SPDEs with rough noise in space: Hölder continuity of the solution

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Abstract

We consider the stochastic wave and heat equations with affine multiplicative Gaussian noise which is white in time and behaves in space like the fractional Brownian motion with index $H \in (\frac{1}{4}, \frac{1}{2})$. The existence and uniqueness of the solution to these equations has been proved recently in [1]. In the present note we show that these solutions have modifications which are Hölder continuous in space of order smaller than H, and Hölder continuous in time of order smaller than γ, where $\gamma = H$ for the wave equation and $\gamma = H/2$ for the heat equation.

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1 Introduction

In this article, we consider the stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{X}(t,x), \quad t \in [0,T],\ x \in \mathbb{R} \\ u(0,x) = u_0(x), \\ \dfrac{\partial u}{\partial t}(0,x) = v_0(x), \end{cases} \tag{SWE}$$

and the stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}(t,x) + \sigma(u(t,x))\dot{X}(t,x), \quad t \in [0,T],\ x \in \mathbb{R} \\ u(0,x) = u_0(x) \end{cases} \tag{SHE}$$

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where $\sigma(x) = ax+b$ with $a, b \in \mathbb{R}$, and u_0 and v_0 are uniformly Hölder continuous of order H. We assume that the noise $\dot{X}$ is white in time and behaves in space like a fractional Brownian motion (fBm) with index $H \in (\frac{1}{4}, \frac{1}{2})$. More precisely, $\dot{X}$ is the formal derivative of a zero-mean Gaussian process $X = \{X(\varphi); \varphi \in C_0^\infty((0,\infty)\times\mathbb{R})\}$ with covariance given by:

$$E[X(\varphi)X(\psi)] = c_H \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}|\xi|^{1-2H} d\xi dt, \tag{1}$$

for any $\varphi, \psi \in C_0^\infty((0,\infty)\times\mathbb{R})$, where $c_H = \Gamma(2H+1)\sin(\pi H)/(2\pi)$. Here $C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ denotes the space of infinitely differentiable functions on $\mathbb{R}_+ \times \mathbb{R}$ with compact support.

Since the Fourier transform of the measure $\mu(d\xi) = c_H|\xi|^{1-2H}d\xi$ is not given by a locally integrable function, the study of equations with this kind of noise does not fall under the general theory of SPDEs with colored noise initiated by [3] and [8]. Instead, the family X can be seen as a random stationary distribution (see [6, 9]), which allows to define stochastic integrals with respect to this type of noise following the ideas of Basse-O'Connor *et al.* [2]. See [1, Sec. 2] for a detailed description of the noise in this setting as well as the construction of the corresponding stochastic integrals.

In [1, Thm. 1.1], we proved the existence of a unique mild solution to equations (SWE) and (SHE) in the space $\mathcal{X}_p$, for any fixed $p \geq 2$, where the latter is defined as the space of $L^2(\Omega)$-continuous and adapted processes $u = \{u(t,x); t \in [0,T], x \in \mathbb{R}\}$ satisfying

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E\big[|u(t,x)|^p\big] < \infty$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\Big(E\big[|u(s,y)-u(s,z)|^p\big]\Big)^{2/p}}{|y-z|^{2-2H}} \, dy\, dz\, ds < \infty. \tag{2}$$

Here, $G_t(x)$ denotes the fundamental solution of the wave (respectively heat) equation, that is

$$G_t(x) = \frac{1}{2} 1_{\{|x|<t\}} \quad \text{for the wave equation,}$$

$$G_t(x) = \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{|x|^2}{2t}\right) \quad \text{for the heat equation.}$$

The method used in [1] to prove existence and uniqueness of solution is based on a Picard iteration scheme. We point out that the term (2) pops up in a quite natural way, for we used some harmonic analysis techniques related to fractional Sobolev spaces.

