents the reflection $w_\alpha \in W$, and hence $\sigma(n_\alpha) \in \langle X_{\rho(\alpha)}, X_{-\rho(\alpha)} \rangle \cap N_{\overline{G}}(\bar{T})$ represents the reflection $w_{\rho(\alpha)} = {}^{\tau}(w_\alpha)$. Since W is generated by the w_α for $\alpha \in \Sigma$, we conclude that σ and τ have the same action on W.

Thus the identification $N_{\overline{G}}(\bar{T})/\bar{T} \cong W$ restricts to the following inclusions:

$$\big(N_G(T) \cap N_{\overline{G}}(\bar{T})\big)/T \le C_{N_{\overline{G}}(\bar{T})}(\sigma)/C_{\bar{T}}(\sigma) \le C_{N_{\overline{G}}(\bar{T})/\bar{T}}(\sigma) \cong C_W(\tau) = W_0\,.$$

If $w \in W_0$ represents the coset $x\bar{T} \subseteq N_{\overline{G}}(\bar{T})$, then $x^{-1}\sigma(x) \in \bar{T}$. By the Lang-Steinberg theorem, each element of $\bar{T}$ has the form $t^{-1}\sigma(t)$ for some $t \in \bar{T}$, and hence we can choose

x such that $\sigma(x) = x$. Then $x \in C_{\bar{G}}(\sigma)$, and hence x normalizes $G = O^{q_0'}(C_{\bar{G}}(\sigma))$ and $T = G \cap \bar{T}$. Since $C_{\bar{G}}(\sigma) = GC_{\bar{T}}(\sigma)$ (see [GLS3, Theorem 2.2.6(g)] or [St3, Corollary 12.3(a)]), some element of $x\bar{T}$ lies in $N_G(T)$. So the above inclusions are equalities. □

The roots in $\bar{G}$ are defined formally as characters of its maximal torus $\bar{T}$. But it will be useful to distinguish the (abstract) root $\alpha \in \Sigma$ from the character $\theta_\alpha \in \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times) \subseteq V$.

For each root $\alpha \in \Sigma \subseteq V$, let $\alpha^\vee \in V^*$ be the corresponding co-root (dual root): the unique element such that $(\alpha^\vee, \alpha) = 2$ and w_α is reflection in the hyperplane $\mathrm{Ker}(\alpha^\vee)$. Since we identify $V = V^*$ via a W-invariant inner product, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Point (c) of the next lemma says that $\alpha^\vee = h_\alpha$, when we regard $h_\alpha \in \mathrm{Hom}(\bar{\mathbb{F}}_{q_0}^\times, \bar{T})$ as an element in V^*.

Lemma 2.4. *Assume we are in the situation of* (A) *and* (B) *in Notation 2.2.*

(a) *We have $C_{\bar{G}}(\bar{T}) = \bar{T}$. In particular, $Z(\bar{G}) \le \bar{T}$, and is finite of order prime to the defining characteristic q_0.*

(b) *The maximal torus $\bar{T}$ in $\bar{G}$ is generated by the elements $h_\alpha(\lambda)$ for $\alpha \in \Pi$ and $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$. If $\bar{G}$ is universal, and $\lambda_\alpha \in \bar{\mathbb{F}}_{q_0}$ are such that $\prod_{\alpha\in\Pi} h_\alpha(\lambda_\alpha) = 1$, then $\lambda_\alpha = 1$ for each $\alpha \in \Pi$. Thus*
$$\bar{T} = \prod_{\alpha\in\Pi} h_\alpha(\bar{\mathbb{F}}_{q_0}^\times),$$
and h_α is injective for each α.

(c) *For each $\beta \in \Sigma$, let $\theta_\beta \in X(\bar{T}) = \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$ be the character such that*
$$^{t}x_\beta(u) = x_\beta(\theta_\beta(t){\cdot}u)$$
for $t \in \bar{T}$ and $u \in \bar{\mathbb{F}}_{q_0}$. Then
$$\theta_\beta(h_\alpha(\lambda)) = \lambda^{(\alpha^\vee,\beta)} \qquad \text{for } \beta, \alpha \in \Sigma,\ \lambda \in \bar{\mathbb{F}}_{q_0}^\times.$$
The product homomorphism $\theta_\Pi = \prod \theta_\beta \colon \bar{T} \longrightarrow \prod_{\beta\in\Pi} \bar{\mathbb{F}}_{q_0}^\times$ is surjective, and $\mathrm{Ker}(\theta_\Pi) = Z(\bar{G})$.

(d) *If $\alpha, \beta_1, \ldots, \beta_k \in \Sigma$ and $n_1, \ldots, n_k \in \mathbb{Z}$ are such that $\alpha^\vee = n_1\beta_1^\vee + \ldots + n_k\beta_k^\vee$, then for each $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$, $h_\alpha(\lambda) = h_{\beta_1}(\lambda^{n_1}) \cdots h_{\beta_k}(\lambda^{n_k})$.*

(e) *For each $w \in W$, $\alpha \in \Sigma$, and $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$, and each $n \in N_{\bar{G}}(\bar{T})$ such that $n\bar{T} = w \in N_{\bar{G}}(\bar{T})/\bar{T} \cong W$, ${}^{n}(\bar{X}_\alpha) = \bar{X}_{w(\alpha)}$ and ${}^{n}(h_\alpha(\lambda)) = h_{w(\alpha)}(\lambda)$. For each $\alpha, \beta \in \Sigma$ and each $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$,*
$$w_\alpha(h_\beta(\lambda)) = h_{w_\alpha(\beta)}(\lambda) = h_\beta(\lambda)h_\alpha(\lambda^{-(\beta^\vee,\alpha)}).$$
Hence $w_\alpha(t) = t \cdot h_\alpha(\theta_\alpha(t))^{-1}$ for each $t \in \bar{T}$.

Proof. **(a)** By [Hu, Proposition 24.1.A], the maximal torus $\bar{T}$ is regular (i.e., contained in only finitely many Borel subgroups). So $C_{\bar{G}}(\bar{T}) = \bar{T}$ by [Hu, Corollary 26.2.A]. Hence $Z(\bar{G}) \le \bar{T}$, it is finite since $\bar{G}$ is assumed simple, and so it has order prime to the defining characteristic q_0.

We claim that it suffices to prove the relations in (c)–(e) in the adjoint group $\bar{G}/Z(\bar{G})$, and hence that we can use the results in [Ca, §§ 7.1–2]. For relations in $\bar{T}$, this holds since $\bar{T}$ is infinitely divisible and $Z(\bar{G})$ is finite (thus each homomorphism to $\bar{T}/Z(\bar{G})$ has at most one lifting to $\bar{T}$). For relations in a root group $\overline{X}_\alpha$, this holds since each element of $\overline{X}_\alpha Z(\overline{G})$ of order q_0 lies in $\overline{X}_\alpha$, since $|Z(\overline{G})|$ is prime to q_0 by (a).

(b) This is stated without proof in [GLS3, Theorem 1.12.5(b)], and with a brief sketch of a proof in [St4, p. 122]. We show here how it follows from the classification of reductive algebraic groups in terms of root data (see, e.g., [Sp, § 10]).

Consider the homomorphism

$$h_\Pi\colon \widetilde{T} \stackrel{\text{def}}{=} \prod_{\alpha\in\Pi} \bar{\mathbb{F}}_{q_0}^\times \xrightarrow{\qquad} \bar{T}$$

which sends $(\lambda_\alpha)_{\alpha\in\Pi}$ to $\prod_\alpha h_\alpha(\lambda_\alpha)$. Then h_Π is surjective with finite kernel (see [Ca, § 7.1]). It remains to show that it is an isomorphism when G is of universal type.

We recall some of the notation used in [Sp, § 7]. To $\overline{G}$ is associated the root datum $\big(X(\bar{T}), \Sigma, X^\vee(\bar{T}), \Sigma^\vee\big)$, where

$$X(\bar{T}) = \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times), \quad X^\vee(\bar{T}) = \mathrm{Hom}(\bar{\mathbb{F}}_{q_0}^\times, \bar{T}), \quad \Sigma^\vee = \{\alpha^\vee = h_\alpha \,|\, \alpha\in\Sigma\} \subseteq X^\vee(\bar{T})\,.$$

As noted before, $X(\bar{T})$ and $X^\vee(\bar{T})$ are groups of algebraic homomorphisms, and are free abelian groups of finite rank dual to each other. Recall that $\Sigma \subseteq X(\bar{T})$, since we identify a root α with the character θ_α.

Set $Y^\vee = \mathbb{Z}\Sigma^\vee \subseteq X^\vee(\bar{T})$, and let $Y \supseteq X(\bar{T})$ be its dual. Then $(Y, \Sigma, Y^\vee, \Sigma^\vee)$ is still a root datum as defined in [Sp, § 7.4]. By [Sp, Proposition 10.1.3] and its proof, it is realized by a connected algebraic group $\widetilde{G}$ with maximal torus $\widetilde{T}$, which lies in a central extension $f\colon \widetilde{G} \longrightarrow \overline{G}$ which extends h_Π. Since $\overline{G}$ is of universal type, f and hence h_Π are isomorphisms.

(c) Let $\mathbb{Z}\Sigma \le V$ be the additive subgroup generated by Σ. In the notation of [Ca, pp. 97–98], for each $\alpha\in\Sigma$ and $\lambda\in\bar{\mathbb{F}}_{q_0}^\times$, $h_\alpha(\lambda) = h(\chi_{\alpha,\lambda})$ where

$$\chi_{\alpha,\lambda}\in \mathrm{Hom}(\mathbb{Z}\Sigma, \bar{\mathbb{F}}_{q_0}^\times) \qquad \text{is defined by} \qquad \chi_{\alpha,\lambda}(v) = \lambda^{2(\alpha,v)/(\alpha,\alpha)} = \lambda^{(\alpha^\vee,v)}.$$

Also, by [Ca, p. 100], for each $\chi\in \mathrm{Hom}(\mathbb{Z}\Sigma, \bar{\mathbb{F}}_{q_0}^\times)$, $\beta\in\Sigma$, and $u\in\bar{\mathbb{F}}_{q_0}$, ${}^{h(\chi)}x_\beta(u) = x_\beta(\chi(\beta){\cdot}u)$. Thus there are homomorphisms $\theta_\beta \in \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$, for each $\beta\in\Sigma$, such that ${}^t x_\beta(u) = x_\beta(\theta_\beta(t){\cdot}u)$, and $\theta_\beta(h(\chi)) = \chi(\beta)$ for each χ. For each $\alpha\in\Sigma$ and $\lambda\in\bar{\mathbb{F}}_{q_0}^\times$,

$$\theta_\beta(h_\alpha(\lambda)) = \theta_\beta(h(\chi_{\alpha,\lambda})) = \chi_{\alpha,\lambda}(\beta) = \lambda^{(\alpha^\vee,\beta)}\ . \tag{1}$$

Assume $t\in\mathrm{Ker}(\theta_\Pi)$. Thus $t\in\mathrm{Ker}(\theta_\alpha)$ for all $\alpha\in\Pi$, and hence for all $\alpha\in\Sigma\subseteq\mathbb{Z}\Pi$. So $[t, X_\alpha] = 1$ for all $\alpha\in\Sigma$, these root subgroups generate $\overline{G}$ (see [Sp, Corollary 8.2.10]), and this proves that $t\in Z(\overline{G})$. The converse is clear: $t\in Z(\overline{G})$ implies $t\in\bar{T}$ by (a), and hence $\theta_\beta(t) = 1$ for all $\beta\in\Pi$ by definition of θ_β.

It remains to show that θ_Π sends $\bar{T}$ onto $\prod_{\beta\in\Pi}\bar{\mathbb{F}}_{q_0}$. Consider the homomorphisms

$$\widetilde{T} \stackrel{\text{def}}{=} \prod_{\alpha\in\Pi}\bar{\mathbb{F}}_{q_0}^\times \xrightarrow{\ h_\Pi\ } \bar{T} \xrightarrow{\ \theta_\Pi\ } \prod_{\beta\in\Pi}\bar{\mathbb{F}}_{q_0}^\times\,, \tag{2}$$

where h_Π was defined in the proof of (b). We just saw that $\theta_\Pi\circ h_\Pi$ has matrix $\big((\alpha^\vee,\beta)\big)_{\alpha,\beta\in\Pi}$, which has nonzero determinant since $\Pi\subseteq V$ and $\Pi^\vee\subseteq V^*$ are bases. Since $\bar{\mathbb{F}}_{q_0}^\times$ is divisible and its finite subgroups are cyclic, this implies that $\theta_\Pi\circ h_\Pi$ is onto, and hence θ_Π is onto.

(d) This follows immediately from (c), where we showed, for $\alpha \in \Sigma$, that $\alpha^\vee$ can be identified with h_α in $\mathrm{Hom}(\bar{\mathbb{F}}_{q_0}^\times, \bar{T}) \subseteq V^*$.

(e) The first statement (${}^n(\bar{X}_\alpha) = \bar{X}_{w(\alpha)}$ and ${}^n(h_\alpha(\lambda)) = h_{w(\alpha)}(\lambda)$) is shown in [Ca, Lemma 7.2.1(ii) & Theorem 7.2.2]. By the usual formula for an orthogonal reflection, $w_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha = \beta - (\alpha^\vee, \beta)\alpha$. Here, we regard w_α as an automorphism of V (not of $\bar{T}$). Since $w_\alpha(\beta)$ and β have the same norm,

$$w_\alpha(\beta)^\vee = \frac{2w_\alpha(\beta)}{(\beta,\beta)} = \frac{2\beta}{(\beta,\beta)} - \frac{2(\alpha,\beta)}{(\beta,\beta)} \cdot \frac{2\alpha}{(\alpha,\alpha)} = \beta^\vee - (\beta^\vee, \alpha) \cdot \alpha^\vee ,$$

and by (d),

$$w_\alpha(h_\beta(\lambda)) = h_{w_\alpha(\beta)}(\lambda) = h_\beta(\lambda) h_\alpha(\lambda^{-(\beta^\vee,\alpha)}) = h_\beta(\lambda) h_\alpha(\theta_\alpha(h_\beta(\lambda))^{-1})$$

where the last equality follows from (c). Since $\bar{T}$ is generated by the $h_\beta(\lambda)$ by (b), this implies that $w_\alpha(t) = t \cdot h_\alpha(\theta_\alpha(t))^{-1}$ for all $t \in \bar{T}$. □

For any algebraic group H, H^0 denotes its identity connected component. The following proposition holds for any connected, reductive group, but we state it only in the context of Notation 2.2. Recall the homomorphisms $\theta_\beta \in \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$, defined for $\beta \in \Sigma$ in Lemma 2.4(c).

Proposition 2.5. *Assume Notation 2.2. For any subgroup $H \le \bar{T}$, $C_{\bar{G}}(H)$ is an algebraic group, $C_{\bar{G}}(H)^0$ is reductive, and*

$$\begin{aligned} C_{\bar{G}}(H)^0 &= \langle \bar{T}, X_\alpha \,|\, \alpha \in \Sigma,\ H \le \mathrm{Ker}(\theta_\alpha) \rangle \\ C_{\bar{G}}(H) &= C_{\bar{G}}(H)^0 \cdot \{ g \in N_{\bar{G}}(\bar{T}) \,|\, [g, H] = 1 \} . \end{aligned} \tag{3}$$

If, furthermore, $\bar{G}$ is of universal type, then $Z(\bar{G}) = C_{\bar{T}}(W)$.

Proof. The description of $C_{\bar{G}}(H)^0$ is shown in [Ca2, Theorem 3.5.3] when H is finite and cyclic, and the proof given there also applies in the more general case. For each $g \in C_{\bar{G}}(H)$, $c_g(\bar{T})$ is another maximal torus in $C_{\bar{G}}(H)^0$, so $gh \in C_{N_{\bar{G}}(\bar{T})}(H)$ for some $h \in C_{\bar{G}}(T)^0$, and thus $C_{\bar{G}}(H) = C_{\bar{G}}(H)^0 \cdot C_{N_{\bar{G}}(\bar{T})}(H)$.

Assume $\bar{G}$ is of universal type. Since $Z(\bar{G}) \le \bar{T}$ by Lemma 2.4(a), we have $Z(\bar{G}) \le C_{\bar{T}}(W)$. Conversely, by Lemma 2.4(b), for each $t \in \bar{T}$ and each $\alpha \in \Sigma$, ${}^t(x_\alpha(u)) = x_\alpha(\theta_\alpha(t)u)$, and $\theta_{-\alpha}(t) = \theta_\alpha(t)^{-1}$. Hence also ${}^t(n_\alpha(1)) = n_\alpha(\theta_\alpha(t))$ (see the formula for $n_\alpha(\lambda)$ in Notation 2.2(B)). If $t \in C_{\bar{T}}(W)$, then $[t, n_\alpha(1)] = 1$ for each α, and since $\bar{G}$ is of universal type, $\langle \bar{X}_\alpha, \bar{X}_{-\alpha} \rangle \cong SL_2(\bar{\mathbb{F}}_{q_0})$. Thus $\theta_\alpha(t) = 1$ for all $\alpha \in \Sigma$, t acts trivially on all root subgroups, and so $t \in Z(\bar{G})$. □

We now look more closely at the lattice $\mathbb{Z}\Sigma^\vee$ generated by the dual roots.

Lemma 2.6. *Assume Notation 2.2(A,B), and also that $\mathbb{G}$ (and hence $\bar{G}$) is of universal type.*

(a) *There is an isomorphism*

$$\Phi \colon \mathbb{Z}\Sigma^\vee \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_{q_0}^\times \longrightarrow \bar{T}$$

with the property that $\Phi(\alpha^\vee \otimes \lambda) = h_\alpha(\lambda)$ for each $\alpha \in \Sigma$ and each $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$.

Fix some $\lambda\in\bar{\mathbb{F}}_{q_0}^\times$*, and set* $m=|\lambda|$*. Set* $\Phi_\lambda=\Phi(-,\lambda)\colon \mathbb{Z}\Sigma^\vee \longrightarrow \bar{T}$*.*

(b) *The map* Φ_λ *is* $\mathbb{Z}[W]$*-linear,* $\operatorname{Ker}(\Phi_\lambda)=m\mathbb{Z}\Sigma^\vee$*, and* $\operatorname{Im}(\Phi_\lambda)=\{t\in\bar{T}\,|\,t^m=1\}$*.*

(c) *Fix* $t\in\bar{T}$ *and* $x\in\mathbb{Z}\Sigma^\vee$ *such that* $\Phi_\lambda(x)=t$*, and also such that*

$$\|x\|<\tfrac{1}{2}m\cdot\min\bigl\{\|\alpha^\vee\|\,\bigm|\,\alpha\in\Pi\bigr\}.$$

Then $C_W(t)=C_W(x)$*.*

(d) *If* $m=|\lambda|\geq 4$*, then for each* $\alpha\in\Sigma$*,* $C_W(h_\alpha(\lambda))=C_W(\alpha)$*.*

Proof. **(a,b)** Identify $\mathbb{Z}\Sigma^\vee$ as a subgroup of $\operatorname{Hom}(\bar{\mathbb{F}}_{q_0}^\times,\bar{T})$, and let

$$\bar{\Phi}\colon \mathbb{Z}\Sigma^\vee\times\bar{\mathbb{F}}_{q_0}^\times \longrightarrow \bar{T}$$

be the evaluation pairing. This is bilinear, hence induces a homomorphism on the tensor product, and $\bar{\Phi}(\alpha^\vee,\lambda)=h_\alpha(\lambda)$ by Lemma 2.4(c). Since $\{\alpha^\vee\,|\,\alpha\in\Pi\}$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\Sigma^\vee$ (since $\Sigma^\vee$ is a root system by [Brb, § VI.1, Proposition 2]), and since $\bar{G}$ is of universal type, Φ is an isomorphism by Lemma 2.4(b).

In particular, for fixed $\lambda\in\bar{\mathbb{F}}_{q_0}^\times$ of order m, $\Phi(-,\lambda)$ induces an isomorphism from the quotient group $\mathbb{Z}\Sigma^\vee/m\mathbb{Z}\Sigma^\vee$ onto the m-torsion subgroup of $\bar{T}$.

(c) Clearly, $C_W(x)\leq C_W(t)$; it remains to prove the opposite inclusion. Fix $w\in C_W(t)$. By (a), $w(x)\equiv x \pmod{m\mathbb{Z}\Sigma^\vee}$.

Set $r=\min\bigl\{\|\alpha^\vee\|\,\bigm|\,\alpha\in\Pi\bigr\}$. For each $\alpha\in\Sigma$, $\|\alpha^\vee\|=\sqrt{k}\cdot r$ for some $k=1,2,3$, and hence $(\alpha^\vee,\alpha^\vee)\in r^2\mathbb{Z}$. For each $\alpha,\beta\in\Sigma$, $2(\alpha^\vee,\beta^\vee)/(\alpha^\vee,\alpha^\vee)\in\mathbb{Z}$ (cf. [Ca, Definition 2.1.1]), and hence $(\alpha^\vee,\beta^\vee)\in\frac{1}{2}r^2\mathbb{Z}$. Thus $(x,x)\in r^2\mathbb{Z}$ for each $x\in\mathbb{Z}\Sigma^\vee$, and in particular, $\min\bigl\{\|x\|\,\bigm|\,0\neq x\in\mathbb{Z}\Sigma^\vee\bigr\}=r$.

By assumption, $\|w(x)\|=\|x\|<mr/2$, so $\|w(x)-x\|<mr$. Since each nonzero element in $m\mathbb{Z}\Sigma^\vee$ has norm at least mr, this proves that $w(x)-x=0$, and hence that $w\in C_W(x)$.

(d) This is the special case of (b), where $x=\alpha^\vee$ and $t=h_\alpha(\lambda)$. □

Lemma 2.7. *Assume Notation 2.2, and assume also that* $\mathbb{G}$ *is of universal type. Let* $\Gamma<\operatorname{Aut}(V)$ *be any finite group of isometries of* (V,Σ)*. Then there is an action of* Γ *on* $\bar{T}$*, where* $g(h_\alpha(u))=h_{g(\alpha)}(u)$ *for each* $g\in\Gamma$*,* $\alpha\in\Sigma$*, and* $u\in\bar{\mathbb{F}}_{q_0}^\times$*. Fix* $m\geq 3$ *such that* $q_0\nmid m$*, and set* $T_m=\{t\in\bar{T}\,|\,t^m=1\}$*. Then* Γ *acts faithfully on* T_m*. If* $1\neq g\in\Gamma$ *and* $\ell\in\mathbb{Z}$ *are such that* $g(t)=t^\ell$ *for each* $t\in T_m$*, then* $\ell\equiv -1 \pmod{m}$*.*

Proof. The action of Γ on $\bar{T}$ is well defined by the relations in Lemma 2.4(d,b).

Now fix $m\geq 3$ prime to q_0, and let $T_m<\bar{T}$ be the m-torsion subgroup. It suffices to prove the rest of the lemma when $m=p$ is an odd prime, or when $m=4$ and $p=2$. Fix $\lambda\in\bar{\mathbb{F}}_{q_0}^\times$ of order m, and let $\Phi_\lambda\colon\mathbb{Z}\Sigma^\vee\longrightarrow\bar{T}$ be the homomorphism of Lemma 2.6(a). By definition of Φ_λ, it commutes with the actions of Γ on $\mathbb{Z}\Sigma^\vee<V$ and on T_m.

Assume $1\neq g\in\Gamma$ and $\ell\in\mathbb{Z}$ are such that $g(t)=t^\ell$ for each $t\in T_m$. Set $r=\dim(V)$, and let $B\in GL_r(\mathbb{Z})$ be the matrix for the action of g on $\mathbb{Z}\Sigma^\vee$, with respect to some $\mathbb{Z}$-basis of $\mathbb{Z}\Sigma^\vee$. Then $|g|=|B|$, and $B\equiv \ell I \pmod{mM_r(\mathbb{Z})}$. If $p=2$ ($m=4$), let $\mu\in\{\pm1\}$ be such that $\ell\equiv\mu \pmod 4$. If p is odd (so $m=p$), then let $\mu\in(\mathbb{Z}_p)^\times$ be such that $\mu\equiv\ell \pmod p$ and $\mu^{p-1}=1$. Set $B'=\mu^{-1}B\in GL_r(\mathbb{Z}_p)$. Thus B' also has finite order, and $B'\equiv I \pmod{mM_r(\mathbb{Z}_p)}$.

The logarithm and exponential maps define inverse bijections

$$I + mM_r(\mathbb{Z}_p) \underset{\exp}{\overset{\ln}{\rightleftarrows}} mM_r(\mathbb{Z}_p)\,.$$

They are not homomorphisms, but they do have the property that $\ln(M^k) = k\ln(M)$ for each $M \in I + mM_r(\mathbb{Z}_p)$ and each $k \geq 1$. In particular, the only element of finite order in $I + mM_r(\mathbb{Z}_p)$ is the identity. Thus $B' = I$, so $B = \mu I$. Since $\mu \in \mathbb{Z}$ and $B \neq I$, we have $\mu = -1$ and $B = -I$. □

The following lemma about the lattice $\mathbb{Z}\Sigma^\vee$ will also be useful when working with the Weyl group action on certain subgroups of $\bar{T}$.

Lemma 2.8. *Assume Notation 2.2(A,B). Set $\Lambda = \mathbb{Z}\Sigma^\vee$: the lattice in V generated by the dual roots. Assume that there are $b \in W$ of order 2, and a splitting $\Lambda = \Lambda_+ \times \Lambda_-$, such that $\Lambda_+, \Lambda_- \neq 0$ and b acts on $\Lambda_\pm$ via $\pm\mathrm{Id}$. Then $\mathbb{G} \cong C_n$ ($= Sp_{2n}$) for some $n \geq 2$.*

Proof. Fix $b \in w$ and a splitting $\Lambda = \Lambda_+ \times \Lambda_-$ as above. When considering individual cases, we use the notation of Bourbaki [Brb, Planches I–IX] to describe the (dual) roots, lattice, and Weyl group.

- If $\mathbb{G} = A_n$ ($n \geq 2$), then $\Lambda = \{(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} \,|\, a_0 + \ldots + a_n = 0\}$, and b exchanges certain coordinates pairwise. Choose $v \in \Lambda$ with coordinates 1, -1, and otherwise 0; where the two nonzero entries are in separate orbits of b of which at least one is nonfixed. Then $v \notin \Lambda_+ \times \Lambda_-$, a contradiction.
- If $\mathbb{G} = G_2$, then as described in [Brb, Planche IX], Λ is generated by the dual fundamental roots $(1, -1, 0)$ and $(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, and does not have an orthogonal basis.
- If $\mathbb{G} = B_n$ ($n \geq 3$), D_n ($n \geq 4$), or F_4, then $\Lambda < \mathbb{Z}^n$ is the sublattice of n-tuples the sum of whose coordinates is even. Also, b acts by permuting the coordinates and changing sign (or we can assume it acts this way in the F_4 case). Choose v with two 1's and the rest 0, where the 1's are in separate b-orbits, of which either at least one is nonfixed, or both are fixed and exactly one is negated. Then $v \notin \Lambda_+ \times \Lambda_-$, a contradiction.
- If $\mathbb{G} = E_8$, then $\Lambda = \Lambda(E_8) < \mathbb{R}^8$ is generated by $\frac{1}{2}(1, 1, \ldots, 1)$ and the n-tuples of integers whose sum is even. We can assume (up to conjugation) that b acts as a signed permutation. Choose v as in the last case.
- If $\mathbb{G} = E_7$, then $\Lambda < \mathbb{R}^8$ is the lattice of all $x = (x_1, \ldots, x_8) \in \Lambda(E_8)$ such that $x_7 = -x_8$. Up to conjugation, b can be again be assumed to act on A via a signed permutation (permuting only the first six coordinates), and v can be chosen as in the last case.
- If $\mathbb{G} = E_6$, then $\Lambda < \mathbb{R}^8$ is the lattice of all $x = (x_1, \ldots, x_8) \in \Lambda(E_8)$ such that $x_6 = x_7 = -x_8$. Also, W contains a subgroup isomorphic to $2^4 : S_5$ with odd index which acts on the remaining five coordinates via signed permutations. So b and v can be taken as in the last three cases. □

We finish the section with a very elementary lemma.

It will be useful to know, in certain situations, that each coset of $\bar{T}$ in $N_{\bar{G}}(\bar{T})$ contains elements of G.

Lemma 2.9. *Assume that we are in the situation of Notation 2.2(A,B). Assume also that σ acts on $\bar{T}$ via $(t \mapsto t^m)$ for some $1 \neq m \in \mathbb{Z}$. Then for each $g \in N_{\bar{G}}(\bar{T})$, $g\bar{T} \cap C_{\bar{G}}(\sigma) \neq \varnothing$.*

Proof. Since $\sigma|_{\bar{T}} \in Z(\mathrm{Aut}(\bar{T}))$, we have $g^{-1}\sigma(g) \in C_{\bar{G}}(\bar{T}) = \bar{T}$, the last equality by Lemma 2.4(a). So for each $t \in \bar{T}$, $\sigma(gt) = gt$ if and only if $g^{-1}\sigma(g) = t^{1-m}$. Since $\bar{T} \cong (\bar{\mathbb{F}}_{q_0}^{\times})^r$ for some r, and $\bar{\mathbb{F}}_{q_0}$ is algebraically complete (and $1 - m \neq 0$), this always has solutions. $\square$

3. Automorphisms of groups of Lie type

Since automorphisms of G play a central role in this paper, we need to fix our notation (mostly taken from [GLS3]) for certain subgroups and elements of $\mathrm{Aut}(G)$. We begin with automorphisms of the algebraic group $\overline{G}$.

Definition 3.1. *Let $\overline{G}$ and its root system Σ be as in Notation 2.2(A,B).*

(a) *When q is any power of q_0 (the defining characteristic of $\overline{G}$), let $\psi_q \in \mathrm{End}(\overline{G})$ be the field endomorphism defined by $\psi_q(x_\alpha(u)) = x_\alpha(u^q)$ for each $\alpha \in \Sigma$ and each $u \in \overline{\mathbb{F}}_{q_0}$. Set $\Phi_{\overline{G}} = \{\psi_{q_0^b} \,|\, b \geq 1\}$: the monoid of all field endomorphisms of $\overline{G}$.*

(b) *Let $\Gamma_{\overline{G}}$ be the group or set of graph automorphisms of $\overline{G}$ as defined in* [GLS3, Definition 1.15.5(e)]. *Thus when $(\mathbb{G}, q_0) \neq (B_2, 2)$, $(G_2, 3)$, nor $(F_4, 2)$, $\Gamma_{\overline{G}}$ is the group of all $\gamma \in \mathrm{Aut}(\overline{G})$ of the form $\gamma(x_\alpha(u)) = x_{\rho(\alpha)}(u)$ (all $\alpha \in \pm\Pi$ and $u \in \overline{\mathbb{F}}_{q_0}$) for some isometry ρ of Σ such that $\rho(\Pi) = \Pi$. If $(\mathbb{G}, q_0) = (B_2, 2)$, $(G_2, 3)$, or $(F_4, 2)$, then $\Gamma_{\overline{G}} = \{1, \psi\}$, where for the angle-preserving permutation ρ of Σ which exchanges long and short roots and sends Π to itself, $\psi(x_\alpha(u)) = x_{\rho(\alpha)}(u)$ when α is a long root and $\psi(x_\alpha(u)) = x_{\rho(\alpha)}(u^{q_0})$ when α is short.*

(c) *A Steinberg endomorphism σ of $\overline{G}$ is "standard" if $\sigma = \psi_q \circ \gamma = \gamma \circ \psi_q$, where q is a power of q_0 and $\gamma \in \Gamma_{\overline{G}}$. A σ-setup $(\overline{G}, \sigma)$ for a finite subgroup $G < \overline{G}$ is standard if σ is standard.*

By [GLS3, Theorem 2.2.3], for any G with σ-setup $(\overline{G}, \sigma)$ as in Notation 2.2, G is $\overline{G}$-conjugate to a subgroup G^* which has a standard σ-setup. This will be made more precise in Proposition 3.6(a).

Most of the time in this paper, we will be working with standard σ-setups. But there are a few cases where we will need to work with setups which are not standard, which is why this condition is not included in Notation 2.2.

Following the usual terminology, we call G a "Chevalley group" if it has a standard σ-setup where $\gamma = \mathrm{Id}$ in the notation of Definition 3.1; i.e., if $G \cong \mathbb{G}(q)$ where q is some power of q_0. In this case, the root groups $X_{\widehat{\alpha}}$ are all abelian and isomorphic to $\mathbb{F}_q$. When G has a standard σ-setup with $\gamma \neq \mathrm{Id}$, we refer to G as a "twisted group", and the different possible structures of its root groups are described in [GLS3, Table 2.4]. We also refer to G as a "Steinberg group" if $\gamma \neq \mathrm{Id}$ and is an algebraic automorphism of $\overline{G}$; i.e., if G is a twisted group and not a Suzuki or Ree group.

The following lemma will be useful in Sections 5 and 6.

Lemma 3.2. *Assume $\overline{G}$ is as in Notation 2.2(A,B). Then for each algebraic automorphism γ of $\overline{G}$ which normalizes $\overline{T}$, there is an orthogonal automorphism τ of V such that $\tau(\Sigma) = \Sigma$, and*

$$\gamma(\overline{X}_\alpha) = \overline{X}_{\tau(\alpha)} \qquad \text{and} \qquad \gamma(h_\alpha(\lambda)) = h_{\tau(\alpha)}(\lambda)$$

for each $\alpha \in \Sigma$ and each $\lambda \in \overline{\mathbb{F}}_{q_0}^\times$. In particular, $\left|\gamma|_{\overline{T}}\right| = |\tau| < \infty$. If, in addition, γ normalizes each of the root groups $\overline{X}_\alpha$ (i.e., $\tau = \mathrm{Id}$), then $\gamma \in \mathrm{Aut}_{\overline{T}}(\overline{G})$.

Proof. By [GLS3, Theorem 1.15.2(b)], and since γ is an algebraic automorphism of $\overline{G}$, $\gamma = c_g \circ \gamma_0$ for some $g \in \overline{G}$ and some $\gamma_0 \in \Gamma_{\overline{G}}$. Furthermore, γ_0 has the form: $\gamma_0(x_\alpha(u)) = x_{\chi(\alpha)}(u)$

for all $\alpha \in \Sigma$ and $u \in \bar{\mathbb{F}}_{q_0}$, and some isometry $\chi \in \mathrm{Aut}(V)$ such that $\chi(\Pi) = \Pi$. Since γ and γ_0 both normalize $\bar{T}$, we have $g \in N_{\bar{G}}(\bar{T})$.

Thus by Lemma 2.4(e), there is $\tau \in \mathrm{Aut}(V)$ such that $\tau(\Sigma) = \Sigma$, and $\gamma(\overline{X}_\alpha) = \overline{X}_{\tau(\alpha)}$ and $\gamma(h_\alpha(\lambda)) = h_{\tau(\alpha)}(\lambda)$ for each $\alpha \in \Sigma$ and $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$. In particular, $\bigl|\gamma|_{\bar{T}}\bigr| = |\tau|$.

If $\tau = \mathrm{Id}$, then $\gamma_0 = \mathrm{Id}$ and $g \in \bar{T}$. Thus $\gamma \in \mathrm{Aut}_{\bar{T}}(\bar{G})$. □

We next fix notation for automorphisms of G.

Definition 3.3. *Let $\bar{G}$ and G be as in Notation 2.2(A,B,C), where in addition, we assume the σ-setup is standard.*

(a) *Set*

$$\mathrm{Inndiag}(G) = \mathrm{Aut}_{\bar{T}}(G)\mathrm{Inn}(G) \qquad \text{and} \qquad \mathrm{Outdiag}(G) = \mathrm{Inndiag}(G)/\mathrm{Inn}(G)\,.$$

(b) *Set $\Phi_G = \bigl\{\psi_q|_G \bigm| q = q_0^b,\ b \ge 1\bigr\}$, the group of field automorphisms of G.*

(c) *If G is a Chevalley group, set $\Gamma_G = \bigl\{\gamma|_G \bigm| \gamma \in \Gamma_{\bar{G}}\bigr\}$, the group of graph automorphisms of G. Set $\Gamma_G = 1$ if G is a twisted group (a Steinberg, Suzuki, or Ree group).*

Note that in [GLS3, Definition 2.5.13], when G has a standard σ-setup $(\bar{G}, \sigma)$, $\mathrm{Inndiag}(G)$ is defined to be the group of automorphisms induced by conjugation by elements of $C_{\bar{G}/Z(\bar{G})}(\sigma)$ (lifted to $\bar{G}$). By [GLS3, Lemma 2.5.8], this is equal to $\mathrm{Inndiag}(G)$ as defined above when $\bar{G}$ is of adjoint form, and hence also in the general case (since $Z(\bar{G}) \le \bar{T}$).

Steinberg's theorem on automorphisms of groups of Lie type can now be stated.

Theorem 3.4 ([St1, § 3])**.** *Let G be a finite group of Lie type. Assume that $(\bar{G}, \sigma)$ is a standard σ-setup for G, where $\bar{G}$ is in adjoint or universal form. Then* $\mathrm{Aut}(G) = \mathrm{Inndiag}(G)\Phi_G\Gamma_G$*, where* $\mathrm{Inndiag}(G) \trianglelefteq \mathrm{Aut}(G)$ *and* $\mathrm{Inndiag}(G) \cap (\Phi_G\Gamma_G) = 1$*.*

Proof. See, e.g., [GLS3, Theorem 2.5.12(a)] (together with [GLS3, Theorem 2.5.14(d)]). Most of this follows from the main result in [St1], and from [St2, Theorems 30 & 36]. □

We also need the following characterizations of $\mathrm{Inndiag}(G)$ which are independent of the choice of σ-setup.

Proposition 3.5. *Assume the hypotheses and notation in 2.2. Then*

(a) *$C_{\bar{G}}(G) = Z(\bar{G})$;*

(b) *$N_{\bar{G}}(G) = GN_{\bar{T}}(G)$;*

(c) $\mathrm{Inndiag}(G) = \mathrm{Aut}_{\bar{T}}(G)\mathrm{Inn}(G) = \mathrm{Aut}_{\bar{G}}(G)$*, and hence* $\mathrm{Outdiag}(G) = \mathrm{Out}_{\bar{T}}(G)$.

In fact, (b) and (c) hold if we replace $\bar{T}$ by any *σ-invariant maximal torus in $\bar{G}$.*

Proof. **(a)** Since the statement is independent of the choice of σ-setup, we can assume that σ is standard. Set $\bar{U} = \prod_{\alpha\in\Sigma_+} \overline{X}_\alpha$ and $\bar{U}^* = \prod_{\alpha\in\Sigma_+} \overline{X}_{-\alpha}$.

Fix $g \in C_{\bar{G}}(G)$. Since $\bar{G}$ has a BN-pair (see [Ca, Proposition 8.2.1]), it has a Bruhat decomposition $\bar{G} = \bar{B}\bar{N}\bar{B} = \bar{U}\bar{N}\bar{U}$ [Ca, Proposition 8.2.2(i)], where $\bar{B} = \bar{T}\bar{U}$ and $\bar{N} = N_{\bar{G}}(\bar{T})$. Write $g = unv$, where $u, v \in \bar{U}$ and $n \in \bar{N}$. For each $x \in \bar{U} \cap G$, ${}^{g}x = {}^{u}({}^{nv}x) \in \bar{U}$ implies that ${}^{nv}x = {}^{n}({}^{v}x) \in \bar{U}$.

Since $n \in N_{\overline{G}}(\overline{T})$, conjugation by n permutes the root groups of $\overline{G}$, in a way determined by the class $w = n\overline{T} \in W = N_{\overline{G}}(\overline{T})/\overline{T}$. Thus w sends each (positive) root in the decomposition of ${}^{v}x$ to a positive root. For each $\alpha \in \Sigma_+$, $\widehat{x}_\alpha(1) \in G$, ${}^{v}(\widehat{x}_\alpha(1))$ has α in its decomposition, and hence $w(\alpha) \in \Sigma_+$.

Thus w sends all positive roots to positive roots, so $w(\Pi) = \Pi$, and $w = 1$ by [Ca, Corollary 2.2.3]. So $n \in \overline{T}$, and $g = unv \in \overline{T}\overline{U}$.

By the same argument applied to the negative root groups, $g \in \overline{T}\overline{U}^*$. Hence $g \in \overline{T}$.

For each $\alpha \in \Sigma$, $g \in \overline{T}$ commutes with $\widehat{x}_\alpha(1) \in G$, and hence g centralizes $\overline{X}_\beta$ for each $\beta \in \widehat{\alpha}$ (Lemma 2.4(c)). Thus g centralizes all root groups in $\overline{G}$, so $g \in Z(\overline{G})$.

(b) Let $\overline{T}^*$ be any σ-invariant maximal torus in $\overline{G}$. Fix $g \in N_{\overline{G}}(G)$. Then $g^{-1} \cdot \sigma(g) \in C_{\overline{G}}(G) = Z(\overline{G}) \le \overline{T}^*$ by (a). By Lang's theorem [GLS3, Theorem 2.1.1], there is $t \in \overline{T}^*$ such that $g^{-1} \cdot \sigma(g) = t^{-1} \cdot \sigma(t)$. Hence $gt^{-1} \in C_{\overline{G}}(\sigma) = G \cdot C_{\overline{T}^*}(\sigma)$, where the last equality holds by [GLS3, Theorem 2.2.6(g)]. So $g \in G\overline{T}^*$, and $g \in GN_{\overline{T}^*}(G)$ since g normalizes G.

(c) By (b), $\mathrm{Aut}_{\overline{G}}(G) = \mathrm{Aut}_{\overline{T}^*}(G)\mathrm{Inn}(G)$ for each σ-invariant maximal torus $\overline{T}^*$. By definition, $\mathrm{Inndiag}(G) = \mathrm{Aut}_{\overline{T}^*}(G)\mathrm{Inn}(G)$ when $\overline{T}^*$ is the maximal torus in a standard σ-setup for G. Hence $\mathrm{Inndiag}(G) = \mathrm{Aut}_{\overline{G}}(G) = \mathrm{Aut}_{\overline{T}^*}(G)\mathrm{Inn}(G)$ for all such $\overline{T}^*$. □

We refer to [GLS3, Definitions 1.15.5(a,e) & 2.5.10] for more details about the definitions of Φ_G and Γ_G. The next proposition describes how to identify these subgroups when working in a nonstandard setup.

Proposition 3.6. *Assume $\overline{G}$, $\overline{T}$, and the root system of $\overline{G}$, are as in Notation 2.2(A,B). Let σ be any Steinberg endomorphism of $\overline{G}$, and set $G = O^{q_0'}(C_{\overline{G}}(\sigma))$.*

(a) *There is a standard Steinberg endomorphism σ^* of $\overline{G}$ such that if we set $G^* = O^{q_0'}(C_{\overline{G}}(\sigma^*))$, then there is $x \in \overline{G}$ such that $G = {}^{x}(G^*)$.*

Fix G^*, σ^*, and x as in (a). Let $\mathrm{Inndiag}(G^*)$, Φ_{G^*}, and Γ_{G^*} be as in Definition 3.3 (with respect to the σ-setup $(\overline{G}, \sigma^*)$). Set $\mathrm{Inndiag}(G) = c_x\mathrm{Inndiag}(G^*)c_x^{-1}$, $\Phi_G = c_x\Phi_{G^*}c_x^{-1}$, and $\Gamma_G = c_x\Gamma_{G^*}c_x^{-1}$, all as subgroups of $\mathrm{Aut}(G)$. Then the following hold.

(b) $\mathrm{Inndiag}(G) = \mathrm{Aut}_{\overline{G}}(G)$.

(c) *For each $\bar{\alpha} \in \Phi_{\overline{G}}\Gamma_{\overline{G}}$ such that $\bar{\alpha}|_{G^*} \in \Phi_{G^*}\Gamma_{G^*}$, and each $\beta \in \bar{\alpha} \cdot \mathrm{Inn}(\overline{G})$ such that $\beta(G) = G$, $\beta|_G \equiv c_x(\bar{\alpha})c_x^{-1}$ (mod $\mathrm{Inndiag}(G)$).*

(d) *If ψ_{q_0} normalizes G, then $\mathrm{Inndiag}(G)\Phi_G = \mathrm{Inndiag}(G)\langle\psi_{q_0}|_G\rangle$.*

Thus the subgroups Φ_G and Γ_G are well defined modulo $\mathrm{Inndiag}(G)$, independently of the choice of standard σ-setup for G.

Proof. **(a)** See, e.g., [GLS3, Theorem 2.2.3]: for any given choice of maximal torus, positive roots, and parametrizations of the root groups, each Steinberg automorphism of $\overline{G}$ is conjugate, by an element of $\mathrm{Inn}(\overline{G})$, to a Steinberg automorphism of standard type.

(b) This follows immediately from Proposition 3.5(c).

(c) By assumption, $\beta \equiv \bar{\alpha} \equiv c_x\bar{\alpha}c_x^{-1}$ (mod $\mathrm{Inn}(\overline{G})$). Since β and $c_x\bar{\alpha}c_x^{-1}$ both normalize G, $\beta|_G \equiv c_x\alpha^*c_x^{-1}$ modulo $\mathrm{Aut}_{\overline{G}}(G) = \mathrm{Inndiag}(G)$.

(d) If ψ_{q_0} normalizes G, then (c), applied with $\bar{\alpha} = \beta = \psi_{q_0}$, implies that as elements of $\mathrm{Aut}(G)/\mathrm{Inndiag}(G)$, $[\psi_{q_0}|_G] = [c_x(\psi_{q_0}|_{G^*})c_x^{-1}]$ generates the image of Φ_G. □

Lemma 3.7. *Assume $\overline{G}$, $\overline{T}$, σ, $G = O^{q_0'}(C_{\overline{G}}(\sigma))$, and the root system of $\overline{G}$, are as in Notation 2.2(A,B). Assume that $\varphi \in \mathrm{Aut}(\overline{T})$ is the restriction of an algebraic automorphism of $\overline{G}$ such that $[\varphi, \sigma|_{\overline{T}}] = 1$. Then there is an algebraic automorphism $\bar{\varphi} \in \mathrm{Aut}(\overline{G})$ such that $\bar{\varphi}|_{\overline{T}} = \varphi$, $[\bar{\varphi}, \sigma] = 1$, and $\bar{\varphi}(G) = G$.*

Proof. By assumption, there is $\bar{\varphi} \in \mathrm{Aut}(\overline{G})$ such that $\bar{\varphi}|_{\overline{T}} = \varphi$. Also, $[\bar{\varphi}, \sigma]$ is an algebraic automorphism of $\overline{G}$ by [GLS3, Theorem 1.15.7(a)], it is the identity on $\overline{T}$, and hence $[\bar{\varphi}, \sigma] = c_t$ for some $t \in \overline{T}$ by Lemma 3.2. Using the Lang-Steinberg theorem, upon replacing $\bar{\varphi}$ by $c_u\bar{\varphi}$ for appropriate $u \in \overline{T}$, we can arrange that $[\bar{\varphi}, \sigma] = 1$. In particular, $\bar{\varphi}(G) = G$. □

The following proposition is well known, but it seems to be difficult to find references where it is proven.

Proposition 3.8. *Fix a prime q_0, and a group $G \in \mathfrak{Lie}(q_0)$ of universal type. Then $Z(G)$ has order prime to q_0, $G/Z(G) \in \mathfrak{Lie}(q_0)$ and is of adjoint type, and $Z(G/Z(G)) = 1$. If $G/Z(G)$ is simple, then each central extension of G by a group of order prime to q_0 splits (equivalently, $H^2(G; \mathbb{Z}/p) = 0$ for all primes $p \neq q_0$).*

Proof. Let $(\overline{G}, \sigma)$ be a σ-setup for G, and choose a maximal torus and positive roots in $\overline{G}$. We can thus assume Notation 2.2. By Lemma 2.4(a), $Z(\overline{G})$ is finite of order prime to q_0. Since $Z(G) \le C_{\overline{G}}(G) = Z(\overline{G})$ by Proposition 3.5(a), $Z(G)$ also has order prime to q_0.

Set $\overline{G}_a = \overline{G}/Z(\overline{G})$ and let $G_a < \overline{G}_a$ be the image of G under projection. Thus $\overline{G}_a$ is an algebraic group of adjoint type, and $G_a = O^{q_0'}(C_{\overline{G}_a}(\sigma_a)) \in \mathfrak{Lie}(q_0)$ where $\sigma_a \in \mathrm{End}(\overline{G}_a)$ is induced by σ. Also, $Z(G_a) \le Z(\overline{G}_a) = 1$ by Proposition 3.5(a) again.

It remains to prove the statement about central extensions. When G is a Chevalley group, this was shown in [St4, Théorème 4.5]. It was shown in [St6, Corollary 6.2] when $G \cong {}^2A_n(q)$ for n even, and in [AG] when $G \cong {}^2G_2(q)$ or $\mathrm{Sz}(q)$. The remaining cases follow by similar arguments (see [St5, 9.4 & 12.4]). (See also [Cu, § 1].) □

The next proposition shows that in most cases, $C_{\overline{G}}(T) = \overline{T}$. In the next section, we will see some conditions which imply that $C_{\overline{G}}(O_p(T)) = \overline{T}$ when p is a prime different from the defining characteristic.

Proposition 3.9. *Let $(\overline{G}, \sigma)$ be a σ-setup for G, where $\overline{G}$ and G are of universal type. Assume Notation 2.2, and in particular, that we have fixed a maximal torus $\overline{T}$ and a root system Σ in $\overline{G}$.*

(a) *Assume that $C_{\overline{G}}(T)^0 \gneqq \overline{T}$, where $(-)^0$ denotes the connected component of the identity. Then there is $\alpha \in \Sigma_+$ such that $\theta_\alpha(T) = 1$. Also, there is $\beta \in \mathrm{Hom}(\overline{T}, \overline{\mathbb{F}}_{q_0}^\times)$ such that $\theta_\alpha = \beta^{-1}\sigma^*(\beta)$; i.e., $\theta_\alpha(t) = \beta(t^{-1}\sigma(t))$ for each $t \in \overline{T}$.*

(b) *If the σ-setup is standard, then $C_{\overline{G}}(T)^0 = \overline{T}$ except possibly when $G \cong {}^r\mathbb{G}(2)$ for some $\mathbb{G}$ and some $r \le 3$, or when $G \cong A_1(3)$, $C_n(3)$ for $n \ge 2$, or ${}^2G_2(3)$.*

(c) *If $C_{\overline{G}}(T)^0 = \overline{T}$, then $N_G(T)/T \cong W_0$.*

Proof. **(a)** Since G is of universal type, $G = C_{\bar{G}}(\sigma)$ and $T = C_{\bar{T}}(\sigma)$. Hence there is a short exact sequence

$$1 \longrightarrow T \longrightarrow \bar{T} \xrightarrow{t \mapsto t^{-1}\sigma(t)} \bar{T}\,,$$

where the last map is onto by the Lang-Steinberg theorem. Upon dualizing, and regarding $\mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$ additively, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times) \xrightarrow{\sigma^* - \mathrm{Id}} \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times) \xrightarrow{\mathrm{restr}} \mathrm{Hom}(T, \bar{\mathbb{F}}_{q_0}^\times)$$

(see also [Ca2, Proposition 3.2.3]), where $\mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$ is the group of *algebraic* homomorphisms. Since θ_α is in the kernel of the restriction map, by assumption, it has the form $\beta^{-1}\sigma^*(\beta)$ for some $\beta \in \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$.

(b) Let $P(\Sigma)$ and $Q(\Sigma)$ be as in [Brb, § VI.1.9] (but with Σ in place of R to denote the root system). Thus $Q(\Sigma) = \mathbb{Z}\Sigma$, the integral lattice generated by Σ, and

$$P(\Sigma) = \{v \in V \mid (v, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \in \Sigma\} \geq Q(\Sigma)\,.$$

For each $v \in P(\Sigma)$, define $\theta_v \in X(\bar{T}) = \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$ by setting $\theta_v(h_\alpha(\lambda)) = \lambda^{(v,\alpha^\vee)}$ for $\alpha \in \Pi$ and $\lambda \in \bar{\mathbb{F}}_{q_0}$. Since G is of universal type, this is a well defined homomorphism by Lemma 2.4(b), and the same formula holds for all $\alpha \in \Sigma$ by Lemma 2.4(d). By Lemma 2.4(c), this extends our definition of θ_β for $\beta \in \Sigma \subseteq P(\Sigma)$.

Recall that $\mathrm{Hom}(\bar{\mathbb{F}}_{q_0}^\times, \bar{\mathbb{F}}_{q_0}^\times) \cong \mathbb{Z}$. For each $\theta \in X(\bar{T})$ and each $\alpha \in \Sigma$, let $n_{\theta,\alpha} \in \mathbb{Z}$ be such that $\theta(h_\alpha(\lambda)) = \lambda^{n_{\theta,\alpha}}$ for all $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$. For given θ, there is $v \in P(\Sigma)$ such that $(v, \alpha^\vee) = n_{\theta,\alpha}$ for all $\alpha \in \Pi$, and hence (by Lemma 2.4(d)) for all $\alpha \in \Sigma$. Then $\theta = \theta_v$ as defined above. In this way, we identify $P(\Sigma)$ with the lattice $X(\bar{T})$ of characters for $\bar{T}$, while identifying $Q(\Sigma)$ with $\mathbb{Z}\Sigma$.

From the appendix to Chapter VI in [Brb] (Planches I–IX), we obtain the following table:

root system Σ	A_n	C_n	$B_n,\ D_n$	G_2	F_4	E_6	E_7	E_8
$\min\{\lVert v\rVert \mid v \in P(\Sigma)\}$	$\sqrt{n/(n{+}1)}$	1	$\min\{\sqrt{n/4}, 1\}$	$\sqrt{2}$	1	$\sqrt{4/3}$	$\sqrt{2}$	$\sqrt{2}$
$\max\{\lVert\alpha\rVert \mid \alpha \in \Sigma\}$	$\sqrt{2}$	2	$\sqrt{2}$	$\sqrt{6}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$

Here, the norms are given with respect to the descriptions of these lattices in [Brb] as subgroups of Euclidean spaces.

Assume $C_{\bar{G}}(T)^0 \gneqq \bar{T}$. By (a), there are $\alpha \in \Sigma_+$ and $\beta \in \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$ such that $\alpha = \beta^{-1}\sigma^*(\beta)$. If we regard α and β as elements in the normed vector space V, then $\lVert\alpha\rVert = \lVert\sigma^*(\beta) - \beta\rVert \geq \lVert\sigma^*(\beta)\rVert - \lVert\beta\rVert$. If $G = {}^\tau\mathbb{G}(q)$ (and σ is a standard setup), then $\lVert\sigma^*(\beta)\rVert = q\lVert\beta\rVert$, except when G is a Suzuki or Ree group in which case $\lVert\sigma^*(\beta)\rVert = \sqrt{q}\lVert\beta\rVert$. Thus

$$\frac{\lVert\alpha\rVert}{\lVert\beta\rVert} + 1 \geq \begin{cases} q & \text{if } G \text{ is a Chevalley or Steinberg group} \\ \sqrt{q} & \text{if } G \text{ is a Suzuki or Ree group.} \end{cases}$$

By the above table, this is possible only if $q = 2$, or if G is isomorphic to one of the groups $A_1(3)$, $B_2(3)$, $C_n(3)$ ($n \geq 3$), ${}^2G_2(3)$, or ${}^2B_2(8)$.

Assume $G \cong {}^2B_2(8) \cong \mathrm{Sz}(8)$. It is most convenient to use the root system for C_2 constructed in [Brb]: $P(\Sigma) = \mathbb{Z}^2$, and $\Sigma = \{(\pm 2, 0), (0, \pm 2), (\pm 1, \pm 1)\}$. Then α and β satisfy the above inequality only if $\lVert\alpha\rVert = 2$, $\lVert\beta\rVert = 1$, and $\lVert\alpha + \beta\rVert = \sqrt{8}$. So $(\alpha, \beta) = \frac{3}{2}$, which is impossible for $\alpha, \beta \in \mathbb{Z}^2$. Hence $C_{\bar{G}}(T)^0 = \bar{T}$ in this case.

(c) If $C_{\overline{G}}(T)^0 = \bar{T}$, then $N_{\overline{G}}(T) \le N_{\overline{G}}(\bar{T})$, and so $N_G(T)/T \cong W_0$ by Lemma 2.3. □

The following, more technical lemma will be needed in Section 6.

Lemma 3.10. *Assume the hypotheses and notation in 2.2, and also that the σ-setup $(\overline{G}, \sigma)$ is standard. Then under the action of W_0 on $\widehat{\Sigma}$, each orbit contains elements of $\widehat{\Pi}$.*

Proof. When $\rho = \mathrm{Id}$, this is [Ca, Proposition 2.1.8]. When $\rho \neq \mathrm{Id}$, it follows from the descriptions of W_0 and $\widehat{\Sigma}$ in [Ca, §§ 13.2–13.3]. □

4. The equicharacteristic case

The following notation will be used in this section.

