Linear transport equations for vector fields with subexponentially integrable divergence

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Abstract

We face the well-posedness of linear transport Cauchy problems

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + c\,u = f & (0,T) \times \mathbb{R}^n, \\ u(0,\cdot) = u_0 \in L^\infty & \mathbb{R}^n, \end{cases}$$

under borderline integrability assumptions on the divergence of the velocity field b. For $W^{1,1}_{\mathrm{loc}}$ vector fields b satisfying $\frac{|b(x,t)|}{1+|x|} \in L^1(0,T;L^1) + L^1(0,T;L^\infty)$ and

$$\operatorname{div} b \in L^\infty + \operatorname{Exp}\left(\frac{L}{\log^\gamma L}\right),$$

we prove existence and uniqueness of weak solutions when $\gamma = 1$. Moreover, optimality is shown by providing examples of non-uniqueness for every $\gamma > 1$. Stability questions and further extensions to the BV setting are also addressed.

1 Introduction

In this paper, we are concerned with the well-posedness (ill-posedness) of the Cauchy problem of the transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0,T) \times \mathbb{R}^n, \\ u(0,\cdot) = u_0 & \mathbb{R}^n. \end{cases} \tag{1}$$

Here $b \in L^1(0,T;W^{1,1}_{\mathrm{loc}})$ or $b \in L^1(0,T;BV_{\mathrm{loc}})$, and $u_0 \in L^\infty$. A function $u \in L^1(0,T;L^1_{\mathrm{loc}})$ is called a *weak solution* to (1) if for each $\varphi \in \mathcal{C}^\infty([0,T] \times \mathbb{R}^n)$ with compact support in $[0,T] \times \mathbb{R}^n$ it holds that

$$-\int_0^T \int_{\mathbb{R}^n} u\,\frac{\partial \varphi}{\partial t}\,dx\,dt - \int_{\mathbb{R}^n} u_0\,\varphi(0,\cdot)\,dx - \int_0^T \int_{\mathbb{R}^n} u\,\operatorname{div}(b\,\varphi)\,dx\,dt = 0.$$

We also say that the problem (1) is *well posed* in $L^\infty(0,T;L^\infty)$ if weak solutions exists and are unique, for any $u \in L^\infty$.

The classical method of characteristics describes, under enough smoothness of the velocity field b, the unique solution to (1) as the composition $u(t,x) = u_0(X(t,x))$, where $X(t,x)$ is the unique solution to

the ODE

$$\begin{cases} \dfrac{d}{dt} X(t,x) = -b(t, X(t,x)), \\ X(0,x) = x. \end{cases} \tag{2}$$

When there is no smoothness, solutions of (2) are more delicate to understand. In the seminal work [DPL89], DiPerna and Lions showed that for $b \in L^1(0,T; W^{1,1}_{\mathrm{loc}})$ satisfying

$$\frac{|b(t,x)|}{1+|x|} \in L^1(0,T; L^1) + L^1(0,T; L^\infty), \tag{3}$$

and

$$\mathrm{div}\, b \in L^1(0,T; L^\infty),$$

the problem (1) is well-posed in $L^\infty(0,T; L^\infty)$. Moreover, the solution is renormalizable, i.e., for each $\beta \in \mathcal{C}^1(\mathbb{R})$, $\beta(u)$ is the unique weak solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} \beta(u) + b \cdot \nabla \beta(u) = 0 & (0,T) \times \mathbb{R}^n, \\ \beta(u)(0, \cdot) = \beta(u_0) & \mathbb{R}^n. \end{cases} \tag{4}$$

Since that, the problem has been found many applications and has been generalized into different settings, let us mention a few below. In a breakthrough paper, Ambrosio [Am04] extended the renormalization property to the setting of bounded variation (or BV) vector fields. Cipriano–Cruzeiro [CiCr05] found nice solutions of (2) for vector fields with exponentially integrable divergence in the setting of Euclidean spaces equipped with Gauss measures. Recently, Mucha [Mu10] established well-posedness for (1) with divergence of the velocity field in BMO with compact support. For more applications and generalizations, we refer to [ACFS09, AF09, CCR06, CDL08, CL02, FL10, Su14] and references therein.

Our primary goal in this paper is to understand to which extent the condition $\mathrm{div}\, b \in L^1(0,T; L^\infty)$ can be relaxed so that the initial value problem (1) remains being well-posed in $L^\infty(0,T; L^\infty)$. As it was already shown by DiPerna–Lions, the assumption

$$\mathrm{div}\, b \in L^1(0,T; L^q), \quad \text{for some } q \in (1, \infty) \tag{5}$$

is *not* sufficient to guarantee uniqueness of solutions $X(t)$ of (2). As a consequence, uniqueness also fails for (1) under (5). However, there is still some room left between L^q and L^∞, e.g., BMO or even spaces of (sub-)exponentially integrable functions.

Mucha [Mu10] recently obtained well-posedness of (1) in $L^\infty(0,T; L^\infty)$ for $W^{1,1}_{\mathrm{loc}}$ vector fields b such that $\frac{|b(t,x)|}{1+|x|} \in L^1(0,T; L^1)$,

$$\mathrm{div}\, b \in L^1(0,T; BMO), \quad \text{and} \quad \mathrm{supp}(\mathrm{div}\, b) \subset B(0,R) \text{ for some } R > 0. \tag{6}$$

Subko [Su14] further generalized Mucha's result by replacing $W^{1,1}_{\mathrm{loc}}$ by the local class BV_{loc} of vector fields with compactly supported BMO divergence. A natural question arises here: is the restriction on the support of $\mathrm{div}\, b$ necessary? At the first sight, one may wonder whether well-posedness holds true if $\mathrm{div}\, b \in L^1(0,T; BMO)$ without any further restriction on the support. We do not know if this is true.

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By using the John-Nirenberg inequality from [JN61], one sees that BMO functions are locally exponentially integrable. Thus, assumption (6) easily gives that

$$\text{div } b \in L^1(0, T; \text{Exp } L),$$

where $\text{Exp } L$ denotes the Orlicz space of globally exponentially integrable functions (see Section 2 for a definition). Nevertheless, it is worth recalling here that no restriction on the support of $\text{div } b$ is needed to get well-posedness if global boundedness is assumed for the divergence, namely $\text{div } b \in L^1(0, T; L^\infty)$. Therefore, it seems reasonable to investigate if the condition

$$\text{div } b \in L^1(0, T; L^\infty) + L^1(0, T; \text{Exp } L)$$

suffices to get well-posedness. Our first result gives a positive answer to this question. Indeed, we prove that an Orlicz space even larger than $\text{Exp } L$ is sufficient for our purpose.

Theorem 1. *Let $T > 0$. Assume that $b \in L^1(0, T; W^{1,1}_{\text{loc}})$ satisfying* (3) *and*

$$\text{div } b \in L^1(0, T; L^\infty) + L^1\left(0, T; \text{Exp}\left(\frac{L}{\log L}\right)\right). \tag{7}$$

Then for each $u_0 \in L^\infty$ there exists a unique weak solution $u \in L^\infty(0, T; L^\infty)$ of the transport problem (1).

See Section 2 for the precise definition of the Orlicz space $\text{Exp}(\frac{L}{\log L})$.

Remark 2. *The conclusion of Theorem 1 still holds if we add reaction and source terms. Namely, in Theorem 1 the same conclusion holds if we replace* (1) *by*

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + cu = f & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 & \mathbb{R}^n, \end{cases}$$

provided that $c, f \in L^1(0, T; L^\infty)$, and $b \in L^1(0, T; W^{1,1}_{\text{loc}})$ satisfies (3) *and* (7). *The proof works similarly.*

Remark 3. *One can still strengthen the borderline a bit more. More precisely, well-posedness still holds if $b \in L^1(0, T; W^{1,1}_{\text{loc}})$ satisfies* (3) *and, at the same time,* (7) *is replaced by the less restrictive condition*

$$\text{div } b \in L^1(0, T; L^\infty) + L^1\left(0, T; \text{Exp}\left(\frac{L}{\log L \, \log\log L \, \ldots \, \underbrace{\log\cdots\log}_{k} L}\right)\right).$$

The proof follows similarly to that of Theorem 1.

At this point, it might bring some light reminding the chain of strict inclusions

$$\text{Exp } L \subset \text{Exp}\left(\frac{L}{\log L}\right) \subset \text{Exp}\left(\frac{L}{\log L \, \log\log L \, \ldots \, \underbrace{\log\cdots\log}_{k} L}\right).$$

In particular, the first one explains the following corollary, which unifies DiPerna–Lions and Mucha's results.

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Corollary 4. *Let $T > 0$. Assume that $b \in L^1(0, T; W^{1,1}_{\text{loc}})$ satisfies (3) and*

$$\operatorname{div} b \in L^1(0, T; L^\infty) + L^1(0, T; \operatorname{Exp} L).$$

Then for each $u_0 \in L^\infty$, there exists a unique weak solution $u \in L^\infty(0, T; L^\infty)$ of the Cauchy problem (1).

The proof of Theorem 1 will be built upon the renormalization property by DiPerna–Lions [DPL89] and properties of Orlicz spaces. A key ingredient is an a priori estimate by using the backward equation, which shows that if $u \in L^\infty(0, T; L^\infty)$ is a solution of (1) with the initial value $u_0 \equiv 0$, then $u \in L^\infty(0, T; L^2 \cap L^\infty)$. See Proposition 15 below. Indeed, the idea behind this is a kind of multiplicative property. That is, if $u_1, u_2 \in L^\infty(0, T; L^\infty)$ satisfy

$$\frac{\partial u_i}{\partial t} + b \cdot \nabla u_i + c_i\, u = 0 \text{ in } (0, T) \times \mathbb{R}^n,$$

then the pointwise multiplication $u_1 u_2$ solves

$$\frac{\partial (u_1 u_2)}{\partial t} + b \cdot \nabla (u_1 u_2) + (c_1 + c_2)\,(u_1 u_2) = 0 \text{ in } (0, T) \times \mathbb{R}^n.$$

See Proposition 19 below for the details.

Notice that our assumption (7) on the divergence is too weak to guarantee the well-posedness of the transport equation in the L^p case for finite values of p. To explain this, let us assume for a while that b generates a flow $X(t) = X(t, x)$ through the ODE (2). Boundedness of $\operatorname{div} b$ guarantees that the image $X(t)_\sharp m$ of Lebesgue measure m is absolutely continuous and has bounded density (see [DPL89]). If $\operatorname{div} b$ is not bounded, but only (sub)-exponentially integrable, then one may still expect $X(t)_\sharp m \ll m$, but boundedness of density might be lost. Thus no control on L^p norms is expected if $p \in [1, \infty)$.

At this point it is worth mentioning that the existence and uniqueness of such a flow $X(t)$ is not an easy issue in our context. Nevertheless, if one assumes $\frac{|b(x,t)|}{1+|x|} \in L^1(0, T; L^\infty)$ and $\operatorname{div} b \in L^1(0, T; L^\infty) + L^1(0, T; \operatorname{Exp} L)$, then a unique flow can be obtained as a consequence of the results from [CiCr05]. We will come back to the flow issue in a forthcoming paper.

We have the following quantitative estimate in $L^p \cap L^\infty$ case under assumption (7). For an easier formulation in the case $\operatorname{div} b \in L^1(0, T; L^\infty) + L^1(0, T; \operatorname{Exp} L)$, see Corollary 17.

