THE RIESZ TRANSFORM, RECTIFIABILITY, AND REMOVABILITY FOR LIPSCHITZ HARMONIC FUNCTIONS

FEDOR NAZAROV, XAVIER TOLSA, AND ALEXANDER VOLBERG

Abstract. We show that, given a set $E \subset \mathbb{R}^{n+1}$ with finite n-Hausdorff measure $\mathcal{H}^n$, if the n-dimensional Riesz transform

$$R_{\mathcal{H}^n\lfloor E} f(x) = \int_E \frac{x-y}{|x-y|^{n+1}} f(y) d\mathcal{H}^n(y)$$

is bounded in $L^2(\mathcal{H}^n\lfloor E)$, then E is n-rectifiable. From this result we deduce that a compact set $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$ is removable for Lipschitz harmonic functions if and only if it is purely n-unrectifiable, thus proving the analog of Vitushkin's conjecture in higher dimensions.

1. Introduction

In this paper we prove that, given a set $E \subset \mathbb{R}^{n+1}$ with finite n-Hausdorff measure $\mathcal{H}^n$, if the n-dimensional Riesz transform is bounded in L^2 with respect to $\mathcal{H}^n\lfloor E$, then E is n-rectifiable. Combined with results from [Vo], it implies that the purely n-unrectifiable compact sets with finite n-Hausdorff measure are removable for Lipschitz harmonic functions.

To state our results in more detail, we need to introduce some notation. Given a (complex) Borel measure ν in $\mathbb{R}^{n+1}$ such that

$$\int_{\mathbb{R}^{n+1}} \frac{d|\nu|(x)}{(1+|x|)^n} < \infty\,, \tag{1.1}$$

the n-dimensional Riesz transform of ν is defined by

$$R\nu(x) = \int \frac{x-y}{|x-y|^{n+1}}\, d\nu(y),$$

for every $x \in \mathbb{R}^{n+1}$ where the integral makes sense. Notice that the kernel inside the integral is vectorial.

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Since the preceding integral may fail to be absolutely convergent for many points $x \in \mathbb{R}^{n+1}$, it is convenient to consider an ε-truncated version, for $\varepsilon > 0$:

$$R_\varepsilon \nu(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}}\, d\nu(y).$$

Given a non-negative measure μ and $f \in L^1_{loc}(\mu)$ such that $\nu = f\mu$ satisfies (1.1), we set $R_\mu f(x) = R(f\,\mu)(x)$ and $R_{\mu,\varepsilon} f(x) = R_\varepsilon(f\,\mu)(x)$. We say that R_μ is bounded in $L^2(\mu)$ if the truncated operators $R_{\mu,\varepsilon}$ are bounded in $L^2(\mu)$ uniformly on $\varepsilon > 0$.

A set $E \subset \mathbb{R}^{n+1}$ is called n-rectifiable if it is contained in a countable union of C^1-manifolds up to a set of zero $\mathcal{H}^n$ measure. On the other hand, E is called purely unrectifiable it does not have any rectifiable subset with positive n-Hausdorff measure.

Our first result in this paper is the following.

Theorem 1.1. *Let $E \subset \mathbb{R}^{n+1}$ be a set such that $\mathcal{H}^n(E) < \infty$. If $R_{\mathcal{H}^n\lfloor E}$ is bounded in $L^2(\mathcal{H}^n\lfloor E)$, then E is n-rectifiable.*

Let us remark that the case $n = 1$ of this theorem has already been known. Indeed, the 1-dimensional Riesz transform and the Cauchy transform coincide modulo a conjugation. So the $L^2(\mu)$ boundedness of R_μ for any non-atomic measure μ implies that the curvature of μ is finite, i.e.,

$$c^2(\mu) = \iiint \frac{1}{R(x,y,z)^2}\, d\mu(x)\, d\mu(y)\, d\mu(z) < \infty\,,$$

where $R(x,y,z)$ stands for the radius of the circumference passing through x, y, z (see [Me] and [MeV]). It remains to refer to the result of David and Léger (see [Lé]) who showed that if $\mathcal{H}^1(E) < \infty$ and $c^2(\mathcal{H}^1\lfloor E) < \infty$, then E is 1-rectifiable.

In the higher dimensional setting, Theorem 1.1 was an open problem (see [Pa2, p. 114], for example). The main reason is that the curvature method is not available because for $n > 1$, there is no relationship between the n-dimensional Riesz transform and any notion as useful as curvature. However, some partial results were known. For example, in [To1] it was shown that the existence of the principal values $\lim_{\varepsilon\to 0} R_\varepsilon(\mathcal{H}^n\lfloor E)(x)$ for $\mathcal{H}^n$-a.e. $x \in E$ implies the n-rectifiability of E (see also [MPr] for a previous result under somewhat stronger assumptions). On the other hand, from some new results in [ENV] it follows that the Riesz transform $R_{\mathcal{H}^n\lfloor E}$ cannot be bounded in $L^2(\mathcal{H}^n\lfloor E)$ if E is a set with finite $\mathcal{H}^n$ measure and vanishing lower n-dimensional density $\theta^n_*(x, \mathcal{H}^n\lfloor E) = \liminf_{r\to 0} r^{-n}\mathcal{H}^n(B(x,r)\cap E)$ for $\mathcal{H}^n$-a.e $x \in E$.