Notation 4.1. *Assume the notation in 2.2, and also that $\rho(\Pi)=\Pi$, $q_0=p$, and $Z(\bar{G})=1$. Thus $\bar{G}=\mathbb{G}(\bar{\mathbb{F}}_p)$ is a connected, simple group over $\bar{\mathbb{F}}_p$ in adjoint form, σ is a Steinberg endomorphism of $\bar{G}$ of standard form, and $G=O^{p'}(C_{\bar{G}}(\sigma))$.*

(D) *Set $\bar{U}=\langle \bar{X}_\alpha \,|\, \alpha\in\Sigma_+\rangle$ and $\bar{B}\overset{\text{def}}{=}N_{\bar{G}}(\bar{U})=\bar{U}\bar{T}$ (the* Borel subgroup *of $\bar{G}$). Set*

$$U=C_{\bar{U}}(\sigma)=\langle X_{\widehat{\alpha}}\,|\,\widehat{\alpha}\in\widehat{\Sigma}_+\rangle\,,\qquad B=N_G(U)\,,\qquad \text{and}\qquad T=\bar{T}\cap G.$$

Thus $U=\prod_{\widehat{\alpha}\in\widehat{\Sigma}_+}X_{\widehat{\alpha}}\in\mathrm{Syl}_p(G)$, and $B=UT$. (See, e.g., [GLS3, Theorems 2.3.4(d) & 2.3.7]*, or* [Ca, Theorems 5.3.3(ii) & 9.4.10] *in the case of Chevalley groups.) When $\widehat{J}\subsetneqq\widehat{\Pi}$ is the image in $\widehat{\Sigma}_+$ of a τ-invariant subset $J\subsetneqq\Pi$, let $U_{\widehat{J}}\le U$ be the subgroup generated by root groups for positive roots in $\Sigma_+\smallsetminus\langle J\rangle$ (the* unipotent radical subgroup *associated to $\widehat{J}$), and set $\mathfrak{P}_{\widehat{J}}=N_G(U_{\widehat{J}})=B\langle X_{-\widehat{\alpha}}\,|\,\alpha\in\langle J\rangle\rangle$ (the* parabolic subgroup *associated to $\widehat{J}$). Thus $U=U_\varnothing$ and $B=\mathfrak{P}_\varnothing$. We also write $U_{\widehat{\alpha}}=U_{\{\widehat{\alpha}\}}$ and $\mathfrak{P}_{\widehat{\alpha}}=\mathfrak{P}_{\{\widehat{\alpha}\}}$ for each $\widehat{\alpha}\in\widehat{\Pi}$.*

(E) *The* height *of a positive root $\alpha=\sum_{\gamma\in\Pi}n_\gamma\gamma\in\Sigma_+$ ($n_\gamma\ge 0$) is defined by* $\mathrm{ht}(\alpha)=\sum_{\gamma\in\Pi}n_\gamma$. *The height $\mathrm{ht}(\widehat{\alpha})$ of a class of roots $\widehat{\alpha}\in\widehat{\Sigma}_+$ is the minimum of the heights of roots in the class $\widehat{\alpha}$.*

(F) *Set $\mathcal{F}=\mathcal{F}_U(G)$ and $\mathcal{L}=\mathcal{L}_U^c(G)$.*

(G) *Set $U_0=\langle X_{\widehat{\alpha}}\,|\,\widehat{\alpha}\in\widehat{\Sigma}_+,\ \widehat{\alpha}\cap\Pi=\varnothing\rangle=\langle X_{\widehat{\alpha}}\,|\,\mathrm{ht}(\widehat{\alpha})\ge 2\rangle$.*

(H) *The* Lie rank *of G is equal to $|\widehat{\Pi}|$; equivalently, to the number of maximal parabolic subgroups containing B.*

For example, assume $\sigma=\psi_q\circ\gamma$, where $\gamma\in\mathrm{Aut}(\bar{G})$ is a graph automorphism which induces $\rho\in\mathrm{Aut}(\Sigma_+)$, and ψ_q is the field automorphism induced by $t\mapsto t^q$. Then for $\widehat{\alpha}\in\widehat{\Sigma}$, $X_{\widehat{\alpha}}\cong\mathbb{F}_q$ when $\widehat{\alpha}=\{\alpha\}$ contains only one root, $X_{\widehat{\alpha}}\cong\mathbb{F}_{q^a}$ if $\widehat{\alpha}=\{\rho^i(\alpha)\}$ is the ρ-orbit of α with length a, and $X_{\widehat{\alpha}}$ is nonabelian if $\widehat{\alpha}$ contains a root α and sums of roots in its ρ-orbit.

We need the following, stronger version of Theorem 3.4.

Theorem 4.2 ([St1, § 3])**.** *Assume G is as in Notation 2.2 and 4.1. If $\alpha\in\mathrm{Aut}(G)$ is such that $\alpha(U)=U$, then $\alpha=c_udfg$ for unique automorphisms $c_u\in\mathrm{Aut}_U(G)$, $d\in\mathrm{Inndiag}(G)=\mathrm{Aut}_{\bar{T}}(G)$, $f\in\Phi_G$, and $g\in\Gamma_G$.*

Proof. Let $N_{\mathrm{Aut}(G)}(U)\le\mathrm{Aut}(G)$ and $N_{\mathrm{Inndiag}(G)}(U)\le\mathrm{Inndiag}(G)$ be the subgroups of those automorphisms which send U to itself. Since $\Phi_G\Gamma_G\le N_{\mathrm{Aut}(G)}(U)$ by definition, Theorem 3.4 implies that $N_{\mathrm{Aut}(G)}(U)=N_{\mathrm{Inndiag}(G)}(U)\cdot(\Phi_G\Gamma_G)$, a semidirect product. Since $\Phi_G\cap\Gamma_G=1$, it remains to show that $N_{\mathrm{Inndiag}(G)}(U)=\mathrm{Aut}_U(G)\mathrm{Aut}_{\bar{T}}(G)$ and $\mathrm{Aut}_U(G)\cap\mathrm{Aut}_{\bar{T}}(G)=1$. The first is immediate: since $\mathrm{Aut}_{\bar{T}}(G)\le N_{\mathrm{Aut}(G)}(U)$ and $N_G(U)=TU$,

$$\begin{aligned}N_{\mathrm{Inndiag}(G)}(U)&=\big(\mathrm{Inn}(G)\mathrm{Aut}_{\bar{T}}(G)\big)\cap N_{\mathrm{Aut}(G)}(U)\\&=\mathrm{Aut}_{N_G(U)}(G)\mathrm{Aut}_{\bar{T}}(G)=\mathrm{Aut}_U(G)\mathrm{Aut}_{\bar{T}}(G)\,.\end{aligned}$$

Finally, if $c_u=c_t\in\mathrm{Aut}(G)$ where $u\in U$ and $t\in\bar{T}$, then $c_u=\mathrm{Id}_G$, since u has p-power order and t has order prime to p. □

Lemma 4.3. *Assume $G \in \mathfrak{Lie}(p)$. Then for $U \in \mathrm{Syl}_p(G)$, $\bar{\kappa}_G$ sends $\mathrm{Out}(G)$ injectively into $\mathrm{Out}(U, \mathcal{F})$.*

Proof. Assume that $\bar{\kappa}_{G/Z(G)}$ is injective. Since $Z(G)$ is a p'-group (since $Z(G) \le \bar{T}$), and since $O^{p'}(G) = G$ by definition of $\mathfrak{Lie}(p)$, $\mathrm{Aut}(G)$ injects into $\mathrm{Aut}(G/Z(G))$, and hence $\bar{\kappa}_G$ is injective. It thus suffices to prove the lemma when G is in adjoint form.

We can thus assume Notation 4.1. By Lemma 1.4, it will suffice to prove that $C_{\mathrm{Aut}(G)}(U) \le \mathrm{Inn}(G)$. Fix $\beta \in \mathrm{Aut}(G)$ such that $\beta|_U = \mathrm{Id}_U$. By Theorem 4.2, there are unique automorphisms $c_u \in \mathrm{Aut}_U(G)$, $d \in \mathrm{Aut}_{\bar{T}}(G)$, $f \in \Phi_G$, and $g \in \Gamma_G$ such that $\beta = c_u d f g$.

If $g \ne \mathrm{Id}$, then it permutes the fundamental root groups nontrivially, while $c_u d f|_U$ sends each such group to itself modulo higher root groups and commutators. Hence $g = \mathrm{Id}$. Similarly, $f = \mathrm{Id}$, since otherwise β would act on the fundamental root groups (modulo higher root groups) via some automorphism other than a translation.

Thus $\beta = c_u d$, where $d = c_t$ for some $t \in N_{\bar{T}}(G)$. Then u has p-power order while t has order prime to p, so $d|_U = c_t|_U = \mathrm{Id}$. By Lemma 2.4(c), c_t sends each root group in $\bar{U}$ to itself via $x_\alpha(u) \mapsto x_\alpha(\theta_\alpha(t){\cdot}u)$ for some character $\theta_\alpha \in \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_p^\times)$ which is linear in α. For each $\widehat{\alpha} \in \widehat{\Sigma}_+$, $c_t|_{X_{\widehat{\alpha}}} = \mathrm{Id}$ implies that $\theta_\alpha(t) = 1$ for all $\alpha \in \widehat{\alpha}$. Thus $\theta_\alpha(t) = 1$ for all $\alpha \in \Sigma_+$, so $c_t = \mathrm{Id}_{\bar{G}}$, and $\beta = c_u \in \mathrm{Inn}(G)$. □

It now remains, when proving Theorem A, to show the surjectivity of κ_G. This will be done case-by-case. We first handle groups of Lie rank at least three, then those of rank one, and finally those of rank two.

For simplicity, we state the next two propositions only for groups of adjoint type, but they also hold without this restriction. The first implies that each element of $\mathrm{Aut}(U, \mathcal{F})$ permutes the subgroups $U_{\widehat{J}}$ (as defined in Notation 4.1), and that each element of $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L}^c_S(G))$ induces an automorphism of the amalgam of parabolics $\mathfrak{P}_{\widehat{J}}$ for $\widehat{J} \subsetneqq \widehat{\Pi}$.

Proposition 4.4. *Assume Notation 4.1. For $1 \ne P \le U$, the following are equivalent:*

(i) $P = U_{\widehat{J}}$ *for some* $\widehat{J} \subsetneqq \widehat{\Pi}$*;*

(ii) $P \trianglelefteq B$*,* $C_U(P) \le P$*, and* $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$*; and*

(iii) $P \trianglelefteq B$*,* $C_G(P) \le P$*, and* $O_p(N_G(P)) = P$*.*

Hence for each $\varphi \in \mathrm{Aut}(U, \mathcal{F})$, φ permutes the subgroups $U_{\widehat{J}}$, and in particular permutes the subgroups $U_{\widehat{\alpha}}$ for $\widehat{\alpha} \in \widehat{\Pi}$.

Proof. **(i) ⟹ (iii):** For each $\widehat{J} \subsetneqq \widehat{\Pi}$, $C_G(U_{\widehat{J}}) = Z(U_{\widehat{J}})$ by [GLS3, Theorem 2.6.5(e)] (recall that G is of adjoint type). Also, $O_p(N_G(U_{\widehat{J}})) = O_p(\mathfrak{P}_{\widehat{J}}) = U_{\widehat{J}}$, and $U_{\widehat{J}}$ is normal in B since $N_G(U_{\widehat{J}}) = \mathfrak{P}_{\widehat{J}} \ge B$.

(iii) ⟹ (ii): This holds since $\mathrm{Out}_{\mathcal{F}}(P) \cong N_G(P)/PC_G(P)$.

(ii) ⟹ (i): In this case, $P \trianglelefteq B$, so $N_G(P) \ge B$, and $N_G(P) = \mathfrak{P}_{\widehat{J}}$ for some $\widehat{J} \subsetneqq \widehat{\Pi}$ (cf. [Ca, Theorem 8.3.2]). Then $P \le O_p(\mathfrak{P}_{\widehat{J}}) = U_{\widehat{J}}$. Also, $U_{\widehat{J}} C_G(P)/PC_G(P) \le O_p(N_G(P)/PC_G(P)) = 1$, so $U_{\widehat{J}} \le PC_G(P)$. Since $U_{\widehat{J}} \le U$, this implies that $U_{\widehat{J}} \le PC_U(P) = P$; i.e., that $P = U_{\widehat{J}}$. So (i) holds.

The last statement follows from the equivalence of (i) and (ii). □

When G has large Lie rank, Theorem A now follows from properties of Tits buildings.

Proposition 4.5. *Assume $G \in \mathfrak{Lie}(p)$ is of adjoint type and has Lie rank at least 3. Fix $U \in \mathrm{Syl}_p(G)$. Then κ_G is split surjective.*

Proof. Set $\mathcal{L} = \mathcal{L}_U^c(G)$. By Proposition 4.4, for each $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$, α permutes the subgroups $U_{\widehat{J}}$ for $\widehat{J} \subsetneqq \widehat{\Pi}$. For each such $\widehat{J}$, $C_G(U_{\widehat{J}}) = Z(U_{\widehat{J}})$, so $\mathrm{Aut}_{\mathcal{L}}(U_{\widehat{J}}) = N_G(U_{\widehat{J}}) = \mathfrak{P}_{\widehat{J}}$. Thus α induces an automorphism of the amalgam of parabolic subgroups $\mathfrak{P}_{\widehat{J}}$. Since G is the amalgamated sum of these subgroups by a theorem of Tits (see [Ti, Theorem 13.5] or [Se, p. 95, Corollary 3]), α extends to a unique automorphism $\bar{\alpha}$ of G.

Thus $\alpha \mapsto \bar{\alpha}$ defines a homomorphism $\widehat{s}\colon \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}) \longrightarrow \mathrm{Aut}(G)$. If $\alpha = c_\gamma$ for $\gamma \in \mathrm{Aut}_{\mathcal{L}}(U) = N_G(U)$, then $\bar{\alpha}$ is conjugation by $\gamma \in G$ and hence lies in $\mathrm{Inn}(G)$. Hence $\widehat{s}$ factors through $s\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \longrightarrow \mathrm{Out}(G)$, $\kappa_G \circ s = \mathrm{Id}_{\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})}$, and thus κ_G is split surjective. □

Before we can handle the rank 1 case, two elementary lemmas are needed.

Lemma 4.6. *Let G be a finite group with normal Sylow p-subgroup $S \trianglelefteq G$. Fix subgroups $1 = S_0 < S_1 < \cdots < S_k = S$ normal in G such that the following hold:*

(i) $S_{k-1} \le \mathrm{Fr}(S)$;

(ii) $C_G(S) \le S$; *and*

(iii) *for each $1 \le i \le k-1$, S_i is characteristic in G, $[S, S_i] \le S_{i-1}$, S_i/S_{i-1} has exponent p, and $\mathrm{Hom}_{\mathbb{F}_p[G/S]}(S/\mathrm{Fr}(S), S_i/S_{i-1}) = 0$ (i.e., no irreducible $\mathbb{F}_p[G/S]$-submodule of S_i/S_{i-1} appears as a submodule of $S/\mathrm{Fr}(S)$).*

Let $\alpha \in \mathrm{Aut}(G)$ be such that $[\alpha, S] \le S_{k-1}$. Then $\alpha \in \mathrm{Aut}_S(G)$.

Proof. For $1 \ne g \in G$ of order prime to p, the conjugation action of g on S is nontrivial since $C_G(S) \le S$, and hence the conjugation action on $S/\mathrm{Fr}(S)$ is also nontrivial (see [G, Theorem 5.3.5]). Thus G/S acts faithfully on $S/\mathrm{Fr}(S)$. Since α induces the identity on $S/\mathrm{Fr}(S)$, α also induces the identity on G/S.

Assume first that $\alpha|_S = \mathrm{Id}$. Since S is a p-group and G/S has order prime to p, $H^1(G/S; Z(S)) = 0$. So by [OV, Lemma 1.2], $\alpha \in \mathrm{Inn}(G)$. If $g \in G$ is such that $\alpha = c_g$, then $[g, S] = 1$ since $\alpha|_S = \mathrm{Id}$, and $g \in S$ since G/S acts faithfully on $S/\mathrm{Fr}(S)$. Thus $\alpha \in \mathrm{Aut}_S(G)$ in this case.

In particular, this proves the lemma when $k = 1$. So assume $k \ge 2$. We can assume inductively that the lemma holds for G/S_1, and hence can arrange (after composing by an appropriate element of $\mathrm{Aut}_S(G)$) that α induces the identity on G/S_1.

Let $\varphi \in \mathrm{Hom}(S, S_1)$ be such that $\alpha(x) = x\varphi(x)$ for each $x \in S$ (a homomorphism since $S_1 \le Z(S)$). Then φ factors through $\bar{\varphi} \in \mathrm{Hom}(S/\mathrm{Fr}(S), S_1)$ since S_1 is elementary abelian, and $\bar{\varphi}$ is a homomorphism of $\mathbb{F}_p[G/S]$-modules since $\alpha(g) \equiv g \pmod{S_1}$ for each $g \in G$ (and $S_1 \le Z(S)$). Thus $\varphi = 1$ since $\mathrm{Hom}_{G/S}(S_k/S_{k-1}, S_1) = 0$ by (iii), so $\alpha|_S = \mathrm{Id}$, and we already showed that this implies $\alpha \in \mathrm{Aut}_S(G)$. □

The next lemma will be useful when checking the hypotheses of Lemma 4.6.

Lemma 4.7. *Fix a prime p and $e \ge 1$, and set $q = p^e$ and $\Gamma = \mathbb{F}_q^\times$. For each $a \in \mathbb{Z}$, set $V_a = \mathbb{F}_q$, regarded as an $\mathbb{F}_p\Gamma$-module with action $\lambda(x) = \lambda^a x$ for $\lambda \in \Gamma$ and $x \in \mathbb{F}_q$.*

(a) *For each a, V_a is $\mathbb{F}_p\Gamma$-irreducible if and only if $a/\gcd(a, q-1)$ does not divide $p^t - 1$ for any $t|e$, $t < e$.*

(b) *For each $a, b \in \mathbb{Z}$, $V_a \cong V_b$ as $\mathbb{F}_p\Gamma$-modules if and only if $a \equiv bp^i \pmod{q-1}$ for some $i \in \mathbb{Z}$.*

Proof. **(a)** Set $d = \gcd(a, q-1)$, and let t be the order of p in $(\mathbb{Z}/\frac{q-1}{d})^\times$. Thus $t|e$ since $\frac{q-1}{d}\big|(p^e-1)$. If $t < e$, then $\lambda^a \in \mathbb{F}_{p^t}$ for each $\lambda \in \mathbb{F}_q$, so $0 \neq \mathbb{F}_{p^t} \subsetneqq V_a$ is a proper $\mathbb{F}_p\Gamma$-submodule, and V_a is reducible.

Conversely, if V_a is reducible, then it contains a proper submodule $0 \neq W \subsetneqq V_a$ of dimension i, some $0 < i < e$. All Γ-orbits in $V_a \smallsetminus 0$, hence in $W \smallsetminus 0$, have length $\frac{q-1}{d}$, so $\frac{q-1}{d}\big|(p^i-1)$, and $t \leq i < e$.

(b) For each $a \in \mathbb{Z}$, let $\bar{V}_a \cong \mathbb{F}_q$ be the $\mathbb{F}_q\Gamma$-module where Γ acts via $\lambda(x) = \lambda^a x$. Then $\mathbb{F}_q \otimes_{\mathbb{F}_p} V_a \cong \bar{V}_a \oplus \bar{V}_{ap} \oplus \cdots \oplus \bar{V}_{ap^{e-1}}$ as $\mathbb{F}_q\Gamma$-modules. Since $\bar{V}_b \cong \bar{V}_a$ if and only if $b \equiv a \pmod{q-1}$, $V_b \cong V_a$ if and only if $b \equiv ap^i \pmod{q-1}$ for some i. □

In principle, we don't need to look at the fusion systems of the simple groups of Lie rank 1 if we only want to prove tameness. Their fusion is controlled by the Borel subgroup, so their fusion systems are tame by Proposition 1.6. But the following proposition is needed when proving Theorem A in its stronger form, and will also be used when working with groups of larger Lie rank.

Proposition 4.8. *Fix a prime p, and a group $G \in \mathfrak{Lie}(p)$ of Lie rank 1. Assume $(G,p) \not\cong (\mathrm{Sz}(2),2)$. Then each $\varphi \in \mathrm{Aut}(U,\mathcal{F})$ extends to an automorphism of G. Also, if $[\varphi,U] \leq [U,U]$, then $\varphi \in \mathrm{Inn}(U)$.*

Proof. If G is of universal form, then $Z(G)$ is cyclic of order prime to p by Proposition 3.8. For each $Z \leq Z(G)$, $\mathrm{Out}(G/Z) \cong \mathrm{Out}(G)$ by [GLS3, Theorem 2.5.14(d)], and $\mathrm{Out}(U,\mathcal{F}_U(G/Z)) \cong \mathrm{Out}(U,\mathcal{F}_U(G))$ since G and G/Z have the same p-fusion systems. It thus suffices to prove the proposition when G has adjoint form.

Assume first $G = PSL_2(q)$. Thus $U \cong \mathbb{F}_q$ (as an additive group), $T \cong C_{(q-1)/\varepsilon}$ where $\varepsilon = \gcd(q-1,2)$, and $\Gamma \overset{\text{def}}{=} \mathrm{Aut}_T(U)$ is the subgroup of index ε in $\mathbb{F}_q^\times$. If $\varphi \in \mathrm{Aut}(U)$ is fusion preserving, then under these identifications, there is $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(u)\varphi(v) = \varphi(uv)$ for each $u \in \Gamma \leq \mathbb{F}_q^\times$ and $v \in \mathbb{F}_q$. After composing with an appropriate diagonal automorphism (conjugation by a diagonal element of $PGL_2(q)$), we can assume that $\varphi(1) = 1$. Hence the above formula (with $v = 1$) implies that $\alpha = \varphi|_\Gamma$, and thus that $\varphi(uv) = \varphi(u)\varphi(v)$ for each $u, v \in \mathbb{F}_q$ with $u \in \Gamma$. If $\varepsilon = 1$, then φ acts as a field automorphism on U, hence is the restriction of a field automorphism of G, and we are done. Otherwise, there is $u \in \Gamma$ such that $\mathbb{F}_q = \mathbb{F}_p(u)$, u and $\varphi(u)$ have the same minimal polynomial over $\mathbb{F}_p$, and there is $\psi \in \mathrm{Aut}(\mathbb{F}_q)$ (a field automorphism) such that $\psi(u) = \varphi(u)$. Thus $\psi(u^i) = \varphi(u^i)$ for each i, so $\psi = \varphi$ since both are additive homomorphisms, and hence φ extends to a field automorphism of G. (Note that this argument also holds when $q = 3$ and $\Gamma = 1$.)

Next assume $G = PSU_3(q)$. Following the conventions in [H, Satz II.10.12(b)], we identify

$$U = \big\{[\![a,b]\!] \,\big|\, a,b \in \mathbb{F}_{q^2},\ b + b^q = -a^{q+1}\big\} \qquad \text{where} \qquad [\![a,b]\!] = \begin{pmatrix} 1 & a & b \\ 0 & 1 & -a^q \\ 0 & 0 & 1 \end{pmatrix};$$
$$T = \big\{d(\lambda) \,\big|\, \lambda \in \mathbb{F}_{q^2}^\times\big\} \qquad \text{where} \qquad d(\lambda) = \mathrm{diag}(\lambda^{-q}, \lambda^{q-1}, \lambda).$$

Here, whenever we write a matrix, we mean its class in $PSU_3(q)$. Then $B = UT = N_G(U) \leq G$ (see [H, Satz II.10.12(b)]), and

$$[\![a,b]\!] \cdot [\![c,d]\!] = [\![a+c, b+d-ac^q]\!] \qquad \text{and} \qquad {}^{d(\lambda)}[\![a,b]\!] = [\![\lambda^{1-2q}a, \lambda^{-1-q}b]\!]\,.$$

Set $\varepsilon = \gcd(2q-1, q^2-1) = \gcd(2q-1, q^2-2q) = \gcd(q+1,3)$. Then $d(\lambda) = 1$ exactly when $\lambda^\varepsilon = 1$, $C_T(U) = 1$, and hence $|T| = |\mathrm{Aut}_B(U/Z(U))| = (q^2-1)/\varepsilon$. If $q > 2$, then $|T|$ does not divide $p^i - 1$ for any power $1 < p^i < q^2$, and by Lemma 4.7(a), $U/Z(U)$ and

$Z(U)$ are both irreducible as $\mathbb{F}_p[T]$-modules. (Note, in particular, the cases $q = 5$ and $q = 8$, where $(U/Z(U), T)$ is isomorphic to $(\mathbb{F}_{25}, C_8)$ and $(\mathbb{F}_{64}, C_{21})$, respectively.)

Fix $\varphi \in \mathrm{Aut}(U, \mathcal{F})$, and extend it to $\alpha \in \mathrm{Aut}(B)$ (Lemma 1.12). Via the same argument as that used when $G = PSL_2(q)$, we can arrange (without changing the class of φ modulo $\mathrm{Im}(\bar{\kappa}_G)$) that $\varphi \equiv \mathrm{Id} \pmod{[U, U]}$. If $q > 2$, then the hypotheses of Lemma 4.6 hold (with $[U, U] < U < B$ in the role of $S_1 < S_2 = S < G$), so $\alpha \in \mathrm{Aut}_U(B)$ and $\varphi \in \mathrm{Inn}(U)$.

If $G \cong PSU_3(2) \cong C_3^2 \rtimes Q_8$ (cf. [Ta, p. 123–124]), then $U \cong Q_8$ and $T = 1$, so $\mathrm{Out}(U, \mathcal{F}) = \mathrm{Out}(U) \cong \Sigma_3$. By Theorem 3.4 (or by direct computation), $\mathrm{Out}(G) = \mathrm{Outdiag}(G)\Phi_G$ has order six, since $|\mathrm{Outdiag}(G)| = \gcd(3, q+1) = 3$ and $|\Phi_G| = 2$. Thus $\bar{\kappa}_G$ is an isomorphism, since it is injective by Lemma 4.3.

The proof when $G = Sz(q)$ is similar. Set $\theta = \sqrt{2q}$. We follow the notation in [HB, § XI.3], and identify U as the group of all $S(a, b)$ for $a, b \in \mathbb{F}_q$ and $T < B = N_G(U)$ as the group of all $d(\lambda)$ for $\lambda \in \mathbb{F}_q^\times$, with relations

$$S(a, b) \cdot S(c, d) = S(a + c, b + d + a^\theta c) \quad \text{and} \quad {}^{d(\lambda)}S(a, b) = S(\lambda a, \lambda^{1+\theta} b)\,.$$

As in the last case, we can arrange that $\varphi \in \mathrm{Aut}(U, \mathcal{F})$ is the identity modulo $[U, U]$. Since $q \geq 8$ ($q \neq 2$ by hypothesis), $Z(U)$ and $U/Z(U)$ are nonisomorphic, irreducible $\mathbb{F}_2 T$-modules by Lemma 4.7(a,b) (and since $Z(U) \cong V_{1+\theta}$ and $U/Z(U) \cong V_1$ in the notation of that lemma). We can thus apply Lemma 4.6 to show that $\varphi \in \mathrm{Inn}(U)$.

It remains to handle the Ree groups ${}^2G_2(q)$, where $q = 3^m$ for some odd $m \geq 1$. Set $\theta = \sqrt{3q}$. We use the notation in [HB, Theorem XI.13.2], and identify $U = (\mathbb{F}_q)^3$ with multiplication given by

$$(x_1, y_1, z_1){\cdot}(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2 + x_1{\cdot}x_2^\theta, z_1 + z_2 - x_1{\cdot}y_2 + y_1{\cdot}x_2 - x_1{\cdot}x_1^\theta{\cdot}x_2)\,.$$

Note that $x^{\theta^2} = x^3$. Let $T \leq B = N_G(U)$ be the set of all $d(\lambda)$ for $\lambda \in \mathbb{F}_q^\times$, acting on U via

$${}^{d(\lambda)}(x, y, z) = (\lambda x, \lambda^{\theta+1} y, \lambda^{\theta+2} z).$$

Again, we first reduce to the case where $\varphi \in \mathrm{Aut}(U, \mathcal{F})$ is such that $[\varphi, U] \leq [U, U]$, and extend φ to $\alpha \in \mathrm{Aut}(B)$. If $q > 3$, then $U/[U, U] \cong V_1$, $[U, U]/Z(U) \cong V_{\theta+1}$, and $Z(U) \cong V_{\theta+2}$ are irreducible and pairwise nonisomorphic as $\mathbb{F}_3 T$-modules by Lemma 4.7 (for V_a as defined in that lemma), since neither $\theta + 1$ nor $\theta + 2$ is a power of 3. So $\varphi \in \mathrm{Inn}(U)$ by Lemma 4.6.

If $q = 3$, then $U = \langle a, b \rangle$, where $|a| = 9$, $|b| = 3$, and $[a, b] = a^3$. Set $Q_i = \langle ab^i \rangle \cong C_9$ ($i = 0, 1, 2$): the three subgroups of U isomorphic to C_9. Let $\mathrm{Aut}^0(U) \leq \mathrm{Aut}(U)$ be the group of those $\alpha \in \mathrm{Aut}(U)$ which send each Q_i to itself. For each such α, the induced action on $U/Z(U)$ sends each subgroup of order three to itself, hence is the identity or $(g \mapsto g^{-1})$, and the latter is seen to be impossible using the relation $[a, b] = a^3$. Thus each $\alpha \in \mathrm{Aut}^0(U)$ induces the identity on $U/Z(U)$ and on $Z(U)$, and has the form $\alpha(g) = g\varphi(g)$ for some $\varphi \in \mathrm{Hom}(U/Z(U), Z(U))$. So $\mathrm{Aut}^0(U) = \mathrm{Inn}(U)$ since they both have order 9 (and clearly $\mathrm{Inn}(U) \leq \mathrm{Aut}^0(U)$). The action of $\mathrm{Aut}(U)$ on $\{Q_0, Q_1, Q_2\}$ thus defines an embedding of $\mathrm{Out}(U)$ into Σ_3, and the automorphisms $(a, b) \mapsto (ab, b)$ and $(a, b) \mapsto (a^{-1}, b)$ show that $\mathrm{Out}(U) \cong \Sigma_3$. Since $|\mathrm{Out}_{\mathcal{F}}(U)| = 2$ and $\mathrm{Aut}_{\mathcal{F}}(U) \trianglelefteq \mathrm{Aut}(U, \mathcal{F})$, it follows that $\mathrm{Out}(U, \mathcal{F}) = 1 = \mathrm{Out}(G)$. (See also [BC, Theorem 2] for more discussion about $\mathrm{Aut}(U)$.) □

It remains to show that κ_G (at the prime p) is surjective when $G \in \mathfrak{Lie}(p)$ has Lie rank 2, with the one exception when $G \cong SL_3(2)$. Our proof is based on ideas taken from the article of Delgado and Stellmacher [DS], even though in the end, we do not actually need to refer to any of their results in our argument. The third author would like to thank Richard Weiss for explaining many of the details of how to apply the results in [DS], and also to Andy Chermak and Sergey Shpectorov for first pointing out the connection.

Fix a prime p, and a finite group $G \in \mathfrak{Lie}(p)$ of Lie rank two. We assume Notation 2.2 and 4.1. In particular, $(\bar{G}, \sigma)$ is a σ-setup for G, $\bar{T} \leq \bar{G}$ is a maximal torus, $U \in \mathrm{Syl}_p(G)$ is generated by the positive root subgroups, and $B = N_G(U)$ is a Borel subgroup. Set $\widehat{\Pi} = \{\widehat{\alpha}_1, \widehat{\alpha}_2\}$, and set $\mathfrak{P}_1 = \mathfrak{P}_{\widehat{\alpha}_1} = \langle B, X_{-\widehat{\alpha}_1} \rangle$ and $\mathfrak{P}_2 = \mathfrak{P}_{\widehat{\alpha}_2} = \langle B, X_{-\widehat{\alpha}_2} \rangle$: the two maximal parabolic subgroups of G containing B. Our proofs are based on the following observation:

Lemma 4.9. *Assume, for $G \in \mathfrak{Lie}(p)$ of rank 2 and its amalgam of parabolics as above, that*

each automorphism of the amalgam $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$ extends to an automorphism of G. $(*)$

Then κ_G is surjective.

Here, by an automorphism of the amalgam, we mean a pair (χ_1, χ_2), where either $\chi_i \in \mathrm{Aut}(\mathfrak{P}_i)$ for $i = 1, 2$ or $\chi_i \in \mathrm{Iso}(\mathfrak{P}_i, \mathfrak{P}_{3-i})$ for $i = 1, 2$, and also $\chi_1|_B = \chi_2|_B$.

Proof. Set $\mathcal{L} = \mathcal{L}_U^c(G)$ and $U_i = O_p(\mathfrak{P}_i)$. By Proposition 4.4, each $\chi \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ either sends U_1 and U_2 to themselves or exchanges them. For each $i = 1, 2$, $C_G(U_i) \leq U_i$, so $\mathrm{Aut}_{\mathcal{L}}(U_i) = N_G(U_i) = \mathfrak{P}_i$. Thus χ induces an automorphism of the amalgam $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$. By assumption, this extends to an automorphism $\bar{\chi}$ of G, and $\kappa_G(\bar{\chi}) = \xi$. □

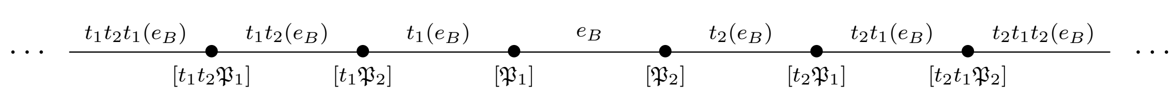
Set $\mathfrak{G} = \mathfrak{P}_1 *_B \mathfrak{P}_2$: the amalgamated free product over B. Let $\rho\colon \mathfrak{G} \longrightarrow G$ be the natural surjective homomorphism. Since each automorphism of the amalgam induces an automorphism of $\mathfrak{G}$, $(*)$ holds if for each automorphism of $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$, the induced automorphism of $\mathfrak{G}$ sends $\mathrm{Ker}(\rho)$ to itself.

Let Γ be the tree corresponding to the amalgam $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$. Thus Γ has a vertex $[g\mathfrak{P}_i]$ for each coset $g\mathfrak{P}_i$ (for all $g \in \mathfrak{G}$ and $i = 1, 2$), and an edge $g(e_B)$ connecting $[g\mathfrak{P}_1]$ to $[g\mathfrak{P}_2]$ for each coset gB in $\mathfrak{G}$. Also, $\mathfrak{G}$ acts on Γ via its canonical action on the cosets, and in particular, it acts on $g(e_B)$ with stabilizer subgroup ${}^gB$.

Similarly, let Γ_G be the graph of G with respect to the same amalgam: the graph with vertex set $(G/\mathfrak{P}_1) \cup (G/\mathfrak{P}_2)$ and edge set G/B. Equivalently, since $\mathfrak{P}_1$, $\mathfrak{P}_2$, and B are self-normalizing, Γ_G is the graph whose vertices are the maximal parabolics in G and whose edges are the Borel subgroups. Let $\widehat{\rho}\colon \Gamma \longrightarrow \Gamma_G$ be the canonical map which sends a vertex $[g\mathfrak{P}_i]$ in Γ to the vertex in Γ_G corresponding to the image of $g\mathfrak{P}_i$ in G.

Fix a subgroup $N \leq G$ such that (B, N) is a BN-pair for G, and such that $B \cap N = T$ and $N/T \cong W_0$ (where T and W_0 are as defined in Notation 2.2). We refer to [Ca, §§ 8.2, 13.5] for the definition of BN-pairs, and the proof that G has a BN-pair (B, N) which satisfies these conditions. In order to stay close to the notation in [DS], we also set T: their notation for the Cartan subgroup. For $i = 1, 2$, choose $t_i \in (N \cap \mathfrak{P}_i) \smallsetminus B = (N \cap \mathfrak{P}_i) \smallsetminus T$. Since $(N \cap \mathfrak{P}_i)/T \cong C_2$ and $N = \langle N \cap \mathfrak{P}_1, N \cap \mathfrak{P}_2 \rangle$, we have $N = T\langle t_1, t_2 \rangle$, consistent with the notation in [DS]. Note that T can be the trivial subgroup. We also regard the $t_i \in \mathfrak{P}_i$ as elements of $\mathfrak{G}$, and $T \leq B$ as a subgroup of $\mathfrak{G}$, when appropriate.

Let $\mathscr{T}$ be the union of the edges in the $T\langle t_1, t_2 \rangle$-orbit of e_B. Thus $\mathscr{T}$ is a path of infinite length in Γ of the following form:



Thus $\widehat{\rho}(\mathscr{T})$ is an apartment in the building Γ_G under Tits's definition and construction of these structures in [Ti, 3.2.6].

A path in Γ is always understood not to double back on itself.

Lemma 4.10. *Let G and Γ be as above. Let $n \in \{3,4,6,8\}$ be such that $W_0 \cong D_{2n}$. Then each path in Γ of length at most $n+1$ is contained in $g(\mathscr{T})$ for some $g \in \mathfrak{G}$.*

Proof. A path of length 1 is an edge, and is in the $\mathfrak{G}$-orbit of e_B which has stabilizer group B. If e_B is extended to a path of length 2 with the edge $t_i(e_B)$ ($i = 1$ or 2), then this path has stabilizer group

$$B \cap {}^{t_i}B = \prod_{\widehat{\alpha} \in \widehat{\Sigma}_+ \smallsetminus \{\widehat{\alpha}_i\}} X_{\widehat{\alpha}} \cdot T\,.$$

(Recall that ${}^{t_i}X_{\widehat{\alpha}_i} = X_{-\widehat{\alpha}_i}$, and $X_{-\widehat{\alpha}_i} \cap B = 1$ by [Ca, Lemma 7.1.2].) Thus the stabilizer subgroup has index p^j in B, where $p^j = |X_{\widehat{\alpha}_i}|$. Furthermore, $|\mathfrak{P}_i/B| = 1 + p^j$, since by [Ca, Proposition 8.2.2(ii)],

$$\mathfrak{P}_i = B \cup (Bt_iB) \quad \text{where} \quad |Bt_iB| = |B| \cdot |B/(B \cap {}^{t_i}B)| = |B| \cdot p^j\,.$$

Hence there are exactly p^j extensions of e_B to a path of length 2 containing the vertex $[\mathfrak{P}_i]$ in the interior, and these are permuted transitively by B.

Upon continuing this argument, we see inductively that for all $2 \leq k \leq n+1$, the paths of length k starting at e_B with endpoint $[\mathfrak{P}_{3-i}]$ are permuted transitively by B, and of them, the one contained in $\mathscr{T}$ has stabilizer subgroup the product of T with $(n+1-k)$ root subgroups in U. (Recall that $B = TU$, and U is the product of n root subgroups.) Since $\mathfrak{G}$ acts transitively on the set of edges in Γ, each path of length k is in the $\mathfrak{G}$-orbit of one which begins with e_B (and with endpoint $[\mathfrak{P}_1]$ or $[\mathfrak{P}_2]$), and hence in the $\mathfrak{G}$-orbit of a subpath of $\mathscr{T}$. □

Proposition 4.11. *Let G, $\mathfrak{G}$, and (T, t_1, t_2) be as above, and let n be such that $W_0 \cong D_{2n}$. Assume that*

for each $(\chi_1, \chi_2) \in \mathrm{Aut}\big(\mathfrak{P}_1 > B < \mathfrak{P}_2\big)$, where $\chi_i \in \mathrm{Aut}(\mathfrak{P}_i)$ or $\chi_i \in \mathrm{Iso}(\mathfrak{P}_i, \mathfrak{P}_{3-i})$ for $i = 1,2$, we have $(\chi_1(t_1)\chi_2(t_2))^n \in \chi_1(T)$. (†)

Then (∗) holds (each automorphism of $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$ extends to an automorphism of G), and hence κ_G is onto.

Proof. Let $\approx$ be the equivalence relation on the set of vertices in Γ generated by setting $x \approx y$ if x and y are of distance $2n$ apart in some path in the $\mathfrak{G}$-orbit of $\mathscr{T}$. Since $T\langle t_1, t_2\rangle/T \cong D_{2n}$ as a subgroup of $N_G(T)/T$, the natural map $\widehat{\rho}\colon \Gamma \longrightarrow \Gamma_G$ sends $\mathscr{T}$ to a loop of length $2n$, and hence sends all apartments in the $\mathfrak{G}$-orbit of $\mathscr{T}$ to loops of length $2n$. Hence $\Gamma \longrightarrow \Gamma_G$ factors through $\Gamma/{\approx}$.

We claim that

Γ_G contains no loops of length strictly less than $2n$; and (1)

each pair of points in $\Gamma/{\approx}$ is connected by a path of length at most n. (2)

Assume (1) does not hold: let L be a loop of minimal length $2k$ ($k < n$), and fix edges $\sigma_i = [x_i, y_i]$ in L ($i = 1,2$) such that the path from x_i to y_{3-i} in L has length $k-1$. Since Γ_G is a building whose apartments are loops of length $2n$ [Ti, 3.2.6], there is an apartment Σ which contains σ_1 and σ_2. By [Ti, Theorem 3.3] or [Br, p. 86], there is a retraction of Γ_G onto Σ. Hence the path from x_i to y_{3-i} in Σ (for $i = 1,2$) has length at most $k-1$, these two paths must be equal to the minimal paths in L since there are no loops of length less than $2k$, and this contradicts the assumption that L and Σ are loops of different lengths. (See also [Br, § IV.3, Exercise 1]. Point (1) also follows since Γ_G is a generalized n-gon in the sense of Tits [Br, p. 117], and hence any two vertices are joined by at most one path of length less than n.)

Now assume (2) does not hold: let x, y be vertices in Γ such that the shortest path between their classes in $\Gamma/\!\approx$ has length $k \geq n+1$. Upon replacing x and y by other vertices in their equivalence classes, if needed, we can assume that the path $[x, y]$ in Γ has length k. Let x^* be the vertex in the path $[x, y]$ of distance $n+1$ from x. By Lemma 4.10, $[x, z]$ is contained in $g(\mathscr{T})$ for some $g \in \mathfrak{G}$; let x' be the vertex in $g(\mathscr{T})$ of distance $2n$ from x and distance $n-1$ from z. Then $x' \approx x$, and $[x', y]$ has length at most $(n-1)+(k-n-1) = k-2$, a contradiction. This proves (2).

Assume the map $(\Gamma/\!\approx) \longrightarrow \Gamma_G$ induced by $\widehat{\rho}$ is not an isomorphism of graphs, and let x and y be distinct vertices in $\Gamma/\!\approx$ whose images are equal in Γ_G. By (2), there is a path from x to y of length at most n, and of even length since the graph is bipartite. This path cannot have length 2 since $\widehat{\rho}\colon \Gamma \longrightarrow \Gamma_G$ preserves valence, so its image in Γ_G is a loop of length at most n, and this contradicts (1). We conclude that $\Gamma_G \cong \Gamma/\!\approx$.

Now let (χ_1, χ_2) be an automorphism of the amalgam $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$. Let $\chi \in \mathrm{Aut}(\mathfrak{G})$ be the induced automorphism of the amalgamated free product, and let $\widehat{\chi} \in \mathrm{Aut}(\Gamma)$ be the automorphism which sends a vertex $[g\mathfrak{P}_i]$ to $[\chi(g\mathfrak{P}_i)]$. Since $(\chi_1(t_1)\chi_2(t_2))^n = 1$ in G by assumption, $\widehat{\rho}(\widehat{\chi}(\mathscr{T}))$ is a loop of length $2n$ in Γ_G. Hence $\widehat{\rho} \circ \widehat{\psi}$ factors through $(\Gamma/\!\approx) \cong \Gamma_G$, and by an argument similar to that used to show that $\Gamma_G \cong \Gamma/\!\approx$, the induced map $\Gamma_G \longrightarrow \Gamma_G$ is an automorphism of Γ_G. So χ sends $\mathrm{Ker}[\mathfrak{G} \xrightarrow{\ \rho\ } G]$ to itself, and thus induces an automorphism of G. The last statement (κ_G is onto) now follows from Lemma 4.9. □

It remains to find conditions under which (†) holds. The following proposition handles all but a small number of cases.

Proposition 4.12. *Assume $N = N_G(T)$ (and hence $N_G(T)/T$ is dihedral of order $2n$). Then (†) holds, and hence each automorphism of the amalgam $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$ extends to an automorphism of G. In particular, (†) and (∗) hold, and hence κ_G is onto, whenever $G = {}^rX_n(q) \in \mathfrak{Lie}(p)$ has Lie rank 2 for $q > 2$ and $G \not\cong Sp_4(3)$.*

Proof. Assume that $N_G(T) = N = T\langle t_1, t_2\rangle$. Then the choices of the t_i are unique modulo T. Also, any two choices of T are B-conjugate, so each automorphism of the amalgam is B-conjugate to one which sends $\mathscr{T}$ to itself. Thus (†) holds, and so (∗) follows from Proposition 4.11.

The last statement now follows from Proposition 3.9. Note that if (†) holds for G of universal type, then it also holds for $G/Z(G)$ of adjoint type. □

What can go wrong, and what does go wrong when $G = SL_3(2)$, is that an automorphism of the amalgam can send t_1, t_2 to another pair of elements whose product (modulo T) has order strictly greater than $2n$. This happens when $\mathscr{T}$ is sent to another path not in the $\mathfrak{G}$-orbit of $\mathscr{T}$: one whose image in Γ_G is a loop of a different length.

Example 4.13. *Assume $G = SL_3(2)$. In particular, $T = 1$. Let B be the group of upper triangular matrices, let t_1 and t_2 be the permutation matrices for $(1\,2)$ and $(2\,3)$, respectively, and set $\mathfrak{P}_i = \langle B, t_i\rangle$.*

Consider the automorphism α of the amalgam which is the identity on $\mathfrak{P}_1$ (hence on B), and which is conjugation by e_{13} (the involution in $Z(B)$) on $\mathfrak{P}_2$. Set $t_i' = \alpha(t_i)$. Thus

$$t_1' = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix} \qquad \text{and} \qquad t_2' = \begin{pmatrix} 1&1&1\\ 0&0&1\\ 0&1&0 \end{pmatrix}.$$

One checks that $t_1't_2'$ has order 4, so that $\langle t_1', t_2'\rangle \cong D_8$ while $\langle t_1, t_2\rangle \cong D_6$. In other words, α sends the lifting (from Γ_G to Γ) of a loop of length 6 to the lifting of a loop of length 8, hence is not compatible with the relation $\approx$, hence does not extend to an automorphism of G.

We are left with seven cases: four cases with $n = 4$, two with $n = 6$, and one with $n = 8$. Those with $n = 4$ are relatively easy to handle.

Proposition 4.14. *Assume G is one of the groups $Sp_4(2)$, $PSp_4(3)$, $PSU_4(2)$, or $PSU_5(2)$. Then (†) holds, and hence (∗) also holds and κ_G is onto.*

Proof. In all cases, we work in the universal groups $Sp_4(q)$ and $SU_n(2)$, but the arguments are unchanged if we replace the subgroups described below by their images in the adjoint group. Recall that p is always the defining characteristic, so the second and third cases are distinct, even though $PSp_4(3) \cong SU_4(2)$ (see [Wi, § 3.12.4] or [Ta, Corollary 10.19]).

Let (χ_1, χ_2) be an automorphism of $(\mathfrak{P}_1 > B < \mathfrak{P}_2)$. Since all subgroups of B isomorphic to T are conjugate to T by the Schur-Zassenhaus theorem, we can also assume that $\chi_i(T) = T$. Set $\chi_0 = \chi_1|_B = \chi_2|_B$ and $t_i^* = \chi_i(t_i)$ for short; we must show that $|t_1^*t_2^*| = n = 4$. Note that $t_1^*t_2^*$ has order at least 4, since otherwise Γ_G would contain a loop of length strictly less than $8 = 2n$, which is impossible by point (1) in the proof of Proposition 4.11.

$\boldsymbol{G = Sp_4(2) \cong \Sigma_6}$ **:** Set $G' = [G, G]$: the subgroup of index 2. The elements $x_\gamma(1)$ for $\gamma \in \Sigma$ are all $\mathrm{Aut}(G)$-conjugate: the long roots and the short roots are all W-conjugate and a graph automorphism exchanges them. Since these elements generate G, none of them are in G'. Hence for $i = 1, 2$, all involutions in

$$\big\langle x_{\alpha_i}(1), x_{-\alpha_i}(1)\big\rangle \cong GL_2(2) \cong \Sigma_3$$

lie in $G{\smallsetminus}G'$, and in particular, $t_i \in G{\smallsetminus}G'$.

Each automorphism of the amalgam sends the focal subgroup to itself (as a subgroup of B), and hence also sends the intersections $\mathfrak{P}_i \cap G'$ to themselves. So $t_1^*, t_2^* \in G{\smallsetminus}G'$, and $t_1^*t_2^* \in G' \cong A_6$. It follows that $|t_1^*t_2^*| \le 5$, and $|t_1^*t_2^*| = 4$ since every dihedral subgroup of order 10 in Σ_6 is contained in A_6.

$\boldsymbol{G = Sp_4(3)}$ **:** In this case, $T \cong C_2^2$, and $N_G(T) \cong SL_2(3) \wr C_2$. Hence $N_G(T)/T \cong A_4 \wr C_2$ contains elements of order 2, 3, 4, and 6, but no dihedral subgroups of order 12. Since $t_1^*t_2^*$ has order at least 4, $|t_1^*t_2^*| = 4$, and condition (†) holds.

$\boldsymbol{G = SU_n(2)}$ **for** $\boldsymbol{n = 4}$ **or 5 :** We regard these as matrix groups via

$$SU_n(2) = \big\{M \in SL_n(4) \,\big|\, \overline{M}^t = M^{-1}\big\} \qquad \text{where} \qquad \overline{(a_{ij})}^t = \big(\overline{a_{n+1-j,n+1-i}}\big)\,,$$

and where $\bar{x} = x^2$ for $x \in \mathbb{F}_4$. We can then take B to be the group of upper triangular matrices in $SU_n(2)$, U the group of strict upper triangular matrices, and T the group of diagonal matrices. We thus have

$$T = \big\{\mathrm{diag}(x, x^{-1}, x^{-1}, x) \,\big|\, x \in \mathbb{F}_4\big\} \cong C_3 \qquad \text{if } n = 4$$
$$T = \big\{\mathrm{diag}(x, y, xy, y, x) \,\big|\, x, y \in \mathbb{F}_4\big\} \cong C_3^2 \qquad \text{if } n = 5.$$

Since $N_G(T)$ must permute the eigenspaces of the action of T on $\mathbb{F}_4^n$, we have $N_{GU_n(2)}(T) \cong GU_2(2) \wr C_2$ (if $n = 4$) or $(GU_2(2) \wr C_2) \times \mathbb{F}_4^\times$ (if $n = 5$). So in both cases,

$$N_G(T)/T \cong PGU_2(2) \wr C_2 \cong \Sigma_3 \wr C_2 \cong C_3^2 \rtimes D_8\,.$$

Set $Q = N_G(T)/O_3(N_G(T)) \cong D_8$, and let $\psi\colon N_G(T) \longrightarrow Q$ be the natural projection. Set $Q_0 = \psi(C_G(T))$. Since $C_G(T)/T \cong \Sigma_3 \times \Sigma_3$ (the subgroup of elements which send each eigenspace to itself), $Q_0 \cong C_2^2$ and $C_G(T) = \psi^{-1}(Q_0)$.

Choose the indexing of the parabolics such that $\mathfrak{P}_1$ is the subgroup of elements which fix an isotropic point and $\mathfrak{P}_2$ of those which fix an isotropic line. Thus

$$\mathfrak{P}_1 = \left\{ \left(\begin{smallmatrix} u & v & x \\ 0 & A & w \\ 0 & 0 & u \end{smallmatrix}\right) \middle|\, A \in GU_{n-2}(2) \right\} \quad \text{and} \quad \mathfrak{P}_2 = \begin{cases} \left\{ \left(\begin{smallmatrix} A & X \\ 0 & (\bar{A}^t)^{-1} \end{smallmatrix}\right) \middle|\, A \in SL_2(4) \right\} & \text{if } n = 4 \\ \left\{ \left(\begin{smallmatrix} A & v & X \\ 0 & u & w \\ 0 & 0 & (\bar{A}^t)^{-1} \end{smallmatrix}\right) \middle|\, A \in GL_2(4) \right\} & \text{if } n = 5 \end{cases}$$

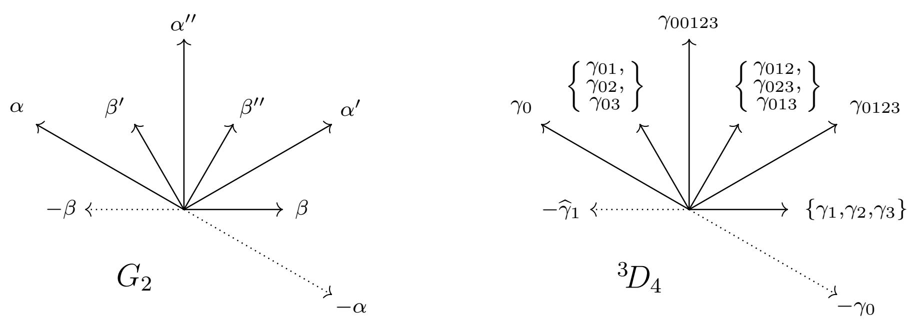
Then $\psi(N_{\mathfrak{P}_1}(T)) \leq Q_0$: no matrix in $\mathfrak{P}_1$ can normalize T and exchange its eigenspaces. Also, $N_B(T)$ contains $C_U(T) = \langle e_{1,n}(1), e_{2,n-1}(1) \rangle$, where $e_{i,j}(u)$ denotes the elementary matrix with unique off-diagonal entry u in position (i,j). Thus $Q_0 \geq \psi(N_{\mathfrak{P}_1}(T)) \geq \psi(N_B(T)) \cong C_2^2$, so these inclusions are all equalities. Also, $\mathfrak{P}_2$ contains the permutation matrix for the permutation $(1\,2)(n{-}1\,n)$, this element exchanges the eigenspaces of rank 2 for T, and so $\psi(N_{\mathfrak{P}_2}(T)) = Q$.

Since $T\langle t_1, t_2 \rangle / T \cong D_8$, $\langle \psi(t_1), \psi(t_2) \rangle = Q$, and so $\psi(t_1) \in Q_0 \smallsetminus Z(Q)$ and $\psi(t_2) \in Q \smallsetminus Q_0$. Since (χ_1, χ_2) induces an automorphism of the amalgam $(Q > Q_0 = Q_0)$, this implies that $\psi(t_1^*) \in Q_0 \smallsetminus Z(Q)$ and $\psi(t_2^*) \in Q \smallsetminus Q_0$. But then $\langle \psi(t_1^*), \psi(t_2^*) \rangle = Q$ since these elements generate modulo $Z(Q)$, so $|t_1^* t_2^*| \in 4\mathbb{Z}$, and $|t_1^* t_2^*| = 4$ since $N_G(T)/T \cong \Sigma_3 \wr C_2$ contains no elements of order 12. □

It remains to handle the groups $G_2(2)$, ${}^3D_4(2)$, and ${}^2F_4(2)$. In the first two cases, if t_i^* is an arbitrary involution in $N_{\mathfrak{P}_i}(T) \smallsetminus N_B(T)$ for $i = 1, 2$, then $t_1^* t_2^*$ can have order 6 or 8 (or order 7 or 12 when $G = G_2(2)$), and there does not seem to be any way to prove condition (†) short of analyzing automorphisms of the amalgam sufficiently to prove (∗) directly.

Let $\{\alpha, \beta\}$ be a fundamental system in the root system of G_2 where α is the long root. Let $\alpha, \alpha', \alpha''$ be the three long positive roots, and β, β', β'' the three short positive roots, as described in (3) below.

Let $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ denote the four fundamental roots in the D_4 root system, where γ_0 is in the center of the Dynkin diagram, and the other three are permuted cyclically by the triality automorphism. Set $\gamma_{ij} = \gamma_i + \gamma_j$ (when it is a root), etc. We identify the six classes of positive roots in ${}^3D_4$ with the roots in G_2 by identifying the following two diagrams:



(3)

The following list gives all nontrivial commutator relations among root subgroups of $G_2(q)$ or ${}^3D_4(q)$ (see [GLS3, Theorems 1.12.1(b) & 2.4.5(b)]):

$$[x_\alpha(u), x_\beta(v)] \equiv x_{\beta'}(\pm uv) x_{\beta''}(\pm uv^{1+q}) \pmod{X_{\alpha'} X_{\alpha''}} \tag{4}$$
$$[x_{\beta'}(u), x_\beta(v)] \equiv x_{\beta''}(\pm(uv^q + u^q v)) \pmod{X_{\alpha'} X_{\alpha''}} \tag{5}$$
$$[x_\alpha(u), x_{\alpha'}(v)] = x_{\alpha''}(\pm uv) \tag{6}$$
$$[x_{\beta'}(u), x_{\beta''}(v)] = x_{\alpha''}(\pm \mathrm{Tr}(uv^q)) \tag{7}$$
$$[x_{\beta''}(u), x_\beta(v)] = x_{\alpha'}(\pm \mathrm{Tr}(u^q v))\,. \tag{8}$$

Again, $\mathrm{Tr}\colon \mathbb{F}_{q^3} \longrightarrow \mathbb{F}_q$ denotes the trace. Note that when $G = G_2(q)$, then $u, v \in \mathbb{F}_q$ in all cases, and hence $u^q = u^{q^2} = u$, $u^{q+q^2} = u^2$, and $\mathrm{Tr}(u) = 3u$. When $G = {}^3D_4(q)$, the notation $x_\beta(-)$, $x_{\beta'}(-)$, and $x_{\beta''}(-)$ is somewhat ambiguous (and formula (5) depends on making the right choice), but this doesn't affect the arguments given below.

Proposition 4.15. *Assume $p = 2$ and $G = G_2(2)$. Then $(*)$ holds: each automorphism of the amalgam $(\mathfrak{P}_\alpha > B < \mathfrak{P}_\beta)$ extends to an automorphism of G. (In fact, each automorphism of the amalgam is conjugation by some element of B.) In particular, κ_G is onto.*

Proof. In this case, $T = 1$, and

$$\mathfrak{P}_\alpha \cong (C_4 \times C_4) \rtimes D_{12} \qquad \text{and} \qquad \mathfrak{P}_\beta \cong (Q_8 \times_{C_2} Q_8) \rtimes \Sigma_3 .$$

Also, $B = U$ has presentation $U = A \rtimes \langle r, t\rangle$, where

$$A = \langle a, b\rangle \cong C_4 \times C_4,\ \langle r, t\rangle \cong C_2^2,\ {}^r a = a^{-1},\ {}^r b = b^{-1},\ {}^t a = b,\ {}^t b = a .$$

In terms of the generators $x_\gamma = x_\gamma(1)$ for $\gamma \in \Sigma_+$, we have $A = \langle x_{\beta'} x_\beta, x_{\beta''} x_\beta\rangle$ and $\Omega_1(A) = \langle x_{\alpha'}, x_{\alpha''}\rangle$, and we can take $r = x_{\beta''}$, $t = x_\alpha$, and $a = x_\beta x_{\beta''}$ (and then $b = {}^t a$). Note that (5) takes the more precise form $[x_{\beta'}, x_\beta] = x_{\alpha'} x_{\alpha''}$ in this case. Also,

$$\begin{aligned} U_\alpha &= A\langle r\rangle \cong (C_4 \times C_4) \rtimes C_2 \\ U_\beta &= \langle ab^{-1}, a^2 t\rangle \times_{\langle a^2 b^2\rangle} \langle ab, a^2 rt\rangle \cong Q_8 \times_{C_2} Q_8 \\ U \cap G' &= A\langle t\rangle \cong C_4 \wr C_2 . \end{aligned}$$

The last formula holds since $G' = [G, G] \cong SU_3(3)$ has index two in G (see [Wi, § 4.4.4] or [Di, pp. 146–150]), since $x_\alpha, x_{\alpha'}, x_{\alpha''} \in G'$ (note that $x_\alpha = [x_{-\beta}, x_{\beta'}]$), and since x_β, $x_{\beta'}$, and $x_{\beta''}$ are all G-conjugate and hence none of them lies in G'.