We recall that a random field $u = \{u(t,x); t \in [0,T], x \in \mathbb{R}\}$ is a solution of (SWE) (respectively (SHE)) if u is predictable and, for any $(t,x) \in [0,T] \in \mathbb{R}$,

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\, \sigma(u(s,y))\, X(ds,dy) \quad \text{a.s.}$$

where the stochastic integral is interpreted in the sense explained in [1, Sec. 2]. In the above expression, $w(t,x)$ denotes the solution of the corresponding homogeneous equation; see the beginning of Section 2 for the precise expression of $w(t,x)$ for wave and heat equations.

The goal of the present note is to show that the solutions of (SWE) and (SHE) have Hölder continuous modifications in space and time. More precisely, we will prove the following result.

Theorem 1.1. *Let $u = \{u(t,x); t \in [0,T], x \in \mathbb{R}\}$ be the solution to equation* (SWE)*, respectively equation* (SHE)*. There exists $h_0 \in (0,1)$ such that, for all $|h| \leq h_0$ and for any $p \geq 2$, we have:*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \Big(E\big[|u(t,x+h)-u(t,x)|^p\big]\Big)^{\frac{1}{p}} \leq C_p\,|h|^H$$

and

$$\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \Big(E\big[|u(t+h,x)-u(t,x)|^p\big]\Big)^{\frac{1}{p}} \leq C_p\,|h|^\gamma,$$

where $C_p > 0$ is a constant depending on p, and $\gamma = H$ for the wave equation and $\gamma = \frac{H}{2}$ for the heat equation. Therefore, the random field u has a modification that has (γ', H')-Hölder-continuous sample paths, for any $\gamma' < \gamma$ and any $H' < H$.

We note that the stochastic heat equation with the same noise $\dot{X}$ as above has been thoroughly studied in the recent preprint [4], in the case of a Lipschitz function σ with Lipschitz derivative and such that $\sigma(0) = 0$. In this article, the authors have obtained an exponential upper bound for the p-th moment of the solution and have shown that that this solution has a Hölder continuous modification of order $(\frac{H}{2} - \varepsilon, H - \varepsilon)$ for any $\varepsilon > 0$ (see Theorem 4.31 of [4]). Moreover, in the case when $\sigma(x) = x$, the authors of [4] have obtained a Feynman-Kac representation for the moments of the solution to the heat equation, which was used to show that these moments grow exponentially in time (Theorems 5.7 and 5.8 of [4]). These impressive investigations were continued in the recent preprint [5] in which the authors computed the exact Lyapunov exponents and the lower and upper growth indices of the solution of the heat equation with noise $\dot{X}$, in the case $\sigma(x) = x$.

One of the key steps which allow the authors of [4] to obtain a solution with σ satisfying the above-mentioned conditions is based on a localization argument which is tied to the parabolic nature of the heat equation. In this sense, an important characteristic of our method in [1] is that we can deal with both heat and wave equations at the same time.

In the next Section we proceed to prove Theorem 1.1.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 will follow from a careful analysis of the p-th moments of the increments of the Picard iteration sequence.

Along this note $w(t,x)$ will denote the solution of the homogeneous wave equation with the same initial conditions as (SWE) (respectively of the homogeneous heat equation), that is :

$$w(t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}\Big(u_0(x+t) + u_0(x-t)\Big) \quad \text{for the wave equation and}$$

$$w(t,x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy \quad \text{for the heat equation.}$$

Let $(u^n)_{n\geq 0}$ be the Picard iteration scheme defined by: $u^0(t,x) = w(t,x)$ and

$$u^{n+1}(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sigma(u^n(s,y))X(ds,dy), \quad n \geq 0.$$

First, in Section 3.2 of [1] we proved the following result.