Theorem 5. *Let $T, M > 0$ and $1 \le p < \infty$. Suppose that $b \in L^1(0, T; W^{1,1}_{\text{loc}})$ satisfies (3) and (7). There exists $\epsilon > 0$ such that, for every $u_0 \in L^p \cap L^\infty$ with $\|u_0\|_{L^\infty} \le M$ and $\|u_0\|^p_{L^p} < \epsilon$, the transport problem (1) has a unique solution u and moreover*

$$\left| \log \log \log \left(\frac{1}{\|u\|_{L^\infty(0,T;L^p)}} \right) - \log \log \log \left(\frac{1}{\|u_0\|_{L^p}} \right) \right| \le 16e \int_0^T \beta(s)\, ds,$$

where $\operatorname{div} b = B_1 + B_2$ and $\beta(t) = \|B_1(t, \cdot)\|_{\operatorname{Exp}(\frac{L}{\log L})} + \|B_2(t, \cdot)\|_{L^\infty}$.

Relying on Ambrosio's seminal result [Am04], Theorem 1 admits an extension to the setting of bounded variation (BV) vector fields.

Theorem 6. *Let $T > 0$. Assume that $b \in L^1(0, T; BV_{\text{loc}})$ satisfying (3) and (7). Then for every $u_0 \in L^\infty$ there exists a unique weak solution $u \in L^\infty(0, T; L^\infty)$ of the transport problem (1).*

Concerning the optimality of (7) in Theorem 1, and after re-analyzing an example from [DPL89, Section 4.1], we can show that the condition

$$b \in L^\infty(0, T; L^\infty), \quad \text{and} \quad \operatorname{div} b \in L^1\left(0, T; \operatorname{Exp}\left(\frac{L}{\log^\gamma L}\right)\right) \quad \text{for some } \gamma > 1$$

is not sufficient to guarantee uniqueness.

Theorem 7. *Let $T > 0$. Given $\gamma \in (1, \infty)$, there exists a vector field $b \in L^1(0, T; W^{1,1}_{\text{loc}})$, that satisfies*

$$b \in L^\infty(0, T; L^\infty), \quad \text{and} \quad \operatorname{div} b \in L^1\left(0, T; \operatorname{Exp}\left(\frac{L}{\log^\gamma L}\right)\right),$$

such that for each $u_0 \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ the Cauchy problem (1) admits infinitely many weak solutions $u \in L^\infty(0, T; L^\infty)$.

The proof is based on DiPerna–Lions' example [DPL89, Section 4.1]. The key point is the construction of an explicit smooth function vanishing exactly at the points of the one third Cantor set.

Remark 8. *Similar examples to that in Theorem 7 can be found in the setting of Remark 3, with a small modification on the smooth function g (see Step 2 of the proof). More precisely, given $\gamma \in (1, \infty)$ and $k \in \{2, 3, \dots\}$, one can find a vector field $b \in L^1(0, T; W^{1,1}_{\text{loc}})$ satisfying (3) and such that*

$$\operatorname{div} b \in L^1\left(0, T; \operatorname{Exp}\left(\frac{L}{(\log L)(\log \log L) \dots \underbrace{(\log \cdots \log L)^\gamma}_{k}}\right)\right),$$

for which the Cauchy problem (1) admits, for every $u_0 \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, infinitely many weak solutions $u \in L^\infty(0, T; L^\infty)$.

Remark 9. *In the context of Corollary 4, and arguing again as in the proof of Theorem 7, one can show that the condition $\operatorname{div} b \in L^1(0, T; \operatorname{Exp} L)$ cannot be replaced by $\operatorname{div} b \in L^1(0, T; \operatorname{Exp}(L^{1/\gamma}))$ if $\gamma > 1$.*

Remark 10. *The example of Theorem 7 admits a further generalization to the setting of the Euclidean space when equipped with Gaussian measure $d\gamma_n$. Namely, one can show that the assumption*

$$\exp\{C|b| + C(|\operatorname{div}_{\gamma_n} b|)^\alpha\} \in L^1(0, T; L^1(\gamma_n)) \qquad \text{for some } \alpha \in (0, 1)$$

does not imply uniqueness of the flow, and therefore uniqueness for (1) also fails. See [CiCr05, AF09, FL10].

The paper is organized as follows. In Section 2 we recall some basic aspects of Orlicz spaces, and prove some technical estimates. In Section 3 we prove Theorem 1. Section 4 is devoted to stability results. In Section 5, we prove Theorem 6. In the last section we prove Theorem 7. Throughout the paper, we denote by C positive constants which are independent of the main parameters, but which may vary from line to line.

2 Orlicz spaces

We will need to use some Orlicz spaces and their duals. For the reader's convenience, we recall here some definitions. See the monograph [RR91] for the general theory of Orlicz spaces. Let

$$P : [0, \infty) \mapsto [0, \infty),$$

be an increasing homeomorphism onto $[0, \infty)$, so that $P(0) = 0$ and $\lim_{t \to \infty} P(t) = \infty$. The Orlicz space L^P is the set of measurable functions f for which the Luxembourg norm

$$\|f\|_{L^P} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} P\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\}$$

is finite. In this paper we will be mainly interested in two particular families of Orlicz spaces. Given $r, s \ge 0$, the first family corresponds to

$$P(t) = t \left(\log^+ t \right)^r \left(\log^+ \log^+ t \right)^s,$$

where $\log^+ t := \max\{1, \log t\}$. The obtained L^P spaces are known as *Zygmund spaces*, and will be denoted from now on by $L \log^r L \log \log^s L$. The second family is at the upper borderline. For $\gamma \ge 0$ we set

$$P(t) = \exp \left\{ \frac{t}{(\log^+ t)^\gamma} \right\} - 1, \qquad t \ge 0. \tag{8}$$

Then we will denote the obtained L^P by $\mathrm{Exp}(\frac{L}{\log^\gamma L})$. If $\gamma = 0$ or $\gamma = 1$, we then simply write $\mathrm{Exp}\, L$ and $\mathrm{Exp}(\frac{L}{\log L})$, respectively. Note that $0 \le \gamma_1 < \gamma_2$ implies $\mathrm{Exp}\, L \subset \mathrm{Exp}(\frac{L}{\log^{\gamma_1} L}) \subset \mathrm{Exp}(\frac{L}{\log^{\gamma_2} L})$. Also, let us observe that compactly supported BMO functions belong to $\mathrm{Exp}\, L$. Similarly, we will denote by

$$\mathrm{Exp} \left(\frac{L}{\log L \, \log \log L \, \dots \, (\underbrace{\log \cdots \log}_{k} L)^\gamma} \right)$$

the Orlicz space corresponding to

$$P(t) = \exp \left\{ \frac{t}{(\log^+ t)\,(\log^+ \log^+ t) \dots (\underbrace{\log^+ \cdots \log^+}_{k} t)^\gamma} \right\} - 1, \qquad t \ge 0.$$

The following technical lemma will be needed at Section 3.

Lemma 11. *If $f \in L \log L \log \log L$ and $g \in \mathrm{Exp}(\frac{L}{\log L})$ then $fg \in L^1$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le 2\|f\|_{L \log L \log \log L}\, \|g\|_{\mathrm{Exp}(\frac{L}{\log L})}.$$

Moreover, if $f \in L^\infty \cap L \log L \log \log L$ then

$$\|f\|_{L \log L \log \log L}$$
$$\le 2e\|f\|_{L^1}\left(\log(e + \|f\|_{L^\infty}) + |\log(\|f\|_{L^1})| \right)\left(\log \log(e^e + \|f\|_{L^\infty}) + |\log|\log(\|f\|_{L^1})|| \right).$$

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Proof. We refer to [RR91, p. 17] for the Hölder inequality. Towards the second estimate, we start by noting that if $f \in L \log L \log \log L$ then $f \in L^1$ and $\|f\|_{L^1} \leq \|f\|_{L \log L \log \log L}$. By setting $M = \|f\|_{L^\infty}$,

$$\lambda = \|f\|_{L^1} \left[\log(e+M) + |\log(\|f\|_{L^1})|\right] \left[\log \log(e^e + M) + |\log|\log(\|f\|_{L^1})||\right],$$

and calling

$$E = \{x \in \mathbb{R}^n : |f(x)| \leq e^e \lambda\},$$

we see that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^+ \left(\frac{|f(x)|}{\lambda}\right) \log^+ \log^+ \left(\frac{|f(x)|}{\lambda}\right) \, dx$$

$$\leq \frac{e}{\lambda} \int_E |f(x)| \, dx + \int_{\mathbb{R}^n \setminus E} \frac{|f(x)|}{\lambda} \log \left(\frac{|f(x)|}{\lambda}\right) \log \log \left(\frac{|f(x)|}{\lambda}\right) \, dx$$

$$\leq e + \int_{\mathbb{R}^n \setminus E} \frac{|f(x)|}{\lambda} \log \left(\frac{e+M}{\|f\|_{L^1}}\right) \log \log \left(\frac{e^e + M}{\|f\|_{L^1}}\right) \, dx$$

$$\leq e + \int_{\mathbb{R}^n \setminus E} \frac{|f(x)| \log \left[\log(e^e + M) + |\log\|f\|_{L^1}|\right]}{\|f\|_{L^1} \left[\log \log(e^e + M) + |\log|\log(\|f\|_{L^1})||\right]} \, dx.$$

Notice that for $x \geq e$ and $y \geq 0$, it holds that

$$\log(x + y) \leq 2 \log x + 2|\log y|,$$

which implies that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^+ \left(\frac{|f(x)|}{\lambda}\right) \log^+ \log^+ \left(\frac{|f(x)|}{\lambda}\right) \, dx$$

$$\leq e + \int_{\mathbb{R}^n \setminus E} \frac{2|f(x)| \left[\log \log(e^e + M) + |\log(|\log\|f\|_{L^1}|)|\right]}{\|f\|_{L^1} \left[\log \log(e^e + M) + |\log|\log(\|f\|_{L^1})||\right]} \, dx \leq 2e.$$

Therefore, we see that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{2e\lambda} \log^+ \left(\frac{|f(x)|}{2e\lambda}\right) \log^+ \log^+ \left(\frac{|f(x)|}{2e\lambda}\right) \, dx$$

$$\leq \int_{\mathbb{R}^n} \frac{|f(x)|}{2e\lambda} \log^+ \left(\frac{|f(x)|}{\lambda}\right) \log^+ \log^+ \left(\frac{|f(x)|}{\lambda}\right) \, dx \leq 1,$$

which gives the desired estimate. $\square$

3 Existence and Uniqueness

The main goal of this section is proving Theorem 1. To this end, we will first prove existence and uniqueness when the initial data is in $L^\infty \cap L^p$ for some $p \in [1, \infty)$ (see Proposition 14 below). Later on, we will use this fact to show in Proposition 15 that any weak solution $u \in L^\infty(0, T; L^\infty)$ to (1) with vanishing initial data $u_0 \equiv 0$ is indeed uniformly square summable, i.e. $u \in L^\infty(0, T; L^2)$. These two steps will make the proof of Theorem 1 almost automatic.

We start with an existence result for initial data in $L^p \cap L^\infty$, $p \in [1, \infty)$, which holds under much milder assumptions on $\operatorname{div} b$.