Fix an automorphism $(\chi_\alpha, \chi_\beta)$ of the amalgam $(\mathfrak{P}_\alpha > B < \mathfrak{P}_\beta)$, and set $\chi_0 = \chi_\alpha|_B = \chi_\beta|_B$. Then $\chi_0 \in \mathrm{Aut}(U)$ sends each of the subgroups U_α, U_β, and $U \cap G'$ to itself. Hence it sends each quaternion factor in U_β to itself, and sends $U_\alpha \cap G' = \langle a, b\rangle$ to itself. After composing by an appropriate element of $\mathrm{Aut}_U(\mathfrak{P}_\beta)$, we can arrange that $\chi_0(ab) = ab$ and $\chi_0(ab^{-1}) = ab^{-1}$. In particular, χ_0 induces the identity on $\Omega_1(A)$ and hence also on $A/\Omega_1(A)$.

Let $g \in \mathfrak{P}_\alpha$ be an element of order 3, chosen so that ${}^g(a^2) = b^2$ and ${}^g(b^2) = a^2 b^2$. The image of $\langle g\rangle$ in $\mathfrak{P}_\alpha / A \cong D_{12}$ is normal, so $\chi_\alpha(g) \in Ag$. Let $x \in \Omega_1(A)$ be such that $\chi_\alpha(b) = \chi_0(b) = ax$. Then ${}^g b \in \langle ab, b^2\rangle \le C_A(\chi_0)$, so ${}^g b = \chi_\alpha({}^g b) = {}^g(bx)$ implies that ${}^g x = 1$ and hence $x = 1$. Thus $\chi_0|_A = \mathrm{Id}$. Also, $\chi_\alpha(\langle g\rangle) \in \mathrm{Syl}_3(\mathfrak{P}_\alpha)$ is conjugate to $\langle g\rangle$ by an element of A, so we can arrange that $\chi_\alpha(\langle g\rangle) = \langle g\rangle$ and hence that $\chi_\alpha|_{A\langle g\rangle} = \mathrm{Id}$. But then χ_α is the identity modulo $C_{\mathfrak{P}_\alpha}(A\langle g\rangle) = Z(A\langle g\rangle) = 1$, so $\chi_\alpha = \mathrm{Id}_{\mathfrak{P}_\alpha}$.

Since $\chi_\beta|_{U_\beta} = \mathrm{Id}$, χ_β induces the identity modulo $C_{\mathfrak{P}_\beta}(U_\beta) = Z(U_\beta) \cong C_2$. It thus has the form $\chi_\beta(x) = x\psi(x)$ for some $\psi \in \mathrm{Hom}(\mathfrak{P}_\beta, Z(U_\beta))$. Hence $\chi_\beta = \mathrm{Id}$, since it is the identity on $U \in \mathrm{Syl}_2(\mathfrak{P}_\beta)$. □

Proposition 4.16. *Assume $p = 2$ and $G = {}^3D_4(2)$. Then $(*)$ holds, and κ_G is onto.*

Proof. In this case, $T \cong \mathbb{F}_8^\times \cong C_7$, $\mathfrak{P}_\alpha / U_\alpha \cong C_7 \times \Sigma_3$, and $\mathfrak{P}_\beta / U_\beta \cong SL_2(8)$. Also, by (6) and (7), U_β is extraspecial with center $X_{\alpha''}$. Fix an automorphism $(\chi_\alpha, \chi_\beta)$ of the amalgam $(\mathfrak{P}_\alpha > B < \mathfrak{P}_\beta)$, and set $\chi_0 = \chi_\alpha|_B = \chi_\beta|_B$. We must show that χ_α and χ_β are the restrictions of some automorphism of G.

By Theorem 3.4, and since $\mathrm{Outdiag}(SL_2(8)) = 1 = \Gamma_{SL_2(8)}$, $\mathrm{Out}(\mathfrak{P}_\beta / U_\beta) \cong \mathrm{Out}(SL_2(8))$ is generated by field automorphisms, and hence automorphisms which are restrictions of field automorphisms of G. So we can compose χ_β and χ_α by restrictions of elements of

$\mathrm{Aut}_B(G)\Phi_G = N_{\mathrm{Aut}_{\mathfrak{P}_\beta}(G)}(U)\Phi_G$, to arrange that χ_β induces the identity on $\mathfrak{P}_\beta/U_\beta$. Then, upon composing them by some element of $\mathrm{Aut}_U(G)$, we can also arrange that $\chi_0(T) = T$. Since $X_{\beta'}$ and $X_{\beta''}$ are dual to each other by (7) and hence nonisomorphic as $\mathbb{F}_2[T]$-modules, χ_0 sends each of them to itself.

Since $\chi_0(T) = T$, χ_0 sends $C_U(T) = X_\alpha X_{\alpha'} X_{\alpha''} \cong D_8$ to itself. It cannot exchange the two subgroups $X_\alpha X_{\alpha''}$ and $X_{\alpha'} X_{\alpha''}$ (the first is not contained in U_α and the second is), so $\chi_0|_{C_U(T)} \in \mathrm{Inn}(C_U(T))$. Hence after composing by an element of $\mathrm{Aut}_{C_U(T)}(G)$, we can arrange that χ_0 is the identity on this subgroup. Also, by applying (4) with $u = 1$, and since $\chi_0|_{X_\beta} \equiv \mathrm{Id} \pmod{U_\beta}$ and $[X_\alpha, U_\beta] \le X_{\alpha''}$, we see that χ_0 is the identity on $X_{\beta'} X_{\beta''}$. We conclude that χ_0 is the identity on U_β.

Since χ_β induces the identity on U_β and on $\mathfrak{P}_\beta/U_\beta$, it has the form $\chi_\beta(x) = x\psi(x)$ (all $x \in \mathfrak{P}_\beta$) for some

$$\psi \in \mathrm{Hom}(\mathfrak{P}_\beta/U_\beta; Z(U_\beta)) \cong \mathrm{Hom}(SL_2(8), C_2) = 1\,.$$

So $\chi_\beta = \mathrm{Id}_{\mathfrak{P}_\beta}$.

Now, $C_{\mathfrak{P}_\alpha}(T) \cong \Sigma_4 \times C_7$, and $\mathrm{Out}(\Sigma_4) = 1$. Hence $\chi_\alpha|_{C_{\mathfrak{P}_\alpha}(T)}$ must be conjugation by some element $z \in Z(C_U(T)) = X_{\alpha''} = Z(\mathfrak{P}_\beta)$. After composing χ_α and χ_β by restrictions of c_z, we can thus assume that χ_α is the identity on $C_{\mathfrak{P}_\alpha}(T)$ (and still $\chi_\beta = \mathrm{Id}_{\mathfrak{P}_\beta}$). Since $\chi_\alpha|_U = \mathrm{Id}$ and $\mathfrak{P}_\alpha = \langle U, C_{\mathfrak{P}_\alpha}(T)\rangle$, we have $\chi_\alpha = \mathrm{Id}_{\mathfrak{P}_\alpha}$. □

It remains only to handle ${}^2F_4(2)$ and the Tits group.

Proposition 4.17. *Assume $G = {}^2F_4(2)'$ or ${}^2F_4(2)$. Then κ_G is an isomorphism.*

Proof. By the pullback square in [AOV, Lemma 2.15] (and since $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is independent of the choice of objects in $\mathcal{L}$ by [AOV, Lemma 1.17]), κ_G is an isomorphism when $G = {}^2F_4(2)$ if it is an isomorphism when G is the Tits group. So from now on, we assume $G = {}^2F_4(2)'$.

We adopt the notation for subgroups of G used by Parrott [Pa]. Fix $T \in \mathrm{Syl}_2(G)$, and set $Z = Z(T) \cong C_2$, $H = C_G(Z)$, and $J = O_2(H)$. Let $z \in Z$ be a generator. Then H is the parabolic subgroup of order $2^{11} \cdot 5$, $|J| = 2^9$, and $H/J \cong C_5 \rtimes C_4$. Set $E = [J, J]$. By [Pa, Lemma 1], $E = Z_2(J) = \mathrm{Fr}(J) \cong C_2^5$, and by the proof of that lemma, the Sylow 5-subgroups of H act irreducibly on $J/E \cong C_2^4$ and on $E/Z \cong C_2^4$. Since each element of $\mathrm{Aut}_{H/J}(J/E)$ sends $C_{J/E}(T/J) \cong C_2$ to itself,

$$\mathrm{Aut}_{H/J}(J/E) = \{\mathrm{Id}_{J/E}\} \quad \text{and} \quad |\mathrm{Hom}_{H/J}(J/E, E/Z)| \le |\mathrm{Hom}_{H/J}(J/E, J/E)| = 2\,. \tag{9}$$

Let $N > T$ be the other parabolic, and set $K = O_2(N)$. Thus $N/K \cong \Sigma_3$, and $[T : K] = 2$.

Fix $P \in \mathrm{Syl}_5(H) \subseteq \mathrm{Syl}_5(G)$ (so $P \cong C_5$). By [Pa, p. 674], $H/E = (J/E) \cdot (N_G(P)/Z)$, where $N_G(P)/Z \cong H/J \cong C_5 \rtimes C_4$. For each $\beta \in \mathrm{Aut}(H)$ such that $\beta(T) = T$, there is $\beta_1 \equiv \beta \pmod{\mathrm{Aut}_J(H)}$ such that $\beta_1(P) = P$. Since each automorphism of H/J which sends $T/J \cong C_4$ to itself is conjugation by an element of T/J, there is $\beta_2 \equiv \beta_1 \pmod{\mathrm{Aut}_{N_T(P)}(H)}$ such that β_2 induces the identity on H/J. By (9), β_2 also induces the identity on J/E, and hence on $H/E = (J/E) \cdot (N_G(P)/Z)$. Thus

$$N_{\mathrm{Aut}(H)}(T) = \mathrm{Aut}_T(H) \cdot \{\beta \in \mathrm{Aut}(H) \mid \beta(P) = P,\ [\beta, H] \le E\}\,. \tag{10}$$

Now set $\mathcal{L} = \mathcal{L}_T^c(G)$ for short, and identify $N = \mathrm{Aut}_{\mathcal{L}}(K)$ and $H = \mathrm{Aut}_{\mathcal{L}}(J)$. For each $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$, let $\alpha_H \in \mathrm{Aut}(H)$ and $\alpha_N \in \mathrm{Aut}(N)$ be the induced automorphisms, and set $\alpha_T = \alpha_H|_T = \alpha_N|_T$. Set

$$\mathcal{A}_0 = \bigl\{\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}) \,\big|\, [\alpha_H, H] \le E \text{ and } \alpha_H|_P = \mathrm{Id}_P\bigr\}\,.$$

By (10), each class in $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ contains at least one automorphism in $\mathcal{A}_0$.

Fix $\alpha \in \mathcal{A}_0$. Since $[\alpha_H, H]$ must be normal in H, we have $[\alpha_H, H] \in \{E, Z, 1\}$. If $[\alpha_H, H] = Z$, then $\alpha_H|_{JP} = \mathrm{Id}$, so $[\alpha_H, K] = [\alpha_N, K] = Z$, which is impossible since Z is not normal in N by [Pa, Lemma 6] (or since $z \notin Z(G)$ and $G = \langle H, N \rangle$). Thus either $\alpha_H = \mathrm{Id}$, or $[\alpha_H, H] = E$.

If $\alpha_H = \mathrm{Id}_H$, then $\alpha_N|_T = \mathrm{Id}$. In this case, α_N determines an element of $H^1(N/K; Z(K))$ whose restriction to $H^1(T/K; Z(K))$ is trivial, and since this restriction map for $H^1(-; Z(K))$ is injective (since $T/K \in \mathrm{Syl}_2(N/K)$), $\alpha_N \in \mathrm{Inn}(N)$ (see, e.g., [OV, Lemma 1.2]). Hence $\alpha_N \in \mathrm{Aut}_Z(N)$ since $\alpha_N|_T = \mathrm{Id}$ (and $Z = Z(T)$). So $\alpha \in \mathrm{Aut}_Z(\mathcal{L})$ in this case, and $[\alpha] = 1 \in \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$.

Set $\overline{H} = H/Z$, and similarly for subgroups of H. Let $\bar{\alpha}_H \in \mathrm{Aut}(\overline{H})$ and $\bar{\alpha}_T \in \mathrm{Aut}(\bar{T})$ be the automorphisms induced by α_H and α_T, and set $\beta = \bar{\alpha}_T|_{\overline{J}}$. Then $\overline{E} = Z(\bar{J})$ since $E = Z_2(J)$, so $\beta(g) = g\varphi(\widehat{g})$ for some $\varphi \in \mathrm{Hom}_{H/J}(J/E, \overline{E})$. If $\varphi = 1$, so that $[\alpha, J] \le Z$, then since $\alpha|_P = \mathrm{Id}$, we have $[\alpha_H, H] < E$ and so $\alpha_H = \mathrm{Id}$.

We have now constructed a homomorphism from $\mathcal{A}_0$ to $\mathrm{Hom}_{H/J}(J/E, \overline{E})$ with kernel $\mathrm{Aut}_Z(\mathcal{L})$. Thus

$$|\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})| \le |\mathcal{A}_0/\mathrm{Aut}_Z(\mathcal{L})| \le |\mathrm{Hom}_{H/J}(J/E, \overline{E})| \le 2\,.$$

where the last inequality holds by (9). Since $|\mathrm{Out}(G)| = 2$ by [GrL, Theorem 2], and since κ_G is injective by Lemma 4.3, this proves that κ_G is an isomorphism.

Alternatively, this can be shown using results in [Fn]. Since $T/[T, T] \cong C_2 \times C_4$ by the above description of T/E (where $E \le [T, T]$), $\mathrm{Aut}(T)$ and hence $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ are 2-groups. So each automorphism of the amalgam $H > T < N$ determines a larger amalgam. Since the only extension of this amalgam is to that of ${}^2F_4(2)$ by [Fn, Theorem 1], $|\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})| = 2$. □

5. The cross characteristic case: I

Throughout this section, we will work with groups $G = C_{\bar{G}}(\sigma)$ which satisfy the conditions in Hypotheses 5.1 below. In particular, 5.1(I) implies that G is not a Suzuki or Ree group. We will see in Section 6 (Proposition 6.8) that while these hypotheses are far from including all finite Chevalley and Steinberg groups, their fusion systems at the prime p do include almost all of those we need to consider.

For any finite abelian group B, we denote its "scalar automorphisms" by

$$\psi_k^B \in \operatorname{Aut}(B), \qquad \psi_k^B(g) = g^k \qquad \text{for all } k \text{ such that } (k, |B|) = 1$$

and define the group of its scalar automorphisms

$$\operatorname{Aut}_{\mathrm{sc}}(B) = \bigl\{\psi_k^B \,\big|\, (k, |B|) = 1\bigr\} \le Z(\operatorname{Aut}(B)) .$$

Hypotheses 5.1. *Assume we are in the situation of Notation 2.2(A,B,C).*

(I) *Let p be a prime distinct from q_0 such that $p\big||W_0|$. Assume also that $\sigma = \psi_q \circ \gamma = \gamma \circ \psi_q \in \operatorname{End}(\bar{G})$, where*

- *q is a power of the prime q_0;*
- *$\psi_q \in \Phi_{\bar{G}}$ is the field automorphism (see Definition 3.1(a)); and*
- *$\gamma \in \operatorname{Aut}(\bar{G})$ is an algebraic automorphism of finite order which sends $\bar{T}$ to itself and commutes with ψ_{q_0} (so that $\psi_{q_0}(G) = G$).*

Also, there is a free $\langle\tau\rangle$-orbit of the form $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ or $\{\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_s\}$ in Σ such that the set $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ is linearly independent in V.

(II) *The algebraic group $\bar{G}$ is of universal type, and $N_G(T)$ contains a Sylow p-subgroup of G.*

(III) *Set $A = O_p(T)$. Assume one of the following holds: either*

(III.1) *$q \equiv 1 \pmod p$, $q \equiv 1 \pmod 4$ if $p = 2$, $|\gamma| \le 2$, and $\gamma \in \Gamma_{\bar{G}}$ (thus $\rho(\Pi) = \Pi$); or*

(III.2) *p is odd, $q \equiv -1 \pmod p$, G is a Chevalley group (i.e., $\gamma \in \operatorname{Inn}(\bar{G})$), and $\gamma(t) = t^{-1}$ for each $t \in \bar{T}$; or*

(III.3) *p is odd, $|\tau| = \operatorname{ord}_p(q) \ge 2$, $C_A(O_{p'}(W_0)) = 1$, $C_S(\Omega_1(A)) = A$, $\operatorname{Aut}_G(A) = \operatorname{Aut}_{W_0}(A)$,*

$$N_{\operatorname{Aut}(A)}(\operatorname{Aut}_{W_0}(A)) \le \operatorname{Aut}_{\mathrm{sc}}(A)\operatorname{Aut}_{\operatorname{Aut}(G)}(A)$$

where $\operatorname{Aut}_{\operatorname{Aut}(G)}(A) = \bigl\{\delta|_A \,\big|\, \delta \in \operatorname{Aut}(G),\ \delta(A) = A\bigr\}$, and

$$\operatorname{Aut}_{W_0}(A) \cap \operatorname{Aut}_{\mathrm{sc}}(A) \le \begin{cases} \bigl\langle \gamma|_A \bigr\rangle & \text{if } 2\big|\operatorname{ord}_p(q) \text{ or } -\mathrm{Id} \notin W \\ \bigl\langle \gamma|_A, \psi_{-1}^A \bigr\rangle & \text{otherwise,} \end{cases}$$

Since W_0 acts on T by Lemma 2.3, it also acts on $A = O_p(T)$.

We will see in Lemma 5.3 that the conditions $C_S(\Omega_1(A)) = A$ (or $C_S(A) = A$ when $p = 2$) and $\operatorname{Aut}_G(A) = \operatorname{Aut}_{W_0}(A)$, both assumed here in (III.3), also hold in cases (III.1) and (III.2).

Recall, in the situation of (III.3), that $|\tau| = |\gamma|_{\bar{T}}|$ by Lemma 3.2.

Note that the above hypotheses eliminate the possibility that G be a Suzuki or Ree group. Since we always assume the Sylow p-subgroups are nonabelian, the only such case which needs to be considered here (when $q_0 \neq p$) is that of ${}^2F_4(q)$ when $p = 3$, and this will be handled separately.

By Lemma 3.2, whenever $\sigma = \psi_q \circ \gamma$, and γ is an algebraic automorphism of $\overline{G}$ which normalizes $\overline{T}$, there is $\tau \in \mathrm{Aut}(V)$ such that $\tau(\Sigma) = \Sigma$ and $\sigma(\overline{X}_\alpha) = \overline{X}_{\tau(\alpha)}$ for each $\alpha \in \Sigma$. So under Hypotheses 5.1, the condition at the beginning of Notation 2.2(C) holds automatically, and with $\rho = \tau|_\Sigma$. To simplify the notation, throughout this section and the next, we write $\tau = \rho$ to denote this induced permutation of Σ.

The following notation will be used throughout this section, in addition to that in Notation 2.2. Note that $\widehat{\Pi}$ and $\widehat{\Sigma}$ are defined in Notation 2.2(C) only when $\rho(\Pi) = \Pi$, and hence only in case (III.1) of Hypotheses 5.1. It will be convenient, in some of the proofs in this section, to extend this definition to case (III.2).

Recall (Notation 2.2) that for $\alpha \in \Sigma$, $w_\alpha \in W$ denotes the reflection in the hyperplane $\alpha^\perp \subseteq V$.

Notation 5.2. *Assume we are in the situation of Notation 2.2 and Hypotheses 5.1.*

(D) *If (III.2) holds, then set $\widehat{\Sigma} = \Sigma$, $\widehat{\Pi} = \Pi$, and $V_0 = V$. Note that $W_0 = W$ in this case.*

(E) *If (III.1) holds, then for each $\widehat{\alpha} \in \widehat{\Sigma}$, let $w_{\widehat{\alpha}} \in W_0$ be the element in $\langle w_\alpha \,|\, \alpha \in \widehat{\alpha}\rangle$ which acts on V_0 as the reflection across the hyperplane $\langle\widehat{\alpha}\rangle^\perp$, and which exchanges the positive and negative roots in the set $\langle\widehat{\alpha}\rangle \cap \Sigma$. (Such an element exists and lies in W_0 by* [Ca, Proposition 13.1.2]*.)*

(F) *If (III.1) or (III.2) holds, then for each $\alpha \in \Sigma$ and each $\widehat{\alpha} \in \widehat{\Sigma}$, set*

$$\overline{K}_\alpha = \langle \overline{X}_\alpha, \overline{X}_{-\alpha}\rangle \qquad \bar{T}_\alpha = h_\alpha(\bar{\mathbb{F}}_{q_0}^\times)$$
$$\overline{K}_{\widehat{\alpha}} = \langle \overline{K}_\alpha \,|\, \alpha \in \widehat{\alpha}\rangle \qquad \bar{T}_{\widehat{\alpha}} = \langle \bar{T}_\alpha \,|\, \alpha \in \widehat{\alpha}\rangle\,.$$

(G) *Set $N = N_G(T)/O_{p'}(T)$, and identify $A = O_p(T)$ with $T/O_{p'}(T) \trianglelefteq N$. If (III.1) or (III.2) holds, then for $\widehat{\alpha} \in \widehat{\Sigma}$, set $A_{\widehat{\alpha}} = A \cap \bar{T}_{\widehat{\alpha}}$.*

(H) *Fix $S \in \mathrm{Syl}_p(G)$ such that $A \le S \le N_G(T)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. (Recall that $N_G(T)$ contains a Sylow p-subgroup of G by Hypotheses 5.1(II).) Set*

$$\mathrm{Aut}(A,\mathcal{F}) = \bigl\{\beta \in \mathrm{Aut}(A) \bigm| \beta = \bar{\beta}|_A,\ \text{some } \bar{\beta} \in \mathrm{Aut}(S,\mathcal{F})\bigr\}\,.$$

Set $\mathrm{Aut}_{\mathrm{diag}}(S,\mathcal{F}) = C_{\mathrm{Aut}(S,\mathcal{F})}(A) = \bigl\{\beta \in \mathrm{Aut}(S,\mathcal{F}) \bigm| \beta|_A = \mathrm{Id}\bigr\}$, *and let* $\mathrm{Out}_{\mathrm{diag}}(S,\mathcal{F})$ *be the image of* $\mathrm{Aut}_{\mathrm{diag}}(S,\mathcal{F})$ *in* $\mathrm{Out}(S,\mathcal{F})$.

Note that when $(\overline{G},\sigma)$ is a standard setup (i.e., in case (III.1)), W_0 acts faithfully on V_0 (see [Ca, Lemma 13.1.1]).

Recall that $N = N_G(T)/O_{p'}(T)$. We identify $A = O_p(T)$ with $T/O_{p'}(T) \trianglelefteq N$.

Lemma 5.3. *Assume Hypotheses 5.1 and Notation 5.2.*

(a) *If condition (III.1) or (III.2) holds, then $C_W(A) = 1$, $C_{\overline{G}}(A) = C_{\overline{G}}(T) = \overline{T}$, $C_G(A) = T$, and $C_S(A) = A$. If p is odd, then $C_W(\Omega_1(A)) = 1$ and $C_S(\Omega_1(A)) = A$.*

(b) *If $C_{\overline{G}}(A)^0 = \overline{T}$ (in particular, if (III.1) or (III.2) holds), then $N_G(A) = N_G(T) \le N_{\overline{G}}(\overline{T})$, and the inclusion of $N_G(T)$ in $N_{\overline{G}}(\overline{T})$ induces isomorphisms $W_0 \cong N_G(T)/T \cong N/A$. Thus* $\mathrm{Aut}_G(A) = \mathrm{Aut}_{W_0}(A)$*.*

Proof. **(a)** Assume condition (III.1) or (III.2) holds. We first prove that $C_W(A) = 1$, and also that $C_W(\Omega_1(A)) = 1$ when p is odd.

If p is odd, set $A_0 = \Omega_1(A)$ and $\widehat{p} = p$. If $p = 2$, set $A_0 = \Omega_2(A)$ and $\widehat{p} = 4$. Thus in all cases, A_0 is the $\widehat{p}$-torsion subgroup of A. Set $\varepsilon = 1$ if we are in case (III.1), or $\varepsilon = -1$ in case (III.2). By assumption, $\widehat{p} | (q - \varepsilon)$. Choose $\lambda \in \mathbb{F}_q^\times$ (or $\lambda \in \mathbb{F}_{q^2}^\times$ if $\varepsilon = -1$) of order $\widehat{p}$. Set $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Fix $w \in C_W(A_0)$.

Assume first $G = \mathbb{G}(q)$, a Chevalley group. Then $T = \{t \in \overline{T} \mid t^{q-\varepsilon} = 1\}$, and A_0 contains all elements of order $\widehat{p}$ in $\overline{T}$. So $w = 1$ by Lemma 2.7.

Now assume that $\mathrm{Id} \neq \gamma \in \Gamma_{\overline{G}}$; i.e., G is one of the Steinberg groups ${}^2A_n(q)$, ${}^2D_n(q)$, or ${}^2E_6(q)$. Then $C_{\overline{G}}(\gamma)$ is a simple algebraic group of type B_m, C_m, or F_4 (cf. [Ca, § 13.1–3]) with root system $\widehat{\Sigma} \subseteq V_0 = C_V(\tau)$, and A_0 contains all $\widehat{p}$-torsion in $C_{\overline{T}}(\gamma)$. By Lemma 2.7 again, $w|_{V_0} = \mathrm{Id}$. Since w and τ are both orthogonal, w also sends the (-1)-eigenspace for the action of τ to itself, and thus $w \in C_W(\tau) = W_0$. But W_0 acts faithfully on V_0 (see, e.g., [Ca, 13.1.1]), so $w = 1$.

Thus $C_W(A_0) = 1$. Hence $C_{\overline{G}}(A_0) = \overline{T}$ by Proposition 2.5, and the other statements follow immediately.

(b) If $C_{\overline{G}}(A)^0 = \overline{T}$, then $N_{\overline{G}}(T) \leq N_{\overline{G}}(A) \leq N_{\overline{G}}(\overline{T})$ (recall that A is the p-power torsion in T). If $g \in N_{\overline{G}}(\overline{T})$ and $\sigma(g) = g$, then g also normalizes $T = C_{\overline{T}}(\sigma)$. Thus $N_G(T) = N_G(A) \leq N_{\overline{G}}(\overline{T})$, and hence $N_G(T)/T \cong W_0$ by Lemma 2.3. The identification $N/A \cong N_G(T)/T$ is immediate from the definition of N. □

Recall (Notation 5.2(F)) that when case (III.1) of Hypotheses 5.1 holds (in particular, when $p = 2$), we set $\overline{K}_{\widehat{\alpha}} = \langle \overline{K}_\alpha \,|\, \alpha \in \widehat{\alpha} \rangle$ for $\widehat{\alpha} \in \widehat{\Sigma}$, where $\overline{K}_\alpha = \langle \overline{X}_\alpha, \overline{X}_{-\alpha} \rangle$. The conditions in (III.1) imply that each class in $\widehat{\Sigma}$ is of the form $\{\alpha\}$, $\{\alpha, \tau(\alpha)\}$, or $\{\alpha, \tau(\alpha), \alpha + \tau(\alpha)\}$ for some α. This last case occurs only when $G \cong SU_n(q)$ for some odd $n \geq 3$ and some $q \equiv 1$ (mod p or mod 4).

Lemma 5.4. *Assume Hypotheses 5.1, case (III.1), and Notation 5.2. For each $\alpha \in \Sigma$, $\overline{K}_\alpha \cong SL_2(\overline{\mathbb{F}}_{q_0})$. For each $\widehat{\alpha} \in \widehat{\Sigma}$, $\overline{K}_{\widehat{\alpha}} \cong SL_2(\overline{\mathbb{F}}_{q_0})$, $SL_2(\overline{\mathbb{F}}_{q_0}) \times SL_2(\overline{\mathbb{F}}_{q_0})$, or $SL_3(\overline{\mathbb{F}}_{q_0})$ whenever the class $\widehat{\alpha}$ has order* 1, 2, *or* 3, *respectively. Also, $G \cap \overline{K}_{\widehat{\alpha}}$ is isomorphic to $SL_2(q)$, $SL_2(q^2)$, or $SU_3(q)$, respectively, in these three cases.*

Proof. By Lemma 3.10, each class in $\widehat{\Sigma}$ is in the W_0-orbit of a class in $\widehat{\Pi}$. So it suffices to prove the statements about $\overline{K}_\alpha$ and $\overline{K}_{\widehat{\alpha}}$ when $\alpha \in \Pi$, and when $\widehat{\alpha} \in \widehat{\Pi}$ is its equivalence class.

By Lemma 2.4(b) (and since $\overline{G}$ is universal), $\overline{K}_\alpha \cong SL_2(\overline{\mathbb{F}}_{q_0})$ for each $\alpha \in \Pi$. So when $\alpha = \tau(\alpha)$ (when $|\widehat{\alpha}| = 1$), $\overline{K}_{\widehat{\alpha}} = \overline{K}_\alpha \cong SL_2(\overline{\mathbb{F}}_{q_0})$.

When $\alpha \neq \tau(\alpha)$ and they are not orthogonal, then $\overline{G} \cong SL_{2n+1}(\overline{\mathbb{F}}_{q_0})$ for some n, and the inclusion of $SL_3(\overline{\mathbb{F}}_{q_0})$ is clear. When $\alpha \perp \tau(\alpha)$, then $[\overline{K}_\alpha, \overline{K}_{\tau(\alpha)}] = 1$, and $\overline{K}_\alpha \cap \overline{K}_{\tau(\alpha)} = 1$ by Lemma 2.4(b) and since G is universal, and since the intersection is contained in the centers of the two factors and hence in the maximal tori. Hence $\overline{K}_{\widehat{\alpha}} = \langle \overline{X}_{\pm\alpha}, \overline{X}_{\pm\tau(\alpha)} \rangle \cong \overline{K}_\alpha \times \overline{K}_{\tau(\alpha)} \cong SL_2(\overline{\mathbb{F}}_{q_0}) \times SL_2(\overline{\mathbb{F}}_{q_0})$.

In all cases, since $\overline{G}$ is universal, $G \cap \overline{K}_{\widehat{\alpha}} = C_{\overline{G}}(\sigma) \cap \overline{K}_{\widehat{\alpha}} = C_{\overline{K}_{\widehat{\alpha}}}(\sigma)$. If $\alpha = \tau(\alpha)$, then γ acts trivially on $\overline{K}_{\widehat{\alpha}}$, and $C_{\overline{K}_{\widehat{\alpha}}}(\sigma) \cong SL_2(q)$. If $\alpha \perp \tau(\alpha)$ then γ exchanges the two factors and $C_{\overline{K}_{\widehat{\alpha}}}(\sigma) \cong SL_2(q^2)$. Finally, if $\alpha \neq \tau(\alpha)$ and they are not orthogonal, then γ is the graph automorphism of $SL_3(\overline{\mathbb{F}}_{q_0})$, so $C_{\overline{K}_{\widehat{\alpha}}}(\sigma) \cong SU_3(q)$. □

We also recall here the definition of the focal subgroup of a saturated fusion system $\mathcal{F}$ over a finite p-group S:

$$\mathfrak{foc}(\mathcal{F}) = \bigl\langle xy^{-1} \bigm| x,y\in S,\ x \text{ is } \mathcal{F}\text{-conjugate to } y\bigr\rangle.$$

By the focal subgroup theorem for groups (cf. [G, Theorem 7.3.4]), if $\mathcal{F}=\mathcal{F}_S(G)$ for some finite group G with $S\in\mathrm{Syl}_p(G)$, then $\mathfrak{foc}(\mathcal{F})=S\cap[G,G]$.

Lemma 5.5. *Assume Hypotheses 5.1, case (III.1) or (III.2), and Notation 5.2. Assume also that $|\widehat{\Pi}|\geq 2$. Then the following hold.*

(a) *If p is odd, then $[w_{\widehat{\alpha}},A]=A_{\widehat{\alpha}}$ for each $\widehat{\alpha}\in\widehat{\Sigma}$. If $p=2$, then for each $\widehat{\alpha}\in\widehat{\Sigma}$, $[w_{\widehat{\alpha}},A]\leq A_{\widehat{\alpha}}$ with index at most 2, and $[w_{\widehat{\alpha}},A]=A_{\widehat{\alpha}}$ with the following exceptions:*

- *$\tau=\mathrm{Id}$, $\mathbb{G}=C_n$ (or B_2), and $\widehat{\alpha}=\{\alpha\}$ where α is a long root; or*
- *$|\tau|=2$, $\mathbb{G}=D_n$ (or A_3), and $\widehat{\alpha}=\{\alpha,\tau(\alpha)\}$ where $\alpha\perp\tau(\alpha)$.*
- *$|\tau|=2$, $\mathbb{G}=A_{2n}$, and $|\widehat{\alpha}|=3$.*

(b) *For each $w\in W_0$ of order 2, $w=w_{\widehat{\alpha}}$ for some $\widehat{\alpha}\in\widehat{\Sigma}$ if and only if $[w,A]$ is cyclic.*

(c) *If $p=2$, then for each $\widehat{\alpha}\in\widehat{\Sigma}$,*

$$C_{\bar{G}}(C_A(w_{\widehat{\alpha}}))=\begin{cases}\bar{T}\bar{K}_{\widehat{\alpha}} & \text{if } |\widehat{\alpha}|\leq 2\\ \bar{T}\bar{K}_{\alpha+\tau(\alpha)} & \text{if } \widehat{\alpha}=\{\alpha,\tau(\alpha),\alpha+\tau(\alpha)\}.\end{cases}$$

If in addition, $|\widehat{\alpha}|\leq 2$, then

$$A_{\widehat{\alpha}}=A\cap\bigl[C_G(C_A(w_{\widehat{\alpha}})),C_G(C_A(w_{\widehat{\alpha}}))\bigr]=A\cap\mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}}))).$$

Proof. **(a)** If we are in case (III.1) of Hypotheses 5.1, then by Lemma 3.10, each orbit of W_0 under its action on $\widehat{\Sigma}$ contains an element of $\widehat{\Pi}$. If we are in case (III.2), then $W_0=W$ and $\widehat{\Sigma}=\Sigma$, so the statement holds by the same lemma. So in either case, it suffices to prove this when $\widehat{\alpha}\in\widehat{\Pi}$.

Fix $\alpha\in\Pi$, and let $\widehat{\alpha}\in\widehat{\Pi}$ be its class. Since $w_{\widehat{\alpha}}\in\langle w_\alpha,w_{\tau(\alpha)}\rangle$, $[w_{\widehat{\alpha}},A]\leq A\cap\bar{T}_{\widehat{\alpha}}=A_{\widehat{\alpha}}$ in all cases by Lemma 2.4(e). By the same lemma, $w_{\widehat{\alpha}}(\widehat{h}_\alpha(\lambda))=\widehat{h}_\alpha(\lambda^{-1})$ for all $\lambda\in\bar{\mathbb{F}}_{q_0}^\times$ if $|\widehat{\alpha}|\leq 2$; and $w_{\widehat{\alpha}}(\widehat{h}_\alpha(\lambda))=\widehat{h}_\alpha(\lambda^{-q})$ for $\lambda\in\mathbb{F}_{q^2}^\times$ if $|\widehat{\alpha}|=3$. So $[w_{\widehat{\alpha}},A]=A_{\widehat{\alpha}}$ if p is odd, and $[w_{\widehat{\alpha}},A]$ has index at most 2 in $A_{\widehat{\alpha}}$ if $p=2$.

Assume now that $p=2$, and hence that $q\equiv 1 \pmod 4$. If $\tau=\mathrm{Id}$ (and hence $\widehat{\alpha}=\{\alpha\}$), then for each $\beta\in\Pi$ and each $\lambda\in\mathbb{F}_q^\times$, Lemma 2.4(e) implies that

$$w_\alpha(h_\beta(\lambda))=\begin{cases}h_\beta(\lambda) & \text{if } \beta\perp\alpha\\ h_\beta(\lambda)h_\alpha(\lambda) & \text{if } \beta\not\perp\alpha,\ \|\beta\|\geq\|\alpha\|\\ h_\beta(\lambda)h_\alpha(\lambda^k) & \text{if } \beta\not\perp\alpha,\ \|\alpha\|=\sqrt{k}\cdot\|\beta\|,\ k=1,2,3.\end{cases}$$

(Note that $w_\alpha(\beta^\vee)=\beta^\vee$, $\beta^\vee+\alpha^\vee$, or $\beta^\vee+k\alpha^\vee$, respectively, in these three cases.) Since T is generated by the $h_\beta(\lambda)$ for $\beta\in\Pi$ and $\lambda\in\mathbb{F}_q^\times$,, it follows that $[w_\alpha,A]$ has index 2 in A_α exactly when there are roots with two lengths and ratio $\sqrt{2}$, α is a long root, and orthogonal to all other long roots in Π. This happens only when $\mathbb{G}\cong C_n$ or B_2.

Now assume $|\tau| = 2$. In particular, all roots in Σ have the same length. By Lemma 2.4(e) again, for each $\beta \in \Pi \smallsetminus \widehat{\alpha}$ such that $\beta \not\perp \alpha$ and with class $\widehat{\beta} \in \widehat{\Pi}$, we have

$$w_{\widehat{\alpha}}(\widehat{h}_\beta(\lambda)) = \begin{cases} \widehat{h}_\beta(\lambda)\widehat{h}_\alpha(\lambda) & \text{if } |\widehat{\beta}| = 1 \text{ and } \lambda \in \mathbb{F}_q^\times \\ \widehat{h}_\beta(\lambda)\widehat{h}_\alpha(\lambda) & \text{if } |\widehat{\beta}| \geq 2,\ |\widehat{\alpha}| = 2, \text{ and } \lambda \in \mathbb{F}_{q^2}^\times \\ \widehat{h}_\beta(\lambda)\widehat{h}_\alpha(\lambda^{q+1}) & \text{if } |\widehat{\beta}| \geq 2,\ |\widehat{\alpha}| = 1 \text{ or } 3, \text{ and } \lambda \in \mathbb{F}_{q^2}^\times \end{cases}$$

Thus $[w_{\widehat{\alpha}}, A] = A_{\widehat{\alpha}}$ exactly when $|\widehat{\alpha}| = 1$, or $|\widehat{\alpha}| = 2$ and there is some $\beta \in \Pi$ such that $\beta \not\perp \alpha$ and $\beta \neq \tau(\beta)$. The only cases where this does not happen are when $\mathbb{G} = D_n$ or A_3 and $|\widehat{\alpha}| = 2$, and when $\mathbb{G} = A_{2n}$ and $|\widehat{\alpha}| \geq 3$.

(b) For each $\widehat{\alpha} \in \widehat{\Sigma}$, $[w_{\widehat{\alpha}}, A] \leq A_{\widehat{\alpha}}$ by (a), and hence is cyclic. It remains to prove the converse.

When we are in case (III.2) (and hence the setup is not standard), it will be convenient to define $V_0 = V$. (Recall that V_0 is defined in Notation 2.2(C) only when $(\overline{G}, \sigma)$ is a standard setup.) Note that by assumption, G is always a Chevalley group in this case.

Let $w \in W_0$ be an element of order 2 which is not equal to $w_{\widehat{\alpha}}$ for any $\widehat{\alpha}$. If G is a Chevalley group (if $W_0 = W$ and $V_0 = V$), then $C_V(w)$ contains no points in the interior of any Weyl chamber, since W permutes freely the Weyl chambers (see [Brb, § V.3.2, Théorème 1(iii)]). Since w is not the reflection in a root hyperplane, it follows that $\dim(V/C_V(w)) \geq 2$. If G is a Steinberg group (thus in case (III.1) with a standard setup), then W_0 acts on V_0 as the Weyl group of a certain root system on V_0 (see [Ca, § 13.3]), so $\dim(V_0/C_{V_0}(w)) \geq 2$ by a similar argument.

Set $\varepsilon = +1$ if we are in case (III.1), or $\varepsilon = -1$ if we are in case (III.2). Set $m = v_p(q - \varepsilon)$, and choose $\lambda \in (\mathbb{F}_{q^2})^\times$ of order p^m. Set $\Lambda = \mathbb{Z}\Sigma^\vee$, regarded as the lattice in V with $\mathbb{Z}$-basis $\Pi^\vee = \{\alpha^\vee \mid \alpha \in \Pi\}$. Let

$$\Phi_\lambda \colon \Lambda/p^m\Lambda \longrightarrow \overline{T}$$

be the $\mathbb{Z}[W]$-linear monomorphism of Lemma 2.6(a) with image the p^m-torsion in $\overline{T}$. Thus $\Phi_\lambda(\alpha^\vee) = h_\alpha(\lambda)$ for each $\alpha \in \Sigma$. Also, $\sigma(h_\alpha(\lambda)) = h_{\tau(\alpha)}(\lambda)$ for each $\alpha \in \Sigma$ ($\lambda^q = \lambda$ by assumption), and thus Φ_λ commutes with the actions of τ on $\Lambda < V$ and of σ on $\overline{T}$.

Set $\Lambda_0 = C_\Lambda(\tau)$ in case (III.1), or $\Lambda_0 = \Lambda$ in case (III.2). Then $C_{\Lambda/p^m\Lambda}(\tau) = \Lambda_0/p^m\Lambda_0$ in case (III.1), since τ permutes the basis $\Pi^\vee$ of Λ. We claim that Φ_λ restricts to a $\mathbb{Z}[W_0]$-linear isomorphism

$$\Phi_0 \colon \Lambda_0/p^m\Lambda_0 \xrightarrow{\ \cong\ } \Omega_m(A)\,,$$

where $\Omega_m(A)$ is the p^m-torsion subgroup of A and hence of $T = C_{\overline{T}}(\sigma)$. If G is a Chevalley group (in either case (III.1) or (III.2)), then $\Lambda_0 = \Lambda$, so $\mathrm{Im}(\Phi_0)$ is the p^m-torsion subgroup of $\overline{T}$ and equal to $\Omega_m(A)$. If G is a Steinberg group, then $\varepsilon = +1$, each element of order dividing p^m in $\overline{T}$ is fixed by ψ^q, and hence lies in $\Omega_m(A)$ if and only if it is fixed by γ (thus in $\Phi_\lambda(C_{\Lambda/p^m\Lambda}(\tau))$).

Thus $[w, A] \geq [w, \Omega_m(A)] \cong [w, \Lambda_0/p^m\Lambda_0]$. Set $B = \Lambda_0/p^m\Lambda_0$ for short; we will show that $[w, B]$ is noncyclic. Set

$$r = \mathrm{rk}(\Lambda_0) = \dim(V_0) \qquad \text{and} \qquad s = \mathrm{rk}(C_{\Lambda_0}(w)) = \dim_{\mathbb{R}}(C_{V_0}(w)) \leq r - 2\,.$$

For each $b \in C_B(w)$, and each $v \in \Lambda_0$ such that $b = v + p^m\Lambda_0$, $v + w(v) \in C_{\Lambda_0}(w)$ maps to $2b \in C_B(w)$. Thus $B \cong (\mathbb{Z}/p^m)^r$, while $\{2b \mid b \in C_B(w)\}$ is contained in $C_{\Lambda_0}(w)/p^mC_{\Lambda_0}(w) \cong (\mathbb{Z}/p^m)^s$. Since $p^m > 2$ by assumption (and $r - s \geq 2$), it follows that $B/C_B(w) \cong [w, B]$ is not cyclic.

(c) Fix $\widehat{\alpha} \in \widehat{\Sigma}$. We set up our notation as follows.

Case (1): $|\widehat{\alpha}| = 1$ or 3. Set $\alpha^* = \alpha$ if $\widehat{\alpha} = \{\alpha\}$ (where $\tau(\alpha) = \alpha$), or $\alpha^* = \alpha + \tau(\alpha)$ if $\widehat{\alpha} = \{\alpha, \tau(\alpha), \alpha+\tau(\alpha)\}$. Set $w_{\widehat{\alpha}} = w_{\alpha^*}$, $W_{\widehat{\alpha}} = \langle w_{\widehat{\alpha}} \rangle$, and $\Delta = \{\pm\alpha^*\} \subseteq \Sigma$.

Case (2): $\widehat{\alpha} = \{\alpha, \tau(\alpha)\}$ where $\alpha \perp \tau(\alpha)$. Set $w_{\widehat{\alpha}} = w_\alpha w_{\tau(\alpha)}$, $W_{\widehat{\alpha}} = \langle w_\alpha, w_{\tau(\alpha)} \rangle$, and $\Delta = \{\pm\alpha, \pm\tau(\alpha)\} \subseteq \Sigma$.

In case (1), by Lemma 2.4(c,e),

$$C_{\bar{T}}(w_{\widehat{\alpha}}) = C_{\bar{T}}(w_{\alpha^*}) = \operatorname{Ker}(\theta_{\alpha^*}) = C_{\bar{T}}(\overline{X}_{\alpha^*}) = C_{\bar{T}}(\overline{X}_{-\alpha^*})\,.$$

Hence $C_{\bar{G}}(C_A(w_{\widehat{\alpha}})) \ge C_{\bar{G}}(C_{\bar{T}}(w_{\widehat{\alpha}})) \ge \bar{T}\langle \overline{X}_{\alpha^*}, \overline{X}_{-\alpha^*} \rangle = \bar{T}\overline{K}_{\alpha^*}$. In case (2), by the same lemma,

$$C_{\bar{T}}(w_{\widehat{\alpha}}) = C_{\bar{T}}(\langle w_\alpha, w_{\tau(\alpha)} \rangle) = C_{\bar{T}}(\langle \overline{X}_\alpha, \overline{X}_{-\alpha}, \overline{X}_{\tau(\alpha)}, \overline{X}_{-\tau(\alpha)} \rangle) = C_{\bar{T}}(\overline{K}_\alpha \overline{K}_{\tau(\alpha)})$$

so that $C_{\bar{G}}(C_A(w_{\widehat{\alpha}})) \ge \bar{T}\overline{K}_{\widehat{\alpha}}$. This proves one of the inclusions in the first statement in (c). By Proposition 2.5, the opposite inclusion will follow once we show that

$$C_W(C_A(w_{\widehat{\alpha}})) \le W_{\widehat{\alpha}}. \tag{1}$$

Fix $w \in C_W(C_A(w_{\widehat{\alpha}}))$.

- Let $\beta \in \Sigma \cap \Delta^\perp$ be such that $\beta = \tau(\beta)$. Then $h_\beta(\lambda) \in C_A(w_{\widehat{\alpha}})$ for $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$ of order 4, so $w(h_\beta(\lambda)) = h_\beta(\lambda)$, and $\beta \in C_V(w)$ by Lemma 2.6(c).
- Let $\beta \in \Sigma \cap \Delta^\perp$ be such that $\beta \ne \tau(\beta)$, and set $\beta' = \tau(\beta)$ for short. Let $r \ge 2$ be such that $q \equiv 1 + 2^r \pmod{2^{r+1}}$, and choose $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$ of order 2^{r+1}. Set $a = 1 - 2^r$, so $\lambda^a = \lambda^q$. Then
$$h_\beta(\lambda)h_{\beta'}(\lambda^a),\ h_\beta(\lambda^a)h_{\beta'}(\lambda) \in C_A(w_{\widehat{\alpha}}) \le C_{\bar{T}}(w)\,.$$
Also, $\|\beta + a\beta'\| = \|a\beta + \beta'\| < (1-a)\|\beta\| = \frac{1}{2}|\lambda|\|\beta\|$ since $a < 0$ and $\beta' \ne -\beta$ (since $\tau(\Sigma_+) = \Sigma_+$). Thus $\beta + a\beta', a\beta + \beta' \in C_V(w)$ by Lemma 2.6(b), so $\beta, \beta' \in C_V(w)$.
- Let $\beta \in \Sigma$ be such that $\beta = \tau(\beta)$ and $\beta \notin \Delta^\perp$, and set $\eta = \beta + w_{\widehat{\alpha}}(\beta)$. Since $w_{\widehat{\alpha}}\tau = \tau w_{\widehat{\alpha}}$ in $\operatorname{Aut}(V)$, $\tau(\eta) = \eta$. Since $\beta \notin \Delta^\perp = C_V(w_{\widehat{\alpha}})$, we have $w_{\widehat{\alpha}}(\beta) \ne \beta$, and hence $\|\eta\| < 2\|\beta\|$. For $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$ of order 4, $t = h_\beta(\lambda)h_{w_{\widehat{\alpha}}(\beta)}(\lambda) \in C_A(w_{\widehat{\alpha}})$, so $w(t) = t$, and $\eta = \beta + w_{\widehat{\alpha}}(\beta) \in C_V(w)$ by Lemma 2.6(b).

Consider the set

$$\Sigma^* = \bigl(\Sigma \cap \Delta^\perp\bigr) \cup \bigl\{\beta + w_{\widehat{\alpha}}(\beta) \,\big|\, \beta \in \Sigma,\ \tau(\beta) = \beta,\ \beta \not\perp \Delta\bigr\} \subseteq V\,.$$

We have just shown that $w(\eta) = \eta$ for each $\eta \in \Sigma^*$, and hence that $w|_{\langle \Sigma^* \rangle} = \mathrm{Id}$. From the description of the root systems in [Brb, Planches I–IX], we see that $\Sigma \cap (\Sigma^*)^\perp = \Delta$, except when $\mathbb{G} \cong A_2$ and $\tau \ne \mathrm{Id}$ (i.e., when $G \cong SU_3(q)$).

Thus when $G \ncong SU_3(q)$, the only reflection hyperplanes which contain $\langle \Sigma^* \rangle$ are those in the set $\{\beta^\perp \mid \beta \in \Delta\}$. Fix a "generic" element $v \in \langle \Sigma^* \rangle$; i.e., one which is not contained in any of these hyperplanes. In case (1), v is contained in only the one reflection hyperplane $\alpha^{*\perp}$, and hence is in the closure of exactly two Weyl chambers for (Σ, W): chambers which are exchanged by $w_{\widehat{\alpha}}$. In case (2), v is contained in the two reflection hyperplanes $\alpha^\perp$ and $\tau(\alpha)^\perp$, and hence in the closure of four Weyl chambers which are permuted freely and transitively by $W_{\widehat{\alpha}} = \langle w_\alpha, w_{\tau(\alpha)} \rangle$. Since W permutes the Weyl chambers freely and transitively (see [Brb, § V.3.2, Théorème 1(iii)]), and since $\langle w, W_{\widehat{\alpha}} \rangle$ permutes the chambers whose closures contain v, we have $w \in W_{\widehat{\alpha}}$.

This proves (1) when $G \ncong SU_3(q)$. If $G \cong SU_3(q)$, then $h_{\alpha^*}(-1) \in C_A(w_{\widehat{\alpha}})$. But no element of order 2 in $\bar{T} < SL_3(\bar{\mathbb{F}}_{q_0})$ centralizes the full Weyl group $W \cong \Sigma_3$, so (1) also holds in this case.

If $|\widehat{\alpha}| \le 2$, then

$$C_G(C_A(w_{\widehat{\alpha}})) = G \cap C_{\overline{G}}(C_A(w_{\widehat{\alpha}})) = T(G \cap \overline{K}_{\widehat{\alpha}})$$

where by Lemma 5.4, $G \cap \overline{K}_{\widehat{\alpha}} \cong SL_2(q)$ or $SL_2(q^2)$. Hence $C_G(C_A(w_{\widehat{\alpha}}))$ has commutator subgroup $G \cap \overline{K}_{\widehat{\alpha}}$, and focal subgroup $A_{\widehat{\alpha}}$. Since $C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}}))$ is the fusion system of $C_G(C_A(w_{\widehat{\alpha}}))$ (cf. [AKO, Proposition I.5.4]), this proves the last statement. □

Lemma 5.6. *Assume Hypotheses 5.1, case (III.1), and Notation 5.2.*

(a) *Assume that all classes in $\widehat{\Sigma}$ have order 1 or 2. (Equivalently, $\tau(\alpha) = \alpha$ or $\tau(\alpha) \perp \alpha$ for each $\alpha \in \Sigma$.) Then $C_{\overline{T}}(W_0) = C_{\overline{T}}(W) = Z(\overline{G})$, and $Z(G) = C_T(W_0)$.*

(b) *Assume that $\widehat{\Sigma}$ contains classes of order 3. Then $\overline{G} \cong SL_{2n-1}(\overline{\mathbb{F}}_{q_0})$ and $G \cong SU_{2n-1}(q)$ for some $n \ge 2$. Also, $C_{\overline{T}}(W_0) \cong \overline{\mathbb{F}}_{q_0}^{\times}$, and $\sigma(t) = t^{-q}$ for all $t \in C_{\overline{T}}(W_0)$.*

Proof. **(a)** Assume that $\tau(\alpha) = \alpha$ or $\tau(\alpha) \perp \alpha$ for each $\alpha \in \Sigma$. We first show, for each $\widehat{\alpha} = \{\alpha, \tau(\alpha)\} \in \widehat{\Pi}$, that $C_{\overline{T}}(w_{\widehat{\alpha}}) = C_{\overline{T}}(w_\alpha, w_{\tau(\alpha)})$. This is clear if $\alpha = \tau(\alpha)$. If $\alpha \perp \tau(\alpha)$, then $w_{\widehat{\alpha}} = w_\alpha w_{\tau(\alpha)}$, so if $t \in C_{\overline{T}}(w_{\widehat{\alpha}})$, then $w_\alpha(t) = w_{\tau(\alpha)}(t)$ and $t^{-1}w_\alpha(t) = t^{-1}w_{\tau(\alpha)}(t)$. Also, $t^{-1}w_\alpha(t) \in \overline{T}_\alpha$ and $t^{-1}w_{\tau(\alpha)}(t) \in \overline{T}_{\tau(\alpha)}$ by Lemma 2.4(e). Since $\overline{T}_\alpha \cap \overline{T}_{\tau(\alpha)} = 1$ by Lemma 2.4(b), $t^{-1}w_\alpha(t) = 1$, and hence $t \in C_{\overline{T}}(w_\alpha, w_{\tau(\alpha)})$.

Since $W = \langle w_\alpha \,|\, \alpha \in \Pi \rangle$, this proves that $C_{\overline{T}}(W_0) = C_{\overline{T}}(W)$. Since $\overline{G}$ is universal, $C_{\overline{T}}(W) = Z(\overline{G})$ by Proposition 2.5. In particular, $C_T(W_0) \le G \cap Z(\overline{G}) \le Z(G)$; while $Z(G) \le C_T(W_0)$ since $C_G(T) = T$ by Lemma 5.3(a).

(b) Assume $\widehat{\Sigma}$ contains a class of order 3. Then by [GLS3, (2.3.2)], $\gamma \ne \mathrm{Id}$, $\mathbb{G} = SL_{2n-1}$, and $G \cong SU_{2n-1}(q)$ (some $n \ge 2$). Also, if we identify

$$\overline{T} = \bigl\{ \mathrm{diag}(\lambda_1, \ldots, \lambda_{2n-1}) \,\big|\, \lambda_i \in \overline{\mathbb{F}}_{q_0}^{\times},\ \lambda_1\lambda_2\cdots\lambda_{2n-1} = 1 \bigr\},$$

and identify $W = \Sigma_{2n-1}$ with its action on $\overline{T}$ permuting the coordinates, then

$$\gamma\bigl(\mathrm{diag}(\lambda_1, \ldots, \lambda_{2n-1})\bigr) = \mathrm{diag}(\lambda_{2n-1}^{-1}, \ldots, \lambda_1^{-1}),$$

and $W_0 \cong C_2 \wr \Sigma_{n-1}$ is generated by the permutations $(i\,2n{-}i)$ and $(i\,j)(2n{-}i\,2n{-}j)$ for $i, j < n$. So $C_{\overline{T}}(W_0)$ is the group of all matrices $\mathrm{diag}(\lambda_1, \ldots, \lambda_{2n-1})$ such that $\lambda_i = \lambda_1$ for all $i \ne n$ and $\lambda_n = \lambda_1^{-(2n-2)}$, and $C_{\overline{T}}(W_0) \cong \overline{\mathbb{F}}_{q_0}^{\times}$. Also, γ inverts $C_{\overline{T}}(W_0)$, so $\sigma(t) = t^{-q}$ for $t \in C_{\overline{T}}(W_0)$. □

Recall (Notation 5.2(H)) that $\mathrm{Aut}(A, \mathcal{F})$ is the group of automorphisms of A which extend to elements of $\mathrm{Aut}(S, \mathcal{F})$. The next result describes the structure of $\mathrm{Aut}(A, \mathcal{F})$ for a group G in the situation of case (III.1) or (III.2) of Hypotheses 5.1. Recall that W_0 acts faithfully on A by Lemma 5.3(a), and hence that $W_0 \cong \mathrm{Aut}_N(A) = \mathrm{Aut}_{N_G(T)}(A)$ by Lemma 5.3(b). It will be convenient to identify W_0 with this subgroup of $\mathrm{Aut}(A)$. Since each element of $\mathrm{Aut}(A, \mathcal{F})$ is fusion preserving, this group normalizes and hence acts on W_0, and $W_0\mathrm{Aut}(A, \mathcal{F})$ is a subgroup of $\mathrm{Aut}(A)$.

For convenience, we set $\mathrm{Aut}_{\mathrm{Aut}(G)}(A) = \bigl\{ \delta|_A \,\big|\, \delta \in \mathrm{Aut}(G),\ \delta(A) = A \bigr\}$.

Lemma 5.7. *Assume that G and $(\overline{G}, \sigma)$ satisfy Hypotheses 5.1, case (III.1) or (III.2). Assume also Notation 5.2.*

(a) $C_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_0) \le W_0\mathrm{Aut}_{\mathrm{sc}}(A)$.

(b) $\mathrm{Aut}(A, \mathcal{F}) \le \mathrm{Aut}_{\mathrm{sc}}(A)\mathrm{Aut}_{\mathrm{Aut}(G)}(A)$, *with the exceptions*

- $(G,p) \cong ({}^2E_6(q),3)$, *or*
- $(G,p) \cong (G_2(q),2)$ *and* $q_0 \neq 3$, *or*
- $(G,p) \cong (F_4(q),3)$ *and* $q_0 \neq 2$.

