On the family $\boldsymbol{QSL}_3$ of quadratic systems with invariant lines of total multiplicity exactly 3

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Abstract

We denote by $\boldsymbol{QSL}_3$ the family of quadratic differential systems possessing invariant straight lines, finite and infinite, of total multiplicity exactly three. In a sequence of papers the complete study of quadratic systems with invariant lines of total multiplicity at least four was achieved. In addition three more families of quadratic systems possessing invariant lines of total multiplicity at least three were also studied, among them the Lotka-Volterra family. However there were still systems in $\boldsymbol{QSL}_3$ missing from all these studies. The goals of this article are: to complete the study of the geometric configurations of invariant lines of $\boldsymbol{QSL}_3$ by studying all the remaining cases and to give the full classification this family modulo their configurations of invariant lines together with their bifurcation diagram. The family $\boldsymbol{QSL}_3$ has a total of 81 distinct configurations of invariant lines. This classification is done in affine invariant terms and we also present the bifurcation diagram of these configurations in the 12-parameter space of coefficients of the systems. This diagram provides an algorithm for deciding for any given system whether it belongs to $\boldsymbol{QSL}_3$ and in case it does, by producing its configuration of invariant straight lines.

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1 Introduction and the statement of the Main Theorem

We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \quad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p, q \in \mathbb{R}[x,y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$. We call *degree* of a system (S) the integer $\deg(S) = \max(\deg(P), \deg(Q))$. We call quadratic (respectively cubic) differential system such a polynomial system of degree two (respectively three). We shall sometimes use quadratic system instead of quadratic differential system. Each such system generates a complex differential vector field when the dependent variables range over $\mathbb{C}$.

Of the three classical problems on these systems, Hilbert's 16th problem, the problem of Poincaré and the problem of the center, only this last one was solved for the family **QS** of quadratic differential systems. Although it is the simplest non-linear class of polynomial systems we are still far from understanding this family. To gain insight into this family, in recent years subfamilies of **QS** began to be studied from a global viewpoint using a variety of methods among them algebraic and geometric or analytical, also numerical or involving substantial symbolic calculations. In particular families of quadratic systems possessing invariant algebraic curves began to be studied, the simplest ones being those possessing invariant lines.

Every system in **QS** possesses an invariant line, the line at infinity. This line could be simple, or multiple in which case producing several distinct lines in perturbations.

The notion of multiplicity of an invariant line of a system (1) has been introduced in [9]. This concept was extended to the notion of multiplicity of an invariant algebraic curve of a differential system. In the fundamental article [6] several notions of multiplicity of an invariant algebraic curve of a polynomial systems were introduced and they were proven to be equivalent in the case of *algebraic solutions* which are algebraic invariant curves defined by polynomials that are irreducible over $\mathbb{C}$. If a system has a finite number of invariant lines $f_i(x, y) = 0$, $i = 1, ..., k$, of respective multiplicities $m_1, ..., m_k$, we call *total multiplicity of the invariant lines* of (S), the number $M = \sum_i m_i + m_\infty$ where m_∞ is the multiplicity of the line at infinity. Since in any system (1) the line at infinity is invariant we always have $m_\infty \geq 1$ and in particular we have this for any system in **QS**.

At the beginning of this century a systematic study of non-degenerated quadratic systems possessing invariant algebraic curves was initiated by Schlomiuk and Vulpe. In the series of articles [9, 11, 13, 14] the authors studied the class $\mathbf{QSL}_{\geq 4}$ of quadratic systems having invariant lines, including the line at infinity, of total multiplicity at least four. We see in [9] that the maximum number of invariant lines, including the line at infinity of non-degenerate quadratic systems is six.

This study was based on the notion of *configuration of invariant lines* of a real polynomial differential system defined in [14]. We recall here this definition.

Definition 1.1. *Consider a real polynomial differential system* (S) *endowed with a finite number of invariant algebraic curves* $f_i(x, y) = 0$, $i = 1, \ldots, k$ *over* $\mathbb{C}$. *We call configuration of invariant curves of* (S) *the set of curves* $f_1 = 0, \ldots, f_k = 0$ *and the line at infinity, each endowed with its own multiplicity, together with all the real singular points of* (S) *situated on these curves, each one of them endowed with its own multiplicity.*

The notion of configuration is an affine invariant which is a powerful classification tool. This was clearly seen in the way the topological classification was obtained for the Lotka-Volterra systems which have a total of 112 phase portraits. The geometry of configurations acts like a guiding light to fray our way through this maze of phase portraits. Thus we first obtained the geometric classification by splitting the class according to their 65 distinct configurations of invariant lines that the systems possess. Then we classified topologically each one of these 65 families.

In order to classify all the configurations of the family $\mathbf{QSL}_3$ we first need to say when two configurations $\mathcal{C}_1$, $\mathcal{C}_2$ of invariant lines of two quadratic systems (S_1) and (S_2) are to be considered as distinct, respectively when two such configurations are to be considered equivalent.

Consider two polynomial differential systems (S_1) and (S_2) such that each has a finite set of singular points and a finite set of invariant lines, including the line at infinity. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be the two configurations of invariant lines of (S_1) and (S_2).

Definition 1.2. *We say that two configurations $\mathcal{C}_1$, $\mathcal{C}_2$, of (S_1) and (S_2) formed by invariant lines (including the line at infinity) are equivalent if and only if there is a bijection ϕ between the two sets of invariant lines sending the line at infinity of $\mathcal{C}_1$ to the line at infinity of $\mathcal{C}_2$, sending a line with coefficients in $\mathbb{R}$ of (S_1) to a line with coefficients in $\mathbb{R}$ of (S_2). In addition the map preserves the multiplicities of the invariant lines, and for each invariant line L of $\mathcal{C}_1$ there is a one to one correspondence ϕ_L between the set of real singular points of (S_1) situated on the line L and the set of real singular points of the system (S_2) situated on the line $\phi(L)$ which preserves the multiplicities of the singular points and sends a real singular point at infinity to a real singular point at infinity. In addition we have the following:*

(i) When we list in a counterclockwise sense the real singular points at infinity on (S_1) starting from a point p on the Poincaré disk, $p_1 = p, ..., p_l$, this correspondence preserves the multiplicities of the singular points and preserves or reverses the orientation.

(ii) We consider the total curves

$$\mathcal{F} : \prod F_j(X;Y;Z)^{m_i} Z^m = 0; \mathcal{F}' : \prod F'_j(X;Y;Z)^{m'_i} Z^m = 0$$

where $F_i(X;Y;Z) = 0$ (respectively $F'_i(X;Y;Z) = 0$) are the projective completions of the lines $\mathcal{L}_i$ (respectively $\mathcal{L}'_i$) and $m_i; m'_i$ are the multiplicities of the curves $F_i = 0$; $F'_i = 0$ and m, m' are respectively the multiplicities of $Z = 0$ in the first and in the second system. Then, there is a one-to-one correspondence between the real singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singular points of the total curves.

After the study of the family $\mathbf{QSL}_{\geq 4}$ mentioned above, the next step is the study of the subfamily $\mathbf{QSL}_3$ of $\mathbf{QS}$ which is the family of all non-degenerate quadratic differential systems with invariant lines of total multiplicity three. The study of this class began with work on the Lotka-Volterra systems (shortly L-V systems), a family important for applications. (Previous literature on L-V systems systems is also mentioned in [16, 17].) This is the class of all quadratic differential systems that have two real invariant lines intersecting at a finite point. In [16, 17] the authors completed the study of this class by giving its bifurcation diagram in the 12-dimensional space of the coefficients of quadratic systems (1).

The family $\mathbf{QSL}_3$ which splits into several subfamilies of $\mathbf{QS}$ according to the geometry of the systems one of them being the L-V systems. Another subfamily of $\mathbf{QSL}_3$ is the family of non-degenerate real quadratic systems possessing two complex invariant lines intersecting at a (real) finite point. The topological classification for this family was done in [19] but without using the configurations of invariant lines. The bifurcation diagram in terms of invariant polynomials was done in [3]. But the configurations of invariant lines for systems in this family and occurring in $\mathbf{QSL}_3$ is presented here for the first time.

In [5] one more subfamily of $\mathbf{QSL}_3$ was studied. More exactly in [5] the study of the family $\mathbf{QSL}^{2p}$ of quadratic systems possessing one of the following defining properties: two parallel invariant lines or a unique affine line that is double, or an affine invariant line and the line at infinity double or the line at infinity triple.

However, we still have quadratic systems in $\mathbf{QSL}_3$ that were not mentioned so far. These are quadratic differential systems in $\mathbf{QSL}_3$ that are limit points of the L-V systems.

Indeed such systems could be obtained from a generic L-V system using the following one of the following three possibilities:

(i) Two simple invariant lines of a L-V-system from the subfamily $\mathbf{QSL}_3$ coalesced obtaining a double invariant line and a multiple real singular point at infinity.

(ii) One simple invariant line of a L-V system from the subfamily $\mathbf{QSL}_3$ coalesced with infinite line $Z = 0$ obtaining a double infinite invariant line with the second invariant line remaining in the finite part of the phase plane.

(iii) Both simple invariant lines of a L-V system from the subfamily $\mathbf{QSL}_3$ coalesced with infinite line $Z = 0$ producing a triple line at infinity.

The goal of this paper is to complete the study of the configurations of invariant lines of family $\boldsymbol{QSL}_3$ and to present all possible configurations of invariant lines which a non-degenerate quadratic system from the class $\mathbf{QSL_3}$ could have. Our main results are summed up in the following theorem:

Main Theorem. *The following statements hold:*

(i) The family $\mathbf{QSL}_3$ possesses a total of 81 distinct configurations of invariant lines given in Figure 1.

(ii) The classification of the family $\mathbf{QSL}_3$ is done using algebraic invariants and hence it is independent of the normal forms in which the systems may be presented.

(iii) The "bifurcation" diagram of the configurations of invariant lines for systems in the family $\mathbf{QSL}_3$ is done in the twelve-dimensional parameter space $\mathbb{R}^{12}$ and it is presented in Diagrams 1 and 2. These diagrams give us an algorithm by determining for any given system if it belongs or not to the family $\mathbf{QSL}_3$ and in case it belongs to this family, it gives us the specific configuration of invariant lines.

2 The main invariant polynomials associated to the class $\boldsymbol{QSL}_3$

We consider the class of real quadratic polynomial differential systems

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(\tilde{a},x,y), \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(\tilde{a},x,y) \end{aligned} \tag{2}$$

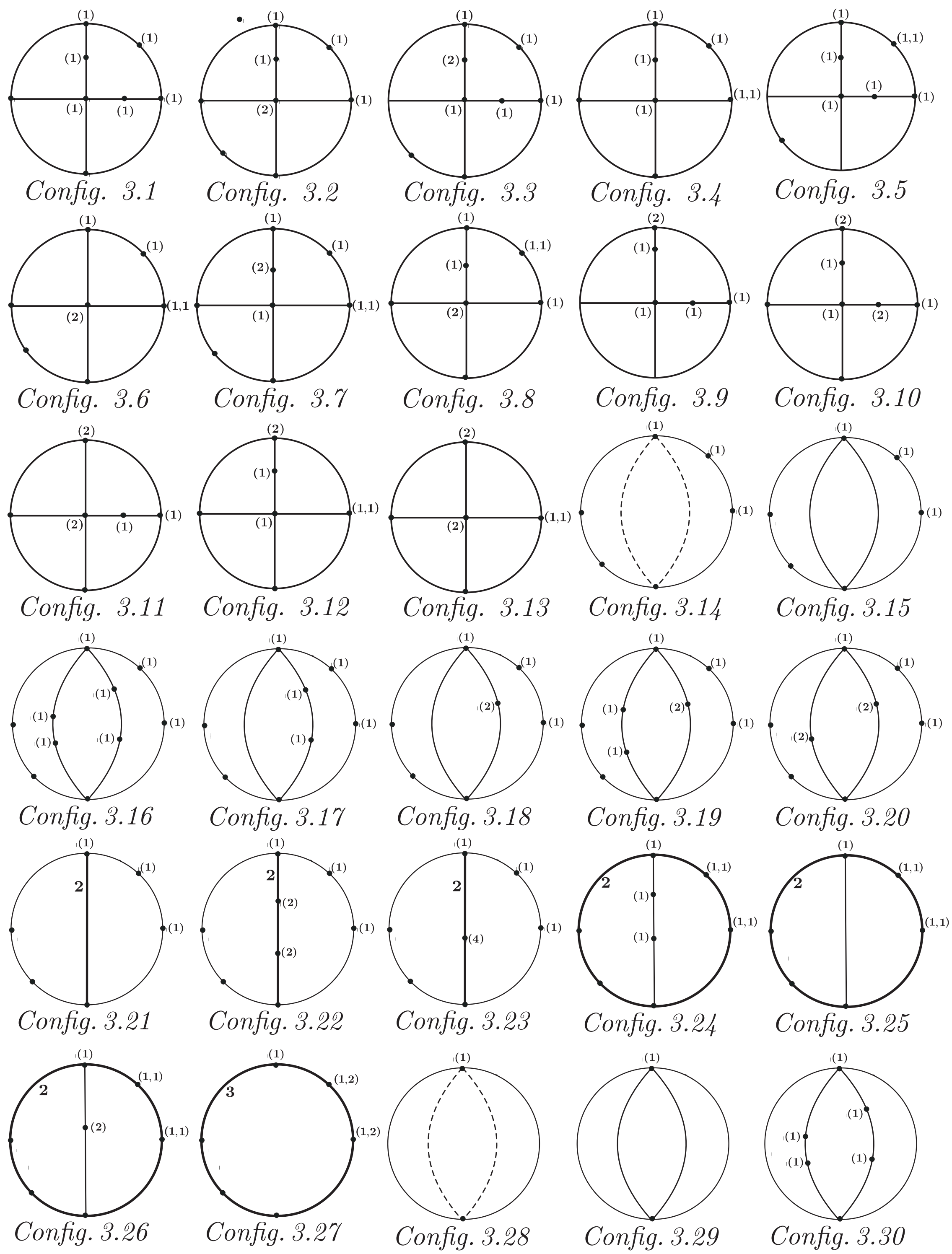

Figure 1: The configurations of quadratic systems in $\mathbf{QSL}_3$

where

$$p_0 = a, \quad p_1(x,y) = cx + dy, \quad p_2(x,y) = gx^2 + 2hxy + ky^2,$$
$$q_0 = b, \quad q_1(x,y) = ex + fy, \quad q_2(x,y) = lx^2 + 2mxy + ny^2$$

