Cellular approximations of fusion systems

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Abstract

In this paper we study the cellularization of classifying spaces of saturated fusion systems over finite p-groups with respect to classifying spaces of finite p-groups. We give explicit criteria to decide when a classifying space is cellular and we explicitly compute the cellularization for a family of exotic examples.

1 Introduction

The transfer plays an important role both in stable homotopy theory of classifying spaces and group cohomology. Given a finite group G and a prime p, if S is a Sylow p-subgroup, the properties of the transfer imply that the mod p cohomology of G injects into the mod p cohomology of S. In stable homotopy theory, the spectrum of $BG^\wedge_p$ is a retract of the spectrum of BS and the splitting is constructed by an idempotent in stable selfmaps of the spectrum of BS.

In 1990s, E. Dror-Farjoun and W. Chachólski generalized the concept of CW-complex, spaces build from spheres by means pointed homotopy colimits. Let A be a pointed space and let $\mathcal{C}(A)$ denote the smallest collection of pointed spaces that contains A and it is closed by weak equivalences and pointed homotopy colimits. A pointed space X is *A-cellular* if $X \in \mathcal{C}(A)$. These concepts are also defined in stable homotopy category.

In the stable context, the fact that $BG^\wedge_p$ is a stable retract of BS implies that that $\Sigma^\infty_+ BG^\wedge_p$ belongs to $\mathcal{C}(\Sigma^\infty_+ BS)$. In unstable homotopy theory of classifying spaces, $BG^\wedge_p$ is not a retract of BS, but we can ask ourselves whether $BG^\wedge_p$ is in the cellular class $\mathcal{C}(BS)$, or more generally, given a finite p-group P, $BG^\wedge_p \in \mathcal{C}(BP)$?

The homotopy type of $BG^\wedge_p$ is determined by the p-local structure of G, the fusion system associated to G. Given a finite p-group S, p a prime, a *fusion system* over S is a subcategory of the category of groups whose objects are the subgroup of S and morphisms are given a set of injective homomorphisms, containing those which are induced by conjugation by elements of S. A fusion system $\mathcal{F}$ is *saturated* if it verifies certain axioms such as would be holded if S were a Sylow p-subgroup of a finite group. These ideas were developed by L. Puig in an unpublished notes. Afterwards, D. Benson suggested the idea of associating a

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"classifying space" to each saturated fusion system (see [Ben98]). The notion of classifying space was formulated by C. Broto, R. Levi and B. Oliver in [BLO03b], where the notion of "centric linking system" (or "p-local finite group") associated to saturated fusion systems appears. At that time, it was not known if every saturated fusion has an associated linking system.

K. Ragnarsson [Rag06] constructed a classifying space spectrum $\mathbb{B}\mathcal{F}$ associated to $\mathcal{F}$ by splitting the spectrum of BS via an idempotent stable selfmap. Analogously to the situation for finite groups, we have then $\mathbb{B}\mathcal{F} \in \mathcal{C}(BS)$.

Recently, A. Chermak [Che13] has proved the existence and uniqueness of centric linking systems, that means, each saturated fusion system $\mathcal{F}$ has a unique (up to isomorphism) centric linking system associated to $\mathcal{F}$, and so a unique (up to homotopy equivalence) classifying space $B\mathcal{F}$. See Section 2 for specific definitions, details and main results which we will use in the rest of the paper about fusion systems.

Previous works in finite groups (see [Flo07], [FS07] and [FF11]) suggest the strong relationship between the cellularity properties of $BG^\wedge_p$ with respect to classifying spaces of finite p-groups and the fusion structure of G at the prime p.

Let P be a finite p-group, we will denote by $Cl_{\mathcal{F}}(P)$ the smallest strongly $\mathcal{F}$-closed subgroup in S which contains all the images of homomorphisms $P \to S$ (see Section 3 below). The main result of this paper is the following theorem.

Theorem 5.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S and let P be a finite p-group. Then $B\mathcal{F}$ is BP-cellular if and only if $S = Cl_{\mathcal{F}}(P)$.*

Corollary 5.2. *Let $(S, \mathcal{F})$ be a saturated fusion system and P a finite p-group.*

(a) The classifying space $B\mathcal{F}$ is BS-cellular.

(b) Let $(S, \mathcal{F}')$ be a saturated fusion system with $\mathcal{F} \subset \mathcal{F}'$. If $B\mathcal{F}$ is BP-cellular then $B\mathcal{F}'$ is also BP-cellular.

(c) Let A be a pointed connected space. If $Cl_{\mathcal{F}}((\pi_1 A)_{ab}) = S$, then $B\mathcal{F}$ is A-cellular.

(d) Let $\Omega_{p^m}(S)$ be the (normal) subgroup of S generated by its elements of order p^i, which $i \leq m$. Then $B\mathcal{F}$ is $B\mathbb{Z}/p^m$-cellular if and only if $S = Cl_{\mathcal{F}}(\Omega_{p^m}(S))$. In particular, there is a non-negative integer $m_0 \geq 0$ such that $B\mathcal{F}$ is $B\mathbb{Z}/p^m$-cellular for all $m \geq m_0$.

There exists an augmented idempotent endofunctor $CW_A\colon Spaces_* \to Spaces_*$ such that for all pointed space X, the space $CW_A X$ is A-cellular and the augmention map $c_X\colon CW_A X \to X$ is an A-equivalence, that means, it is induced a weak equivalence in pointed mapping space $(c_X)_*\colon \mathrm{map}_*(A, CW_A X) \xrightarrow{\simeq} \mathrm{map}_*(A, X)$. Roughly speaking, $CW_A X$ is the best A-cellular approximation of X. We will say that $CW_A X$ is the *A-cellularization* of X and the map $c_X\colon CW_A X \to X$ is the *A-cellular approximation* of X. See [DF96] for more details about the construction and main properties of the functor CW_A.

The strategy of proof of Theorem 5.1 is the analysis of the Chachólski fibration to compute $CW_{BP}(B\mathcal{F})$ which is described in [Cha96]. Let C be the homotopy cofibre of the evaluation map $ev\colon \bigvee_{[BP,B\mathcal{F}]_*} BP \to B\mathcal{F}$, then $CW_{BP}(B\mathcal{F})$ is the homotopy fibre of the composite $r\colon B\mathcal{F} \to C \to P_{\Sigma BP}C$, where $P_{\Sigma BP}$ denotes the ΣBP-nullification functor defined by A. K. Bousfield in [Bou94]. We prove that $CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F}$ if and only if the map $r_p^\wedge$ is null-homotopic (see the proof of Theorem 5.1). Therefore, the BP-cellularity of $B\mathcal{F}$ is equivalent to the homotopy nullity of the map $r_p^\wedge$.