(c) *In all cases,* $\mathrm{Aut}(A,\mathcal{F}) \cap \mathrm{Aut}_{\mathrm{sc}}(A)\mathrm{Aut}_{\mathrm{Aut}(G)}(A)$ *has index at most* 2 *in* $\mathrm{Aut}(A,\mathcal{F})$.

Proof. Recall that in Notation 2.2(C), V_0, $\widehat{\Sigma}$, and $\widehat{\Pi}$ are defined when $\rho(\Pi)=\Pi$, and hence in case (III.1) of Hypotheses 5.1. In case (III.2), we defined $V_0=V$, $\widehat{\Sigma}=\Sigma$, and $\widehat{\Pi}=\Pi$ in Notation 5.2(D). So under the hypotheses of the lemma (and since G is always a Chevalley group in case (III.2)), we have $V_0=V$ and $\widehat{\Pi}=\Pi$ if and only if G is a Chevalley group.

If $\alpha\in\Pi$ and $\alpha+\tau(\alpha)\in\Sigma$, then $\overline{T}_{\alpha+\tau(\alpha)} \le \overline{T}_\alpha\overline{T}_{\tau(\alpha)}$ by Lemma 2.4(d). Hence $\overline{T}_{\widehat{\alpha}} = \overline{T}_\alpha\overline{T}_{\tau(\alpha)}$ (the maximal torus in $\overline{K}_{\widehat{\alpha}}$) for each $\alpha\in\Pi$. So by Lemma 2.4(b), in all cases,

$$T = C_{\overline{T}}(\psi_q\gamma) = \prod_{\widehat{\alpha}\in\widehat{\Pi}} C_{\overline{T}_{\widehat{\alpha}}}(\psi_q\gamma) \qquad \text{and hence} \qquad A = \prod_{\widehat{\alpha}\in\widehat{\Pi}} A_{\widehat{\alpha}}\,. \tag{2}$$

Set

$$\varepsilon = \begin{cases} +1 & \text{if } q\equiv 1 \pmod p \quad \text{(case (III.1))} \\ -1 & \text{if } q\equiv -1 \pmod p \text{ and } p \text{ is odd} \quad \text{(case (III.2))} \end{cases} \qquad \text{and} \qquad m = v_p(q-\varepsilon)\,.$$

By assumption, $\varepsilon=1$ if G is a Steinberg group or if $p=2$, and $m>0$ in all cases.

If G is a Chevalley group, then $\tau=\varepsilon\cdot\mathrm{Id}_V$, so $W_0=W$. Also, $\sigma=\gamma\psi_q$ acts on $\overline{T}$ via $\sigma(t)=t^{\varepsilon q}$, so $T=\{t\in\overline{T}\,|\,t^{q-\varepsilon}=1\}$, and $A=\{t\in\overline{T}\,|\,t^{p^m}=1\}$. Thus for each $\alpha\in\Sigma$, $T_\alpha\cong C_{q-\varepsilon}$ and $A_\alpha\cong C_{p^k}$.

Now assume G is a Steinberg group ($\tau\neq\mathrm{Id}$). For each $\widehat{\alpha}\in\widehat{\Pi}$, either

- $\widehat{\alpha}\cap\Pi=\{\alpha,\tau(\alpha)\}$ for some $\alpha\in\Pi$ such that $\alpha\neq\tau(\alpha)$, in which case $\sigma=\psi_q\gamma$ acts on $\overline{T}_{\widehat{\alpha}}=\overline{T}_\alpha\times\overline{T}_{\tau(\alpha)}$ by sending (a,b) to (b^q,a^q), and so $C_{\overline{T}_{\widehat{\alpha}}}(\psi_q\gamma)\cong C_{q^2-1}$; or
- $\widehat{\alpha}=\{\alpha\}$ for some $\alpha\in\Pi$ such that $\alpha=\tau(\alpha)$, in which case $\psi_q\gamma$ acts on $\overline{T}_{\widehat{\alpha}}=\overline{T}_\alpha$ via $(a\mapsto a^q)$, and $C_{\overline{T}_{\widehat{\alpha}}}(\psi_q\gamma)\cong C_{q-1}$.

Since $v_p(q^2-1)=m$ (p odd) or $m+1$ ($p=2$), we have now shown that in all cases,

$$A_{\widehat{\alpha}}\cong C_{p^m} \text{ if } p \text{ is odd;} \qquad A_{\widehat{\alpha}}\cong\begin{cases} C_{2^m} & \text{if } p=2 \text{ and } |\widehat{\alpha}|=1 \\ C_{2^{m+1}} & \text{if } p=2 \text{ and } |\widehat{\alpha}|\ge 2. \end{cases} \tag{3}$$

Step 1: We first prove that

$$\varphi\in C_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_0) \quad\Longrightarrow\quad \varphi(A_{\widehat{\alpha}})=A_{\widehat{\alpha}} \text{ for all } \widehat{\alpha}\in\widehat{\Sigma}. \tag{4}$$

If p is odd, then $A_{\widehat{\alpha}}=[w_{\widehat{\alpha}},A]$ by Lemma 5.5(a), so (4) is immediate.

Next assume that $p=2$, and also that $|\widehat{\alpha}|\le 2$. Write $\varphi=w\circ\varphi_0$, where $w\in W_0$ and $\varphi_0\in\mathrm{Aut}(A,\mathcal{F})$. Then $\varphi_0(C_A(w_{\widehat{\alpha}}))=w^{-1}(C_A(w_{\widehat{\alpha}}))=C_A(w_{\widehat{\beta}})$, where $\widehat{\beta}=w^{-1}(\widehat{\alpha})$. By definition of $\mathrm{Aut}(A,\mathcal{F})$ (Notation 5.2), $\varphi_0=\overline{\varphi}_0|_A$ for some $\overline{\varphi}_0\in\mathrm{Aut}(S,\mathcal{F})$. Since $\overline{\varphi}_0$ is fusion preserving, it sends $\mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}})))$ onto $\mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\beta}})))$. Since these focal subgroups are $A_{\widehat{\alpha}}$ and $A_{\widehat{\beta}}$, respectively, by Lemma 5.5(c), $\varphi(A_{\widehat{\alpha}})=w(A_{\widehat{\beta}})=A_{w(\widehat{\beta})}=A_{\widehat{\alpha}}$ also in this case (the second equality by Lemma 2.4(e)).

It remains to consider the case where $p=2$ and $|\widehat{\alpha}|=3$, and thus where $G\cong SU_{2n+1}(q)$ for some $n\ge 1$. There is a subgroup $(H_1\times\cdots\times H_n)\rtimes\Sigma_n < G$ of odd index, where

$H_i \cong GU_2(q)$. Fix $S_i \in \mathrm{Syl}_2(H_i)$; then $S_i \cong SD_{2^k}$ where $k = v_2(q^2-1)+1 \geq 4$. Let $A_i, Q_i < S_i$ denote the cyclic and quaternion subgroups of index 2 in S_i. Then we can take $A = A_1 \times \cdots \times A_n \cong (C_{2^{k-1}})^n$, $N = (S_1 \times \cdots \times S_n) \rtimes \Sigma_n$, and $S \in \mathrm{Syl}_2(N)$.

There are exactly n classes $\widehat{\alpha}_1, \dots, \widehat{\alpha}_n \in \widehat{\Sigma}_+$ of order 3, which we label so that $[w_{\widehat{\alpha}_i}, A] \leq A_i$ ($[w_{\widehat{\alpha}_i}, A] = A \cap Q_i$). Equivalently, these are chosen so that $w_{\widehat{\alpha}_i}$ acts on A via conjugation by an element of $S_i \smallsetminus A_i$. Let $\alpha_i^* \in \Sigma_+$ be the root in the class $\widehat{\alpha}_i$ which is the sum of the other two.

Write $\varphi = w \circ \varphi_0$, where $w \in W_0$ and $\varphi_0 \in \mathrm{Aut}(A, \mathcal{F})$, and let $\bar{\varphi}_0 \in \mathrm{Aut}(S, \mathcal{F})$ be such that $\varphi_0 = \bar{\varphi}_0|_A$. For each $1 \leq i \leq n$, $\varphi_0(C_A(w_{\widehat{\alpha}_i})) = w^{-1}(C_A(w_{\widehat{\alpha}_i})) = C_A(w_{\widehat{\alpha}_{f(i)}})$, where $f \in \Sigma_n$ is such that $\widehat{\alpha}_{f(i)} = w^{-1}(\widehat{\alpha}_i)$. Since $\bar{\varphi}_0$ is fusion preserving, it sends $\mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}_i})))$ onto $\mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}_{f(i)}})))$. By Lemma 5.5(c), $C_G(C_A(w_{\widehat{\alpha}_i})) = G \cap (\overline{T}\,\overline{K}_{\alpha_i^*})$, its commutator subgroup is $G \cap \overline{K}_{\alpha_i^*} \cong SL_2(q)$, and hence $\mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}_i}))) = Q_i$. Thus $\bar{\varphi}_0(Q_i) = Q_{f(i)}$.

For each i, set $Q_i^* = \langle Q_j \,|\, j \neq i \rangle$. Then $C_G(Q_i^*)$ is the product of $G \cap \overline{K}_{\widehat{\alpha}_i} \cong SL_3(q)$ (Lemma 5.4) with $Z(Q_i^*)$. Thus $\bar{\varphi}_0$ sends $\mathfrak{foc}(C_{\mathcal{F}}(Q_i^*)) = S_i$ to $\mathfrak{foc}(C_{\mathcal{F}}(Q_{f(i)}^*)) = S_{f(i)}$, and hence $\varphi_0(A_i) = A_{f(i)}$. So $\varphi(A_i) = w(A_{f(i)}) = A_i$ for each i where $A_i = A_{\widehat{\alpha}_i}$, and this finishes the proof of (4).

Step 2: We next prove point (a): that $C_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_0) \leq W_0\mathrm{Aut}_{\mathrm{sc}}(A)$. Let $\varphi \in W_0\mathrm{Aut}(A, \mathcal{F})$ be an element which centralizes $\mathrm{Aut}_N(A) \cong N/A \cong W_0$. By (4), $\varphi(A_{\widehat{\alpha}}) = A_{\widehat{\alpha}}$ for each $\widehat{\alpha} \in \widehat{\Sigma}$. Since $A_{\widehat{\alpha}}$ is cyclic for each $\widehat{\alpha} \in \widehat{\Sigma}_+$ by (3), $\varphi|_{A_{\widehat{\alpha}}}$ is multiplication by some unique $u_{\widehat{\alpha}} \in (\mathbb{Z}/q_{\widehat{\alpha}})^\times$, where $q_{\widehat{\alpha}} = |A_{\widehat{\alpha}}|$. We must show that $u_{\widehat{\alpha}}$ is independent of $\widehat{\alpha}$.

Assume first that $\tau = \mathrm{Id}$. By (3), $|A_\alpha| = p^m$ for each $\alpha \in \Pi$. Fix $\alpha_1, \alpha_2 \in \Pi$ and $\beta \in \Sigma_+$ such that $\frac{1}{k}\beta = \frac{1}{k}\alpha_1 + \alpha_2$, where either

- $k = 1$ and all three roots have the same length; or
- $k \in \{2,3\}$ and $\|\beta\| = \|\alpha_1\| = \sqrt{k} \cdot \|\alpha_2\|$.

The relation between the three roots is chosen so that $h_\beta(\lambda) = h_{\alpha_1}(\lambda)h_{\alpha_2}(\lambda)$ for all $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$ by Lemma 2.4(d). Hence $u_{\alpha_1} \equiv u_\beta \equiv u_{\alpha_2} \pmod{p^m}$ by (2). By the connectivity of the Dynkin diagram, the u_α for $\alpha \in \Pi$ are all equal, and $\varphi \in \mathrm{Aut}_{\mathrm{sc}}(A)$.

Now assume $|\tau| = 2$; the argument is similar but slightly more complicated. By assumption, $\mathbb{G}$ is of type A_n, D_n, or E_n; i.e., all roots have the same length. Set $m' = v_p(q^2-1)$; then $m' = m$ if p is odd, and $m' = m+1$ if $p = 2$. Fix $\alpha_1, \alpha_2 \in \Pi$ such that $\alpha_1 \neq \tau(\alpha_2)$ and $\beta \overset{\mathrm{def}}{=} \alpha_1 + \alpha_2 \in \Sigma_+$. Choose $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$ of order $p^{m'}$.

If $\alpha_1 \neq \tau(\alpha_1)$ and $\alpha_2 \neq \tau(\alpha_2)$, then $|A_{\widehat{\alpha}_1}| = |A_{\widehat{\alpha}_2}| = p^{m'}$ by (3), and

$$\widehat{h}_{\alpha_1}(\lambda)\widehat{h}_{\alpha_2}(\lambda) = h_{\alpha_1}(\lambda)h_{\tau(\alpha_1)}(\lambda^q)h_{\alpha_2}(\lambda)h_{\tau(\alpha_2)}(\lambda^q) = h_\beta(\lambda)h_{\tau(\beta)}(\lambda^q) = \widehat{h}_\beta(\lambda) \in A_{\widehat{\beta}}\,.$$

Hence

$$\big(\widehat{h}_{\alpha_1}(\lambda)\widehat{h}_{\alpha_2}(\lambda)\big)^{u_{\widehat{\beta}}} = \varphi\big(\widehat{h}_{\alpha_1}(\lambda)\widehat{h}_{\alpha_2}(\lambda)\big) = \widehat{h}_{\alpha_1}(\lambda)^{u_{\widehat{\alpha}_1}} \cdot \widehat{h}_{\alpha_2}(\lambda)^{u_{\widehat{\alpha}_2}}\,,$$

and together with (2), this proves that $u_{\widehat{\alpha}_1} \equiv u_{\widehat{\beta}} \equiv u_{\widehat{\alpha}_2} \pmod{p^{m'}}$.

If $\tau(\alpha_i) = \alpha_i$ for $i = 1, 2$, then a similar argument shows that $u_{\widehat{\alpha}_1} \equiv u_{\widehat{\beta}} \equiv u_{\widehat{\alpha}_2} \pmod{p^m}$. It remains to handle the case where $\alpha_1 \neq \tau(\alpha_1)$ and $\alpha_2 = \tau(\alpha_2)$. In this case, $|A_{\widehat{\alpha}_1}| = p^{m'}$ and $|A_{\widehat{\alpha}_2}| = p^m$ by (3), and these groups are generated by $\widehat{h}_{\alpha_1}(\lambda) = h_{\alpha_1}(\lambda)h_{\tau(\alpha_1)}(\lambda^q)$ and $h_{\alpha_2}(\lambda^{q+1})$, respectively. Then

$$\widehat{h}_{\alpha_1}(\lambda)\widehat{h}_{\alpha_2}(\lambda^{q+1}) = h_{\alpha_1}(\lambda)h_{\tau(\alpha_1)}(\lambda^q)h_{\alpha_2}(\lambda^{q+1}) = h_\beta(\lambda)h_{\tau(\beta)}(\lambda^q) = \widehat{h}_\beta(\lambda) \in A_{\widehat{\beta}}\,,$$

so

$$\big(\widehat{h}_{\alpha_1}(\lambda)\widehat{h}_{\alpha_2}(\lambda^{q+1})\big)^{u_{\widehat{\beta}}} = \varphi\big(\widehat{h}_{\alpha_1}(\lambda)\widehat{h}_{\alpha_2}(\lambda^{q+1})\big) = \widehat{h}_{\alpha_1}(\lambda)^{u_{\widehat{\alpha}_1}} \cdot h_{\alpha_2}(\lambda^{q+1})^{u_{\widehat{\alpha}_2}} ,$$

and $u_{\widehat{\alpha}_1} \equiv u_{\widehat{\beta}} \equiv u_{\widehat{\alpha}_2} \pmod{p^m}$ by (2) again.

Since the Dynkin diagram is connected, and since the subdiagram of nodes in free orbits in the quotient diagram is also connected, this shows that the $u_{\widehat{\alpha}}$ are all congruent for $\widehat{\alpha} \in \widehat{\Pi}$ (modulo p^m or $p^{m'}$, depending on where they are defined), and hence that $\varphi \in \mathrm{Aut_{sc}}(A)$.

Step 3: Consider the subset $W_{\widehat{\Pi}} = \{w_{\widehat{\alpha}} \,|\, \widehat{\alpha} \in \widehat{\Pi}\}$. We need to study the subgroup $N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$: the group of elements of $W_0\mathrm{Aut}(A,\mathcal{F})$ which permute the set $W_{\widehat{\Pi}}$. Note that $W_0 = \langle W_{\widehat{\Pi}} \rangle$ (see, e.g., [Ca, Proposition 13.1.2], and recall that $W_0 = W$ and $\widehat{\Pi} = \Pi$ in case (III.2)). We first show that

$$\mathrm{Aut}(A,\mathcal{F}) \le W_0 N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}}) . \tag{5}$$

Write $\widehat{\Pi} = \{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_k\}$, ordered so that for each $2 \le i \le k$, $\widehat{\alpha}_i$ is orthogonal to all but one of the $\widehat{\alpha}_j$ for $j < i$. Here, $\widehat{\alpha}_i \perp \widehat{\alpha}_j$ means orthogonal as vectors in V_0. Thus $w_{\widehat{\alpha}_i}$ commutes with all but one of the $w_{\widehat{\alpha}_j}$ for $j < i$. By inspection of the Dynkin diagram of $\mathbb{G}$ (or the quotient of that diagram by τ), this is always possible.

Fix $\varphi \in \mathrm{Aut}(A,\mathcal{F})$. In particular, φ normalizes W_0 (recall that we identify $W_0 = \mathrm{Aut}_{W_0}(A)$) since φ is fusion preserving. (Recall that $\mathrm{Aut}_G(A) = \mathrm{Aut}_{W_0}(A)$ by Lemma 5.3(b).) We must show that some element of φW_0 normalizes the set $W_{\widehat{\Pi}}$.

By definition of $\mathrm{Aut}(A,\mathcal{F})$ (Notation 5.2), $\varphi = \bar{\varphi}|_A$ for some $\bar{\varphi} \in \mathrm{Aut}(S,\mathcal{F})$. Since $\bar{\varphi}$ is fusion preserving, φ normalizes $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_G(A)$, where $\mathrm{Aut}_G(A) \cong N/A \cong W_0$ since $C_N(A) = A$ by Lemma 5.3(a). Thus there is a unique automorphism $\widehat{\varphi} \in \mathrm{Aut}(W_0)$ such that $\widehat{\varphi}(w) = \varphi \circ w \circ \varphi^{-1}$ for each $w \in W_0$.

For each i, since $|\widehat{\varphi}(w_{\widehat{\alpha}_i})| = 2$ and $[\widehat{\varphi}(w_{\widehat{\alpha}_i}), A] \cong [w_{\widehat{\alpha}_i}, A]$ is cyclic, $\widehat{\varphi}(w_{\widehat{\alpha}_i}) = w_{\widehat{\alpha}'_i}$ for some $\widehat{\alpha}'_i \in \widehat{\Sigma}$ by Lemma 5.5(b), where $\widehat{\alpha}'_i$ is uniquely determined only up to sign. For $i \ne j$,

$$\widehat{\alpha}_i \perp \widehat{\alpha}_j \iff [w_{\widehat{\alpha}_i}, w_{\widehat{\alpha}_j}] = 1 \iff [\widehat{\varphi}(w_{\widehat{\alpha}_i}), \widehat{\varphi}(w_{\widehat{\alpha}_j})] = 1 \iff \widehat{\alpha}'_i \perp \widehat{\alpha}'_j .$$

So using the assumption about orthogonality, we can choose successively $\widehat{\alpha}'_1, \widehat{\alpha}'_2, \ldots, \widehat{\alpha}'_k$ so that $\widehat{\varphi}(w_{\widehat{\alpha}_i}) = w_{\widehat{\alpha}'_i}$ for each i, and $\langle \widehat{\alpha}'_i, \widehat{\alpha}'_j \rangle \le 0$ for $i \ne j$.

For each $i \ne j$, since $|w_{\widehat{\alpha}_i} w_{\widehat{\alpha}_j}| = |w_{\widehat{\alpha}'_i} w_{\widehat{\alpha}'_j}|$, the angle (in V_0) between $\widehat{\alpha}_i$ and $\widehat{\alpha}_j$ is equal to that between $\widehat{\alpha}'_i$ and $\widehat{\alpha}'_j$ (by assumption, both angles are between $\pi/2$ and π). The roots $\widehat{\alpha}'_i$ for $1 \le i \le k$ thus generate $\widehat{\Sigma}$ as a root system on V_0 with Weyl group W_0, and hence are the fundamental roots for another Weyl chamber for $\widehat{\Sigma}$. (Recall that $\widehat{\Sigma} = \Sigma$, $V_0 = V$, and $W_0 = W$ in case (III.2).) Since W_0 permutes the Weyl chambers transitively [Brb, § VI.1.5, Theorem 2(i)], there is $w \in W_0$ which sends the set $\{w_{\widehat{\alpha}_i}\}$ onto $\{\widehat{\varphi}(w_{\widehat{\alpha}_i})\}$. Thus $c_w^{-1} \circ \varphi \in N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$, so $\varphi \in W_0 N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$, and this proves (5).

Step 4: Set $\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}}) = N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}}) / C_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$: the group of permutations of the set $W_{\widehat{\Pi}}$ which are induced by elements of $W_0\mathrm{Aut}(A,\mathcal{F})$. By (a) (Step 2) and (5), and since $W_0 = \langle W_{\widehat{\Pi}} \rangle$, there is a surjection

$$\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}}) \xrightarrow{\text{onto}} \frac{W_0 N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})}{W_0 C_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})} = \frac{W_0\mathrm{Aut}(A,\mathcal{F})}{W_0\mathrm{Aut_{sc}}(A)} . \tag{6}$$

To finish the proof of the lemma, we must show that each element of $\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$ is represented by an element of $\mathrm{Aut}_{\mathrm{Aut}(G)}(A)$ (i.e., the restriction of an automorphism of G), with the exceptions listed in point (b).

In the proof of Step 3, we saw that each element of $\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$ preserves angles between the corresponding elements of $\widehat{\Pi}$, and hence induces an automorphism of the Coxeter diagram for $(V_0,\widehat{\Sigma})$ (i.e., the Dynkin diagram without orientation on the edges).

Case 1: Assume $G=\mathbb{G}(q)$ is a Chevalley group. The automorphisms of the Coxeter diagram of $\mathbb{G}$ are well known, and we have

$$\bigl|\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})\bigr| \le \begin{cases} 6 & \text{if } \mathbb{G}=D_4 \\ 2 & \text{if } \mathbb{G}=A_n\ (n\ge 2),\ D_n\ (n\ge 5),\ E_6,\ B_2,\ G_2, \text{ or } F_4 \\ 1 & \text{otherwise.} \end{cases} \tag{7}$$

In case (III.1) (i.e., when the setup is standard), all of these automorphisms are realized by restrictions of graph automorphisms in Γ_G (see [Ca, §§ 12.2–4]), except possibly when $G\cong B_2(q)$, $G_2(q)$, or $F_4(q)$. In case (III.2), with the same three exceptions, each such automorphism is realized by some graph automorphism $\varphi\in\Gamma_{\bar{G}}$, and $\varphi|_{\bar{T}}$ commutes with $\sigma|_{\bar{T}}\in Z(\mathrm{Aut}(\bar{T}))$. Hence by Lemma 3.7, $\varphi|_T$ extends to an automorphism of G whose restriction to A induces the given symmetry of the Coxeter diagram. Together with (6), this proves the lemma for Chevalley groups, with the above exceptions.

If $G\cong B_2(q)$ or $F_4(q)$ and $p\ne 2$, then $\bigl|\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})\bigr|=2$, and the nontrivial element is represented by an element of $\mathrm{Aut}_{\Gamma_G}(A)$ exactly when $q_0=2$. This proves the lemma in these cases, and a similar argument holds when $G\cong G_2(q)$ and $p\ne 3$.

It remains to check the cases where $(G,p)\cong(B_2(q),2)$, $(G_2(q),3)$, or $(F_4(q),2)$. We claim that $\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})=1$ in these three cases; then the three groups in (6) are trivial, and so $\mathrm{Aut}(A,\mathcal{F})\le W_0\mathrm{Aut}_{\mathrm{sc}}(A)$. If $(\mathbb{G},p)=(B_2,2)$ or $(G_2,3)$, then with the help of Lemma 2.4(d,b), one shows that the subgroups $\Omega_1(A_\alpha)$ are all equal for α a short root, and are all distinct for the distinct (positive) long roots. More precisely, of the $p+1$ subgroups of order p in $\Omega_1(A)\cong C_p^2$, one is equal to A_α when α is any of the short roots in Σ_+, while each of the other p is equal to A_α for one distinct long root α. Since $\Omega_1(A_\alpha)=\Omega_1([w_\alpha,A])$ for each α, no element of $N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$ can exchange the long and short roots, so $\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})=1$.

Now assume $(\mathbb{G},p)=(F_4,2)$. Let $\alpha,\beta\in\Pi$ be such that α is long, β is short, and $\alpha\not\perp\beta$. Then α and β generate a root system of type B_2, and by the argument in the last paragraph, no element of $N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$ can exchange them. Thus no element in $N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$ can exchange the long and short roots in $\mathbb{G}$, so again $\mathrm{Aut}_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})=1$.

Case 2: Assume G is a Steinberg group. In particular, we are in case (III.1). The Coxeter diagram for the root system $(V_0,\widehat{\Sigma})$ has type B_n, C_n, or F_4 (recall that we excluded the triality groups ${}^3D_4(q)$ in Hypotheses 5.1), and hence has a nontrivial automorphism only when it has type B_2 or F_4. It thus suffices to consider the groups $G={}^2A_3(q)$, ${}^2A_4(q)$, and ${}^2E_6(q)$.

For these groups, the elements $\widehat{h}_\alpha(\lambda)$ for $\lambda\in\mathbb{F}_q^\times$, and hence the $(q-1)$-torsion in the subgroups $T_{\widehat{\alpha}}$ for $\widehat{\alpha}\in\widehat{\Sigma}_+$, have relations similar to those among the corresponding subgroups of T when $G=B_2(q)$ or $F_4(q)$. This follows from Lemma 2.6(a): if $\lambda\in\mathbb{F}_q^\times$ is a generator, then Φ_λ restricts to an isomorphism from $C_{\mathbb{Z}\Sigma^\vee}(\tau)/(q-1)$ to the $(q-1)$-torsion in T, and the elements in $\widehat{\Pi}$ can be identified in a natural way with a basis for $C_{\mathbb{Z}\Sigma^\vee}(\tau)$. Hence when $p=2$, certain subgroups $\Omega_1(A_{\widehat{\alpha}})$ are equal for distinct $\widehat{\alpha}\in\widehat{\Sigma}_+$, proving that no element in $N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})$ can exchange the two classes of roots. Thus the same argument as that used in Case 1 when $(G,p)=(B_2(q),2)$ or $(F_4(q),2)$ applies to prove that $N_{W_0\mathrm{Aut}(A,\mathcal{F})}(W_{\widehat{\Pi}})=\mathrm{Aut}_{\mathrm{sc}}(A)$ in these cases.

Since $p\big||W_0|$ by Hypotheses 5.1(I), we are left only with the case where $p = 3$ and $G = {}^2E_6(q)$ for some $q \equiv 1 \pmod 3$. Then $(V_0, \widehat{\Sigma})$ is the root system of F_4, so $\mathrm{Aut}(A, \mathcal{F}) \cap W_0\mathrm{Aut}_{\mathrm{sc}}(A)$ has index at most 2 in $\mathrm{Aut}(A, \mathcal{F})$ by (6) and (7). Thus (c) holds in this case. (In fact, the fusion system of G is isomorphic to that of $F_4(q)$ by [BMO, Example 4.4], and does have an "exotic" graph automorphism.) □

We now look at groups which satisfy any of the cases (III.1), (III.2), or (III.3) in Hypotheses 5.1. Recall that $\bar{\kappa}_G = \mu_G \circ \kappa_G \colon \mathrm{Out}(G) \longrightarrow \mathrm{Out}(S, \mathcal{F})$.

Lemma 5.8. *Assume Hypotheses 5.1 and Notation 5.2. Then each $\varphi \in \mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F})$ is the restriction of a diagonal automorphism of G. More precisely, $\bar{\kappa}_G$ restricts to an epimorphism from $\mathrm{Outdiag}(G)$ onto $\mathrm{Out}_{\mathrm{diag}}(S, \mathcal{F})$ whose kernel is the p'-torsion subgroup. Also, $C_A(W_0) = O_p(Z(G))$.*

Proof. In general, whenever H is a group and $B \trianglelefteq H$ is a normal abelian subgroup, we let $\mathrm{Aut}_{\mathrm{diag}}(H, B)$ be the group of all $\varphi \in \mathrm{Aut}(H)$ such that $\varphi|_B = \mathrm{Id}_B$ and $[\varphi, H] \le B$, and let $\mathrm{Out}_{\mathrm{diag}}(H, B)$ be the image of $\mathrm{Aut}_{\mathrm{diag}}(H, B)$ in $\mathrm{Out}(H)$. There is a natural isomorphism $\mathrm{Aut}_{\mathrm{diag}}(H, B)/\mathrm{Aut}_B(H) \xrightarrow[\cong]{\eta_{H,B}} H^1(H/B; B)$ (cf. [Sz1, 2.8.7]), and hence $H^1(H/B; B)$ surjects onto $\mathrm{Out}_{\mathrm{diag}}(H, B)$. If B is centric in H (if $C_H(B) = B$), then $\mathrm{Out}_{\mathrm{diag}}(H, B) \cong H^1(H/B; B)$ since $\mathrm{Aut}_B(H) = \mathrm{Inn}(H) \cap \mathrm{Aut}_{\mathrm{diag}}(H, B)$.

In particular, $\mathrm{Out}_{\mathrm{diag}}(S, A)$ is a p-group since $H^1(S/A; A)$ is a p-group. Also, $C_S(A) = A$ by Lemma 5.3(a) (or by assumption in case (III.3)), and hence we have $\mathrm{Out}_{\mathrm{diag}}(S, A) \cong \mathrm{Aut}_{\mathrm{diag}}(S, A)/\mathrm{Aut}_A(S)$. So $\mathrm{Aut}_{\mathrm{diag}}(S, A)$ is a p-group, and its subgroup $\mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F})$ is a p-group. It follows that

$$\mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F}) \cap \mathrm{Aut}_G(S) = \mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F}) \cap \mathrm{Inn}(S) = \mathrm{Aut}_A(S)\,,$$

and thus $\mathrm{Out}_{\mathrm{diag}}(S, \mathcal{F}) \cong \mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F})/\mathrm{Aut}_A(S)$.

Since $\mathrm{Outdiag}(G) = \mathrm{Out}_{\bar{T}}(G)$ by Proposition 3.5(c), $\bar{\kappa}_G(\mathrm{Outdiag}(G)) \le \mathrm{Out}_{\mathrm{diag}}(S, \mathcal{F})$, and in particular, $\bar{\kappa}_G$ sends all torsion prime to p in $\mathrm{Outdiag}(G)$ to the identity. It remains to show that it sends $O_p(\mathrm{Outdiag}(G))$ isomorphically to $\mathrm{Out}_{\mathrm{diag}}(S, \mathcal{F})$.

Consider the following commutative diagram of automorphism groups and cohomology groups:

$$\begin{array}{ccc}
\mathrm{Out}_{\mathrm{diag}}(S, \mathcal{F}) \cong \mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F})/\mathrm{Aut}_A(S) & \xrightarrow{\ \chi\ } & H^1(\mathrm{Aut}_G(A); A) \\
\Big\downarrow{\scriptstyle \mathrm{incl}} & & \Big\downarrow{\scriptstyle \rho_2} \\
\mathrm{Out}_{\mathrm{diag}}(S, A) \cong \mathrm{Aut}_{\mathrm{diag}}(S, A)/\mathrm{Aut}_A(S) & \xrightarrow[\cong]{\ \eta_{S,A}\ } & H^1(\mathrm{Aut}_S(A); A)\,.
\end{array} \tag{8}$$

Here, ρ_2 is induced by restriction, and is injective by [CE, Theorem XII.10.1] and since $\mathrm{Aut}_S(A) \in \mathrm{Syl}_p(\mathrm{Aut}_G(A))$ (since $A \trianglelefteq S \in \mathrm{Syl}_p(G)$). For each $\omega \in \mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F})$, since ω is fusion preserving, $\eta_{S,A}([\omega]) \in H^1(\mathrm{Aut}_S(A); A)$ is stable with respect to $\mathrm{Aut}_G(A)$-fusion, and hence by [CE, Theorem XII.10.1] is the restriction of a unique element $\chi([\omega]) \in H^1(\mathrm{Aut}_G(A); A)$.

The rest of the proof splits into two parts, depending on which of cases (III.1), (III.2), or (III.3) in Hypotheses 5.1 holds. Recall that $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_G(A) = \mathrm{Aut}_{W_0}(A)$: the second equality by Lemma 5.3(b) in cases (III.1) or (III.2), or by assumption in case (III.3).

Cases (III.2) and (III.3): We show that in these cases, $\mathrm{Outdiag}(G)$, $\mathrm{Out}_{\mathrm{diag}}(S, \mathcal{F})$, $Z(G)$, and $C_A(W_0)$ all have order prime to p. Recall that p is odd in both cases. By hypothesis in case (III.3), and since $\gamma|_{\bar{T}} \in O_{p'}(W_0)$ inverts $\bar{T}$ in case (III.2), $C_A(O_{p'}(W_0)) = 1$. In particular, $C_A(W_0) = 1$. Since $Z(G) \le Z(\bar{G})$ by Proposition 3.5(a), and $Z(\bar{G}) \le \bar{T}$ by Lemma 2.4(a), $Z(G) \le G \cap C_{\bar{T}}(W) \le C_T(W_0)$, so $O_p(Z(G)) \le C_A(W_0) = 1$. This proves the last statement.

Now, $O_p(\mathrm{Outdiag}(G)) = 1$ since $\mathrm{Outdiag}(G) \cong Z(G)$ (see [GLS3, Theorem 2.5.12(c)]) and $O_p(Z(G)) = 1$. Also,

$$H^1(\mathrm{Aut}_G(A); A) = H^1(\mathrm{Aut}_{W_0}(A); A) \cong H^1(\mathrm{Aut}_{W_0}(A)/\mathrm{Aut}_{O_{p'}(W_0)}(A); C_A(O_{p'}(W_0))) = 0$$

since A is a p-group and $C_A(O_{p'}(W_0)) = 1$. Hence $\mathrm{Out}_{\mathrm{diag}}(S,\mathcal{F}) = 1$ by diagram (8).

Case (III.1): Since $C_W(A) = 1$ by Lemma 5.3(a) (and since $\mathrm{Aut}_G(A) = \mathrm{Aut}_{W_0}(A)$), we can identify $H^1(\mathrm{Aut}_G(A); A) = H^1(W_0; A)$. Consider the following commutative diagram of automorphism groups and cohomology groups

$$\begin{array}{ccccc} O_p(\mathrm{Outdiag}(G)) & \xrightarrow{R} & O_p(\mathrm{Out}_{\mathrm{diag}}(N_G(T),T)) & \xrightarrow[\cong]{\eta_{N(T),T}} & H^1(W_0;T)_{(p)} \\ \Big\downarrow{\scriptstyle \bar{\kappa}_G} & & \cong\Big\downarrow{\scriptstyle\sigma_1} & & \cong\Big\downarrow{\scriptstyle\sigma_2} \\ & {\scriptstyle\chi}\nearrow & \mathrm{Out}_{\mathrm{diag}}(N,A) & \xrightarrow[\cong]{\eta_{N,A}} & H^1(W_0;A) \\ & {\scriptstyle\chi_0}\nearrow & \Big\downarrow{\scriptstyle\rho_1} & & \Big\downarrow{\scriptstyle\rho_2} \\ \mathrm{Out}_{\mathrm{diag}}(S,\mathcal{F}) & \xrightarrow{\mathrm{incl}} & \mathrm{Out}_{\mathrm{diag}}(S,A) & \xrightarrow[\cong]{\eta_{S,A}} & H^1(S/A;A) \end{array} \qquad (9)$$

where R is induced by restriction to $N_G(T)$. By Lemma 5.3(a), T is centric in $N_G(T)$ and A is centric in N, so the three η's are well defined and isomorphisms (i.e., $\mathrm{Out}_{\mathrm{diag}}(N,A) = \mathrm{Aut}_{\mathrm{diag}}(N,A)/\mathrm{Aut}_A(N)$, etc.). The maps σ_i are induced by dividing out by $O_{p'}(T)$, and are isomorphisms since $A = O_p(T)$. The maps ρ_i are induced by restriction, and are injective since $S/A \in \mathrm{Syl}_p(W_0)$ (see [CE, Theorem XII.10.1]).

Consider the short exact sequence

$$1 \longrightarrow T \longrightarrow \bar{T} \xrightarrow{\ \Psi\ } \bar{T} \longrightarrow 1,$$

where $\Psi(t) = t^{-1} \cdot \gamma\psi_q(t) = t^{-1}\gamma(t^q)$ for $t \in \bar{T}$. Let

$$1 \longrightarrow C_T(W_0) \longrightarrow C_{\bar{T}}(W_0) \xrightarrow{\ \Psi_*\ } C_{\bar{T}}(W_0) \xrightarrow{\ \delta\ } H^1(W_0;T) \xrightarrow{\ \theta\ } H^1(W_0;\bar{T}) \qquad (10)$$

be the induced cohomology exact sequence for the W_0-action, and recall that $H^1(W_0;A) \cong H^1(W_0;T)_{(p)}$ by (9). We claim that

(11) $|O_p(\mathrm{Outdiag}(G))| = |\mathrm{Im}(\delta)_{(p)}| = |O_p(Z(G))| = |C_A(W_0)|$;

(12) R is injective; and

(13) $\chi(\mathrm{Out}_{\mathrm{diag}}(S,\mathcal{F})) \le \mathrm{Ker}(\theta)$.

These three points will be shown below. It then follows from the commutativity of diagram (9) (and since $\mathrm{Im}(\delta) = \mathrm{Ker}(\theta)$) that $\bar{\kappa}_G$ sends $O_p(\mathrm{Outdiag}(G))$ isomorphically onto $\mathrm{Out}_{\mathrm{diag}}(S,\mathcal{F})$.

Proof of (11) and (12): Assume first that $\gamma \ne \mathrm{Id}$ and $\mathbb{G} = SL_{2n-1}$ (some $n \ge 1$). Thus $G \cong SU_{2n-1}(q)$. By [St1, 3.4], $\mathrm{Outdiag}(G)$ and $Z(G)$ are cyclic of order $(q+1, 2n-1)$, and hence have no p-torsion (recall $p|(q-1)$). By Lemma 5.6(b), $C_{\bar{T}}(W_0) \cong \bar{\mathbb{F}}_{q_0}^\times$, and $\sigma(u) = u^{-q}$ for $u \in C_{\bar{T}}(W_0)$. Thus $\Psi_*(u) = u^{-1}\sigma(u) = u^{-1-q}$ for $u \in C_{\bar{T}}(W_0)$, so Ψ_* is onto, and $\mathrm{Im}(\delta) = 1 \cong O_p(\mathrm{Outdiag}(G))$ in this case. Also, $C_T(W_0) = \mathrm{Ker}(\Psi_*)$ has order $q+1$, so $C_A(W_0) = O_p(C_T(W_0)) = 1$.

Now assume $\gamma = \mathrm{Id}$ or $\mathbb{G} \neq SL_{2n-1}$. By Lemma 5.6, in all such cases,

$$C_{\overline{T}}(W_0) = C_{\overline{T}}(W) = Z(\overline{G}) \qquad \text{and} \qquad C_T(W_0) = Z(G)\ . \tag{14}$$

In particular, these groups are all finite, and hence $|\mathrm{Im}(\delta)| = |Z(G)|$ by the exactness of (10). By [GLS3, Theorem 2.5.12(c)], $\mathrm{Outdiag}(G) \cong Z(G)$ in all cases, and hence $|\mathrm{Outdiag}(G)| = |\mathrm{Im}(\delta)|$.

If $[\varphi] \in \mathrm{Ker}(R)$, then we can assume that it is the class of $\varphi \in \mathrm{Aut}_{\overline{T}}(G)$. Thus $\varphi = c_x$ for some $x \in N_{\overline{T}}(G)$, and $\varphi|_{N_G(T)} = c_y$ for some $y \in N_G(T)$ which centralizes A. Then $y \in C_G(A) = T$ by Lemma 5.3(a), and upon replacing φ by $c_y^{-1} \circ \varphi$ and x by $y^{-1}x$ (without changing the class $[\varphi]$), we can arrange that $\varphi|_{N_G(T)} = \mathrm{Id}$. Then $x \in C_{\overline{T}}(W_0)$ since it centralizes $N_G(T)$ (and since $N_G(T)/T \cong W_0$ by Lemma 5.3(b)), so $x \in Z(\overline{G})$ by (14), and hence $\varphi = \mathrm{Id}_G$. Thus R is injective.

Proof of (13): Fix $\varphi \in \mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F})$. Choose $\bar{\varphi} \in \mathrm{Aut}_{\mathrm{diag}}(N, A)$ such that $\bar{\varphi}|_S = \varphi$ (i.e., such that $[\bar{\varphi}] = \chi_0([\varphi])$ in diagram (9)). Recall that $W_0 \cong N/A$ by Lemma 5.3(b). Let $\underline{c}\colon W_0 \cong N/A \longrightarrow A$ be such that $\bar{\varphi}(g) = \underline{c}(gA){\cdot}g$ for each $g \in N$; thus $\eta_{N,A}([\varphi]) = [\underline{c}]$. We must show that $\theta([\underline{c}]) = 1$: that this is a consequence of φ being fusion preserving.

For each $\widehat{\alpha} \in \widehat{\Pi}$, set $u_{\widehat{\alpha}} = \underline{c}(w_{\widehat{\alpha}})$. Thus for $g \in N$, $\bar{\varphi}(g) = u_{\widehat{\alpha}}g$ if $g \in w_{\widehat{\alpha}}$ (as a coset of A in N). Since $w_{\widehat{\alpha}}^2 = 1$, $g^2 = \bar{\varphi}(g^2) = (u_{\widehat{\alpha}}g)^2$, and hence $w_{\widehat{\alpha}}(u_{\widehat{\alpha}}) = u_{\widehat{\alpha}}^{-1}$. We claim that $u_{\widehat{\alpha}} \in A_{\widehat{\alpha}} = A \cap \overline{K}_{\widehat{\alpha}}$ for each $\widehat{\alpha} \in \widehat{\Pi}$.

- If p is odd, then $u_{\widehat{\alpha}} \in A_{\widehat{\alpha}}$, since $A_{\widehat{\alpha}} = \{a \in A \,|\, w_{\widehat{\alpha}}(a) = a^{-1}\}$ by Lemma 2.4(e).
- If $p = 2$, $w_{\widehat{\alpha}} \in S/A$, and $|\widehat{\alpha}| \leq 2$, choose $g_{\widehat{\alpha}} \in S \cap \overline{K}_{\widehat{\alpha}}$ such that $w_{\widehat{\alpha}} = g_{\widehat{\alpha}}A$. (For example, if we set $g = \prod_{\alpha\in\widehat{\alpha}} n_\alpha(1)$ (see Notation 2.2(B)), then $g \in N_G(T)$ represents the class $w_{\widehat{\alpha}} \in W_0$, and is T-conjugate to an element of $S \cap \overline{K}_{\widehat{\alpha}}$.) By Lemma 5.5(c), $C_G(C_A(w_{\widehat{\alpha}})) = G \cap \overline{T}\overline{K}_{\widehat{\alpha}}$, where $G \cap \overline{K}_{\widehat{\alpha}} \cong SL_2(q)$ or $SL_2(q^2)$ by Lemma 5.4. Hence
$$\mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}}))) = \mathfrak{foc}(C_G(C_A(w_{\widehat{\alpha}}))) = S \cap [G \cap \overline{T}\overline{K}_{\widehat{\alpha}}, G \cap \overline{T}\overline{K}_{\widehat{\alpha}}] = S \cap \overline{K}_{\widehat{\alpha}}$$
(see the remarks before Lemma 5.5), and $g_{\widehat{\alpha}}$ lies in this subgroup. Since φ is fusion preserving, $\varphi(g_{\widehat{\alpha}}) \in \mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}})))$. By Lemma 5.5(c) again,
$$u_{\widehat{\alpha}} = \varphi(g_{\widehat{\alpha}}) \cdot g_{\widehat{\alpha}}^{-1} \in A \cap \mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}}))) = A_{\widehat{\alpha}}\,.$$
- If $p = 2$, $w_{\widehat{\alpha}} \in S/A$, and $\widehat{\alpha} = \{\alpha, \tau(\alpha), \alpha^*\}$ where $\alpha^* = \alpha + \tau(\alpha)$, then $w_{\widehat{\alpha}} = w_{\alpha^*}$. Choose $g_{\widehat{\alpha}} \in S \cap \overline{K}_{\alpha^*}$ such that $g_{\widehat{\alpha}}A = w_{\widehat{\alpha}} \in N/A$. (For example, there is such a $g_{\widehat{\alpha}}$ which is T-conjugate to $n_{\alpha^*}(1)$.) By Lemma 5.5(c), $C_G(C_A(w_{\widehat{\alpha}})) = G \cap \overline{T}\overline{K}_{\alpha^*}$, $G \cap \overline{K}_{\alpha^*} \cong SL_2(q)$, and hence $g_{\widehat{\alpha}} \in \mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}})))$. So $\varphi(g_{\widehat{\alpha}}) \in \mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}})))$ since $\varphi|_S$ is fusion preserving. By Lemma 5.5(c),
$$u_{\widehat{\alpha}} = \varphi(g_{\widehat{\alpha}}) \cdot g_{\widehat{\alpha}}^{-1} \in A \cap \mathfrak{foc}(C_{\mathcal{F}}(C_A(w_{\widehat{\alpha}}))) = A \cap \overline{K}_{\alpha^*} \leq A_{\widehat{\alpha}}\,.$$
- If $p = 2$ and $w_{\widehat{\alpha}} \notin S/A \in \mathrm{Syl}_2(W_0)$, then it is W_0-conjugate to some other reflection $w_{\widehat{\beta}} \in S/A$ (for $\widehat{\beta} \in \widehat{\Sigma}_+$), $\underline{c}(w_{\widehat{\beta}}) \in A_{\widehat{\beta}}$ by the above argument, and hence $u_{\widehat{\alpha}} = \underline{c}(w_{\widehat{\alpha}}) \in A_{\widehat{\alpha}}$.

Consider the homomorphism

$$\Phi = (\Phi_\alpha)_{\alpha\in\Pi}\colon \overline{T} \xrightarrow{\qquad} \prod_{\alpha\in\Pi} \overline{T}_\alpha \qquad \text{where} \qquad \Phi_\alpha(t) = t^{-1}w_\alpha(t) \quad \forall\ t \in \overline{T},\ \alpha \in \Pi.$$

Since $W = \langle w_\alpha \,|\, \alpha \in \Pi\rangle$, we have $\mathrm{Ker}(\Phi) = C_{\overline{T}}(W) = Z(\overline{G})$ is finite (Proposition 2.5). Thus Φ is (isomorphic to) a homomorphism from $(\overline{\mathbb{F}}_{q_0}^\times)^r$ to itself with finite kernel (where $r = |\Pi|$), and any such homomorphism is surjective since $\overline{\mathbb{F}}_{q_0}^\times$ has no subgroups of finite index.

Choose elements $v_\alpha \in \bar{T}_\alpha$ for $\alpha \in \Pi$ as follows.

- If $\widehat{\alpha} = \{\alpha\}$ where $\tau(\alpha) = \alpha$, we set $v_\alpha = u_{\widehat{\alpha}}$.
- If $\widehat{\alpha} = \{\alpha, \tau(\alpha)\}$, where $\alpha \perp \tau(\alpha)$, then $\bar{T}_{\widehat{\alpha}} = \bar{T}_\alpha \times \bar{T}_{\tau(\alpha)}$, and we let $v_\alpha, v_{\tau(\alpha)}$ be such that $v_\alpha v_{\tau(\alpha)} = u_{\widehat{\alpha}}$.
- If $\widehat{\alpha} = \{\alpha, \tau(\alpha), \alpha^*\}$ where $\alpha^* = \alpha + \tau(\alpha)$, then $u_{\widehat{\alpha}} = h_\alpha(\lambda)h_{\tau(\alpha)}(\lambda')$ for some $\lambda, \lambda' \in \bar{\mathbb{F}}_{q_0}^\times$,
$$w_{\widehat{\alpha}}(h_\alpha(\lambda)h_{\tau(\alpha)}(\lambda')) = h_\alpha(\lambda'^{-1})h_{\tau(\alpha)}(\lambda^{-1})$$
by Lemma 2.4(e), and $\lambda = \lambda'$ since $w_{\widehat{\alpha}}(u_{\widehat{\alpha}}) = u_{\widehat{\alpha}}^{-1}$. Set $v_\alpha = h_\alpha(\lambda)$ and $v_{\tau(\alpha)} = 1$. (This depends on the choice of $\alpha \in \widehat{\alpha} \cap \Pi$.)

Let $t \in \bar{T}$ be such that $\Phi(t) = (v_\alpha)_{\alpha\in\Pi}$. We claim that $t^{-1}w_{\widehat{\alpha}}(t) = u_{\widehat{\alpha}}$ for each $\widehat{\alpha} \in \widehat{\Pi}$. This is clear when $|\widehat{\alpha}| \leq 2$. If $\widehat{\alpha} = \{\alpha, \tau(\alpha), \alpha^*\}$ and λ are as above, then
$$\begin{aligned} w_{\widehat{\alpha}}(t) = w_{\alpha^*}(t) = w_{\tau(\alpha)}w_\alpha w_{\tau(\alpha)}(t) = w_{\tau(\alpha)}(w_\alpha(t)) = w_{\tau(\alpha)}\big(t \cdot h_\alpha(\lambda)\big) \\ = t \cdot w_{\tau(\alpha)}(h_\alpha(\lambda)) = t \cdot h_{\alpha^*}(\lambda) = t \cdot u_{\widehat{\alpha}}. \end{aligned}$$

Thus $\underline{c}(w_{\widehat{\alpha}}) = dt(w_{\widehat{\alpha}})$ for each $\widehat{\alpha} \in \widehat{\Pi}$. Since $W_0 = \langle w_{\widehat{\alpha}} \,|\, \widehat{\alpha} \in \widehat{\Pi}\rangle$ (and since $\underline{c}$ and dt are both cocycles), this implies that $\underline{c} = dt$, and hence that $[\underline{c}] = 0$ in $H^1(W_0; \bar{T})$. □

As one consequence of Lemma 5.8, the Z^*-theorem holds for these groups. This is known to hold for all finite groups (see [GLS3, § 7.8]), but its proof for odd p depends on the classification of finite simple groups, which we prefer not to assume here.

Corollary 5.9. *Assume that $G \in \mathfrak{Lie}(q_0)$, $p \neq q_0$, and $S \in \mathrm{Syl}_p(G)$ satisfy Hypotheses 5.1. Then $Z(\mathcal{F}_S(G)) = O_p(Z(G))$.*

Proof. By Lemma 5.8, $O_p(Z(G)) = C_A(W_0)$, where $C_S(A) = A$ and $\mathrm{Aut}_G(A) = \mathrm{Aut}_{W_0}(A)$ by Lemma 5.3(a,b) or by hypothesis (in Case 5.1(III.3)). Hence $Z(\mathcal{F}_S(G)) \leq O_p(Z(G))$, while the other inclusion is clear. □

We now need the following additional hypotheses, in order to be able to compare $\mathrm{Aut}_{\mathrm{sc}}(A)$ with the group of field automorphisms of G.

Hypotheses 5.10. *Fix a prime p and a prime power q. Assume that $q = q_0^b$ where q_0 is prime, $b \geq 1$, $q_0 \neq p$, and*

(i) $q_0 \equiv \pm 3$ *(mod 8) if $p = 2$;*

(ii) *the class of q_0 generates $(\mathbb{Z}/p^2)^\times$ if p is odd; and*

(iii) $b|(p-1)p^\ell$ *for some $\ell \geq 0$.*

We will also say that "G satisfies Hypotheses 5.10" (for a given prime p) if $G \cong {}^t\mathbb{G}(q)$ for some t and $\mathbb{G}$, and some q which satisfies the above conditions.

By Hypothesis 5.1(I), $\psi_{q_0}(G) = G$, and thus all field endomorphisms of $\bar{G}$ normalize G. When G has a *standard* σ-setup, Φ_G was defined to be the group of restrictions of such endomorphisms $\psi_{q_0^a} \in \Phi_{\bar{G}}$ for $a \geq 0$. Under our Hypotheses 5.1, this applies only when we are in case (III.1) (although Proposition 3.6 describes the relation between Φ_G and ψ_{q_0} in the other cases). In what follows, it will be useful to set
$$\widehat{\Phi}_G = \langle \psi_{q_0}|_G \rangle \leq \mathrm{Aut}(G).$$

By Proposition 3.6(d), $\mathrm{Inndiag}(G)\widehat{\Phi}_G = \mathrm{Inndiag}(G)\Phi_G$, although $\widehat{\Phi}_G$ can be strictly larger than Φ_G ($\widehat{\Phi}_G \cap \mathrm{Inndiag}(G)$ need not be trivial). Note that since each element of this group acts on $\bar{T}$ via $(t \mapsto t^r)$ for some r, $\widehat{\Phi}_G$ normalizes T and each of its subgroups.

Recall that $\tau \in \mathrm{Aut}(V)$ is the automorphism induced by σ, and also denotes the induced permutation of Σ.

Lemma 5.11. *Assume Hypotheses 5.1 and 5.10 and Notation 5.2. Let*

$$\chi_0 \colon \widehat{\Phi}_G \longrightarrow \mathrm{Aut}(A, \mathcal{F})$$

be the homomorphism induced by restriction from G to A. Set $m = |\tau| = \big|\gamma|_{\bar{T}}\big|$. Then the following hold.

(a) *Either T has exponent $q^m - 1$; or p is odd, $m = \mathrm{ord}_p(q)$, m is even, and $(q^{m/2} + 1)\big|\mathrm{expt}(T)\big|(q^m - 1)$.*

(b) *If p is odd, then $\chi_0(\widehat{\Phi}_G) = \mathrm{Aut}_{\mathrm{sc}}(A)$. If $p = 2$, then $\chi_0(\widehat{\Phi}_G)$ has index 2 in $\mathrm{Aut}_{\mathrm{sc}}(A)$, and $\mathrm{Aut}_{\mathrm{sc}}(A) = \mathrm{Im}(\chi_0)\langle \psi_{-1}^A \rangle$.*

(c) *If $p = 2$, then χ_0 is injective. If p is odd, then*

$$\mathrm{Ker}(\chi_0) = \begin{cases} \langle \psi_q|_G \rangle = \langle \gamma|_G \rangle & \text{in case (III.1)} \\ \langle (\psi_q|_G)^m \rangle = \langle \gamma^m|_G \rangle = \widehat{\Phi}_G \cap \mathrm{Aut}_{\bar{T}}(G) & \text{in cases (III.2) and (III.3).} \end{cases}$$

Proof. We first recall some of the assumptions in cases (III.1–3) of Hypotheses 5.1:

case (III.1)	$\mathrm{ord}_p(q) = 1,\ m = \|\gamma\|,$ and $m \leq 2$	
case (III.2)	$\mathrm{ord}_p(q) = m = 2$	p is odd
case (III.3)	$\mathrm{ord}_p(q) = m$	p is odd

(15)

(Recall that γ is a graph automorphsm in case (III.1), so $|\gamma| = |\tau| = m$.) In all of these cases, $p|(q^m - 1)$ since $\mathrm{ord}_p(q)|m$.

(a) For each $t \in T = C_{\bar{T}}(\psi_q \circ \gamma)$, $t^q = \psi_q(t) = \gamma^{-1}(t)$. Hence $t = \gamma^{-m}(t) = (\psi_q)^m(t) = t^{q^m}$, and $t^{q^m-1} = 1$. Thus $\mathrm{expt}(T)\big|(q^m - 1)$.

By Hypotheses 5.1(I), there is a linearly independent subset $\Omega = \{\alpha_1, \ldots, \alpha_s\} \subseteq \Sigma$ such that either Ω or $\pm\Omega = \{\pm\alpha_1, \ldots, \pm\alpha_s\}$ is a free $\langle\tau\rangle$-orbit in Σ. Assume Ω is a free orbit (this always happens in case (III.1)). In particular, $m = |\tau| = s$. For each $1 \neq \lambda \in \bar{\mathbb{F}}_{q_0}^\times$ such that $\lambda^{q^s-1} = 1$, the element

$$t(\lambda) = \prod_{i=0}^{m-1} h_{\tau^i(\alpha_1)}(\lambda^{q^i})$$

is fixed by $\sigma = \psi_q \circ \gamma$ (recall $\sigma(h_\beta(\lambda)) = h_{\tau(\beta)}(\lambda^q)$ for each $\beta \in \Sigma$ by Lemma 3.2). Hence $t(\lambda) \in T$, and $t(\lambda) \neq 1$ when $\lambda \neq 1$ by Lemma 2.4(d,b). Thus T contains the subgroup $\{t(\lambda)\,|\,\lambda^{q^m-1} = 1\}$ of order $q^m - 1$, this subgroup is cyclic (isomorphic to a subgroup of $\bar{\mathbb{F}}_{q_0}^\times$), and hence $\mathrm{expt}(T) = q^m - 1$.

Assume now that $\pm\Omega$ is a free $\langle\tau\rangle$-orbit (thus $m = |\tau| = 2s$). In particular, we are not in case (III.1), so p is odd and $m = \mathrm{ord}_p(q)$. Then $\tau^i(\alpha_1) = -\alpha_1$ for some $0 < i < 2s$, and $i = s$ since $\tau^{2i}(\alpha_1) = \alpha_1$. For each $1 \neq \lambda \in \bar{\mathbb{F}}_{q_0}^\times$ such that $\lambda^{q^s+1} = 1$,

$$t(\lambda) = \prod_{i=0}^{s-1} h_{\tau^i(\alpha_1)}(\lambda^{q^i})$$

is fixed by $\sigma = \psi_q \circ \gamma$ by Lemma 3.2 and since $h_{\tau^s(\alpha_1)}(\lambda^{q^s}) = h_{-\alpha_1}(\lambda^{-1}) = h_{\alpha_1}(\lambda)$. Hence $t(\lambda) \in T$, and $t(\lambda) \neq 1$ when $\lambda \neq 1$ by Lemma 2.4 again. Thus $\{t(\lambda) \mid \lambda^{q^e+1} = 1\} \leq T$ is cyclic of order q^s+1, and so $(q^s+1)\big|\mathrm{expt}(T)$.