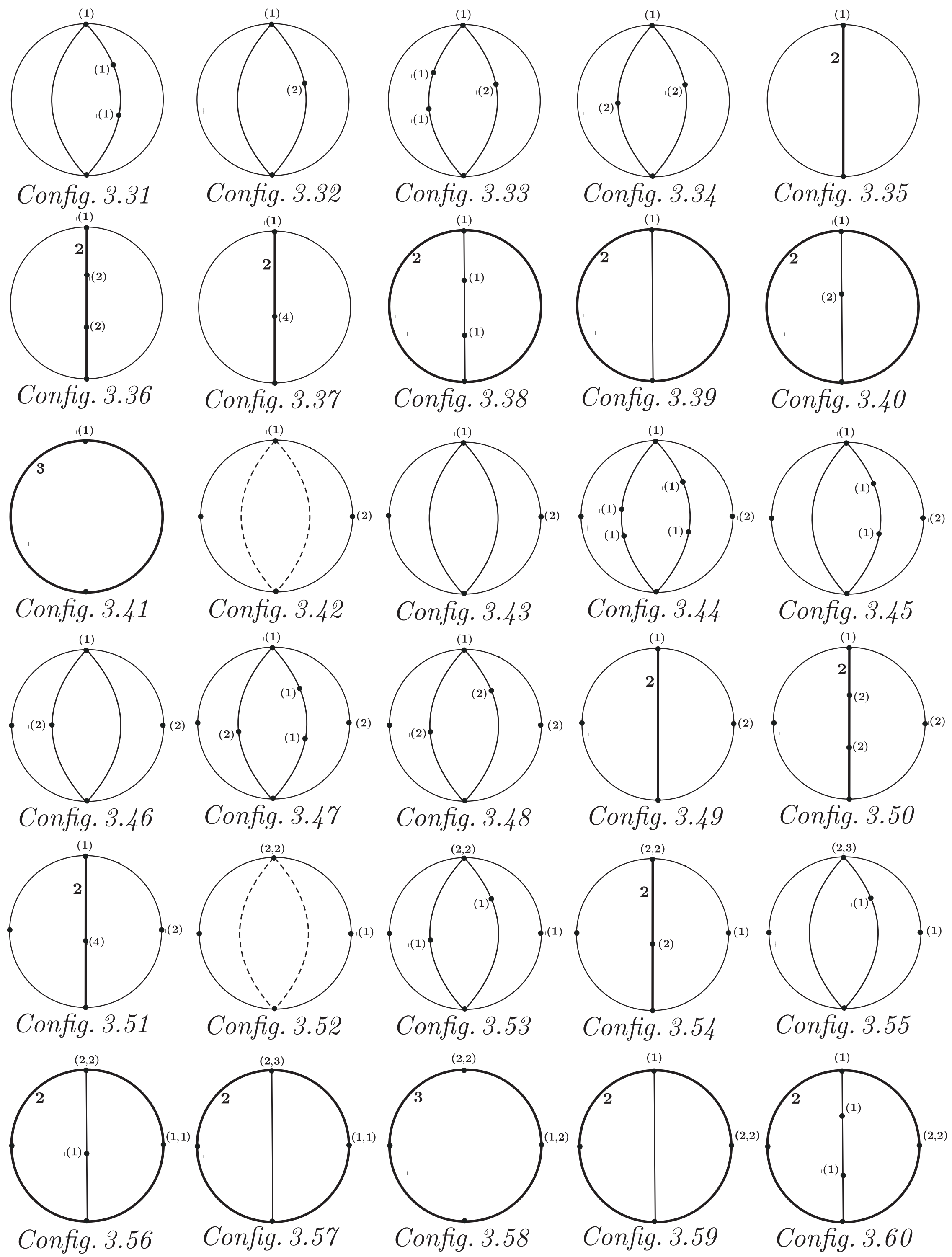

Figure 1 *(continuation)* The configurations of quadratic systems in $\mathbf{QSL}_3$

and with $\max(\deg(p), \deg(q)) = 2$. It is known that on the set $\boldsymbol{QS}$ acts the group $Aff(2, \mathbb{R})$ of affine transformations on the plane (cf. [10]). For every subgroup $G \subseteq Aff(2, \mathbb{R})$ we have an induced action of G on $\boldsymbol{QS}$. We can identify the set $\boldsymbol{QS}$ of systems (2) with

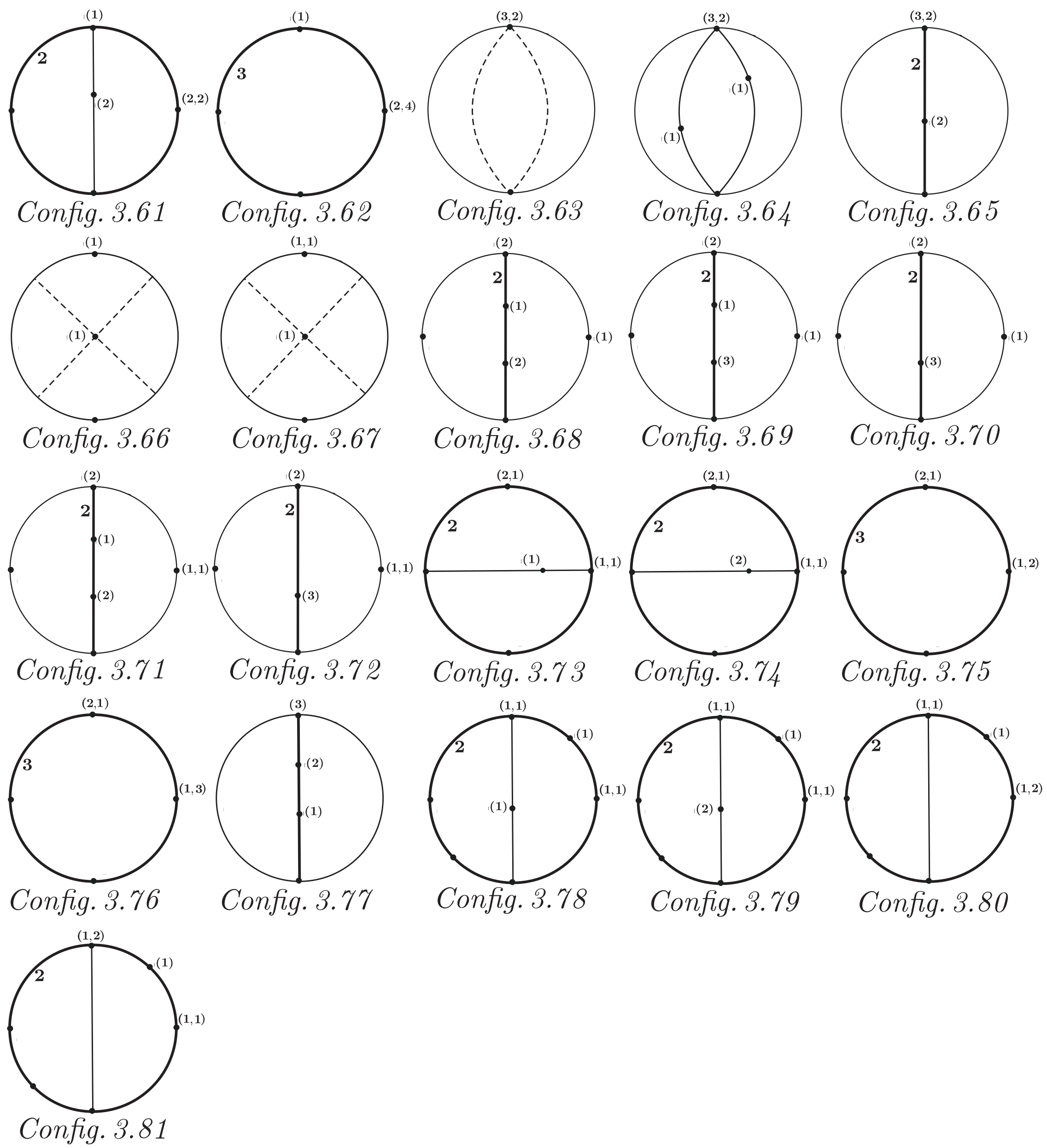

Figure 1 *(continuation)* The configurations of quadratic systems in $\mathbf{QSL}_3$

a subset of $\mathbb{R}^{12}$ via the map $\boldsymbol{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL–comitants (GL–invariants), the T–comitants (affine invariants) and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [10] (see also [1]).

According to [1] (see also [4]) we apply the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ acting

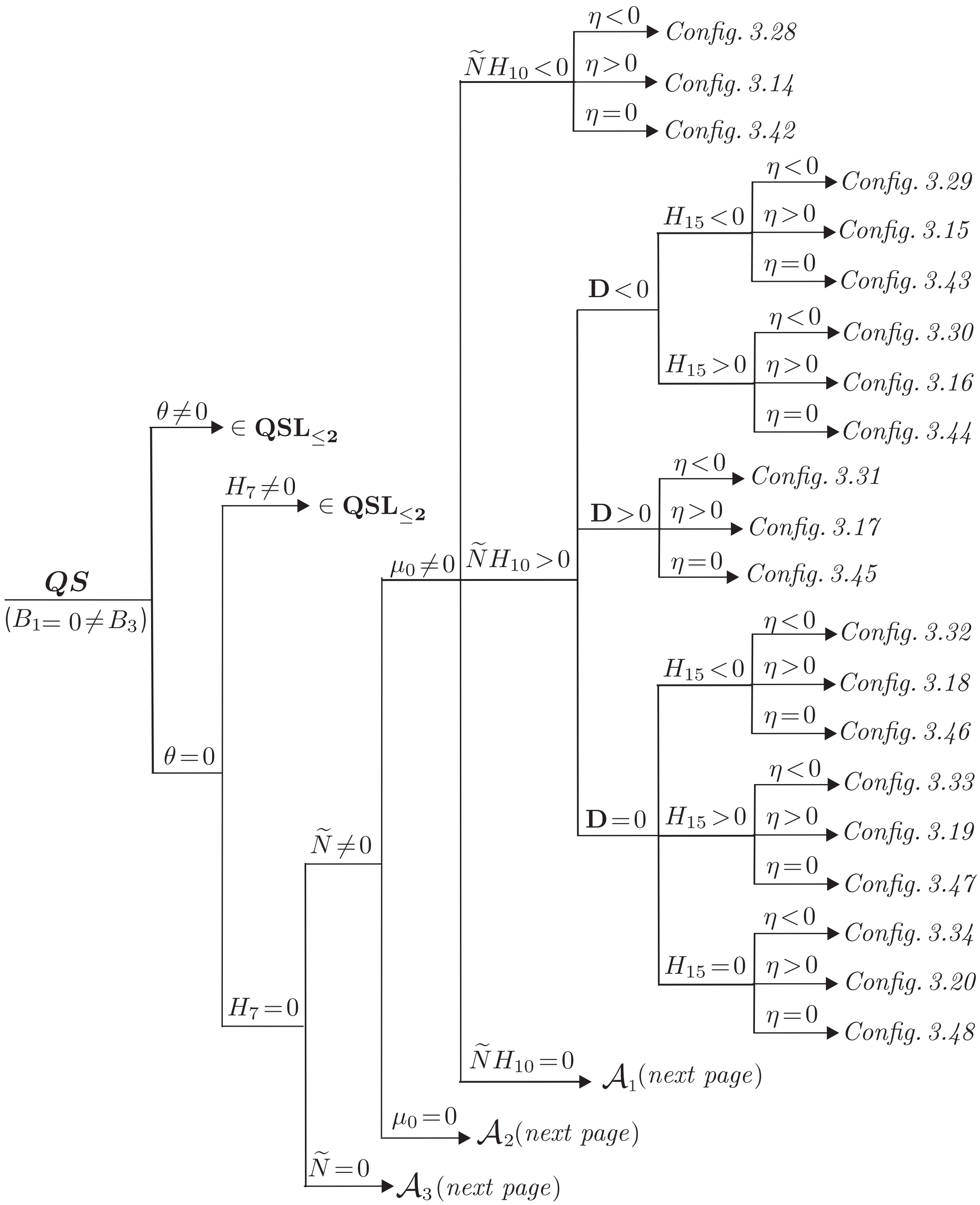

Diagram 1: The configurations of systems in **QSL** with $B_1 = 0$ and $B_2 \neq 0$

on $\mathbb{R}[\tilde{a}, x, y]$ with

$$\begin{aligned}\mathbf{L}_1 =& 2a\frac{\partial}{\partial c} + c\frac{\partial}{\partial g} + \frac{1}{2}d\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial e} + e\frac{\partial}{\partial l} + \frac{1}{2}f\frac{\partial}{\partial m},\\ \mathbf{L}_2 =& 2a\frac{\partial}{\partial d} + d\frac{\partial}{\partial k} + \frac{1}{2}c\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial f} + f\frac{\partial}{\partial n} + \frac{1}{2}e\frac{\partial}{\partial m},\end{aligned}$$

to construct several needed invariant polynomials. More precisely using this operator and