Very recently, the proof of the of the so called David-Semmes conjecture in the codimension 1 case was completed. The conjecture follows from the results of our new paper [NToV] and deep results by David and Semmes in [DS]. The assertion is that if $\mathcal{H}^n\lfloor E \subset \mathbb{R}^{n+1}$ is Ahlfors–David regular, and $R_{\mathcal{H}^n\lfloor E}$ is bounded in $L^2(\mathcal{H}^n\lfloor E)$, then E is uniformly n-rectifiable and thus, in particular, n-rectifiable. Recall that a measure μ is called Ahlfors–David (AD) regular if there exists some constant c such that

$$c^{-1}r^n \leq \mu(\cap B(x,r)) \leq c\,r^n \qquad \text{for all } x \in E,\, r > 0\,.$$

A nice covering theorem due to Pajot [Pa1] will allow us to reduce Theorem 1.1 to the combination of this result with the one from [ENV].

Let us remark that we do not know if Theorem 1.1 can be extended to codimensions higher than 1, that is, to the n-dimensional Riesz transform and sets $E \subset \mathbb{R}^d$ with $0 < \mathcal{H}^n(E) < \infty$, $d > n+1$. This is due to the fact that the corresponding analogs of the results from [NToV] and [ENV] are also open in this case.

Theorem 1.1 has a corollary regarding the removability of singularities for Lipschitz harmonic functions. Recall that a subset $E \subset \mathbb{R}^{n+1}$ is removable for Lipschitz harmonic functions if, for each open set $\Omega \subset \mathbb{R}^{n+1}$, every Lipschitz function $f : \Omega \to \mathbb{R}$ that is harmonic in $\Omega \setminus E$ is harmonic in the whole Ω. By combining Theorem 1.1 with the results on the Lipschitz harmonic capacity from [Vo], one gets the proof of the following analog of Vitushkin's conjecture in higher dimensions.

Theorem 1.2. *Let $E \subset \mathbb{R}^{n+1}$ be a compact set such that $\mathcal{H}^n(E) < \infty$. Then, E is removable for Lipschitz harmonic functions in $\mathbb{R}^{n+1}$ if and only if E is purely n-unrectifiable.*

Again, this theorem was already known in the case $n = 1$. It was proved by David and Mattila in [DM]. The analogous result for the removable singularities for bounded analytic functions is the celebrated solution of Vitushkin's conjecture by David [Da2]. It is also worth mentioning that in [NTrV2], a Tb type theorem suitable to prove the analytic part of Vitushkin's conjecture was obtained very shortly after David's proof. The arguments in [NTrV2] also work in the case $n > 1$ and are an essential tool in the work [Vo] about Lipschitz harmonic capacity.

2. The main lemma

2.1. Statement of the Main Lemma. We say that a Borel measure μ in $\mathbb{R}^d$ has growth of degree n if there exists some constant c such that

$$\mu(B(x,r)) \leq c\, r^n \qquad \text{for all } x \in \mathbb{R}^d,\ r > 0.$$

We define the upper and lower n-dimensional densities by

$$\theta^{n,*}(x,\mu) = \limsup_{r\to 0} r^{-n}\mu(B(x,r)) \text{ and } \theta^n_*(x,\mu) = \liminf_{r\to 0} r^{-n}\mu(B(x,r)),$$

respectively.

If μ and σ are Borel measures on $\mathbb{R}^d$, the notation $\mu \leq \sigma$ means that $\mu(A) \leq \sigma(A)$ for all Borel sets $A \subset \mathbb{R}^d$.

Lemma 2.1 (Main Lemma)**.** *Let μ be a compactly supported finite Borel measure in $\mathbb{R}^d$ with growth of degree n such that $\theta^n_*(x,\mu) > 0$ for μ-a.e. $x \in \mathbb{R}^d$. Suppose that R_μ is bounded in $L^2(\mu)$. Then there are finite Borel measures μ_k, $k \geq 1$, such that*

(a) $\mu \leq \sum_{k\geq 1} \mu_k$
(b) *μ_k is AD-regular for each $k \geq 1$ (with the AD-regularity constant depending on k), and*
(c) *for each $k \geq 1$, R_{μ_k} is bounded in $L^2(\mu_k)$.*

Before proving the Main Lemma, we will show how it allows one to reduce Theorem 1.1 to the results in [ENV] and [NToV].

As usual in harmonic analysis, the letter c stands for some fixed constant (quite often an absolute constant), which may change its value at different occurrences. On the other hand, constants with subscripts, such as c_1, are assumed to keep their values in the whole paper.

2.2. Proof of Theorem 1.1 using the Main Lemma 2.1. It is immediate to check that to prove the theorem, we may assume E to be bounded. So let $E \subset \mathbb{R}^{n+1}$ be a bounded set with $\mathcal{H}^n(E) < \infty$. Set $\mu = \mathcal{H}^n\lfloor E$, and suppose that R_μ is bounded in $L^2(\mu)$.

Let E_0 be the subset of those $x \in E$ for which $\theta^n_*(x,\mu) = 0$. We set

$$\mu_0 = \mu\lfloor E_0.$$

Then,

$$\theta^n_*(x,\mu_0) \leq \theta^n_*(x,\mu) = 0 \ \text{ for } \mu_0\text{-a.e. } x \in \mathbb{R}^{n+1},$$

and, moreover, R_{μ_0} is bounded in $L^2(\mu_0)$. Then, by the main theorem of [ENV] (applied to the codimension 1 case) we deduce that $\mu_0 = 0$. That is,

$$\theta^n_*(x,\mu) > 0 \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^{n+1}.$$

So the measure μ satisfies the assumptions of Main Lemma 2.1, and thus we may consider measures μ_k as in the statement of the Main Lemma.