(b) By definition, $\mathrm{Im}(\chi_0) = \chi_0(\widehat{\Phi}_G)$ is generated by $\chi_0(\psi_{q_0}) = \psi_{q_0}|_A$, which acts on A via $(a \mapsto a^{q_0})$. If p is odd, then by Hypotheses 5.10(ii), the class of q_0 generates $(\mathbb{Z}/p^2)^\times$, and hence generates $(\mathbb{Z}/p^k)^\times$ for each $k>0$. So $\mathrm{Im}(\chi_0) = \mathrm{Aut}_{\mathrm{sc}}(A)$ in this case.

If $p=2$, then $q_0 \equiv \pm 3 \pmod 8$ by Hypotheses 5.10(i). So for each $k \geq 2$, $\langle q_0 \rangle$ has index 2 in $(\mathbb{Z}/2^k)^\times = \langle q_0, -1 \rangle$. Hence $\mathrm{Im}(\chi_0) = \langle \psi_{q_0}|_A \rangle$ has index 2 in $\mathrm{Aut}_{\mathrm{sc}}(A) = \langle \psi_{q_0}|_A, \psi_{-1}^A \rangle$.

(c) Set $\phi_0 = \psi_{q_0}|_G$, a generator of $\widehat{\Phi}_G$. Then $(\phi_0)^b = \psi_q|_G = (\gamma|_G)^{-1}$ since $G = C_{\overline{G}}(\psi_q \circ \gamma)$, and so $\big|\phi_0|_T\big|$ divides $b|\gamma|_{\overline{T}}| = bm$. Also, $(\phi_0)^{bm} = (\gamma|_G)^{-m} \in \mathrm{Aut}_{\overline{T}}(G)$ by Lemma 3.2.

By (a), either $\mathrm{expt}(T) = q^m - 1$; or m is even, p is odd, $\mathrm{ord}_p(q) = m$, and $(q^{m/2}+1)\big|\mathrm{expt}(T)\big|(q^m-1)$. In the latter case, $v_p(q^{m/2}+1) = v_p(q^m-1) > 0$ since $p \nmid (q^{m/2}-1)$. Thus

$$\mathrm{expt}(A) = p^e \qquad \text{where} \qquad e = v_p(q^m-1) = v_p({q_0}^{bm}-1) > 0\,. \tag{16}$$

If $p=2$, then we are in case (III.1). In particular, $q = q_0^b \equiv 1 \pmod 4$, and $m \leq 2$. Also, b (and hence bm) is a power of 2 by Hypotheses 5.10(iii). If $bm=1$, then $q = q_0 \equiv 5 \pmod 8$, so $e = v_2(q-1) = 2$. If bm is even, then $e = v_2(q_0^{bm}-1) = v_2(q_0^2-1)+v_2(bm/2) = 3+v_2(bm/2)$ by Lemma 1.13. Thus in all cases, $e = 2+v_2(bm)$. So $\mathrm{Im}(\chi_0) \leq \mathrm{Aut}_{\mathrm{sc}}(A) \cong (\mathbb{Z}/2^e)^\times$ has order $2^{e-2} = bm$. Since $(\psi_{q_0}|_G)^{bm} = (\psi_q|_G)^m = (\gamma^{-1}|_G)^m = \mathrm{Id}_G$ (recall $m = |\gamma|$ in case (III.1)), χ_0 is injective.

Now assume p is odd, and set $m_0 = \mathrm{ord}_p(q)$. Then $b|(p-1)p^\ell$ for some $\ell \geq 0$ by Hypotheses 5.10(iii), and $q = q_0^b$ where the class of q_0 generates $(\mathbb{Z}/p^k)^\times$ for each $k \geq 1$. For $r \in \mathbb{Z}$, $q^r = q_0^{br} \equiv 1 \pmod p$ if and only if $(p-1)|br$. Hence $bm_0 = b \cdot \mathrm{ord}_p(q) = (p-1)p^\ell$ for some $\ell \geq 0$. Since $v_p(q_0^{p-1}-1) = 1$, and since $m = m_0$ or $2m_0$, Lemma 1.13 implies that

$$e = v_p(q^m-1) = v_p(q_0^{bm}-1) = v_p(q_0^{bm_0}-1) = 1 + v_p(p^\ell) = 1+\ell\,.$$

Thus $\ell = e-1$, where $p^e = \mathrm{expt}(A)$ by (16), so $|\mathrm{Aut}_{\mathrm{sc}}(A)| = (p-1)p^{e-1} = bm_0$. Since χ_0 sends the generator ϕ_0 of $\widehat{\Phi}_G$ to the generator $\chi_0(\phi_0)$ of $\mathrm{Aut}_{\mathrm{sc}}(A)$, this proves that $\mathrm{Ker}(\chi_0) = \langle \psi_q^{m_0}|_G \rangle = \langle \gamma^{m_0}|_G \rangle$. The descriptions in the different cases now follow immediately. Note that in cases (III.2) and (III.3) (where $m = m_0$), $\phi_0^{bm} = \gamma^{-m}|_G \in \mathrm{Aut}_{\overline{T}}(G)$ by Lemma 3.2. The converse is immediate: $\widehat{\Phi}_G \cap \mathrm{Aut}_{\overline{T}}(G) \leq \mathrm{Ker}(\chi_0)$. □

Before applying these results to describe $\mathrm{Out}(S,\mathcal{F})$ and the homomorphism $\bar{\kappa}_G$, we need to know in which cases the subgroup A is characteristic in S.

Proposition 5.12. *Assume Hypotheses 5.1 and Notation 5.2.*

(a) *If $p=2$, then A is characteristic in S, and is the unique abelian subgroup of S of order $|A|$, except when $q \equiv 5$ (mod 8) and $G \cong Sp_{2n}(q)$ for some $n \geq 1$.*

(b) *If p is odd, then A is characteristic in S, and $\Omega_1(A)$ is the unique elementary abelian subgroup of S of maximal rank, except when $p=3$, $q \equiv 1$ (mod 3), $v_3(q-1)=1$, and $G \cong SU_3(q)$ or $G_2(q)$.*

In all cases, each normal subgroup of S isomorphic to A is $N_G(S)$-conjugate to A.

Proof. If p is odd, then by [GL, 10-2(1,2)], there is a unique elementary p-subgroup $E \leq S$ of rank equal to that of A (denoted r_{m_0} in [GL]), except when $p=3$ and G is isomorphic to one of the groups $SL_3(q)$ ($q \equiv 1 \pmod 3$), $SU_3(q)$ ($q \equiv -1 \pmod 3$), or $G_2(q)$, ${}^3D_4(q)$, or ${}^2F_4(q)$ ($q \equiv \pm 1 \pmod 3$). When there is a unique such subgroup E, then $A = C_S(E)$ by Lemma 5.3(a) (or by assumption in case (III.3)), and hence A is characteristic in S.

Among the exceptions, $SL_3(q)$ and $G_2(q)$ are the only ones which satisfy Hypotheses 5.1. In both cases, S is an extension of $A \cong (C_{3^\ell})^2$ by C_3, where $\ell = v_3(q-1)$, and where $Z(S) = C_A(S)$ has order 3. If $\ell > 1$, then A is the unique abelian subgroup of index p in S. If $\ell = 1$, then S is extraspecial of order 3^3 and exponent 3. By Theorem 1.8(a), we can assume $q = 4$ without changing the isomorphism type of the fusion system, so G contains $SU_3(2)$. This is a semidirect product $S \rtimes Q_8$ (cf. [Ta, p. 123–124]), and hence the four subgroups of S of order 9 are $N_G(S)$-conjugate.

It remains to prove the proposition when $p = 2$. We use [O4, § 2] as a reference for information about best offenders, since this contains what we need in a brief presentation. Assume A is not the unique abelian subgroup of S of order $|A|$. Then there is an abelian subgroup $1 \neq B \leq W_0$ such that $|B| \cdot |C_A(B)| \geq |A|$. In other words, the action of the Weyl group W_0 on A has a nontrivial best offender [O4, Definition 2.1(b)]. Hence by Timmesfeld's replacement theorem [O4, Theorem 2.5], there is a quadratic best offender $1 \neq B \leq W_0$: an offender such that $[B, [B, A]] = 1$.

We consider three different cases.

Case 1: $G \cong \mathbb{G}(q)$ is a Chevalley group, where either $q \equiv 1 \pmod 8$, or $G \ncong Sp_{2n}(q)$ for any $n \geq 1$. Set $n = \mathrm{rk}(A) = \mathrm{rk}(T)$: the Lie rank of G (or of $\mathbb{G}$). Set $\ell = v_2(q-1) \geq 2$. Then $A \cong (C_{2^\ell})^n$ is the group of all 2^ℓ-torsion elements in T (or in $\bar{T}$). Since the result is clear when $n = 1$ ($G \cong SL_2(q) \cong Sp_2(q)$, $A \cong C_{2^\ell}$, and $S \cong Q_{2^{\ell+1}}$), we assume $n \geq 2$.

Let $\Lambda = \mathbb{Z}\Sigma^\vee$ be the lattice in V generated by the dual roots. By Lemma 2.6(a), there are $\mathbb{Z}[W]$-linear isomorphisms $A \cong \Lambda/2^\ell\Lambda$ and $\Omega_1(A) \cong \Lambda/2\Lambda$.

Assume first that B acts faithfully on $\Omega_1(A)$. Since B has quadratic action, it is elementary abelian [O4, Lemma 2.4]. Set $k = \mathrm{rk}(B)$; thus $B \cong C_2^k$ and $|A/C_A(B)| \leq 2^k$.

Since the B-action on V is faithful, the characters $\chi \in \mathrm{Hom}(B, \{\pm 1\})$ which have nontrivial eigenspace on V generate the dual group B^*. So we can choose a basis $\chi_1, \ldots, \chi_k$ for B^* such that each χ_i has nontrivial eigenspace. Let $b \in B$ be the unique element such that $\chi_i(b) = -1$ for each $i = 1, \ldots, k$. Let V_+, V_- be the ± 1-eigenspaces for the b-action on V, and set $\Lambda_\pm = \Lambda \cap V_\pm$. By construction, $\dim(V_-) \geq k$.

Let $v \in \Lambda$ be an element whose class modulo $2^\ell\Lambda$ is fixed by b, and write $v = v_+ + v_-$ where $v_\pm \in V_\pm$. Then $2v_- = v - b(v) \in 2^\ell\Lambda \cap V_- = 2^\ell\Lambda_-$, so $v_- \in 2^{\ell-1}\Lambda_-$ and $v_+ = v - v_- \in \Lambda \cap V_+ = \Lambda_+$. Thus $C_{\Lambda/2^\ell\Lambda}(b) = (\Lambda_+ \times 2^{\ell-1}\Lambda_-)/2^\ell\Lambda$. Set $r = \mathrm{rk}(\Lambda_-) = \dim(V_-) \geq k$; then

$$\begin{aligned}2^k \geq |A/C_A(B)| \geq |A/C_A(b)| = |\Lambda/(\Lambda_+ \times 2^{\ell-1}\Lambda_-)| &= 2^{r(\ell-1)} \cdot |\Lambda/(\Lambda_+ \times \Lambda_-)| \\ &\geq 2^{k(\ell-1)} \cdot |\Lambda/(\Lambda_+ \times \Lambda_-)|.\end{aligned}$$

In particular, $\Lambda = \Lambda_+ \times \Lambda_-$. But then b acts trivially on $\Lambda/2\Lambda$, hence on $\Omega_1(A)$, which contradicts our assumption.

Thus B does not act faithfully on $\Omega_1(A)$. Set $B_0 = C_B(\Omega_1(A)) \cong C_B(\Lambda/2\Lambda) \neq 1$. If $-\mathrm{Id}_V \in B_0$, then it inverts A, $[B, \Omega_1(A)] \leq [B, [B_0, A]] = 1$ since B acts quadratically, so $B = B_0$, and $|B_0| \geq |A/C_A(B)| \geq |A/\Omega_1(A)| = 2^{(\ell-1)n}$. If $b \in B_0$ is such that $b^2 = -\mathrm{Id}_V$, then b defines a $\mathbb{C}$-vector space structure on V, and hence does not induce the identity on $\Lambda/2\Lambda$, a contradiction.

Thus there is $b \in B_0$ which does not act on V via $\pm\mathrm{Id}$. Let $V_\pm \neq 0$ be the ± 1-eigenspaces for the b-action on V, and set $\Lambda_\pm = \Lambda \cap V_\pm$. For each $v \in \Lambda$, $v - b(v) \in 2\Lambda$ since b acts trivially on $\Omega_1(A) \cong \Lambda/2\Lambda$. Set $v = v_+ + v_-$, where $v_\pm \in V_\pm$. Then $2v_- = v - b(v) \in 2\Lambda \cap V_- = 2\Lambda_-$ implies that $v_- \in \Lambda_-$, and hence $v_+ \in \Lambda_+$. Thus $V \in \Lambda_+ \times \Lambda_-$, so by Lemma 2.8, $\mathbb{G} = C_n$. By assumption, $q \equiv 1 \pmod 8$, so $\ell \geq 3$, and $[b, [b, \Lambda/2^\ell\Lambda]] \geq 4\Lambda_-/2^\ell\Lambda_- \neq 1$, contradicting the assumption that B acts quadratically on A.

Case 2: $G \cong Sp_{2n}(q)$ for some $n \geq 1$ and some $q \equiv 5 \pmod 8$. Fix subgroups $H_i \leq G$ ($1 \leq i \leq n$) and $K < G$ such that $H_i \cong Sp_2(q)$ for each i, $K \cong \Sigma_n$ is the group of permutation matrices (in 2×2 blocks), and K normalizes $H = H_1 \times \cdots \times H_n$ and permutes the factors in the obvious way. We can also fix isomorphisms $\chi_i\colon H_i \xrightarrow{\cong} Sp_2(q)$ such that the action of K on the H_i commutes with the χ_i.

Fix subgroups $\widehat{A} < \widehat{Q} < Sp_2(q)$, where $\widehat{Q} \cong Q_8$ (a Sylow 2-subgroup), and $\widehat{A} \cong C_4$ is contained in the maximal torus. Set $Q_i = \chi_i^{-1}(\widehat{Q})$ and $A_i = \chi_i^{-1}(\widehat{A})$, and set $Q = Q_1Q_2\cdots Q_n$ and $A = A_1A_2\cdots A_n$. Thus $A = O_2(T)$ is as in Hypotheses 5.1(III): the 2-power torsion in the maximal torus of G. By [CF, § I], $S = QR$ for some $R \in \mathrm{Syl}_2(K)$. Also, $W \cong QK/A \cong C_2 \wr \Sigma_n$ acts on A via signed permutations of the coordinates.

Let B be any nontrivial best offender in W on A. Consider the action of B on the set $\{1, 2, \dots, n\}$, let $X_1, \dots, X_k$ be the set of orbits, and set $d_i = |X_i|$. For $1 \leq i \leq k$, let $A_i \leq A$ be the subgroup of elements whose coordinates vanish except for those in positions in X_i; thus $A_i \cong (C_4)^{d_i}$ and $A = A_1 \times \cdots \times A_k$. Set $B_i = B/C_B(A_i)$; then $|B| \leq \prod_{i=1}^k |B_i|$. Since B is abelian, either $|B_i| = d_i$ and B_i permutes the coordinates freely, or $|B_i| = 2d_i$ and there is a unique involution in B_i which inverts all coordinates in A_i. In the first case, $|C_{A_i}(B_i)| = 4$, and so $|B_i| \cdot |C_{A_i}(B_i)| = d_i \cdot 4 \leq 4^{d_i} = |A_i|$ with equality only if $d_i = 1$. In the second case, $|C_{A_i}(B_i)| = 2$, and again $|B_i| \cdot |C_{A_i}(B_i)| = 2d_i \cdot 2 \leq 4^{d_i} = |A_i|$ with equality only if $d_i = 1$. Since

$$\prod_{i=1}^k |A_i| = |A| \leq |B| \cdot |C_A(B)| = |B| \cdot \prod_{i=1}^k |C_{A_i}(B_i)| \leq \prod_{i=1}^k \bigl(|B_i| \cdot |C_{A_i}(B_i)|\bigr),$$

we conclude that $d_i = 1$ for all i, and hence that B acts only by changing signs in certain coordinates.

For each $1 \leq i \leq n$, let $\mathrm{pr}_i\colon Q \longrightarrow Q_i$ be the projection onto the i-th factor. If $A^* \leq S$ is abelian of order 4^n, then A^*A/A is a best offender in W on A, and hence $A^* \leq Q$ by the last paragraph. Also, $\mathrm{pr}_i(A^*)$ is cyclic of order at most 4 for each i, and since $|A^*| = 4^n$, $\mathrm{pr}_i(A^*) \cong C_4$ for each i and $A^* = \prod_{i=1}^n \mathrm{pr}_i(A^*)$. Thus there are exactly 3^n such subgroups.

Now assume $A^* \trianglelefteq S$, and set $A_i^* = \mathrm{pr}_i(A^*) \leq Q_i$ for short. Since A^* is normal, the subgroups $\chi_i(A_i^*) \leq \widehat{Q} < Sp_2(q)$ are equal for all i lying in any R-orbit of the set $\{1, 2, \dots, n\}$. Hence we can choose elements $x_1, x_2, \dots, x_n$, where $x_i \in N_{H_i}(Q_i) \cong SL_2(3)$ and ${}^{x_i}(A_i) = A_i^*$ for each i, and such that $\chi_i(x_i) \in Sp_2(q)$ is constant on each R-orbit. Set $x = x_1x_2\cdots x_n$; then ${}^xA = A^*$, and $x \in N_G(S)$.

Case 3: G is a Steinberg group. Assume $\gamma \in \Gamma_{\overline{G}}$ is a graph automorphism of order 2, and that $G = C_{\overline{G}}(\sigma)$ where $\sigma = \gamma\psi_q$. Set $G_0 = C_{\overline{G}}(\gamma, \psi_q)$; thus $G_0 \leq G$. Set $\ell = v_2(q-1) \geq 2$. We must again show that the action of W_0 on A has no nontrivial best offenders.

If $G \cong {}^2E_6(q)$ or $\mathrm{Spin}_{2n}^-(q)$ ($n \geq 4$), then $G_0 \cong F_4(q)$ or $\mathrm{Spin}_{2n-1}(q)$, respectively, and W_0 is the Weyl group of G_0. If $1 \neq B \leq W_0$ is a best offender in W_0 on A, then it is also a best offender on $\Omega_\ell(A) \leq G_0$, which is impossible by Case 1.

If $G \cong SU_{2n+1}(q) \cong {}^2A_{2n}(q)$, then $S \cong (SD_{2^{\ell+2}})^n \rtimes R$ for some $R \in \mathrm{Syl}_2(\Sigma_n)$ [CF, pp. 143–144]. Thus $A \cong (C_{2^{\ell+1}})^n$, $W_0 \cong C_2 \wr \Sigma_n$, $\Sigma_n < W_0$ acts on A by permuting the coordinates, and the subgroup $W_1 \cong (C_2)^n$ in W_0 has a basis each element of which acts on one coordinate by $(a \mapsto a^{2^\ell-1})$. If $B \le W_0$ is a nontrivial quadratic best offender on A, then it is also a best offender on $\Omega_\ell(A)$ [O4, Lemma 2.2(a)], hence is contained in W_1 by the argument in Case 2, which is impossible since no nontrivial element in this subgroup acts quadratically. Thus A is characteristic in this case.

It remains to consider the case where $G \cong SU_{2n}(q) \cong {}^2A_{2n-1}(q)$. Since the case $SU_2(q) \cong Sp_2(q)$ has already been handled, we can assume $n \ge 2$. Set $\widehat{G} = GU_{2n}(q) > G$, set $G_0 = GU_2(q) \times \cdots \times GU_2(q) \le \widehat{G}$, and set $G_1 = N_{\widehat{G}}(G_0) \cong GU_2(q) \wr \Sigma_n$. Then G_1 has odd index in $\widehat{G}$ [CF, pp. 143–144], so we can assume $S \le G_1 \cap G$. Fix $H_0 \in \mathrm{Syl}_2(G_0)$; thus $H_0 \cong (SD_{2^{\ell+2}})^n$. Since $v_2(q+1) = 1$, and since the Sylow 2-subgroups of $SU_2(q)$ are quaternion,

$$G \cap H_0 = \mathrm{Ker}\bigl[H_0 \cong (SD_{2^{\ell+2}})^n \xrightarrow{\ \chi^n\ } C_2^n \xrightarrow{\ \text{sum}\ } C_2\bigr]\,,$$

where $\chi\colon SD_{2^{\ell+2}} \longrightarrow C_2$ is the surjection with quaternion kernel. As in the last case, $W_0 \cong C_2 \wr \Sigma_n$ with normal subgroup $W_1 \cong C_2^n$. If $B \le W_0$ is a nontrivial quadratic best offender on A, then it is also a best offender on $\Omega_\ell(A)$ [O4, Lemma 2.2(a)], so $B \le W_1$ by the argument used in Case 2. Since no nontrivial element in W_1 acts quadratically on A, we conclude that A is characteristic in this case. □

The next lemma is needed to deal with the fact that not all fusion preserving automorphisms of A lie in $\mathrm{Aut}(A,\mathcal{F})$ (since they need not extend to automorphisms of S).

Lemma 5.13. *Let G be any finite group, fix $S \in \mathrm{Syl}_p(G)$, and let $S_0 \trianglelefteq S$ be a normal subgroup. Let $\varphi \in \mathrm{Aut}(G)$ be such that $\varphi(S_0) = S_0$ and $\varphi|_{S_0} \in N_{\mathrm{Aut}(S_0)}(\mathrm{Aut}_S(S_0))$. Then there is $\varphi' \in \mathrm{Aut}(G)$ such that $\varphi'|_{S_0} = \varphi|_{S_0}$, $\varphi'(S) = S$, and $\varphi' \equiv \varphi$ (mod $\mathrm{Inn}(G)$).*

Proof. Since $\varphi|_{S_0}$ normalizes $\mathrm{Aut}_S(S_0)$, and $c_{\varphi(g)} = \varphi c_g \varphi^{-1}$ for each $g \in G$, we have $\mathrm{Aut}_{\varphi(S)}(S_0) = {}^{\varphi}\mathrm{Aut}_S(S_0) = \mathrm{Aut}_S(S_0)$. Hence $\varphi(S) \le C_G(S_0)S$. Since S normalizes $C_G(S_0)$ and $S \in \mathrm{Syl}_p(C_G(S_0)S)$, we have $\varphi(S) = {}^xS$ for some $x \in C_G(S_0)$. Set $\varphi' = c_x^{-1} \circ \varphi \in \mathrm{Aut}(G)$; then $\varphi'(S) = S$ and $\varphi'|_{S_0} = \varphi|_{S_0}$. □

In the next two propositions, we will be referring to the short exact sequence

$$1 \longrightarrow \mathrm{Aut}_{\mathrm{diag}}(S,\mathcal{F}) \longrightarrow N_{\mathrm{Aut}(S,\mathcal{F})}(A) \xrightarrow{\ R\ } \mathrm{Aut}(A,\mathcal{F}) \longrightarrow 1\,. \qquad (17)$$

Here, R is induced by restriction, and $\mathrm{Aut}(A,\mathcal{F}) = \mathrm{Im}(R)$ and $\mathrm{Aut}_{\mathrm{diag}}(S,\mathcal{F}) = \mathrm{Ker}(R)$ by definition of these two groups (Notation 5.2(H)). By Proposition 5.12, in all cases, each class in $\mathrm{Out}(S,\mathcal{F})$ is representated by elements of $N_{\mathrm{Aut}(S,\mathcal{F})}(A)$.

Proposition 5.14. *Assume Hypotheses 5.1 and 5.10 and Notation 5.2. Then $\bar{\kappa}_G$ is surjective, except in the following cases:*

- $(G,p) \cong ({}^2E_6(q),3)$, *or*
- $(G,p) \cong (G_2(q),2)$ *and* $q_0 \ne 3$, *or*
- $(G,p) \cong (F_4(q),3)$ *and* $q_0 \ne 2$.

In the exceptional cases, $|\mathrm{Coker}(\bar{\kappa}_G)| \le 2$.

Proof. We first claim that for $\varphi \in \mathrm{Aut}(S,\mathcal{F})$,

$$\varphi(A) = A \quad\text{and}\quad \varphi|_A \in \mathrm{Aut}_{\mathrm{sc}}(A)\mathrm{Aut}_{\mathrm{Aut}(G)}(A) \qquad\Longrightarrow\qquad [\varphi] \in \mathrm{Im}(\bar{\kappa}_G)\,. \qquad (18)$$

To see this, fix such a φ. By Lemma 5.11(b), each element of $\mathrm{Aut}_{\mathrm{sc}}(A)$, or of $\mathrm{Aut}_{\mathrm{sc}}(A)/\langle\psi_{-1}^A\rangle$ if $p = 2$, is the restriction of an element of $\widehat{\Phi}_G$. If $p = 2$, then we are in case (III.1), the σ-setup is standard, and hence the inversion automorphism ψ_{-1}^A is the restriction of an inner automorphism of G (if $-\mathrm{Id}_V \in W$) or an element of $\mathrm{Inn}(G)\Gamma_G$. Thus $\varphi|_A$ extends to an automorphism of G.

Now, $\varphi|_A$ normalizes $\mathrm{Aut}_S(A)$ since $\varphi(S) = S$. So by Lemma 5.13, $\varphi|_A$ is the restriction of an automorphism of G which normalizes S, and hence is the restriction of an element $\psi \in \mathrm{Aut}(S, \mathcal{F})$ such that $[\psi] \in \mathrm{Im}(\bar{\kappa}_G)$. Then $\varphi\psi^{-1} \in \mathrm{Ker}(R) = \mathrm{Aut}_{\mathrm{diag}}(S, \mathcal{F})$ by the exactness of (17), and $[\varphi\psi^{-1}] \in \mathrm{Im}(\bar{\kappa}_G)$ by Lemma 5.8. So $[\varphi] \in \mathrm{Im}(\bar{\kappa}_G)$, which proves (18).

By Proposition 5.12, each class in $\mathrm{Out}(S, \mathcal{F})$ is represented by an element of $N_{\mathrm{Aut}(S,\mathcal{F})}(A)$. Hence by (18), $|\mathrm{Coker}(\bar{\kappa}_G)|$ is at most the index of $\mathrm{Aut}(A, \mathcal{F}) \cap \mathrm{Aut}_{\mathrm{sc}}(A)\mathrm{Aut}_{\mathrm{Aut}(G)}(A)$ in $\mathrm{Aut}(A, \mathcal{F})$. So by Lemma 5.7, $|\mathrm{Coker}(\bar{\kappa}_G)| \le 2$, and $\bar{\kappa}_G$ is surjective with the exceptions listed above. □

We now want to refine Proposition 5.14, and finish the proof of Theorem B, by determining $\mathrm{Ker}(\bar{\kappa}_G)$ in each case where 5.1 and 5.10 hold and checking whether it is split. In particular, we still want to show that each of these fusion systems is tamely realized by some finite group of Lie type (and not just an extension of such a group by outer automorphisms).

Since $O_{p'}(\mathrm{Outdiag}(G)) \le \mathrm{Ker}(\bar{\kappa}_G)$ in all cases by Lemma 5.8, $\bar{\kappa}_G$ induces a quotient homomorphism

$$\mathring{\kappa}_G \colon \mathrm{Out}(G)/O_{p'}(\mathrm{Outdiag}(G)) \longrightarrow \mathrm{Out}(S, \mathcal{F}),$$

and it is simpler to describe $\mathrm{Ker}(\mathring{\kappa}_G)$ than $\mathrm{Ker}(\bar{\kappa}_G)$. The projection of $\mathrm{Out}(G)$ onto the quotient $\mathrm{Out}(G)/O_{p'}(\mathrm{Outdiag}(G))$ is split: by Steinberg's theorem (Theorem 3.4), it splits back to $O_p(\mathrm{Outdiag}(G))\Phi_G\Gamma_G$. Hence $\mathring{\kappa}_G$ is split surjective if and only if $\bar{\kappa}_G$ is split surjective.

Proposition 5.15. *Assume Hypotheses 5.1 and 5.10 and Notation 5.2. Assume also that none of the following hold: neither*

- $(G, p) \cong ({}^2E_6(q), 3)$, *nor*
- $(G, p) \cong (G_2(q), 2)$ *and* $q_0 \neq 3$, *nor*
- $(G, p) \cong (F_4(q), 3)$ *and* $q_0 \neq 2$.

(a) *If* $p = 2$, *then* $\mathring{\kappa}_G$ *is an isomorphism, and* $\bar{\kappa}_G$ *is split surjective.*

(b) *Assume that* p *is odd, and that we are in the situation of case (III.1) of Hypotheses 5.1. Then* $\sqrt{q} \in \mathbb{N}$, *and*

$$\mathrm{Ker}(\mathring{\kappa}_G) = \begin{cases} \langle [\psi_{\sqrt{q}}] \rangle \cong C_2 & \text{if } \gamma = \mathrm{Id} \text{ and } -\mathrm{Id} \in W \\ \langle [\gamma_0\psi_{\sqrt{q}}] \rangle \cong C_2 & \text{if } \gamma = \mathrm{Id} \text{ and } -\mathrm{Id} \notin W \\ \langle [\psi_{\sqrt{q}}] \rangle \cong C_4 & \text{if } \gamma \neq \mathrm{Id} \ (G \text{ is a Steinberg group}) \end{cases}$$

where in the second case, $\gamma_0 \in \Gamma_G$ *is a graph automorphism of order* 2. *Hence* $\bar{\kappa}_G$ *and* $\mathring{\kappa}_G$ *are split surjective if and only if either* $\gamma = \mathrm{Id}$ *and* $-\mathrm{Id} \notin W$, *or* $p \equiv 3 \pmod 4$.

(c) *Assume that* p *is odd, and that we are in the situation of case (III.2) or (III.3) of Hypotheses 5.1. Assume also that* G *is a Chevalley group* ($\gamma \in \mathrm{Inn}(\overline{G})$), *and that* $\mathrm{ord}_p(q)$ *is even or* $-\mathrm{Id} \notin W_0$. *Let* $\Phi_G, \Gamma_G \le \mathrm{Aut}(G)$ *be as in Proposition 3.6. Then* $\Phi_G \cap \mathrm{Ker}(\mathring{\kappa}_G) = 1$, *so* $|\mathrm{Ker}(\mathring{\kappa}_G)| \le |\Gamma_G|$, *and* $\bar{\kappa}_G$ *and* $\mathring{\kappa}_G$ *are split surjective.*

(d) *Assume that* p *is odd, and that we are in the situation of case (III.3) of Hypotheses 5.1. Assume also that* G *is a Steinberg group* ($\gamma \notin \mathrm{Inn}(\overline{G})$), *and that* $\mathrm{ord}_p(q)$ *is even. Then*

$$\mathrm{Ker}(\mathring{\kappa}_G) = \begin{cases} \langle [\gamma|_G] \rangle \cong C_2 & \text{if } \gamma|_A \in \mathrm{Aut}_{W_0}(A) \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\bar{\kappa}_G$ and $\mathring{\kappa}_G$ are split surjective if and only if q is an odd power of q_0 or $\operatorname{Ker}(\bar{\kappa}_G) = O_{p'}(\operatorname{Outdiag}(G))$. *If $\bar{\kappa}_G$ is not split surjective, then its kernel contains a graph automorphism of order* 2 *in* $\operatorname{Out}(G)/\operatorname{Outdiag}(G)$.

Proof. In all cases, κ_G is surjective by Proposition 5.14 (with the three exceptions listed above).

By definition and Proposition 5.12,

$$\operatorname{Out}(S,\mathcal{F}) = \operatorname{Aut}(S,\mathcal{F})/\operatorname{Aut}_{\mathcal{F}}(S) \cong N_{\operatorname{Aut}(S,\mathcal{F})}(A)\big/N_{\operatorname{Aut}_{\mathcal{F}}(S)}(A)\,.$$

Also, $\operatorname{Out}_{\mathrm{diag}}(S,\mathcal{F})$ is the image in $\operatorname{Out}(S,\mathcal{F})$ of $\operatorname{Aut}_{\mathrm{diag}}(S,\mathcal{F})$. Since $N_{\operatorname{Aut}_{\mathcal{F}}(S)}(A)$ is the group of automorphisms of S induced by conjugation by elements in $N_G(S)\cap N_G(A)$, the short exact sequence (17) induces a quotient exact sequence

$$1 \longrightarrow \operatorname{Out}_{\mathrm{diag}}(S,\mathcal{F}) \longrightarrow \operatorname{Out}(S,\mathcal{F}) \xrightarrow{\ \bar{R}\ } \operatorname{Aut}(A,\mathcal{F})/\operatorname{Aut}_{N_G(S)}(A) \longrightarrow 1\,. \quad (19)$$

We claim that

$$\operatorname{Aut}_{N_G(S)}(A) = \operatorname{Aut}(A,\mathcal{F}) \cap \operatorname{Aut}_G(A)\,. \quad (20)$$

That $\operatorname{Aut}_{N_G(S)}(A)$ is contained in the two other groups is clear. Conversely, assume $\alpha \in \operatorname{Aut}(A,\mathcal{F})\cap\operatorname{Aut}_G(A)$. Then $\alpha = c_g|_A$ for some $g\in N_G(A)$, and $\alpha \in N_{\operatorname{Aut}(A)}(\operatorname{Aut}_S(A))$ since it is the restriction of an element of $\operatorname{Aut}(S,\mathcal{F})$. Hence g normalizes $SC_G(A)$, and since $S\in\operatorname{Syl}_p(SC_G(A))$, there is $h\in C_G(A)$ such that $hg\in N_G(S)$. Thus $\alpha = c_g|_A = c_{hg}|_A \in \operatorname{Aut}_{N_G(S)}(A)$, and this finishes the proof of (20).

By Lemma 5.8, $\bar{\kappa}_G$ sends $\operatorname{Outdiag}(G)$ onto $\operatorname{Out}_{\mathrm{diag}}(S,\mathcal{F})$ with kernel $O_{p'}(\operatorname{Outdiag}(G))$. Hence by the exactness of (19), restriction to A induces an isomorphism

$$\begin{aligned}\operatorname{Ker}(\mathring{\kappa}_G) \xrightarrow[\cong]{\ \bar{R}_0\ } &\operatorname{Ker}\big[\operatorname{Out}(G)/\operatorname{Outdiag}(G) \longrightarrow \operatorname{Aut}(A,\mathcal{F})/\operatorname{Aut}_{N_G(S)}(A)\big] \\ &= \operatorname{Ker}\big[\operatorname{Out}(G)/\operatorname{Outdiag}(G) \longrightarrow N_{\operatorname{Aut}(A)}(\operatorname{Aut}_G(A))/\operatorname{Aut}_G(A)\big]\,, \quad (21)\end{aligned}$$

where the equality holds by (20).

Recall that for each ℓ prime to p, $\psi_\ell^A \in \operatorname{Aut}_{\mathrm{sc}}(A)$ denotes the automorphism $(a\mapsto a^\ell)$.

(a,b) Under either assumption (a) or (b), we are in case (III.1) of Hypotheses 5.1. In particular, $(\bar{G},\sigma)$ is a standard σ-setup for G. Set $k = v_p(q-1)$; then $k\ge 1$, and $k\ge 2$ if $p=2$.

If p is odd, then by Hypotheses 5.10(b), the class of q_0 generates $(\mathbb{Z}/p)^\times$. Since $q = q_0^b \equiv 1 \pmod p$, this implies that $(p-1)|b$. In particular, b is even, and $\sqrt{q} = q_0^{b/2}\in\mathbb{N}$.

Since $\operatorname{Out}(G)/\operatorname{Outdiag}(G)\cong \Phi_G\Gamma_G$ by Theorem 3.4, where $\Phi_G\Gamma_G$ normalizes T and hence A, (21) takes the form

$$\operatorname{Ker}(\mathring{\kappa}_G) \cong \big\{\varphi\in\Phi_G\Gamma_G \,\big|\, \varphi|_A \in \operatorname{Aut}_{W_0}(A)\big\}\,. \quad (22)$$

In fact, when $\operatorname{Ker}(\mathring{\kappa}_G)$ has order prime to p (which is the case for all examples considered here), the isomorphism in (22) is an equality since $\operatorname{Outdiag}(G)/O_{p'}(\operatorname{Outdiag}(G))$ is a p-group.

Assume first that $G = \mathbb{G}(q)$ is a Chevalley group. Thus $\sigma = \psi_q$ where $q\equiv 1 \pmod p$, and $A = \big\{t\in\bar{T}\,\big|\, t^{p^k}=1\big\}$. By Lemma 2.7 (applied with $m = p^k\ge 3$), the group $\operatorname{Aut}_W(\bar{T})\operatorname{Aut}_{\Gamma_{\bar{G}}}(\bar{T})$ acts faithfully on A, and its action intersects $\operatorname{Aut}_{\mathrm{sc}}(A)$ only in $\langle\psi_{-1}^A\rangle$. By Lemma 5.11(b,c), restriction to A sends $\widehat{\Phi}_G$ isomorphically onto $\operatorname{Aut}_{\mathrm{sc}}(A)$ if p is odd, and with index 2 if $p = 2$. So when $\mathbb{G}$ is not one of the groups B_2, F_4, or G_2, then $\Phi_G\Gamma_G$ acts faithfully on A, and

$$\{\varphi \in \Phi_G\Gamma_G \,|\, \varphi|_A \in \mathrm{Aut}_{W_0}(A)\} = \begin{cases} 1 & \text{if } p = 2 \\ \langle\psi_{\sqrt{q}}\rangle & \text{if } p \text{ is odd and } -\mathrm{Id} \in W \\ \langle\gamma_0\psi_{\sqrt{q}}\rangle & \text{if } p \text{ is odd and } -\mathrm{Id} \notin W \end{cases}$$

where in the last case, $\gamma_0 \in \Gamma_G$ is a graph automorphism such that the coset $\gamma_0 W$ contains $-\mathrm{Id}$. (Note that $b = (p-1)p^\ell$ for some $\ell \geq 0$ by Hypotheses 5.10(b,c) and since $p|(q-1)$. Hence $\sqrt{q} \equiv -1$ modulo $p^k = \mathrm{expt}(A)$, and $\psi_{\sqrt{q}}|_A = \psi_{-1}^A$.)

Thus by (22), $\mathring{\kappa}_G$ is injective if $p = 2$, and $|\mathrm{Ker}(\mathring{\kappa}_G)| = 2$ if p is odd. When p is odd, since $\mathrm{Ker}(\mathring{\kappa}_G)$ is normal of order prime to p in $\mathrm{Out}(G)$ (hence of order prime to $|O_p(\mathrm{Outdiag}(G))|$), $\mathrm{Ker}(\mathring{\kappa}_G)$ is generated by $[\psi_{\sqrt{q}}]$ if $-\mathrm{Id} \in W$ (i.e., if there is an inner automorphism which inverts $\bar{T}$ and hence A), or by $[\gamma_0\psi_{\sqrt{q}}]$ otherwise for γ_0 as above. In the latter case, $\mathring{\kappa}_G$ is split since it sends $O_p(\mathrm{Outdiag}(G))\Phi_G O_3(\Gamma_G)$ isomorphically onto $\mathrm{Out}(S,\mathcal{F})$ (recall $\Gamma_G \cong C_2$ or Σ_3). When $\mathrm{Ker}(\mathring{\kappa}_G) = \langle[\psi_{\sqrt{q}}]\rangle$, the map is split if and only if $4 \nmid |\Phi_G| = b$, and since $b = (p-1)p^m$ for some m, this holds exactly when $p \equiv 3 \pmod 4$.

If $(G,p) \cong (B_2(q),2)$, $(F_4(q),2)$, or $(G_2(q),3)$, then since $q_0 \neq p$, $\Gamma_G = 1$. So similar arguments show that $\mathrm{Ker}(\mathring{\kappa}_G) = 1$, 1, or $\langle[\psi_{\sqrt{q}}]\rangle \cong C_2$, respectively, and that $\mathring{\kappa}_G$ is split in all cases.

Next assume $G = G_2(q)$, where $p = 2$, $q = 3^b$, and b is a power of 2. Then $b \geq 2$ since $q \equiv 1 \pmod 4$. The above argument shows that Φ_G injects into $\mathrm{Out}(S,\mathcal{F})$. Since $\mathrm{Out}(G)$ is cyclic of order $2b$, generated by a graph automorphism whose square generates Φ_G (and since $2|b$), $\mathrm{Out}(G)$ injects into $\mathrm{Out}(S,\mathcal{F})$.

If $G = F_4(q)$, where $p = 3$, $q = 2^b$, and $b = 2\cdot 3^\ell$ for some $\ell \geq 0$, then the same argument shows that Φ_G injects into $\mathrm{Out}(S,\mathcal{F})$. Since $\mathrm{Out}(G)$ is cyclic of order $2b = 4\cdot 3^\ell$, generated by a graph automorphism whose square generates Φ_G, $\mathrm{Out}(G)$ injects into $\mathrm{Out}(S,\mathcal{F})$.

It remains to handle the Steinberg groups. Let $\mathbb{H}$ be such that $C_{\bar{G}}(\gamma) = \mathbb{H}(\bar{\mathbb{F}}_{q_0})$: a simple algebraic group by [GLS3, Theorem 1.15.2(d)]. In particular, $G \geq H = \mathbb{H}(q)$. Also, W_0 is the Weyl group of $\mathbb{H}$ by [GLS3, Theorem 1.15.2(d)] (or by the proof of [St3, Theorem 8.2]). By Lemma 2.7 applied to $\mathbb{H}(\bar{\mathbb{F}}_{q_0})$, W_0 acts faithfully on $A \cap H = \Omega_k(A)$, and intersects $\mathrm{Aut}_{\mathrm{sc}}(A)$ at most in $\langle\psi_{-1}^A\rangle$.

If $p = 2$, then by Lemma 5.11(b), ψ_{-1}^A is not the restriction of an element in Φ_G. Also, $\Phi_G \cong C_{2b}$ is sent injectively into $\mathrm{Aut}_{\mathrm{sc}}(A)$ by Lemma 5.11(c), so $\mathring{\kappa}_G$ is injective by (22).

If p is odd, then $\psi_{q_0}|_A$ has order b in $\mathrm{Aut}_{\mathrm{sc}}(A)$ by Lemma 5.11(c). Since $(\psi_{q_0})^{b/2} = \psi_{\sqrt{q}}$ where $\sqrt{q} \equiv -1 \pmod p$ (recall $b|(p-1)p^\ell$ for some ℓ by Hypotheses 5.10(iii)), $\psi_{q_0}|_A$ has order $b/2$ modulo $\mathrm{Aut}_{W_0}(A)$. So by (22) and the remark afterwards, and since Φ_G is cyclic of order $2b$, $\mathrm{Ker}(\mathring{\kappa}_G) = \langle[\psi_{\sqrt{q}}]\rangle \cong C_4$. In particular, $\mathring{\kappa}_G$ is split only if $b/2$ is odd; equivalently, $p \equiv 3 \pmod 4$.

(c,d) In both of these cases, p is odd, $\mathrm{ord}_p(q)$ is even or $-\mathrm{Id} \notin W$, and we are in the situation of case (III.2) or (III.3) in Hypothesis 5.1. Then $\gamma|_G = (\psi_q|_G)^{-1}$ since $G \leq C_{\bar{G}}(\gamma\psi_q)$. Also, $\psi_{q_0}(G) = G$ by 5.1(I), and hence $\gamma(G) = G$. Since ψ_{q_0} and γ both normalize $\bar{T}$ by assumption or by construction, they also normalize $T = G \cap \bar{T}$ and $A = O_p(T)$. By Proposition 3.6(d), $[\psi_{q_0}]$ generates the image of Φ_G in $\mathrm{Out}(G)/\mathrm{Outdiag}(G)$.

We claim that in all cases,

$$\mathrm{Aut}_G(A)=\mathrm{Aut}_{W_0}(A) \qquad \text{and} \qquad \mathrm{Aut}_G(A)\cap \mathrm{Aut}_{\mathrm{sc}}(A)\le \langle \gamma|_A\rangle\,. \tag{23}$$

This holds by assumption in case (III.3), and since $\mathrm{ord}_p(q)$ is even or $-\mathrm{Id}\notin W_0$. In case (III.2), the first statement holds by Lemma 5.3(b), and the second by Lemma 2.7 (and since $W_0=W$ and A contains all p^k-torsion in $\bar{T}$).

(c) Assume in addition that G is a Chevalley group. Thus $\gamma\in\mathrm{Inn}(\bar{G})$, so $\gamma|_G\in \mathrm{Inndiag}(G)=\mathrm{Inn}(G)\mathrm{Aut}_{\bar{T}}(G)$ by Proposition 3.6(b), and hence $\gamma|_A\in\mathrm{Aut}_G(A)$. Also, $\gamma|_A=(\psi_q|_A)^{-1}=(\psi_{q_0}|_A)^{-b}$ since $\sigma=\gamma\psi_q$ centralizes $G\ge A$. Since $\psi_{q_0}|_A$ has order $b\cdot\mathrm{ord}_p(q)$ in $\mathrm{Aut}_{\mathrm{sc}}(A)$ by Lemma 5.11(c), its class in $N_{\mathrm{Aut}(A)}(\mathrm{Aut}_G(A))/\mathrm{Aut}_G(A)$ has order b by (23).

Thus by (21), $\mathring{\kappa}_G$ sends $O_p(\mathrm{Outdiag}(G))\Phi_G$ injectively into $\mathrm{Out}(S,\mathcal{F})$. Since Γ_G is isomorphic to 1, C_2, or Σ_3 (and since $\mathring{\kappa}_G$ is onto by Proposition 5.14), $\mathring{\kappa}_G$ and $\bar{\kappa}_G$ are split.

(d) Assume G is a Steinberg group and $\mathrm{ord}_p(q)$ is even. In this case, $\gamma\notin\mathrm{Inn}(\bar{G})$, and $\mathrm{Out}(G)/\mathrm{Outdiag}(G)\cong\Phi_G$ is cyclic of order $2b$, generated by the class of $\psi_{q_0}|_G$. Hence by (21), $\mathrm{Ker}(\mathring{\kappa}_G)$ is isomorphic to the subgroup of those $\psi\in\Phi_G$ such that $\psi|_A\in\mathrm{Aut}_G(A)$. By (23) and since $\psi_q|_A=\gamma^{-1}|_A$, $\mathrm{Aut}_G(A)\cap\mathrm{Aut}_{\mathrm{sc}}(A)\le\langle\psi_q^A\rangle$. Thus $|\mathrm{Ker}(\mathring{\kappa}_G)|\le 2$, and

$$|\mathrm{Ker}(\mathring{\kappa}_G)|=2 \qquad\Longleftrightarrow\qquad \gamma|_A\in\mathrm{Aut}_G(A)=\mathrm{Aut}_{W_0}(A)\,.$$

When $\mathrm{Ker}(\mathring{\kappa}_G)\ne 1$, $\mathring{\kappa}_G$ is split if and only if $4\nmid|\Phi_G|=2b$; i.e., when b is odd. □

In the situation of Proposition 5.15(c), if $-\mathrm{Id}\notin W$, then $\mathrm{Ker}(\mathring{\kappa}_G)=\big\langle[\gamma_0\psi_{\sqrt{q}}]\big\rangle$ where γ_0 is a nontrivial graph automorphism. If $-\mathrm{Id}\in W$ (hence $\mathrm{ord}_p(q)$ is even), then $\mathring{\kappa}_G$ is always injective: either because $\Gamma_G=1$, or by the explicit descriptions in the next section of the setups when $\mathrm{ord}_p(q)=2$ (Lemma 6.4), or when $\mathrm{ord}_p(q)>2$ and $\mathbb{G}=D_{2n}$ (Lemma 6.5).

The following examples help to illustrate some of the complications in the statement of Proposition 5.15.

Example 5.16. *Set $p=5$. If $G=\mathrm{Spin}^-_{4k}(3^4)$, $Sp_{2k}(3^4)$, or $SU_k(3^4)$ ($k\ge5$), then by Proposition 5.15(b), $\bar{\kappa}_G$ is surjective but not split. (These groups satisfy case (III.1) of Hypotheses 5.1 by Lemma 6.1.) The fusion systems of the last two are tamely realized by $Sp_{2\ell}(3^2)$ and $SL_n(3^2)$, respectively (these groups satisfy case (III.2) by Lemma 6.4, hence Proposition 5.15(c) applies). The fusion system of $\mathrm{Spin}^-_{4k}(3^4)$ is also realized by $\mathrm{Spin}^-_{4k}(3^2)$, but not tamely (Example 6.6(b)). It is tamely realized by $\mathrm{Spin}_{4k-1}(3^2)$ (see Propositions 1.9(c) and 5.15(c)).*

6. The cross characteristic case: II

In Section 5, we established certain conditions on a finite group G of Lie type in characteristic q_0, on a σ-setup for G, and on a prime $p \neq q_0$, and then proved that the p-fusion system of G is tame whenever those conditions hold. It remains to prove that for each G of Lie type and each p different from the characteristic, there is another group G^* whose p-fusion system is tame by the results of Section 5, and is isomorphic to that of G.

We first list the groups which satisfy case (III.1) of Hypotheses 5.1.

Lemma 6.1. *Fix a prime p and a prime power $q \equiv 1 \pmod p$, where $q \equiv 1 \pmod 4$ if $p = 2$. Assume $G \cong \mathbb{G}(q)$ for some simple group scheme $\mathbb{G}$ over $\mathbb{Z}$ of universal type, or $G \cong {}^2\mathbb{G}(q)$ for $\mathbb{G} \cong A_n$, D_n, or E_6 of universal type. Then G has a σ-setup $(\bar{G}, \sigma)$ such that Hypotheses 5.1, case (III.1) holds.*

Proof. Set $\bar{G} = \mathbb{G}(\bar{\mathbb{F}}_q)$, and let $\psi_q \in \Phi_{\bar{G}}$ be the field automorphism. Set $\sigma = \gamma\psi_q \in \mathrm{End}(\bar{G})$, where $\gamma = \mathrm{Id}$ if $G \cong \mathbb{G}(q)$, and $\gamma \in \Gamma_{\bar{G}}$ has order 2 if $G \cong {}^2\mathbb{G}(q)$.

***$N_G(T)$* contains a Sylow *p*-subgroup of *G*.** If $\gamma = \mathrm{Id}$, then by [Ca, Theorem 9.4.10] (and since G is in universal form), $|G| = q^N \prod_{i=1}^r (q^{d_i} - 1)$ for some integers $N, d_1, \ldots, d_r$ ($r = \mathrm{rk}(\mathbb{G})$), where $d_1 d_2 \cdots d_r = |W|$ by [Ca, Theorem 9.3.4]. Also, $|T| = (q-1)^r$, $N_G(T)/T \cong W$, and so

$$v_p(|G|) = \sum_{i=1}^r v_p(q^{d_i} - 1) = \sum_{i=1}^r \bigl(v_p(q-1) + v_p(d_i)\bigr) = v_p(|T|) + v_p(|W|) = v_p(N_G(T)),$$

where the second equality holds by Lemma 1.13.

If $|\gamma| = 2$, then by [Ca, §§ 14.2–3], for N and d_i as above, there are $\varepsilon_i, \eta_i \in \{\pm 1\}$ for $1 \leq i \leq r$ such that $|G| = q^N \prod_{i=1}^r (q^{d_i} - \varepsilon_i)$ and $|T| = \prod_{i=1}^r (q - \eta_i)$. (More precisely, the η_i are the eigenvalues of the γ-action on V, and polynomial generators $I_1, \ldots, I_r \in \mathbb{R}[x_1, \ldots, x_r]^W$ can be chosen such that $\deg(I_i) = d_i$ and $\tau(I_i) = \varepsilon_i I_i$.) By [Ca, Proposition 14.2.1],

$$|W_0| = \lim_{t \to 1} \prod_{i=1}^r \Bigl(\frac{1 - \varepsilon_i t^{d_i}}{1 - \eta_i t}\Bigr) \quad \Longrightarrow \quad \begin{array}{l} \bigl|\{1 \leq i \leq r \,|\, \varepsilon_i = 1\}\bigr| = \bigl|\{1 \leq i \leq r \,|\, \eta_i = 1\}\bigr| \\ \text{and} \quad |W_0| = \prod \{d_i \,|\, \varepsilon_i = +1\}. \end{array}$$

Also, $v_p(q^d + 1) = v_p(q + 1)$ for all $d \geq 1$: they are both 0 if p is odd, and both 1 if $p = 2$. Hence

$$v_p(|G|) - v_p(|T|) = \sum_{\substack{i=1 \\ \varepsilon_i = +1}}^r v_p\Bigl(\frac{q^{d_i} - 1}{q - 1}\Bigr) = \sum_{\substack{i=1 \\ \varepsilon_i = +1}}^r v_p(d_i) = v_p(|W_0|) = v_p(|N_G(T)|) - v_p(|T|)$$

by Lemma 1.13 again, and so $N_G(T)$ contains a Sylow p-subgroup of G.

The free $\langle\gamma\rangle$-orbit $\{\alpha\}$ (if $\gamma = \mathrm{Id}$) or $\{\alpha, \tau(\alpha)\}$ (if $|\gamma| = 2$ and $\alpha \neq \tau(\alpha)$), for any $\alpha \in \Sigma$, satisfies the hypotheses of this condition.

$[\boldsymbol{\gamma}, \boldsymbol{\psi_{q_0}}] = \mathbf{Id}$ since $\gamma \in \Gamma_{\bar{G}}$. □

We are now ready to describe the reduction, when $p = 2$, to groups with σ-setups satisfying Hypotheses 5.1.

Proposition 6.2. *Assume $G \in \mathfrak{Lie}(q_0)$ is of universal type for some odd prime q_0. Fix $S \in \mathrm{Syl}_2(G)$, and assume S is nonabelian. Then there is an odd prime q_0^*, a group $G^* \in \mathfrak{Lie}(q_0^*)$ of universal type, and $S^* \in \mathrm{Syl}_2(G^*)$, such that $\mathcal{F}_S(G) \cong \mathcal{F}_{S^*}(G^*)$, and G^* has a*

σ-setup which satisfies case (III.1) of Hypotheses 5.1 and also Hypotheses 5.10. Moreover, if $G^* \cong G_2(q^*)$ where q^* is a power of q_0^*, then we can arrange that either $q^* = 5$ or $q_0^* = 3$.

Proof. Since q_0 is odd, and since the Sylow 2-subgroups of ${}^2G_2(3^{2k+1})$ are abelian for all $k \geq 1$ [Ree, Theorem 8.5], G must be a Chevalley or Steinberg group. If $G \cong {}^3D_4(q)$, then $\mathcal{F}$ is also the fusion system of $G_2(q)$ by [BMO, Example 4.5]. So we can assume that $G \cong {}^\tau\mathbb{G}(q)$ for some odd prime power q, some $\mathbb{G}$, and some graph automorphism τ of order 1 or 2.

Let $\varepsilon \in \{\pm 1\}$ be such that $q \equiv \varepsilon \pmod 4$. By Lemma 1.11, there is a prime q_0^* and $k \geq 0$ such that $\overline{\langle q \rangle} = \overline{\langle \varepsilon \cdot (q_0^*)^{2^k} \rangle}$, where either $q_0^* = 5$ and $k = 0$, or $q_0^* = 3$ and $k \geq 1$.

If $\varepsilon = 1$, then set $G^* = {}^\tau\mathbb{G}((q_0^*)^{2^k})$, and fix $S^* \in \mathrm{Syl}_2(G^*)$. Then $\mathcal{F}_{S^*}(G^*) \cong \mathcal{F}_S(G)$ by Theorem 1.8(a), G^* satisfies case (III.1) of Hypotheses 5.1 by Lemma 6.1 (and since $(q_0^*)^{2^k} \equiv 1 \pmod 4$), and G^* also satisfies Hypotheses 5.10.

Now assume $\varepsilon = -1$. If $-\mathrm{Id}$ is in the Weyl group of G, then set $G^* = {}^\tau\mathbb{G}((q_0^*)^{2^k})$. If $-\mathrm{Id}$ is not in the Weyl group, then $\mathbb{G} = A_n$, D_n for n odd, or E_6, and we set $G^* = \mathbb{G}((q_0^*)^{2^k})$ if $\tau \neq \mathrm{Id}$, and $G^* = {}^2\mathbb{G}((q_0^*)^{2^k})$ if $G = \mathbb{G}(q)$. In all cases, for $S^* \in \mathrm{Syl}_p(G^*)$, $\mathcal{F}_{S^*}(G^*) \cong \mathcal{F}_S(G)$ by Theorem 1.8(c,d), G^* satisfies case (III.1) of Hypotheses 5.1 by Lemma 6.1 again, and also satisfies Hypotheses 5.10.

By construction, if $\mathbb{G} = G_2$, then either $q_0^* = 3$ or $(q_0^*)^{2^k} = 5$. □

When $G \cong G_2(5)$ and $p = 2$, G satisfies Hypotheses 5.1 and 5.10, but $\bar{\kappa}_G$ is not shown to be surjective in Proposition 5.14 (and in fact, it is not surjective). Hence this case must be handled separately.

Proposition 6.3. *Set $G = G_2(5)$ and $G^* = G_2(3)$, and fix $S \in \mathrm{Syl}_2(G)$ and $S^* \in \mathrm{Syl}_2(G^*)$. Then $\mathcal{F}_{S^*}(G^*) \cong \mathcal{F}_S(G)$ as fusion systems, and $\bar{\kappa}_{G^*} = \mu_{G^*} \circ \kappa_{G^*}$ is an isomorphism from $\mathrm{Out}(G^*) \cong C_2$ onto $\mathrm{Out}(S^*, \mathcal{F}_{S^*}(G^*))$.*

Proof. The first statement follows from Theorem 1.8(c). Also, $|\mathrm{Out}(G)| = 2$ and $|\mathrm{Out}(G^*)| = 1$ by Theorem 3.4, and since G and G^* have no field automorphisms and all diagonal automorphisms are inner (cf. [St1, 3.4]), and $G = G_2(3)$ has a nontrivial graph automorphism while $G^* = G_2(5)$ does not [St1, 3.6]. Since G satisfies Hypotheses 5.1 and 5.10, $|\mathrm{Coker}(\bar{\kappa}_G)| \leq 2$ by Proposition 5.14, so $|\mathrm{Out}(S, \mathcal{F}_S(G))| \leq 2$.