Theorem 2.1. *Let $p \geq 2$ be fixed. Then, for any $n \geq 0$,*

$$\left.\begin{array}{l} u^n(t,x) \text{ is well-defined for any } (t,x) \in [0,T]\times\mathbb{R}, \\ \displaystyle\sup_{(t,x)\in[0,T]\times\mathbb{R}} E|u^n(t,x)|^p < \infty, \quad \text{and} \\ \displaystyle\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G_{t-s}^2(x-y) \frac{\Big(E|u^n(s,y)-u^n(s,z)|^p\Big)^{2/p}}{|y-z|^{2-2H}} dy\, dz\, ds < \infty \end{array}\right\} \quad \text{(P)}$$

and, for any $h \in \mathbb{R}$ with $|h| < 1$,

$$\left.\begin{array}{r} \displaystyle\sup_{(t,x)\in[0,T]\times\mathbb{R}} E|u^n(t,x+h)-u^n(t,x)|^2 \leq C_n|h|^{2H} \\ \displaystyle\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} E|u^n(t+h,x)-u^n(t,x)|^2 \leq C_n|h|^{\beta}, \end{array}\right\} \quad \text{(Q)}$$

where $\beta = 2H$ for the wave equation, and $\beta = H$ for the heat equation. Here C_n is a constant which depends on n (and also on H, T, σ, u_0 and v_0).

On the other hand, an immediate consequence of [1, Thm 3.9] is that, for all $p \geq 2$,

$$\sup_{n\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}} E|u^n(t,x)|^p < \infty. \tag{3}$$

Now, we aim to improve property (Q) above in the following sense.

Proposition 2.2. *Let $p \geq 2$ and $h_0 \in (0,1)$. Then, for any $n \geq 0$,*

$$\left.\begin{array}{r} \displaystyle\sup_{(t,x)\in[0,T]\times\mathbb{R}} \Big(E[|u^n(t,x+h)-u^n(t,x)|^p]\Big)^{\frac{1}{p}} \leq C_n|h|^{H} \\ \displaystyle\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \Big(E[|u^n(t+h,x)-u^n(t,x)|^p]\Big)^{\frac{1}{p}} \leq C_n|h|^{\gamma}, \end{array}\right\} \quad \text{(Q')}$$

for all $|h| \leq h_0$, where $\gamma = H$ for the wave equation, and $\gamma = \frac{H}{2}$ for the heat equation, and the constant C_n satisfies

$$C_n \leq C\left(c(h_0) + \bar{c}(h_0)C_{n-1}\right).$$

The functions $c, \bar{c} : \mathbb{R} \to \mathbb{R}$ are non-negative and $\lim_{h_0\to 0} \bar{c}(h_0) = 0$. By definition, $C_{-1} = 0$.

Proof. We split the proof in four steps. We will only develop in detail the computations which are relevant to attain our main objective, so that the reader will be directed to [1] for similar arguments or computations.

Step 1. The case $n = 0$ follows from the first part of the proof of [1, Thm 3.7]. More precisely, for the wave equation we proved that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |w(t,x+h) - w(t,x)| \leq C\,|h|^H \tag{4}$$

and

$$\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} |w(t+h,x) - w(t,x)| \leq C\left(|h|^H + |h|\right) \leq C\left(1 + h_0^{1-H}\right)|h|^H.$$

On the other hand, for the heat equation we obtained the same estimate (4) for the space increments, and

$$\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} |w(t+h,x) - w(t,x)| \leq C\,|h|^{\frac{H}{2}}.$$

Thus, we obtain condition (Q') with $C_0 := C\left(1 + h_0^{1-H}\right) \leq C$.

Step 2. Induction step. We first consider the space increments of u^{n+1}. We have, thanks to a [Burkholder-Davis-Gundy]-type inequality for stochastic integrals with respect to our fractional noise X (see [1, Thm. 2.9]),

$$\left(E[|u^{n+1}(t,x+h) - u^{n+1}(t,x)|^p]\right)^{\frac{1}{p}} \leq C(I_0 + I_1 + I_2),$$