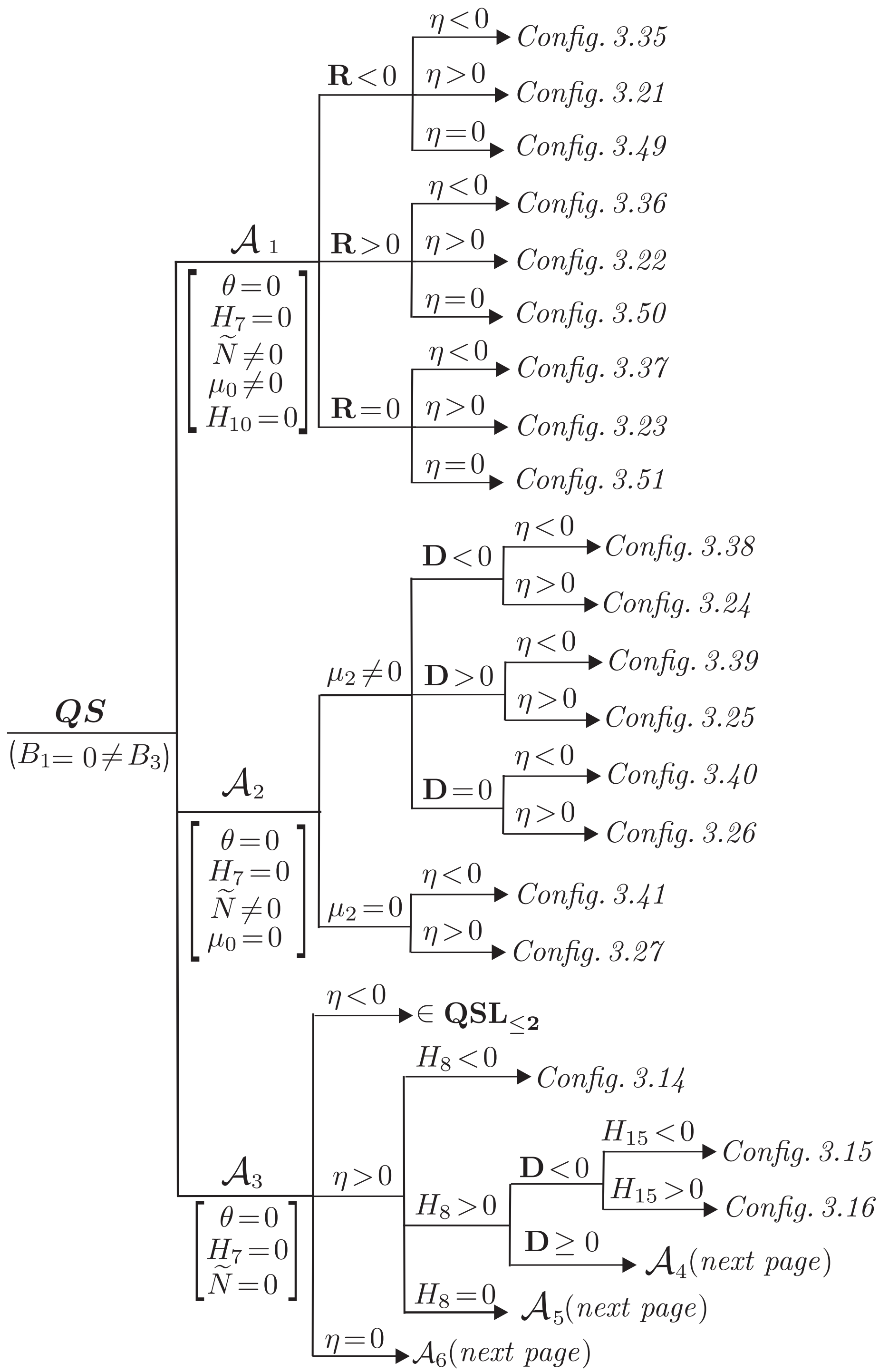

Diagram 1 *(continuation)*: The configurations of systems in **QSL** with $B_1=0$ and $B_2\neq 0$

the affine invariant $\mu_0 = \text{Res}_x\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ \ i=1,..,4, \quad \text{where} \quad \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).$$

Using these invariant polynomials we define some new ones, which according to [1] are

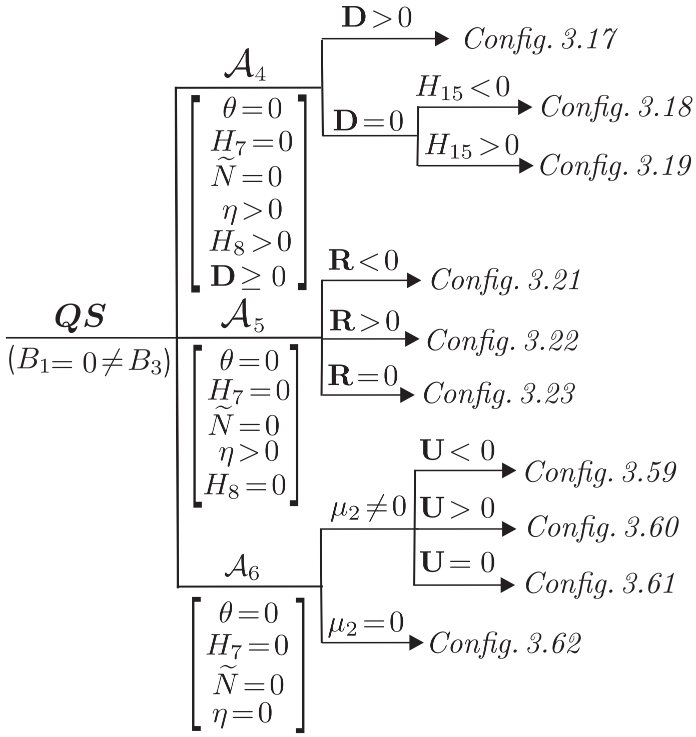

Diagram 1 *(continuation)*: The configurations of systems in **QSL** with $B_1 = 0$ and $B_2 \neq 0$

responsible for the number and multiplicities of the finite singular points of (2):

$$\begin{aligned}
\mathbf{D} &= \Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\
\mathbf{P} &= 12\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\\
\mathbf{R} &= 3\mu_1^2 - 8\mu_0\mu_2,\\
\mathbf{S} &= \mathbf{R}^2 - 16\mu_0^2\mathbf{P},\\
\mathbf{T} &= 18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 2\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR},\\
\mathbf{U} &= \mu_3^2 - 4\mu_2\mu_4.
\end{aligned}$$

In what follows we also need the so-called *transvectant of order k* (see [7], [8]) of two polynomials $f,\ g \in \mathbb{R}[\tilde{a}, x, y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

In order to construct the remaining invariant polynomials contained in the set (**??**) we first need to define some elementary bricks which help us to construct these elements of the set.

We remark that the following polynomials in $\mathbb{R}[\tilde{a}, x, y]$ are the simplest invariant polynomials of degree one with respect to the coefficients of the differential systems (2) which are GL-comitants:

$$C_i(x,y) = y p_i(x,y) - x q_i(x,y),\ \ i = 0,1,2;$$
$$D_i(x,y) = \frac{\partial}{\partial x} p_i(x,y) + \frac{\partial}{\partial y} q_i(x,y),\ \ i = 1,2.$$

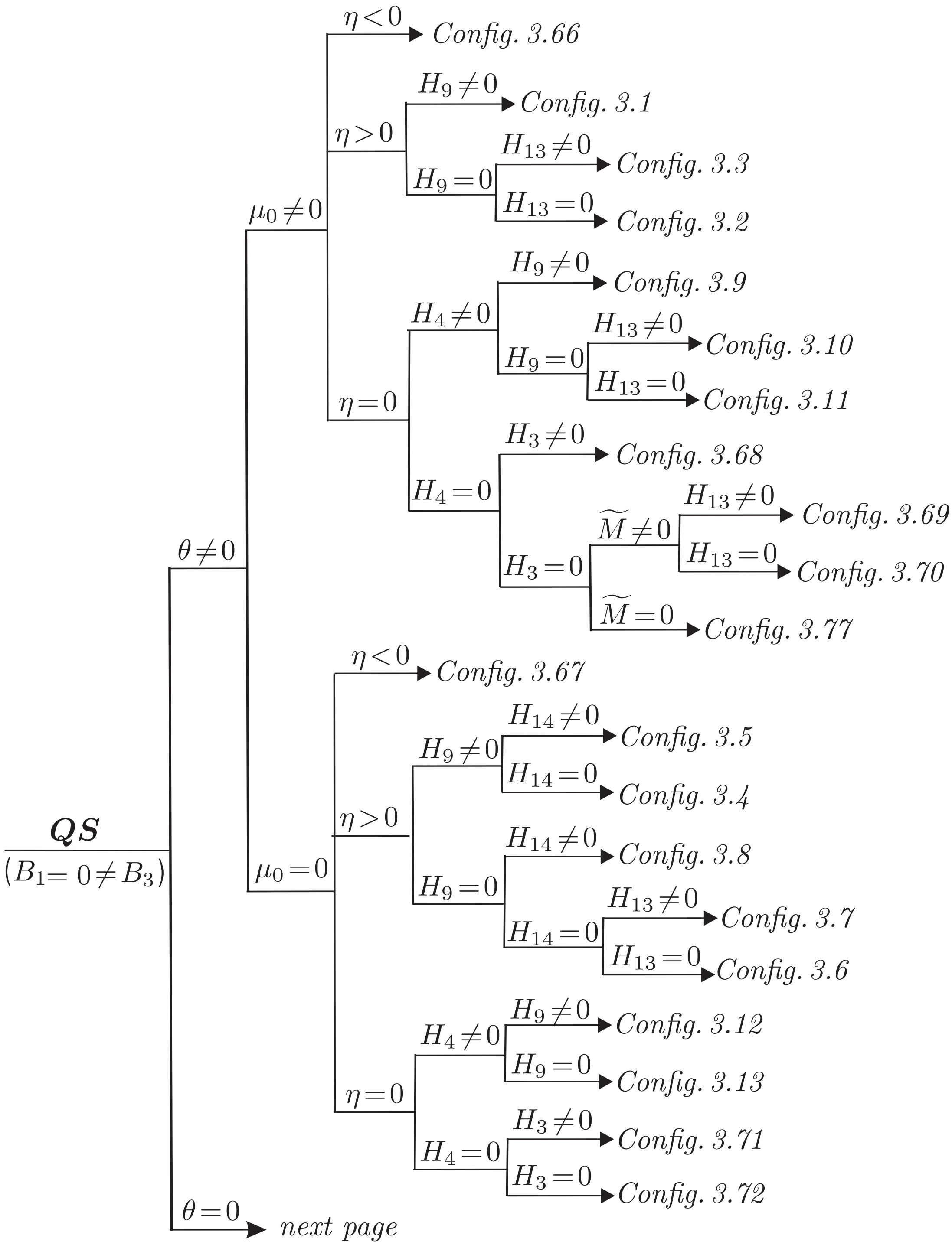

Diagram 2: The configurations of systems in **QSL** with $B_2 = 0$ and $B_3 \neq 0$

Apart from these simple invariant polynomials we shall also make use of the following nine

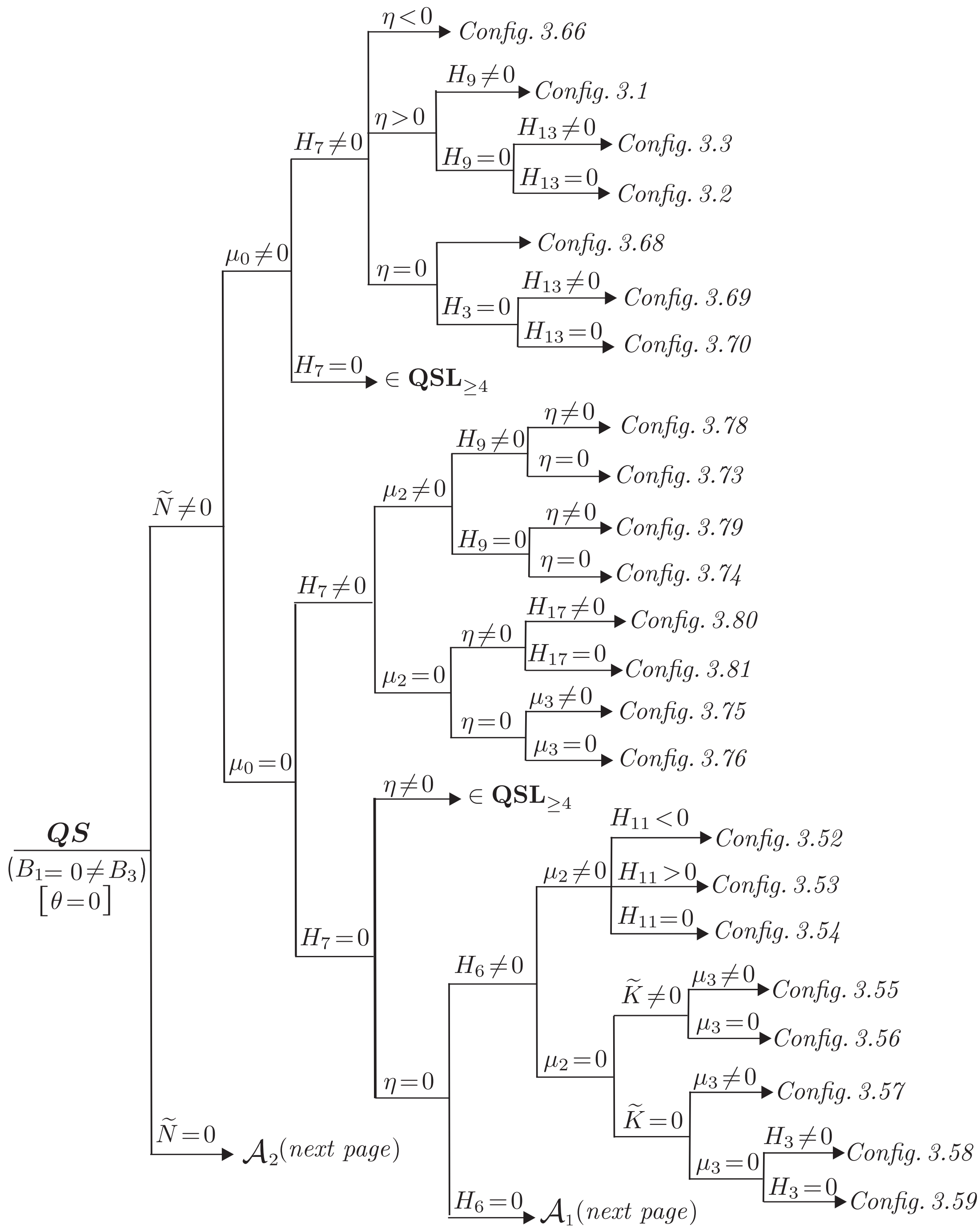

Diagram 2 *(continuation)*: The configurations of systems in **QSL** with $B_2 = 0$ and $B_3 \neq 0$

GL-invariant polynomials:

$$T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)},$$
$$T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)},$$
$$T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}.$$

These are of degree two with respect to the coefficients of systems (2).