By [O6, Proposition 4.2], S^* contains a unique subgroup $Q \cong Q_8 \times_{C_2} Q_8$ of index 2. Let $x \in Z(Q) = Z(S^*)$ be the central involution. Set $\bar{G} = G_2(\bar{\mathbb{F}}_3) > G^*$. Then $C_{\bar{G}}(x)$ is connected since $\bar{G}$ is of universal type [St3, Theorem 8.1], so $C_{\bar{G}}(x) \cong SL_2(\bar{\mathbb{F}}_3) \times_{C_2} SL_2(\bar{\mathbb{F}}_3)$ by Proposition 2.5. Furthermore, any outer (graph) automorphism which centralizes x exchanges the two central factors $SL_2(\bar{\mathbb{F}}_3)$. Hence for each $\alpha \in \mathrm{Aut}(G^*) \smallsetminus \mathrm{Inn}(G^*)$ which normalizes S^*, α exchanges the two factors Q_8, and in particular, does not centralize S^*. Thus $\bar{\kappa}_{G^*}$ is injective, and hence an isomorphism since $|\mathrm{Out}(G^*)| = 2$ and $|\mathrm{Out}(S^*, \mathcal{F}_{S^*}(G^*))| = |\mathrm{Out}(S, \mathcal{F}_S(G))| \leq 2$. □

We now turn to case (III.2) of Hypotheses 5.1.

Lemma 6.4. *Fix an odd prime p, and an odd prime power q prime to p such that $q \equiv -1 \pmod p$. Let G be one of the groups $Sp_{2n}(q)$, $\mathrm{Spin}_{2n+1}(q)$, $\mathrm{Spin}_{4n}^+(q)$ ($n \geq 2$), $G_2(q)$, $F_4(q)$, $E_7(q)$, or $E_8(q)$ (i.e., $G = \mathbb{G}(q)$ for some $\mathbb{G}$ whose Weyl group contains $-\mathrm{Id}$), and assume that the Sylow p-subgroups of G are nonabelian. Then G has a σ-setup $(\bar{G}, \sigma)$ such that Hypotheses 5.1, case* (III.2)*, hold.*

Proof. Assume $q = q_0^b$ where q_0 is prime and $b \geq 1$. Set $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_{q_0})$, and let $\overline{T} < \overline{G}$ be a maximal torus. Set $r = \operatorname{rk}(\overline{T})$ and $k = v_p(q+1)$.

In all of these cases, $-\mathrm{Id} \in W$, so there is a coset $w_0 \in N_{\overline{G}}(\overline{T})/\overline{T}$ which inverts $\overline{T}$. Fix $g_0 \in N_{\overline{G}}(\overline{T})$ such that $g_0\overline{T} = w_0$ and $\psi_{q_0}(g_0) = g_0$ (Lemma 2.9). Set $\gamma = c_{g_0}$ and $\sigma = \gamma \circ \psi_q$. We identify $G = O^{q_0'}(C_{\overline{G}}(\sigma))$, $T = G \cap \overline{T}$, and $A = O_p(T)$. Since $\sigma(t) = t^{-q}$ for each $t \in \overline{T}$, $T \cong (C_{q+1})^r$ is the $(q+1)$-torsion subgroup of $\overline{T}$, and $A \cong (C_{p^k})^r$.

$N_G(T)$ contains a Sylow p-subgroup of G. In all cases, by [Ca, Theorem 9.4.10] (and since G is in universal form), $|G| = q^N \prod_{i=1}^r (q^{d_i} - 1)$, where $d_1 d_2 \cdots d_r = |W|$ by [Ca, Theorem 9.3.4]. Also, the d_i are all even in the cases considered here (see [St2, Theorem 25] or [Ca, Corollary 10.2.4 & Proposition 10.2.5]). Hence by Lemma 1.13 and since p is odd,

$$\begin{aligned} v_p(|G|) &= \sum_{i=1}^r v_p(q^{d_i} - 1) = \sum_{i=1}^r v_p\big((q^2)^{d_i/2} - 1\big) = \sum_{i=1}^r \big(v_p(q^2-1) + v_p(d_i/2)\big) \\ &= r \cdot v_p(q+1) + \sum_{i=1}^r v_p(d_i) = v_p(|T|) + v_p(|W|) = v_p(|N_G(T)|)\,. \end{aligned}$$

$[\gamma, \psi_{q_0}] = \mathrm{Id}$ since $\gamma = c_{g_0}$ and $\psi_{q_0}(g_0) = g_0$.

A free $\langle\gamma\rangle$-orbit in Σ. For each $\alpha \in \Sigma$, $\{\pm\alpha\}$ is a free $\langle\gamma\rangle$-orbit. □

We now consider case (III.3) of Hypotheses 5.1. By [GL, 10-1,2], when p is odd, each finite group of Lie type has a σ-setup for which $N_G(T)$ contains a Sylow p-subgroup of G. Here, we need to construct such setups explicitly enough to be able to check that the other conditions in Hypotheses 5.1 hold.

When p is a prime, A is a finite abelian p-group, and $\mathrm{Id} \neq \xi \in \operatorname{Aut}(A)$ has order prime to p, we say that ξ is a *reflection* in A if $[A, \xi]$ is cyclic. In this case, there is a direct product decomposition $A = [A, \xi] \times C_A(\xi)$, and we call $[A, \xi]$ the *reflection subgroup* of ξ. This terminology will be used in the proofs of the next two lemmas.

Lemma 6.5. *Fix an odd prime p, and an odd prime power q prime to p such that $q \not\equiv 1 \pmod p$. Let G be one of the classical groups $SL_n(q)$, $Sp_{2n}(q)$, $\mathrm{Spin}_{2n+1}(q)$, or $\mathrm{Spin}_{2n}^{\pm}(q)$, and assume that the Sylow p-subgroups of G are nonabelian. Then G has a σ-setup $(\overline{G}, \sigma)$ such that case (III.3) of Hypotheses 5.1 holds.*

Proof. Set $m = \operatorname{ord}_p(q)$; $m > 1$ by assumption. We follow Notation 2.2, except that we have yet to fix the σ-setup for G. Thus, for example, q_0 is the prime of which q is a power.

When defining and working with the σ-setups for the spinor groups, it is sometimes easier to work with orthogonal groups than with their 2-fold covers. For this reason, throughout the proof, we set $\mathbb{G}_c = SO_\ell$ when $\mathbb{G} = \mathrm{Spin}_\ell$, set $\overline{G}_c = SO_\ell(\overline{\mathbb{F}}_{q_0})$ when $\overline{G} = \mathrm{Spin}_\ell(\overline{\mathbb{F}}_{q_0})$, and let $\chi\colon \overline{G} \longrightarrow \overline{G}_c$ be the natural surjection. We then set $G_c = C_{\overline{G}_c}(\sigma) \cong SO_\ell^{\pm}(q)$, once σ has been chosen so that $G = C_{\overline{G}}(\sigma) \cong \mathrm{Spin}_\ell^{\pm}(q)$, and set $\overline{T}_c = \chi(\overline{T})$ and $T_c = C_{\overline{T}_c}(\sigma)$. Also, in order to prove the lemma without constantly considering these groups as a separate case, we set $\overline{G}_c = \overline{G}$, $G_c = G$, $\chi = \mathrm{Id}$, etc. when G is linear or symplectic. Thus G_c and $\overline{G}_c$ are classical groups in all cases.

Regard $\overline{G}_c$ as a subgroup of $\operatorname{Aut}(\overline{V}, \mathfrak{b})$, where $\overline{V}$ is an $\overline{\mathbb{F}}_{q_0}$-vector space of dimension n, $2n$, or $2n+1$, and $\mathfrak{b}$ is a bilinear form. Explicitly, $\mathfrak{b} = 0$ if $\mathbb{G} = SL_n$, and $\mathfrak{b}$ has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{\oplus n}$

if $\mathbb{G} = Sp_{2n}$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{\oplus n}$ if $\mathbb{G} = \mathrm{Spin}_{2n}$, or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^{\oplus n} \oplus (1)$ if $\mathbb{G} = \mathrm{Spin}_{2n+1}$. Let $\bar{T}_c$ be the group of diagonal matrices in $\bar{G}_c$, and set

$$[\lambda_1, \dots, \lambda_n] = \begin{cases} \mathrm{diag}(\lambda_1, \dots, \lambda_n) & \text{if } \mathbb{G} = SL_n \\ \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}) & \text{if } \mathbb{G} = Sp_{2n} \text{ or } \mathrm{Spin}_{2n} \\ \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}, 1) & \text{if } \mathbb{G} = \mathrm{Spin}_{2n+1}. \end{cases}$$

In this way, we identify the maximal torus $\bar{T}_c < \bar{G}_c$ with $(\bar{\mathbb{F}}_{q_0}^\times)^n$ in the symplectic and orthogonal cases, and with a subgroup of $(\bar{\mathbb{F}}_{q_0}^\times)^n$ in the linear case.

Set $W^* = W$ (the Weyl group of $\mathbb{G}$ and of $\mathbb{G}_c$), except when $\mathbb{G} = \mathrm{Spin}_{2n}$, in which case we let $W^* < \mathrm{Aut}(\bar{T}_c)$ be the group of all automorphisms which permute and invert the coordinates. Thus in this last case, $W^* \cong \{\pm 1\} \wr \Sigma_n$, while W is the group of signed permutations which invert an even number of coordinates (so $[W^* : W] = 2$). Since W^* induces a group of isometries of the root system for Spin_{2n} and contains W with index 2, it is generated by W and the restriction to $\bar{T}_c$ of a graph automorphism of order 2 (see, e.g., [Brb, § VI.1.5, Proposition 16]).

We next introduce some notation in order to identify certain elements in W^*. For each r, s such that $rs \le n$, let $\boldsymbol{\tau}_r^s \in \mathrm{Aut}(\bar{T}_c)$ be the Weyl group element induced by the permutation $(1 \cdots r)(r{+}1 \cdots 2r) \cdots ((s{-}1)r + 1 \cdots sr)$; i.e.,

$$\boldsymbol{\tau}_r^s([\lambda_1, \dots, \lambda_n]) = [\lambda_r, \lambda_1, \dots, \lambda_{r-1}, \lambda_{2r}, \lambda_{r+1}, \dots, \lambda_{sr}, \lambda_{(s-1)r+1}, \dots, \lambda_{sr-1}, \lambda_{sr+1}, \dots].$$

For $1 \le i \le n$, let $\boldsymbol{\xi}_i \in \mathrm{Aut}(\bar{T})$ be the automorphism which inverts the i-th coordinate. Set $\boldsymbol{\tau}_{r,+1}^s = \boldsymbol{\tau}_r^s$ and $\boldsymbol{\tau}_{r,-1}^s = \boldsymbol{\tau}_r^s \boldsymbol{\xi}_r \boldsymbol{\xi}_{2r} \cdots \boldsymbol{\xi}_{sr}$. Thus for $\theta = \pm 1$,

$$\boldsymbol{\tau}_{r,\theta}^s([\lambda_1, \dots, \lambda_n]) = [\lambda_r^\theta, \lambda_1, \dots, \lambda_{r-1}, \lambda_{2r}^\theta, \lambda_{r+1}, \dots \lambda_{sr}^\theta, \lambda_{(s-1)r+1}, \dots, \lambda_{sr-1}, \lambda_{sr+1}, \dots].$$

Recall that $m = \mathrm{ord}_p(q)$. Define parameters μ, θ, k, and κ as follows:

if m is odd :	$\mu = m$	$\theta = 1$	$k = [n/m]$	$\kappa = [n/\mu] = [n/m]$
if m is even :	$\mu = m/2$	$\theta = -1$		$\kappa = [n/\mu] = [2n/m]$.

We can now define our σ-setups for G and G_c. Recall that we assume $m > 1$. In Table 6.1, we define an element $w_0 \in W^*$, and then describe $T_c = C_{\bar{T}_c}(w_0 \circ \psi_q)$ and $W_0^* = C_{W^*}(w_0)$ (where $W_0 = C_W(w_0)$ has index at most 2 in W_0^*). In all cases, we choose $\gamma \in \mathrm{Aut}(\bar{G}_c)$ as follows. Write $w_0 = w_0' \circ \gamma_0|_{\bar{T}_c}$ for some $w_0' \in W$ and $\gamma_0 \in \Gamma_{\bar{G}_c}$ (possibly $\gamma_0 = \mathrm{Id}$). Choose $g_0 \in N_{\bar{G}_c}(\bar{T}_c)$ such that $g_0 \bar{T}_c = w_0'$ and $\psi_{q_0}(g_0) = g_0$ (Lemma 2.9), and set $\gamma = c_{g_0} \circ \gamma_0$. Then $[\gamma, \psi_{q_0}] = \mathrm{Id}_{\bar{G}_c}$, since c_{g_0} and γ_0 both commute with ψ_{q_0}, and we set $\sigma = \gamma \circ \psi_q = \psi_q \circ \gamma$. When $\mathbb{G} = \mathrm{Spin}_{2n}$ or Spin_{2n+1}, since $\bar{G}$ is a perfect group and $\mathrm{Ker}(\chi) \le Z(\bar{G})$, γ and σ lift to unique endomorphisms of $\bar{G}$ which we also denote γ and σ (and still $[\gamma, \psi_{q_0}] = 1$ in $\mathrm{Aut}(\bar{G})$).

Thus $G \cong C_{\bar{G}}(\sigma)$ and $G_c \cong C_{\bar{G}_c}(\sigma)$ in all cases, and we identify these groups. Set $T = C_{\bar{T}}(\sigma)$, $T_c = C_{\bar{T}_c}(\sigma)$, $W_0^* = C_{W^*}(\gamma)$, and $W_0 = C_W(\gamma)$. If $\mathbb{G} = \mathrm{Spin}_{2n+1}$ or Spin_{2n}, then $\chi(T)$ is the kernel of the homomorphism $T_c \to \mathrm{Ker}(\chi)$ which sends $\chi(t)$ to $t^{-1}\sigma(t)$, and thus has index at most 2 in T_c. Since p is odd, this proves the statement in the last line of Table 6.1.

In the description of W_0^* in Table 6.1, H always denotes a direct factor of order prime to p. The first factor in the description of W_0^* acts on the first factor in that of T, and H acts on the other factors.

<table>
<tr><th>G_c</th><th>conditions</th><th>$w_0 = \gamma|_{\bar{T}_c}$</th><th>$T_c$</th><th>$W_0^*$</th></tr>
<tr><td>$SL_n(q)$</td><td></td><td>$\boldsymbol{\tau}_m^k$</td><td>$(C_{q^m-1})^k \times C_{q-1}^{n-mk-1}$</td><td>$(C_m \wr \Sigma_k) \times H$</td></tr>
<tr><td>$Sp_{2n}(q)$</td><td rowspan="2"></td><td rowspan="3">$\boldsymbol{\tau}_{\mu,\theta}^{\kappa}$</td><td rowspan="3">$(C_{q^\mu-\theta})^\kappa \times C_{q-1}^{n-\kappa\mu}$</td><td rowspan="3">$(C_{2\mu} \wr \Sigma_\kappa) \times H$</td></tr>
<tr><td>$SO_{2n+1}(q)$</td></tr>
<tr><td rowspan="4">$SO_{2n}^{\varepsilon}(q)$</td><td>$\varepsilon = \theta^\kappa$</td></tr>
<tr><td>$\varepsilon \neq \theta^\kappa,\ \mu\nmid n$</td><td>$\boldsymbol{\tau}_{\mu,\theta}^{\kappa}\,\boldsymbol{\xi}_n$</td><td>$(C_{q^\mu-\theta})^\kappa \times C_{q-1}^{n-\kappa\mu-1} \times C_{q+1}$</td><td>$(C_{2\mu} \wr \Sigma_\kappa) \times H$</td></tr>
<tr><td>$\varepsilon \neq \theta^\kappa,\ \mu|n$
$\theta = -1$</td><td>$\boldsymbol{\tau}_{\mu,\theta}^{\kappa-1}$</td><td>$(C_{q^\mu-\theta})^{\kappa-1} \times C_{q-1}^{\mu}$</td><td rowspan="2">$(C_{2\mu} \wr \Sigma_{\kappa-1}) \times H$</td></tr>
<tr><td>$\varepsilon \neq \theta^\kappa,\ \mu|n$
$\theta = +1$</td><td>$\boldsymbol{\tau}_{\mu,\theta}^{\kappa-1}\,\boldsymbol{\xi}_n$</td><td>$(C_{q^\mu-\theta})^{\kappa-1} \times C_{q-1}^{\mu-1} \times C_{q+1}$</td></tr>
<tr><td colspan="5">In all cases, $T \xrightarrow{\chi} T_c$ has kernel and cokernel of order ≤ 2, and so $A = O_p(T) \cong O_p(T_c)$.</td></tr>
</table>

TABLE 6.1.

When $G_c = SL_n(q)$ and $m|n$, the second factor C_{q-1}^{-1} in the description of T doesn't make sense. It should be interpreted to mean that T is "a subgroup of index $q-1$ in the first factor $(C_{q^m-1})^k$".

Recall that $T_c = C_{\bar{T}_c}(\gamma \circ \psi_q)$. When $U = (\bar{\mathbb{F}}_{q_0}^\times)^\mu$, then

$$C_U(\boldsymbol{\tau}_{\mu,\theta}^1 \circ \psi_q) = \bigl\{(\lambda, \lambda^q, \lambda^{q^2}, \ldots, \lambda^{q^{\mu-1}}) \bigm| (\lambda^{q^{\mu-1}})^{q\theta} = \lambda\bigr\}$$
$$= \bigl\{(\lambda, \lambda^q, \lambda^{q^2}, \ldots, \lambda^{q^{\mu-1}}) \bigm| \lambda^{q^\mu-\theta} = 1\bigr\} \cong C_{q^\mu-\theta}\,.$$

This explains the description of T_c in the symplectic and orthogonal cases: it is always the direct product of $(C_{q^\mu-\theta})^\kappa$ or $(C_{q^\mu-\theta})^{\kappa-1}$ with a group of order prime to p. (Note that $p|(q+1)$ only when $m=2$; i.e., when $\theta=-1$ and $1=\mu|n$.)

Since the cyclic permutation $(1\,2\,\cdots\,\mu)$ generates its own centralizer in Σ_μ, the centralizer of $\boldsymbol{\tau}_{\mu,\theta}^1$ in $\{\pm1\} \wr \Sigma_\mu < \mathrm{Aut}((\bar{\mathbb{F}}_{q_0}^\times)^\mu)$ is generated by $\boldsymbol{\tau}_{\mu,\theta}^1$ and $\psi_{-1}^{\bar{T}}$. If $\theta=-1$, then $(\boldsymbol{\tau}_{\mu,\theta}^1)^\mu = \psi_{-1}^{\bar{T}}$, while if $\theta=1$, then $\boldsymbol{\tau}_{\mu,\theta}^1$ has order μ. Since $m=\mu$ is odd in the latter case, the centralizer is cyclic of order 2μ in both cases. This is why, in the symplectic and orthogonal cases, the first factor in W_0^* is always a wreath product of $C_{2\mu}$ with a symmetric group.

We are now ready to check the conditions in case (III.3) of Hypotheses 5.1.

$N_G(T)$ contains a Sylow p-subgroup of G. Set

$$e = v_p(q^m-1) = v_p(q^\mu-\theta)\,.$$

The second equality holds since if $2|m$, then $p\nmid(q^\mu-1)$ and hence $e = v_p(q^\mu+1)$. Recall also that $m|(p-1)$, so $v_p(m)=0$. Consider the information listed in Table 6.2, where the formulas for $v_p(|T|) = v_p(|T_c|)$ and $v_p(|W_0|)$ follow from Table 6.1, and those for $|G|$ are shown in [St2, Theorems 25 & 35] and also in [Ca, Corollary 10.2.4, Proposition 10.2.5 & Theorem 14.3.2].

For all $i>0$, we have

$$v_p(q^i-1) = \begin{cases} e + v_p(i/m) & \text{if } m|i \\ 0 & \text{if } m\nmid i. \end{cases}$$

<table>
<tr><th>G</th><th>cond.</th><th>$v_p(|G|)$</th><th>$v_p(|T|)$</th><th>$v_p(|W_0|)$</th></tr>
<tr><td>$SL_n(q)$</td><td></td><td>$\sum_{i=2}^{n} v_p(q^i-1)$</td><td>$ke$</td><td>$v_p(k!)$</td></tr>
<tr><td>$Sp_{2n}(q)$</td><td rowspan="2"></td><td rowspan="2">$\sum_{i=1}^{n} v_p(q^{2i}-1)$</td><td rowspan="4">$\kappa e$</td><td rowspan="4">$v_p(\kappa!)$</td></tr>
<tr><td>$\mathrm{Spin}_{2n+1}(q)$</td></tr>
<tr><td rowspan="3">$\mathrm{Spin}^{\varepsilon}_{2n}(q)$</td><td>$\varepsilon=\theta^\kappa$</td><td rowspan="3">$v_p(q^n-\varepsilon)$
$+\sum_{i=1}^{n-1} v_p(q^{2i}-1)$</td></tr>
<tr><td>$\varepsilon\neq\theta^\kappa,\ \mu\nmid n$</td></tr>
<tr><td>$\varepsilon\neq\theta^\kappa,\ \mu|n$</td><td>$(\kappa-1)e$</td><td>$v_p((\kappa-1)!)$</td></tr>
</table>

TABLE 6.2.

The first case follows from Lemma 1.13, and the second case since $m=\mathrm{ord}_p(q)$. Using this, we check that $v_p(q^{2i}-1)=v_p(q^i-1)$ for all i whenever m is odd, and that

$$v_p(q^n-\varepsilon)=\begin{cases} e+v_p(2n/m) & \text{if } m|2n \text{ and } \varepsilon=(-1)^{2n/m} \\ 0 & \text{otherwise.}\end{cases}$$

So in all cases, $v_p(|G|)=v_p(|T|)+v_p(|W_0|)$ by the above relations and the formulas in Table 6.2. Since $N_G(T)/T\cong W_0$ by Lemma 5.3(b), this proves that $v_p(|G|)=v_p(|N_G(T)|)$, and hence that $N_G(T)$ contains a Sylow p-subgroup of G.

$\boldsymbol{|\gamma|_{\bar{T}}|=|\tau|=\mathrm{ord}_p(q)\geq 2}$ **and** $\boldsymbol{[\gamma,\psi_{q_0}]=\mathrm{Id}}$ by construction. Note, when G is a spinor group, that these relations hold in $\bar{G}$ if and only if they hold in $\bar{G}_c$.

$\boldsymbol{C_S(\Omega_1(A))=A}$ by Table 6.1 and since $p\nmid|H|$.

$\boldsymbol{C_A(O_{p'}(W_0))=1.}$ By Table 6.1, in all cases, there are $r,t\geq 1$ and $1\neq s|(p-1)$ such that $A\cong(C_{p^t})^r$, and $\mathrm{Aut}_{W_0^*}(A)\cong C_s\wr\Sigma_r$ acts on A by acting on and permuting the cyclic factors. In particular, $\mathrm{Aut}_{O_{p'}(W_0)}(A)$ contains a subgroup of index at most 2 in $(C_s)^r$, this subgroup acts nontrivially on each of the cyclic factors in A, and hence $C_A(O_{p'}(W_0))=1$.

A free $\boldsymbol{\langle\gamma\rangle}$-orbit in $\boldsymbol{\Sigma}$. This can be defined as described in Table 6.3. In each case, we use the notation of Bourbaki [Brb, pp. 250–258] for the roots of $\mathbb{G}$. Thus, for example, the roots of SL_n are the $\pm(\varepsilon_i-\varepsilon_j)$ for $1\leq i<j\leq n$, and the roots of SO_{2n} the $\pm\varepsilon_i\pm\varepsilon_j$. Note that since S is assumed nonabelian, $p\big||W_0|$, and hence $n\geq pm$ in the linear case, and $n\geq p\mu$ in the other cases.

<table>
<tr><th>G</th><th>$\theta=1$</th><th>$\theta=-1$</th></tr>
<tr><td>$SL_n(q)$</td><td colspan="2">$\{\varepsilon_i-\varepsilon_{m+i}\,|\,1\leq i\leq m\}$</td></tr>
<tr><td>$Sp_{2n}(q)$</td><td>$\{2\varepsilon_i\,|\,1\leq i\leq\mu\}$</td><td>$\{\pm 2\varepsilon_i\,|\,1\leq i\leq\mu\}$</td></tr>
<tr><td>$\mathrm{Spin}_{2n+1}(q)$</td><td>$\{\varepsilon_i\,|\,1\leq i\leq\mu\}$</td><td>$\{\pm\varepsilon_i\,|\,1\leq i\leq\mu\}$</td></tr>
<tr><td>$\mathrm{Spin}^{\varepsilon}_{2n}(q)$</td><td>$\{\varepsilon_i-\varepsilon_{\mu+i}\,|\,1\leq i\leq\mu\}$</td><td>$\{\pm(\varepsilon_i-\varepsilon_{\mu+i})\,|\,1\leq i\leq\mu\}$</td></tr>
</table>

TABLE 6.3.

$$\mathbf{Aut}_{W_0}(A)\cap\mathbf{Aut}_{\mathrm{sc}}(A)\le\begin{cases}\langle\gamma|_A\rangle & \textbf{if } \mathbf{ord}_p(q) \textbf{ even or } -\mathbf{Id}\notin W_0\\ \langle\gamma|_A,\psi_{-1}^A\rangle & \textbf{otherwise.}\end{cases}$$

Set $K^*=\mathrm{Aut}_{W_0^*}(A)\cap\mathrm{Aut}_{\mathrm{sc}}(A)$ and $K=\mathrm{Aut}_{W_0}(A)\cap\mathrm{Aut}_{\mathrm{sc}}(A)$ for short. By Table 6.1, $|K^*|=m$ if $G\cong SL_n(q)$, and $|K^*|=2\mu$ otherwise. Also, $\langle\gamma|_A\rangle=\langle\psi_q^{-1}|_A\rangle$ has order $\mathrm{ord}_p(q)$. Thus $K\le K^*=\langle\gamma|_A\rangle$ except when G is symplectic or orthogonal and $m=\mathrm{ord}_p(q)$ is odd. In this last case, $K=K^*$ (so $|K|=2\mu=2m$) if W_0 contains an element which inverts A (hence which inverts T and $\bar{T}$); and $|K^*/K|=2$ ($|K|=m$) otherwise.

$\mathbf{Aut}_{G}(A)=\mathbf{Aut}_{W_0}(A)$. Since $A=O_p(T)\cong O_p(T_c)$ by Table 6.1, it suffices to prove this for G_c. Fix $g\in N_{G_c}(A)$. Since ${}^g\bar{T}_c$ is a maximal torus in the algebraic group $C_{\bar{G}_c}(A)$ (Proposition 2.5), there is $b\in C_{\bar{G}_c}(A)$ such that ${}^b\bar{T}_c={}^g\bar{T}_c$. Set $a=b^{-1}g\in N_{\bar{G}_c}(\bar{T}_c)$; thus $c_a=c_g\in\mathrm{Aut}(A)$. Set $w=a\bar{T}_c\in W=N_{\bar{G}_c}(\bar{T}_c)/\bar{T}_c$; thus $w\in N_W(A)$, and $w|_A=c_g|_A$.

By the descriptions in Table 6.1, we can factor $\bar{T}_c=\bar{T}_1\times\bar{T}_2$, where γ and each element of $N_W(A)$ send each factor to itself, $\gamma|_{\bar{T}_2}=\mathrm{Id}$, $A\le\bar{T}_1$, and $[C_W(A),\bar{T}_1]=1$. Since $\sigma(g)=g$, $\sigma(a)\equiv a \pmod{C_{\bar{G}_c}(A)}$, and so $\tau(w)\equiv w \pmod{C_W(A)}$. Thus $\tau(w)|_{\bar{T}_1}=w|_{\bar{T}_1}$ since $C_W(A)$ acts trivially on this factor, $\tau(w)|_{\bar{T}_2}=w|_{\bar{T}_2}$ since $\gamma|_{\bar{T}_2}=\mathrm{Id}$, and so $w\in W_0=C_W(\tau)$.

$N_{\mathbf{Aut}(A)}(\mathbf{Aut}_{W_0}(A))\le\mathbf{Aut}_{\mathrm{sc}}(A)\mathbf{Aut}_{\mathbf{Aut}(G)}(A)$. By Table 6.1, for some $r,t\ge1$, $A=A_1\times\cdots\times A_r$, where $A_i\cong C_{p^t}$ for each i. Also, for some $1\ne s|(p-1)$, $\mathrm{Aut}_{W_0^*}(A)\cong C_s\wr\Sigma_r$ acts on A via faithful actions of C_s on each A_i and permutations of the A_i.

Let $\mathrm{Aut}^0_{W_0^*}(A)\trianglelefteq\mathrm{Aut}_{W_0^*}(A)$ and $\mathrm{Aut}^0_{W_0}(A)\trianglelefteq\mathrm{Aut}_{W_0}(A)$ be the subgroups of elements which normalize each cyclic subgroup A_i. Thus $\mathrm{Aut}^0_{W_0^*}(A)\cong(C_s)^r$, and contains $\mathrm{Aut}^0_{W_0}(A)$ with index at most 2.

Case 1: Assume first that $\mathrm{Aut}^0_{W_0}(A)$ is characteristic in $\mathrm{Aut}_{W_0}(A)$. Fix some $\alpha\in N_{\mathrm{Aut}(A)}(\mathrm{Aut}_{W_0}(A))$. We first show that $\alpha\in\mathrm{Aut}_{W_0^*}(A)\mathrm{Aut}_{\mathrm{sc}}(A)$.

Since α normalizes $\mathrm{Aut}_{W_0}(A)$, it also normalizes $\mathrm{Aut}^0_{W_0}(A)$. For each $\beta\in\mathrm{Aut}^0_{W_0}(A)$, $[\beta,A]$ is a product of A_i's. Hence the factors A_i are characterized as the minimal nontrivial intersections of such $[\beta,A]$, and are permuted by α. So after composing with an appropriate element of $\mathrm{Aut}_{W_0^*}(A)$, we can assume that $\alpha(A_i)=A_i$ for each i.

After composing α by an element of $\mathrm{Aut}_{\mathrm{sc}}(A)$, we can assume that $\alpha|_{A_1}=\mathrm{Id}$. Fix $i\ne1$ ($2\le i\le r$), let $u\in\mathbb{Z}$ be such that $\alpha|_{A_i}=\psi_u^{A_i}=(a\mapsto a^u)$, and choose $w\in\mathrm{Aut}_{W_0}(A)$ such that $w(A_1)=A_i$. Then $w^{-1}\alpha w\alpha^{-1}\in\mathrm{Aut}_{W_0}(A)$ since α normalizes $\mathrm{Aut}_{W_0}(A)$, and $\left(w^{-1}\alpha w\alpha^{-1}\right)\big|_{A_1}=\psi_u^{A_1}$. Hence $u^s\equiv1 \pmod{p^t=|A_1|}$, and since this holds for each i, $\alpha\in\mathrm{Aut}_{W_0^*}(A)$.

Thus $N_{\mathrm{Aut}(A)}(\mathrm{Aut}_{W_0}(A))\le\mathrm{Aut}_{W_0^*}(A)\mathrm{Aut}_{\mathrm{sc}}(A)$. By Table 6.1, each element of $\mathrm{Aut}_{W_0^*}(A)$ extends to some $\varphi\in\mathrm{Aut}_{W^*}(\bar{T})$ which commutes with $\sigma|_{\bar{T}}$. So $\mathrm{Aut}_{W_0^*}(A)\le\mathrm{Aut}_{\mathrm{Aut}(G)}(A)$ by Lemma 3.7, and this finishes the proof of the claim.

Case 2: Now assume that $\mathrm{Aut}^0_{W_0}(A)$ is not characteristic in $\mathrm{Aut}_{W_0}(A)$. Then $r\le4$, and since $p\le r$, we have $p=3$ and $r=3,4$. Also, $s=2$ since $s|(p-1)$ and $s\ne1$. Thus $r=4$, since $\mathrm{Aut}^0_{W_0}(A)=O_2(\mathrm{Aut}_{W_0}(A))$ if $r=3$. Thus $\mathrm{Aut}_{W_0}(A)\cong C_2^3\rtimes\Sigma_4$: the Weyl group of D_4. Also, $m=2$ since $p=3$, so (in the notation used in the tables) $\mu=1$, $\theta=-1$, and $\kappa=n$. By Table 6.1, $G\cong SO_8(q)$ for some $q\equiv2 \pmod 3$ (and $W_0=W$).

Now, $O_2(W)\cong Q_8\times_{C_2}Q_8$, and so $\mathrm{Out}(O_2(W))\cong\Sigma_3\wr C_2$. Under the action of $W/O_2(W)\cong\Sigma_3$, the elements of order 3 act on both central factors and those of order 2 exchange the factors. (This is seen by computing their centralizers in $O_2(W)$.) It follows that

$N_{\mathrm{Out}(O_2(W))}(\mathrm{Out}_W(O_2(W)))/\mathrm{Out}_W(O_2(W)) \cong \Sigma_3 \cong \Gamma_G$, and all classes in this quotient extend to graph automorphisms of $G \cong \mathrm{Spin}_8(q)$. So for each $\alpha \in N_{\mathrm{Aut}(A)}(\mathrm{Aut}_W(A))$, after composing with a graph automorphism of G we can arrange that α commutes with $O_2(W)$, and in particular, normalizes $\mathrm{Aut}^0_W(A)$. Hence by the same argument as used in Case 1, $\alpha \in \mathrm{Aut}_{\mathrm{sc}}(A)\mathrm{Aut}_{\mathrm{Aut}(G)}(A)$.

This finishes the proof that this σ-setup for G satisfies case (III.3) of Hypotheses 5.1. □

Example 6.6. *Fix distinct odd primes p and q_0, and a prime power $q = q_0^b$ where b is even and $\mathrm{ord}_p q$ is even. Set $G = \mathrm{Spin}^-_{4k}(q)$ for some $k \geq 2$. Let $(\bar{G}, \sigma)$ be the setup for G of Lemma 6.5, where $\sigma = \psi_q \gamma$ for $\gamma \in \mathrm{Aut}(\bar{G})$. In the notation of Table 6.1, $m = \mathrm{ord}_p(q)$, $\mu = m/2$, $\theta = -1 = \varepsilon$, $n = 2k$, and $\kappa = [2k/\mu] = [4k/m]$. There are three cases to consider:*

(a) *If $q^{2k} \equiv -1$ (mod p); equivalently, if $m|4k$ and $\kappa = 4k/m$ is odd, then $\varepsilon = \theta^\kappa$, $w_0 = \gamma|_{\bar{T}_c} = \boldsymbol{\tau}^\kappa_{\mu,\theta}$, $\mathrm{rk}(A) = \kappa$, and $W_0^* \cong C_m \wr \Sigma_\kappa$. Then W_0^* acts faithfully on A while $w_0 \in W_0^* \smallsetminus W_0$, and so $\gamma|_A \notin \mathrm{Aut}_{W_0}(A)$. Hence by Proposition 5.15(d), $\bar{\kappa}_G$ is split.*

(b) *If $q^{2k} \equiv 1$ (mod p); equivalently, if $m|4k$ and $\kappa = 4k/m$ is even, then $\varepsilon \neq \theta^\kappa$, $\gamma|_{\bar{T}_c} = \boldsymbol{\tau}^{\kappa-1}_{\mu,\theta}$, $\mathrm{rk}(A) = \kappa - 1$, and $W_0^* \cong (C_m \wr \Sigma_{\kappa-1}) \times H$ where $H \cong (C_2 \wr \Sigma_\mu)$. Then H acts trivially on A and contains elements in $W_0^* \smallsetminus W_0$, so $\gamma|_A \in \mathrm{Aut}_{W_0}(A)$. Hence $\bar{\kappa}_G$ is not split.*

(c) *If $q^{2k} \not\equiv \pm 1$ (mod p); equivalently, if $m \nmid 4k$, then in either case (κ even or odd), the factor H in the last column of Table 6.1 is nontrivial, acts trivially on A, and contains elements in $W_0^* \smallsetminus W_0$. Hence $\gamma|_A \in \mathrm{Aut}_{W_0}(A)$ in this case, and $\bar{\kappa}_G$ is not split.*

We also need the following lemma, which handles the only case of a Chevalley group of exceptional type which we must show satisfies case (III.3) of Hypotheses 5.1.

Lemma 6.7. *Set $p = 5$, let q be an odd prime power such that $q \equiv \pm 2$ (mod 5), and set $G = E_8(q)$. Then G has a σ-setup which satisfies Hypotheses 5.1 (case (III.3)).*

Proof. We use the notation in 2.2, where q is a power of the odd prime q_0, and $\bar{G} = E_8(\bar{\mathbb{F}}_{q_0})$.

By [Brb, Planche VII], the of roots of E_8 can be given the following form, where $\{\varepsilon_1, \dots, \varepsilon_8\}$ denotes the standard orthonormal basis of $\mathbb{R}^8$:

$$\Sigma = \Big\{ \pm\varepsilon_i \pm \varepsilon_j \,\Big|\, 1 \leq i < j \leq 8 \Big\} \cup \Big\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{m_i} \varepsilon_i \,\Big|\, \sum_{i=1}^{8} m_i \text{ even} \Big\} \subseteq \mathbb{R}^8 .$$

By the same reference, the Weyl group W is the group of all automorphisms of $\mathbb{R}^8$ which permute Σ ($A(R) = W(R)$ in the notation of [Brb]). Give $\mathbb{R}^8$ a complex structure by setting $i\varepsilon_{2k-1} = \varepsilon_{2k}$ and $i\varepsilon_{2k} = -\varepsilon_{2k-1}$, and set $\varepsilon_k^* = \varepsilon_{2k-1}$ for $1 \leq k \leq 4$. Multiplication by i permutes Σ, and hence is the action of an element $w_0 \in W$. Upon writing the elements of Σ with complex coordinates, we get the following equivalent subset $\Sigma^* \subseteq \mathbb{C}^4$:

$$\Sigma^* = \Big\{ (\pm 1 \pm i)\varepsilon_k^* \,\Big|\, 1 \leq k \leq 4 \Big\} \cup \Big\{ i^m \varepsilon_k^* + i^n \varepsilon_\ell^* \,\Big|\, 1 \leq k < \ell \leq 4,\ m, n \in \mathbb{Z} \Big\} \cup \Big\{ \frac{1+i}{2} \sum_{k=1}^{4} i^{m_k} \varepsilon_k^* \,\Big|\, \sum m_k \text{ even} \Big\} .$$

Let $\mathbb{Z}\Sigma \subseteq \mathbb{R}^8$ be the lattice generated by Σ. By Lemma 2.4(d) (and since $(\alpha, \alpha) = 2$ for all $\alpha \in \Sigma$), we can identify $\bar{T} \cong \mathbb{Z}\Sigma \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_{q_0}^\times$ by sending $h_\alpha(\lambda)$ to $\alpha \otimes \lambda$ for $\alpha \in \Sigma$ and $\lambda \in \bar{\mathbb{F}}_{q_0}^\times$.

Set $\Lambda_0 = \mathbb{Z}\Sigma \cap \mathbb{Z}^8$, a lattice in $\mathbb{R}^8$ of index 2 in $\mathbb{Z}\Sigma$ and in $\mathbb{Z}^8$. The inclusions of lattices induce homomorphisms

$$\bar{T} \cong \mathbb{Z}\Sigma \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_{q_0}^\times \xleftarrow{\ \chi_1\ } \Lambda_0 \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_{q_0}^\times \xrightarrow{\ \chi_2\ } \mathbb{Z}^8 \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_{q_0}^\times \cong (\bar{\mathbb{F}}_{q_0}^\times)^8$$

each of which is surjective with kernel of order 2 (since $\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/2, \bar{\mathbb{F}}_{q_0}^\times) \cong \mathbb{Z}/2$). We can thus identify $\bar{T} = (\bar{\mathbb{F}}_{q_0}^\times)^8$, modulo 2-power torsion, in a way so that

$$\alpha = \sum_{i=1}^{8} k_i\varepsilon_i \in \Sigma,\ \lambda \in \bar{\mathbb{F}}_{q_0}^\times \qquad \Longrightarrow \qquad h_\alpha(\lambda) = (\lambda^{k_1}, \ldots, \lambda^{k_8}).$$

Under this identification, by the formula in Lemma 2.4(c),

$$\beta = \sum_{i=1}^{8} \ell_i\varepsilon_i \in \Sigma \qquad \Longrightarrow \qquad \theta_\beta(\lambda_1, \ldots, \lambda_8) = \lambda_1^{\ell_1} \cdots \lambda_8^{\ell_8} \tag{1}$$

for $\lambda_1, \ldots, \lambda_8 \in \bar{\mathbb{F}}_{q_0}^\times$. Also,

$$w_0(\lambda_1, \ldots, \lambda_8) = (\lambda_2^{-1}, \lambda_1, \lambda_4^{-1}, \lambda_3, \ldots, \lambda_8^{-1}, \lambda_7)$$

for each $(\lambda_1, \ldots, \lambda_8)$.

Choose $g_0 \in N_{\bar{G}}(\bar{T})$ such that $g_0\bar{T} = w_0$ and $\psi_{q_0}(g_0) = g_0$ (Lemma 2.9), and set $\gamma = c_{g_0} \in \mathrm{Inn}(\bar{G})$. Thus $\sigma = \psi_q \circ \gamma = \gamma \circ \psi_q$, $G = C_{\bar{G}}(\sigma)$, and $T = C_{\bar{T}}(\sigma)$. By the Lang-Steinberg theorem [St3, Theorem 10.1], there is $h \in \bar{G}$ such that $g = h\psi_q(h^{-1})$; then $\sigma = c_h\psi_q c_h^{-1}$ and $G \cong C_{\bar{G}}(\psi_q) = E_8(q)$. It remains to check that the setup $(\bar{G}, \sigma)$ satisfies the list of conditions in Hypotheses 5.1.

We identify $W_0 = C_W(w_0)$ with the group of $\mathbb{C}$-linear automorphisms of $\mathbb{C}^4$ which permute Σ^*. The order of W_0 is computed in [Ca3, Table 11] (the entry $\Gamma = D_4(a_1)^2$), but since we need to know more about its structure, we describe it more precisely here. Let $W_2 \le GL_4(\mathbb{C})$ be the group of monomial matrices with nonzero entries ± 1 or $\pm i$, and with determinant ± 1. Then $W_2 \le W_0$, $|W_2| = \frac{1}{2} \cdot 4^4 \cdot 4! = 2^{10} \cdot 3$, and W_2 acts on Σ^* with three orbits corresponding to the three subsets in the above description of Σ^*. The (complex) reflection of order 2 in the hyperplane orthogonal to $\frac{1+i}{2}(\varepsilon_1^* + \varepsilon_2^* + \varepsilon_3^* + \varepsilon_4^*)$ sends $(1+i)\varepsilon_1^*$ to $\frac{1+i}{2}(\varepsilon_1^* - \varepsilon_2^* - \varepsilon_3^* - \varepsilon_4^*)$, and it sends $(\varepsilon_1^* + i\varepsilon_2^*)$ to $\frac{1+i}{2}(i^3\varepsilon_1^* + i\varepsilon_2^* - \varepsilon_3^* - \varepsilon_4^*)$. Thus W_0 acts transitively on Σ^*.

Let $\bar{\Sigma} \subseteq P(\mathbb{C}^4)$ be the set of projective points representing elements of Σ^*, and let $[\alpha] \in \bar{\Sigma}$ denote the class of $\alpha \in \Sigma^*$. To simplify notation, we also write $[x] = [\alpha]$ for $x \in \mathbb{C}^4$ representing the same point, also when $x \notin \Sigma^*$. Let $\sim$ denote the relation on $\bar{\Sigma}$: $[\alpha] \sim [\beta]$ if $\alpha = \beta$, or if $\alpha \perp \beta$ and the projective line $\langle [\alpha], [\beta] \rangle \subseteq P(\mathbb{C}^4)$ contains four other points in $\bar{\Sigma}$. By inspection, $[\varepsilon_j^*] \sim [\varepsilon_k^*]$ for all $j, k \in \{1,2,3,4\}$, and these are the only elements $[\alpha]$ such that $[\alpha] \sim [\varepsilon_j^*]$ for some j. Since this relation is preserved by W_0, and W_0 acts transitively on $\bar{\Sigma}$, we see that $\sim$ is an equivalence relation on $\bar{\Sigma}$ with 15 classes of four elements each. Set $\Delta = \bar{\Sigma}/{\sim}$, and let $[\alpha]_\Delta$ denote the class of $[\alpha]$ in Δ. Thus $|\bar{\Sigma}| = \frac{1}{4}|\Sigma| = 60$ and $|\Delta| = 15$. Since W_2 is the stabilizer subgroup of $[\varepsilon_1^*]_\Delta$ under the transitive W_0-action on Δ, we have $|W_0| = |W_2| \cdot 15 = 2^{10} \cdot 3^2 \cdot 5$.

Let $W_1 \trianglelefteq W_0$ be the subgroup of elements which act trivially on Δ. By inspection, $W_1 \le W_2$, $|W_1| = 2^6$, and W_1 is generated by $w_0 = \mathrm{diag}(i,i,i,i)$, $\mathrm{diag}(1,1,-1,-1)$, $\mathrm{diag}(1,-1,1,-1)$, and the permutation matrices for the permutations $(1\,2)(3\,4)$ and $(1\,3)(2\,4)$. Thus $W_1 \cong C_4 \times_{C_2} D_8 \times_{C_2} D_8$.

By the above computations, $|W_0/W_1| = 2^4 \cdot 3^2 \cdot 5 = |Sp_4(2)|$. There is a bijection from Δ to the set of maximal isotropic subspaces in $W_1/Z(W_1)$ which sends a class $[\alpha]_\Delta$ to the subgroup of those elements in W_1 which send each of the four projective points in $[\alpha]_\Delta$ to itself. Hence for each $w \in C_{W_0}(W_1)$, w acts via the identity on Δ, and so $w \in W_1$ by definition. Thus W_0/W_1 injects into $\mathrm{Out}(W_1) \cong \Sigma_6 \times C_2$, and injects into the first factor since $Z(W_1) = Z(W_0)$ ($\cong C_4$). So by counting, $W_0/W_1 \cong \Sigma_6$. Also, $W_1 = O_2(W_0)$.

Set $a = v_5(q^4-1) = v_5(q^2+1)$, and fix $u \in \bar{\mathbb{F}}_{q_0}^\times$ of order 5^a. Let A be as in Notation 5.2(G): the subgroup of elements in T of 5-power order. Thus

$$A = \bigl\{(u_1, u_1^q, u_2, u_2^q, u_3, u_3^q, u_4, u_4^q) \bigm| u_1, u_2, u_3, u_4 \in \langle u \rangle\bigr\} \cong (C_{5^a})^4 \,. \qquad (2)$$

By (2) and (1), there is no $\beta \in \Sigma$ such that $A \le \mathrm{Ker}(\theta_\beta)$. Hence $C_{\bar{G}}(A)^0 = \bar{T}$ by Proposition 2.5. So by Lemma 5.3(b),

$$N_G(A) = N_G(T) \quad \text{and} \quad N_G(T)/T = W_0\,. \qquad (3)$$

We are now ready to check the conditions in Case (III.3) of Hypotheses 5.1.

$N_G(T)$ contains a Sylow p-subgroup of G. Let S be a Sylow p-subgroup of $N_G(T)$ which contains A. Since $N_G(T)/T = W_0$ by (3), $A \cong (C_{5^a})^4$, and $W_0/O_2(W_0) \cong \Sigma_6$, $|S| = 5^{4a+1}$. By [St2, Theorem 25] or [Ca, Corollary 10.2.4 & Proposition 10.2.5], and since $v_5(q^k-1) = 0$ when $4 \nmid k$ and $v_5(q^{4\ell}-1) = a + v_5(\ell)$ (Lemma 1.13),

$$v_5(|G|) = v_5\bigl((q^{24}-1)(q^{20}-1)(q^{12}-1)(q^8-1)\bigr) = 4a+1\,.$$

Thus $S \in \mathrm{Syl}_p(G)$.

$\bigl|\gamma|_{\bar{T}}\bigr| = \mathrm{ord}_p(q) \ge 2$ and $[\gamma, \psi_{q_0}] = \mathrm{Id}$. The first is clear, and the second holds since $\gamma = c_{g_0}$ where $\psi_{q_0}(g_0) = g_0$.

$C_S(\Omega_1(A)) = A$ by the above description of the action of W_0 on A.

$C_A(O_{p'}(W_0)) = 1$ since $w_0 \in O_{5'}(W_0)$ and $C_A(w_0) = 1$.

A free $\langle\gamma\rangle$-orbit in Σ. The subset $\{\pm(\varepsilon_1+\varepsilon_3), \pm(\varepsilon_2+\varepsilon_4)\} \subseteq \Sigma$ is a free $\langle\gamma\rangle$-orbit.

$\mathrm{Aut}_{W_0}(A) \cap \mathrm{Aut}_{\mathrm{sc}}(A) \le \langle \gamma|_A \rangle$. Recall that $\bigl|\gamma|_{\bar{T}}\bigr| = 4$ and $|\mathrm{Aut}_{\mathrm{sc}}(A)| = 4 \cdot 5^k$ for some k, and W_0 acts faithfully on A. So if this is not true, then there is an element of order 5 in $Z(W_0)$, which is impossible by the above description of W_0.

$\mathrm{Aut}_G(A) = \mathrm{Aut}_{W_0}(A)$ by (3).

$N_{\mathrm{Aut}(A)}(\mathrm{Aut}_{W_0}(A)) \le \mathrm{Aut}_{\mathrm{sc}}(A)\mathrm{Aut}_{W_0}(A)$. For $j = 1, 2, 3, 4$, let $A_j < A$ be the cyclic subgroup of all elements as in (2) where $u_k = 1$ for $k \ne j$. The group W_0 contains as subgroup $C_2 \wr \Sigma_4$: the group which permutes pairs of coordinates up to sign. So each of the four subgroups A_j is the reflection subgroup of some reflection in W_0.

For each $\varphi \in C_{\mathrm{Aut}(A)}(\mathrm{Aut}_{W_0}(A))$, $\varphi(A_j) = A_j$ for each j, and $\varphi(a) = a^{n_j}$ for some $n_j \in (\mathbb{Z}/5^a)^\times$. Also, $n_1 = n_2 = n_3 = n_4$ since the A_j are permuted transitively by elements of W_0, and hence $\varphi \in \mathrm{Aut}_{\mathrm{sc}}(A)$.

Now assume $\varphi \in N_{\mathrm{Aut}(A)}(\mathrm{Aut}_{W_0}(A))$. Since φ centralizes $Z(W_1) = \langle w_0 \rangle = \langle \mathrm{diag}(i,i,i,i) \rangle$ (since $\mathrm{diag}(i,i,i,i) \in Z(\mathrm{Aut}(A))$), $c_\varphi|_{W_1} \in \mathrm{Inn}(W_1)$, and we can assume (after composing by an appropriate element of W_1) that $[\varphi, W_1] = 1$. So $c_\varphi \in \mathrm{Aut}(W_0)$ has the form $c_\varphi(g) = g\chi(\bar{g})$, where $\bar{g} \in W_0/W_1 \cong \Sigma_6$ is the class of $g \in W_0$, and where $\chi \in \mathrm{Hom}(W_0/W_1, Z(W_1)) \cong \mathrm{Hom}(\Sigma_6, C_4) \cong C_2$ is some homomorphism. Since $(w_0)^2$ inverts the torus T, composition with $(w_0)^2$ does not send reflections (in A) to reflections, and so we must have $c_\varphi = \mathrm{Id}_{W_0}$. Thus $\varphi \in C_{\mathrm{Aut}(A)}(\mathrm{Aut}_{W_0}(A)) = \mathrm{Aut}_{\mathrm{sc}}(A)$ (modulo $\mathrm{Aut}_{W_0}(A)$). □

The following lemma now reduces the proof of Theorem B to the cases considered in Section 5, together with certain small cases handled at the end of this section. As before, when p is a prime and $p \nmid n$, $\mathrm{ord}_p(n)$ denotes the multiplicative order of n in $\mathbb{F}_p^\times$.

Proposition 6.8. *Fix an odd prime p, and assume $G \in \mathfrak{Lie}(q_0)$ is of universal type for some prime $q_0 \neq p$. Fix $S \in \mathrm{Syl}_p(G)$, and assume S is nonabelian. Then there is a prime $q_0^* \neq p$, a group $G^* \in \mathfrak{Lie}(q_0^*)$ of universal type, and $S^* \in \mathrm{Syl}_p(G^*)$, such that $\mathcal{F}_S(G) \cong \mathcal{F}_{S^*}(G^*)$, and one of the following holds: either*

- (a) *G^* has a σ-setup which satisfies Hypotheses 5.1 and 5.10, $G^* \cong \mathbb{G}(q^*)$ or ${}^2\mathbb{G}(q^*)$ where q^* is a power of q_0^*, and*
  - (a.1) *$-\mathrm{Id} \notin W$ and G^* is a Chevalley group, or*
  - (a.2) *$-\mathrm{Id} \in W$ and $\mathrm{ord}_p(q^*)$ is even, or*
  - (a.3) *$p \equiv 3$ (mod p) and $\mathrm{ord}_p(q^*) = 1$*

  where W is the Weyl group of $\mathbb{G}$; or
- (b) *$p = 3$, $q_0^* = 2$, $G \cong {}^3D_4(q)$ or ${}^2F_4(q)$ for q some power of q_0, and $G^* \cong {}^3D_4(q^*)$ or ${}^2F_4(q^*)$ for q^* some power of 2.*

Moreover, if $p = 3$ and $G^* = F_4(q^*)$ where q^* is a power of q_0^*, then we can assume $q_0^* = 2$. In all cases, we can choose G^* to be either one of the groups listed in Proposition 1.10(a–e), or one of $E_7(q^*)$ or $E_8(q^*)$ for some $q^* \equiv -1$ (mod p).

Proof. We can assume that $G = \mathbb{G}(q)$ is one of the groups listed in one of the five cases (a)–(e) of Proposition 1.10. In all cases except 1.10(c), we can also assume that G satisfies Hypotheses 5.10, with $q_0 = 2$ if $p = 3$ and $\mathbb{G} = F_4$, and with q_0 odd in cases (a) and (b) of 1.10. If $G = SL_n(q)$ or $\mathrm{Spin}_{2n}^\pm(q)$ where $p|(q-1)$, or G is in case (d), then G satisfies Hypotheses 5.1 by Lemma 6.1. If $G \cong SL_n(q)$ or $\mathrm{Spin}_{2n}^\pm(q)$ where $p \nmid (q-1)$, then G satisfies Hypotheses 5.1 by Lemma 6.5. This leaves only case (c) in Proposition 1.10, which corresponds to case (b) here, and case (e) ($p = 5$, $G = E_8(q)$, $q \equiv \pm 2$ (mod 5)) where G^* satisfies Hypotheses 5.1 by Lemma 6.7.

We next show, in cases (a,b,d,e) of Proposition 1.10, that we can arrange for one of the conditions (a.1), (a.2), or (a.3) to hold. If $-\mathrm{Id} \notin W$, then $\mathbb{G} = A_n$, D_n for n odd, or E_6, and G is a Chevalley group by the assumptions in cases (a,b,d) of Proposition 1.10. So (a.1) holds. If $-\mathrm{Id} \in W$ and $\mathrm{ord}_p(q)$ is even, then (a.2) holds, while if $p \equiv 3$ (mod 4) and $\mathrm{ord}_p(q) = 1$, then (a.3) holds.

By inspection, we are left with the following two cases:

(b′) $G = \mathbb{G}(q)$, where $\mathbb{G} = \mathrm{Spin}_{2n}^+$, $n \geq p$ is even, $\mathrm{ord}_p(q)$ is odd, and $q^n \equiv 1$ (mod p); or

(d′) $G = \mathbb{G}(q)$ where $\mathbb{G} = G_2$, F_4, E_6, E_7, or E_8, $p \equiv 1$ (mod 4), $p|(q-1)$, and $p \bigm| |W(\mathbb{G})|$.

In case (d′), the above conditions leave only the possibility $p = 5$ and $\mathbb{G} = E_7$ or E_8 (see the computations of $|W|$ in, e.g., [Brb, § VI.4]). In either case, by Lemma 1.11(a), we can choose a prime power $q^\vee$ which satisfies Hypotheses 5.10 and such that $\overline{\langle q^\vee \rangle} = \overline{\langle -q \rangle}$ in $\mathbb{Z}_p^\times$, and set $G^\vee = \mathbb{G}(q^\vee)$. Then $G^\vee \sim_p G$ by Theorem 1.8(c), $\mathrm{ord}_p(q^\vee)$ is even, so (a.2) holds, and $G^\vee$ satisfies Hypotheses 5.1 by Lemma 6.4 or 6.5. □

We now consider the two families of groups which appear in Proposition 6.8(b): those not covered by Hypotheses 5.1.

Proposition 6.9. *Let G be one of the groups ${}^3D_4(q)$ where q is a prime power prime to 3, ${}^2F_4(2^{2m+1})$ for $m \geq 0$, or ${}^2F_4(2)'$. Then the 3-fusion system of G is tame. If $G \cong {}^3D_4(2^n)$*

($n \geq 1$), ${}^2F_4(2^{2m+1})$ ($m \geq 0$), or ${}^2F_4(2)'$, then κ_G is split surjective, and $\mathrm{Ker}(\kappa_G)$ *is the subgroup of field automorphisms of order prime to* 3.

Proof. Fix $S \in \mathrm{Syl}_3(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$.

If G is the Tits group ${}^2F_4(2)'$, then S is extraspecial of order 3^3 and exponent 3, so $\mathrm{Out}(S) \cong GL_2(3)$. Also, $\mathrm{Out}_G(S) \cong D_8$ and $\mathrm{Out}_{\mathrm{Aut}(G)}(S) \cong SD_{16}$, since the normalizer in ${}^2F_4(2)$ of an element of order 3 (the element t_4 in [Sh]) has the form $SU_3(2) : 2 \cong 3_+^{1+2} : SD_{16}$ by [Sh, Table IV] or [Ma, Proposition 1.2]. Hence $\mathrm{Out}(S, \mathcal{F}) \leq N_{\mathrm{Out}(S)}(\mathrm{Out}_G(S))/\mathrm{Out}_G(S)$ has order at most 2, and $\bar{\kappa}_G$ sends $\mathrm{Out}(G) \cong C_2$ ([GrL, Theorem 2]) isomorphically to $\mathrm{Out}(S, \mathcal{F})$. If $G = {}^2F_4(2)$, then $\mathrm{Out}_G(S) \cong SD_{16}$, so $\mathrm{Out}(S, \mathcal{F}) = 1$ by a similar argument, and κ_G is an isomorphism between trivial groups.