Proposition 12. *Let $p \in [1, \infty)$ and $b \in L^1(0, T; W_{\mathrm{loc}}^{1,1})$ be such that*

$$\operatorname{div} b \in L^1(0, T; L^1) + L^1(0, T; L^\infty).$$

Assume also that $c \in L^1(0, T; L^\infty)$. Then for every $u_0 \in L^\infty \cap L^p$, there is a weak solution $u \in L^\infty(0, T; L^p \cap L^\infty)$ to the transport problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + c\, u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases}$$

Proof. We will follow the usual method of regularization. Let $0 \leq \rho \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \rho(x)\, dx = 1$. For each $\epsilon > 0$, set $\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho(x/\epsilon)$, and $b_\epsilon = b * \rho_\epsilon$, $c_\epsilon = c * \rho_\epsilon$, $u_{0,\epsilon} = u_0 * \rho_\epsilon$. Since

$$\operatorname{div} b \in L^1(0, T; L^1) + L^1(0, T; L^\infty),$$

we have for each $\epsilon > 0$,

$$\operatorname{div} b_\epsilon = (\operatorname{div} b) * \rho_\epsilon \in L^1(0, T; L^\infty).$$

Therefore, b_ϵ and c_ϵ satisfy the requirements from DiPerna–Lions [DPL89, Proposition 2.1, Theorem 2.2]. Since $u_{0,\epsilon} \in L^p \cap L^\infty$, it follows that there exists a unique solution $u_\epsilon \in L^\infty(0, T; L^p \cap L^\infty)$ to the transport equation

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + b_\epsilon \cdot \nabla u_\epsilon + c_\epsilon u_\epsilon = 0 & (0, T) \times \mathbb{R}^n, \\ u_\epsilon(0, \cdot) = u_{0,\epsilon} & \mathbb{R}^n. \end{cases}$$

Moreover, we can bound u_ϵ in $L^\infty(0, T; L^\infty)$ as

$$\begin{aligned} \|u_\epsilon\|_{L^\infty(0,T;L^\infty)} &\leq \|u_{0,\epsilon}\|_{L^\infty} \exp\left\{ \int_0^T \|c_\epsilon(t)\|_{L^\infty}\, dt \right\} \\ &\leq \|u_0\|_{L^\infty} \exp\left\{ \int_0^T \|c(t)\|_{L^\infty}\, dt \right\} =: M. \end{aligned} \tag{9}$$

Therefore, by extracting a subsequence, $\{\epsilon_k\}_k$, we know that u_{ϵ_k} converges to some u in the weak-$*$ topology of $L^\infty(0, T; L^\infty)$.

Now, the smoothness allows us to say that

$$\frac{\partial |u_\epsilon|^p}{\partial t} + b_\epsilon \cdot \nabla |u_\epsilon|^p + p\, c_\epsilon\, |u_\epsilon|^p = 0.$$

But we also know that $\int_{\mathbb{R}^n} \operatorname{div}(b_\epsilon |u_\epsilon|^p)\, dx = 0$, since $b_\epsilon |u_\epsilon|^p \in W^{1,1}$. Thus, integrating on $\mathbb{R}^n$ we get that

$$\frac{\partial}{\partial t} \int_{\mathbb{R}^n} |u_\epsilon|^p\, dx - \int_{\mathbb{R}^n} |u_\epsilon|^p \operatorname{div} b_\epsilon\, dx + \int_{\mathbb{R}^n} p c_\epsilon |u_\epsilon|^p\, dx = 0. \tag{10}$$

Our assumptions on $\operatorname{div} b$ allow us to decompose $\operatorname{div} b = B_1 + B_2$, where $B_1 \in L^1(0, T; L^1)$ and $B_2 \in L^1(0, T; L^\infty)$. By letting $B_{1,\epsilon} = B_1 * \rho_\epsilon$ and $B_{2,\epsilon} = B_2 * \rho_\epsilon$, we get from (10) that

$$\|u_\epsilon(T)\|_{L^p}^p \leq \|u_{0,\epsilon}\|_{L^p}^p + M^p \int_0^T \|B_{1,\epsilon}\|_{L^1}\, dt + \int_0^T \|B_{2,\epsilon} - p\, c_\epsilon\|_{L^\infty} \|u_\epsilon\|_{L^p}^p\, dt.$$

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We then see that

$$\|u_\epsilon(T)\|_{L^p}^p \le \left\{\|u_0\|_{L^p}^p + M^p \int_0^T \|B_1\|_{L^1}\, dt\right\} \exp\left\{\int_0^T \|B_2 - p\,c\|_{L^\infty}\, dt\right\}, \tag{11}$$

i.e., $\{u_\epsilon\}$ is uniformly bounded in $L^\infty(0,T;L^p)$. Hence, there exists a subsequence of $\{\epsilon_k\}_k$, $\{\epsilon_{k_j}\}_{k_j}$, such that $u_{\epsilon_{k_j}}$ weakly converges to some $\tilde u \in L^\infty(0,T;L^p)$, if $p > 1$. For $p = 1$, notice that, since $\{u_\epsilon\} \in L^\infty(0,T;L^\infty)$ with a uniform upper bound independent of ϵ, $\{u_\epsilon\}$ is weakly relative compact in $L^\infty(0,T;L^1_{\mathrm{loc}})$. Therefore, there exists a subsequence of $\{\epsilon_k\}_k$, $\{\epsilon_{k_j}\}_{k_j}$, such that $u_{\epsilon_{k_j}}$ weakly converges to some $\tilde u \in L^\infty(0,T;L^1)$. By using a duality argument, it is easy to see that $u = \tilde u$ a.e., which is the required solution. Moreover, from (9) and (11), we see that

$$\|u\|_{L^\infty(0,T;L^\infty)} \le \|u_0\|_{L^\infty} \exp\left\{\int_0^T \|c(t)\|_{L^\infty}\, dt\right\}, \tag{12}$$

and

$$\|u\|_{L^\infty(0,T;L^p)}^p \le \left\{\|u_0\|_{L^p}^p + M^p \int_0^T \|B_1\|_{L^1}\, dt\right\} \exp\left\{\int_0^T \||B_2 - p\,c|\|_{L^\infty}\, dt\right\}, \tag{13}$$

which completes the proof. $\qquad\square$

The following commutator estimate is a special case of DiPerna–Lions [DPL89, Theorem 2.1].

Lemma 13 (DiPerna–Lions). *Let $u \in L^\infty(0,T;L^\infty)$ be a solution to the transport equation*

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + c\,u = 0 & (0,T) \times \mathbb{R}^n, \\ u(0,\cdot) = u_0 & \mathbb{R}^n. \end{cases}$$

Here $b \in L^1(0,T;W^{1,1}_{\mathrm{loc}})$ and $c \in L^1(0,T;L^1_{\mathrm{loc}})$. Let $0 \le \rho \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \rho\, dx = 1$, and $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$. Then, $u_\epsilon = u * \rho_\epsilon$ satisfies

$$\dfrac{\partial u_\epsilon}{\partial t} + b \cdot \nabla u_\epsilon + c\,u_\epsilon = r_\epsilon,$$

where $r_\epsilon \to 0$ in $L^1(0,T;L^1_{\mathrm{loc}})$ as $\epsilon \to 0$.

Lemma 13 above allows us to prove uniqueness when the initial value is in $L^p \cap L^\infty$.

Proposition 14. *Let $p \in [1,\infty)$ and assume that $b \in L^1(0,T;W^{1,1}_{\mathrm{loc}})$ satisfies (3) and (7). Assume also that $c \in L^1(0,T;L^\infty)$. Then, for every $u_0 \in L^\infty \cap L^p$ the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + cu = 0 & (0,T) \times \mathbb{R}^n, \\ u(0,\cdot) = u_0 & \mathbb{R}^n, \end{cases}$$

admits a unique weak solution $u \in L^\infty(0,T;L^p \cap L^\infty)$.

Proof. Since $\mathrm{Exp}(\frac{L}{\log L}) \subset L^p$ for every $p \in [1,\infty)$, we know by Proposition 12 that there exists a weak solution $u \in L^\infty(0,T;L^p \cap L^\infty)$. In order to get uniqueness, it suffices to assume that $u_0 \equiv 0$,

because the equation is linear. We start by regularizing the Cauchy problem as in Lemma 13 so we get a regularized problem

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t} + b \cdot \nabla u_\epsilon + c u_\epsilon = r_\epsilon & (0,T) \times \mathbb{R}^n, \\ u_\epsilon(0, \cdot) = 0 & \mathbb{R}^n. \end{cases}$$

Also, $r_\epsilon \to 0$ as $\epsilon \to 0$ in the $L^1(0,T; L^1_{\mathrm{loc}})$ sense, by Lemma 13. Now, for each $R > 0$, let $\psi_R \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ be a cutoff function, so that

$$0 \leq \psi_R \leq 1, \ \psi_R(x) = 1 \text{ whenever } |x| \leq R,$$

$$\psi_R(x) = 0 \text{ whenever } |x| \geq 2R, \ \text{and } |\nabla \psi_R(x)| \leq \frac{C}{R}. \tag{14}$$

By noticing that

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u_\epsilon|^p \psi_R \, dx + \int_{\mathbb{R}^n} b \cdot \nabla |u_\epsilon|^p \psi_R \, dx + \int_{\mathbb{R}^n} pc|u_\epsilon|^p \psi_R \, dx = \int_{\mathbb{R}^n} r_\epsilon p |u_\epsilon|^{p-1} \psi_R \, dx, \tag{15}$$

and integrating over time, we see that for every $t \leq T$

$$\int_{\mathbb{R}^n} |u_\epsilon(t, \cdot)|^p \, \psi_R \, dx$$
$$= \int_0^t \int_{\mathbb{R}^n} (\operatorname{div} b - pc) \, |u_\epsilon|^p \psi_R \, dx \, ds + \int_0^t \int_{\mathbb{R}^n} b \cdot \nabla \psi_R |u_\epsilon|^p \, dx \, ds + \int_0^t \int_{\mathbb{R}^n} r_\epsilon p |u_\epsilon|^{p-1} \psi_R \, dx \, ds.$$

By Lemma 13 and dominated convergence theorem, letting $\epsilon \to 0$ yields

$$\int_{\mathbb{R}^n} |u(t, \cdot)|^p \, \psi_R \, dx = \int_0^t \int_{\mathbb{R}^n} (\operatorname{div} b - pc) \, |u|^p \, \psi_R \, dx \, ds + \int_0^t \int_{\mathbb{R}^n} b \cdot \nabla \psi_R \, |u|^p \, dx \, ds. \tag{16}$$

Using the assumption (3) on b, and the facts $|u|^p \in L^\infty(0,T; L^1 \cap L^\infty)$ and $|\nabla \psi_R| \leq C/R$, one obtains

$$\lim_{R \to \infty} \left| \int_0^t \int_{\mathbb{R}^n} b \cdot \nabla \psi_R |u|^p \, dx \, ds \right| \leq \lim_{R \to \infty} \left| \int_0^t \int_{B(0,2R) \backslash B(0,R)} \frac{|b(s,x)|}{1 + |x|} |u|^p \, dx \, ds \right| = 0,$$

which kills the second term on the right hand side of (16). For the first term, write

$$\operatorname{div} b = B_1 + B_2,$$

where $B_1 \in L^1(0,T; \operatorname{Exp}(\frac{L}{\log L}))$ and $B_2 \in L^1(0,T; L^\infty)$. Letting $R \to \infty$ in (16) yields

$$\int_{\mathbb{R}^n} |u(t,x)|^p \, dx \leq \int_0^t \int_{\mathbb{R}^n} [|\operatorname{div} b| + p|c|] |u|^p \, dx \, ds$$

$$\leq \int_0^t \int_{\mathbb{R}^n} |B_1| |u|^p \, dx \, ds + \int_0^t \||B_2| + pc\|_{L^\infty} \int_{\mathbb{R}^n} |u|^p \, dx \, ds. \tag{17}$$