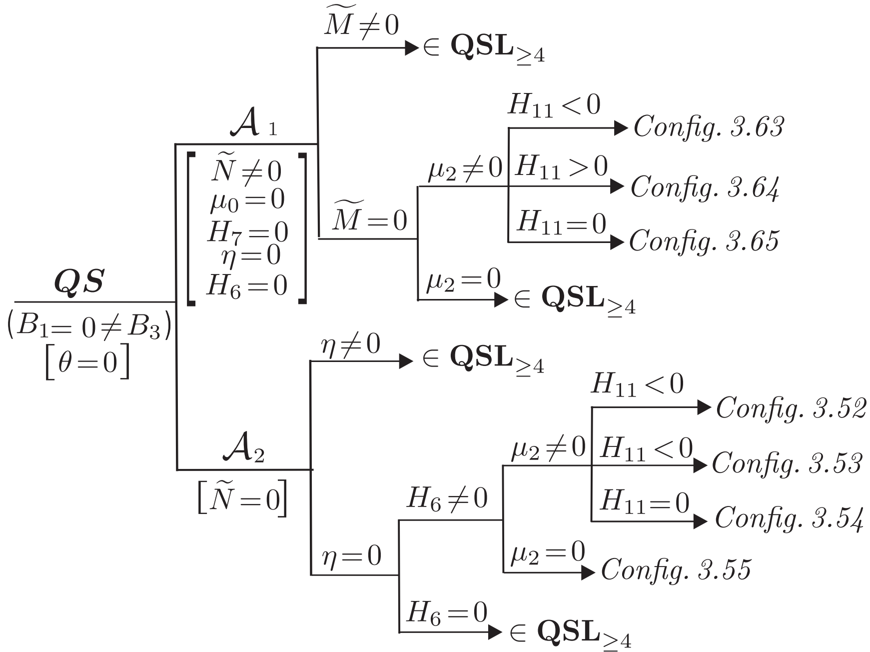

Diagram 2 *(continuation)*: The configurations of systems in **QSL** with $B_2 = 0$ and $B_3 \neq 0$

We next define a list of T-comitants:

$$
\begin{aligned}
\hat{A}(\tilde{a}) =\ & (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144,\\
\widehat{B}(\tilde{a}, x, y) =\ & \Big\{16D_1(D_2, T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2, T_9)^{(1)}(3D_1D_2\\
& - 5T_6 + 9T_7) + 2(D_2, T_9)^{(1)}\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0, T_5)^{(1)}\big)\\
& + 6(D_2, T_7)^{(1)}\big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5)\\
& + C_2(9T_4 + 96T_3)\big] + 6(D_2, T_6)^{(1)}\big[32C_0T_9 - C_1(12T_7 + 52D_1D_2)\\
& - 32C_2D_1^2\big] + 48D_2(D_2, T_1)^{(1)}(2D_2^2 - T_8) + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9)\\
& - 32D_1T_8(D_2, T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7) - 16D_1(C_2, T_8)^{(1)}(D_1^2 + 4T_3)\\
& + 12D_1(C_1, T_8)^{(2)}(C_1D_2 - 2C_2D_1) + 12D_1(C_1, T_8)^{(1)}(T_7 + 2D_1D_2)\\
& + 96D_2^2\big[D_1(C_1, T_6t)^{(1)} + D_2(C_0, T_6)^{(1)}\big] - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9)\\
& - 16D_1D_2T_3(2D_2^2 + 3T_8) + 6D_1^2D_2^2(7T_6 + 2T_7) - 252D_1D_2T_4T_9\Big\}/(2^8 3^3),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{D}(\tilde{a}, x, y) =\ & \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6) - (C_1, T_5)^{(1)} - 9D_1^2C_2\\
& + 6D_1(C_1D_2 - T_5)\big]/36,\\
\widehat{E}(\tilde{a}, x, y) =\ & \big[D_1(2T_9 - T_8) - 3(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\big]/72,\\
\widehat{F}(\tilde{a}, x, y) =\ & \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2, T_9)^{(1)} - 9D_2^2T_4\\
& + 288D_1\widehat{E} - 24(C_2, \widehat{D})^{(2)} + 120(D_2, \widehat{D})^{(1)} - 36C_1(D_2, T_7)^{(1)}\\
& + 8D_1(D_2, T_5)^{(1)}\big]/144,\\
\widehat{K}(\tilde{a}, x, y) =\ & (T_8 + 4T_9 + 4D_2^2)/72,\\
\widehat{H}(\tilde{a}, x, y) =\ & (-T_8 + 8T_9 + 2D_2^2)/72,
\end{aligned}
$$

as well as the following affine invariants (which serve as bricks for constructing the needed invariant polynomials):

$$
\begin{array}{ll}
A_2(\tilde{a}) = (C_2, \widehat{D})^{(3)}/12, & A_{17}(\tilde{a}) = \big(((\widehat{D},\widehat{D})^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)}/64,\\
A_{18}(\tilde{a}) = \big((\widehat{D},\widehat{F})^{(2)}, D_2\big)^{(1)}/16, & A_{19}(\tilde{a}) = \big((\widehat{D},\widehat{D}\big)^{(2)}, \widehat{H}\big)^{(2)}/16,\\
A_{20}(\tilde{a}) = \big((C_2,\widehat{D})^{(2)}, \widehat{F}\big)^{(2)}/16. &
\end{array}
$$

Next we present here the list of invariant polynomials which are necessary for the classification of the configurations of invariant lines for the family $\mathbf{QSL}_3$:

$$
\begin{aligned}
\widetilde{K}(\tilde{a},x,y) &= 4\widehat{K} \equiv \mathrm{Jacob}\,\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big),\\
\widetilde{M}(\tilde{a},x,y) &= (C_2,C_2)^{(2)} \equiv 2\mathrm{Hess}\,\big(C_2(\tilde{a},x,y)\big),\\
\widetilde{N}(\tilde{a},x,y) &= \widetilde{K} - 4\widehat{H},\\
\widetilde{D}(\tilde{a},x,y) &= \widehat{D},\\
\eta(\tilde{a}) &= (\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \mathrm{Discrim}\,\big(C_2(\tilde{a},x,y)\big),\\
\theta(\tilde{a}) &= -\,(\widetilde{N},\widetilde{N})^{(2)}/2 \equiv \mathrm{Discrim}\,\big(\widetilde{N}(\tilde{a},x,y)\big);\\
B_1(\tilde{a}) &= \mathrm{Res}_x\left(C_2,\widetilde{D}\right)/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)},\\
B_2(\tilde{a},x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,\widetilde{D})^{(3)},\\
B_3(\tilde{a},x,y) &= (C_2,\widetilde{D})^{(1)} \equiv \mathrm{Jacob}\left(C_2,\widetilde{D}\right),\\
H_1(\tilde{a}) &= -\,\big((C_2,C_2)^{(2)},C_2)^{(1)},\widetilde{D}\big)^{(3)},\\
H_3(\tilde{a},x,y) &= (C_2,\widetilde{D})^{(2)},\\
H_4(\tilde{a}) &= \big((C_2,\widetilde{D})^{(2)},(C_2,D_2)^{(1)}\big)^{(2)},\\
H_6(\tilde{a},x,y) &= 16N^2(C_2,\widetilde{D})^{(2)} + H_2^2(C_2,C_2)^{(2)},\\
H_7(\tilde{a}) &= (\widetilde{N},C_1)^{(2)},\\
H_8(\tilde{a}) &= 9\big((C_2,\widetilde{D})^{(2)},(\widetilde{D},D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2,\widetilde{D})^{(3)}\Big]^2,\\
H_9(\tilde{a}) &= -\,[\![\widetilde{D},\widetilde{D})^{(2)},\widetilde{D},)^{(1)},\widetilde{D})^{(3)},\\
H_{10}(\tilde{a}) &= \big((\widetilde{N},\widetilde{D})^{(2)},\ D_2\big)^{(1)},\\
H_{11}(\tilde{a},x,y) &= 8\widehat{H}\big[(C_2,\widetilde{D})^{(2)} + 8(\widetilde{D},D_2)^{(1)}\big] + 3\big[(C_1,\,2\widehat{H}-\widetilde{N})^{(1)} - 2D_1\widetilde{N}\big]^2,\\
H_{13}(\tilde{a},x,y) &= A_1A_2 - A_{14} - A_{15},\\
H_{14}(\tilde{a},x,y) &= A_2(156A_5 - 20A_3 - 33A_4) + 4(99A_1A_6 - 5A_{22} + 42A_{23} - 21A_{24}),\\
H_{15}(\tilde{a}) &= \big((\widetilde{D},\widetilde{D})^{(2)},\ \widetilde{H}\big)^{(1)},\\
H_{17}(\tilde{a}) &= 2A_2^2 - 16A_{17} - 16A_{18} + 12A_{19} - 2A_{20},\\
N_1(\tilde{a},x,y) &= C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},\\
N_2(\tilde{a},x,y) &= D_1(C_1,C_2)^{(2)} - \big((C_2,C_2)^{(2)},C_0\big)^{(1)},\\
N_3(\tilde{a},x,y) &= (C_2,C_1)^{(1)},
\end{aligned}
$$

$$N_4(\tilde{a},x,y) = 4\,(C_2,C_0)^{(1)} - 3C_1D_1,$$
$$N_5(\tilde{a},x,y) = \big[(D_2,C_1)^{(1)} + D_1D_2\big]^2 - 4\big(C_2,C_2\big)^{(2)}\big(C_0,D_2\big)^{(1)},$$
$$N_6(\tilde{a},x,y) = 8D + C_2\big[8(C_0,D_2)^{(1)} - 3(C_1,C_1)^{(2)} + 2D_1^2\big].$$

3 Preliminary results involving the use of polynomial invariants

The following two lemmas reveal the geometrical meaning of the invariant polynomials B_1, B_2, B_3, θ and $\widetilde{N}$.

Lemma 3.1 ([9]). *For the existence of an invariant straight line in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).*

Lemma 3.2 ([9]). *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a system* (2) *corresponding to $\boldsymbol{a} \in \mathbb{R}^{12}$ is the condition $\theta(\boldsymbol{a}) = 0$ (respectively, $\widetilde{N}(\boldsymbol{a},x,y) = 0$).*

We remark that the invariant polynomials $\mu_i(\tilde{a},x,y)$ $(i = 0,1,\dots,4)$ defined earlier (see page 9) are responsible for the total multiplicity of the finite singularities of quadratic systems (2). Moreover they detect whether a quadratic system is degenerate or not as well as the coordinates of infinite singularities that result after the coalescence of finite singularities with an infinite one. More exactly according to [1, Lemma 5.2] we have the following lemma.

Lemma 3.3. *Consider a quadratic system (S) with coefficients $\boldsymbol{a} \in \mathbb{R}^{12}$. Then:*

(i) The total multiplicity of the finite singularities of this system is $4-k$ if and only if for every i such that $0 \le i \le k-1$ we have $\mu_i(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_k(\boldsymbol{a},x,y) \neq 0$. In this case the factorization $\mu_k(\boldsymbol{a},x,y) = \prod_{i=1}^{k}(u_ix - v_iy) \neq 0$ over $\mathbb{C}$ yields the coordinates $[v_i : u_i : 0]$ of points at infinity that have multiplicity greater than one, this being the result of coalescence of finite and infinite singularities. Moreover the number of distinct expressions $u_ix - v_iy$ in this factorization is less than or equal to three (the maximum number of infinite singularities of a quadratic system), and the multiplicity of each one of the expressions $u_ix - v_iy$ gives us the number of the finite singularities of the system (S) that have coalesced with the infinite singular point $[v_i : u_i : 0]$.

(ii) Let the point $M_0(0,0)$ be a singular point for the quadratic system (S). Then the point $M_0(0,0)$ is a singular point of multiplicity k $(1 \le k \le 4)$ if and only if for every i such that $0 \le i \le k-1$ we have $\mu_{4-i}(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_{4-k}(\boldsymbol{a},x,y) \neq 0$.

(iii) The system (S) is degenerate (i.e. $\gcd(p,q) \neq$ constant) if and only if $\mu_i(\boldsymbol{a},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i = 0,1,2,3,4$.

On the other hand the invariant polynomials η, $\widetilde{M}$ and C_2 govern the number of real and complex infinite singularities. More precisely, according to [18] (see also [10]) we have the next result.

Lemma 3.4. *The number of infinite singularities (real and complex) of a quadratic system in* **QS** *is determined by the following conditions:*

(i) 3 *real if* $\eta > 0$*;*

(ii) 1 *real and* 2 *imaginary if* $\eta < 0$*;*

(iii) 2 *real if* $\eta = 0$ *and* $\widetilde{M} \neq 0$*;*

(iv) 1 *real if* $\eta = \widetilde{M} = 0$ *and* $C_2 \neq 0$*;*

(v) ∞ *if* $\eta = \widetilde{M} = C_2 = 0$.

Moreover, the quadratic systems (2)*, for each one of these cases, can be brought via a linear transformation to the corresponding case of the following canonical systems* $(\mathbf{S}_I) - (\mathbf{S}_V)$*:*

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a + cx + dy + x^2, \\ \dot{y} &= b + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_V)$$

Remark 3.1. *In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [9]. Thus we denote by "(a, b)" the ordered couple of a, respectively b where a (respectively b) is the maximum number of infinite (respectively finite) singularities which can be obtained by perturbation of a multiple infinite singular point.*

Now we define the affine comitants which are responsible for the existence of invariant lines for a non-degenerate quadratic system (2).