To approach this question, we study the kernel of $r_p^\wedge$ in the sense of D. Notbohm introduced in [Not94] for maps from classifying spaces of compact Lie groups (Section 3). Given a map $f\colon B\mathcal{F} \to Z$, where Z is a connected p-complete and $\Sigma B\mathbb{Z}/p$-null space (that is, $Z_p^\wedge \simeq Z$ and $\mathrm{map}_*(\Sigma B\mathbb{Z}/p, Z) \simeq *$), then $\ker(f) := \{x \in S \mid f|_{B\langle x\rangle} \simeq *\}$. We show that $\ker(f)$ is a strongly $\mathcal{F}$-closed subgroup of S (Proposition 3.5) with the following main property.

Theorem 3.6. *Let $(S, \mathcal{F})$ be a saturated fusion system. Let Z be a connected p-complete and $\Sigma B\mathbb{Z}/p$-null space. A map $f\colon B\mathcal{F} \to Z$ is null-homotopic if and only if* $\ker(f) = S$.

Furthermore, any strongly $\mathcal{F}$-closed subgroup $K \leq S$ is the kernel of a map from $B\mathcal{F}$ to $B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge$ for certain $m \geq 0$ (Proposition 3.7).

In Section 4 we study homotopy properties of $CW_{BP}(B\mathcal{F})$ needed for the proof of the main theorem. Section 5 contains the proof of Theorem 5.1. A key step is the computation of the kernel of $r_p^\wedge$.

Proposition 5.5. *Let $(S, \mathcal{F})$ be a saturated fusion system. Then* $\ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$.

The last two sections are devoted to give explicit examples. Concretely, in Section 6 we describe a strategy to compute the BP-cellularization of $B\mathcal{F}$ when $S \neq Cl_{\mathcal{F}}(P)$. This is the case when we have a homotopy factorization of $r_p^\wedge\colon B\mathcal{F} \to (P_{\Sigma BP}C)_p^\wedge$ by a map $\tilde{r}_p^\wedge\colon B\mathcal{F}' \to (P_{\Sigma BP}C)_p^\wedge$ with trivial kernel (verifying certain technical conditions in Proposition 6.2). This is the case when $Cl_{\mathcal{F}}(S)$ is normal in $\mathcal{F}$.

Corollary 6.5. *Let $(S, \mathcal{F})$ be a fusion system and let P be a finite p-group. If $Cl_{\mathcal{F}}(P) \lhd \mathcal{F}$, then $CW_{BP}(B\mathcal{F})$ is homotopy equivalent to the homotopy fibre of $B\mathcal{F} \to B(\mathcal{F}/Cl_{\mathcal{F}}(P))$.*

This result allow us compute, for all $r \geq 1$, the $B\mathbb{Z}/p^r$-cellularation of the classifying space of $\mathbb{Z}/p^n \wr \mathbb{Z}/q$, with $p \neq q$, and of the Suzuki group $\mathrm{Sz}(2^n)$, with n an odd integer at least 3 (Example 6.6).

The last section contains an explicit description of the $B\mathbb{Z}/3^l$-cellularation of the classifying space of a family of exotic fusion systems over a finite 3-group given in [DRV07].

Corollary 7.1. *Let $\mathcal{F}$ be an exotic fusion system over* $B(3, r; 0, \gamma, 0)$ *such that $\mathcal{F}$ has at least one $\mathcal{F}$-Alperin rank two elementary abelian* 3*-subgroup given in [DRV07, Theorem 5.10]. Then*

(i) If $\gamma = 0$, then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$-cellular for all $l \geq 1$.

(ii) Assume $\gamma \neq 0$. Then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$-cellular if and only if $l \geq 2$. If $l = 1$, $Cl_{\mathcal{F}}(\mathbb{Z}/3) = \langle s, s_2\rangle$.

Moreover, when $\gamma \neq 0$ and $l = 1$, we show that $CW_{B\mathbb{Z}/3}(B\mathcal{F})$ is the homotpy fibre of a map $B\mathcal{F} \to (B\Sigma_3)_3^\wedge$.

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2 Preliminaries on the homotopy theory of fusion systems

A saturated fusion system is a small subcategory of the category of groups which encodes fusion/conjugacy data between subgroups of a fixed finite p-group S, as formalized by L. Puig (see [Pui06] and also [AKO11]). Such objects have classifying spaces which satisfy many of the rigid homotopy theoretic properties of p-completed classifying spaces of finite groups, as generalized by C. Broto, R. Levi and B. Oliver (see [BLO03b]).

Definition 2.1. Let S be a finite p-group. A *saturated fusion system on S* is a subcategory $\mathcal{F}$ of the category of groups with $\mathrm{Ob}(\mathcal{F})$ the set of all subgroups of S and such that it satisfies the following properties. For all $P, Q \leq S$:

(f.1) $\mathrm{Hom}_S(P,Q) \subset \mathrm{Hom}_{\mathcal{F}}(P,Q) \subset \mathrm{Inj}(P,Q)$; and

(f.2) each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion.

For all $P \leq S$ and all $P' \leq S$ which is *$\mathcal{F}$-conjugate to P* (P and P' are isomorphic as objects in $\mathcal{F}$):

(s.1) For all $P \leq S$ which is *fully normalized in $\mathcal{F}$* (i.e. $|N_S(P)| \geq |N_S(P')|$ for all P' $\mathcal{F}$-conjugate to P), P is also *fully centralized in $\mathcal{F}$* ($|C_S(P)| \geq |C_S(P')|$ for all P' $\mathcal{F}$-conjugate to P), and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$.

(s.2) Let $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ be such that $\varphi(P)$ is fully centralized. If we set

$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\},$$

then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

The standard example is given by the fusion category of a finite group G. Given a finite group G with a fixed Sylow p-subgroup S, let $\mathcal{F}_S(G)$ be the category with $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for all $P,Q \leq S$, where $\mathrm{Hom}_G(P,Q) = \{\varphi \in \mathrm{Hom}(P,Q) \mid \varphi = c_g \text{ for some } g \in G\}$. This category $\mathcal{F}_S(G)$ satisfies the saturation axioms (see [BLO03b, Proposition 1.3]).