Recall that $B_1 \in L^1(0,T; \operatorname{Exp}(\frac{L}{\log L})) \subset L^1(0,T; L^p)$ for each $p \in [1, \infty)$. This, together with $u \in L^\infty(0,T; L^p \cap L^\infty)$, further implies that there exists $T_1 > 0$, such that

$$\int_{\mathbb{R}^n} |u(t,x)|^p \, dx < \exp \left\{ -\exp \left\{ e + \|u\|_{L^\infty(0,T;L^\infty)} \right\} \right\} \tag{18}$$

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for each $t \in (0, T_1)$. For convenience, in what follows we denote by $\alpha(t)$, $\beta_1(t)$, $\beta_2(t)$ the quantities $\|u(t, \cdot)\|_{L^p}^p$, $\|B_1\|_{\mathrm{Exp}(\frac{L}{\log L})}$ and $\||B_2| + pc\|_{L^\infty}$, respectively. Denote $\|u\|_{L^\infty(0,T;L^\infty)}$ by M. From the first estimate of Lemma 11, we find that

$$\int_{\mathbb{R}^n} |B_1| |u|^p \, dx \leq 2\|B_1\|_{\mathrm{Exp}(\frac{L}{\log L})} \||u|^p\|_{L \log L \log \log L}. \tag{19}$$

By the second estimate of Lemma 11, the factor $\||u|^p\|_{L \log L \log \log L}$ is bounded by

$$2e\|B_1\|_{\mathrm{Exp}(\frac{L}{\log L})} \alpha(t) \Big(\log(e + M) + |\log \alpha(t)| \Big) \Big(\log \log(e^e + M) + |\log |\log(\alpha(t))|| \Big). \tag{20}$$

Notice that by (18) we have

$$\log(e + M) \leq |\log(\alpha(t))| = \log \frac{1}{\alpha(t)}$$

and

$$\log \log(e^e + M) \leq |\log (|\log \alpha(t)|)| = \log \log \frac{1}{\alpha(t)}$$

for $t \in (0, T_1)$. This fact, together with the inequalities (16), (19) and (20), gives

$$\alpha(t) \leq 16e \int_0^t \beta_1(s)\alpha(s) \log \frac{1}{\alpha(s)} \log \log \frac{1}{\alpha(s)} + \beta_2(s)\alpha(s) \, ds$$
$$\leq 16e \int_0^t \beta(s)\alpha(s) \log \frac{1}{\alpha(s)} \log \log \frac{1}{\alpha(s)} \, ds, \tag{21}$$

where we denote by $\beta(s)$ the quantity $\beta_1(s) + \beta_2(s)$. We will now use a Gronwall type argument. For each $s \in (0, T]$, let

$$\alpha^*(s) = \exp \left\{ -\exp \left\{ \exp \left\{ \log \log \log \frac{1}{\epsilon} - 16e \int_0^s \beta(s) \, ds \right\} \right\} \right\},$$

where $\epsilon > 0$ is chosen small enough so that

$$\alpha^*(T) = \exp \left\{ -\exp \left\{ \exp \left\{ \log \log \log \frac{1}{\epsilon} - 16e \int_0^T \beta(s) \, ds \right\} \right\} \right\} < \exp \left\{ -\exp\{e + M\} \right\}.$$

From the definition, we see that α^* is Lipschitz continuous and increasing on $[0, T]$. Moreover, for every $t \in [0, T]$ it holds that

$$\alpha^*(t) = \epsilon + 16e \int_0^t \beta(s)\alpha^*(s) \log \frac{1}{\alpha^*(s)} \log \log \frac{1}{\alpha^*(s)} \, ds.$$

Also, we see from (18) that $\alpha(t)$ takes values on $[0, \exp\{-e^{e+M}\})$ if $t \in (0, T_1)$, and the function $s \mapsto s|\log s||\log(|\log s|)|$ is increasing on that interval. From this, the definition of α^* and (21), we conclude that for each $t \in [0, T_1]$,

$$0 \leq \alpha(t) \leq \alpha^*(t) \leq \exp \left\{ -\exp \left\{ \exp \left\{ \log \log \log \frac{1}{\epsilon} - 16e \int_0^T \beta(s) \, ds \right\} \right\} \right\}.$$

By letting $\epsilon \to 0$, we conclude that $\alpha(t) \equiv 0$ for each $t \in (0, T_1]$, which means

$$u(t, x) \equiv 0 \quad \text{in } (0, T_1) \times \mathbb{R}^n.$$

The proof is therefore completed. $\qquad \square$

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We now give in the following proposition an apriori estimate for solutions $u \in L^{\infty}(0, T; L^{\infty})$ to the transport equation subject to a vanishing initial value. This estimate is the key of the proof of Theorem 1.

Proposition 15. *Let $T > 0$, and assume that $b \in L^1(0, T; W^{1,1}_{\text{loc}})$ satisfies (3) and*

$$\operatorname{div} b \in L^1(0, T; L^1 \cap L^2) + L^1(0, T; L^{\infty}).$$

Let $u \in L^{\infty}(0, T; L^{\infty})$ be a weak solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = 0 & \mathbb{R}^n. \end{cases}$$

Then $u \in L^{\infty}(0, T; L^2)$.

Proof. Once more, we write $\operatorname{div} b = B_1 + B_2$, where now $B_1 \in L^1(0, T; L^1 \cap L^2)$ and $B_2 \in L^1(0, T; L^{\infty})$. Let us begin with the following backward transport problem

$$\begin{cases} \dfrac{\partial v}{\partial t} + b \cdot \nabla v + B_2 \, v = 0 & (0, T_0) \times \mathbb{R}^n, \\ v(T_0, \cdot) = \chi_K \, u(T_0, \cdot) & \mathbb{R}^n, \end{cases}$$

where $T_0 \in (0, T]$, and χ_K is the characteristic function of an arbitrary compact set $K \subset \mathbb{R}^n$. By Proposition 12, we see that this problem admits a solution $v \in L^{\infty}(0, T_0; L^1 \cap L^{\infty})$, because certainly $\chi_K \, u(T_0, \cdot)$ belongs to $L^1 \cap L^{\infty}$. Now we regularize both backward and forward problems with the help of a mollifier $0 \le \rho \in \mathcal{C}^{\infty}_c(\mathbb{R}^n)$, and obtain two regularized problems,

$$\begin{cases} \dfrac{\partial u_{\epsilon}}{\partial t} + b \cdot \nabla u_{\epsilon} = r_{u, \epsilon} & (0, T) \times \mathbb{R}^n, \\ u_{\epsilon}(0, \cdot) = 0 & \mathbb{R}^n, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial v_{\epsilon}}{\partial t} + b \cdot \nabla v_{\epsilon} + B_2 \, v_{\epsilon} = r_{v, \epsilon} & (0, T_0) \times \mathbb{R}^n, \\ v_{\epsilon}(T_0, \cdot) = (\chi_K \, u(T_0, \cdot)) * \rho_{\epsilon} & \mathbb{R}^n, \end{cases}$$

where $u_{\epsilon} = u * \rho_{\epsilon}$, $v_{\epsilon} = v * \rho_{\epsilon}$, and $r_{u, \epsilon}$, $r_{v, \epsilon}$ converge to 0 in the $L^1(0, T; L^1_{\text{loc}})$ sense, see Lemma 13. Choose now $\psi_R \in \mathcal{C}^{\infty}_c(\mathbb{R}^n)$ with $\psi_R \equiv 1$ on $B(0, R)$, $\operatorname{supp} \psi_R \subset B(0, 2R)$ and $|\nabla \psi_R| \le C/R$. Then we multiply the first equation by $v_{\epsilon} \psi_R$, and integrate over time and space. We conclude that

$$
\begin{aligned}
0 &= \int_0^{T_0} \int_{\mathbb{R}^n} \left(\frac{\partial u_{\epsilon}}{\partial t} + b \cdot \nabla u_{\epsilon} - r_{u, \epsilon} \right) v_{\epsilon} \psi_R \, dx \, dt \\
&= \int_{\mathbb{R}^n} u_{\epsilon}(T_0, x)(\chi_K \, u(T_0, \cdot)) * \rho_{\epsilon}(x) \, \psi_R(x) \, dx \\
&\quad - \int_0^{T_0} \int_{\mathbb{R}^n} \left[u_{\epsilon} \left(\frac{\partial v_{\epsilon}}{\partial t} + b \cdot \nabla v_{\epsilon} + v_{\epsilon} \operatorname{div} b \right) \psi_R + u_{\epsilon} v_{\epsilon} b \cdot \nabla \psi_R + r_{u, \epsilon} v_{\epsilon} \psi_R \right] dx \, dt \\
&= \int_{\mathbb{R}^n} u_{\epsilon}(T_0, x)(u(T_0)\chi_K) * \rho_{\epsilon}(x) \psi_R(x) \, dx \\
&\quad - \int_0^{T_0} \int_{\mathbb{R}^n} \left[(u_{\epsilon} v_{\epsilon} B_1 + v_{\epsilon} r_{u, \epsilon} + u_{\epsilon} r_{v, \epsilon}) \psi_R + u_{\epsilon} v_{\epsilon} b \cdot \nabla \psi_R \right] dx \, dt.
\end{aligned}
$$

Notice that $u_\epsilon \in L^\infty(0,T; L^\infty)$, $v_\epsilon \in L^\infty(0,T; L^1 \cap L^\infty)$ for each $\epsilon > 0$, and $r_{u,\epsilon}, r_{v,\epsilon} \to 0$ as $\epsilon \to 0$ in $L^1(0,T; L^1_{\text{loc}})$ by Lemma 13. Letting $\epsilon \to 0$ in the above equality yields that

$$\int_{\mathbb{R}^n} u^2(T_0,x)\chi_K(x)\psi_R(x)\, dx \leq \int_0^{T_0} \int_{\mathbb{R}^n} |uvB_1|\psi_R + |uv||b \cdot \nabla\psi_R|\, dx\, dt.$$

Using the fact $\frac{b(t,x)}{1+|x|} \in L^1(0,T; L^1) + L^1(0,T; L^\infty)$, $uv \in L^\infty(0,T; L^1 \cap L^\infty)$, and letting $R \to \infty$, yields

$$\int_{\mathbb{R}^n} u^2(T_0,x)\chi_K(x)\, dx \leq \int_0^{T_0} \int_{\mathbb{R}^n} |uvB_1|\, dx\, dt. \tag{22}$$

Denote by M, $\widetilde{M}$ the quantities $\|u\|_{L^\infty(0,T_0;L^\infty)}$ and $\|B_1\|_{L^1(0,T;L^1)} + \|B_1\|_{L^1(0,T;L^2)}$, respectively. Since v is a solution to the transport equation, by using (13), we see that

$$\|v\|_{L^\infty(0,T;L^2)}^2 \leq \left\{ \|u(T_0,\cdot)\chi_K\|_{L^2}^2 + M^2 \int_0^{T_0} \int_{\mathbb{R}^n} |B_1|\, dx\, dt \right\} \exp\left\{ 2\int_0^{T_0} \|B_2\|_{L^\infty}\, dt \right\}.$$

By this, the fact $T_0 \in (0,T]$, and using the Hölder inequality, we get from (22) that

$$\begin{aligned}
\int_{\mathbb{R}^n} u^2(T_0,x)\chi_K(x)\, dx &\leq M \int_0^{T_0} \int_{\mathbb{R}^n} |v||B_1|\, dx\, dt \leq M \int_0^{T_0} \|v\|_{L^2}\|B_1\|_{L^2}\, dt \\
&\leq M \left(\int_0^{T_0} \|B_1\|_{L^2}\, dt \right) \exp\left\{ \int_0^{T_0} \|B_2\|_{L^\infty}\, dt \right\} \\
&\quad \times \left\{ \|u(T_0,\cdot)\chi_K\|_{L^2}^2 + M^2 \int_0^{T_0} \int_{\mathbb{R}^n} |B_1|\, dx\, dt \right\}^{1/2} \\
&\leq M\widetilde{M} \exp\left\{ \int_0^T \|B_2\|_{L^\infty}\, dt \right\} \left\{ \|u(T_0,\cdot)\chi_K\|_{L^2} + M\left(\widetilde{M}\right)^{1/2} \right\}.
\end{aligned}$$

An application of the Young inequality gives us that

$$\int_{\mathbb{R}^n} u^2(T_0,x)\chi_K(x)\, dx \leq 2M^2 \left((\widetilde{M})^{3/2} + (\widetilde{M})^2 \right) \exp\left\{ 2\int_0^T \|B_2\|_{L^\infty}\, dt \right\},$$

where the right hand side is independent of K and T_0. By using the Fatou Lemma, we can finally conclude that

$$\|u\|_{L^\infty(0,T;L^2)} \leq 2M^2 \left((\widetilde{M})^{3/2} + (\widetilde{M})^2 \right) \exp\left\{ 2\int_0^T \|B_2\|_{L^\infty}\, dt \right\},$$

i.e., $u \in L^\infty(0,T; L^2)$, which completes the proof. $\qquad\square$

We can now complete the proof of Theorem 1.