where $I_0 = |w(t,x+h) - w(t,x)|$,

$$I_1 = \left(E\left|\int_0^t \int_{\mathbb{R}^2} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 \frac{|\sigma(u^n(s,y)) - \sigma(u^n(s,z))|^2}{|y-z|^{2-2H}}\,dy\,dz\,ds\right|^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\begin{aligned} I_2 = \Bigg(E\Bigg|\int_0^t \int_{\mathbb{R}^2} \frac{|\sigma(u^n(s,z))|^2}{|y-z|^{2-2H}}\, &|(G_{t-s}(x+h-y) - G_{t-s}(x-y)) \\ &- (G_{t-s}(x+h-z) - G_{t-s}(x-z))|^2\,dy\,dz\,ds\Bigg|^{\frac{p}{2}}\Bigg)^{\frac{1}{p}}. \end{aligned}$$

We have already proved that $I_0 \leq C_0|h|^H$. Let us treat I_1. By Minkowski's inequality and using that σ is Lipschitz, we have

$$\begin{aligned} I_1^2 \leq C \int_0^t \int_{\mathbb{R}} &|G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 \\ &\times \left(\int_{\mathbb{R}} \frac{\left(E[|u^n(s,y+z) - u^n(s,y)|^p]\right)^{\frac{2}{p}}}{|z|^{2-2H}}\,dz\right)\,dy\,ds \\ =: C\,(I_1' + I_1''), \end{aligned}$$

where I_1' and I_1'' denote the integrals corresponding to the regions $\{|z| > h_0\}$, respectively $\{|z| \le h_0\}$, in the dz integral. By (3) and taking into account that $\int_{|z|>h_0} |z|^{2H-2} dz = C h_0^{2H-1}$, we have (as in page 22 of [1])

$$\begin{aligned} I_1' &\le C\, h_0^{2H-1} \int_0^t \int_{\mathbb{R}} |G_{t-s}(x+h-y) - G_{t-s}(x-y)|^2 \, dy \, ds \\ &\le C\, h_0^{2H-1} |h| = C\, h_0^{2H-1} |h|^{1-2H} |h|^{2H} \le C\, |h|^{2H}. \end{aligned}$$

On the other hand, by the induction hypothesis and using that $\int_{|z|\le h_0} |z|^{4H-2} dz = C\, h_0^{4H-1}$, it holds

$$I_1'' \le C\, C_n^2\, h_0^{4H-1} |h| \le C\, C_n^2\, h_0^{2H} |h|^{2H}.$$

Thus $I_1 \le C\,(1 + h_0^H C_n)|h|^H$.

In order to deal with I_2, we apply again Minkowski's inequality, the linear growth on σ and (3), and we argue as in the last part of page 22 in [1]:

$$\begin{aligned} I_2^2 &\le C \int_0^t \int_{\mathbb{R}} (1 - \cos(h|\xi|))\, |\mathcal{F} G_{t-s}(\xi)|^2\, |\xi|^{1-2H} \, d\xi \, ds \\ &\le C\, |h|^{2H}, \end{aligned}$$

which implies that $I_2 \le C\, |h|^H$. Hence, putting together the estimates for I_0, I_1 and I_2, we have proved that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \left(E[|u^{n+1}(t,x+h) - u^{n+1}(t,x)|^p] \right)^{\frac{1}{p}} \le C\, \left(C_0 + h_0^H C_n \right) |h|^H. \tag{5}$$

Step 3. Let us now consider the time increments. We consider the case $h \ge 0$, being similar the case $h < 0$. We have that

$$\left(E[|u^{n+1}(t+h,x) - u^{n+1}(t,x)|^p] \right)^{\frac{1}{p}} \le C(J_0 + J_1 + J_2),$$

where $J_0 = |w(t+h,x) - w(t,x)|$,

$$J_1 = \left(E \left| \int_t^{t+h} \int_{\mathbb{R}^2} \frac{|G_{t+h-s}(x-y)\sigma(u^n(s,y)) - G_{t+h-s}(x-z)\sigma(u^n(s,z))|^2}{|y-z|^{2-2H}} \, dy \, dz \, ds \right|^{\frac{p}{2}} \right)^{\frac{1}{p}},$$

$$\begin{aligned} J_2 = \Bigg(E \Bigg| \int_0^t \int_{\mathbb{R}^2} &|(G_{t+h-s}(x-y) - G_{t-s}(x-y))\sigma(u^n(s,y)) \\ &- (G_{t+h-s}(x-z) - G_{t-s}(x-z))\sigma(u^n(s,z))|^2 |y-z|^{2H-2} \, dy \, dz \, ds \Bigg|^{\frac{p}{2}} \Bigg)^{\frac{1}{p}}. \end{aligned}$$