In order to recover the homotopy type of the Bousfiel-Kan p-completion of the classifying space $BG^\wedge_p$, Broto-Levi-Oliver [BLO03a] introduce a new category defined using the group G. A p-subgroup $P \leq G$ is p-centric if $Z(P')$ is a Sylow p-subgroup of $C_G(P')$ for all P' G-conjugate to P. Let $\mathcal{L}_S(G)$ be the category whose objects are p-centric subgroups of G with $\mathrm{Mor}_{\mathcal{L}_S(G)}(P,Q) = \{x \in G \mid xP^{-1}x \leq Q\}/O^p(C_G(P))$ for all $P,Q \leq S$. In [BLO03a] the authors proved that the classifying space of this p-local structure is $|\mathcal{L}_S(G)|^\wedge_p \simeq BG^\wedge_p$. This new structure one can associate to a finite group can also be generalized in the context of abstract fusion systems.

Definition 2.2. [BLO03b] Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the *$\mathcal{F}$-centric subgroups of* S (i.e., the subgroups $P \leq S$ such that $C_S(P) = Z(P)$), together with a functor $\pi\colon \mathcal{L} \to \mathcal{F}^c$, and "distinguished" monomorphisms $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfies the following conditions:

(l.1) π is the identity on objects and surjective on morphisms. For each pair of objects $P, Q \leq \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(l.2) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(l.3) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, $f \circ \delta_P(g) = \delta_Q(\pi f(g)) \circ f$.

Recently, in 2013, A. Chermak [Che13] proved the existence and uniqueness of centric linking systems associated to saturated fusion systems (see also [Oli13]).

Theorem 2.3. *([Che13],[Oli13]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there exists a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$. Moreover, $\mathcal{L}$ is uniquely determined by $\mathcal{F}$ up to isomorphism.*

Definition 2.4. The *classifying space $B\mathcal{F}$* of a saturated fusion system $(S, \mathcal{F})$ is the Bousfield-Kan p-completion of the nerve of the associated centric linking system $|\mathcal{L}|^\wedge_p$. We denote by $\Theta\colon BS \to B\mathcal{F}$ the map induced by the distinguished monomorphism δ_S.

Given a map $f\colon B\mathcal{F} \to X$ and $P \leq S$, we denote by $f|_{BP}$ the composite $f \circ \Theta \circ Bi$ where $i\colon P \leq S$.

We describe some results concerning the homotopy type of the classifying space $B\mathcal{F}$ and mapping spaces $\mathrm{map}(BP, B\mathcal{F})$ which will be used in the rest of the paper.

Proposition 2.5 ([BLO03b],[BCG+07],[CL09])**.** *For any satured fusion system $(S, \mathcal{F})$, $B\mathcal{F}$ is a p-complete space and $\pi_i(B\mathcal{F})$ are finite p-groups for all $i \geq 1$. The fundamental group $\pi_1(B\mathcal{F}) \cong S/O^p_{\mathcal{F}}(S)$, where $O^p_{\mathcal{F}}(S) := \langle [Q, O^p(\mathrm{Aut}_{\mathcal{F}}(Q))] \mid Q \leq S \rangle$.*

Proof. The description of the fundamental group $\pi_1(B\mathcal{F})$ is given in [BCG+07, Theorem B]. The fact that $B\mathcal{F}$ is p-complete follows from [BK72, Proposition I.5.2] since the nerve of the associated linking system $|\mathcal{L}|$ is p-good by [BLO03b, Proposition 1.12]. Finally, $\pi_i(B\mathcal{F})$ are finite p-groups for all $i \geq 1$ by [CL09, Lemma 7.6]. □

Proposition 2.6. *[BLO03b, Theorem 6.3] Let $(S, \mathcal{F})$ be a saturated fusion system. If P is a finite p-group and $\rho\colon P \to S$ a group homomorphism such that $\rho(P)$ is fully $\mathcal{F}$-centralized, there is a saturated fusion system $(C_S(\rho(P)), C_{\mathcal{F}}(\rho(P)))$ and a homotopy equivalence $BC_{\mathcal{F}}(\rho(P)) \xrightarrow{\simeq} \mathrm{map}(BP, B\mathcal{F})_\rho$. In particular, the evaluation $\mathrm{map}(BP, B\mathcal{F})_c \to B\mathcal{F}$ is a homotopy equivalence.*

Moreover, any finite covering of the classifying space of a saturated fusion system is again the classifying space of a saturated fusion system.

Theorem 2.7 ([BCG$^+$07, Theorem A])**.** *Let $(S, \mathcal{F})$ be a saturated fusion system and $\mathcal{L}$ be the associated linking system. Then there is a normal subgroup $H \lhd \pi_1|\mathcal{L}|$ which is minimal among all those whose quotient is finite and p-solvable. Any covering space of the geometric realization $|\mathcal{L}|$ whose fundamental group contains H is homotopy equivalent to $|\mathcal{L}'|$ for some linking system $\mathcal{L}'$ associated to a saturated fusion system $(S', \mathcal{F}')$, where $S' \leq S$ and $\mathcal{F}' \leq \mathcal{F}$.*

In this work it is important to understand the homotopy properties of mapping spaces between classifying paces. This question was already studied in [BLO03b]. One of the standard techniques used when studying maps between p-completed classifying spaces of finite groups is to replace them by the p-completion of a homotopy colimit of classifying spaces of subgroups.

Definition 2.8. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. The *orbit category* of $\mathcal{F}$ is the category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S, and whose morphisms are defined by

$$\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P, Q) = \mathrm{Rep}_{\mathcal{F}}(P, Q) := \mathrm{Inn}(Q)/\,\mathrm{Hom}_{\mathcal{F}}(P, Q).$$

We let $\mathcal{O}^c(\mathcal{F})$ denote the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S. If $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$, then $\tilde{\pi}$ denotes the composite functor

$$\tilde{\pi} \colon \mathcal{L} \xrightarrow{\pi} \mathcal{F}^c \twoheadrightarrow \mathcal{O}^c(\mathcal{F}).$$

The homotopy type of the nerve of a centric linking system can be described as a homotopy colimit over the orbit category.