Proof of Theorem 1. By [DPL89, Proposition 2.1], we know that there exists a weak solution $u \in L^\infty(0,T; L^\infty)$. Let us prove uniqueness. Suppose that $u \in L^\infty(0,T; L^\infty)$ is a solution to the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0,T) \times \mathbb{R}^n, \\ u(0,\cdot) = 0 & \mathbb{R}^n. \end{cases}$$

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Notice that, since $\operatorname{div} b \in L^1(0,T;\operatorname{Exp}(\frac{L}{\log L})) + L^1(0,T;L^\infty)$, we have

$$\operatorname{div} b \in L^1(0,T;L^1 \cap L^2) + L^1(0,T;L^\infty).$$

By Proposition 15, we see that $u \in L^\infty(0,T;L^2)$, and so $u \in L^\infty(0,T;L^2 \cap L^\infty)$. Then since $\operatorname{div} b \in L^1(0,T;\operatorname{Exp}(\frac{L}{\log L})) + L^1(0,T;L^\infty)$, we can apply Proposition 14 and obtain that such weak solution $u \in L^\infty(0,T;L^2 \cap L^\infty)$ is unique. It is obvious that $u \equiv 0$ is such a unique weak solution. The proof is completed. $\qquad\square$

4   Stability

In this section, we prove Theorem 5, and provide some stability result for the transport equation for vector fields having exponentially integrable divergence.

Theorem 16. *Let $T > 0$. Assume that $b \in L^1(0,T;W_{\mathrm{loc}}^{1,1})$ satisfies (3) and (7). Suppose that $u_0 \in L^\infty$ and $\{u_0^k\}_k \in L^\infty$ have uniform upper bound in L^∞, and $u_0^k - u_0 \to 0$ as $k \to \infty$ in L^p. Let u, $u^k \in L^\infty(0,T;L^\infty)$ be the solutions of the transport equation*

$$\frac{\partial u}{\partial t} + b \cdot \nabla u = 0 \text{ in } (0,T) \times \mathbb{R}^n,$$

subject to the initial values u_0, u_0^k, respectively. Then

$$u^k - u \to 0, \quad \text{in } L^\infty(0,T;L^p)$$

as $k \to \infty$.

Proof. **Step 1.** Let $v_0^k = u_0^k - u_0$, and $v^k = u^k - u$ for each k. Denote by

$$M = \sup_k \|v_0^k\|_{L^\infty} = \sup_k \|u_0^k - u_0\|_{L^\infty}.$$

Notice that v^k is the unique solution in $L^\infty(0,T;L^\infty)$ of

$$\begin{cases} \dfrac{\partial v^k}{\partial t} + b \cdot \nabla v^k = 0 & (0,T) \times \mathbb{R}^n, \\ v^k(0,\cdot) = v_0^k & \mathbb{R}^n. \end{cases}$$

On the other hand, by Proposition 14, there exists a unique solution $\tilde{v}^k \in L^\infty(0,T;L^p \cap L^\infty)$, since $v_0^k \in L^p \cap L^\infty$. By the uniqueness, we see that $v^k = \tilde{v}^k \in L^\infty(0,T;L^\infty)$, and hence, $v^k \in L^\infty(0,T;L^p \cap L^\infty)$. Write $\operatorname{div} b = B_1 + B_2$, where $B_1 \in L^1(0,T;\operatorname{Exp}(\frac{L}{\log L}))$ and $B_2 \in L^1(0,T;L^\infty)$. Then, from the estimates (12) and (13) of Proposition 12, we see that

$$\|v^k\|_{L^\infty(0,T;L^\infty)} \le \|v_0^k\|_{L^\infty} \le M, \tag{23}$$

and

$$\|v^k\|_{L^\infty(0,T;L^p)}^p \le \left\{ \|v_0^k\|_{L^p}^p + M^p \int_0^T \|B_1\|_{L^1}\, dt \right\} \exp\left\{ \int_0^T \|B_2\|_{L^\infty}\, dt \right\}. \tag{24}$$

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For each $R > 0$, let $\psi_R \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be as in (14). Arguing as in (16) we see that

$$\int_{\mathbb{R}^n} (|v^k(t_1)|^p - |v^k(t_0)|^p)\psi_R \, dx = \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \operatorname{div} b \cdot |v^k|^p \psi_R \, dx \, ds + \int_{t_0}^{t_1} \int_{\mathbb{R}^n} b \cdot \nabla \psi_R |v^k|^p \, dx \, ds,$$

for any $0 \leq t_0 < t_1 \leq T$. For the second term above, we use $|v^k|^p \in L^\infty(0, T; L^1 \cap L^\infty)$ and $\frac{b(t,x)}{1+|x|} \in L^1(0, T; L^1) + L^1(0, T; L^\infty)$ to see that

$$\lim_{R \to \infty} \left| \int_{t_0}^{t_1} \int_{\mathbb{R}^n} b \cdot \nabla \psi_R |v^k|^p \, dx \, ds \right| \leq \lim_{R \to \infty} \left| \int_{t_0}^{t_1} \int_{B(0,2R) \setminus B(0,R)} \frac{|b(s,x)|}{1+|x|} |v^k|^p \, dx \, ds \right| = 0.$$

Thus, with the help of (16) we get that

$$\begin{aligned} \|v^k(t_1)\|_{L^p}^p &\leq \|v^k(t_0)\|_{L^p}^p + \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |\operatorname{div} b||v^k|^p \, dx \, ds \\ &\leq \|v^k(t_0)\|_{L^p}^p + \int_{t_0}^{t_1} \int_{\mathbb{R}^n} |B_1||v^k|^p \, dx \, ds + \int_{t_0}^{t_1} \|B_2\|_{L^\infty} \|v^k\|_{L^p}^p \, ds. \end{aligned} \tag{25}$$

Notice that $B_1 \in L^1(0, T; \operatorname{Exp}(\frac{L}{\log L})) \subset L^1(0, T; L^1)$, and $B_2 \in L^1(0, T; L^\infty)$. This, together with (24), (23) and the fact that $\|v_0^k\|_{L^p} \to 0$, further implies that there exist $i, K_1 \in \mathbb{N}$ and $0 = T_0 < T_1 < \cdots < T_i < T_{i+1} = T$ such that

$$\int_{T_j}^{T_{j+1}} \int_{\mathbb{R}^n} |B_1||v^k|^p \, dx \, ds + \int_{T_j}^{T_{j+1}} \|B_2\|_{L^\infty} \|v^k\|_{L^p}^p \, ds \leq \frac{1}{2} \exp(-e^{e+M}), \tag{26}$$

for each $0 \leq j \leq i$ and $k \geq K_1$. For convenience, in what follows we denote by $\alpha_k(t)$, $\beta_1(t)$, $\beta_2(t)$ the quantities $\|v^k(t, \cdot)\|_{L^p}^p$, $\|B_1(t, \cdot)\|_{\operatorname{Exp}(\frac{L}{\log L})}$ and $\|B_2(t, \cdot)\|_{L^\infty}$, respectively. Denote also $\beta_1(t) + \beta_2(t)$ by $\beta(t)$.

Step 2. Let us introduce a continuous function as, for each $s \in (0, T]$,

$$\alpha^*(s) = \exp\left\{ -\exp\left\{ \exp\left\{ \log \log \log \frac{1}{\epsilon} - 16e \int_0^s \beta(s) \, ds \right\} \right\} \right\},$$

where $\epsilon > 0$ is small enough so that

$$\alpha^*(T) = \exp\left\{ -\exp\left\{ \exp\left\{ \log \log \log \frac{1}{\epsilon} - 16e \int_0^T \beta(s) \, ds \right\} \right\} \right\} < \frac{1}{2} \exp(-e^{e+M}).$$

From the definition, we see that α^* is Lipschitz smooth and increasing on $[0, T]$.

Step 3. Using again that $\|v_0^k\|_{L^p} \to 0$, we find that there exists $K_\epsilon \geq K_1$ such that

$$\alpha_k(0) = \|v_0^k\|_{L^p}^p < \epsilon, \qquad \text{whenever } k \geq K_\epsilon.$$

Using this fact, and equations (25) and (26), we conclude that

$$\begin{aligned} \alpha_k(t) &< \epsilon + \int_0^{T_1} \int_{\mathbb{R}^n} |B_1||v^k|^p \, dx \, ds + \int_0^{T_1} \|B_2\|_{L^\infty} \|v^k\|_{L^p}^p \, ds \\ &< \exp(-e^{e+M}), \end{aligned}$$

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for each $t \in (0, T_1]$. Therefore, if $t \in (0, T_1]$ and $k \geq K_\epsilon$ we have

$$\log(e + M) \leq |\log \alpha_k(t)| = \log \frac{1}{\alpha_k(t)}$$

and

$$\log \log(e^e + M) \leq |\log(|\log \alpha_k(t)|)| = \log \log \frac{1}{\alpha_k(t)}.$$

By using the first part of Lemma 11, we find for $t \in (0, T_1]$ and $k \geq K_\epsilon$ that

$$\int_{\mathbb{R}^n} |B_1||v^k|^p \, dx \leq 2\|B_1\|_{\mathrm{Exp}(\frac{L}{\log L})} \||v^k|^p\|_{L \log L \log \log L}$$

$$\leq 4e\|B_1\|_{\mathrm{Exp}(\frac{L}{\log L})} \alpha_k(t) \left[\log(e + M) + |\log \alpha_k(t)|\right] \left[\log \log(e^e + M) + |\log|\log(\alpha_k(t))||\right]$$

$$\leq 16e\|B_1\|_{\mathrm{Exp}(\frac{L}{\log L})} \alpha_k(t) \log\left(\frac{1}{\alpha_k(t)}\right) \log \log \left(\frac{1}{\alpha_k(t)}\right),$$

which together with (25) yields

$$\alpha_k(t) \leq 16e \int_0^t \beta_1(s) \alpha_k(s) \log \frac{1}{\alpha_k(s)} \log \log \frac{1}{\alpha_k(s)} + \beta_2(s) \alpha_k(s) \, ds + \|v_0^k\|_{L^p}^p$$

$$\leq 16e \int_0^t \beta(s) \alpha_k(s) \log \frac{1}{\alpha_k(s)} \log \log \frac{1}{\alpha_k(s)} \, ds + \epsilon.$$