By the results of [NToV]and [DS], $\operatorname{supp}\mu_k$ is n-rectifiable. Therefore,

$$F = \bigcup_{k\geq 1} \operatorname{supp}\mu_k$$

is also n-rectifiable. Since

$$\mathcal{H}^n(E \setminus F) = \mu(\mathbb{R}^{n+1} \setminus F) \leq \sum_k \mu_k(\mathbb{R}^d \setminus F) = 0,$$

we infer that E is n-rectifiable too.

3. Proof of the Main Lemma 2.1

For the proof of the Main Lemma 2.1 we will need the following proposition.

Proposition 3.1. *Let μ and σ be Borel measures with growth of degree n in $\mathbb{R}^d$ such that R_μ is bounded in $L^2(\mu)$ and R_σ is bounded in $L^2(\sigma)$. Then, $R_{\mu+\sigma}$ is bounded in $L^2(\mu+\sigma)$.*

Proof. The boundedness of R_μ in $L^2(\mu)$ implies the boundedness of R from the space of real measures $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu)$. In other words, the following inequality holds for any $\nu \in M(\mathbb{R}^d)$ uniformly on $\varepsilon > 0$:

$$\mu\big\{x \in \mathbb{R}^d : |R_\varepsilon\nu(x)| > \lambda\big\} \leq c\,\frac{\|\nu\|}{\lambda} \qquad \text{for all } \lambda > 0.$$

For the proof, see Theorem 9.1 of [NTrV1]. Analogously, the same bound holds with μ replaced by σ. As a consequence, we infer that for all $\lambda > 0$,

$$(\mu+\sigma)\big\{x \in \mathbb{R}^d : |R_\varepsilon\nu(x)| > \lambda\big\} \leq c\,\frac{\|\nu\|}{\lambda}.$$

That is, R is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mu+\sigma)$. In particular, $R_{\mu+\sigma}$ is of weak type $(1,1)$ with respect to $\mu+\sigma$. This implies that $R_{\mu+\sigma}$ is bounded in $L^2(\mu+\sigma)$. For the proof, based on interpolation, see Theorem 10.1 of [NTrV1] (an alternative argument based on a good lambda inequality can be also found in Chapter 2 of the book [To2]). □

Let us remark that the preceding proposition and its proof remain valid for more general Calderón-Zygmund operators. However, we will need it only for the Riesz transforms.

In the proof of the Main Lemma 2.1 it will be convenient to work with an ε-regularized version $\widetilde{R}_{\mu,\varepsilon}$ of the Riesz transform R_μ. We set

$$\widetilde{R}_{\mu,\varepsilon} f(x) = \int \frac{x-y}{\max(|x-y|,\varepsilon)^{n+1}}\, f(y)\, d\mu(y).$$

It is easy to check that

$$|\widetilde{R}_{\mu,\varepsilon} f(x) - R_{\mu,\varepsilon} f(x)| \leq c\, M_\mu f(x) \qquad \text{for all } x \in \mathbb{R}^d,$$

where c is independent of ε and M_μ is the centered maximal Hardy-Littlewood operator with respect to μ:

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu.$$

Since M_μ is bounded in $L^2(\mu)$, it turns out that R_μ is bounded in $L^2(\mu)$ if and only if the operators $\widetilde{R}_{\mu,\varepsilon}$ are bounded in $L^2(\mu)$ uniformly on $\varepsilon > 0$. The advantage of $\widetilde{R}_{\mu,\varepsilon}$ over $R_{\mu,\varepsilon}$ is that the kernel

$$K_\varepsilon(x) = \frac{x}{\max(|x|,\varepsilon)^{n+1}}$$

satisfies the smoothness condition

$$|\nabla \widetilde{K}_\varepsilon(x)| \leq \frac{c}{|x|^{n+1}}$$

(with c independent of ε), which implies that $K_\varepsilon(x-y)$ is a Calderón-Zygmund kernel (with constants independent of ε), unlike the kernel of $R_{\mu,\varepsilon}$.

Proof of the Main Lemma 2.1. We follow an idea of H. Pajot (see Theorem 10 of [Pa2]), where some measures μ_k satisfying (a) and (b) are constructed. For the reader's convenience, we will repeat the arguments of the construction and of the proof of (b), and subsequently we will show that the statement (c) holds.

Consider the subset $F \subset \operatorname{supp}\mu$ of those $x \in \mathbb{R}^d$ for which $\theta^n_*(x,\mu) > 0$, so that $\mu(\mathbb{R}^d \setminus F) = 0$. For positive integers p, s, we denote

$$F_p = \left\{ x \in F : \text{ for } 0 < r \leq D,\ \mu(B(x,r)) \geq \tfrac{1}{p}\, r^n \right\},$$
$$F_{p,s} = \left\{ x \in F_p : \text{ for } 0 < r \leq D,\ \mu(F_p \cap B(x,r)) \geq \tfrac{1}{ps}\, r^n \right\},$$

where $D = \operatorname{diam}(\operatorname{supp}\mu)$. From the definitions of F and F_p, it is clear that

$$F = \bigcup_{p \geq 1} F_p.$$

Also, $\theta^n_*(x,\mu) = \theta^n_*(x,\mu\lfloor F_p)$ for μ-a.e. $x \in F_p$ by the Lebesgue differentiation theorem, and thus

$$\mu\Big(F_p \setminus \bigcup_{s\geq 1} F_{p,s}\Big) = 0.$$

So we have

$$\mu \leq \sum_{p,s\geq 1} \mu\lfloor F_{p,s}.$$

The strategy of the construction consists in adding a measure $\sigma_{p,s}$ to each $\mu\lfloor F_{p,s}$ so that the resulting measure is AD-regular, for each p, s.