We have already seen that $J_0 \le C_0 |h|^\gamma$.

As far as J_1 is concerned, we apply Minkowski's inequality and we add and subtract the term $G_{t+h-s}(x-y)\sigma(u^n(s,z))$. We obtain

$$\begin{aligned} J_1^2 &\le \int_t^{t+h} \int_{\mathbb{R}^2} \frac{\left(E[|G_{t+h-s}(x-y)\sigma(u^n(s,y)) - G_{t+h-s}(x-z)\sigma(u^n(s,z))|^p]\right)^{\frac{2}{p}}}{|y-z|^{2-2H}}\, dy\, dz\, ds \\ &\le C\,(J_{11} + J_{12}), \end{aligned}$$

where

$$J_{11} = \int_t^{t+h} \int_{\mathbb{R}^2} G_{t+h-s}^2(x-y) \frac{\left(E[|\sigma(u^n(s,y)) - \sigma(u^n(s,z))|^p]\right)^{\frac{2}{p}}}{|y-z|^{2-2H}}\, dz\, dy\, ds,$$

$$J_{12} = \int_t^{t+h} \int_{\mathbb{R}^2} \left(E[|\sigma(u^n(s,z))|^p]\right)^{\frac{2}{p}} \frac{|G_{t+h-s}(x-y) - G_{t+h-s}(x-z)|^2}{|y-z|^{2-2H}}\, dy\, dz\, ds.$$

First, we have

$$\begin{aligned} J_{11} &\le C \int_t^{t+h} \int_{\mathbb{R}} G_{t+h-s}^2(x-y) \left(\int_{\mathbb{R}} \frac{\left(E[|u_n(s,y+z) - u_n(s,z)|^p]\right)^{\frac{2}{p}}}{|z|^{2-2H}}\, dz \right) dy\, ds \\ &= C\,(J_{11}' + J_{11}''), \end{aligned}$$

where the latter are defined by splitting the dz integrals into two integrals corresponding to the regions $\{|z| > h_0\}$ and, $\{|z| \le h_0\}$, respectively. By (3) and using that $\int_{|z|>h_0} |z|^{2H-2} dz = C\, h_0^{2H-1}$, we obtain (as in page 23 of [1])

$$\begin{aligned} J_{11}' &\le C\, h_0^{2H-1} \int_0^h \int_{\mathbb{R}} G_s^2(s,y)\, dy\, ds \\ &\le C\, h_0^{2H-1} |h|^{\frac{\gamma}{H}} = C\, h_0^{2H-1+(\frac{1}{H}-2)\gamma} |h|^{2\gamma}, \end{aligned}$$

where we recall that $\gamma = H$ for the wave equation and $\gamma = \frac{H}{2}$ for the heat equation.

On the other hand, by the induction hypothesis, and using that $\int_{|z|\le h_0} |z|^{4H-2} dz = C\, h_0^{4H-1}$, and that $\frac{\gamma}{H} - 2\gamma > 0$ for any $\gamma > 0$, we have

$$J_{11}'' \le C\, h_0^{4H-1}\, C_n^2\, |h|^{\frac{\gamma}{H}} \le C\, C_n^2\, h_0^{4H-1+\frac{\gamma}{H}-2\gamma} |h|^{2\gamma}.$$

Observe that the quantity $4H - 1 + \frac{\gamma}{H} - 2\gamma$ is always positive.