Notice that by the definition of $\alpha^*(t)$, we find that

$$\alpha^*(t) = \epsilon + 16e \int_0^t \beta(s) \alpha^*(s) \log \frac{1}{\alpha^*(s)} \log \log \frac{1}{\alpha^*(s)} \, ds.$$

Then for each $t \in [0, T_1]$ and $k \geq K_\epsilon$, by the fact $\alpha_k(t) < e^{-e}$, and the function on $t|\log t||\log(|\log t|)|$ is increasing on $[0, e^{-e}]$, we see that

$$0 \leq \alpha_k(t) \leq \alpha^*(t).$$

This together with the fact that $\alpha^*(t)$ is increasing on $[0, T_1]$ implies

$$0 \leq \alpha_k(t) \leq \alpha^*(T_1) < \frac{1}{2} \exp\left\{-\exp\{e + M\}\right\}.$$

Step 4. We can now iterate the approach to get the desired estimates. By the choice of T_i (see (26)) and Step 3, we see that for each $t \in (T_1, T_2]$ and $k \geq K_\epsilon$,

$$\alpha_k(t) \leq \alpha_k(T_1) + \int_{T_1}^{T_2} \int_{\mathbb{R}^n} |B_1||v^k|^p \, dx \, ds + \int_{T_1}^{T_2} \|B_2\|_{L^\infty} \|v^k\|_{L^p}^p \, ds$$

$$< \exp\left\{-\exp\{e + M\}\right\}.$$

Hence, for all $t \in (0, T_2]$ and $k \geq K_\epsilon$, we have

$$\alpha_k(t) \leq 16e \int_0^{T_2} \beta_1(s) \alpha_k(s) \log \frac{1}{\alpha_k(s)} \log \log \frac{1}{\alpha_k(s))} + \beta_2(s) \alpha_k(s) \, ds + \int_{\mathbb{R}^n} |v^k|^p(0, x) \, dx$$

$$\leq 16e \int_0^{T_2} \beta(s) \alpha_k(s) \log \frac{1}{\alpha_k(s)} \log \log \frac{1}{\alpha_k(s)} \, ds + \alpha_k(0),$$

and, by the definition of α^*,

$$\alpha^*(t) = \epsilon + 16e \int_0^t \beta(s) \alpha^*(s) \log \frac{1}{\alpha^*(s)} \log \log \frac{1}{\alpha^*(s)} \, ds.$$

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Therefore, the proof of Step 3 works well for $(0, T_2]$, and hence, we see that

$$0 \leq \alpha_k(t) \leq \alpha^*(t)$$

for all $t \in (0, T_2]$ and $k \geq K_\epsilon$. Repeating this argument $i - 1$ times more, we can conclude that for all $t \in (0, T]$ and $k \geq K_\epsilon$, it holds

$$\|v^k\|_{L^\infty(0,T;L^p)}^p \leq \alpha^*(T) = \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\epsilon} - 16e \int_0^T \beta(s)\,ds\right\}\right\}\right\}, \qquad (27)$$

which gives the desired estimate, and completes the proof. $\square$

We next prove Theorem 5.

Proof of Theorem 5. Let $\epsilon > 0$ be chosen small enough such that

$$\exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\epsilon} - 32e \int_0^T \beta(s)\,ds\right\}\right\}\right\} < \frac{1}{2}\exp\left\{-\exp\{e + M\}\right\}.$$

Then by (27), we know that if $\|u_0\|_{L^p} < \epsilon$ and $\|u_0\|_{L^\infty} < M$, then the solution u satisfies

$$\|u\|_{L^\infty(0,T;L^p)}^p \leq \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\epsilon} - 16e \int_0^T \beta(s)\,ds\right\}\right\}\right\},$$

and hence,

$$\|u\|_{L^\infty(0,T;L^p)}^p \leq \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\|u_0\|_{L^p}} - 16e \int_0^T \beta(s)\,ds\right\}\right\}\right\}. \qquad (28)$$

Notice that

$$\exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\|u(T)\|_{L^p}^p} - 16e \int_0^T \beta(s)\,ds\right\}\right\}\right\}$$

$$\leq \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\epsilon} - 32e \int_0^T \beta(s)\,ds\right\}\right\}\right\} < \frac{1}{2}\exp\left\{-\exp\{e + M\}\right\}.$$

Therefore, by considering the backward equation and using the estimate (27) again, we obtain

$$\|u\|_{L^\infty(0,T;L^p)}^p \leq \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\|u(T)\|_{L^p}^p} - 16e \int_0^T \beta(s)\,ds\right\}\right\}\right\},$$

which implies that

$$\|u_0\|_{L^p}^p \leq \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\|u\|_{L^\infty(0,T;L^p)}^p} - 16e \int_0^T \beta(s)\,ds\right\}\right\}\right\}.$$

Combining this and (28) we get the desired estimate and complete the proof. $\square$

Similarly, by considering vector fields with exponentially integrable divergence, we arrive at the following quantitative estimate. Since the proof is rather identical to the above theorem, we will skip the proof.

Corollary 17. *Let $T, M > 0$ and $1 \le p < \infty$. Suppose that $b \in L^1(0, T; W_{\mathrm{loc}}^{1,1})$ satisfies (3) and*

$$\mathrm{div}\, b \in L^1(0, T; L^\infty) + L^1(0, T; \mathrm{Exp}\, L).$$

Then there exists $\epsilon > 0$ with the following property: if $u_0 \in L^p \cap L^\infty$ satisfies $\|u_0\|_{L^\infty} \le M$ and $\|u_0\|_{L^p}^p < \epsilon$, then the problem (1) has exactly one weak solution u satisfying

$$\left| \log\log \frac{1}{\|u\|_{L^\infty(0,T;L^p)}^p} - \log\log \frac{1}{\|u_0\|_{L^p}} \right| \le 16e \int_0^T \beta(s)\, ds,$$

where $\mathrm{div}\, b = B_1 + B_2$ and $\beta(t) = \|B_1(t, \cdot)\|_{\mathrm{Exp}\, L} + \|B_2(t, \cdot)\|_{L^\infty}$.

5 Extension to BV vector fields

In this section, we shall prove Theorem 6. Let us begin by recalling the renormalization result of Ambrosio [Am04]. An L^1 function is said to belong to BV if its first order distributional derivatives are finite Radon measures. By a BV_{loc} function we mean any L_{loc}^1 function whose first order distributional derivatives are locally finite Radon measures. See [AFP00] fore more on BV functions.

Theorem 18 (Ambrosio). *Let $u \in L^\infty(0, T; L^\infty)$ be a solution of the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + c\, u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases}$$

Here $b \in L^1(0, T; BV_{\mathrm{loc}})$ with $\mathrm{div}\, b \in L^1(0, T; L_{\mathrm{loc}}^1)$ and $c \in L^1(0, T; L_{\mathrm{loc}}^1)$. Then, for each $\beta \in \mathcal{C}^1(\mathbb{R})$ the composition $\beta(u)$ is a weak solution to the transport problem

$$\begin{cases} \dfrac{\partial\, \beta(u)}{\partial t} - b \cdot \nabla \beta(u) + c\, u\, \beta'(u) = 0 & (0, T) \times \mathbb{R}^n, \\ \beta(u)(0, \cdot) = \beta(u_0) & \mathbb{R}^n. \end{cases}$$

Proof. See the proof of [Am04, Theorem 3.5]; see also [Cr09]. □

As explained in the introduction, the following result is a kind of multiplicative property for solutions of the transport equation.

Proposition 19. *Let $T > 0$, $b \in L^1(0, T; BV_{\mathrm{loc}})$ with $\mathrm{div}\, b \in L^1(0, T; L_{\mathrm{loc}}^1)$, and $c_1, c_2 \in L^1(0, T; L_{\mathrm{loc}}^1)$. Suppose that $u, v \in L^\infty(0, T; L^\infty)$ are solutions of the transport equation*

$$\frac{\partial u}{\partial t} + b \cdot \nabla u + c_i\, u = 0 \text{ in } (0, T) \times \mathbb{R}^n,$$

$i = 1, 2$, respectively. Then, the pointwise multiplication $u\, v$ is a solution of

$$\frac{\partial u}{\partial t} + b \cdot \nabla u + c_1\, u + c_2\, u = 0 \text{ in } (0, T) \times \mathbb{R}^n.$$

Proof. Let $0 \le \rho \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be an even function, such that $\int_{\mathbb{R}^n} \rho \, dx = 1$. For each $\epsilon > 0$, set $\rho_\epsilon(x) = \epsilon^{-n} \rho(x/\epsilon)$. We can use ρ to mollify both equations, and obtain that

$$\frac{\partial u * \rho_\epsilon}{\partial t} + b \cdot \nabla u * \rho_\epsilon + c_1 \, u * \rho_\epsilon = r_\epsilon, \text{ and}$$

$$\frac{\partial v * \rho_\epsilon}{\partial t} + b \cdot \nabla v * \rho_\epsilon + c_2 \, v * \rho_\epsilon = s_\epsilon,$$

where

$$r_\epsilon = b \cdot \nabla u * \rho_\epsilon - (b \cdot \nabla u) * \rho_\epsilon + c_1 \, u * \rho_\epsilon - (c_1 \, u) * \rho_\epsilon$$

and

$$s_\epsilon = b \cdot \nabla v * \rho_\epsilon - (b \cdot \nabla v) * \rho_\epsilon + c_2 \, v * \rho_\epsilon - (c_2 \, v)_\epsilon.$$

Therefore, we see that

$$\frac{\partial(u * \rho_\epsilon v * \rho_\epsilon)}{\partial t} + b \cdot \nabla(u * \rho_\epsilon \, v * \rho_\epsilon) + (c_1 + c_2) \, u * \rho_\epsilon v * \rho_\epsilon = v * \rho_\epsilon \, r_\epsilon + u * \rho_\epsilon \, s_\epsilon.$$

Since $u, v \in L^\infty(0, T; L^\infty)$, by using the commutator estimate from [Am04, Theorem 3.2], we see that for each compact set $K \subset \mathbb{R}^n$,

$$\limsup_{\epsilon \to 0} \int_0^T \int_K |v * \rho_\epsilon \, r_\epsilon + u * \rho_\epsilon \, s_\epsilon| \, dx \, dt < \infty.$$

Letting $\epsilon \to 0$, we obtain that $\frac{\partial (u \, v)}{\partial t} + b \cdot \nabla(u \, v) + (c_1 + c_2) \, u \, v$ is a signed measure with finite total variation in $(0, T) \times K$. Then arguing as Step 2 and Step 3 of the proof of [Am04, Tehorem 3.5], we see that $u \, v$ is a solution of

$$\frac{\partial (u \, v)}{\partial t} + b \cdot \nabla(u \, v) + (c_1 + c_2) \, u \, v = 0.$$

The proof is completed. □

With the aid of Theorem 18, we next outline the proof of Theorem 6, which is similar to that of Theorem 1.

Proof of Theorem 6. The proof of existence is rather standard, and is similar to that of Proposition 12, which will be omitted. Uniqueness follows by combining the following steps which are analogues of Propositions 12, 14 and 15.

Step 1. In this step, we show that if $p \in [1, \infty)$ and $c \in L^1(0, T; L^\infty)$, then for each $u_0 \in L^\infty \cap L^p$ there is a unique weak solution $u \in L^\infty(0, T; L^p \cap L^\infty)$ of the transport problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u + c \, u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases}$$

The proof of existence is the same as that of Proposition 12.