It is easy to check that all the sets F_p and $F_{p,s}$ are compact. Fix p, s and denote

$$d(x) = \frac{1}{10}\operatorname{dist}(x, F_{p,s}).$$

Notice that $d(y) > 0$ if $x \notin F_{p,s}$, as $F_{p,s}$ is closed. Now we cover $F_p \setminus F_{p,s}$ by a family of balls of the form $B(x, d(x))$, with $x \in F_p \setminus F_{p,s}$, using Besicovitch's covering theorem. So there exists a family of points $H_{p,s} \subset F_p \setminus F_{p,s}$, at most countable, such that

$$F_p \setminus F_{p,s} \subset \bigcup_{x\in H_{p,s}} B(x, d(x)),$$

and

$$\sum_{x\in H_{p,s}} \chi_{B(x,d(x))} \leq C_d\,.$$

Moreover, we can split $H_{p,s} = \bigcup_{i=1}^{N_d} H^i_{p,s}$ so that for each i, the balls from $\{B(x, d(x))\}_{x\in H^i_{p,s}}$ are pairwise disjoint. Here C_d, N_d are some constants depending on d only.

To define $\sigma_{p,s}$, for each $x \in H_{p,s}$ we consider an arbitrary n-plane Π_x containing x and set $P_x = \Pi_x \cap B(x, \frac{1}{2}d(x))$. Then we define

$$\sigma_{p,s} = \mathcal{H}^n\lfloor\Pi_{p,s} + \sum_{x\in H_{p,s}} \mathcal{H}^n\lfloor P_x$$

where $\Pi_{p,s}$ is an arbitrary n-plane in $\mathbb{R}^d$ intersecting $F_{p,s}$. We set

$$\mu_{p,s} = \sigma_{p,s} + \mu\lfloor F_{p,s}.$$

We also denote

$$\sigma^i_{p,s} = \sum_{x\in H^i_{p,s}} \mathcal{H}^n\lfloor P_x,$$

so that $\sigma_{p,s} = \mathcal{H}^n\lfloor\Pi + \sum_{i=1}^{N_d} \sigma^i_{p,s}$. We will show now that $\mu_{p,s}$ is AD-regular.

Upper AD-regularity of $\mu_{p,s}$. We have to show that $\mu_{p,s}$ has growth of degree n. Since $\mu\lfloor F_{p,s}$ and $\mathcal{H}^n\lfloor\Pi$ have growth of degree n, it is enough to show that so does $\sigma^i_{p,s}$, for each $i = 1, \dots, N_d$.

We set $r(x) = \frac{1}{2}\, d(x)$ for $x \in H^i_{p,s}$. So $\sigma^i_{p,s}$ is a measure supported on the union of the closed balls

$$B_x := B(x, r(x)) = B(x, \frac{1}{2}\, d(x)), \qquad x \in H^i_{p,s}, \tag{3.1}$$

coinciding with $\mathcal{H}^n\lfloor P_x$ inside B_x. Notice also that the balls $2B_x$, $x \in H^i_{p,s}$ are pairwise disjoint.

Let Δ be some fixed closed ball of radius $r(\Delta)$. Let H_a be the subset of points $x \in H^i_{p,s}$ such that $B_x \cap \Delta \neq \varnothing$ and $2r(\Delta) < r(x)$, and let $H_b \subset H^i_{p,s}$ be the subset of points such that $B_x \cap \Delta \neq \varnothing$ and $2r(\Delta) \geq r(x)$. We have

$$\sigma^i_{p,s}(\Delta) = \sum_{x\in H_a} \mathcal{H}^n(P_x \cap \Delta) + \sum_{x\in H_b} \mathcal{H}^n(P_x \cap \Delta),$$

It is immediate that for $x \in H_a$ we have $\Delta \subset 2B_x$. Thus, since the balls $2B_x$ are pairwise disjoint, H_a contains at most one point z, and so,

$$\sum_{x\in H_a} \mathcal{H}^n(P_x \cap \Delta) = \mathcal{H}^n(P_z \cap \Delta) \leq c\, r(\Delta)^n.$$

On the other hand, for $x \in H_b$, we have $B_x \subset 5\Delta$. Recall also that, since B_x is centered at some point from F_p,

$$\mu(B_x) \geq \frac{1}{p}\, r(x)^n \tag{3.2}$$

(recall that $r(x) = \frac{1}{2}d(x) \leq D$ because $d(x)$ is the distance between two points in $\operatorname{supp}\mu$). As a consequence,

$$\sum_{x\in H_b} \mathcal{H}^n(P_x\cap\Delta) \leq c \sum_{x\in H_b} r(x)^n \leq cp \sum_{x\in H_b} \mu(B_x) \leq cp\, \mu(5\Delta) \leq cp\, r(\Delta)^n,$$

by the growth of degree n of μ.

Lower AD-regularity of $\mu_{p,s}$. Consider an arbitrary closed ball Δ centered at $\operatorname{supp}\mu_{p,s}$. Suppose first that it is centered at some point $z \in F_{p,s}$.