Proposition 2.9 ([BLO03b, Proposition 2.2])**.** *Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S and an associated centric linking system $\mathcal{L}$, and let $\tilde{\pi} \to \mathcal{O}^c(\mathcal{F})$ be the projection functor. Let $\tilde{B} \colon \mathcal{O}^c(\mathcal{F}) \to \mathrm{Top}$ be the left homotopy Kan extension over $\tilde{\pi}$ of the constant functor $\mathcal{L} \xrightarrow{*} \mathrm{Top}$. Then $\tilde{B}$ is a lift of the classifying space functor $P \mapsto BP$ to the category of topological spaces, and*

$$|\mathcal{L}| \simeq \mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}).$$

Given fusion systems $\mathcal{F}$ and $\mathcal{F}'$ on S and S' respectively, a homomorphism $\psi \colon S \to S'$ is called *fusion preserving* if for every $\varphi \in \mathcal{F}(P, Q)$ there exists some $\varphi' \in \mathcal{F}'(\psi(P), \psi(Q))$ such that $\psi \circ \varphi = \varphi' \circ \psi$.

Theorem 2.10 ([CL09, Theorem 1.3])**.** *Let $(S, \mathcal{F})$ and $(S', \mathcal{F}')$ be saturated fusion systems. Suppose that $\rho \colon S \to S'$ is a fusion preserving homomorphism. Then there exists some $m \geq 0$ and a map $\tilde{f} \colon B\mathcal{F} \to B(\mathcal{F}' \wr \Sigma_{p^m})$ such that the diagram below commutes up to homotopy*

$$\begin{array}{ccccc}
BS & \xrightarrow{\Theta} & B\mathcal{F} & & \\
\Big\downarrow{\scriptstyle B\rho} & & & \searrow^{\tilde{f}} & \\
BS' & \xrightarrow[\Theta']{} & B\mathcal{F}' & \xrightarrow[\Delta_p^\wedge]{} & B(\mathcal{F}' \wr \Sigma_{p^m}).
\end{array}$$

where $\mathcal{F}' \wr \Sigma_n$ is the saturated fusion system whose classifying space is $((B\mathcal{F}')^n_{h\Sigma_n})^\wedge_p$.

3 The kernel of a map from a classifying space

Given a connected space A, we say that a space X is A-null if the evaluation map $ev\colon \mathrm{map}(A, X) \to X$ is a weak equivalence (see [DF96]). If X is connected, X is A-null iff $\mathrm{map}_*(A, X)$ is weakly contractible. There is a nullification functor $P_A\colon \mathit{Spaces} \to \mathit{Spaces}$ with a natural transformation $\eta_X\colon X \to P_A(X)$ which is initial with respect to maps into A-null spaces. We say that X is A-acyclic if $P_A X \simeq *$.

The kernel of a map $f\colon BG^\wedge_p \to Z$, where G is a compact Lie group and Z is a connected p-complete $\Sigma B\mathbb{Z}/p$-null space, is defined by D. Notbohm in [Not94]. We adapt his definition to our context.

Definition 3.1. Let $(S, \mathcal{F})$ be a saturated fusion system and let Z be a connected p-complete $\Sigma B\mathbb{Z}/p$-null space. If $f\colon B\mathcal{F} \to Z$ is a pointed map, we define the kernel of f

$$\ker(f) := \{g \in S \mid f|_{B\langle g\rangle} \simeq *\}.$$

Remark 3.2. By Proposition 2.6, we have $\mathrm{map}(B\mathbb{Z}/p, B\mathcal{F})_c \simeq B\mathcal{F}$. It follows then by looping that $\Omega B\mathcal{F}$ is $B\mathbb{Z}/p$-null, or equivalently, that $B\mathcal{F}$ is $\Sigma B\mathbb{Z}/p$-null ([DF96, 3.A.1]).

Remark 3.3. If X is a $B\mathbb{Z}/p$-null space, then X is BP-null for any finite p-group P. There are weak equivalences $\mathrm{map}_*(BP, X) \simeq \mathrm{map}_*(P_{B\mathbb{Z}/p}(BP), X) \simeq *$, where the last equivalence follows from Lemma 6.13 in [Dwy96] which states that BP is $B\mathbb{Z}/p$-acyclic. A direct proof can be obtained by induction using the central extension of a p-group and Zabrodsky's Lemma [Dwy96, Proposition 3.4].

We will show that $\ker(f)$ is a subgroup of S with some important properties.

Definition 3.4. Let $\mathcal{F}$ be a fusion system over a finite p-group S. Then a subgroup $K \leq S$ is *strongly $\mathcal{F}$-closed* if for all $P \leq K$ and all morphism $\varphi\colon P \to S$ in $\mathcal{F}$ we have $\varphi(P) \leq K$.

If G is a finite group and $S \in \mathrm{Syl}_p(G)$, $K \leq S$ is strongly $\mathcal{F}_S(G)$-closed if and only if K is *strongly closed in* G, i.e., if for all $k \in K$ and $g \in G$ such that $c_g(s) \in S$, then $c_g(s) \in K$.

Since the intersection of strongly $\mathcal{F}$-closed subgroups is again strongly $\mathcal{F}$-closed, given a finite p-group P, we define $Cl_{\mathcal{F}}(P)$ to be the smallest strongly $\mathcal{F}$-closed subgroup of S that contains $f(P)$ for all $f \in \mathrm{Hom}(P, S)$.

Proposition 3.5. *Let $f\colon B\mathcal{F} \to Z$ be a pointed map as in Definition 3.1. The kernel* $\ker(f)$ *is a strongly $\mathcal{F}$-closed subgroup of S.*

Proof. Since $\langle x\rangle = \langle x^{-1}\rangle$, in order to show that $\ker(f)$ is a subgroup of S it is sufficient to prove that if $x, y \in \ker(f)$, then $xy \in \ker(f)$. The composite $B\langle x, y\rangle \to BS \to B\mathcal{F} \to Z$ is null-homotopic by [Not94, Proposition 2.4]. Since $\langle xy\rangle \hookrightarrow \langle x, y\rangle$, $f|_{B\langle xy\rangle} \simeq *$.

Let $P \leq \ker(f)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. We have the following homotopy commutative diagram

$$\begin{array}{ccccc} BP & \xrightarrow{B\delta_P} & & & \\ {\scriptstyle B\varphi}\downarrow{\scriptstyle \simeq} & & B\mathcal{F} & \xrightarrow{f} & Z \\ B\varphi(P) & \xrightarrow{B\delta_{\varphi(P)}} & & & \end{array}$$

which shows that $f|_{BP}$ is null-homotopic if and only if $f|_{B\varphi(P)}$ is null-homotopic. □

W. Dwyer shows in [Dwy96, Theorem 5.1] that if we have a finite group G, a map $f\colon BG^\wedge_p \to Z$ as in Definition 3.1 is null-homotopic if and only if $\ker(f) = S$. This statement is also true for classifying spaces of saturated fusion systems.