Step 2. In this step, we show that, if $c \in L^1(0, T; L^\infty)$, then for each $u_0 \in L^2 \cap L^\infty$, there is at most one weak solution $u \in L^\infty(0, T; L^2 \cap L^\infty)$ to the transport problem

$$\begin{cases} \frac{\partial u}{\partial t} + b \cdot \nabla u + c\, u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases}$$

For uniqueness, let us suppose the initial value $u_0 \equiv 0$. For each $R > 0$, let $\psi_R \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be a cut-off function as in (14). By using Theorem 18 and integrating over time and space, we see that

$$\int_{\mathbb{R}^n} |u(t, x)|^2\, \psi_R(x)\, dx = \int_0^t \int_{\mathbb{R}^n} (\operatorname{div} b - p\,c)\, |u|^2\, \psi_R\, dx\, ds + \int_0^t \int_{\mathbb{R}^n} b \cdot \nabla \psi_R\, |u|^2\, dx\, ds.$$

Then the rest proof is the same as that of Proposition 14.

Step 3. In this step, we show that if $u \in L^\infty(0, T; L^\infty)$ is a solution to the transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = 0 & \mathbb{R}^n, \end{cases}$$

then $u \in L^\infty(0, T; L^2)$. To this end, we write $\operatorname{div} b = B_1 + B_2$, where $B_1 \in L^1(0, T; \operatorname{Exp}(\frac{L}{\log L}))$ and $B_2 \in L^1(0, T; L^\infty)$. Now, we consider the following backward transport equation, given as

$$\begin{cases} \dfrac{\partial v}{\partial t} + b \cdot \nabla v + B_2 v = 0 & (0, T) \times \mathbb{R}^n, \\ v(T_0, \cdot) = \chi_K\, u(T_0, \cdot) & \mathbb{R}^n, \end{cases}$$

where $T_0 \in (0, T]$, and K is an arbitrary compact subset in $\mathbb{R}^n$. By Step 1, we see that the above admits a solution $v \in L^\infty(0, T_0, L^1 \cap L^\infty)$, since $\chi_K\, u(T_0, \cdot)$ belongs to $L^1 \cap L^\infty$. By Proposition 19, we know that the product $u\,v$ satisfies

$$\frac{\partial(u\,v)}{\partial t} + b \cdot \nabla(u\,v) + B_2\, u\,v = 0 \text{ in } (0, T) \times \mathbb{R}^n.$$

For each $R > 0$, let $\psi_R \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be a cut-off function as before in (14). Then we deduce that

$$\begin{aligned} 0 &= \int_0^{T_0} \int_{\mathbb{R}^n} \left(\frac{\partial(u\,v)}{\partial t} + b \cdot \nabla(u\,v) + B_2\, u\,v \right) \psi_R\, dx\, dt \\ &= \int_{\mathbb{R}^n} u(T_0, x)^2\, \chi_K(x)\, \psi_R(x)\, dx + \int_0^{T_0} \int_{\mathbb{R}^n} (u\,v\psi_R\, B_2 - u\,v\,\psi_R, \operatorname{div} b - u\,v\,b \cdot \nabla \psi_R)\, dx\, dt \\ &= \int_{\mathbb{R}^n} u(T_0, x)^2\, \chi_K(x)\, \psi_R(x)\, dx - \int_0^{T_0} \int_{\mathbb{R}^n} (u\,v\,\psi_R\, B_1 + u\,v\,b \cdot \nabla \psi_R)\, dx\, dt, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}^n} u^2(T_0, x) \chi_K(x) \psi_R(x)\, dx \leq \int_0^{T_0} \int_{\mathbb{R}^n} |uvB_1| \psi_R + |uv| |b \cdot \nabla \psi_R|\, dx\, dt.$$

Once more, the rest of the proof is the same of Proposition 15.

Step 4. In this step, we finish the proof of the theorem. If $u \in L^\infty(0, T; L^\infty)$ is a solution of (1) with initial value $u_0 = 0$, then from Step 3 we know that $u \in L^\infty(0, T; L^2 \cap L^\infty)$. Using Step 2, we see that such a solution u must be zero, which completes the proof. □

6 Counterexamples

In this section, we give the proof of Theorem 7, i.e., we wish to show that the condition

$$\operatorname{div} b \in L^1\left(0, T; \operatorname{Exp}\left(\frac{L}{\log^\gamma L}\right)\right) \qquad \text{for some } \gamma > 1$$

is not enough to guarantee the uniqueness. We only give the example in $\mathbb{R}^2$, which easily can be generalized to higher dimensions.

Let us begin with recalling an example from DiPerna–Lions [DPL89]. Let K be a Cantor set in $[0, 1]$, let $g \in \mathcal{C}^\infty(\mathbb{R})$ be such that $0 \le g < 1$ on $\mathbb{R}$, and $g(x) = 0$ if and only if $x \in K$. For all $x \in \mathbb{R}$, we set

$$f(x) := \int_0^x g(t)\, dt.$$

Since $0 < g(x) < 1$ at points $x \in \mathbb{R} \setminus K$, we see that f is a $\mathcal{C}^\infty$ homeomorphism from $\mathbb{R}$ into itself.

Denote by $\mathcal{M}$ the set of atom-free, nonnegative, finite measures on K. For any measure $m \in \mathcal{M}$, the equation

$$f_m(x + m([K \cap [0, x]])) = f(x), \quad x \in \mathbb{R}$$

defines a function $f_m : \mathbb{R} \to \mathbb{R}$. One now sets

$$b(x) = (1, f'(f^{-1}(x_2))), \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2.$$

It follows from [DPL89] that, for every fixed $m \in \mathcal{M}$, the function

$$X_m(t, x) = (x_1 + t, f_m(t + f_m^{-1}(x_2))), \quad \forall\, t \in \mathbb{R}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

satisfies

$$\frac{\partial}{\partial t} X_m(t, x) = b(X_m(t, x)).$$

We proceed now to prove Theorem 7.

Proof of Theorem 7. We divide the proof into the following three steps. Based on the example of DiPerna–Lions [DPL89], the main point left is to construct an explicit smooth function g.

Step 1: A minor modification of DiPerna–Lions' vector field [DPL89]. We start with K, f and f_m as introduced above. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be such that $0 \le \phi \le 1$ and $\operatorname{supp}\phi \subset [-1, 2]$, which equals one on $[0, 1]$. We choose the vector field $\tilde{b}$ as

$$\tilde{b}(x) := (0, \phi(x_1) f'(f^{-1}(x_2))), \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

As $g \in \mathcal{C}^\infty(\mathbb{R})$ and $0 \le g < 1$, it follows readily that $\tilde{b} \in L^1(0, T; W^{1,1}_{\text{loc}}(\mathbb{R}^2))$ and

$$|\tilde{b}(x)| \in L^\infty(0, T; L^\infty(\mathbb{R}^2)).$$

For each $m \in \mathcal{M}$, let

$$\widetilde{X}_m(t,x) = (x_1, f_m(t\phi_1(x) + f_m^{-1}(x_2))), \quad t \in \mathbb{R}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Then we see that

$$\frac{\partial}{\partial t}\widetilde{X}_m(t,x) = (0, \phi(x_1)f_m'(t\phi_1(x) + f_m^{-1}(x_2))) = (0, \phi((\widetilde{X}_m)_1)f_m'(f_m^{-1}((\widetilde{X}_m)_2))).$$

From [DPL89], we know that, for each $t \in \mathbb{R}$,

$$f_m'(f_m^{-1}(t)) = f'(f^{-1}(t)),$$

which, together with the above equality, yields that

$$\frac{\partial}{\partial t}\widetilde{X}_m(t,x) = (0, \phi((\widetilde{X}_m)_1)f'(f^{-1}((\widetilde{X}_m)_2))) = \tilde{b}(\widetilde{X}_m).$$

Hence, the vector field $\tilde{b}$ generates infinitely many flows. It is easy to show that, for each $u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ and $m \in \mathcal{M}$, the function

$$u_m(t,x) = u_0(\widetilde{X}_m(t,x))$$

is a solution to the transport problem with initial data u_0.

Step 2: Construction of the function g. In order to check the integrability of $\operatorname{div}\tilde{b}$, we need to describe g explicitly. To do this, we now fix K to be a one third Cantor set on $[0,1]$. By $\{C_{k_j}\}_{j=1}^{2^{k-1}}$ we denote the collection of open sets removed in the k-th generation, and $\{y_{k_j}\}_{j=1}^{2^{k-1}}$ be their centers. For each C_{k_j} we associate it with a smooth function as

$$\psi_{k_j}(x) := \begin{cases} \exp\left\{-\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\right\}\right\} & x \in C_{k_j}; \\ 0 & x \in \mathbb{R} \setminus C_{k_j}. \end{cases}$$

We next choose the function for $(-\infty, 0)$ and $(1, \infty)$ as

$$g_1(x) := \begin{cases} \exp\left\{-\exp\left\{\exp\left\{\frac{1}{x^2}\right\}\right\}\right\} & x \in (-\infty, 0); \\ \exp\left\{-\exp\left\{\exp\left\{\frac{1}{(x-1)^2}\right\}\right\}\right\} & x \in (1, \infty); \\ 0 & x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

It is obvious that $\psi_{k_j}, g_1 \in \mathcal{C}^\infty(\mathbb{R})$. Now we set

$$g(x) = \sum_{k \geq 1} \sum_{j=1}^{2^{k-1}} \psi_{k_j}(x) + g_1(x).$$

It is readily seen that g is smooth on $\mathbb{R}$, $g(x) = 0$ for each $x \in K$ and $0 < g < 1$ for each $x \in \mathbb{R} \setminus K$. Therefore, g satisfies the requirements from the example of DiPerna–Lions [DPL89] as recalled above.

Step 3: $\operatorname{div}\tilde{b} \in L^1(0, T; \operatorname{Exp}(\frac{L}{\log^\gamma L}))$ whenever $\gamma > 1$. Notice that, for $1 < \gamma_1 < \gamma_2 < \infty$, it holds that

$$\exp\left(\frac{t}{(\log^+ t)^{\gamma_2}}\right) \leq \exp\left(\frac{t}{(\log^+ t)^{\gamma_1}}\right)$$

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for each $t \geq 0$. Therefore, we only need to show that $\operatorname{div} \tilde{b} \in L^1(0, T; \operatorname{Exp}(\frac{L}{\log^\gamma L}))$ for each $\gamma > 1$ close to one. Let us fix $\gamma \in (1, 2)$.