First we claim that if $2B_x$, $x \in H_{p,s}$, intersects $\frac{1}{2}\Delta$, then $2B_x \subset \Delta$. Indeed, recalling that the function $d(\cdot)$ is $\frac{1}{10}$-Lipschitz, we get

$$10\, d(x) = \operatorname{dist}(x, F_{p,s}) \leq |x - z| \leq \frac{1}{2}\, r(\Delta) + 2r(x) = \frac{1}{2}\, r(\Delta) + d(x).$$

Thus,

$$r(2B_x) = 2r(x) = d(x) \leq \frac{1}{18}\, r(\Delta),$$

which clearly implies that $2B_x \subset \Delta$.

From the claim above and the definition of $\mu_{p,s}$ we deduce that

$$\mu_{p,s}(\Delta) \geq \mu(F_{p,s} \cap \Delta) + c^{-1} \sum_{x \in H_{p,s}: 2B_x \cap \frac{1}{2}\Delta \neq \varnothing} r(x)^n.$$

Since μ has growth of degree n, and since

$$\bigcup_{x \in H_{p,s}: 2B_x \cap \frac{1}{2}\Delta \neq \varnothing} 2B_x \supset F_p \cap \frac{1}{2}\Delta \setminus F_{p,s}\,,$$

we derive

$$\begin{aligned}\mu_{p,s}(\Delta) &\geq \mu(F_{p,s} \cap \Delta) + c^{-1} \sum_{x \in H_{p,s}: 2B_x \cap \frac{1}{2}\Delta \neq \varnothing} \mu(2B_x) \\ &\geq \mu\Big(F_{p,s} \cap \frac{1}{2}\Delta\Big) + c^{-1}\mu\Big(F_p \cap \frac{1}{2}\Delta \setminus F_{p,s}\Big) \geq c^{-1}\mu\Big(F_p \cap \frac{1}{2}\Delta\Big).\end{aligned}$$

Since Δ is centered at some point from $F_{p,s}$,

$$\mu\Big(F_p \cap \frac{1}{2}\Delta\Big) \geq c^{-1}\, \frac{1}{ps}\, r(\Delta)^n,$$

provided that $r(\Delta) \leq 2D$. On the other hand, if $r(\Delta) \geq 2D$ then $\operatorname{dist}(z, \Pi_{p,s}) \leq D$, and, thereby,

$$\mu_{p,s}(\Delta) \geq \mathcal{H}^n(B(z', \frac{1}{2} r(\Delta)) \geq c\, r(\Delta)^n\,,$$

where z' is the nearest to z point of $\Pi_{p,s}$.

If Δ is centered on $\Pi_{p,s}$, the lower bound is trivial.

Suppose now that Δ is centered at some point $z \in P_x$, for some $x \in H_{p,s}$. If $r(\Delta) \leq 40r(B_x)$, from the lower AD-regularity of $\mathcal{H}^n \lfloor P_x$ we infer that $\mu(\Delta) \geq c^{-1}\, r(\Delta)^n$. Assume now that $r(\Delta) > 40r(x) = 20d(x)$. In this case, by the definition of $d(x)$, there exists some $y \in F_{p,s}$ satisfying

$$|z-y| \leq |z-x| + |x-y| \leq r(x) + 10d(x) = \frac{21}{2}\, d(x) \leq \frac{21}{40}\, r(\Delta).$$

Thus, Δ contains the ball $B(y, \frac{1}{10}\, r(\Delta))$. Then, since

$$\mu\Big(B\Big(y, \frac{1}{10}\, r(\Delta)\Big)\Big) \geq c^{-1}\, \frac{1}{ps}\, r(\Delta)^n,$$

we are done.

Boundedness of $R_{\mu_{p,s}}$ in $L^2(\mu_{p,s})$. Taking into account that $R_{\mu\lfloor F_{p,s}}$ is bounded in $L^2(\mu\lfloor F_{p,s})$, and that $R_{\mathcal{H}^n\lfloor \Pi_{p,s}}$ is bounded in $L^2(\mathcal{H}^n\lfloor \Pi_{p,s})$, it is enough to show that $R_{\sigma^i_{p,s}}$ is bounded in $L^2(\sigma^i_{p,s})$ for each $i = 1, \ldots, N_d$. Then the repeated application of Proposition 3.1 yields the result.

To simplify notation, for fixed p, s, i, we denote $\sigma = \sigma^i_{p,s}$, $H = H^i_{p,s}$.

Now we define

$$\nu = \sum_{x\in H} c_x\, \mu\lfloor B_x,$$

with $c_x = \mathcal{H}^n(P_x)/\mu(B_x)$. Observe that the constants c_x, $x \in H$, are uniformly bounded by some constant depending on p, because of (3.2), and thus R_ν is bounded in $L^2(\nu)$. Further, $\nu(B_x) = \sigma(B_x)$ for each $x \in H$. Recall also that, by construction both σ and ν are supported on the union of the balls $B_x, x \in H$, and the double balls $2B_x$ are pairwise disjoint.