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(\tilde{a}, x, y)$ and $q(\tilde{a}, x, y)$. We obtain $\hat{p}(\hat{a}(\tilde{a}, x_0, y_0), x', y') = p(\tilde{a}, x' + x_0, y' + y_0)$, $\hat{q}(\hat{a}(\tilde{a}, x_0, y_0), x', y') = q(\tilde{a}, x' + x_0, y' + y_0)$. Let us construct the following polynomials

$$\Gamma_i(\tilde{a}, x_0, y_0) \equiv \mathrm{Res}_{x'}\Big(C_i\big(\hat{a}(\tilde{a}, x_0, y_0), x', y'\big), C_0\big(\hat{a}(\tilde{a}, x_0, y_0), x', y'\big)\Big)/(y')^{i+1},$$
$$\Gamma_i(\tilde{a}, x_0, y_0) \in \mathbb{R}[\tilde{a}, x_0, y_0], \ (i = 1, 2).$$

Notation 3.1.

$$\tilde{\mathcal{E}}_i(\tilde{a}, x, y) = \Gamma_i(\tilde{a}, x_0, y_0)|_{\{x_0 = x, \ y_0 = y\}} \in \mathbb{R}[\tilde{a}, x, y] \quad (i = 1, 2).$$

Observation 3.1. *We note that the constructed polynomials* $\tilde{\mathcal{E}}_1(\tilde{a}, x, y)$ *and* $\tilde{\mathcal{E}}_2(\tilde{a}, x, y)$ *are affine comitants of systems* (2) *and are homogeneous polynomials in the coefficients* $a, \dots, n$ *and non-homogeneous in* x, y *and*

$$\deg_{\tilde{a}} \tilde{\mathcal{E}}_1 = 3,\ \deg_{(x,y)} \tilde{\mathcal{E}}_1 = 5,\quad \deg_{\tilde{a}} \tilde{\mathcal{E}}_2 = 4,\ \deg_{(x,y)} \tilde{\mathcal{E}}_2 = 6.$$

Notation 3.2. *Let* $\mathcal{E}_i(\tilde{a}, X, Y, Z)$ $(i = 1, 2)$ *be the homogenization of* $\tilde{\mathcal{E}}_i(\tilde{a}, x, y)$*, i.e.*

$$\mathcal{E}_1(\tilde{a}, X, Y, Z) = Z^5 \tilde{\mathcal{E}}_1(\tilde{a}, X/Z, Y/Z), \qquad \mathcal{E}_2(\tilde{a}, X, Y, Z) = Z^6 \tilde{\mathcal{E}}_2(\tilde{a}, X/Z, Y/Z)$$

and $\mathcal{H}(\tilde{a}, X, Y, Z) = \gcd\Big(\mathcal{E}_1(\tilde{a}, X, Y, Z),\ \mathcal{E}_2(\tilde{a}, X, Y, Z)\Big)$ *in* $\mathbb{R}[\tilde{a}, X, Y, Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [9]):

Lemma 3.5 ([9])**.** *The straight line* $\mathcal{L}(x, y) \equiv ux + vy + w = 0$*,* $u, v, w \in \mathbb{C}$*,* $(u, v) \neq (0, 0)$ *is an invariant line for a quadratic system* (2) *if and only if the polynomial* $\mathcal{L}(x, y)$ *is a common factor of the polynomials* $\tilde{\mathcal{E}}_1(\boldsymbol{a}, x, y)$ *and* $\tilde{\mathcal{E}}_2(\boldsymbol{a}, x, y)$ *over* $\mathbb{C}$*, i.e.*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)\widetilde{W}_i(x, y) \quad (i = 1, 2),$$

where $\widetilde{W}_i(x, y) \in \mathbb{C}[x, y]$.

Lemma 3.6. *1) If* $\mathcal{L}(x, y) \equiv ux + vy + w = 0$*,* $u, v, w \in \mathbb{C}$*,* $(u, v) \neq (0, 0)$ *is an invariant straight line of multiplicity* k *for a quadratic system* (2) *then* $[\mathcal{L}(x, y)]^k \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ *in* $\mathbb{C}[x, y]$*, i.e. there exist* $W_i(\boldsymbol{a}, x, y) \in \mathbb{C}[x, y]$ $(i = 1, 2)$ *such that*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)^k W_i(\boldsymbol{a}, x, y), \quad i = 1, 2.$$

2) If the line $l_\infty : Z = 0$ *is of multiplicity* $k > 1$ *then* $Z^{k-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$*, in other words we have* $Z^{k-1} \mid \mathcal{H}(\boldsymbol{a}, X, Y, Z)$.

In what follows the following Lemma will be useful.

Lemma 3.7. *The non-singular invariant line at infinity for a non-degenerate quadratic system has the multiplicity greater or equal to two if and only if the the condition* $\widetilde{K} = 0$.

Proof: Considering Lemma 3.6 (see statement 2) we deduce, that the line at infinity of a quadratic system is of multiplicity > 1 if and only if $Z \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$. In other words Z is a common factor of the polynomials $\mathcal{E}_1(X, Y, Z)$ and $\mathcal{E}_2(X, Y, Z)$ (see Notation 3.2).

Taking into account the definition of the invariant polynomials $\mathcal{E}_1(X, Y, Z)$ and $\mathcal{E}_2(X, Y, Z)$ (see Notations 3.1 and 3.2) for systems (2) we calculate

$$\begin{aligned}
\mathcal{E}_1(X, Y, Z) =& \frac{1}{2} C_2(X, Y)\widetilde{K}(X, Y) + \phi_1(X, Y)Z + \phi_2(X, Y)Z^2 + \ldots + \phi_5(X, Y)Z^5, \\
\mathcal{E}_2(X, Y, Z) =& C_2(X, Y)\Psi(X, Y) + \psi_1(X, Y)Z + \psi_2(X, Y)Z^2 + \ldots + \psi_6(X, Y)Z^6,
\end{aligned}$$

where

$$\begin{aligned}
C_2(X, Y) =& -lX^3 + (g - 2m)X^2Y + (2h - n)XY^2 + kY^3, \\
\widetilde{K}(X, Y) =& 4\big[(gm - hl)X^2 + (gn - kl)XY + (hn - km)Y^2\big] \equiv 4\big[\alpha X^2 + \beta XY + \gamma Y^2\big], \\
\Psi(X, Y) =& (2g\alpha + l\beta)X^3 + \big[(4h + 2n)\alpha + g\beta + 4l\gamma\big]X^2Y \\
& + \big[2k\alpha + (2h + n)\beta + 4m\gamma\big]XY^2 + (k\beta + 2n\gamma)Y^3.
\end{aligned}$$

Therefore we conclude that the invariant polynomials $\mathcal{E}_1(X,Y,Z)$ and $\mathcal{E}_2(X,Y,Z)$ have the common factor Z if and only if the conditions $C_2(X,Y)\widetilde{K}(X,Y) = C_2(X,Y)\Psi(X,Y) = 0$ hold. Since $C_2 = 0$ leads to systems with the line at infinity filled up with singularities (see Lemma 3.4) clearly the condition $C_2 \neq 0$ has to be satisfied.

On the other hand we observe that the condition $\widetilde{K}(X,Y) = 0$ implies $\alpha = \beta = \gamma = 0$ and then $\Psi(X,Y) = 0$. Therefore we obtain that the condition $\widetilde{K}(X,Y) = 0$ is necessary and sufficient for a quadratic system to have the invariant line at infinity of multiplicity at least 2. This completes the proof of Lemma 3.7. ∎

4 The quadratic systems belonging to the family $\mathbf{QSL}_3$

As it is mentioned in Introduction some of the configurations of the quadratic systems in the family $\mathbf{QSL}_3$ are determined early in other papers. More exactly in [16] the configurations *Config. 3.1–Config. 3.13* are constructed. In a recent published article [5] the family of systems possessing two parallel invariant lines is considered and the configurations *Config. 3.14–Config. 3.65* are determined.

In this section we complete the investigation of the family $\mathbf{QSL}_3$ and prove that there exists 16 possible new configurations *Config. 3.66–Config. 3.81*.

First of all we prove some necessary conditions for a quadratic system to belong to the family $\mathbf{QSL}_3$. We have the following lemma.

Lemma 4.1. *Assume that a non-degenerate quadratic system belongs to the class* $\mathbf{QSL}_3$. *Then for this system the conditions* $B_1 = 0$ *and* $B_3 \neq 0$ *haves to be fulfilled.*

Proof: According to Lemma 3.1 if for a quadratic system the condition $B_1 \neq 0$ holds then this system could not have any invariant affine line going in some direction. On the other hand if a system belong to the class $\mathbf{QSL}_3$ then there either exists at least one invariant affine line or the line at infinity is triple. However in the second case there must exist a perturbation such that the perturbed system necessarily possesses at least one invariant affine line and this means that for this system we must have $B_1 = 0$. So we deduce that this condition must be satisfied for the non-perturbed system, too.

Therefore we obtain that for a system in $\mathbf{QSL}_3$ the condition $B_1 = 0$ have to be satisfied. In order to complete the proof of Lemma 4.1 we have to show that for a system in $\mathbf{QSL}_3$ the condition $B_3 \neq 0$ is also necessary. We prove the following lemma.

Lemma 4.2. *Assume that for a non-degenerate quadratic system the condition* $B_3 = 0$ *holds. Then this system belongs to the class* $\mathbf{QSL}_{\geq 4}$. *Moreover any system in this class could have a configuration of invariant lines given in Diagram 3 if and only if the corresponding conditions are satisfied, respectively.*

Proof: Assume that for a non-degenerate quadratic system the condition $B_3 = 0$ is fulfilled. In the articles [9] and [11] the families of quadratic systems possessing invariant line of total multiplicity at least four are investigated and the corresponding possible configurations of invariant lines are determined.

So considering Tables 2 and 4 from [9] as well as Table 2 from [11] it is not too difficult to convince ourselves that the conditions given in these tables for the corresponding configurations are equivalent to the respective conditions presented in Diagram 3.

We observe that this diagram gives us a complete partition of the whole set $\mathbf{QSL}_{\{B_3=0\}}$. This completes the proof of Lemma 4.2 as well as the proof of Lemma 4.1. ∎

4.1 Configurations of systems belonging to the subfamily $\mathbf{QSL}_3 \cap \mathbf{QS}_{2cIL}$

In paper [2] (see also [19]) the phase portraits of the family of quadratic systems possessing two complex invariant lines intersecting at a real finite point are considered. We denote this family by $\mathbf{QS}_{2cIL}$. A result in [2] determined 20 topologically distinct phase portraits. However the problem of how many configurations of invariant lines could have systems in the family $\mathbf{QS}_{2cIL}$ remains open.

Here we are interested in the configurations of the quadratic systems belonging to the subfamily $\mathbf{QSL}_3 \cap \mathbf{QS}_{2cIL}$. We prove the following theorem.

Theorem 4.1. *An arbitrary non-degenerate quadratic system belongs to the subfamily* $\mathbf{QSL}_3 \cap \mathbf{QS}_{2cIL}$ *if and only if the conditions* $\eta < 0$, $B_2 = 0$ *and* $B_3\widetilde{N} \neq 0$ *hold. Moreover this system possesses the configuration Config. 3.66 if* $\mu_0 \neq 0$ *and Config. 3.67 if* $\mu_0 = 0$.

Proof: According to [2, Theorem 1] a non-degenerate quadratic system possesses two complex invariant lines meeting at a finite real point if and only if one of the following two sets of conditions are satisfied:

$$(i)\ \ \eta < 0, \quad B_2 = 0; \qquad (ii)\ \ C_2 = 0, \quad \mathbf{D} > 0.$$

By [15] quadratic systems with $C_2 = 0$ possess in the finite part of the phase plane invariant lines of total multiplicity three. Therefore we obtain that a system with $C_2 = 00$ could not belong to the class $\boldsymbol{QSL}_3$. Moreover we deduce that for $C_2 \neq 0$ the conditions $\eta < 0$ and $B_2 = 0$ are necessary and sufficient for a system to belong to the family $\boldsymbol{QS}_{2cIL}$.

Since we are interested in the determinations of the configurations of the quadratic systems in the subclass $\boldsymbol{QSL}_3 \cap \boldsymbol{QS}_{2cIL}$ we consider that for a non-degenerate quadratic system the conditions $\eta < 0$ and $B_2 = 0$ are satisfied. Thus according to what is mentioned above we conclude that in order to complete the proof of Theorem 4.1 it is sufficient to prove that if for a quadratic system we have $\eta < 0$ and $B_2 = 0$ then the condition $B_3\widetilde{N} \neq 0$ guarantees that this system belongs to the class $\boldsymbol{QSL}_3$. Moreover we have also to determine the possible configurations of invariant lines of these systems.

According to [20] if a quadratic system possesses two complex invariant lines intersecting at a real finite singular point then via an affine transformation this system takes the following form:

$$\begin{aligned}\frac{dx}{dt} &= (\alpha x - \beta y)(ax + by + c) + k(x^2 + y^2) \equiv P(x, y),\\ \frac{dy}{dt} &= (\beta x + \alpha y)(ax + by + c) \equiv Q(x, y)\end{aligned} \tag{3}$$

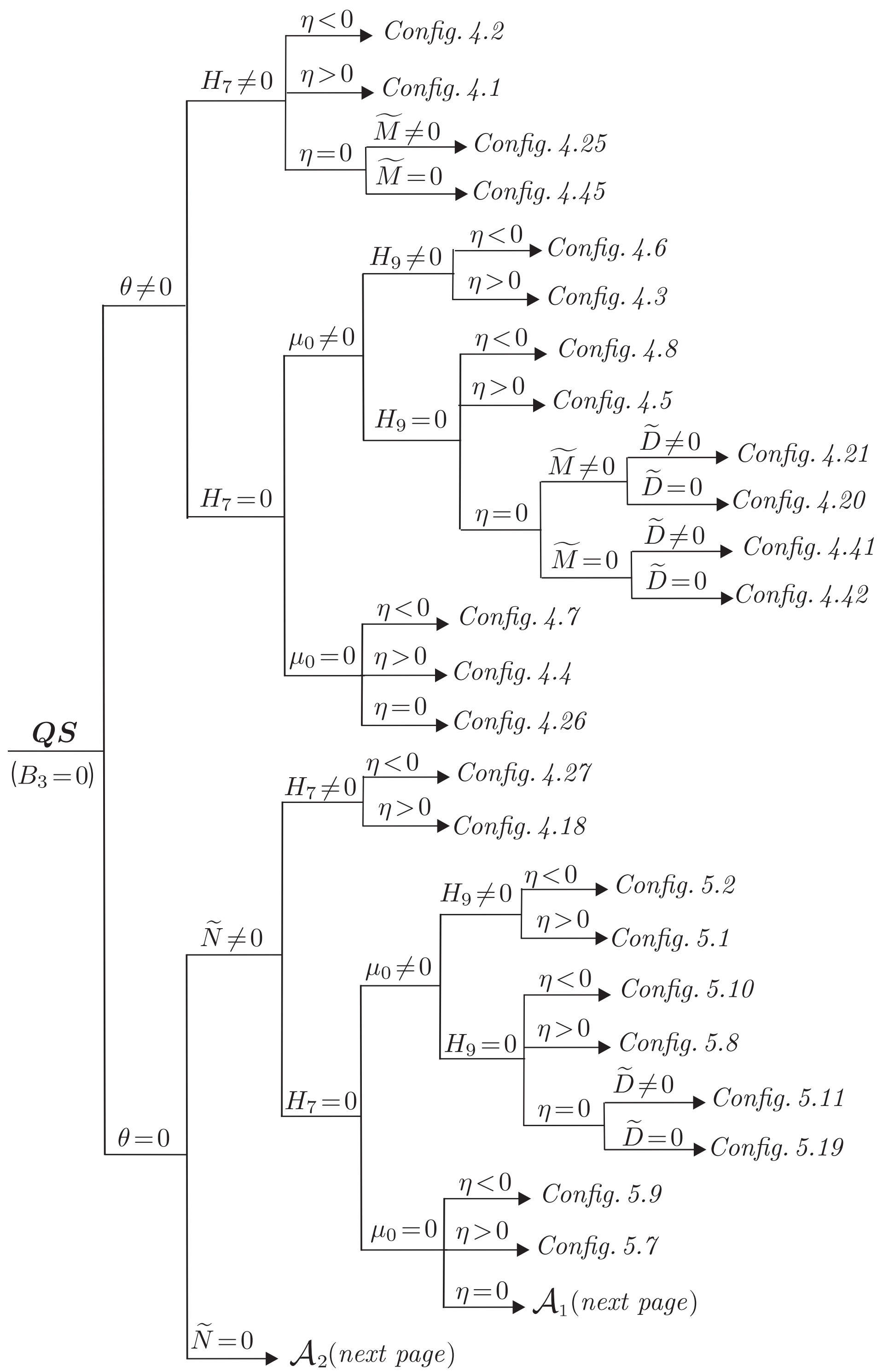

Diagram 3: The configurations of systems in **QSL** with $B_3=0$

where $\alpha, \beta, a, b, c, k$ are arbitrary real parameters. These systems possess the complex invariant lines $x \pm iy = 0$ and we calculate

$$\eta = -4\big[(k-b\beta)^2 + a^2\beta^2\big]^2 < 0, \quad B_2 = 0, \quad B_3 = 3ac^2k\beta(\alpha^2+\beta^2)(x^2+y^2)^2.$$