Assume now that $G \cong {}^2F_4(2^n)$ for odd $n \geq 3$ or $G \cong {}^3D_4(q)$ where $3 \nmid q$. In order to describe the Sylow 3-subgroups of these groups, set $\zeta = e^{2\pi i/3}$, $R = \mathbb{Z}[\zeta]$, and $\mathfrak{p} = (1 - \zeta)R$. Let S_k be the semidirect product $R/\mathfrak{p}^k \rtimes C_3$, where the quotient acts via multiplication by ζ. Explicitly, set

$$S_k = \{(x, i) \,|\, x \in R/\mathfrak{p}^k,\ i \in \mathbb{Z}/3\} \qquad \text{and} \qquad A_k = R/\mathfrak{p}^k \times \{0\},$$

where $(x, i)(y, j) = (x + \zeta^i y, i + j)$. Thus $|S_k| = 3^{k+1}$. Set $s = (0, 1)$, so that $s(x, 0)s^{-1} = (\zeta x, 0)$ for each $x \in R/\mathfrak{p}^k$.

Assume $k \geq 3$, so that A_k is the unique abelian subgroup of index three in S_k. Set $S = S_k$ and $A = A_k$ for short. We want to describe $\mathrm{Out}(S)$. Define automorphisms ξ_a ($a \in (R/\mathfrak{p}^k)^\times$), ω, η, and ρ by setting

$$\xi_a(x, i) = (xa, i), \quad \eta = \xi_{-1}, \quad \omega(x, i) = (-\bar{x}, -i), \quad \rho(x, i) = (x + \lambda(i), i). \tag{4}$$

Here, $x \mapsto \bar{x}$ means complex conjugation, and $\lambda(i) = 1 + \zeta + \ldots + \zeta^{i-1}$. Note, when checking that ρ is an automorphism, that $\lambda(i) + \zeta^i \lambda(j) = \lambda(i + j)$. Note that $\rho^3 \in \mathrm{Inn}(S)$: it is (left) conjugation by $(1 - \zeta^2, 0)$.

Let $\mathrm{Aut}^0(S) \trianglelefteq \mathrm{Aut}(S)$ be the subgroup of automorphisms which induce the identity on $S/[S, S] = S/[s, A]$, and set $\mathrm{Out}^0(S) = \mathrm{Aut}^0(S)/\mathrm{Inn}(S)$. Each element in $s{\cdot}[s, A]$ is conjugate to s, and thus each class in $\mathrm{Out}^0(S)$ is represented by an automorphism which sends s to itself, which is unique modulo $\langle c_s \rangle$. If $\varphi \in \mathrm{Aut}(S)$ and $\varphi(s) = s$, then $\varphi|_A$ commutes with c_s, thus is R-linear under the identification $A \cong R/\mathfrak{p}^k$, and hence $\varphi = \xi_a$ for some $a \in 1 + \mathfrak{p}/\mathfrak{p}^k$. Moreover, since

$$(1 + \mathfrak{p}/\mathfrak{p}^k)^\times = (1 + \mathfrak{p}^2/\mathfrak{p}^k)^\times \times \langle \zeta \rangle = (1 + 3R/\mathfrak{p}^k)^\times \times \langle \zeta \rangle$$

as multiplicative groups (just compare orders, noting that the groups on the right have trivial intersection), each class in $\mathrm{Out}^0(S)$ is represented by ξ_a for some unique $a \in 1 + 3R/\mathfrak{p}^k$.

Since the images of η, ω, and ρ generate $\mathrm{Aut}(S)/\mathrm{Aut}^0(S)$ (the group of automorphisms of $S/[s, A] \cong C_3^2$ which normalize $A/[s, A] \cong C_3$), this shows that $\mathrm{Out}(S)$ is generated by the classes of the automorphisms in (4). In fact, a straightforward check of the relations among them shows that

$$\mathrm{Out}(S) \cong \Big(\mathrm{Out}^0(S) \rtimes \underset{[\omega]}{C_2}\Big) \times \underset{[\rho],[\eta]}{\Sigma_3} \qquad \text{where} \qquad \mathrm{Out}^0(S) = \big\{[\xi_a] \,\big|\, a \in (1 + 3R/\mathfrak{p}^k)^\times\big\}.$$

Also, $\omega \xi_a \omega^{-1} = \xi_{\bar{a}}$ for $a \in (1 + 3R/\mathfrak{p}^k)^\times$.

For each $x \in 1 + 3R$ such that $\bar{x} \equiv x \pmod{\mathfrak{p}^k}$, we can write $x = r + s\zeta$ with $r, s \in \mathbb{Z}$, and then $s(\zeta - \overline{\zeta}) \in \mathfrak{p}^k$, so $s \in \mathfrak{p}^{k-1}$, and $x \in r + s + \mathfrak{p}^k \subseteq 1 + 3\mathbb{Z} + \mathfrak{p}^k$. This proves that

$$C_{\mathrm{Out}(S)}(\omega) = \big\{[\xi_a] \,\big|\, a \in \mathbb{Z}\big\} \times \big\langle[\omega]\big\rangle \times \big\langle[\rho], [\eta]\big\rangle.$$

For any group G with $S \in \mathrm{Syl}_3(G)$ and $S \cong S_k$, $\mathrm{Out}_G(S)$ has order prime to 3, and hence is a 2-group and conjugate to a subgroup of $\langle \omega, \eta \rangle \in \mathrm{Syl}_2(\mathrm{Out}(S))$. If $|\mathrm{Out}_G(S)| = 4$, then we can identify S with S_k in a way so that $\mathrm{Out}_G(S) = \big\langle [\omega], [\eta] \big\rangle$. Then

$$\begin{aligned}\mathrm{Out}(S, \mathcal{F}) &\leq N_{\mathrm{Out}(S)}\big(\big\langle [\omega], [\eta] \big\rangle\big) / \big\langle [\omega], [\eta] \big\rangle \\ &= C_{\mathrm{Out}(S)}\big(\big\langle [\omega], [\eta] \big\rangle\big) / \big\langle [\omega], [\eta] \big\rangle = \big\{ [\xi_a] \,\big|\, a \in \mathbb{Z} \big\} = \big\langle [\xi_2] \big\rangle ,\end{aligned}$$

where the first equality holds since $O_3(\mathrm{Out}(S))$ has index four in $\mathrm{Out}(S)$.

We are now ready to look at the individual groups. Assume $G = {}^2F_4(q)$, where $q = 2^n$ and $n \geq 3$ is odd. By [St1, 3.2–3.6], $\mathrm{Out}(G)$ is cyclic of order n, generated by the field automorphism ψ_2. By the main theorem in [Ma], there is a subgroup $\mathcal{N}_G(T_8) \cong (C_{q+1})^2 \rtimes GL_2(3)$, the normalizer of a maximal torus, which contains a Sylow 3-subgroup. Hence if we set $k = v_3(q+1) = v_3(4^n - 1) = 1 + v_3(n)$ (Lemma 1.13), we have $S \cong S_{2k} \cong (C_{3^k})^2 \rtimes C_3$, and $\mathrm{Out}_G(S) = \langle \omega, \eta \rangle$ up to conjugacy. So $\mathrm{Out}(S, \mathcal{F})$ is cyclic, generated by $\xi_2 = \kappa_G(\psi_2)$. Since $A \cong (C_{3^k})^2$, and since $\xi_{-1} \in \mathrm{Out}_G(S)$, $|\mathrm{Out}(S, \mathcal{F})| = |[\xi_2]| = 3^{k-1}$ where $k - 1 = v_3(n)$. Thus $\bar{\kappa}_G$ is surjective, and is split since the Sylow 3-subgroup of $\mathrm{Out}(G) \cong C_n$ is sent isomorphically to $\mathrm{Out}(S, \mathcal{F})$.

Next assume $G = {}^3D_4(q)$, where $q = 2^n$ for $n \geq 1$. By [St1, 3.2–3.6], $\mathrm{Out}(G)$ is cyclic of order $3n$, generated by the field automorphism ψ_2 (and where the field automorphism ψ_{2^n} of order three is also a graph automorphism). Set $k = v_3(q^2 - 1) = v_3(2^{2n} - 1) = 1 + v_3(n)$ (Lemma 1.13). Then $S \cong S_{2k+1}$: this follows from the description of the Sylow structure in G in [GL, 10-1(4)], and also from the description (based on [Kl]) of its fusion system in [O5, Theorem 2.8] (case (a.ii) of the theorem). Also, $\mathrm{Out}_G(S) = \langle \omega, \eta \rangle$ up to conjugacy. So $\mathrm{Out}(S, \mathcal{F})$ is cyclic, generated by $\xi_2 = \kappa_G(\psi_2)$. Since $A \cong C_{3^k} \times C_{3^{k+1}}$, and since $\xi_{-1} \in \mathrm{Out}_G(S)$, $|\mathrm{Out}(S, \mathcal{F})| = |[\xi_2]| = 3^k$. Thus $\bar{\kappa}_G$ is surjective, and is split since the Sylow 3-subgroup of $\mathrm{Out}(G) \cong C_{3n}$ is sent isomorphically to $\mathrm{Out}(S, \mathcal{F})$.

By Theorem 1.8(b) and Lemma 1.11(a), for each prime power q with $3 \nmid q$, the 3-fusion system of ${}^3D_4(q)$ is isomorphic to that of ${}^3D_4(2^n)$ for some n. By [O1, Theorem C], μ_G is injective in all cases. Thus the 3-fusion systems of all of these groups are tame. □

Appendix A. Injectivity of μ_G
Bob Oliver

Recall that for any finite group G and any $S \in \mathrm{Syl}_p(G)$,

$$\mu_G\colon \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G)) \longrightarrow \mathrm{Out}(S, \mathcal{F}_S(G))$$

is the homomorphism which sends the class of $\beta \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L}_S^c(G))$ to the class of $\beta_S|_S$, where β_S is the induced automorphism of $\mathrm{Aut}_{\mathcal{L}_S^c(G)}(S) = N_G(S)/O_{p'}(C_G(S))$. We need to develop tools for computing $\mathrm{Ker}(\mu_G)$, taking as starting point [AOV, Proposition 4.2].

As usual, for a finite group G and a prime p, a proper subgroup $H < G$ is *strongly p-embedded* in G if $p\big||H|$, and $p \nmid |H \cap {}^gH|$ for $g \in G \smallsetminus H$. The following properties of groups with strongly embedded subgroups will be needed.

Lemma A.1. *Fix a prime p and a finite group G.*

(a) *If G contains a strongly p-embedded subgroup, then $O_p(G) = 1$.*

(b) *If $H < G$ is strongly p-embedded, and $K \trianglelefteq G$ is a normal subgroup of order prime to p such that $KH < G$, then HK/K is strongly p-embedded in G/K.*

Proof. **(a)** See, e.g., [AKO, Proposition A.7(c)].

(b) Assume otherwise. Thus there is $g \in G \smallsetminus HK$ such that $p\big||({}^gHK/K) \cap (HK/K)|$, and hence $x \in {}^gHK \cap HK$ of order p. Then $H \cap K\langle x\rangle$ and ${}^gH \cap K\langle x\rangle$ have order a multiple of p, so there are elements $y \in H$ and $z \in {}^gH$ of order p such that $y \equiv x \equiv z \pmod{K}$.

Since $\langle y\rangle, \langle z\rangle \in \mathrm{Syl}_p(K\langle x\rangle)$, there is $k \in K$ such that $\langle y\rangle = {}^k\langle z\rangle$. Then $y \in H \cap {}^{kg}H$, and $kg \notin H$ since $k \in K$ and $g \notin HK$. But this is impossible, since H is strongly p-embedded. □

For the sake of possible future applications, we state the next proposition in terms of abstract fusion and linking systems. We refer to [AOV], and also to Chapters I.2 and III.4 in [AKO], for the basic definitions. Recall that if $\mathcal{F}$ is a fusion system over a finite p-group S, and $P \le S$, then

- P is *$\mathcal{F}$-centric* if $C_S(Q) \le Q$ for each Q which is $\mathcal{F}$-conjugate to P;
- P is *fully normalized in $\mathcal{F}$* if $|N_S(P)| \ge |N_S(Q)|$ whenever Q is $\mathcal{F}$-conjugate to P; and
- P is *$\mathcal{F}$-essential* if $P < S$, P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and if $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.

For any saturated fusion system $\mathcal{F}$ over a finite p-group S, set

$$\widehat{\mathcal{Z}}(\mathcal{F}) = \bigl\{E \le S \bigm| E \text{ elementary abelian, fully normalized in } \mathcal{F}, \\ E = \Omega_1(Z(C_S(E))),\ \mathrm{Aut}_{\mathcal{F}}(E) \text{ has a strongly } p\text{-embedded subgroup}\bigr\}.$$

The following proposition is our main tool for proving that $\mu_{\mathcal{L}}$ is injective in certain cases. Point (a) will be used to handle the groups $\mathrm{Spin}_n^{\pm}(q)$, point (c) the linear and symplectic groups, and point (b) the exceptional Chevalley groups.

Proposition A.2. *Fix a saturated fusion system $\mathcal{F}$ over a p-group S and an associated centric linking system $\mathcal{L}$. Let $E_1, \ldots, E_k \in \widehat{\mathcal{Z}}(\mathcal{F})$ be such that each $E \in \widehat{\mathcal{Z}}(\mathcal{F})$ is $\mathcal{F}$-conjugate to E_i for some unique i. For each i, set $P_i = C_S(E_i)$ and $Z_i = Z(P_i)$. Then the following hold.*

(a) *If $k = 0$ ($\widehat{\mathcal{Z}}(\mathcal{F}) = \varnothing$), then* $\mathrm{Ker}(\mu_{\mathcal{L}}) = 1$.

(b) *If* $k = 1$, $E_1 \trianglelefteq S$, *and* $\mathrm{Aut}_{\mathcal{F}}(\Omega_1(Z(S))) = 1$, *then* $\mathrm{Ker}(\mu_{\mathcal{L}}) = 1$.

(c) *Assume, for each* $(g_i)_{i=1}^k \in \prod_{i=1}^k C_{Z_i}(\mathrm{Aut}_S(P_i))$, *that there is* $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ *such that* $g_i \in g \cdot C_{Z_i}(\mathrm{Aut}_{\mathcal{F}}(P_i))$ *for each* i. *Then* $\mathrm{Ker}(\mu_{\mathcal{L}}) = 1$.

Proof. We first show that (a) and (b) are special cases of (c), and then prove (c). That (a) follows from (c) is immediate.

(b) If $k = 1$, $E_1 \trianglelefteq S$, and $\mathrm{Aut}_{\mathcal{F}}(\Omega_1(Z(S))) = 1$, then the group $\mathrm{Out}_{\mathcal{F}}(S)$ of order prime to p acts trivially on $\Omega_1(Z(S))$, and hence acts trivially on $Z(S)$ (cf. [G, Theorem 5.2.4]). Also, $P_1 = C_S(E_1) \trianglelefteq S$, so $C_{Z_1}(\mathrm{Aut}_S(P_1)) = Z(S) = C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$, and $\mathrm{Ker}(\mu_{\mathcal{L}}) = 1$ by (c).

(c) Fix a class $[\alpha] \in \mathrm{Ker}(\mu_{\mathcal{L}})$. By [AOV, Proposition 4.2], there is an automorphism $\alpha \in \mathrm{Aut}_{\mathrm{typ}}^I(\mathcal{L})$ in the class $[\alpha]$ such that $\alpha_S = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(S)}$. By the same proposition, there are elements $g_P \in C_{Z(P)}(\mathrm{Aut}_S(P))$, defined for each $P \le S$ which is fully normalized and $\mathcal{F}$-centric, such that

(i) $\alpha_P \in \mathrm{Aut}(\mathrm{Aut}_{\mathcal{L}}(P))$ is conjugation by $\delta_P(g_P)$;

(ii) $\alpha_P = \mathrm{Id}$ if and only if $g_P \in C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$; and

(iii) if $Q < P$ are both fully normalized and $\mathcal{F}$-centric, then

$$g_P \equiv g_Q \pmod{C_{Z(Q)}(N_{\mathrm{Aut}_{\mathcal{F}}(P)}(Q))}.$$

Furthermore,

(iv) $[\alpha] = 1 \in \mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ if and only if there is $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ such that $g_P \in g \cdot C_{Z(P)}(\mathrm{Aut}_{\mathcal{F}}(P))$ for each $P < S$ such that P is $\mathcal{F}$-essential and $P = C_S(\Omega_1(Z(P)))$.

Set $g_i = g_{P_i}$ $(1 \le i \le k)$ for short.

By hypothesis, we can assume there is an element $g \in C_{Z(S)}(\mathrm{Aut}_{\mathcal{F}}(S))$ such that $g_i \in g{\cdot}C_{Z(P_i)}(\mathrm{Aut}_{\mathcal{F}}(P_i))$ for each i. Upon replacing α by its composite with $c_{\delta_S(g)}^{-1}$, we can assume $g_i \in C_{Z(P_i)}(\mathrm{Aut}_{\mathcal{F}}(P_i))$, and hence $\alpha_{P_i} = \mathrm{Id}_{\mathrm{Aut}_{\mathcal{L}}(P_i)}$ for each i.

We claim that $\alpha_P = \mathrm{Id}$ for all $P \le S$, and hence that $[\alpha] = 1$ by (iv) and (ii). Assume otherwise, and choose $Q < S$ which is fully normalized and of maximal order among all subgroups such that $\alpha_Q \ne \mathrm{Id}$. Thus $\alpha_R = \mathrm{Id}$ for all $R \le S$ with $|R| > |Q|$. By Alperin's fusion theorem (cf. [AKO, Theorem I.3.6]), Q is $\mathcal{F}$-essential, and α is the identity on $\mathrm{Mor}_{\mathcal{L}}(P, P^*)$ for all $P, P^* \in \mathrm{Ob}(\mathcal{L})$ such that $|P|, |P^*| > |Q|$. Also, for each $Q^* \in Q^{\mathcal{F}}$, there is (by Alperin's fusion theorem again) an isomorphism $\chi \in \mathrm{Iso}_{\mathcal{L}}(Q, Q^*)$ which is a composite of isomorphisms each of which extends to an isomorphism between strictly larger subgroups, and hence $\alpha_{Q,Q^*}(\chi) = \chi$. Thus

$$Q^* \in Q^{\mathcal{F}},\ Q^* \text{ fully normalized} \qquad \Longrightarrow \qquad \alpha_{Q^*} \ne \mathrm{Id}\,. \tag{1}$$

Set $E = \Omega_1(Z(Q))$. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(E), S)$ be such that $\varphi(E)$ is fully normalized (cf. [AKO, Lemma I.2.6(c)]). Then $N_S(Q) \le N_S(E)$, so $|N_S(\varphi(Q))| \ge |N_S(Q)|$, and $\varphi(Q)$ is fully normalized since Q is. Since $\alpha_{Q^*} \ne \mathrm{Id}$ by (1), we can replace Q by Q^* and E by E^*, and arrange that Q and E are both fully normalized in $\mathcal{F}$ (and Q is still $\mathcal{F}$-essential).

Set $\Gamma = \mathrm{Aut}_{\mathcal{F}}(Q)$, and set

$$\Gamma_0 = C_\Gamma(E) = \big\{\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q) \,\big|\, \varphi|_E = \mathrm{Id}_E\big\}$$
$$\Gamma_1 = \big\langle \varphi \in \mathrm{Aut}_{\mathcal{F}}(Q) \,\big|\, \varphi = \bar\varphi|_Q \text{ for some } \bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, S),\ R > Q \big\rangle\,.$$

Let $\pi_Q\colon \mathrm{Aut}_{\mathcal{L}}(Q) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(Q)$ be the homomorphism induced by the functor π. For each $\varphi \in \Gamma = \mathrm{Aut}_{\mathcal{F}}(Q)$, and each $\psi \in \pi_Q^{-1}(\varphi)$, we have

$$\varphi(g_Q) = g_Q \quad\iff\quad [\psi, \delta_Q(g_Q)] = \mathrm{Id} \quad\iff\quad \alpha_Q(\psi) = \psi \ : \tag{2}$$

the first by axiom (C) in the definition of a linking system (see, e.g., [AKO, Definition III.4.1]) and since δ_Q is injective, and the second by point (i) above.

Now, $\mathrm{Aut}_S(Q) \le \Gamma_1$, since each element of $\mathrm{Aut}_S(Q)$ extends to $N_S(Q)$ and $N_S(Q) > Q$ (see [Sz1, Theorem 2.1.6]). Hence

$$\Gamma_0\Gamma_1 \le O^p(\Gamma_0)\cdot \mathrm{Aut}_S(Q)\cdot \Gamma_1 = O^p(\Gamma_0)\Gamma_1\,.$$

For each $\varphi \in \Gamma_0$ of order prime to p, $\varphi|_{Z(Q)} = \mathrm{Id}_{Z(Q)}$ since φ is the identity on $E = \Omega_1(Z(Q))$ (cf. [G, Theorem 5.2.4]). Thus $g_Q \in C_{Z(Q)}(O^p(\Gamma_0))$. If $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ extends to $\bar\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,S)$ for some $R > Q$, then by the maximality of Q, $\alpha(\bar\psi) = \bar\psi$ for each $\bar\psi \in \mathrm{Mor}_{\mathcal{L}}(R,S)$ such that $\pi(\bar\psi) = \bar\varphi$, and since α commutes with restriction (it sends inclusions to themselves), α_Q is the identity on $\bar\psi|_{Q,Q} \in \pi_Q^{-1}(\varphi)$. So by (2), $\varphi(g_Q) = g_Q$. Thus $\varphi(g_Q) = g_Q$ for all $\varphi \in \Gamma_1$. Since $\alpha_Q \ne \mathrm{Id}$ by assumption, there is some $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\varphi(g_Q) \ne g_Q$ (by (2) again), and we conclude that

$$g_Q \in C_{Z(Q)}(\Gamma_0\Gamma_1) \qquad\text{and}\qquad \Gamma_0\Gamma_1 < \Gamma = \mathrm{Aut}_{\mathcal{F}}(Q)\,. \tag{3}$$

Set $Q^* = N_{C_S(E)}(Q) \ge Q$. Then $\mathrm{Aut}_{Q^*}(Q) = \Gamma_0 \cap \mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\Gamma_0)$ since $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\Gamma)$, and by the Frattini argument, $\Gamma = N_\Gamma(\mathrm{Aut}_{Q^*}(Q))\Gamma_0$. If $Q^* > Q$, then for each $\varphi \in N_\Gamma(\mathrm{Aut}_{Q^*}(Q))$, φ extends to $\bar\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q^*)$ by the extension axiom. Thus $N_\Gamma(\mathrm{Aut}_{Q^*}(Q)) \le \Gamma_1$ in this case, so $\Gamma = \Gamma_1\Gamma_0$, contradicting (3). We conclude that $Q^* = Q$.

The homomorphism $\Gamma = \mathrm{Aut}_{\mathcal{F}}(Q) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(E)$ induced by restriction is surjective by the extension axiom, so $\mathrm{Aut}_{\mathcal{F}}(E) \cong \Gamma/\Gamma_0$. By [AKO, Proposition I.3.3(b)], $\Gamma_1/\mathrm{Inn}(Q)$ is strongly p-embedded in $\Gamma/\mathrm{Inn}(Q) = \mathrm{Out}_{\mathcal{F}}(Q)$; and $\Gamma_0\Gamma_1 < \Gamma$ by (3). Also, $p \nmid |\Gamma_0/\mathrm{Inn}(Q)|$, since otherwise we would have $\Gamma_1 \ge N_\Gamma(T)$ for some $T \in \mathrm{Syl}_p(\Gamma_0)$, in which case $\Gamma_1\Gamma_0 \ge N_\Gamma(T)\Gamma_0 = \Gamma$ by the Frattini argument. Thus $\Gamma_1\Gamma_0/\Gamma_0$ is strongly p-embedded in $\Gamma/\Gamma_0 \cong \mathrm{Aut}_{\mathcal{F}}(E)$ by Lemma A.1(b).

Now, $C_S(E) = Q$ since $N_{C_S(E)}(Q) = Q$ (cf. [Sz1, Theorem 2.1.6]). Thus $\Omega_1(Z(C_S(E))) = \Omega_1(Z(Q)) = E$, and we conclude that $E \in \widehat{\mathcal{E}}(\mathcal{F})$. Then $E \in (E_i)^{\mathcal{F}}$ for some unique $1 \le i \le k$, and $Q \in (P_i)^{\mathcal{F}}$ by the extension axiom (and since E and E_i are both fully centralized). But then $\alpha_{P_i} \ne \mathrm{Id}$ by (1), contradicting the original assumption about α_{P_i}. We conclude that $\alpha = \mathrm{Id}$. □

A.1. **Classical groups of Lie type in odd characteristic.**

Throughout this subsection, we fix an odd prime power q and an integer $n \ge 1$. We want to show $\mathrm{Ker}(\mu_G) = 1$ when G is one of the quasisimple classical groups of universal type over $\mathbb{F}_q$. By Theorem 1.8(d), we need not consider the unitary groups.

Proposition A.3. *Fix an odd prime power q. Let G be isomorphic to one of the quasisimple groups $SL_n(q)$, $Sp_n(q)$ ($n = 2m$), or $\mathrm{Spin}_n^{\pm}(q)$ ($n \ge 3$). Then* $\mathrm{Ker}(\mu_G) = 1$.

Proof. Let V, $\mathfrak{b}$, and $\widehat{G} = \mathrm{Aut}(V,\mathfrak{b})$ be such that $G = [\widehat{G},\widehat{G}]$ if $G \cong Sp_n(q)$ or $G \cong SL_n(q)$, and $G/\langle z\rangle = [\widehat{G},\widehat{G}]$ for some $z \in Z(\widehat{G})$ if $G \cong \mathrm{Spin}_n^{\pm}(q)$ (where $z \in Z(G)$). Thus V is a vector space of dimension n over the field $K = \mathbb{F}_q$, $\mathfrak{b}$ is a trivial symplectic, or quadratic form, and $\widehat{G}$ is one of the groups $GL_n(q)$, $Sp_{2n}(q)$, or $O_n^{\pm}(q)$.

Fix $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Set $\widehat{\mathcal{Z}} = \widehat{\mathcal{Z}}(\mathcal{F})$ for short.

Case 1: Assume $G = \mathrm{Spin}(V, \mathfrak{b})$, where $\mathfrak{b}$ is nondegenerate and symmetric. Set $Z = Z(G)$, and let $z \in Z$ be such that $G/\langle z\rangle = \Omega(V, \mathfrak{b})$. We claim that $\widehat{\mathcal{Z}} = \varnothing$ in this case, and hence that $\mathrm{Ker}(\mu_G) = 1$ by Proposition A.2(a).

Fix an elementary abelian 2-subgroup $E \le G$ where $E \ge Z$. Let $V = \bigoplus_{i=1}^{m} V_i$ be the decomposition as a sum of eigenspaces for the action of E on V. Fix indices $j, k \in \{1, \ldots, m\}$ such that either $\dim(V_j) \ge 2$, or the subspaces have the same discriminant (modulo squares). (Since $\dim(V) \ge 3$, this can always be done.) Then there is an involution $\gamma \in SO(V, \mathfrak{b})$ such that $\gamma(V_i) = V_i$ for all i, $\gamma|_{V_i} = \mathrm{Id}$ for $i \ne j, k$, $\det(\gamma|_{V_j}) = \det(\gamma|_{V_k}) = -1$, and such that the (-1)-eigenspace of γ has discriminant a square. This last condition ensures that $\gamma \in \Omega(V, \mathfrak{b})$ (cf. [LO, Lemma A.4(a)]), so we can lift it to $g \in G$. Then for each $x \in E$, $c_g(x) = x$ if x has the same eigenvalues on V_j and V_k, and $c_g(x) = zx$ otherwise (see, e.g., [LO, Lemma A.4(c)]). Since z is fixed by all elements of $\mathrm{Aut}_{\mathcal{F}}(E)$, $c_g \in O_2(\mathrm{Aut}_{\mathcal{F}}(E))$, and hence $\mathrm{Aut}_{\mathcal{F}}(E)$ has no strongly 2-embedded subgroups by Lemma A.1(a). Thus $E \notin \widehat{\mathcal{Z}}$.

Case 2: Now assume G is linear or symplectic, and fix $S \in \mathrm{Syl}_2(G)$. For each $\mathcal{V} = \{V_1, \ldots, V_k\}$ such that $V = \bigoplus_{i=1}^{k} V_i$, and such that $V_i \perp V_j$ for $i \ne j$ if G is symplectic, set

$$E(\mathcal{V}) = \bigl\{\varphi \in G \bigm| \varphi|_{V_i} = \pm\mathrm{Id} \ \text{ for each } i\bigr\}.$$

We claim that each subgroup in $\widehat{\mathcal{Z}}$ has this form. To see this, fix $E \in \widehat{\mathcal{Z}}$, and let $\mathcal{V} = \{V_1, \ldots, V_k\}$ be the eigenspaces for the nonzero characters of E. Then $E \le E(\mathcal{V})$, $V = \bigoplus_{i=1}^{k} V_i$, and this is an orthogonal decomposition if G is symplectic. Also, $C_{\widehat{G}}(E)$ is the product of the groups $\mathrm{Aut}(V_i, \mathfrak{b}|_{V_i})$. Since $E = \Omega_1(Z(P))$ where $P = C_S(E)$, E contains the 2-torsion in the center of $C_G(E)$, and thus $E = E(\mathcal{V})$. Furthermore, the action of S on each V_i must be irreducible (otherwise $\Omega_1(Z(C_S(E))) > E$), so $\dim(V_i)$ is a power of 2 for each i.

Again assume $E = E(\mathcal{V}) \in \widehat{\mathcal{Z}}$ for some $\mathcal{V}$. Then $\mathrm{Aut}_{\widehat{G}}(E)$ is a product of symmetric groups: if $\mathcal{V}$ contains n_i subspaces of dimension i for each $i \ge 1$, then $\mathrm{Aut}_{\widehat{G}}(E(\mathcal{V})) \cong \prod_{i\ge 1} \Sigma_{n_i}$. Each such permutation can be realized by a self map of determinant one (if G is linear), so $\mathrm{Aut}_G(E) = \mathrm{Aut}_{\widehat{G}}(E)$. Since $\mathrm{Aut}_G(E)$ contains a strongly 2-embedded subgroup by definition of $\widehat{\mathcal{Z}}$ (and since a direct product of groups of even order contains no strongly 2-embedded subgroup), $\mathrm{Aut}_G(E) = \mathrm{Aut}_{\widehat{G}}(E) \cong \Sigma_3$.

Write $n = \dim(V) = 2^{k_0} + 2^{k_1} + \ldots + 2^{k_m}$, where $0 \le k_0 < k_1 < \cdots < k_m$. There is an (orthogonal) decomposition $V = \bigoplus_{i=0}^{m} V_m$, where S acts irreducibly on each V_i, and where $\dim(V_i) = 2^{k_i}$ (see [CF, Theorem 1]). For each $1 \le i \le m$, fix an (orthogonal) decomposition $\mathcal{W}_i$ of V_i whose components have dimensions $2^{k_{i-1}}, 2^{k_{i-1}}, 2^{k_{i-1}+1}, \ldots, 2^{k_i-1}$, and set

$$\mathcal{V}_i = \{V_j \,|\, j \ne i\} \cup \mathcal{W}_i$$

and $E_i = E(\mathcal{V}_i)$. Thus $\mathcal{V}_i$ contains exactly three subspaces of dimension $2^{k_{i-1}}$, and the dimensions of the other subspaces are distinct. Hence $\mathrm{Aut}_G(E_i) \cong \Sigma_3$, and $E_i \in \widehat{\mathcal{Z}}$. Conversely, by the above analysis (and since the conjugacy class of $E \in \widehat{\mathcal{Z}}$ is determined by the dimensions of its eigenspaces), each subgroup in $\widehat{\mathcal{Z}}$ is G-conjugate to one of the E_i.

For each $1 \le i \le m$, set $P_i = C_S(E_i)$ and $Z_i = Z(P_i)$ (so $E_i = \Omega_1(Z_i)$). Since each element of $N_G(P_i) = N_G(E_i)$ permutes members of $\mathcal{V}_i$ of equal dimension, and the elements of $N_S(P_i)$ do so only within each of the V_j, we have

$$\begin{aligned} Z_i &= \bigl\{z\in G \bigm| z|_X = \lambda_X^{(z)}\mathrm{Id}_X \text{ for all } X\in\mathcal{V}_i, \text{ some } \lambda_X^{(z)}\in O_2(\mathbb{F}_q^\times)\bigr\} \\ C_{Z_i}(\mathrm{Aut}_S(P_i)) &= \bigl\{z\in Z_i \bigm| \lambda_{X_i}^{(z)} = \lambda_{X_i'}^{(z)}\bigr\} \\ C_{Z_i}(\mathrm{Aut}_G(P_i)) &= \bigl\{z\in Z_i \bigm| \lambda_{X_i}^{(z)} = \lambda_{X_i'}^{(z)} = \lambda_{V_{i-1}}^{(z)}\bigr\}, \end{aligned} \tag{4}$$

where X_i, X_i', and V_{i-1} are the three members of the decomposition $\mathcal{V}_i$ of dimension $2^{k_{i-1}}$ (and $X_i, X_i'\in\mathcal{W}_i$).

Fix $(g_i)_{i=1}^m\in\prod_{i=1}^m C_{Z_i}(\mathrm{Aut}_S(P_i))$. Then $g_i\in C_{Z_i}(\mathrm{Aut}_G(P_i))$ if and only if $\lambda_{V_{i-1}}^{(g_i)} = \lambda_{X_i}^{(g_i)}$. Choose $g\in\widehat{G}$ such that $g|_{V_i} = \eta_i\cdot\mathrm{Id}$ for each i, where the $\eta_i\in O_2(\mathbb{F}_q^\times)$ are chosen so that $\eta_i/\eta_{i-1} = \lambda_{X_i}^{(g_i)}/\lambda_{V_{i-1}}^{(g_i)}$ for each $1\le i\le m$. If G is linear, then $\det(g)=\theta^{2^{k_0}}$ for some $\theta\in O_2(\mathbb{F}_q^\times)$, and upon replacing g by $g\circ\theta^{-2^{k_0}/n}\mathrm{Id}_V$ (recall $k_0 = v_2(n)$) we can assume $g\in G$. Then $g\in C_{Z(S)}(\mathrm{Aut}_G(S))$ since it is a multiple of the identity on each V_i and has 2-power order. By construction and (4), $g\equiv g_i \pmod{C_{Z_i}(\mathrm{Aut}_G(P_i))}$ for each i; so $\mathrm{Ker}(\mu_G)=1$ by Proposition A.2(c). □

A.2. **Exceptional groups of Lie type in odd characteristic.**

Throughout this subsection, q_0 is an odd prime, and q is a power of q_0. We show that $\mathrm{Ker}(\mu_G)=1$ when G is one of the groups $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, or $E_8(q)$ and is of universal type.

The following proposition is a special case of [GLS3, Theorem 2.1.5], and is stated and proven explicitly in [O2, Proposition 8.5]. It describes, in many cases, the relationship between conjugacy classes and normalizers in a connected algebraic group and those in the subgroup fixed by a Steinberg endomorphism.

Proposition A.4. *Let $\bar{G}$ be a connected algebraic group over $\bar{\mathbb{F}}_{q_0}$, let σ be a Steinberg endomorphism of $\bar{G}$, and set $G = C_{\bar{G}}(\sigma)$. Let $H\le G$ be any subgroup, and let $\mathcal{H}$ be the set of G-conjugacy classes of subgroups $\bar{G}$-conjugate to H. Let $N_{\bar{G}}(H)$ act on $\pi_0(C_{\bar{G}}(H))$ by sending g to $xg\sigma(x)^{-1}$ (for $x\in N_{\bar{G}}(H)$). Then there is a bijection*

$$\omega\colon \mathcal{H} \xrightarrow{\ \cong\ } \pi_0(C_{\bar{G}}(H))/N_{\bar{G}}(H),$$

defined by setting $\omega([{}^xH]) = [x^{-1}\sigma(x)]$ whenever ${}^xH\le C_{\bar{G}}(\sigma)$. Also, for each $x\in\bar{G}$ such that ${}^xH\le G$, $\mathrm{Aut}_G({}^xH)$ is isomorphic to the stabilizer of $[x^{-1}\sigma(x)]\in\pi_0(C_{\bar{G}}(H))/C_{\bar{G}}(H)$ under the action of $\mathrm{Aut}_{\bar{G}}(H)$ on this set.

Since we always assume $\bar{G}$ is of universal type in this section, the group $G = C_{\bar{G}}(\sigma)$ of Proposition A.4 is equal to the group $G = O^{q_0'}(C_{\bar{G}}(\sigma))$ of Definition 2.1 and Notation 2.2.

The following definitions will be useful when applying Proposition A.4. For any finite group G, set

$$\begin{aligned} \mathcal{SE}(G) &= \bigl\{H\le G \bigm| H \text{ has a strongly 2-embedded subgroup}\bigr\} \\ \delta(G) &= \begin{cases} \min\bigl\{[G:H] \bigm| H\in\mathcal{SE}(G)\bigr\} & \text{if } \mathcal{SE}(G)\ne\varnothing \\ \infty & \text{if } \mathcal{SE}(G)=\varnothing. \end{cases} \end{aligned}$$

Thus by Proposition A.4, if $H<\bar{G}$ is such that $|\pi_0(C_{\bar{G}}(H))| > \delta(\mathrm{Out}_{\bar{G}}(H))$, then no subgroup $H^*\le C_{\bar{G}}(\sigma)$ which is $\bar{G}$-conjugate to H has the property that $\mathrm{Aut}_{C_{\bar{G}}(\sigma)}(H^*)$ has a strongly 2-embedded subgroup. The next lemma provides some tools for finding lower bounds for $\delta(G)$.

Lemma A.5. (a) *For any finite group G, $\delta(G) \ge |O_2(G)| \cdot \delta(G/O_2(G))$.*

(b) *If $G = G_1 \times G_2$ is finite, and $\delta(G_i) < \infty$ for $i = 1, 2$, then*

$$\delta(G) = \min\bigl\{\delta(G_1) \cdot \eta(G_2)\,,\ \delta(G_2) \cdot \eta(G_1)\bigr\}\,,$$

where $\eta(G_i)$ is the smallest index of any odd order subgroup of G_i.

(c) *If $\delta(G) < \infty$, and there is a faithful $\mathbb{F}_2[G]$-module V of rank n, then $2^{v_2(|G|)-[n/2]}\big|\delta(G)$.*

(d) *More concretely, $\delta(GL_3(2)) = 28$, $\delta(GL_4(2)) = 112$, $\delta(GL_5(2)) = 2^8 \cdot 7 \cdot 31$, and $\delta(SO_4^+(2)) = 2 = \delta(SO_4^-(2))$. Also, $2^4 \le \delta(SO_6^+(2)) < \infty$ and $2^6 \le \delta(SO_7(2)) < \infty$.*

Proof. **(a)** If $H \in \mathcal{SE}(G)$, then $H \cap O_2(G) = 1$ by Lemma A.1(a). Hence there is a subgroup $H^* \le G/O_2(G)$ isomorphic to H, and

$$[G : H] = |O_2(G)| \cdot [G/O_2(G) : H^*] \ge |O_2(G)| \cdot \delta(G/O_2(G))\,.$$

(b) If a finite group H has a strongly 2-embedded subgroup, then so does its direct product with any odd order group. Hence $\delta(G) \le \delta(G_i)\eta(G_{3-i})$ for $i = 1, 2$.

Assume $H \le G$ has a strongly 2-embedded subgroup $K < H$. Set $H_i = H \cap G_i$ for $i = 1, 2$. Since all involutions in H are H-conjugate (see [Sz2, 6.4.4]), H_1 and H_2 cannot both have even order. Assume $|H_2|$ is odd. Let pr_1 be projection onto the first factor. If $\mathrm{pr}_1(K) = \mathrm{pr}_1(H)$, then there is $x \in (H \smallsetminus K) \cap H_2$, and this commutes with all Sylow 2-subgroups of H since they lie in G_1, contradicting the assumption that K is strongly 2-embedded in H. Thus $\mathrm{pr}_1(K) < \mathrm{pr}_1(H)$. Then $\mathrm{pr}_1(H)$ has a strongly 2-embedded subgroup by Lemma A.1(b), and hence

$$[G : H] = [G_1 : \mathrm{pr}_1(H)] \cdot [G_2 : H_2] \ge \delta(G_1) \cdot \eta(G_2)\,.$$

So $\delta(G) \ge \delta(G_i)\eta(G_{3-i})$ for $i = 1$ or 2.

(c) This follows from [OV, Lemma 1.7(a)]: if $H < G$ has a strongly 2-embedded subgroup, $T \in \mathrm{Syl}_2(H)$, and $|T| = 2^k$, then $\dim(V) \ge 2k$.

(d) The formulas for $\delta(SO_4^{\pm}(2))$ hold since $SO_4^+(2) \cong \Sigma_3 \wr C_2$ contains a subgroup isomorphic to $C_3^2 \rtimes C_4$ and $SO_4^-(2) \cong \Sigma_5$ a subgroup isomorphic to A_5. Since $4|\delta(GL_3(2))$ by (c), and since $7|\delta(GL_3(2))$ (there are no subgroups of order 14 or 42), we have $28|\delta(GL_3(2))$, with equality since Σ_3 has index 28. The last two (very coarse) estimates follow from (c), and the 6- and 7-dimensional representations of these groups.

Fix $n = 4, 5$, and set $G_n = GL_n(2)$. Assume $H \le G_n$ has a strongly embedded subgroup, where $7\big||H|$ or $31\big||H|$. By (c), $2^4|\delta(G_4)$ and $2^8|\delta(G_5)$, and thus $8 \nmid |H|$. If H is almost simple, then $H \cong A_5$ by Bender's theorem (see [Sz2, Theorem 6.4.2]), contradicting the assumption about $|H|$. So by the main theorem in [A1], H must be contained in a member of one of the classes $\mathcal{C}_i$ ($1 \le i \le 8$) defined in that paper. One quickly checks that since $(7 \cdot 31, |H|) \ne 1$, H is contained in a member of $\mathcal{C}_1$. Thus H is reducible, and since $O_2(H) = 1$, either H is isomorphic to a subgroup of $GL_3(2) \times GL_{n-3}(2)$, or $n = 5$ and $H < GL_4(2)$. By (b) and since $7\big||\delta(GL_3(2))$, we must have $H \cong \Sigma_3 \times (C_7 \rtimes C_3)$, in which case $|H| < 180 = |GL_2(4)|$. Thus $7|\delta(G_n)$ for $n = 4, 5$, and $31|\delta(G_5)$. Since $GL_4(2)$ contains a subgroup isomorphic to $GL_2(4) \cong C_3 \times A_5$, we get $\delta(G_4) = 2^4 \cdot 7$ and $\delta(G_5) = 2^8 \cdot 7 \cdot 31$. □

We illustrate the use of the above proposition and lemma by proving the injectivity of μ_G when $G = G_2(q)$.

Proposition A.6. *If $G = G_2(q)$ for some odd prime power q, then* $\mathrm{Ker}(\mu_G) = 1$.

Proof. Assume q is a power of the prime q_0, set $\overline{G} = G_2(\overline{\mathbb{F}}_{q_0})$, and fix a maximal torus $\overline{T}$. We identify $G = C_{\overline{G}}(\psi_q)$, where ψ_q is the field automorphism, and acts via $(t \mapsto t^q)$ on $\overline{T}$. Fix $S \in \mathrm{Syl}_2(G)$, and set $\widehat{\mathcal{Z}} = \widehat{\mathcal{Z}}(\mathcal{F}_S(G))$.

Let $E \cong C_2^2$ be the 2-torsion subgroup of $\overline{T}$. By Proposition 2.5, $C_{\overline{G}}(V) = \overline{T}\langle\theta\rangle$ where $\theta \in N_{\overline{G}}(\overline{T})$ inverts the torus. Thus by Proposition A.4, there are two G-conjugacy classes of subgroups $\overline{G}$-conjugate to E, represented by $E^{\pm}$ ($E^+ = E$), where $\mathrm{Aut}_G(E^{\pm}) = \mathrm{Aut}(E^{\pm}) \cong \Sigma_3$ and $C_G(E^{\pm}) = (C_{q\mp 1})^2 \rtimes C_2$. The subgroups in one of these classes have centralizer in S isomorphic to C_2^3, hence are not in $\widehat{\mathcal{Z}}$, while those in the other class do lie in $\widehat{\mathcal{Z}}$. The latter also have normalizer of order $12(q \pm 1)^2$ and hence of odd index in G, and thus are normal in some choice of Sylow 2-subgroup.

By [Gr, Table I], for each nontoral elementary abelian 2-subgroup $E \le \overline{G}$, $\mathrm{rk}(E) = 3$, $C_{\overline{G}}(E) = E$, and $\mathrm{Aut}_{\overline{G}}(E) \cong GL_3(2)$. By Proposition A.4, and since $\delta(\mathrm{Aut}_{\overline{G}}(E)) = 28 > |C_{\overline{G}}(E)|$ by Lemma A.5, $\mathrm{Aut}_G(E)$ contains no strongly 2-embedded subgroup, and thus $E \notin \widehat{\mathcal{Z}}$.

Thus $\widehat{\mathcal{Z}}$ is contained in a unique G-conjugacy class of subgroups of rank 2, and $\mathrm{Ker}(\mu_G) = 1$ by Proposition A.2(b). □

Throughout the rest of this section, fix an odd prime power q, and let $\mathbb{G}$ be one of the groups F_4, E_6, E_7, or E_8.

Hypotheses A.7. *Assume $\overline{G} = \mathbb{G}(\overline{\mathbb{F}}_{q_0})$ and $G \cong \mathbb{G}(q)$, where q is a power of the odd prime q_0, and where $\mathbb{G} = F_4$, E_6, E_7, or E_8 and is of universal type. Fix a maximal torus $\overline{T} < \overline{G}$.*

(I) *Set $T_{(2)} = \{t \in \overline{T} \mid t^2 = 1\}$. Let* **2A** *and* **2B** *denote the two $\overline{G}$-conjugacy classes of noncentral involutions in $\overline{G}$, as defined in* [Gr, Table VI], except *that when $\mathbb{G} = E_7$, they denote the classes labelled* **2B** *and* **2C**, *respectively, in that table. For each elementary abelian 2-subgroup $E < \overline{G}$, define*

$$\mathfrak{q}_E \colon E \longrightarrow \mathbb{F}_2$$

by setting $\mathfrak{q}(x) = 0$ if $x \in \mathbf{2B} \cup \{1\}$, and $\mathfrak{q}(x) = 1$ if $x \in \mathbf{2A} \cup (Z(\overline{G}) \smallsetminus 1)$.

(II) *Assume $G = C_{\overline{G}}(\psi_q)$, where ψ_q is the field endomorphism with respect to some root structure with maximal torus $\overline{T}$. Thus $\psi_q(t) = t^q$ for all $t \in \overline{T}$. Fix $S \in \mathrm{Syl}_2(G)$, and set $\widehat{\mathcal{Z}} = \widehat{\mathcal{Z}}(\mathcal{F}_S(G))$.*

By [Gr, Lemma 2.16], $\mathfrak{q}_{T_{(2)}}$ is a quadratic form on $T_{(2)}$ in all cases, and hence $\mathfrak{q}_E$ is quadratic for each $E \le T_{(2)}$. In general, $\mathfrak{q}_E$ need not be quadratic when E is not contained in a maximal torus. In fact, Griess showed in [Gr, Theorems 7.3, 8.2, & 9.2] that in many (but not all) cases, E is contained in a torus if and only if $\mathfrak{q}_E$ is quadratic ($\mathrm{cx}(E) \le 2$ in his terminology).

With the above choices of notation for noncentral involutions, all of the inclusions $F_4 \le E_6 \le E_7 \le E_8$ restrict to inclusions of the classes **2A** and of the classes **2B**. This follows since the forms are quadratic, and also (for $E_7 < E_8$) from [Gr, Lemma 2.16(iv)].

Lemma A.8. *Assume Hypotheses A.7, and let $\mathfrak{b}$ be the bilinear form associated to $\mathfrak{q}$. Define*

$$V_0 = \{v \in T_{(2)} \mid \mathfrak{b}(v, T_{(2)}) = 0,\ \mathfrak{q}(v) = 0\}$$
$$\gamma_x = \bigl(v \mapsto v + \mathfrak{b}(v,x)x\bigr) \in \mathrm{Aut}(T_{(2)}, \mathfrak{q}) \qquad \text{for } x \in T_{(2)} \text{ with } \mathfrak{q}(x) = 1,\ q \not\perp T_{(2)}$$

Then the following hold.

(a) $\mathrm{Aut}_{\bar{G}}(T_{(2)}) = \mathrm{Aut}(T_{(2)}, \mathfrak{q})$.

(b) *For each nonisotropic $x \in T_{(2)} \smallsetminus T_{(2)}^{\perp}$, γ_x is the restriction to $T_{(2)}$ of a Weyl reflection on $\bar{T}$. If $\alpha \in \Sigma$ is such that $\gamma_x = w_\alpha|_{T_{(2)}}$, then $\theta_\alpha(v) = (-1)^{\mathfrak{b}(x,v)}$ for each $v \in T_{(2)}$.*

(c) *If $\mathbb{G} = E_r$ ($r = 6, 7, 8$), then $\mathfrak{q}$ is nondegenerate ($V_0 = 0$), and the restriction to $T_{(2)}$ of each Weyl reflection is equal to γ_x for some nonisotropic $x \in T_{(2)} \smallsetminus T_{(2)}^{\perp}$.*

(d) *If $\mathbb{G} = F_4$, then $\dim(V_0) = 2$, and $\mathfrak{q}(v) = 1$ for all $v \in T_{(2)} \smallsetminus V_0$.*

Proof. **(a)** Since $\mathrm{Aut}_{\bar{G}}(T_{(2)})$ has to preserve $\bar{G}$-conjugacy classes, it is contained in $\mathrm{Aut}(T_{(2)}, \mathfrak{q})$. Equality will be shown while proving (c) and (d).

(c) If $\mathbb{G} = E_r$ for $r = 6, 7, 8$, then $\mathfrak{q}$ is nondegenerate by [Gr, Lemma 2.16]. Hence the only orthogonal transvections are of the form γ_x for nonisotropic x, and each Weyl reflection restricts to one of them. By a direct count (using the tables in [Brb]), the number of pairs $\{\pm\alpha\}$ of roots in $\mathbb{G}$ (hence the number of Weyl reflections) is equal to 36, 63, or 120, respectively. This is equal to the number of nonisotropic elements in $T_{(2)} \smallsetminus T_{(2)}^{\perp} = T_{(2)} \smallsetminus Z(\bar{G})$ (see the formula in [Ta, Theorem 11.5] for the number of isotropic elements). So all transvections are restrictions of Weyl reflections, and $\mathrm{Aut}_{\bar{G}}(T_{(2)}) = \mathrm{Aut}(T_{(2)}, \mathfrak{q})$.

(d) Assume $\mathbb{G} = F_4$. Then $\dim(V_0) = 2$ and $\mathfrak{q}^{-1}(1) = T_{(2)} \smallsetminus V_0$ by [Gr, Lemma 2.16]. Thus $|\mathrm{Aut}(T_{(2)}, \mathfrak{q})| = 4^2 \cdot |GL_2(2)|^2 = 2^6 \cdot 3^2 = \frac{1}{2}|W|$ (see [Brb, Planche VIII]), so $\mathrm{Aut}_W(T_{(2)}) = \mathrm{Aut}(T_{(2)}, \mathfrak{q})$ since W also contains $-\mathrm{Id}$.

There are three conjugacy classes of transvections $\gamma \in \mathrm{Aut}(T_{(2)}, \mathfrak{q})$: one of order 36 containing those where $\gamma|_{V_0} \neq \mathrm{Id}$ (and hence $[\gamma, T_{(2)}] \leq V_0$), and two of order 12 containing those where $\gamma|_{V_0} = \mathrm{Id}$ (one where $[\gamma, T_{(2)}] \leq V_0$ and one where $[\gamma, T_{(2)}] \not\leq V_0$). Since there are two W-orbits of roots (long and short), each containing 12 pairs $\pm\alpha$, the corresponding Weyl reflections must restrict to the last two classes of transvections, of which one is the set of all γ_x for $x \in T_{(2)} \smallsetminus V_0$.

(b) We showed in the proofs of (c) and (d) that each orthogonal transvection γ_x is the restriction of a Weyl reflection. If $\gamma_x = w_\alpha|_{T_{(2)}}$ for some root $\alpha \in \Sigma$, then $\theta_\alpha \in \mathrm{Hom}(\bar{T}, \bar{\mathbb{F}}_{q_0}^\times)$ (Lemma 2.4(c)), so $[T_{(2)} : \mathrm{Ker}(\theta_\alpha|_{T_{(2)}})] \leq 2$. Also, $\mathrm{Ker}(\theta_\alpha) \leq C_{\bar{T}}(w_\alpha)$ by Lemma 2.4(e), so $\mathrm{Ker}(\theta_\alpha|_{T_{(2)}}) \leq C_{T_{(2)}}(w_\alpha) = C_{T_{(2)}}(\gamma_x) = x^\perp$, with equality since $[T_{(2)} : x^\perp] = 2$. Since $\theta_\alpha(T_{(2)}) \leq \{\pm 1\}$, it follows that $\theta_\alpha(v) = (-1)^{\mathfrak{b}(x,v)}$ for each $v \in T_{(2)}$. □

We are now ready to list the subgroups in $\widehat{\mathscr{Z}}(\mathbb{G}(q))$ in all cases. The proof of the following lemma will be given at the end of the section.

Lemma A.9. *Let $\bar{G} = \mathbb{G}(\bar{\mathbb{F}}_{q_0})$ and $G = \mathbb{G}(q)$ be as in Hypotheses A.7. Assume $E \in \widehat{\mathscr{Z}}(G)$. Then either $\mathbb{G} \neq E_7$, $\mathrm{rk}(E) = 2$, and $\mathfrak{q}_E = 0$; or $\mathbb{G} = E_7$, $Z = Z(\bar{G}) \cong C_2$, and $E = Z \times E_0$ where $\mathrm{rk}(E_0) = 2$ and $\mathfrak{q}_{E_0} = 0$. In all cases, $\mathrm{Aut}_{\bar{G}}(E) \cong \Sigma_3$.*

Proof. This will be shown in Lemmas A.14 and A.15. □

The next two lemmas will be needed to apply Proposition A.2(b) to these groups. The first is very elementary.

Lemma A.10. *Let V be an $\mathbb{F}_2$-vector space of dimension k, and let $\mathfrak{q}\colon V \longrightarrow \mathbb{F}_2$ be a quadratic form on V. For $m \geq 1$ such that $k > 2m$, the number of totally isotropic subspaces of dimension m in V is odd.*

Proof. This will be shown by induction on m, starting with the case $m = 1$. Since $k \geq 3$, there is an orthogonal splitting $V = V_1 \perp V_2$ where $V_1, V_2 \neq 0$. Let k_i be the number of isotropic elements in V_i (including 0), and set $n_i = |V_i|$. The number of isotropic elements in V is then $k_1k_2 + (n_1 - k_1)(n_2 - k_2)$, and is even since the n_i are even. The number of 1-dimensional isotropic subspaces is thus odd.

Now fix $m > 1$ (such that $k > 2m$), and assume the lemma holds for subspaces of dimension $m - 1$. For each isotropic element $x \in V$, a subspace $E \leq V$ of dimension m containing x is totally isotropic if and only if $E \leq x^\perp$ *and* $E/\langle x\rangle$ is isotropic in $x^\perp/\langle x\rangle$ with the induced quadratic form. By the induction hypothesis, and since

$$2 \cdot \dim(E/\langle x\rangle) = 2(m - 1) < k - 2 \leq \dim(x^\perp/\langle x\rangle),$$

the number of isotropic subspaces of dimension m which contain x is odd. Upon taking the sum over all x, and noting that each subspace has been counted $2^m - 1$ times, we see that the number of isotropic subspaces of dimension m is odd. □

Lemma A.11. *Assume Hypotheses A.7(I). Let σ be a Steinberg endomorphism of $\overline{G}$ such that for some $\varepsilon = \pm 1$, $\sigma(t) = t^{\varepsilon q}$ for each $t \in \overline{T}$. Set $G = C_{\overline{G}}(\sigma)$. Fix $E \leq T_{(2)}$ of rank 2 such that $\mathfrak{q}(E) = 0$. Then the set of subgroups of G which are $\overline{G}$-conjugate to E, and the set of subgroups which are G-conjugate to E, both have odd order and contain all totally isotropic subgroups of rank 2 in $T_{(2)}$.*

Proof. Let $\overline{\mathfrak{X}} \supseteq \mathfrak{X}$ be the sets of subgroups of G which are $\overline{G}$-conjugate to E or G-conjugate to E, respectively. Let $\mathfrak{X}_0$ be the subset of all totally isotropic subgroups of $T_{(2)}$ of rank 2. If $\mathfrak{q}$ is nondegenerate, then by Witt's theorem (see [Ta, Theorem 7.4]), $\mathrm{Aut}_W(T_{(2)}) = \mathrm{Aut}(T_{(2)}, \mathfrak{q})$ permutes $\mathfrak{X}_0$ transitively, and hence all elements in $\mathfrak{X}_0$ are G-conjugate to E by Lemma 2.9. If in addition, $\dim(T_{(2)}) \geq 5$, then $|\mathfrak{X}_0|$ is odd by Lemma A.10. Otherwise, by Lemma A.8(c,d), $\mathbb{G} = F_4$ and $\mathfrak{X}_0 = \{E\}$. Thus in all cases, $\mathfrak{X}_0 \subseteq \mathfrak{X}$ and $|\mathfrak{X}_0|$ is odd.