It is clear that, in a sense, ν can be considered as an approximation of σ (and conversely). To prove the boundedness of R_σ in $L^2(\sigma)$, we will prove that $\widetilde{R}_{\sigma,\varepsilon}$ is bounded in $L^2(\sigma)$ uniformly on $\varepsilon > 0$ by comparing it to $\widetilde{R}_{\nu,\varepsilon}$. First we need to introduce some local and non local operators: given $z \in \bigcup_{x\in H} B_x$, we denote by $B(z)$ the ball B_x, $x \in H$, that contains z. Then we write, for $z \in \bigcup_{x,x\in H} B_x$,

$$R^{loc}_{\nu,\varepsilon} f(z) = \widetilde{R}_{\nu,\varepsilon}(f\chi_{B(z)})(z), \qquad R^{nl}_{\nu,\varepsilon} f(z) = \widetilde{R}_{\nu,\varepsilon}(f\chi_{\mathbb{R}^d\setminus B(z)})(z).$$

We define analogously $R^{loc}_{\sigma,\varepsilon} f$ and $R^{nl}_{\sigma,\varepsilon} f$. It is straightforward to check that $R^{loc}_{\nu,\varepsilon}$ is bounded in $L^2(\nu)$, and that $R^{loc}_{\sigma,\varepsilon}$ is bounded in $L^2(\sigma)$, both uniformly on ε (in other words, R^{loc}_ν is bounded in $L^2(\nu)$ and R^{loc}_σ is bounded in $L^2(\sigma)$). Indeed,

$$\|R^{loc}_{\sigma,\varepsilon} f\|^2_{L^2(\sigma)} = \sum_{x\in H} \|\chi_{B_x} \widetilde{R}_{\sigma,\varepsilon}(f\chi_{B_x})\|^2_{L^2(\sigma)} \leq c \sum_{x\in H} \|f\chi_{B_x}\|^2_{L^2(\sigma)} = c\|f\|^2_{L^2(\sigma)},$$

by the boundedness of the n-Riesz transforms on n-planes. Using the boundedness of R_ν in $L^2(\nu)$, one derives the $L^2(\nu)$ boundedness of $R^{loc}_{\nu,\varepsilon}$ analogously.

Boundedness of R^{nl}_σ in $L^2(\sigma)$. We must show that R^{nl}_σ is bounded in $L^2(\sigma)$. Observe first that, since $R^{nl}_{\nu,\varepsilon} = \widetilde{R}_{\nu,\varepsilon} - R^{loc}_{\nu,\varepsilon}$, and both $\widetilde{R}_{\nu,\varepsilon}$ and $R^{loc}_{\nu,\varepsilon}$ are bounded in $L^2(\nu)$, it turns out that $R^{nl}_{\nu,\varepsilon}$ is bounded in $L^2(\nu)$ (all uniformly on $\varepsilon > 0$).

We will prove below that, for all $f \in L^2(\nu)$ and $g \in L^2(\sigma)$ satisfying

$$\int_{B_x} f\, d\nu = \int_{B_x} g\, d\sigma \qquad \text{for all } x \in H, \tag{3.3}$$

we have

$$(3.4)\quad I(f,g) := \int |R^{nl}_{\nu,\varepsilon}f - R^{nl}_{\sigma,\varepsilon}g|^2\, d(\nu+\sigma) \le c\,(\|f\|^2_{L^2(\nu)} + \|g\|^2_{L^2(\sigma)}),$$

uniformly on ε. Let us see how the boundedness of R^{nl}_σ in $L^2(\sigma)$ follows from this estimate. As a preliminary step, we show that $R^{nl}_\sigma : L^2(\sigma) \to L^2(\nu)$ is bounded. To this end, given $g \in L^2(\sigma)$, we consider a function $f \in L^2(\nu)$ satisfying (3.3) that is constant on each ball B_x. It is straightforward to check that

$$\|f\|_{L^2(\nu)} \le \|g\|_{L^2(\sigma)}.$$

Then from the $L^2(\nu)$ boundedness of R^{nl}_ν and (3.4), we obtain

$$\|R^{nl}_{\sigma,\varepsilon}g\|_{L^2(\nu)} \le \|R^{nl}_{\nu,\varepsilon}f\|_{L^2(\nu)} + I(f,g)^{1/2} \le c\|f\|_{L^2(\nu)} + c\,\|g\|_{L^2(\sigma)} \le c\,\|g\|_{L^2(\sigma)},$$

which proves that $R^{nl}_\sigma : L^2(\sigma) \to L^2(\nu)$ is bounded.

Notice that R^{nl}_ε is antisymmetric. Indeed, its kernel is

$$\left[1 - \sum_{x\in H} \chi_{B_x}(z)\chi_{B_x}(y)\right] \frac{z-y}{\max(|z-y|,\varepsilon)^{n+1}}\,.$$

Then, by duality, we deduce that $R^{nl}_\nu : L^2(\nu) \to L^2(\sigma)$ is bounded. To prove now the $L^2(\sigma)$ boundedness of R^{nl}_σ, we consider an arbitrary function $g \in L^2(\sigma)$, and we construct $f \in L^2(\nu)$ satisfying (3.3) which is constant in each ball B_x. Again, we have $\|f\|_{L^2(\nu)} \le \|g\|_{L^2(\sigma)}$.

Using the boundedness of $R^{nl}_\nu : L^2(\nu) \to L^2(\sigma)$ together with (3.4), we obtain

$$\|R^{nl}_{\sigma,\varepsilon}g\|_{L^2(\sigma)} \le \|R^{nl}_{\nu,\varepsilon}f\|_{L^2(\sigma)} + I(f,g)^{1/2} \le c\|f\|_{L^2(\nu)} + c\,\|g\|_{L^2(\sigma)} \le c\,\|g\|_{L^2(\sigma)},$$

as wished.