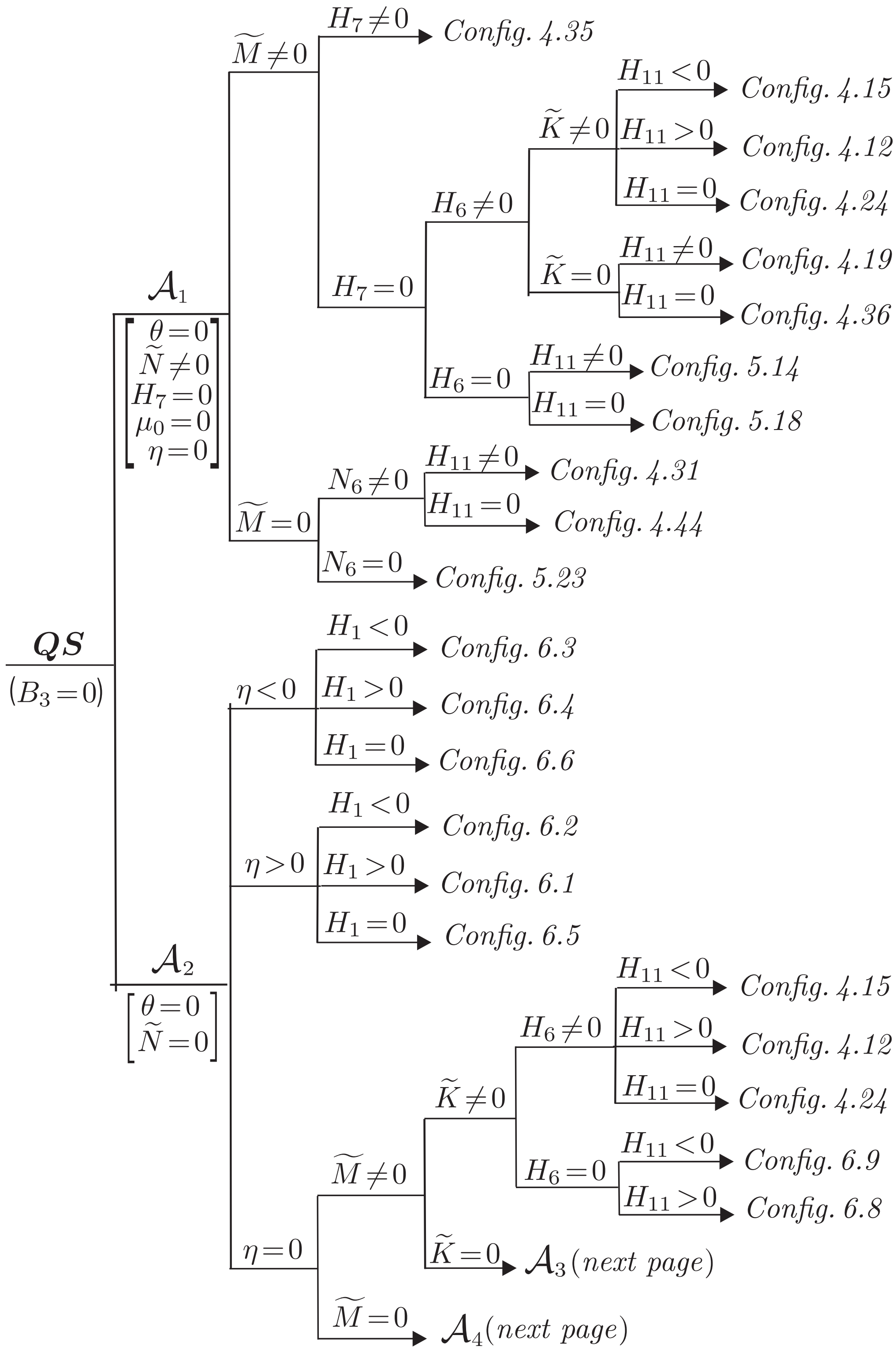

Diagram 3 *(continuation)*: The configurations of systems in **QSL** with $B_3 = 0$

According to Lemma 4.1 for a system (3) to belong to the class $\boldsymbol{QSL}_3$ the condition $B_3 \neq 0$ is necessary. The question which appears is the following: which conditions must

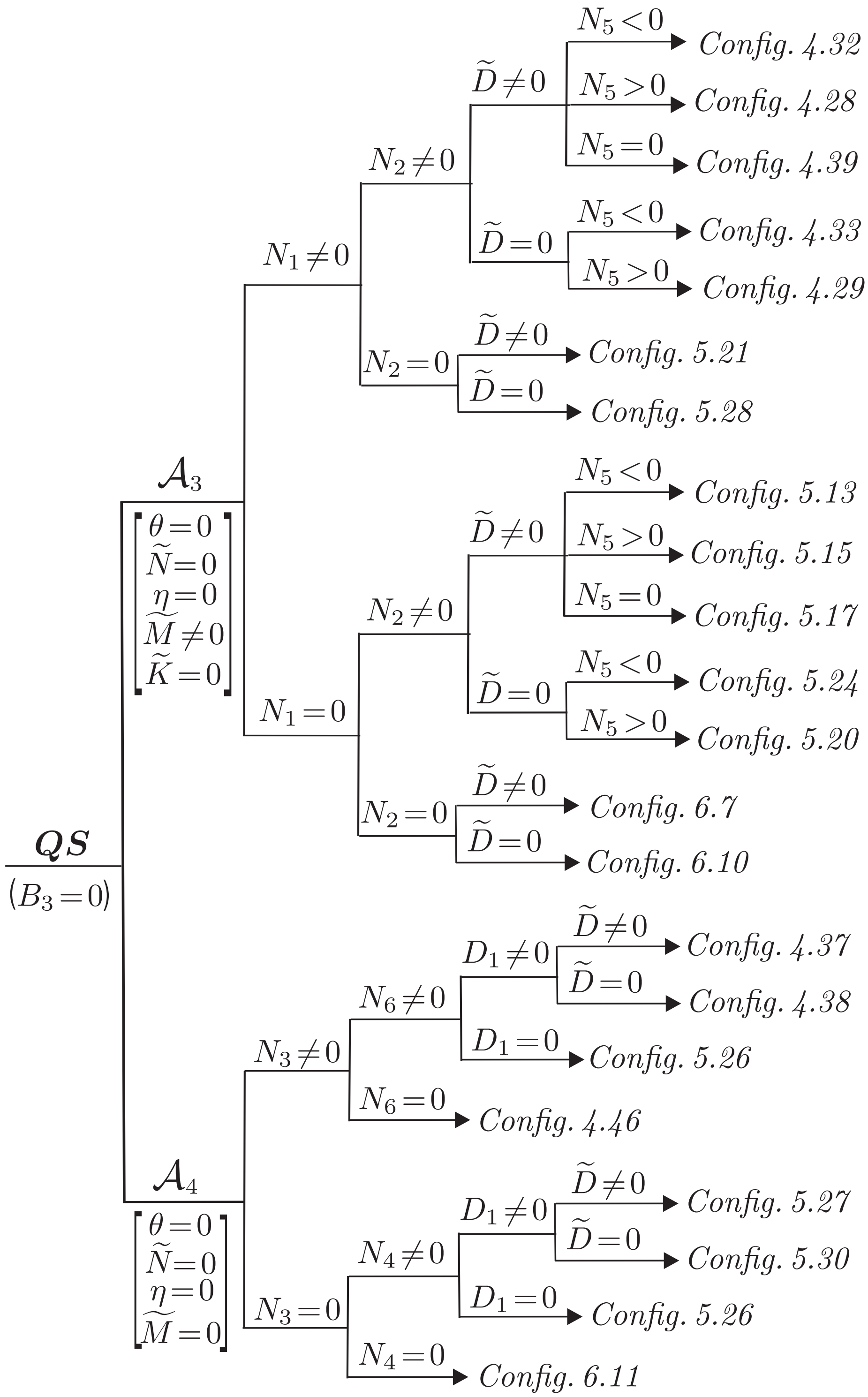

Diagram 3 *(continuation)*: The configurations of systems in **QSL** with $B_3 = 0$

be added in order to get the necessary and sufficient ones?

Providing the conditions $\eta < 0$ and $B_3 \neq 0$ to be fulfilled for a system (3) we examine what additional conditions could increase the total multiplicity of the invariant lines of this system.

Assume that a system (3) possesses invariant lines of total multiplicity exactly four. In [11] the family of systems belonging to $\boldsymbol{QSL}_4$ has been investigated and in Table 2 necessary and sufficient conditions for the realization of each one of the possible 46 configurations of invariant lines for this class are given. Considering Table 2 from [11] we detect that systems with the condition $\eta < 0$ (i.e. having 2 complex and one real infinite singularities) could possess only one of the following 4 configurations: *Config.4.2* and *Config.4.6–Config.4.8*. However for all these configurations the condition $B_3 = 0$ has to be satisfied and hence we get a contradiction to Lemma 4.1.

Thus we conclude that a system (3) could not belong to the class $\boldsymbol{QSL}_4$.

Suppose now that a system (3) possesses invariant lines of total multiplicity at least five. According to [9] (see Theorem 50, statement (ii)) for having invariant lines of total multiplicity 6 the condition $B_3 = 0$ is necessary for any quadratic system. So we conclude that a system (3) could not belong to the class $\boldsymbol{QSL}_6$.

It remains to consider the possibility when a system (3) with $\eta \neq 0$ (i.e. $\eta < 0$) and $B_3 \neq 0$ belongs to the class $\boldsymbol{QSL}_5$. In this case we consider Table 4 from [9] and we detect that subject of these conditions we could have the unique configuration *Config.5.6*. However to obtain this configuration the condition $\widetilde{N} = 0$ must be satisfied.

Thus we conclude that a system (3) with $\eta < 0$ and $B_3 \neq 0$ belongs to the class $\boldsymbol{QSL}_3$ if $\widetilde{N} \neq 0$ and to the class $\boldsymbol{QSL}_5$ if $\widetilde{N} = 0$. This means that the conditions provided by Theorem 4.1 for a quadratic systems to belong to the subclass $\boldsymbol{QSL}_3 \cap \boldsymbol{QS}_{2cIL}$ are satisfied.

Next we determine the configurations which a system (3) from the class $\boldsymbol{QSL}_3$ could possess. For this we have to determine the position of the singularities of this system with respect to the invariant lines.

A straightforward calculation gives us the following finite singularities of systems (3):

$$M_1(0,0), \quad M_2 = \Big(-\frac{c\alpha}{k + a\alpha - b\beta}, \frac{c\beta}{k + a\alpha - b\beta}\Big), \quad M_{3,4} = \Big(-\frac{c}{a \pm ib}, -\frac{c}{b \mp ia}\Big).$$

Since the condition $B_3 \neq 0$ implies $ack\beta \neq 0$ we conclude that the singular points M_2 and $M_{3,4}$ could not coalesce with M_1. Moreover the singular point M_2 exists if $k + a\alpha - b\beta \neq 0$, otherwise it goes to infinity coalescing with the real infinite singularity.

On the other hand for systems (3) we calculate

$$\mu_0 = (a^2 + b^2)k(k + a\alpha - b\beta)(\alpha^2 + \beta^2)$$

and hence for $\mu_0 \neq 0$ these systems possess two real and two complex finite singular points and we arrive at the configuration given by *Config.3.66*.

Assume now $\mu_0 = 0$. Due to the condition $B_3 \neq 0$ (i.e. $ack\beta \neq 0$) we get $k = b\beta - a\alpha \neq 0$ and hence we calculate

$$\mu_1 = c(a^2 + b^2)(a\alpha - b\beta)(\alpha^2 + \beta^2)(\beta x + \alpha y).$$

We observe that $\mu_1 \neq 0$ due to the condition $ack\beta(b\beta - a\alpha) \neq 0$. Since $\mu_0 = 0$, according to Lemma 3.3 one finite singular point went to infinity and coalesced with the infinite real singularity $N_1[\alpha, -\beta, 0]$ (see the factor of the invariant polynomial $\mu_1(x, y)$). In this case we arrive at the configuration given by *Config.3.67*.

As all the cases are examined we conclude that Theorem 4.1 is proved. ∎

4.2 Configurations of quadratic systems that are limit points of the family of Lotka-Volterra systems

It turn out that a quadratic system could have invariant lines of total multiplicity 3 which are not included in one of the following three classes: (i) Lotka-Volterra systems, or (ii) systems with two parallel invariant lines, or (iii) systems with two complex lines meeting at a finite singularity.

Indeed such kind of configurations could be obtained from an L-V system using the following two possibilities:

($\boldsymbol{\alpha}$) Two simple invariant affine lines of a L-V system belonging to the subclass $\mathbf{QSL}_3$ coalesced and we obtain a double invariant affine line and a multiple real singular point at infinity.

($\boldsymbol{\beta}$) One (or two) simple invariant affine lines of an L-V system in $\mathbf{QSL}_3$ coalesced with infinite line $Z = 0$ obtaining a double (or a triple) infinite invariant line.

Since we are in the class of L-V systems by Lemma 3.1 it is clear that the condition $B_2 = 0$ must be satisfied in both these cases. Moreover in the case **($\boldsymbol{\alpha}$)** the condition $\eta = 0$ has to be fulfilled, because we have a double (or triple) singular point at infinity.

On the other hand considering Lemma 3.7 we conclude that in the case **($\boldsymbol{\beta}$)** the condition $\widetilde{K}(a, x, y) = 0$ is necessary.