Notice that the function $t \mapsto \frac{t}{(\log^+ t)^\gamma}$ is not monotonic. Indeed, it is increasing on $[0, e]$ and $[e^\gamma, \infty)$ and decreasing on $[e, e^\gamma]$. However, it is not hard to see that if $0 < t < s < \infty$, then

$$\frac{t}{(\log^+ t)^\gamma} \leq \frac{e\gamma^\gamma}{e^\gamma} \frac{s}{(\log^+ s)^\gamma}.$$

A direct calculation shows that

$$\operatorname{div} \tilde{b}(x) = \phi(x_1) \frac{g'(f^{-1}(x_2))}{g(f^{-1}(x_2))}, \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Recall that $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, $\operatorname{supp}\phi \subset [-1, 2]$, and $\phi = 1$ on $[0, 1]$. Therefore,

$$\operatorname{div} \tilde{b}(x) \equiv 0, \quad \forall x \in (\mathbb{R} \setminus (-1, 2)) \times \mathbb{R}, \tag{29}$$

and

$$\int_{\mathbb{R}^2} \left[\exp\left\{ \frac{C \left| \operatorname{div} \tilde{b}(x) \right|}{\left[\log^+ \left(\left| C \operatorname{div} \tilde{b}(x) \right| \right) \right]^\gamma} \right\} - 1 \right] dx$$

$$= \int_{\mathbb{R}^2} \left[\exp\left\{ \frac{C \left| \phi(x_1) \frac{g'(f^{-1}(x_2))}{g(f^{-1}(x_2))} \right|}{\left[\log^+ \left(\left| C\phi(x_1) \frac{g'(f^{-1}(x_2))}{g(f^{-1}(x_2))} \right| \right) \right]^\gamma} \right\} - 1 \right] dx_1\, dx_2$$

$$= \int_{[-1,2] \times \mathbb{R}} \left[\exp\left\{ \frac{C \left| \phi(x_1) \frac{g'(t)}{g(t)} \right|}{\left[\log^+ \left(\left| C\phi(x_1) \frac{g'(t)}{g(t)} \right| \right) \right]^\gamma} \right\} - 1 \right] g(t)\, dx_1\, dt$$

$$\leq \int_{\mathbb{R}} 3 \left[\exp\left\{ \frac{Ce^{1-\gamma}\gamma^\gamma \left| \frac{g'(t)}{g(t)} \right|}{\left[\log^+ \left(\left| C\frac{g'(t)}{g(t)} \right| \right) \right]^\gamma} \right\} - 1 \right] g(t)\, dt. \tag{30}$$

By the above inequality, in order to show $\operatorname{div} \tilde{b} \in L^1(0, T; \operatorname{Exp}(\frac{L}{\log^\gamma L}))$, it is sufficient to show that

$$\int_{\mathbb{R}} \left[\exp\left\{ \frac{Ce^{1-\gamma}\gamma^\gamma \left| \frac{g'(t)}{g(t)} \right|}{\left[\log^+ \left(\left| C\frac{g'(t)}{g(t)} \right| \right) \right]^\gamma} \right\} - 1 \right] g(t)\, dt < \infty, \tag{31}$$

for some $C > 0$.

Claim 1: For $A = \frac{(\gamma - 1)^2}{2^3}$, one has

$$\int_{\mathbb{R} \setminus [-1,2]} \left[\exp\left\{ \frac{Ae^{1-\gamma}\gamma^\gamma \left| \frac{g'(t)}{g(t)} \right|}{\left[\log^+ \left(\left| A\frac{g'(t)}{g(t)} \right| \right) \right]^\gamma} \right\} - 1 \right] g(t)\, dt < \infty. \tag{32}$$

By symmetry of the function g on $(-\infty, -1) \cup (2, \infty)$ and the fact $0 \leq g < 1$, we only need to show that

$$\int_{(-\infty, -1)} \left[\exp\left\{ \frac{Ae^{1-\gamma}\gamma^\gamma \left| \frac{g'(t)}{g(t)} \right|}{\left[\log^+ \left(\left| Ae^{1-\gamma}\gamma^\gamma \frac{g'(t)}{g(t)} \right| \right) \right]^\gamma} \right\} - 1 \right] dt < \infty. \tag{33}$$

By noticing that

$$\frac{g'(t)}{g(t)} = \exp\left\{\exp\left\{\frac{1}{t^2}\right\}\right\}\exp\left\{\frac{1}{t^2}\right\}\frac{2}{t^3}, \quad \text{if } t \in (-\infty, -1),$$

and

$$\log^+\left(\left|A\frac{g'(t)}{g(t)}\right|\right) \geq 1,$$

we conclude by using the Taylor expansion that

$$\int_{(-\infty,-1)}\left[\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\left|\frac{g'(t)}{g(t)}\right|}{\left[\log^+\left(\left|A\frac{g'(t)}{g(t)}\right|\right)\right]^\gamma}\right\} - 1\right]dt \leq \int_{(-\infty,-1)}\left[\exp\left\{\frac{\widetilde{A}}{t^3}\right\} - 1\right]dt$$

$$\leq \int_{(-\infty,-1)}\sum_{l=1}^{\infty}\frac{(\widetilde{A})^l}{l!}\frac{1}{t^{3l}}\,dt = \sum_{l=1}^{\infty}\frac{(\widetilde{A})^l}{l!}\frac{1}{3l-1} < \infty,$$

where $\widetilde{A} = 2Ae^{1-\gamma}\gamma^\gamma\exp\{e^{1+e}\}$. This implies (33) and hence, (32) holds.

Claim 2. If A is as above, then

$$\int_{[-1,2]}\left[\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\left|\frac{g'(t)}{g(t)}\right|}{\left[\log^+\left(\left|A\frac{g'(t)}{g(t)}\right|\right)\right]^\gamma}\right\} - 1\right]g(t)\,dt < \infty. \tag{34}$$

Since $0 \leq g < 1$, the above inequality will follow from

$$\int_{[-1,2]}\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\left|\frac{g'(t)}{g(t)}\right|}{\left[\log^+\left(\left|A\frac{g'(t)}{g(t)}\right|\right)\right]^\gamma}\right\}g(t)\,dt < \infty. \tag{35}$$

Notice that, for each $x \in C_{k_j}$,

$$\frac{g'(x)}{g(x)} = -\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2}\right\}\right\}\exp\left\{\frac{1}{\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2}\right\}\frac{2(x-y_{k_j})}{[\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2]^2},$$

and hence,

$$\frac{|g'(x)|}{g(x)} \leq \exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2}\right\}\right\}\exp\left\{\frac{1}{\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2}\right\}\frac{\frac{1}{3^k}}{[\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2]^2}$$

$$\leq \exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2}\right\}\right\}\exp\left\{\frac{\frac{1+\gamma}{2}}{\frac{1}{(2\cdot3^k)^2} - (x-y_{k_j})^2}\right\}\frac{2}{3^k}\frac{2^2}{(\gamma-1)^2}.$$

Notice that the function $\frac{t}{(\log^+ t)^\gamma}$ is increasing on $(0, e) \cup (e^\gamma, \infty)$ and decreasing on (e, e^γ). If $A\frac{|g'(x)|}{g(x)} < e^\gamma$, then

$$\frac{A\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma} \leq e^\gamma; \tag{36}$$

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while for $A\frac{|g'(x)|}{g(x)} \geq e^\gamma$, by the choice of A, we have

$$\frac{A\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma} \leq \frac{A\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\right\}\exp\left\{\frac{\frac{1+\gamma}{2}}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\frac{2}{3^k}\frac{2^2}{(\gamma-1)^2}}{\left[\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}+\frac{\frac{1+\gamma}{2}}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}+\log(A\frac{2}{3^k}\frac{2^2}{(\gamma-1)^2})\right]^\gamma}$$

$$\leq \frac{1}{3^k}\frac{\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\right\}\exp\left\{\frac{\frac{1+\gamma}{2}}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}}{\left[\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}+(2\cdot 3^k)^2+\log(\frac{1}{3^k})\right]^\gamma}$$

$$\leq \frac{1}{3^k}\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\right\}\exp\left\{\frac{\frac{1-\gamma}{2}}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}$$

$$\leq \frac{1}{3^k}\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\right\}\exp\left\{2\cdot 3^{2k}(1-\gamma)\right\}$$

$$\leq \frac{1}{3}\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\right\}. \tag{37}$$

Combining (36) and (37), we deduce that for each $x \in C_{k_j}$,

$$\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma}\right\}g(x) \leq \exp\left\{e\gamma^\gamma - \left[1-\frac{e^{1-\gamma}\gamma^\gamma}{3}\right]\exp\left\{\exp\left\{\frac{1}{\frac{1}{(2\cdot 3^k)^2}-(x-y_{k_j})^2}\right\}\right\}\right\}$$

$$\leq \exp\left\{e\gamma^\gamma\right\}, \tag{38}$$

since by assumption $1 < e^{1-\gamma}\gamma^\gamma < e$. Indeed, from (37) and (38), we can further see that the function

$$\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma}\right\}g(x)$$

is smooth on $\overline{C_{k_j}}$, and equals 0 on the boundary of C_{k_j}.

On the other hand, notice that for each $x \in [-1, 0)$, it holds that

$$\frac{|g'(x)|}{g(x)} = \exp\left\{\exp\left\{\frac{1}{x^2}\right\}\right\}\exp\left\{\frac{1}{x^2}\right\}\frac{2}{|x|^3} \leq \exp\left\{\exp\left\{\frac{1}{x^2}\right\}\right\}\exp\left\{\frac{1+\gamma}{2x^2}\right\}\frac{2\cdot 2^2}{(\gamma-1)^2}.$$

If $A\frac{|g'(x)|}{g(x)} < e^\gamma$, then

$$\frac{A\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma} \leq e^\gamma,$$

while for $A\frac{|g'(x)|}{g(x)} \geq e^\gamma$,

$$\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma}\right\}g(x) \leq \exp\left\{e^{1-\gamma}\gamma^\gamma\frac{\exp\left\{\exp\left\{\frac{1}{x^2}\right\}\right\}\exp\left\{\frac{1+\gamma}{2x^2}\right\}}{\exp\left\{\frac{\gamma}{x^2}\right\}}\right\}g(x)$$

$$\leq \exp\left\{\exp\left\{\exp\left\{\frac{1}{x^2}\right\}\right\}\left[e^{1-\gamma}\gamma^\gamma\exp\left\{\frac{1-\gamma}{2x^2}\right\}-1\right]\right\}$$

$$\leq \begin{cases} 1, & x \in \left(-\frac{\sqrt{\gamma-1}}{2}, 0\right); \\ \exp\left\{\exp\left\{\exp\left\{\frac{4}{\gamma-1}\right\}\right\}\exp\left\{1+\frac{1-\gamma}{2}\right\}\right\}, & x \in \left[-1, -\frac{\sqrt{\gamma-1}}{2}\right) \end{cases}$$

$$\leq \exp\left\{\exp\left\{1+\exp\left\{\frac{4}{\gamma-1}\right\}\right\}\right\},$$

since $1 < \gamma < 2$. The above two inequalities imply that

$$\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma}\right\}g(x) \leq \exp\left\{\exp\left\{1+\exp\left\{\frac{4}{\gamma-1}\right\}\right\}\right\},$$

and, similarly, for each $x \in (1, 2]$,

$$\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|A\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma}\right\}g(x) \leq \exp\left\{\exp\left\{1+\exp\left\{\frac{4}{\gamma-1}\right\}\right\}\right\}.$$

Therefore, from these two estimates together with (38), we conclude that

$$\int_{[-1,2]}\exp\left\{\frac{Ae^{1-\gamma}\gamma^\gamma\frac{|g'(x)|}{g(x)}}{\left[\log^+\left(\left|C\frac{g'(x)}{g(x)}\right|\right)\right]^\gamma}\right\}g(x)\,dx \leq 2C(\gamma) + \sum_k\sum_{j=1}^{2^{k-1}}\frac{\exp\left\{e\gamma^\gamma\right\}}{3^k} < \infty,$$

where $C(\gamma) = \exp\left\{\exp\left\{1+\exp\left\{\frac{4}{\gamma-1}\right\}\right\}\right\}$. This, together with (35), yields (34). Combining the inequalities (32) and (34) yields

$$\int_{\mathbb{R}^2}\left[\exp\left\{\frac{A\left|\operatorname{div}\tilde{b}(x)\right|}{\left[\log^+\left(\left|A\operatorname{div}\tilde{b}(x)\right|\right)\right]^\gamma}\right\}-1\right]dx < \infty,$$

via (30) and (31), where $A = \frac{(\gamma-1)^2}{2^3}$. Therefore, the proof of the theorem is completed. $\square$

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