It remains to prove that (3.4) holds for $f \in L^2(\nu)$ and $g \in L^2(\sigma)$ satisfying (3.3). For $z \in \bigcup_{x\in H} B_x$, we have

$$|R^{nl}_{\nu,\varepsilon}f(z) - R^{nl}_{\sigma,\varepsilon}g(z)| \le \sum_{x\in H: z\notin B_x} \left| \int_{B_x} K_\varepsilon(z-y)(f(y)\,d\nu(y) - g(y)\,d\sigma(y)) \right|,$$

where $K_\varepsilon(z)$ is the kernel of the ε-regularized n-Riesz transform. By standard estimates, using (3.3), the fact that the balls $2B_x$, $x \in H$, are pairwise disjoint, and the smoothness of K_ε, it follows that

$$\begin{aligned}\left|\int_{B_x} K_\varepsilon(z-y)(f(y)\,d\nu(y)-g(y)\,d\sigma(y))\right| \\ &= \left|\int_{B_x} (K_\varepsilon(z-y)-K_\varepsilon(z-x))(f(y)\,d\nu(y)-g(y)\,d\sigma(y))\right| \\ &\leq c\int_{B_x} \frac{|x-y|}{|x-y|^{n+1}}(|f(y)|\,d\nu(y)+|g(y)|\,d\sigma(y)) \\ &\approx \frac{r(x)}{\operatorname{dist}(B(z),B_x)^{n+1}}\int_{B_x}(|f|\,d\nu+|g|\,d\sigma).\end{aligned}$$

Recall that $B(z)$ stands for the ball $B_x, x\in H$, that contains z.

We consider the operators

$$T_\nu(f)(z)=\sum_{x\in H:z\notin B_x}\frac{r(x)}{\operatorname{dist}(B(z),B_x)^{n+1}}\int_{B_x} f\,d\nu\,,$$

and T_σ is defined in the same way with ν replaced by σ. Observe that

$$\begin{aligned}I(f,g) &\leq c\|T_\nu(|f|)+T_\sigma(|g|)\|^2_{L^2(\nu+\sigma)} \\ &\leq c\|T_\nu(|f|)\|^2_{L^2(\nu+\sigma)}+c\|T_\sigma(|g|)\|^2_{L^2(\nu+\sigma)} \\ &= 2c\|T_\nu(|f|)\|^2_{L^2(\nu)}+2c\|T_\sigma(|g|)\|^2_{L^2(\sigma)},\end{aligned}$$

where, for the last equality, we took into account that both $T_\nu(|f|)$ and $T_\sigma(|g|)$ are constant on each ball B_x and that $\nu(B_x)=\sigma(B_x)$ for all $x\in H$.

To finish the proof of (3.4) it is enough to show that T_ν is bounded in $L^2(\nu)$ and T_σ in $L^2(\sigma)$. We only deal with T_ν, since the arguments for T_σ are analogous. We argue by duality again. So we consider non-negative functions $f,h\in L^2(\nu)$. We have

$$\begin{aligned}\int T_\nu(f)\,h\,d\nu &\approx \int\left(\sum_{x\in H:z\notin B_x}\frac{r(x)}{\operatorname{dist}(z,B_x)^{n+1}}\int_{B_x} f\,d\nu\right)h(z)\,d\nu(z) \\ &= \sum_{x\in H} r(x)\int_{B_x} f\,d\nu\int_{\mathbb{R}^d\setminus B_x}\frac{1}{\operatorname{dist}(z,B_x)^{n+1}}\,h(z)\,d\nu(z).\end{aligned}$$

From the growth of degree n of ν and the fact that the balls $2B_x$ are disjoint, it follows easily that

$$\int_{\mathbb{R}^d\setminus B_x}\frac{1}{\operatorname{dist}(z,B_x)^{n+1}}\,h(z)\,d\nu(z)\leq\frac{c}{r(x)}\,M_\nu h(y),$$

for all $y \in B_x$, where M_ν stands for the (centered) maximal Hardy-Littlewood operator (with respect to ν). Then we deduce that

$$\int T_\nu(f)\, h\, d\nu \lesssim \sum_{x\in H} \int_{B_x} f(y)\, M_\nu h(y)\, d\nu(y) \lesssim \|f\|_{L^2(\nu)} \|h\|_{L^2(\nu)},$$

by the $L^2(\nu)$ boundedness of M_ν. Thus T_ν is bounded in $L^2(\nu)$. □

4. Proof of Theorem 1.2

It is already known that if $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$ is removable for Lipschitz harmonic functions, then it must be purely n-unrectifiable (see [MPa]). So we only have to show that if E is non removable, then it is not purely n-unrectifiable. The proof follows from Theorem 1.1 and results from [Vo], by standard arguments. However, for the reader's convenience we show all the details.

Let $E \subset \mathbb{R}^{n+1}$ be a compact set such that $\mathcal{H}^n(E) < \infty$. Suppose that it is not removable for Lipschitz harmonic functions. Then, by Theorem 2.2 from [Vo], there exists some measure μ supported on E with growth of degree n such that R_μ is bounded in $L^2(\mu)$.