Theorem 3.6. *Let $(S,\mathcal{F})$ be a saturated fusion system. Let Z be a connected p-complete $\Sigma B\mathbb{Z}/p$-null space. Then a map $f\colon B\mathcal{F} \to Z$ is null-homotopic if and only if* $\ker(f) = S$.

Proof. If $f \simeq *$, then $f|_{BS} \simeq *$ and therefore $\ker(f) = S$. Now assume that $f|_{BS} \simeq *$, we will show that $f \simeq *$.

Step 1: Assume that $\pi_1(Z)$ is abelian. By Proposition 2.9, $B\mathcal{F} \simeq (\mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})} \tilde{B}P)^\wedge_p$, where $\tilde{B}P \simeq BP$ for $P \in \mathcal{F}^c$. Since any map $BP \to B\mathcal{F}$ factors through $\Theta\colon BS \to B\mathcal{F}$ by [BLO03b, Theorem 4.4], $f|_{BP} \simeq *$ for all $P \in \mathcal{F}^c$. Therefore we have two maps

$$\mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}P) \longrightarrow (\mathrm{hocolim}_{\mathcal{O}^c(\mathcal{F})}(\tilde{B}P))^\wedge_p \overset{f}{\underset{*}{\rightrightarrows}} Z\ ,$$

such that both are nullhomotopic when restricted to BP for all $P \in \mathcal{F}^c$.

The obstructions for these maps to be homotopic are in $\lim^i_{\mathcal{O}^c(\mathcal{F})} \pi_i(\mathrm{map}(BP, Z)_c)$, for $i \geq 1$ (see [Woj87]). Since a $B\mathbb{Z}/p$-null space is BQ-null for any finite p-group Q (Remark 3.3), Z is ΣBP-null and hence $\mathrm{map}_*(BP, Z)$ is homotopically discrete, therefore $\mathrm{map}_*(BP, Z)_c \simeq *$ and, from the fibration $\mathrm{map}_*(BP, Z)_c \to \mathrm{map}(BP, Z)_c \to Z$, we obtain $\mathrm{map}(BP, Z)_c \simeq Z$.

Given a morphism $\varphi\colon P \to Q$ in $\mathcal{F}^c$, $B\varphi$ is a pointed map and induces a commutative diagram

$$\begin{array}{ccc} \mathrm{map}(BQ, Z)_c & \xrightarrow{B\varphi^*} & \mathrm{map}(BP, Z)_c \\ \simeq \big\downarrow ev & & ev \big\downarrow \simeq \\ Z & \xrightarrow[id]{} & Z \end{array}$$

which shows that the obstructions are in $\lim^i_{\mathcal{O}^c(\mathcal{F})} \pi_i Z$, where $\pi_* Z$ is a constant functor in $\mathcal{O}^c(\mathcal{F})$. Since Z is p-complete and $\pi_1(Z)$ is abelian, the constant functors $\pi_i(Z)$ all take values in $\mathbb{Z}_{(p)} - \mathrm{Mod}$.

Let $F := \pi_* Z$ be the constant functor. Fix P in $\mathcal{O}^c(\mathcal{F})^{\mathrm{op}}$ and consider the atomic functors $F_P\colon \mathcal{O}^c(\mathcal{F})^{\mathrm{op}} \to \mathbb{Z}_{(p)} - \mathrm{Mod}$

$$F_P(Q) := \begin{cases} \pi_* Z & \text{, if } Q = P, \\ 0 & \text{, if } Q \neq P. \end{cases}$$

and $\tilde{F}_P(Q) := F(Q)/F_P(Q)$. Filtering F as a series of extensions of functors F_P, one for each object P, and using the long exact sequence for higher limits (see proof of Theorem 1.10 of [JMO95]), if $\lim^i F_P = 0$ for all P and $i > 0$, then $\lim^i F = 0$ for $i > 0$. By [JMO92, Proposition 6.1 (i),(ii)], if $p \nmid |\mathrm{Out}_{\mathcal{F}}(P)|$, then

$$\lim{}^i F_P = \begin{cases} (\pi_* Z)^{\mathrm{Out}_{\mathcal{F}}(P)} = \pi_* Z & \text{, if } i = 0, \\ 0 & \text{, if } i > 0. \end{cases}$$

and if $p \mid |\mathrm{Out}_{\mathcal{F}}(P)|$, then $\lim^i F_P = 0$ for $i \geq 0$. Therefore $\lim^i_{\mathcal{O}^c(\mathcal{F})} \pi_i Z = 0$ for $i > 0$, and finally $f \simeq *$.

Step 2: We will prove that a map $f\colon B\mathcal{F} \to BG$ is null-homotopic if $f|_{BS}\colon BS \to BG$ is null-homotopic where G is a discrete group.

We will apply Zabrodsky's lemma [Dwy96, Proposition 3.5] to the map $f|_{BS}$ and the fibre sequence $F \to BS \to B\mathcal{F}$ where F is connected since $\pi_1(\Theta)$ is an epimorphism (Lemma 2.5). Note that $\mathrm{map}_*(F, BG)$ is homotopically discrete, then Zabrodsky's lemma shows that there is a homotopy equivalence $\mathrm{map}(B\mathcal{F}, BG) \simeq \mathrm{map}(BS, BG)_{[c]}$ where that last mapping space corresponds to the components which are null-homotopic when restricted to F. The bijection between components implies that $f \simeq *$.

Finally, we can prove the theorem by looking at the universal cover $\tilde{Z}$ of Z. Let $p\colon Z \to B\pi_1(Z)$. The composite $p \circ f$ is null-homotopic by Step 2. Therefore there is a lift $\tilde{f}\colon B\mathcal{F} \to \tilde{Z}$ to the universal cover of Z. In order to apply Step 1, we need to check that $\tilde{f}|_{BS}\colon BS \to \tilde{Z}$ is null-homotopic. Since both Z and $B\pi_1(Z)$ are $\Sigma B\mathbb{Z}/p$-null, applying mapping spaces from BS to the universal cover fibration shows that $\mathrm{map}(BS, \tilde{Z})_{\{c\}} \simeq \tilde{Z}$ where $\{c\}$ is the set of maps which are nullhomotopic when postcomposed with $\tilde{Z} \to Z$. Note that $[\tilde{f}|_{BS}] \in \{c\}$. Since $\tilde{Z}$ is connected, the set $\{c\}$ only consists on the constant map and then $\tilde{f}|_{BS}$ is nullhomotopic. □

Given a strongly $\mathcal{F}$-closed subgroup $K \leq S$ in a saturated fusion system $\mathcal{F}$, there exists a map between classifying spaces $f\colon B\mathcal{F} \to BG^\wedge_p$ for a finite group G such that $\ker(f) = K$.