In what follows assuming the condition $B_2 = 0$ to be fulfilled we examine each one of the cases we mentioned above and determine the possible configurations of invariant lines as well as the corresponding affine invariant conditions for their realization.

($\boldsymbol{\alpha}$) In this case for a quadratic system the condition $\eta = 0$ has to be satisfied. We examine two cases: $\widetilde{M} \neq 0$ and $\widetilde{M} = 0$.

1: **The case $\widetilde{M} \neq 0$.** According to Lemma 3.4 a quadratic system could be brought via a linear transformation to the canonical form $(\mathbf{S}_{III})$, i.e. we have to examine the family systems

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + hxy, \\ \dot{y} =& b + ex + fy + (g-1)xy + hy^2. \end{aligned} \tag{4}$$

For these systems calculations yield:

$$\theta = 8h^2(1-g), \quad \mu_0 = gh^2, \quad C_2 = x^2y, \quad \widetilde{N} = (g^2-1)x^2 + 2h(g-1)xy + h^2y^2. \tag{5}$$

Since $C_2 = x^2y$ we conclude that these systems possess two infinite singularities: $N_1[1:0:0]$ (simple) and $N_2[0:1:0]$ (double). We discuss two subcases: $\theta \neq 0$ and $\theta = 0$.

1.1: **The subcase $\theta \neq 0$.** The condition $\theta \neq 0$ yields $h(g-1) \neq 0$ and we may consider $d = e = 0$ due to a translation. Moreover, since $h \neq 0$ we may assume $h = 1$ due to the rescaling $y \to y/h$. Thus we obtain the family of systems

$$\dot{x} = a + cx + gx^2 + xy, \quad \dot{y} = b + fy + (g-1)xy + y^2,$$

for which we calculate Coefficient$[B_2, y^4] = -648a^2$. Hence the necessary condition $B_2 = 0$ yields $a = 0$ and then

$$B_2 = -648b(b + c^2 - cf)(g-1)^2x^4, \quad H_4 = 48(b + c^2 - cf), \quad \theta = 8(1-g),$$
$$B_3 = -3\big[b(g-1)^2x^2 - (b + c^2 - cf)y^2\big]x^2$$

We shall consider two possibilities: $H_4 \neq 0$ and $H_4 = 0$.

1.1.1: **The possibility $H_4 \neq 0$.** In this case the condition $B_2 = 0$ yields $b = 0$ and we arrive to the family of systems

$$\dot{x} = x(c + gx + y), \quad \dot{y} = y[f + (g-1)x + y],$$

possessing the invariant lines $x = 0$ and $y = 0$. So we obtain LV-systems, i.e. no new configurations could de detected.

1.1.2: **The possibility $H_4 = 0$.** Then we have $b = c(f - c)$ and this leads to the family of systems

$$\dot{x} = x(c + gx + y), \quad \dot{y} = c(f-c) + fy + (g-1)xy + y^2, \tag{6}$$

possessing the invariant line $x = 0$ which is double because $\mathcal{H}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = X^2$ (see Notation 3.2). So, these systems possess invariant lines of total multiplicity at least 3. However for these systems the condition $B_3 = B_3 = 3c(c-f)(g-1)^2x^4 \neq 0$ is necessary and therefore by Lemma 3.1 we could not have an additional invariant line in the direction $y = 0$.

Thus we deduce that in the case $B_3 \neq 0$ systems (6) possess invariant lines of total multiplicity exactly 3. More exactly we have a double invariant affine line $x = 0$, on which there are located two finite singularities: $M_1(0, -c)$ and $M_2(0, c - f)$. The third finite singularity $M_3(x_3, y_3)$ of systems (6) has the coordinate

$$x_3 = -\frac{cg + c - fg}{g}, \quad y_3 = (c - f)g.$$

Since for systems (6) we have $\mu_0 = g$ we conclude that for $\mu_0 \neq 0$ all the finite singularities are on the plane and this means that one of the mentioned finite singularities is double. We claim that the double singularity is $M_1(0, -c)$. Indeed after translation of the origin of coordinates to the singular point M_1 we obtain the systems

$$\dot{x} = x(gx + y), \quad \dot{y} = c(1-g)x + (f - 2c)y + (g-1)xy + y^2 \tag{7}$$

possessing a double singular point at the origin because the determinant of the linear part equals zero. So these systems have the finite singular points

$$M_1(0,0), \quad M_2(0, 2c - f), \quad M_3\big(-(c + cg - fg)/g, c + cg - fg\big)$$

and we observe that M_3 goes to infinity if $g = 0$. Moreover it is clear that M_2 coalesces with M_1 if $2c - f = 0$ and M_3 coalesces with M_1 if $c + cg - fg = 0$.

On the other hand for systems (7) calculations yield:

$$\mu_0 = g, \; H_3 = 8(2c-f)(c + cg - fg)x^2, \quad H_{13} = -288c(2c-f)^2(g-1),$$
$$B_3 = 3c(c-f)(g-1)^2x^4$$

and we observe that due to $B_3 \neq 0$ the condition $H_{13} = 0$ is equivalent to $f = 2c$. So we consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1.1.2.1: *The case* $\mu_0 \neq 0$. Then $g \neq 0$ and the finite singularity M_3 remains in the finite plane. So if $H_3 \neq 0$ none of the singular points could coalesced and we arrive at the configuration *Config. 3.68* (see Figure **??**)

Assume now $H_3 = 0$, i.e. $(2c - f)(c + cg - fg) = 0$. Then evidently we obtain *Config. 3.69* if $H_{13} \neq 0$ and *Config. 3.70* if $H_{13} = 0$.

We point out that all three finite singularities could not coalesced due to $B_3 \neq 0$ (i.e. $c \neq 0$).

1.1.2.2: *The case* $\mu_0 = 0$. Then $g = 0$ and systems (7) become

$$\dot{x} = xy, \quad \dot{y} = cx + (f - 2c)y - xy + y^2$$

possessing the following two finite singularities: $M_1(0, 0)$ and $M_2(0, 2c - f)$. Since for the above systems we have $\mu_0 = \mu_1 = 0$ and $\mu_2 = -cy \neq 0$ (otherwise we get degenerate systems), according to Lemma 3.3 the singular point M_3 of systems (7) has gone to infinity and coalesced with the infinite singular point $N_1[1 : 0 : 0]$ which becomes of multiplicity 2 of the type $(1, 1)$.

On the other hand the finite singularity M_2 could coalesce with M_1 if the condition $f = 2c$ holds. For the above systems we calculate

$$B_3 = 3c(c - f)x^4 \neq 0, \quad H_3 = 8c(2c - f)x^2$$

and therefore we arrive at the configuration *Config. 3.71* if $H_3 \neq 0$ and *Config. 3.72* if $H_3 = 0$.

1.2: **The subcase** $\theta = 0$**.** Considering (5) this condition gives $h(g - 1) = 0$ and since $\mu_0 = gh^2$ we examine two possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

1.2.1: **The possibility** $\mu_0 \neq 0$**.** Then $h \neq 0$ and hence the condition $\theta = 0$ yields $g = 1$. Therefore we may consider $h = 1$ due to the rescaling $y \to y/h$ and $d = f = 0$ due to a translation. Thus we obtain the family of systems

$$\dot{x} = a + cx + x^2 + xy, \quad \dot{y} = b + ex + y^2,$$

for which we have Coefficient$[B_2, y^4] = -648a^2$ and therefore the condition $B_2 = 0$ implies $a = 0$. Then we calculate

$$B_2 = -648(b + c^2)e^2x^4, \quad H_7 = -4e.$$

and clearly if $e = 0$ (i.e. $H_7 = 0$) then the above systems with $a = e = 0$ possess three invariant affine lines $x = 0$ and $y^2 + b = 0$. Therefore we could not obtain new configurations apart from the ones already known.

Assuming $H_7 \neq 0$ we get the conditions $b = -c^2$ and this leads to the family of systems

$$\dot{x} = x(c + x + y), \quad \dot{y} = -c^2 + ex + y^2,$$

possessing the invariant line $x = 0$ which is double because $\mathcal{H}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = X^2$ (see Notation 3.2). These systems have three finite singularities $M_1(0, -c)$, $M_2(0, c)$ and $M_3(-2c-e, c+e)$.

We observe that the singular point M_1 is double because after the translation $(x,y) \to (x, y+c)$ we arrive at the systems

$$\dot{x} = x(x+y), \quad \dot{y} = ex - 2cy + y^2, \tag{8}$$

possessing a double singularity $M_1(0,0)$ at the origin of coordinates (since the determinant of linear part vanishes) and two elemental singularities $M_2(0, 2c)$ and $M_3(-2c-e, 2c+e)$. It is clear that in the case $e = -2c$ the singular point M_3 coalesces with the double point M_1 whereas for $c = 0$ the singularity M_2 coalesces with M_1.

On the other hand for the above systems we calculate

$$B_3 = -3e^2x^4, \quad H_3 = 16c(2c+e)x^2, \quad H_{13} = 2c^2e.$$

and due to $B_3 \neq 0$ (i.e. $e \neq 0$), by Lemma 3.1 systems (8) could not possess invariant lines in the direction $y = 0$. Therefore we deduce that in this case systems (8) possess invariant lines of total multiplicity 3.

Thus considering the condition $H_7 \neq 0$ (i.e. $e \neq 0$) it is not to difficult to determine that we get the configuration *Config. 3.68* if $H_3 \neq 0$; *Config. 3.69* if $H_3 = 0$ and $H_{13} \neq 0$, and *Config. 3.70* if $H_3 = H_{13} = 0$.

1.2.2: **The possibility $\mu_0 = 0$.** Considering (5) we get $h = 0$ and therefore for systems (4) we obtain $\widetilde{N} = (g^2-1)x^2$.

So we discuss two cases: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

1.2.2.1: *The case* $\widetilde{N} \neq 0$. Then $g - 1 \neq 0$ and assuming $e = f = 0$ (due to a translation) we arrive at the systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy,$$

for which we calculate

$$H_7 = 4d(g^2-1), \quad \widetilde{N} = (g^2-1)x^2, \quad \text{Coefficient}[B_2, y^4] = -648d^4g^2.$$

We observe that for $d = 0$ the above systems possess two parallel invariant lines $a + cx + gx^2 = 0$ and hence no new configurations could be obtained in this case.

Since $\widetilde{N} \neq 0$ we obtain that the condition $d = 0$ is equivalent to $H_7 = 0$ and in what follows we assume $H_7 \neq 0$. Then the condition $B_2 = 0$ implies $g = 0$ and then we obtain

$$B_2 = -648bcdx^4, \quad H_7 = -4d, \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = -cdxy$$

and we discuss two subcases: $\mu_2 \neq 0$ and $\mu_2 = 0$.

1.2.2.1.1: *The subcase* $\mu_2 \neq 0$. Then we have $c \neq 0$ and the condition $B_2 = 0$ gives $b = 0$ and we obtain the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = -xy, \tag{9}$$

possessing the invariant line $y = 0$. Moreover for these systems we calculate $\mathcal{H}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = YZ$ and by Lemma 3.6 we deduce that the infinite invariant line is double. In other words we have invariant lines of total multiplicity 3.

Since $\mu_0 = \mu_1 = 0$ and $\mu_2 = -cdxy \neq 0$, according to Lemma 3.3 we deduce that two finite singular points have gone to infinity and coalesced with infinite singular points $N_1[1:0:0]$ and $N_2[0:1:0]$, respectively. So at infinity we get two multiple singularities of multiplicities $(1,1)$ and $(2,1)$ (see Remark 3.1), correspondingly.

On the other hand due to $\mu_2 \neq 0$ (i.e. $cd \neq 0$) systems (9) possess two finite singularities $M_1(0,-a/d)$ and $M_1(-a/c,0)$ both simple (i.e. of multiplicity one). We observe that M_2 is located on the invariant line $y=0$ and these singularities coalesce if and only if $a=0$.

Since this condition is captured by the invariant polynomial $H_9 = -576a^2c^2d^2$ we arrive at the configuration *Config. 3.73* if $H_9 \neq 0$ and *Config. 3.74* if $H_9 = 0$.

1.2.2.1.2: *The subcase* $\mu_2 = 0$. Since $d \neq 0$ (due to $H_7 \neq 0$) we obtain $c=0$ and this leads to the systems

$$\dot{x} = a + dy, \quad \dot{y} = b - xy,$$

for which we have

$$B_2 = 0, \quad B_3 = -3bx^4, \quad H_7 = -4d \neq 0, \quad \mu_0 = \mu_1 = \mu_2 = 0, \quad \mu_3 = adxy^2.$$

For for these systems we calculate $\mathcal{H}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = Z^2$ and by Lemma 3.6 we deduce that the infinite invariant line is triple, i.e. we have invariant lines of total multiplicity 3. It is clear that we remain in this class due to the condition $B_3 \neq 0$.

It $\mu_3 = adxy^2 \neq 0$ then by Lemma 3.3 we deduce that two finite singular points have gone to infinity and coalesced with the infinite singularity $N_1[1:0:0]$ producing a triple point of the multiplicity $(1,2)$. At the same time one finite singularity has coalesced with $N_2[0:1:0]$ and we obtain a triple infinite singularity of multiplicity $(2,1)$. As a result we obtain the configuration *Config. 3.75.*

Assume now $\mu_3 = 0$. Then due to $H_7 \neq 0$ (i.e. $d \neq 0$) we get $a=0$ and hence we arrive at the systems

$$\dot{x} = dy, \quad \dot{y} = b - xy,$$

for which we have

$$B_2 = 0, \quad B_3 = -3bx^4 \neq 0, \quad H_7 = -4d \neq 0, \quad \mu_0 = \mu_1 = \mu_2 = \mu_3 = 0, \quad \mu_4 = -bd^2xy^3.$$

We observe that $\mu_4 = -bd^2x \neq 0$ (due to $B_3H_7 \neq 0$) and therefore according to Lemma 3.3 in the same manner as it was described above these systems possess at infinity the singularities $N_1[1:0:0]$ and $N_2[0:1:0]$ of multiplicities $(2,1)$ and $(1,3)$, respectively. In this case we obtain the configuration *Config. 3.76.*

1.2.2.2: *The case* $\widetilde{N} = 0$. In this case $g^2 - 1 \neq 0$ and since for systems (4) with $h=0$ we have $\widetilde{K} = 2g(g-1)x^2$ we consider two subcases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

1.2.2.2.1: *The subcase* $\widetilde{K} \neq 0$. Then $g \neq 1$ and the condition $\widetilde{N} = 0$ gives $g = -1$. Then we may assume in systems (4) $e = f = 0$ and we arrive at the systems

$$\dot{x} = a + cx + dy - x^2, \quad \dot{y} = b - 2xy.$$

for which we have Coefficient$[B_2(\boldsymbol{a},x,y),y^4] = -648d^4y^4$ and hence the condition $B_2 = 0$ implies $d = 0$. However in this case we obtain two parallel invariant lines $a + cx - x^2 = 0$ and this class of systems is already investigated in [5].