Assume $\mathbb{G} = E_6$. Then $C_{\overline{G}}(T_{(2)}) = \overline{T}$ by Proposition 2.5. Consider the conjugation action of $T_{(2)}$ on $\overline{\mathfrak{X}}$, and let $\mathfrak{X}_1$ be its fixed point set. Since $T_{(2)} \leq G$ by the assumptions on σ, this action also normalizes $\mathfrak{X}$. For $F \in \mathfrak{X}_1$, either the action of $T_{(2)}$ fixes F pointwise, in which case $F \in \mathfrak{X}_0$, or there are $x, y \in F$ such that $[x, T_{(2)}] = 1$ and $[y, T_{(2)}] = \langle x\rangle$. In particular, $c_y \in \mathrm{Aut}_{\overline{G}}(T_{(2)}) = SO(T_{(2)}, \mathfrak{q})$. For each $v \in T_{(2)}$ such that $[y, v] = x$, $\mathfrak{q}(v) = \mathfrak{q}(vx)$ and $\mathfrak{q}(x) = 0$ imply $x \perp v$, so $x \perp T_{(2)}$ since $T_{(2)}$ is generated by those elements. This is impossible since $\mathfrak{q}$ is nondegenerate by Lemma A.8(c), and thus $\mathfrak{X}_1 = \mathfrak{X}_0$.

Now assume $\mathbb{G} = F_4$, E_7, or E_8. Then $-\mathrm{Id} \in W$, so there is $\theta \in N_{\overline{G}}(\overline{T})$ which inverts $\overline{T}$. Then $C_{\overline{G}}(T_{(2)}) = \overline{T}\langle\theta\rangle$. By the Lang-Steinberg theorem, there is $g \in \overline{G}$ such that $g^{-1}\sigma(g) \in \theta\overline{T}$; then $\sigma(gtg^{-1}) = gt^{\mp q}g^{-1}$ for $t \in \overline{T}$, and thus σ acts on $g\overline{T}g^{-1}$ via $t \mapsto t^{\mp q}$. We can thus assume $\overline{T}$ was chosen so that $G \cap \overline{T} = C_{\overline{T}}(\sigma)$ contains the 4-torsion subgroup $\overline{T}_{(4)} \leq \overline{T}$. Let $\mathfrak{X}_1 \subseteq \overline{\mathfrak{X}}$ be the fixed point set of the conjugation action of $\overline{T}_{(4)}$ on $\overline{\mathfrak{X}}$. For $F \in \mathfrak{X}_1$, either the action of $\overline{T}_{(4)}$ fixes F pointwise, in which case $F \in \mathfrak{X}_0$, or there are $x, y \in F$ such that $[x, \overline{T}_{(4)}] = 1$ and $[y, \overline{T}_{(4)}] = \langle x\rangle$. But then $[F, \overline{T}^*_{(4)}] = 1$ for some

$\bar{T}^*_{(4)} < \bar{T}_{(4)}$ of index two, $[F, T_{(2)}] = 1$ implies $F \le T_{(2)}\langle\theta\rangle$; and $F \le T_{(2)}$ since no element in $\bar{T}_{(4)} \smallsetminus T_{(2)}$ commutes with any element of $T_{(2)}\theta$. So $\mathfrak{X}_1 = \mathfrak{X}_0$ in this case.

Thus in both cases, $\mathfrak{X}_0$ is the fixed point set of an action of a 2-group on $\bar{\mathfrak{X}}$ which normalizes $\mathfrak{X}$. Since $|\mathfrak{X}_0|$ is odd, so are $|\bar{\mathfrak{X}}|$ and $|\mathfrak{X}|$. □

We are now ready to prove:

Proposition A.12. *Fix an odd prime power q. Assume G is a quasisimple group of universal type isomorphic to $G_2(q)$, $F_4(q)$, $E_6(q)$, $E_7(q)$, or $E_8(q)$. Then* $\mathrm{Ker}(\mu_G) = 1$.

Proof. This holds when $G \cong G_2(q)$ by Proposition A.6, so we can assume Hypotheses A.7. Let $\mathfrak{X}$ be the set of all elementary abelian 2-subgroups $E \le G$ such that either $\mathbb{G} \ne E_7$, $\mathrm{rk}(E) = 2$, and $\mathfrak{q}_E = 0$; or $\mathbb{G} = E_7$, $\mathrm{rk}(E) = 3$, and $E = Z(G) \times E_0$ where $\mathfrak{q}_{E_0} = 0$. By Lemma A.11, $|\mathfrak{X}|$ is odd. In all cases, by Lemma A.9, $\widehat{\mathscr{Z}}(G) \subseteq \mathfrak{X}$. By Proposition A.2(a,b), to prove μ_G is injective, it remains to show that if $\widehat{\mathscr{Z}}(G) \ne \varnothing$, then $\widehat{\mathscr{Z}}(G)$ has odd order and is contained in a single G-conjugacy class, and $\mathrm{Aut}_G(Z(S)) = 1$.

Fix $E \in \mathfrak{X}$ such that $E \le T_{(2)}$. We first claim that if $\mathbb{G} = F_4$, E_6, or E_7, then $C_{\bar{G}}(E)$ is connected, and hence all elements in $\mathfrak{X}$ are G-conjugate to E by Proposition A.4. If $\mathbb{G} = E_7$, then $C_{\bar{G}}(E)$ is connected by [Gr, Proposition 9.5(iii)(a)]. If $\mathbb{G} = F_4$ or E_6, then for $x \in E$, $C_{\bar{G}}(x) \cong \mathrm{Spin}_9(\bar{\mathbb{F}}_{q_0})$ or $\bar{\mathbb{F}}_{q_0} \times_{C_4} \mathrm{Spin}_{10}(\bar{\mathbb{F}}_{q_0})$, respectively (see [Gr, Table VI]). Since the centralizer of each element in the simply connected groups $\mathrm{Spin}_9(\bar{\mathbb{F}}_{q_0})$ and $\mathrm{Spin}_{10}(\bar{\mathbb{F}}_{q_0})$ is connected [St3, Theorem 8.1], $C_{\bar{G}}(E)$ is connected in these cases.

Now assume $\mathbb{G} = E_8$. We can assume $G = C_{\bar{G}}(\psi_q)$, where ψ_q is the field automorphism; in particular, $\psi_q(t) = t^q$ for $t \in \bar{T}$. Fix $x, y \in E$ such that $E = \langle x, y\rangle$. By [Gr, Lemma 2.16(ii)], $(T_{(2)}, \mathfrak{q})$ is of positive type (has a 4-dimensional totally isotropic subspace). Hence $E^\perp = E \times V_1 \times V_2$, where $\dim(V_i) = 2$ and $\mathfrak{q}(V_i \smallsetminus 1) = 1$ for $i = 1, 2$, and $V_1 \perp V_2$. Thus $(\mathfrak{q}_{E^\perp})^{-1}(1) = \bigcup_{i=1}^2 \big((V_i \smallsetminus 1) \times E\big)$, and by Lemma A.8(b,c), these are the restrictions to $T_{(2)}$ of Weyl reflections w_α for $\alpha \in \Sigma$ such that $E \le \mathrm{Ker}(\theta_\alpha)$. Also, $C_W(E) \cong W(D_4) \wr C_2$. By Proposition 2.5, $C_{\bar{G}}(E)^0$ has type $D_4 \times D_4$ and $|\pi_0(C_{\bar{G}}(E))| = 2$. More precisely, $C_{\bar{G}}(E) = (\bar{H}_1 \times_E \bar{H}_2)\langle\delta\rangle$, where $\bar{H}_i \cong \mathrm{Spin}_8(\bar{\mathbb{F}}_{q_0})$ and $Z(\bar{H}_i) = E$ for $i = 1, 2$, and conjugation by $\delta \in N_{\bar{G}}(\bar{T})$ exchanges V_1 and V_2 and hence exchanges $\bar{H}_1$ and $\bar{H}_2$.

By Proposition A.4, the two connected components in the centralizer give rise to two G-conjugacy classes of subgroups which are $\bar{G}$-conjugate to E, represented by E and gEg^{-1} where $g^{-1}\sigma(g)$ lies in the nonidentity component of $C_{\bar{G}}(E)$. Then $C_G(E)$ contains a subgroup $\mathrm{Spin}_8^+(q) \times_{C_2^2} \mathrm{Spin}_8^+(q)$ with index 8 (the extension by certain pairs of diagonal automorphisms of the $\mathrm{Spin}_8^+(q)$-factors, as well as an automorphism which switches the factors). So $E = Z(T)$ for $T \in \mathrm{Syl}_2(C_G(E))$, and $E \in \widehat{\mathscr{Z}}(G)$. Also, $gyg^{-1} \in C_G(gEg^{-1})$ if and only if $y \in C_{\bar{G}}(E)$ and $\widetilde{\tau}(y) = y$ where $\widetilde{\tau} = c_{g^{-1}\sigma(g)} \circ \sigma$. Then $\widetilde{\tau}$ switches the central factors in $C_{\bar{G}}(E)$, and the group $C_{C_{\bar{G}}(E)}(\widetilde{\tau})$ splits as a product of E times the group of elements which are invariant after lifting $\widetilde{\tau}$ to the 4-fold cover $\mathrm{Spin}_8(\bar{\mathbb{F}}_{q_0}) \wr C_2$. Since gEg^{-1} intersects trivially with the commutator subgroup of $C_G(gEg^{-1})$, $\Omega_1(Z(T)) > gEg^{-1}$ for any $T \in \mathrm{Syl}_2(C_G(gEg^{-1}))$ (since $Z(T) \cap [T, T] \ne 1$); and thus $gEg^{-1} \notin \widehat{\mathscr{Z}}(G)$. Thus $\widehat{\mathscr{Z}}(G)$ is the G-conjugacy class of E, and has odd order by Lemma A.11.

Thus, in all cases, if $\widehat{\mathscr{Z}}(G)$ is nonempty, it has odd order and is contained in one G-conjugacy class. Also, $Z(S) \le C_E(\mathrm{Aut}_S(E)) < E$ for $E \in \widehat{\mathscr{Z}}(G)$, so either $|Z(S)| = 2$, or

$\mathbb{G} = E_7$, $Z(S) \cong C_2^2$, and the three involutions in $Z(S)$ belong to three different $\overline{G}$-conjugacy classes. Hence $\mathrm{Aut}_G(Z(S)) = 1$. □

It remains to prove Lemma A.9, which is split into the two Lemmas A.14 and A.15. The next proposition will be used to show that certain elementary abelian subgroups are not in $\widehat{\mathcal{Z}}$.

Proposition A.13. *Assume Hypotheses A.7. Let $E \le T_{(2)}$ and $x \in T_{(2)} \smallsetminus E$ be such that the orbit of x under the $C_W(E)$-action on $T_{(2)}$ has odd order. Then no subgroup of S which is $\overline{G}$-conjugate to E is in $\widehat{\mathcal{Z}}$. More generally, if $\overline{E} \ge E$ is also elementary abelian, and is such that x is not $C_{\overline{G}}(E)$-conjugate to any element of $\overline{E}$, then for any $L \trianglelefteq G$ which contains $\{gxg^{-1} \mid g \in \overline{G}\} \cap G$, no subgroup of S which is $\overline{G}$-conjugate to $\overline{E}$ is in $\widehat{\mathcal{Z}}$.*

Proof. In [O2], an elementary abelian p-subgroup $E < G$ is called *pivotal* if $O_p(\mathrm{Aut}_G(E)) = 1$, and $E = \Omega_1(Z(P))$ for some $P \in \mathrm{Syl}_p(C_G(E))$. In particular, by Lemma A.1(a), the subgroups in $\widehat{\mathcal{Z}}$ are all pivotal. Note that $T_{(2)} \le G$ by Hypotheses A.7. By [O2, Proposition 8.9], no subgroup satisfying the above conditions can be pivotal, and hence they cannot be in $\widehat{\mathcal{Z}}$. □

In the next two lemmas, we show that in all cases, $E \in \widehat{\mathcal{Z}}$ implies $\mathrm{rk}(E) = 2$ and $\mathfrak{q}_E = 0$ if $\mathbb{G} \ne E_7$, with a similar result when $\mathbb{G} = E_7$. We first handle those subgroups which are *toral* (contained in a maximal torus in $\overline{G}$), and then those which are not toral. By a $\mathbf{2A}^k$-subgroup or subgroup of type $\mathbf{2A}^k$ ($\mathbf{2B}^k$-subgroup or subgroup of type $\mathbf{2B}^k$) is meant an elementary abelian 2-subgroup of rank k all of whose nonidentity elements are in class $\mathbf{2A}$ (class $\mathbf{2B}$).

Lemma A.14. *Assume Hypotheses A.7. Fix some $E \in \widehat{\mathcal{Z}}$ which is contained in a maximal torus of $\overline{G}$. Then either $\mathbb{G} \ne E_7$, $\mathrm{rk}(E) = 2$, and $\mathfrak{q}_E = 0$; or $\mathbb{G} = E_7$, $Z = Z(\overline{G}) \cong C_2$, and $E = Z \times E_0$ where $\mathrm{rk}(E_0) = 2$ and $\mathfrak{q}_{E_0} = 0$. In all cases, $\mathrm{Aut}_{\overline{G}}(E) \cong \Sigma_3$.*

Proof. Set $Z = O_2(Z(\overline{G})) \le T_{(2)}$. Thus $|Z| = 2$ if $\mathbb{G} = E_7$, and $|Z| = 1$ otherwise. Recall that $\mathrm{Aut}_G(T_{(2)}) = \mathrm{Aut}_{\overline{G}}(T_{(2)}) = \mathrm{Aut}(T_{(2)}, \mathfrak{q})$ by Lemmas 2.9 and A.8(a).

The following notation will be used to denote isomorphism types of quadratic forms over $\mathbb{F}_2$. Let $[\boldsymbol{n}]^{\pm}$ denote the isomorphism class of a nondegenerate form of rank n. When n is even, $[\boldsymbol{n}]^{+}$ denotes the hyperbolic form (with maximal Witt index), and $[\boldsymbol{n}]^{-}$ the form with nonmaximal Witt index. Finally, a subscript "$(\boldsymbol{k})$" denotes sum with a k-dimensional trivial form. By [Gr, Lemma 2.16], $\mathfrak{q}_{T_{(2)}}$ has type $[\mathbf{2}]^{-}_{(\mathbf{2})}$, $[\mathbf{6}]^{-}$, $[\mathbf{7}]$, or $[\mathbf{8}]^{+}$ when $\mathbb{G} = F_4$, E_6, E_7, or E_8, respectively.

Fix $E \le T_{(2)}$; we want to determine whether E can be $\overline{G}$-conjugate to an element of $\widehat{\mathcal{Z}}$. Set $E_1 = E \cap E^{\perp}$ (the orthogonal complement taken with respect to $\mathfrak{q}$), and set $E_0 = \mathrm{Ker}(\mathfrak{q}_{E_1})$. Note that $E_1 > E_0$ if $\mathbb{G} = E_7$ ($E \ge Z$).

Assume first that $E_0 = 1$. If $\mathbb{G} = F_4$, then $T_{(2)} \cap \mathbf{2B}$ is a $C_W(E)$-orbit of odd order. If $\mathbb{G} = E_r$ and $E_1 = 1$, then $E \times E^{\perp}$, $E^{\perp}$ is $C_W(E)$-invariant, and hence there is $1 \ne x \in E^{\perp}$ whose $C_W(E)$-orbit has odd order. If $\mathbb{G} = E_r$ and $\mathrm{rk}(E_1) = 1$, then $E \cap E^{\perp} = E_1$, there is an odd number of involutions in $E^{\perp} \smallsetminus E_1$ of each type (isotropic or not), and again there is $1 \ne x \in E^{\perp}$ whose $C_W(E)$-orbit has odd order. In all cases, x has the property that $C_W(\langle E, x \rangle)$ has odd index in $C_W(E)$. So by Proposition A.13, no subgroup of G which is $\overline{G}$-conjugate to E can be in $\widehat{\mathcal{Z}}$.

Thus $E_0 \neq 1$. Set $k = \mathrm{rk}(E_0)$. Then

$$\begin{aligned}\bigl|\pi_0(C_{\overline{G}}(E))\bigr| &= \bigl|C_W(E)/\langle w_\alpha \,|\, \alpha\in\Sigma,\ E\le \mathrm{Ker}(\theta(\alpha))\rangle\bigr| && \text{(Proposition 2.5)}\\ &\le \bigl|C_W(T_{(2)})\bigr|\cdot\bigl|C_{SO(T_{(2)},\mathfrak{q})}(E)/\langle \gamma_v \,|\, v\in \mathbf{2A}\cap E^\perp\rangle\bigr| && \text{(Lemma A.8(a,b))}\\ &\le \bigl|C_W(T_{(2)})\bigr|\cdot\bigl|C_{SO(T_{(2)},\mathfrak{q})}(E_0^\perp)\bigr|\cdot\bigl|C_{SO(E_0^\perp,\mathfrak{q})}(E)/\langle \gamma_v \,|\, v\in \mathbf{2A}\cap E^\perp\rangle\bigr|. && (5)\end{aligned}$$

The first factor is easily described:

$$\bigl|C_W(T_{(2)})\bigr| = 2^\varepsilon \qquad \text{where} \quad \varepsilon = \begin{cases}1 & \text{if } -\mathrm{Id}\in W \ (\text{if } \mathbb{G}=F_4,\ E_7,\ E_8)\\ 0 & \text{if } -\mathrm{Id}\notin W \ (\text{if } \mathbb{G}=E_6).\end{cases} \tag{6}$$

We next claim that

$$\bigl|C_{SO(T_{(2)},\mathfrak{q})}(E_0^\perp)\bigr| \le 2^{\binom{k}{2}}, \tag{7}$$

with equality except possibly when $\mathbb{G} = F_4$. To see this, let $F_1 < T_{(2)}$ be a subspace complementary to $E_0^\perp$. Each $\alpha \in C_{\mathrm{Aut}(T_{(2)})}(E_0^\perp)$ has the form $\alpha(x) = x\psi(x)$ for some $\psi\in \mathrm{Hom}(F_1, E_0)$, and α is orthogonal if and only if $x \perp \psi(x)$ for each x. The space of such homomorphisms has dimension at most $\binom{k}{2}$ (corresponding to symmetric $k\times k$ matrices with zeros on the diagonal); with dimension equal to $\binom{k}{2}$ if $\dim(F_1) = \dim(E_0)$ (which occurs if $\mathfrak{q}$ is nondegenerate).

Write $(E_0)^\perp = E \times F_2$, where $E^\perp = E_0 \times F_2$ and the form $\mathfrak{q}_{F_2}$ is nondegenerate. By [Ta, Theorem 11.41], $SO(F_2, \mathfrak{q}_{F_2})$ is generated by transvections unless $\mathfrak{q}_{F_2}$ is of type $[\mathbf{4}]^+$, in which case the reflections generate a subgroup of $SO(F_2, \mathfrak{q}_{F_2}) \cong \Sigma_3 \wr C_2$ isomorphic to $\Sigma_3\times\Sigma_3$. Also, F_2 is generated by nonisotropic elements except when $\mathfrak{q}_{F_2}$ is of type $[\mathbf{2}]^+$, and when this is the case, all automorphisms of $(E_0)^\perp$ which induce the identity on E and on $(E_0)^\perp/E_0$ are composites of transvections. (Look at the composites $\gamma_{vx}\circ\gamma_v$ for $v\in F_2$ and $x\in E_0$.) Hence

$$\bigl|C_{SO(E_0^\perp,\mathfrak{q})}(E)/\langle \gamma_v \,|\, v\in \mathbf{2A}\cap E^\perp\rangle\bigr| \le 2^\eta$$

where $\eta = 1$ if $\mathfrak{q}_{E^\perp}$ has type $[\mathbf{4}]^+_{(\boldsymbol{k})}$, $\eta = k$ if $\mathfrak{q}_{E^\perp}$ has type $[\mathbf{2}]^+_{(\boldsymbol{k})}$, and $\eta = 0$ otherwise. Together with (5), (6), and (7), this proves that

$$|\pi_0(C_{\overline{G}}(E))| \le 2^{\binom{k}{2}+\varepsilon+\eta} \qquad\qquad \text{where } \varepsilon\le 1. \tag{8}$$

Now, $N_{\overline{G}}(E) \le C_{\overline{G}}(E)^0 N_{\overline{G}}(\overline{T})$ by the Frattini argument: each maximal torus which contains E lies in $C_{\overline{G}}(E)^0$ and hence is $C_{\overline{G}}(E)^0$-conjugate to $\overline{T}$. So each element of $\mathrm{Aut}_{\overline{G}}(E)$ is represented by a coset of $\overline{T}$ in $N_{\overline{G}}(\overline{T})$, and can be chosen to lie in G by Lemma 2.9. Thus the action described in Proposition A.4 which determines the automizers $\mathrm{Aut}_G(E^*)$ for E^* $\overline{G}$-conjugate to E is the conjugation action of $\mathrm{Aut}_{\overline{G}}(E)$ on the set of conjugacy classes in $\pi_0(C_{\overline{G}}(E))$. In particular, this action is not transitive, since the identity is fixed.

Set $\ell = \mathrm{rk}(E/E_0) - 1$ if $\mathbb{G} = E_7$ and $\ell = \mathrm{rk}(E/E_0)$ otherwise. Every automorphism of E which induces the identity on E_0Z and on E/E_0 is orthogonal, and hence the restriction of an element of $O_2(C_W(E))$. Thus $|O_2(\mathrm{Out}_{\overline{G}}(E))| \ge 2^{k\ell}$. If $E^*\in\mathcal{Z}$ is $\overline{G}$-conjugate to E, then since $\mathrm{Aut}_G(E^*)$ has a strongly 2-embedded subgroup, $2^{k\ell} \le \delta(\mathrm{Aut}_{\overline{G}}(E)) < \bigl|\pi_0(C_{\overline{G}}(E))\bigr|$ by Proposition A.4 and Lemma A.5(a), with strict inequality since the action of $N_{\overline{G}}(E)$ on $\pi_0(C_{\overline{G}}(E))$ is not transitive. Together with (8), and since $\varepsilon\le 1$, this implies that $k\ell \le \binom{k}{2}+\eta \le \binom{k}{2}+k$. Thus $\ell\le \frac{k+1}{2}$, and $\ell \le \frac{k-1}{2}$ if $\eta = 0$. By definition, $\eta = 0$ whenever $\mathrm{rk}(E_1/E_0) = 1$, which is the case if $\mathbb{G} = E_7$ or ℓ is odd. Since $2k+\ell\le 8$, we are thus left with the following possibilities.

- If $(k,\ell)=(3,2)$, then $\mathbb{G}=E_8$, E has form of type $[\mathbf{2}]^+_{(\mathbf{3})}$, so $E^\perp=E_0$ has trivial form, and $\eta=0$. Thus $k\ell\not\leq\binom{k}{2}+\eta$, so this cannot occur.
- If $(k,\ell)=(3,1)$, then $\mathbb{G}=E_8$, E has form of type $*+3$, and $\mathrm{rk}(E)=\mathrm{rk}(E_1)=4$. Then $\mathrm{Aut}_{\overline{G}}(E)\cong C_2^3\rtimes GL_3(2)$, so $\delta(\mathrm{Aut}_{\overline{G}}(E))\geq 2^3\cdot 28$ by Lemma A.5(a,d). Since $|\pi_0(C_{\overline{G}}(E))|\leq 16$, this case is also impossible.
- If $(k,\ell)=(4,0)$, then $\mathbb{G}=E_8$ and $E=E_0$ is isotropic of rank 4. By Proposition 2.5 and Lemma A.8(c), $C_{\overline{G}}(E)^0=\overline{T}$. By [CG, Proposition 3.8(ii)], $\pi_0(C_{\overline{G}}(E))$ is extraspecial of order 2^7 and $\mathrm{Aut}_{\overline{G}}(E)\cong GL_4(2)$. (This is stated for subgroups of $E_8(\mathbb{C})$, but the same argument applies in our situation.) In particular, $\pi_0(C_{\overline{G}}(E))$ has just 65 conjugacy classes. Since $\delta(GL_4(2))=112$ by Lemma A.5(d), Proposition A.4 implies that $\mathrm{Aut}_G(E^*)$ cannot have a strongly 2-embedded subgroup.
- If $(k,\ell)=(3,0)$, then $E=Z\times E_0$ where $\dim(E_0)=3$, and $\mathrm{Aut}_{\overline{G}}(E)\cong GL_3(2)$. If $\mathbb{G}=E_6$ or E_7, then $E^\perp=E$, and $|\pi_0(C_{\overline{G}}(E))|\leq 16$ by (8).

  If $\mathbb{G}=E_8$, then $(E^\perp,\mathfrak{q}_{E^\perp})$ has type $[\mathbf{2}]^+_{(\mathbf{3})}$. By the arguments used to prove (8),
$$|C_W(E)|=|C_W(T_{(2)})|\cdot|C_{SO(T_{(2)},\mathfrak{q})}(E_0^\perp)|\cdot|C_{SO(E_0^\perp,\mathfrak{q})}(E)|=2\cdot 2^3\cdot 2^7=2^{11}.$$
  Also, $E^\perp$ contains exactly 8 nonisotropic elements, they are pairwise orthogonal, and hence determine 8 pairwise commuting transvections on $T_{(2)}$. These extend to 8 Weyl reflections which are pairwise commuting since no two can generate a dihedral subgroup of order 8 (this would imply two roots of different lengths). Hence by Proposition 2.5, $C_{\overline{G}}(E)^0$ has type $(A_1)^8$ and $|\pi_0(C_{\overline{G}}(E))|=2^{11}/2^8=2^3$. Since $\delta(GL_3(2))=28$ by Lemma A.5(c), this case cannot occur.
- If $(k,\ell)=(2,0)$, then $E=Z\times E_0$ where $\dim(E_0)=2$. Then E is as described in the statement of the lemma. □

It remains to handle the nontoral elementary abelian subgroups.

Lemma A.15. *Assume Hypotheses A.7. Let $E\leq G$ be an elementary abelian 2-group which is not contained in a maximal torus of $\overline{G}$. Then $E\notin\widehat{\mathcal{Z}}$.*

Proof. To simplify notation, we write $\mathbb{K}=\overline{\mathbb{F}}_{q_0}$. Set $Z=O_2(Z(\overline{G}))\leq T_{(2)}$. Thus $|Z|=2$ if $\mathbb{G}=E_7$, and $|Z|=1$ otherwise. The maximal nontoral subgroups of $\overline{G}$ are described in all cases by Griess [Gr].

(A) If $\mathbb{G}=F_4$ or E_6, then by [Gr, Theorems 7.3 & 8.2], $\overline{G}$ contains a unique conjugacy class of maximal nontoral elementary abelian 2-subgroups, represented by W_5 of rank five. There is a subgroup $W_2\leq W_5$ of rank two such that $W_5\cap\mathbf{2A}=W_5\smallsetminus W_2$. Also, $\mathrm{Aut}_{\overline{G}}(E_5)=\mathrm{Aut}(E_5,\mathfrak{q}_{E_5})$: the group of all automorphisms of W_5 which normalize W_2. A subgroup $E\leq W_5$ is nontoral if and only if it contains a $\mathbf{2A}^3$-subgroup.

   When $\mathbb{G}=F_4$, we can assume $W_5=T_{(2)}\langle\theta\rangle$, where $\theta\in N_{\overline{G}}(\overline{T})$ inverts the torus.

(B) If $\mathbb{G}=E_7$, then by [Gr, Theorem 9.8(i)], $\overline{G}$ contains a unique maximal nontoral elementary abelian 2-subgroup W_6, of rank six. For any choice of $E_6(\mathbb{K})<G$, $W_5<E_6(\mathbb{K})$ (as just described) has rank 5, is nontoral since it contains a $\mathbf{2A}^3$-subgroup, and so we can take $W_6=Z\times W_5$.

   Each coset of Z of involutions in $\overline{G}\smallsetminus Z$ contains one element of each class $\mathbf{2A}$ and $\mathbf{2B}$. Together with the above description of E_5, this shows that all $\mathbf{2A}^2$-subgroups of W_6 are contained in W_5. Hence for each nontoral subgroup $E\leq W_6$ which contains

Z, $E\cap W_5$ is the subgroup generated by $\mathbf{2A}^2$-subgroups of E, thus is normalized by $\mathrm{Aut}_{\overline{G}}(E)$, and so

$$\begin{aligned}\mathrm{Aut}_{\overline{G}}(E)&\cong \mathrm{Aut}_{\overline{G}}(E\cap W_5)=\mathrm{Aut}(E\cap W_5,\mathfrak{q}_{E\cap W_5})\cong \mathrm{Aut}(E,\mathfrak{q}_E)\\ \mathrm{Aut}_{\overline{G}}(W_6)&\cong \mathrm{Aut}(W_6,\mathfrak{q}_{W_6})\cong C_2^6\rtimes(\Sigma_3\times GL_3(2))\end{aligned}$$

For $Z\le E\le W_6$, the subgroup E is nontoral exactly when it contains a $\mathbf{2A}^3$-subgroup. This is immediate from the analogous statement in (A) for $E_6(\mathbb{K})$.

(C) If $\mathbb{G}=E_8$, then by [Gr, Theorem 2.17], $\overline{G}$ contains two maximal elementary abelian subgroups W_8 and W_9, neither of which is toral [Gr, Theorem 9.2]. An elementary abelian 2-subgroup $E\le\overline{G}$ is nontoral if and only if $\mathfrak{q}_E$ is not quadratic or E has type $\mathbf{2B}^5$ [Gr, Theorem 9.2].

We refer to [Gr, Theorem 2.17] for descriptions of W_8 and W_9. There are subgroups $F_0\le F_1,F_2\le W_8$ such that $\mathrm{rk}(F_0)=2$, $\mathrm{rk}(F_1)=\mathrm{rk}(F_2)=5$, $F_1\cap F_2=F_0$, and $W_8\cap\mathbf{2A}=(F_1\smallsetminus F_0)\cup(F_2\smallsetminus F_0)$. Also, $\mathrm{Aut}_{\overline{G}}(W_8)$ is the group of those automorphisms of W_8 which leave F_0 invariant, and either leave F_1 and F_2 invariant or exchange them.

We can assume that $W_9=T_{(2)}\langle\theta\rangle$, where $\theta\in N_{\overline{G}}(\overline{T})$ inverts $\overline{T}$. Also, $W_9\smallsetminus T_{(2)}\subseteq\mathbf{2B}$. Hence $T_{(2)}=\langle W_9\cap\mathbf{2A}\rangle$ is $\mathrm{Aut}_{\overline{G}}(W_9)$-invariant. Each automorphism of W_9 which is the identity on $T_{(2)}$ is induced by conjugation by some element of order 4 in $\overline{T}$, and thus $\mathrm{Aut}_{\overline{G}}(W_9)$ is the group of all automorphisms whose restriction to $T_{(2)}$ lies in $\mathrm{Aut}_{\overline{G}}(T_{(2)})$.

We next list other properties of elementary abelian subgroups of $\overline{G}$, and of their centralizers and normalizers, which will be needed in the proof.

(D) *If $\mathbb{G}=E_8$, $E\le\overline{G}$, $E\cong C_2^r$, and $|E\cap\mathbf{2A}|=m$, then* $\dim(C_{\overline{G}}(E))=2^{8-r}+2^{5-r}m-8$.

This follows from character computations: if $\mathfrak{g}$ denotes the Lie algebra of $\overline{G}=E_8(\mathbb{K})$, then $\dim(C_{\overline{G}}(E))=\dim(C_{\mathfrak{g}}(E))=|E|^{-1}\sum_{x\in E}\chi_{\mathfrak{g}}(x)$. By [Gr, Table VI], $\chi_{\mathfrak{g}}(1)=\dim(\overline{G})=248$, and $\chi_{\mathfrak{g}}(x)=24$ or -8 when $x\in\mathbf{2A}$ or $\mathbf{2B}$, respectively.

(E) *If $\mathbb{G}=E_8$, $E\le\overline{G}$ is an elementary abelian 2-group, and $E_t<E$ has index 2 and is such that $E\smallsetminus E_t\subseteq\mathbf{2B}$, then there is $g\in\overline{G}$ such that ${}^gE\le W_9=T_{(2)}\langle\theta\rangle$ and ${}^gE_t\le T_{(2)}$.*

It suffices to prove this when E is maximal among such such pairs $E_t<E$. We can assume that E is contained in W_8 or W_9.

If $E\le W_8$, then in the notation of (C), $F_0\le E$ (since E is maximal), and either $\mathrm{rk}(E\cap F_i)=3$ for $i=1,2$ and $\mathrm{rk}(E)=6$, or $\mathrm{rk}(E\cap F_i)=4$ for $i=1,2$ and $\mathrm{rk}(E)=7$. These imply that $|E\cap\mathbf{2A}|=8$ or 24, respectively, and hence by (D) that $\dim(C_{\overline{G}}(E_t))=8$ ($C_{\overline{G}}(E_t)^0=\overline{T}$) and $\dim(C_{\overline{G}}(E))=0$. Hence in either case, if $g\in\overline{G}$ is such that ${}^gE_t\le T_{(2)}$, then ${}^gE\smallsetminus{}^gE_t\subseteq\theta\overline{T}$, and there is $t\in\overline{T}$ such that ${}^{tg}E\le T_{(2)}\langle\theta\rangle=W_9$.

If $E\le W_9$, set $E_2=\langle E\cap\mathbf{2A}\rangle$. Then $E_2\le E\cap T_{(2)}$ and $E_2\le E_t$, so there is nothing to prove unless $\mathrm{rk}(E/E_2)\ge2$. In this case, from the maximality of E, we see that $E_t=E_a\times E_b$, where $E_a\cong C_2^2$ has type $\mathbf{2ABB}$, E_b is a $\mathbf{2B}^3$-group, and $E_a\perp E_b$ with respect to the form $\mathfrak{q}$. Thus $\mathrm{rk}(E)=6$, $|E\cap\mathbf{2A}|=8$, and the result follows by the same argument as in the last paragraph.

(F) *If $\mathbb{G}=E_8$, and $E\le\overline{G}$ is a nontoral elementary abelian 2-group, then either E contains a $\mathbf{2A}^3$-subgroup, or E is $\overline{G}$-conjugate to a subgroup of W_9.*

Assume $E \le W_8$ is nontoral and contains no $\mathbf{2A}^3$-subgroup. We use the notation $F_0 < F_1, F_2 < W_8$ of (C). Set $E_i = E \cap F_i$ for $i = 0,1,2$. Then $\mathfrak{q}_{E_1E_2}$ is quadratic: it is the orthogonal direct sum of $\mathfrak{q}_{E_0}$, $\mathfrak{q}_{E_1/E_0}$, and $\mathfrak{q}_{E_2/E_0}$, each of which is quadratic since $\mathrm{rk}(E_i/E_0) \le 2$ for $i = 1,2$ (E has no $\mathbf{2A}^3$-subgroup). Hence $E > E_1E_2 \ge \langle E \cap \mathbf{2A}\rangle$ since E is nontoral, so E is conjugate to a subgroup of W_9 by (E).

(G) *Let $E \le \overline{G}$ be an elementary abelian 2-subgroup, and let $E_t \le E$ be maximal among toral subgroups of E. Assume that $E_t \cap E_t^\perp \cap \mathbf{2B} = \varnothing$, and that either $\mathrm{rk}(\overline{T}) - \mathrm{rk}(E_t) \ge 2$ or $E_t \cap E_t^\perp = 1$. Then $E \notin \widehat{\mathcal{Z}}$.*

To see this, choose $F \ge F_t$ which is $\overline{G}$-conjugate to $E \ge E_t$ and such that $F_t = F \cap T_{(2)}$. By maximality, no element of $F \smallsetminus F_t$ is $C_{\overline{G}}(F_t)$-conjugate to an element of $\overline{T}$. If $F_t \cap F_t^\perp = 1$, then some $C_W(F_t)$-orbit in $F_t^\perp \smallsetminus 1$ has odd order. Otherwise, since $\mathfrak{q}$ is linear on $F_t \cap F_t^\perp$, we have $F_t \cap F_t^\perp = \langle y\rangle$ for some $y \in \mathbf{2A}$, in which case $|\mathfrak{q}_{F_t^\perp}^{-1}(0)| = |F_t^\perp|/2$ is even since $\mathrm{rk}(F_t^\perp) \ge \mathrm{rk}(\overline{T}) - \mathrm{rk}(F_t) \ge 2$. So again, some $C_W(F_t)$-orbit in $F_t^\perp \smallsetminus 1$ has odd order in this case. Point (G) now follows from Proposition A.13.

(H) *Assume $\mathbb{G} = E_8$. Let $1 \ne E_0 \le E \le \overline{G}$ be elementary abelian 2-subgroups, where $\mathrm{rk}(E) = 3$, and $E \cap \mathbf{2A} = E_0 \smallsetminus 1$. Then*

$$C_{\overline{G}}(E) \cong \begin{cases} E \times F_4(\mathbb{K}) & \text{if } \mathrm{rk}(E_0) = 3 \\ E \times PSp_8(\mathbb{K}) & \text{if } \mathrm{rk}(E_0) = 2 \\ E \times PSO_8(\mathbb{K}) & \text{if } \mathrm{rk}(E_0) = 1. \end{cases}$$

To see this, fix $1 \ne y \in E_0$, and identify $C_{\overline{G}}(y) \cong SL_2(\mathbb{K}) \times_{C_2} E_7(\mathbb{K})$. For each $x \in E \smallsetminus \langle y\rangle$, since x and xy are $\overline{G}$-conjugate, $x \ne (1,b)$ for $b \in E_7(\mathbb{K})$. Thus $x = (a,b)$ for some $a \in SL_2(\mathbb{K})$ and $b \in E_7(\mathbb{K})$ both of order 4, and (in the notation of [Gr, Table VI]) b is in class $\mathbf{4A}$ or $\mathbf{4H}$ since $b^2 \in Z(E_7(\mathbb{K}))$. By (D) and [Gr, Table VI],

$$\dim(C_{\overline{G}}(E)) = \begin{cases} 80 = \dim(C_{E_7(\mathbb{K})}(\mathbf{4H})) + 1 & \text{if } E \text{ has type } \mathbf{2AAA} \\ 64 = \dim(C_{E_7(\mathbb{K})}(\mathbf{4A})) + 1 & \text{if } E \text{ has type } \mathbf{2ABB}, \end{cases}$$

and thus $x \in \mathbf{2A}$ if $b \in \mathbf{4H}$ and $x \in \mathbf{2B}$ if $b \in \mathbf{4A}$. Thus if $E = \langle y, x_1, x_2\rangle$, and $x_i = (a_i, b_i)$, then $\langle a_1, a_2\rangle \le SL_2(\mathbb{K})$ and $\langle b_1, b_2\rangle \le E_7(\mathbb{K})$ are both quaternion of order 8. Point (H) now follows using the description in [Gr, Proposition 9.5(i)] of centralizers of certain quaternion subgroups of $E_7(\mathbb{K})$. When combined with the description in [Gr, Table VI] of $C_{E_7(\mathbb{K})}(\mathbf{4A})$, this also shows that

$$F \cong C_2^2 \text{ of type } \mathbf{2ABB} \implies C_{\overline{G}}(F)^0 \text{ is of type } A_7T^1 \tag{9}$$

(i.e., $C_{\overline{G}}(F)^0 \cong (SL_8(\mathbb{K}) \times \mathbb{K}^\times)/Z$, for some finite subgroup $Z \le Z(SL_8(\mathbb{K})) \times \mathbb{K}^\times$).

(I) *If $U < \overline{G}$ is a $\mathbf{2A}^3$-subgroup, then $C_{\overline{G}}(U) = U \times H$, where H is as follows:*

$\mathbb{G}$	F_4	E_6	E_7	E_8
H	$SO_3(\mathbb{K})$	$SL_3(\mathbb{K})$	$Sp_6(\mathbb{K})$	$F_4(\mathbb{K})$

When $\mathbb{G} = E_8$, this is a special case of (H). For $x \in \mathbf{2A} \cap F_4(\mathbb{K})$, $C_{E_8(\mathbb{K})}(x) \cong SL_2(\mathbb{K}) \times_{C_2} E_7(\mathbb{K})$ by [Gr, 2.14]. Since $C_{F_4(\mathbb{K})}(x) \cong SL_2(\mathbb{K}) \times_{C_2} Sp_6(\mathbb{K})$, this shows that $C_{E_7(\mathbb{K})}(U) \cong U \times Sp_6(\mathbb{K})$.

Similarly, $C_{E_8(\mathbb{K})}(y) \cong SL_3(\mathbb{K}) \times_{C_3} E_6(\mathbb{K})$ by [Gr, 2.14] again (where y is in class **3B** in his notation). There is only one class of element of order three in $F_4(\mathbb{K})$ whose centralizer contains a central factor $SL_3(\mathbb{K})$ — $C_{F_4(\mathbb{K})}(y) \cong SL_3(\mathbb{K}) \times_{C_3} SL_3(\mathbb{K})$ for y of type **3C** in $F_4(\mathbb{K})$ — and thus $C_{E_6(\mathbb{K})}(U) \cong U \times SL_3(\mathbb{K})$.

If $\mathbb{G} = F_4$, then by [Gr, 2.14], for $y \in$ **3C**, $C_{\overline{G}}(y) \cong SL_3(\mathbb{K}) \times_{C_3} SL_3(\mathbb{K})$. Also, the involutions in one factor must all lie in the class **2A** and those in the other in **2B**. This, together with Proposition 2.5, shows that for $U_2 < U$ of rank 2, $C_{\overline{G}}(U_2) \cong (T^2 \times_{C_3} SL_3(\mathbb{K}))\langle\theta\rangle$, where θ inverts a maximal torus. Thus $C_{\overline{G}}(U) = U \times C_{SL_3(\mathbb{K})}(\theta)$, where by [Gr, Proposition 2.18], $C_{SL_3(\mathbb{K})}(\theta) \cong SO_3(\mathbb{K})$. This finishes the proof of (I).

For the rest of the proof, we fix a nontoral elementary abelian 2-subgroup $E < \overline{G}$. We must show that $E \notin \widehat{\mathcal{Z}}$. In almost all cases, we do this either by showing that the hypotheses of (G) hold, or by showing that $\delta(\mathrm{Aut}_{\overline{G}}(E)) > |\pi_0(C_{\overline{G}}(E))|$ (where $\delta(-)$ is as in Lemma A.5), in which case $\mathrm{Aut}_G(E)$ has no strongly 2-embedded subgroup by Proposition A.4, and hence $E \notin \widehat{\mathcal{Z}}$.

By (A), (B), and (F), either E contains a $\mathbf{2A}^3$-subgroup of rank three, or $\mathbb{G} = E_8$ and E is $\overline{G}$-conjugate to a subgroup of W_9. These two cases will be handled separately.

Case 1: Assume first that E contains a $\mathbf{2A}^3$-subgroup $U \leq E$. From the lists in (A,B,C) of maximal nontoral subgroups, there are the following possibilities.

$\mathbb{G} = F_4$**,** E_6**, or** E_7**:** By (A,B), we can write $E = U \times E_0 \times Z$, where E_0 is a $\mathbf{2B}^k$ subgroup (some $k \leq 2$) and $UE_0 \smallsetminus E_0 \subseteq$ **2A** (and where $Z = 1$ unless $\mathbb{G} = E_7$). If $k = 0$, then $E \notin \widehat{\mathcal{Z}}$ by (G), so assume $k \geq 1$. By (I), and since each elementary abelian 2-subgroup of $SL_3(\mathbb{K})$ and of $Sp_6(\mathbb{K})$ has connected centralizer, $\pi_0(C_{\overline{G}}(E)) \cong U$ if $\mathbb{G} = E_6$ or E_7. If $\mathbb{G} = F_4$, then by (I) again, and since the centralizer in $SO_3(\mathbb{K}) \cong PSL_2(\mathbb{K})$ of any C_2^k has 2^k components, $|\pi_0(C_{\overline{G}}(E))| = 2^{3+k}$.

By (A,B) again, $\mathrm{Aut}_{\overline{G}}(E)$ is the group of all automorphisms which normalize E_0 and UE_0 and fix Z. Hence

$$|O_2(\mathrm{Aut}_{\overline{G}}(E))| = 2^{3k} \qquad \text{and} \qquad \mathrm{Aut}_{\overline{G}}(E)/O_2(\mathrm{Aut}_{\overline{G}}(E)) \cong GL_3(2) \times GL_k(2)\,.$$

So $\delta(\mathrm{Aut}_{\overline{G}}(E)) \geq 2^{3k+3} > |\pi_0(C_{\overline{G}}(E))|$ by Lemma A.5, and $E \notin \widehat{\mathcal{Z}}$.

$\mathbb{G} = E_8$**:** By (I), $C_{\overline{G}}(U) = U \times H$ where $H \cong F_4(\mathbb{K})$. Set $E_2 = E \cap H$, and let $E_0 = \langle E_2 \cap \mathbf{2B}\rangle$. Set $k = \mathrm{rk}(E_0)$ and $\ell = \mathrm{rk}(E_2/E_0)$.

If $k = 0$, then E_2 has type $\mathbf{2A}^\ell$, and $E \smallsetminus (U \cup E_2) \subseteq$ **2B**. So each maximal toral subgroup $E_t < E$ has the form $E_t = U_1 \times U_2$, where $\mathrm{rk}(U_1) = 2$, $\mathrm{rk}(U_2) \leq 2$, and $E_t \cap \mathbf{2A} = (U_1 \cup U_2) \smallsetminus 1$. The hypotheses of (G) thus hold, and so $E^* \notin \widehat{\mathcal{Z}}$.

Thus $k = 1, 2$. If $\ell \leq 2$, then E_2 is toral, and $|\pi_0(C_{\overline{G}}(E))| = 8 \cdot |\pi_0(C_H(E_2))| \leq 2^{3+k}$ by formula (8) in the proof of Lemma A.14. (Note that $\varepsilon = 1$ and $\eta = 0$ in the notation of that formula.) If $\ell = 3$, then $|\pi_0(C_{\overline{G}}(E))| = 2^{6+k}$ by the argument just given for $F_4(\mathbb{K})$. Also, $\mathrm{Aut}_{\overline{G}}(E)$ contains all automorphisms of E which normalize E_0, and either normalize UE_0 and E_2 or (if $\ell = 3$) exchange them: since in the notation of (C), each such automorphism extends to an automorphism of W_8 which normalizes F_1 and F_2. So $|O_2(\mathrm{Aut}_{\overline{G}}(E))| \geq 2^{k(3+\ell)}$, and $\mathrm{Aut}_{\overline{G}}(E)/O_2(\mathrm{Aut}_{\overline{G}}(E)) \cong GL_3(2) \times GL_k(2) \times GL_\ell(2)$ or (if $\ell = 3$) $(GL_3(2) \wr C_2) \times GL_k(2)$. In all cases, $\delta(\mathrm{Aut}_{\overline{G}}(E)) \geq 2^{3k+\ell k+3} > |\pi_0(C_{\overline{G}}(E))|$, so $E \notin \mathcal{Z}$.

Case 2: Now assume that $\mathbb{G} = E_8$, and that E is $\overline{G}$-conjugate to a subgroup of W_9. To simplify the argument, we assume that $E \leq W_9$, and then prove that no subgroup $E^* \in \widehat{\mathcal{Z}}$

can be $\bar{G}$-conjugate to E. Recall that $W_9 = T_{(2)}\langle\theta\rangle$, where $\theta \in N_{\bar{G}}(\bar{T})$ inverts the torus and $\theta T_{(2)} \subseteq \mathbf{2B}$.

If $E \cap \mathbf{2A} = \varnothing$, then $\mathrm{rk}(E) = 5$. In this case, $\mathrm{Aut}_{\bar{G}}(E) \cong GL_5(2)$ and $|C_{\bar{G}}(E)| = 2^{15}$ [CG, Proposition 3.8]. (Cohen and Griess work in $E_8(\mathbb{C})$, but their argument also holds in our situation.) Since $\delta(GL_5(2)) > 2^{15}$ by Lemma A.5(d), no $E^* \in \widehat{\mathscr{Z}}$ can be $\bar{G}$-conjugate to E.

Now assume E has $\mathbf{2A}$-elements, and set $E_2 = \langle E \cap \mathbf{2A}\rangle$. Then $E_2 \le T_{(2)}$ (hence $\mathfrak{q}_{E_2}$ is quadratic) by the above remarks. Set $E_1 = E_2^\perp \cap E_2$ and $E_0 = \mathrm{Ker}(\mathfrak{q}_{E_1})$. If $E_0 = 1$ and $\mathrm{rk}(E_2) \ne 7$, then by (G), no subgroup of S which is $\bar{G}$-conjugate to E lies in $\widehat{\mathscr{Z}}$.

It remains to consider the subgroups E for which $E_0 \ne 1$ or $\mathrm{rk}(E_2) = 7$. Information about $|O_2(\mathrm{Aut}_{\bar{G}}(E))|$ and $|\pi_0(C_{\bar{G}}(E))|$ for such E is summarized in Table A.1. By the "type of $\mathfrak{q}_E$" is meant the type of quadratic form, in the notation used in the proof of Lemma A.14.

Case nr.	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathrm{rk}(E/E_2)$	1	1	1	1	1	1	1	1	1	1	2	2	2
$\mathrm{rk}(E_2/E_0)$	7	6^+	5	4^+	4^+	4^-	3	3	2^-	2^-	1	1	1
$\mathrm{rk}(E_0)$	0	1	1	2	1	1	2	1	2	1	3	2	1
type of $\mathfrak{q}_{E_2}$	$[\mathbf{7}]$	$[\mathbf{6}]^+_{(\mathbf{1})}$	$[\mathbf{5}]_{(\mathbf{1})}$	$[\mathbf{4}]^+_{(\mathbf{2})}$	$[\mathbf{4}]^+_{(\mathbf{1})}$	$[\mathbf{4}]^-_{(\mathbf{1})}$	$[\mathbf{3}]_{(\mathbf{2})}$	$[\mathbf{3}]_{(\mathbf{1})}$	$[\mathbf{2}]^-_{(\mathbf{2})}$	$[\mathbf{2}]^-_{(\mathbf{1})}$	$[\mathbf{1}]_{(\mathbf{3})}$	$[\mathbf{1}]_{(\mathbf{2})}$	$[\mathbf{1}]_{(\mathbf{1})}$
$\lvert\pi_0(C_{\bar{G}}(E^*))\rvert \le$	2^9	2^9	2^{10}	2^{10}	2^8	2^7	2^6	2^5	2^5	2^4	2^{12}	2^8	2^5
$\lvert O_2(\mathrm{Aut}_{\bar{G}}(E^*))\rvert$	2^7	2^{13}	2^{11}	2^{14}	2^9	2^9	2^{11}	2^7	2^8	2^5	2^{11}	2^8	2^5
$\delta(\mathrm{Aut}_{\bar{G}}(E^*)) \ge$	2^{13}	2^{17}	$3{\cdot}2^{13}$	2^{16}	2^{10}	2^{10}	2^{12}	2^7	2^9	2^5	2^{14}	2^9	2^5

TABLE A.1.

We first check that the table includes all cases. If $\mathrm{rk}(E/E_2) = 1$, then $E_2 = E \cap T_{(2)}$, and the table lists all types which the form $\mathfrak{q}_{E_2}$ can have. Note that since E_2 is generated by nonisotropic vectors, $\mathfrak{q}_{E_2}$ cannot have type $[\mathbf{2}]^+_{(\boldsymbol{k})}$. If $\mathrm{rk}(E/E_2) = 2$, then $\mathfrak{q}_{E_2}$ is linear, and must be one of the three types listed. Since $\mathfrak{q}_{E\cap T_{(2)}}$ is quadratic and $\mathfrak{q}_E$ is not, E_2 has index at most 2 in $E \cap T_{(2)}$.

We claim that

$$E, F < W_9,\ \alpha \in \mathrm{Iso}(E,F) \text{ such that } \alpha(E \cap T_{(2)}) = F \cap T_{(2)} \text{ and } \alpha(E \cap \mathbf{2A}) = F \cap \mathbf{2A} \implies \alpha = c_{tg} \text{ for some } t \in \bar{T} \text{ and some } g \in N_G(\bar{T}) = G \cap N_{\bar{G}}(\bar{T}). \tag{10}$$

By (C) and Witt's theorem (see [Ta, Theorem 7.4]), there is $g \in N_{\bar{G}}(\bar{T})$ such that $\alpha|_{E\cap T_{(2)}} = c_g$, and we can assume $g \in G$ by Lemma 2.9. Then ${}^gE \smallsetminus {}^g(E \cap T_{(2)}) \le \theta\bar{T}$ since $\theta\bar{T} \in Z(N_{\bar{G}}(\bar{T}))/\bar{T}$, so $\alpha = c_{tg}$ for some $t \in \bar{T}$. This proves (10). In particular, any two subgroups of W_9 which have the same data as listed in the first three rows of Table A.1 are $\bar{G}$-conjugate.

By (10), together with (E) when $\mathrm{rk}(E/E_2) = 2$, we have $\mathrm{Aut}_{\bar{G}}(E) = \mathrm{Aut}(E, \mathfrak{q}_E)$ in all cases. Thus $\mathrm{Aut}_{\bar{G}}(E)$ is the group of all automorphisms of E which normalize E_0 and E_2 and preserve the induced quadratic form on E_2/E_0. This gives the value for $|O_2(\mathrm{Aut}_{\bar{G}}(E))|$ in the table, and the lower bounds for $\delta(\mathrm{Aut}_{\bar{G}}(E))$ then follow from Lemma A.5.

In cases 1–6, the upper bounds for $|\pi_0(C_{\bar{G}}(E))|$ given in the table are proven in [O2, p. 78–79]. In all cases, $|\pi_0(C_{\bar{G}}(E_2))|$ is first computed, using Proposition 2.5 or the upper bound given in formula (8) in the proof of the last lemma, and then [O2, Proposition 8.8] is used to compute an upper bound for $|\pi_0(C_{\overline{G}}(E))|/|\pi_0(C_{\overline{G}}(E_2))|$. There is in fact an error in the table on [O2, p. 79] (the group $C_G(E_0)^0_s$ in the third-to-last column should be $SL_2 \times SL_2$ up to finite cover), but correcting this gives in fact a better estimate $|\pi_0(C_{\overline{G}}(E))| \leq 2^9$.

Case nr. 11 can be handled in a similar way. Set $E_t = E \cap T_{(2)} < E$, so that $|E/E_t| = 2 = |E_t/E_2|$. The form $\mathfrak{q}_{E_t}$ has type $[\mathbf{2}]^+_{(\mathbf{3})}$, while $E_t^\perp$ has type $\mathbf{2B}^3$. Hence $|\pi_0(C_{\overline{G}}(E_t))| \leq 2^4$ by (8). By [O2, Proposition 8.8], $|\pi_0(C_{\overline{G}}(E))| \leq 2^{4+r}$, where $r = \dim(\overline{T}) = 8$.

To handle the remaining cases, fix rank 2 subgroups $F_1, F_2 \leq T_{(2)} < \overline{G}$ with involutions of type **AAA** and **ABB**, respectively, and consider the information in Table A.2. The

i	$C_{\overline{G}}(F_i\langle\theta\rangle)$	$\dim(C_{\overline{G}}(F_i)\langle\theta, g\rangle)$ for g as follows:				
		$-I_4 \oplus I_4$	$-I_2 \oplus I_6$	order 4	**2A**	**2B**
1	$F_1\langle\theta\rangle \times PSp_8(\mathbb{K})$	20	24	16	16	20
2	$F_2\langle\theta\rangle \times PSO_8(\mathbb{K})$	12	16	16	16	12

TABLE A.2.

description of $C_{\overline{G}}(F_i\langle\theta\rangle)$ follows from (H). The third through fifth columns give dimensions of centralizers of $F_i\langle\theta\rangle\langle g\rangle$, for g as described after lifting to $Sp_8(\mathbb{K})$ or $SO_8(\mathbb{K})$. (Here, I_m denotes the $m \times m$ identity matrix.) The last two columns do this for $g \in \mathbf{2A}$ or $\mathbf{2B}$, respectively, when $g \in T_{(2)}$ is orthogonal to F_i with respect to the form $\mathfrak{q}$, and the dimensions follow from (D). Thus elements of class **2B** lift to involutions in $Sp_8(\mathbb{K})$ or $SO_8(\mathbb{K})$ with 4-dimensional (-1)-eigenspace, while for $i = 1$ at least, elements of class **2A** lift to elements of order 4 in $Sp_8(\mathbb{K})$.

Thus in all of the cases nr. 7–13 in Table A.1, we can identify $E = F_i\langle\theta\rangle \times F^*$, where $i = 1$ in nr. 7–10 or $i = 2$ in nr. 11–13, and where F^* lifts to an abelian subgroup of $Sp_8(\mathbb{K})$ or $SO_8(\mathbb{K})$ (elementary abelian except for nr. 7–8). This information, together with the following:

$$H \text{ a group, } Z \leq Z(H),\ |Z| = p,\ Z \leq P \leq H \text{ a } p\text{-subgroup} \implies \big|C_{H/Z}(P)/C_H(P)/Z\big| \leq |P/\mathrm{Fr}(P)|$$

(applied with $H = Sp_8(\mathbb{K})$ or $SO_8(\mathbb{K})$), imply the remaining bounds in the last line of Table A.1.

In all but the last case in Table A.1, $\delta(\mathrm{Aut}_{\overline{G}}(E)) > |\pi_0(C_{\overline{G}}(E))|$, so no $E^* \in \widehat{\mathscr{Z}}$ is $\overline{G}$-conjugate to E by Proposition A.4. In the last case, by the same proposition, E can be $\overline{G}$-conjugate to some $E^* \in \widehat{\mathscr{Z}}$ only if $\mathrm{Aut}_{\overline{G}}(E)$ acts transitively on $\pi_0(C_{\overline{G}}(E)) \cong C_2^5$ with point stabilizers isomorphic to Σ_3. By (10), each class in $O_2(\mathrm{Aut}_{\overline{G}}(E))$ is represented by some element $tg \in N_{\overline{G}}(E)$, where $g \in N_G(\overline{T})$ and $t \in \overline{T}$. In particular, $(tg)\sigma(tg)^{-1} = t\sigma(t)^{-1} \in \overline{T}$. So each class in the $O_2(\mathrm{Aut}_{\overline{G}}(E))$-orbit of $1 \in \pi_0(C_{\overline{G}}(E))$ has nonempty intersection with $\overline{T}$. But by (9), $C_{\overline{G}}(F_2)^0 \cap \theta\overline{T} = \varnothing$, so $\theta C_{\overline{G}}(E)^0 \cap \overline{T} = \varnothing$. Thus the action is not transitive on $\pi_0(C_{\overline{G}}(E))$, and hence $E^* \notin \widehat{\mathscr{Z}}$. □

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