Proposition 3.7. *Let $(S, \mathcal{F})$ be a saturated fusion system and K be a strongly $\mathcal{F}$-closed subgroup and $\mathcal{L}$ be the associated centric linking system. Let ρ be the composite $S \xrightarrow{\pi} S/K \xrightarrow{reg} \Sigma_{|S/K|}$, where π is the quotient homomorphism and reg is the regular representation of S/K. Then there is a non-negative integer $m \geq 0$ and a map $f\colon |\mathcal{L}| \to B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p$ such that the following diagram*

$$\begin{array}{ccc} BS & \xrightarrow{(\Delta B\rho)^\wedge_p} & (B(\Sigma_{|S/K|})^{p^m})^\wedge_p \\ {\scriptstyle \iota_S}\downarrow & & \downarrow{\scriptstyle \Delta^\wedge_p} \\ |\mathcal{L}| & \xrightarrow[f]{} & B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p \end{array}$$

is commutative up to homotopy. Moreover, $\ker(f^\wedge_p) = K$.

Proof. Let $n = |S/K|$. According to [CL09, Theorem 1.2], if ρ is fusion invariant then there is a non-negative integer $m \geq 0$ and a map $f\colon |\mathcal{L}| \to B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p$ such that $f|_{BS}$ is homotopic to the composite $BS \xrightarrow{(\Delta B\rho)^\wedge_p} (B(\Sigma_n)^{p^m})^\wedge_p \xrightarrow{\Delta^\wedge_p} B(\Sigma_n \wr \Sigma_{p^m})^\wedge_p$. Therefore, it is sufficient to show that ρ is fusion invariant: for all $P \leq S$ and $\varphi\colon P \to S$ in $\mathcal{F}$ there is $\omega \in \Sigma_n$ such that $\rho|_{\varphi(P)} \circ \varphi = c_\omega \circ \rho|_P$.

The homomorphisms $\rho|_P$ and $c_\omega \rho|_P$ equip S/K with a structure of a P-set which are isomorphic. Hence, to prove the above equality, we only need to show that $(S/K, \leq)$ and $(S/K, \leq_\varphi)$ are equivalent as P-sets.

Note that for any $\varphi\colon P \to S \in \mathcal{F}$,

$$(S/K, \leq_\varphi) \cong \mathrm{Iso}^*(\varphi)\,\mathrm{Res}^S_{\varphi(P)}(S/K) \cong \mathrm{Iso}^*(\varphi)\,\mathrm{Res}^S_{\varphi(P)}\,\mathrm{Ind}^S_K(*).$$

Applying the Mackey formula to $\mathrm{Res}^S_{\varphi(P)}\,\mathrm{Ind}^S_K$, we get

$$\begin{aligned}(S/K, \leq_\varphi) &\cong \coprod_{[x]\in\varphi(P)\backslash S/K} \mathrm{Iso}^*(\varphi)\,\mathrm{Ind}^{\varphi(P)}_{\varphi(P)\cap K^x}\,\mathrm{Iso}^*(c_x)\,\mathrm{Res}^K_{(\varphi(P)\cap K)^x}(*) \\ &= \coprod_{[x]\in\varphi(P)\backslash S/K} \mathrm{Ind}^P_{\varphi^{-1}(\varphi(P)\cap K)}\,\mathrm{Iso}^*(\varphi)\,\mathrm{Iso}^*(c_x)\,\mathrm{Res}^K_{(\varphi(P)\cap K)^x}(*).\end{aligned}$$

where the second equality comes from the commutativity of isogation and induction and, where $K^x = K$ because K is strongly $\mathcal{F}$-closed and $c_x\colon K \to S$ is in $\mathcal{F}$, hence $c_x(K) = K^x \leq K$ and since c_x is an isomorphism, $K^x = K$.

Next we prove that $\varphi^{-1}(\varphi(P)\cap K) = P \cap K$. On the one hand, $\varphi^{-1}(\varphi(P)\cap K) \leq P\cap K$ because $\varphi^{-1}|_{\varphi(P)\cap K}\colon \varphi(P)\cap K \to S$ is in $\mathcal{F}$, $\varphi(P)\cap K \leq K$, K is strongly $\mathcal{F}$-closed and $\varphi^{-1}(\varphi(P)\cap K) \leq P$. The equality will follow if $|\varphi^{-1}(\varphi(P)\cap K)| = |P\cap K|$. Since φ is an isomorphism, it is enough to check $|\varphi(P)\cap K| = |\varphi(P\cap K)|$. We already know that $|\varphi(P)\cap K| = |\varphi^{-1}(\varphi(P)\cap K)| \leq |P\cap K| = |\varphi(P\cap K)|$. Since $\varphi|_{P\cap K}\colon P\cap K \to S$ is in $\mathcal{F}$, $P\cap K\leq K$ and K is strongly $\mathcal{F}$-closed, $\varphi(P\cap K)\leq K$ but also $\varphi(P\cap K)\leq\varphi(P)$, hence $\varphi(P\cap K)\leq\varphi(P)\cap K$ and therefore $|\varphi(P\cap K)| \leq |\varphi(P)\cap K|$.

Therefore in the above formula, since $\mathrm{Iso}^*(\varphi)\,\mathrm{Iso}^*(c_x)\,\mathrm{Res}^K_{(\varphi(P)\cap K)^x}(*) = *$ as $(P\cap K)$-set and $\varphi^{-1}(\varphi(P)\cap K) = P\cap K$, we get for all $\varphi\colon P\to S\in\mathcal{F}$

$$(S/K, \leq_\varphi) \cong \coprod_{[x]\in\varphi(P)\backslash S/K} \mathrm{Ind}^P_{P\cap K}(*) \cong \coprod_{l_\varphi} P/P\cap K,$$

where $l_\varphi = |\varphi(P)\backslash S/K| = |S/\varphi(P)\cdot K| = |S|/|\varphi(P)\cdot K|$ since $K \triangleleft S$. In particular, if $\varphi = id_P$, then

$$(S/K, \leq) \cong \coprod_{[x]\in P\backslash S/K} \mathrm{Ind}^P_{P\cap K}(*) \cong \coprod_{l} P/P\cap K,$$

where $l = |S|/|P\cdot K|$.