The growth condition on μ implies that $\mu \ll \mathcal{H}^n\lfloor E$. Indeed, let $A \subset \mathbb{R}^{n+1}$ be a Borel set. Consider $\varepsilon > 0$ and any covering $A \subset \bigcup_i A_i$ with $\mathrm{diam} A_i \leq \varepsilon$. For each $i \geq 1$, let B_i be a closed ball of radius equal to $\mathrm{diam}(A_i)$ centered at some point from A_i. Since $A \subset \bigcup_i B_i$, we have

$$\mu(A) \leq \sum_i \mu(B_i) \leq c_0 \sum_i \mathrm{diam}(A_i)^n.$$

Taking the infimum over all the coverings $\bigcup_i A_i$ as above, we obtain

$$\mu(A) \leq c_0\, \mathcal{H}^n(A) \quad \text{for any Borel set } A \subset \mathbb{R}^d.$$

As a consequence,

$$\mu(A) = \mu(A\cap E) \leq c_0\, \mathcal{H}^n(A \cap E) \quad \text{for any Borel set } A \subset \mathbb{R}^d. \tag{4.1}$$

That is, $\mu \ll \mathcal{H}^n\lfloor E$.

From the Radon-Nykodim theorem we infer that there exists some function $g \in L^1(\mathcal{H}^n\lfloor E)$ such that $\mu = g\, \mathcal{H}^n\lfloor E$. In fact, from (4.1) and the Lebesgue differentiation theorem we deduce that $\|g\|_\infty \leq c_0$. Take $\eta > 0$ small enough so that

$$F = \{x \in E : g(x) > \eta\}$$

satisfies $\mu(F) > 0$ (and thus $\mathcal{H}^n(F) > 0$ too). Since $\mu\lfloor F = g\, \mathcal{H}^n\lfloor F$, we have

$$\mathcal{H}^n\lfloor F = g^{-1} \mu\lfloor F.$$

As g^{-1} is bounded by η^{-1} in F, we deduce that $R_{\mathcal{H}^n\lfloor F}$ is bounded in $L^2(\mathcal{H}^n\lfloor F)$. As a consequence, F is n-rectifiable, by Theorem 1.1. That is, E is not purely n-unrectifiable.

References

[Da1] G. David, *Wavelets and singular integrals on curves and surfaces,* Lecture Notes in Math. 1465, Springer-Verlag, Berlin, 1991.

[Da2] G. David, *Unrectifiable* 1*-sets have vanishing analytic capacity,* Revista Mat. Iberoamericana 14(2) (1998), 369–479.

[DM] G. David and P. Mattila. *Removable sets for Lipschitz harmonic functions in the plane.* Rev. Mat. Iberoamericana 16(1) (2000), 137–215.

[DS] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets,* Mathematical Surveys and Monographs, 38, American Mathematical Society, Providence, RI, 1993.

[ENV] V. Eiderman, F. Nazarov and A. Volberg. *The s-Riesz transform of an s-dimensional measure in* $\mathbb{R}^2$ *is unbounded for* $1 < s < 2$*,* to appear in J. Anal. Math.

[Lé] J.C. Léger, *Menger curvature and rectifiability,* Ann. of Math. 149 (1999), 831–869.

[Ma] P. Mattila. *Geometry of sets and measures in Euclidean spaces,* Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.

[MPa] P. Mattila and P.V. Paramonov. *On geometric properties of harmonic* Lip_1*-capacity,* Pacific J. Math. 171:2 (1995), 469–490.

[MPr] P. Mattila and D. Preiss, *Rectifiable measures in Rn and existence of principal values for singular integrals,* J. London Math. Soc. (2) 52 (1995), no. 3, 482?-496.

[Me] M.S. Melnikov, *Analytic capacity: discrete approach and curvature of a measure,* Sbornik: Mathematics 186(6) (1995), 827–846.

[MeV] M.S. Melnikov and J. Verdera, *A geometric proof of the* L^2 *boundedness of the Cauchy integral on Lipschitz graphs.* Internat. Math. Res. Notices (1995), 325–331.

[NTrV1] F. Nazarov, S. Treil and A. Volberg, *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators in nonhomogeneous spaces,* Int. Math. Res. Notices **9** (1998), 463–487.

[NTrV2] F. Nazarov, S. Treil and A. Volberg, *The Tb-theorem on non-homogeneous spaces that proves a conjecture of Vitushkin* , CRM preprint No. 519 (2002), pp. 1–84.

[NToV] F. Nazarov, X. Tolsa and A. Volberg, *On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension* 1, Preprint (2012).

[Pa1] H. Pajot, *Théorème de recouvrement par des ensembles Ahlfors-réguliers et capacité analytique.* (French) C. R. Acad. Sci. Paris Sr. I Math. 323 (1996), no. 2, 133?-135.

[Pa2] H. Pajot, *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral,* Lecture Notes in Math. 1799, Springer, 2002.

[Par] P.V. Paramonov, *On harmonic approximation in the* $\mathcal{C}^1$ *norm*, Math. USSR-Sb., 71 (1992), 183–207.

[To1] X. Tolsa, *Principal values for Riesz transforms and rectifiability.* J. Funct. Anal., vol. 254(7) (2008), 1811-1863.

[To2] X. Tolsa, *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory.* To appear (2012).

[Vo] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces.* CBMS Regional Conf. Ser. in Math. 100, Amer. Math. Soc., Providence, 2003.

Fedor Nazarov, Department of Mathematical Sciences, Kent State University, Kent, Ohio, USA

Xavier Tolsa, ICREA/Universitat Autònoma de Barcelona, Barcelona, Catalonia

Alexander Volberg, Department of mathematics, Michigan State University, East Lansing, Michigan, USA