Let us now deal with J_{12}. Indeed, by (3) and [1, Prop. 2.8] we have

$$J_{12} \le C \int_0^h \int_{\mathbb{R}} |\mathcal{F}G_r(\xi)|^2\, |\xi|^{1-2H}\, d\xi dr.$$

By Lemma 3.1 in [1], the last integral is equal to Ch^{2H+1} for the wave equation, and Ch^H for the heat equation. Hence,

$$J_{12} \le C\,\big(1 + h_0)|h|^{2\gamma}.$$

Thus, we have proved that

$$J_1 \leq C \left(1 + h_0^{\frac{1}{2}} + h_0^{H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma} + C_n\, h_0^{2H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma}\right)|h|^{\gamma}.$$

Now we treat the term J_2. Arguing as in page 23 of [1] and applying Minkowski's inequality, we can infer that $J_2^2 \leq C\,(J_{21} + J_{22})$, where

$$\begin{aligned} J_{21} &= \int_0^t \int_{\mathbb{R}^2} |G_{t+h-s}(x-y) - G_{t-s}(x-y)|^2 \\ &\qquad \left(E[|\sigma(u^n(s,y)) - \sigma(u^n(s,z))|^p]\right)^{\frac{2}{p}} |y-z|^{2H-2}\, dy\, dz\, ds \\ &= \int_0^t \int_{\mathbb{R}^2} |G_{t+h-s}(x-y) - G_{t-s}(x-y)|^2 \\ &\qquad \left(E[|\sigma(u^n(s,y)) - \sigma(u^n(s,y+z))|^p]\right)^{\frac{2}{p}} |z|^{2H-2}\, dy\, dz\, ds, \end{aligned}$$

$$\begin{aligned} J_{22} &= \int_0^t \int_{\mathbb{R}^2} \left(E[|\sigma(u^n(s,z))|^p]\right)^{\frac{2}{p}} |(G_{t+h-s}(x-y) - G_{t-s}(x-y)) - \\ &\qquad (G_{t+h-s}(x-z) - G_{t-s}(x-z))|^2\, |y-z|^{2H-2}\, dy\, dz\, ds. \end{aligned}$$

Similarly as before, we have $J_{21} \leq C\,(J_{21}' + J_{21}'')$, where J_{21}' and J_{21}'' are integrals corresponding to the regions $\{|z| > h_0\}$, respectively $\{|z| \leq h_0\}$. Then,

$$\begin{aligned} J_{21}' &\leq C\, h_0^{2H-1} \int_0^t \int_{\mathbb{R}} |G_{t+h-s}(x-y) - G_{t-s}(x-y)|^2 dy\, ds \\ &\leq C\, h_0^{2H-1} |h|^{\frac{\gamma}{H}} \leq C\, h_0^{2H-1+(\frac{1}{H}-2)\gamma} |h|^{2\gamma}. \end{aligned}$$

On the other hand, by Lipschitz condition and the induction hypothesis, we have

$$J_{21}'' \leq C\, h_0^{4H-1}\, C_n^2\, |h|^{\frac{\gamma}{H}} \leq C\, C_n^2\, h_0^{4H-1+\frac{\gamma}{H}-2\gamma} |h|^{2\gamma}.$$

Finally, using that $E[|\sigma(u^n(s,z))|^p]$ is uniformly bounded on s, z and n, and Proposition 2.8 of [1],

$$\begin{aligned} J_{22} &\leq C \int_0^t \int_{\mathbb{R}} |\mathcal{F}G_{t+h-s}(\xi) - \mathcal{F}G_{t-s}(\xi)|^2\, |\xi|^{1-2H}\, d\xi\, ds \\ &\leq C\, |h|^{2\gamma}. \end{aligned}$$

Hence, we have obtained that

$$J_2 \leq C \left(1 + h_0^{H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma} + C_n\, h_0^{2H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma}\right)|h|^{\gamma}.$$