Therefore, ρ will be fusion invariant if $l = l_\varphi$. It is enough to show $|\varphi(P)\cap K| = |P\cap K|$ and, in fact, this equality has been already proved in a paragraph above.

Finally, $f|_{BS}$ is the induced map on classifying spaces of the homomorphism

$$S \xrightarrow{\pi} S/K \xrightarrow{reg} \Sigma_{|S/K|} \longrightarrow \Sigma_{|S/K|} \wr \Sigma_{p^m}$$

whose kernel is K by construction. □

Question: Given a saturated fusion system $(S,\mathcal{F})$ and a map $f\colon B\mathcal{F} \to Z$ where Z is a connected $\Sigma B\mathbb{Z}/p$-null p-complete space. Does f factors, up to homotopy, through $\tilde{f}\colon B\mathcal{F}' \to Z$ with trivial kernel, where $\mathcal{F}'$ is a saturated fusion system?

Under some hypothesis we can give a positive answer to the previous question.

Proposition 3.8. *Let $(S,\mathcal{F})$ be a saturated fusion system with associated linking system $\mathcal{L}$. Let K be the kernel of $f\colon B\mathcal{F} \to Z$, where Z is $\Sigma B\mathbb{Z}/p$-null p-complete space.*

If K is normal in $\mathcal{F}$, then there exist a saturated fusion system $(S/K,\mathcal{F}/K)$ with associated linking system $\mathcal{L}/K$ and a map $pr\colon |\mathcal{L}| \to |\mathcal{L}/K|$, whose homotopy fibre is BK, such that f factors via $\tilde{f}\colon B(\mathcal{F}/K) \to Z$ with trivial kernel.

Proof. In [OV07, Section 2], the authors prove that if K is normal in $\mathcal{F}$, then there is a saturated fusion system $(S/K,\mathcal{F}/K)$ with linking system $\mathcal{L}/K$ and a map $pr\colon |\mathcal{L}| \to |\mathcal{L}/K|$ whose homotopy fibre is BK.

Since Z is p-complete, one can consider the composite $g\colon |\mathcal{L}| \to B\mathcal{F} \to Z$ such that $g^\wedge_p \simeq f$. By assumption Z is $\Sigma B\mathbb{Z}/p$-null, and also BK is $B\mathbb{Z}/p$-acyclic, then we get a unique factorization, up to homotopy, $\tilde{g}\colon |\mathcal{L}/K| \to Z$ of g by applying Zabrodsky lemma (see [Dwy96, Proposition 3.4]) to the homotopy fibre sequence $BK \to |\mathcal{L}| \to |\mathcal{L}/K|$ and the map g. Take $\tilde{f} = \tilde{g}^\wedge_p$. The same argument applied to the fibration $BK \to BS \xrightarrow{\pi} B(S/K)$ and $g|_{BS}$ shows that $\tilde{f}|_{B(S/K)} \circ B\pi \simeq f|_{BS}$, by uniqueness.

Let $[x] \in \ker(\tilde{f}) \leq S/K$, then $\tilde{f}|_{B\langle [x]\rangle} \simeq *$ implies that $f|_{B\langle x\rangle} \simeq *$ and finally $x \in K$. □

Corollary 3.9. *Let $(S,\mathcal{F})$ be a saturated fusion system with associated linking system $\mathcal{L}$. Let K be the kernel of $f\colon B\mathcal{F} \to Z$, where Z is a connected $\Sigma B\mathbb{Z}/p$-null p-complete space.*

If K is abelian or S is resistant, then there exist a saturated fusion system $(S/K,\mathcal{F}/K)$ with associated linking system $\mathcal{L}/K$ and a map $pr\colon |\mathcal{L}| \to |\mathcal{L}/K|$ whose homotopy fibre is BK and a homotopy factorization $\tilde{f}\colon B\mathcal{F}/K \to Z$ with trivial kernel.

Proof. Recall that K is strongly $\mathcal{F}$-closed by Proposition 3.5. If K is abelian and strongly $\mathcal{F}$-closed, then it is normal in $\mathcal{F}$ by [Cra10, Proposition 3.14]. If S is resistant, each strongly $\mathcal{F}$-closed subgroup is also normal in $\mathcal{F}$. Finally, apply Proposition 3.8. □

4 Homotopy properties of BP-cellular approximations

In this context, it is natural to study homotopy properties of $CW_{BP}X$ which are inherited by those of X. We will show in this section that $CW_{BP}(B\mathcal{F})$ is a p-good nilpotent space whose fundamental group is a finite p-group.

Proposition 4.1. *Let $(S,\mathcal{F})$ be a saturated fusion system. Then both $B\mathcal{F}$ and $CW_{BP}(B\mathcal{F})$ are nilpotent. Furthermore, there exists a homotopy fibre sequence*

$$CW_{BP}(B\mathcal{F}) \to CW_{BP}(B\mathcal{F})^\wedge_p \to (CW_{BP}(B\mathcal{F})^\wedge_p)_{\mathbb{Q}}.$$

Proof. The classifying space $B\mathcal{F}$ is p-complete and $\pi_i(B\mathcal{F})$ are finite p-groups by Lemma 2.5, hence $B\mathcal{F}$ is nilpotent according to [BK72, Proposition VII.4.3(ii)]. Tthen $CW_{BP}(B\mathcal{F})$ is also nilpotent [CF15, Corollary 3.2].

Moreover, it follows from [CF15, Lemma 2.8] that $R_\infty CW_{BP}(B\mathcal{F}) \simeq *$ for $R = \mathbb{Q}$ and $R = \mathbb{F}_q$, $q \neq p$, since $\tilde{H}_*(BP;R) = 0$. Finally, the Sullivan's arithmetic square applied to the nilpotent space $CW_{BP}(B\mathcal{F})$ gives the desired result. □

The R-completion functor is a homological localization functor when we restrict to R-good spaces (see [BK72, p. 205]). By [BK72, Proposition VII.5.1], if the fundamental group of a pointed space X is finite, then X is p-good for any prime p. Therefore, in order to show that $CW_{BP}(B\mathcal{F})$ is a p-good space, we will prove that its fundamental group is a finite p-group.