1.2.2.2.2: *The subcase* $\widetilde{K} = 0$. Then the condition $\widetilde{N} = 0$ gives $g = 1$ and we may assume $c = 0$ in systems (4) with $h = 0$ ad $g = 1$. This leads to the family of systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + fy$$

for which we have $B_2 = -648d^4y^4$ Therefore the condition $B_2 = 0$ yields $d = 0$ obtaining two invariant affine lines $x^2 + a = 0$. So we get two parallel invariant lines and we conclude that in this case we also could not have new configurations.

2: **The case $\widetilde{M} = 0$.** According to Lemma 3.4 a quadratic system in this class could be brought via a linear transformation to the canonical form $(\mathbf{S}_{IV})$, i.e. we have to examine the family systems

$$\begin{aligned}\dot{x} =& a + cx + dy + gx^2 + hxy,\\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2,\end{aligned} \tag{10}$$

for which calculations yield:

$$\theta = 8h^3, \quad \mu_0 = -h^3, \quad C_2 = x^3.$$

Since $C_2 = x^3$ we conclude that these systems possess only one infinite singularity $N_1[0 : 1 : 0]$ which is triple. We discuss two subcases: $\theta \neq 0$ and $\theta = 0$.

2.1: **The subcase $\theta \neq 0$.** Then $h \neq 0$ and we may assume $c = d = 0$ due to a translation. So we obtain the systems

$$\begin{aligned}\dot{x} =& a + gx^2 + hxy,\\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2,\end{aligned}$$

for which we calculate Coefficient$[B_2, y^4] = -3888a^2h^4x^2y^2$ and therefore the condition $B_2 = 0$ implies $a = 0$ due to $h \neq 0$. In this case we obtain $B_2 = -648b^2h^4x^4 = 0$ which implies $b = 0$ and we get the systems

$$\dot{x} = x(gx + hy), \quad \dot{y} = ex + fy - x^2 + gxy + hy^2. \tag{11}$$

For these systems following Notation 3.2 we calculate $\mathcal{H}(\mathcal{X},\mathcal{Y},\mathcal{Z}) = X^2$, i.e. by Lemma 3.6 the invariant line $x = 0$ of systems (11) has the multiplicity 2.

On the other hand due to $\theta \neq 0$ (i.e. $h \neq 0$) the above systems possess the following three finite singularities:

$$M_1(0,0), \quad M_2(0,-f/h), \quad M_3((eh - fg)/h, g(fg - eh)/h^2).$$

It is clear that M_1 is double because the first equation of systems (11) does not have linear terms (nor constant one).

For systems (11) we have $B_2 = 0$ and by Lemma 4.1 in order to remain in the class $\mathbf{QSL}_3$ the condition $B_3 = -3f(fg - eh)x^4 \neq 0$ is necessary.

Then considering the information pointed out above about the multiplicity of finite and infinite singularities of systems (11) we arrive at the configuration *Config. 3.77*.

2.2: **The subcase $\theta = 0$.** This condition gives $h = 0$ and then for systems (10) we have

$$\text{Coefficient}[B_2, x^2y^2] = -3888d^4g^2, \quad \widetilde{N} = g^2x^2.$$

We observe that in the case $\widetilde{N} \neq 0$ (i.e. $g \neq 0$) the condition $B_2 = 0$ implies $d = 0$ and then systems (10) possess two parallel invariant lines $gx^2 + cx + a = 0$. Since this family of systems was already investigated we have to impose the condition $\widetilde{N} = 0$ which yields $g = 0$. However in this case we get $B_2 = -648d^4x^4 = 0$, i.e. $d = 0$ and again we conclude that no new configurations could be obtained in this case.

Thus in the case $\widetilde{M} = 0$ and $B_2 = 0$ we have exactly one new configuration *Config. 3.77* and for this it is necessary $\theta \neq 0$.

($\boldsymbol{\beta}$) It was mentioned earlier (see page 24) that in this case for a quadratic system apart from the condition $B_2 = 0$ the condition $\widetilde{K} = 0$ has to be satisfied. According to Lemma 3.7 the infinite invariant line is of multiplicity at least two. This case contains both possibilities: either $Z = 0$ is double and we have an additional invariant affine line or $Z = 0$ is triple. Clearly in both cases we are in the class $\mathbf{QSL}_3$.

In the previous case **($\boldsymbol{\alpha}$)** when $\eta = 0$ we have examined all the possibilities when the invariant line $Z = 0$ is either simple or double or triple. So we have to investigate the cases $\eta < 0$ and $\eta > 0$ when in addition we have the multiple invariant line at infinity.

1: **The case $\eta < 0$.** We prove the following lemma.

Lemma 4.3. *If for a quadratic system the conditions $\eta < 0$ and $B_2 = \widetilde{K}(a, x, y) = 0$ hold, than this system possesses invariant lines af total multiplicity at least 4.*

Proof: Assume that for a quadratic system the condition $\eta < 0$ holds. Then according to Lemma 3.4 a quadratic system in this class could be brought via a linear transformation to the canonical form $(\mathbf{S}_{II})$, i.e. we have to examine the family systems

$$\begin{aligned}\dot{x} =& a + cx + dy + gx^2 + (h+1)xy,\\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2,\end{aligned}$$

for which calculations yield:

$$C_2 = x(x^2 + y^2), \quad \widetilde{K} = 2(1 + g^2 + h)x^2 + 4ghxy + 2h(1+h)y^2.$$

Evidently the condition $\widetilde{K} = 0$ is equivalent to $g = 0$ and $h = -1$ and therefore applying an additional translation which gives $e = f = 0$ we get the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b - x^2 - y^2.$$

For these systems we have

$$B_2 = -648\big[16a^2 + (c^2 + d^2 - 4b)^2\big]x^4 = 0 \;\Rightarrow\; a = 0,\; b = (c^2 + d^2)/4$$

and we arrive at the systems

$$\dot{x} = cx + dy, \quad \dot{y} = (c^2 + d^2)/4 - x^2 - y^2$$

which possess the double invariant line $Z = 0$ and two complex invariant affine lines

$$(c \pm id \mp 2ix + 2y) = 0.$$

So the above systems have invariant lines of total multiplicity at least four and this completes the proof of Lemma 4.3. ∎

2: **The case $\eta > 0$.** By Lemma 3.4 a quadratic system in this class could be brought via a linear transformation to the canonical form $(\mathbf{S}_I)$, i.e. we have to examine the family systems

$$\begin{aligned}\dot{x} &= a + cx + dy + gx^2 + (h-1)xy,\\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2,\end{aligned} \tag{12}$$

for which we calculate:

$$C_2 = xy(x - y), \quad \widetilde{K} = 2g(g-1)x^2 + 4ghxy + 2h(h-1)y^2.$$

Therefore the condition $\widetilde{K} = 0$ implies $gh = g(g-1) = h(h-1) = 0$. Evidently we can assume $g = 0$, otherwise we apply the change

$$(x, y, a, b, c, d, e, f, g, h) \mapsto (y, x, b, a, f, e, d, c, h, g) \tag{13}$$

which conserves systems (12). In this case we have either $g = h = 0$ or $g = 0$ and $h = 1$. We claim that the second case can be reduced to the first one via a transformation. Indeed assuming $g = h = 0$ we get the family of systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = b + ex + fy - xy \tag{14}$$

whereas for $g = 0$ in the case $h = 1$ we arrive at the systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + ex + fy - xy + y^2 \tag{15}$$

Then applying the linear transformation $x_1 = y$, $y_1 = y - x$ to systems (15) we arrive at the family of systems

$$\dot{x}_1 = a' + c'x_1 + d'y_1 - x_1y_1, \quad \dot{y}_1 = b' + e'x_1 + f'y_1 - x_1y_1$$

where

$$a' = b,\ c' = e + f,\ d' = -e,\ b' = b - a,\ e' = e + f - c - d,\ f' = c - e.$$

Comparing the above system with (14) we deduce that our claim is proved.

Thus $g = h = 0$ and for systems (14) we calculate

$$B_2 = -648(a + cd)(b + ef)(x - y)^4 = 0.$$

Due to the change (13) we may assume $b = -ef$. Then we arrive at the systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = (f - x)(y - e), \tag{16}$$

which besides the double infinite invariant line $Z = 0$ possess the invariant affine line $y = e$. We point out that for these systems we have

$$B_3 = 3(a + cd)(x - y)^2 y^2, \quad H_7 = -4(c + d - e - f)$$

and by Lemma 4.1 the condition $B_3 \neq 0$ has to be satisfied. We claim that in order to be in the class $\mathbf{QSL}_3$ we must force also the condition $H_7 \neq 0$ to be fulfilled. Indeed supposing $H_7 = 0$ we obtain $f = c + d - e$ and we arrive at the family of systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = (c + d - e - x)(y - e)$$

possessing the following two invariant affine lines:

$$y = e, \quad a + ce + de - e^2 + (c - e)(x - y) = 0,$$

i.e. the above systems belong to the class $\mathbf{QSL}_{\geq 4}$ and this completes the proof of our claim.

Next we examine configurations of invariant lines for the family of systems (16) in the case $B_3 H_7 \neq 0$. We determine that these systems possess the singular points $M_i(x_i, y_i)$ with the coordinates:

$$x_1 = f, \quad y_1 = -\frac{a + cf}{d - f}; \quad x_2 = -\frac{a + de}{c - e}, \quad y_2 = e$$

and evidently these finite singularities exist if and only if $(c - e)(d - f) \neq 0$. Moreover the singularity M_2 is located on the invariant line $y = e$ of systems (16).

On the other hand for systems (16) we calculate

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = -(c - e)(d - f)xy, \quad H_9 = -576(c - e)^2(d - f)^2(a + de + cf - ef)^2.$$

and hence if $\mu_2 \neq 0$ we have two finite singularities M_1 and M_2, where M_2 is located on the invariant line $y = e$. Moreover we observe that in the case $a + de + cf - ef = 0$ the singular point M_1 coalesced with M_2 obtaining the double singularity $M_{1,2}(f, e)$ on the invariant line $y = e$. We examine two possibilities: $\mu_2 \neq 0$ and $\mu_2 = 0$.

2.1: *The possibility* $\mu_2 \neq 0$. In this case the condition $H_9 = 0$ is equivalent to $a + de + cf - ef = 0$. Then taking into account the factorization of the invariant polynomial $\mu_2(x, y)$ by Lemma 3.3 we obtain that at infinity both the singular points $N_1[1 : 0 : 0]$ and $N_2[0 : 1 : 0]$ are double of the type $(1, 1)$. Therefore we arrive at the configuration *Config. 3.78* if $H_9 \neq 0$ and at *Config. 3.79* if $H_9 = 0$.

2.2: *The possibility* $\mu_2 = 0$. This condition implies $(c - e)(d - f) = 0$ and since we have

$$\mu_3 = -(c - e)(a - de + cf + ef)x^2 y + (d - f)(a + de - cf + ef)xy^2$$

by Lemma 3.3 we deduce that for $d = f$ the finite singularity M_1 coalesced with infinite singular point $N_1[1 : 0 : 0]$ which becomes of the multiplicity $(1, 2)$. This leads to the configuration *Config. 3.80.*

In the case $c = e$ the finite singularity M_2 coalesced with infinite singular point $N_1[0 : 1 : 0]$ located at the "end" of the invariant affine line $y = e$ and we get the configuration *Config. 3.81*.

On the other hand for systems (16) we have

$$H_{17} = 9(a + cd)(c - e)^2$$

and therefore in the case $\mu_2 = 0$ (i.e. $(c - e)(d - f) = 0$) we get $d = f$ if $H_{17} \neq 0$ (*Config. 3.80*) and we obtain $c = e$ if $H_{17} = 0$ (*Config. 3.81*).

We point out that we could not have simultaneously $d = f$ and $c = e$ because in this case we get $H_7 = -4(c + d - e - f) = 0$ and this contradicts our assumption $H_7 \neq 0$.

On the other hand in the case $d = f$ we obtain

$$\mu_3 = (e - c)(a + cf)x^2y, \quad B_3 = 3(a + cf)(x - y)^2y^2, \quad H_7 = 4(e - c)$$

whereas for $c = e$ we have

$$\mu_3 = (d - f)(a + de)xy^2, \quad B_3 = 3(a + de)(x - y)^2y^2, \quad H_7 = -4(d - f).$$

We observe that in both cases the condition $B_3H_7 \neq 0$ implies $\mu_3 \neq 0$ and therefore we could not have other new configurations in the case under consideration.

It remains to show that **i)** the 16 new configurations are distinct among themselves and **ii)** that they are distinct from all the other 65 configurations.

We first show **i)**. First we split these 16 configurations in four subsets with distinct geometrical properties:

a) Two configurations with complex invariant affine lines: *Config. 3.66* and *Config. 3.67*;

b) Six configurations with the line at infinity double: *Config. 3.73*, *Config. 3.74* and *Config. 3.78*–*Config. 3.81*;

c) Six configurations with the unique invariant affine line which is double: *Config. 3.68*–*Config. 3.72* and *Config. 3.77*;

d) Two configurations with the line at infinity triple: *Config. 3.75* and *Config. 3.76*.

We now prove that each of these four subsets has distinct configurations.

1: *The group a).* The two configurations in this group are of course distinct from all the other 14 new configurations because the affine invariant lines are complex. They are also distinct from one another because the multiplicities of the unique singular point at infinity are different.

2: *The group b).* The six configurations in this group are distinguished by the number of the singular points at infinity, by their multiplicities and by the multiplicities of the singular points on the invariant affine line.

3: *The group c).* The six configurations in this group are clearly distinguished by the number of singularities on the double invariant line as well as their multiplicities.

4: The group d). The two configurations in this group are distinguished by the multiplicities of the singular points on this triple line. In *Config. 3.75* we have two such singularities each triple while in *Config. 3.76* we have two such singular points, one triple and one quadruple.

Entirely similar arguments apply also for the proof of **ii)**.

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