Putting together the bounds for J_0, J_1 and J_2, we get

$$\begin{aligned} \sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} & \left(E[|u^{n+1}(t+h,x) - u^{n+1}(t,x)|^p]\right)^{\frac{1}{p}} \\ &\leq C \left(1 + C_0 + h_0^{\frac{1}{2}} + h_0^{H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma} + C_n\, h_0^{2H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma}\right)|h|^{\gamma}. \end{aligned} \tag{6}$$

Step 4. Finally, by estimates (5) and (6) we have property (Q') with a constant $C_{n+1} := C\left(c(h_0) + \bar{c}(h_0)C_n\right)$, where

$$c(h_0) = 1 + C_0 + h_0^{\frac{1}{2}} + h_0^{H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma}$$

and

$$\bar{c}(h_0) = h_0^{2H-\frac{1}{2}+(\frac{1}{2H}-1)\gamma}.$$

Observe that for any $\gamma > 0$ it holds that $2H - \frac{1}{2} + (\frac{1}{2H} - 1)\gamma > 0$. Hence $\lim_{h_0 \to 0} \bar{c}(h_0) = 0$. This concludes the proof. □

Now, we are in position to prove our main result.

Proof of Theorem 1.1. Let us first prove that, choosing a small enough $h_0 \in (0,1)$, the sequence of constants $(C_n)_{n\geq 0}$ in property (Q') is bounded. Indeed, using the recursion for the constant C_{n+1}, one easily verifies that, for all $n \geq 0$,

$$\begin{aligned} C_{n+1} &= Cc(h_0)\big(1 + C\bar{c}(h_0) + [C\bar{c}(h_0)]^2 + \cdots + [C\bar{c}(h_0)]^n\big) + [C\bar{c}(h_0)]^{n+1}C_0 \\ &\leq \max(Cc(h_0), C_0) \sum_{k=0}^{n+1} [C\bar{c}(h_0)]^k, \end{aligned}$$

where we recall that $C_0 = C\left(1 + h_0^{1-H}\right)$. Thus, choosing h_0 small enough such that $C\bar{c}(h_0) < 1$, we get that $\sup_{n\geq 1} C_n < +\infty$.

Hence, we have obtained the validity of property (Q') with the constant C_n replaced by a constant C, which does not depend on n. At this point, taking limits as n tends to infinity in (Q'), one gets the first part of the statement, since we already know that u^n converges to u in $L^2(\Omega)$, uniformly in time and space.

The second part of the statement follows from a d-dimensional parameter version of Kolmogorov criterion of continuity; see, for instance, Theorem 1.4.1 of [7]. The proof is complete. □

Remark 2.3. An important consequence of Theorem 1.1 is that, in fact, the solutions of our SPDEs belong to a smaller space than the space $\mathcal{X}_p$ defined in the Introduction (see also [1, Def. 3.6]). Precisely, for any $p \geq 2$, they belong to the space of adapted random fields $\{u(t,x); t \in [0,T], x \in \mathbb{R}\}$ satisfying the following three conditions:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} E[|u(t,x)|^p] < \infty,$$

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \big(E|u(t,x+h) - u(t,x)|^p\big)^{\frac{1}{p}} \leq C_p\,|h|^H$$

and

$$\sup_{(t,x)\in[0,T\wedge(T-h)]\times\mathbb{R}} \big(E|u(t+h,x) - u(t,x)|^p\big)^{\frac{1}{p}} \leq C_p\,|h|^\gamma.$$

Indeed, it is easy to see, using the usual argument of splitting the dz integral, that the above conditions imply that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \int_0^t \int_{\mathbb{R}^2} G^2_{t-s}(x-y) \frac{\Big(E|u(s,y)-u(s,z)|^p\Big)^{2/p}}{|y-z|^{2-2H}}\, dy\, dz\, ds < \infty.$$

On the other hand, the processes belonging to the intersection for all $p \geq 2$ of these spaces have versions with (γ', H')-Hölder continuous paths for any $\gamma' < \gamma$ and any $H' < H$.

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