Proposition 4.2. *Let $(S, \mathcal{F})$ be a saturated fusion system and let P be a finite p-group. Then $\pi_1 CW_{BP}(B\mathcal{F})$ is a finite p-group. Therefore, $CW_{BP}(B\mathcal{F})$ is a p-good space.*

Proof. The proof will be divided in several steps. Let C be the homotopy cofibre of the evaluation map $ev\colon \bigvee_{[BP,B\mathcal{F}]_*} BP \to B\mathcal{F}$.

Step 1: Assume C is simply connected. The Chacholski's homotopy fibre sequence $CW_{BP}(B\mathcal{F}) \to B\mathcal{F} \to P_{\Sigma BP}C$ induces a long exact sequence of homotopy groups

$$\ldots \to \pi_2(P_{\Sigma BP}C) \to \pi_1 CW_{BP}(B\mathcal{F}) \to \pi_1(B\mathcal{F}) \to \ldots$$

where $\pi_1(B\mathcal{F})$ is a finite p-group by Lemma 2.5. Therefore, we are reduced to prove that $\pi_2(P_{\Sigma BP}C)$ is a finite p-group.

Hurewicz's theorem shows that $H_2(P_{\Sigma BP}C;\mathbb{Z}) \cong \pi_2(P_{\Sigma BP}C)$, since $P_{\Sigma BP}C$ is simply connected ([Bou94, 2.9]). Moreover, since ΣBP is 1-connected, we obtain an epimorphism $H_2(C;\mathbb{Z}) \twoheadrightarrow H_2(P_{\Sigma BP}C;\mathbb{Z})$ by [CGR15, Proposition 3.2]. Then it is enough to prove that $H_2(C;\mathbb{Z})$ is a finite p-group.

The cofibration sequence $\bigvee_{[BP,B\mathcal{F}]_*} BP \to B\mathcal{F} \to C$ induces a long exact sequence of homology groups

$$\ldots \longrightarrow H_2(B\mathcal{F};\mathbb{Z}) \xrightarrow{f_1} H_2(C;\mathbb{Z}) \xrightarrow{f_2} H_1(\textstyle\bigvee_{[BP,B\mathcal{F}]_*} BP;\mathbb{Z}) \longrightarrow \ldots$$

which allows to describe $H_2(C;\mathbb{Z})$ by a short exact sequence

$$0 \to \ker f_2 \to H_2(C;\mathbb{Z}) \to \operatorname{Im} f_2 \to 0.$$

By exactness, $\ker f_2 = \operatorname{Im} f_1$ is a quotient of $H_2(B\mathcal{F};\mathbb{Z})$, where $H_2(B\mathcal{F};\mathbb{Z})$ is a finite p-group since $H^*(B\mathcal{F};\mathbb{Z}) \hookrightarrow H^*(BS;\mathbb{Z})$ by [BLO03b, Theorem B]. Hence $\ker f_2$ is a finite p-group.

Now note that

$$\operatorname{Im} f_2 \subset H_1(\bigvee_{[BP,B\mathcal{F}]_*} BP;\mathbb{Z}) \cong \pi_1(\bigvee_{[BP,B\mathcal{F}]_*} BP)_{ab} \cong \oplus_{[BP,B\mathcal{F}]_*} P_{ab},$$

where $[BP, B\mathcal{F}]_*$ is a finite set because there is an epimorphism of sets $[BP, BS]_* \twoheadrightarrow [BP, B\mathcal{F}]_*$ by [BLO03b, Theorem 4.4], and $[BP, BS]_* \cong \operatorname{Hom}(P, S)$ is finite. Finally, $H_2(C;\mathbb{Z})$ is a finite p-group.

Step 2: We will show that there exists a saturated fusion system $(S', \mathcal{F}')$ and a BP-equivalence $f\colon B\mathcal{F}' \to B\mathcal{F}$ such that the homotopy cofibre of $ev\colon \bigvee_{[BP,B\mathcal{F}']_*} BP \to B\mathcal{F}'$ is 1-connected.

Let N be the normal subgroup of $\pi_1 B\mathcal{F}$ generated by the image of homomorphisms $P \to \pi_1 B\mathcal{F}$ such that the induced pointed map $BP \to B\pi_1 B\mathcal{F}$ lifts to $B\mathcal{F}$. Let X be the pullback of the universal cover $\widetilde{B\mathcal{F}} \to B\mathcal{F} \to B\pi_1 B\mathcal{F}$ along $BN \to B\pi_1 B\mathcal{F}$. Then we have a homotopy commutative diagram

$$\begin{array}{ccccc} \widetilde{B\mathcal{F}} & = & \widetilde{B\mathcal{F}} & & \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{f} & B\mathcal{F} & \longrightarrow & B(\pi_1 B\mathcal{F}/N) \\ \downarrow & & \downarrow & & \| \\ BN & \to & B\pi_1 B\mathcal{F} & \to & B(\pi_1 B\mathcal{F}/N) \end{array}$$

where the vertical arrows are homotopy fibrations and the horizontal arrows are principal homotopy fibrations. By Theorem 2.7, there is a satured fusion system $(S', \mathcal{F}')$ such that $S' \leq S$, $\mathcal{F}' \leq \mathcal{F}$ and $X \simeq B\mathcal{F}'$. Furthermore, $f\colon B\mathcal{F}' \simeq X \to B\mathcal{F}$ is a BP-equivalence by Proposition 4.3.

Let C be the homotopy cofibre of $ev\colon \bigvee_{[BP,B\mathcal{F}']_*} BP \to B\mathcal{F}'$. By Seifert-Van Kampen's theorem, C will be 1-connected if $\pi_1(ev)\colon *_{[BP,B\mathcal{F}']_*} P \to \pi_1(B\mathcal{F}') \cong N$ is an epimorphism.

Since N is generated by the image of homomorphisms $f\colon P \to \pi_1 B\mathcal{F}$ such that the induced map $Bf\colon BP \to B\pi_1 B\mathcal{F}$ lifts to $\tilde{f}\colon BP \to B\mathcal{F}$, it is enough to show that for each of those f, $\mathrm{Im}(f) \leq \mathrm{Im}(\pi_1(ev))$.

By definition, there is a factorization $f\colon P \to N \leq \pi_1 B\mathcal{F}$ which lifts to $B\mathcal{F}$. There exists a unique $\beta\colon BP \to B\mathcal{F}'$ (up to homotopy) such that the diagram

$$\begin{array}{ccc} BP & \xrightarrow{\tilde{f}} & B\mathcal{F} \\ & \overset{\beta}{\searrow} \; B\mathcal{F}' \longrightarrow & B\mathcal{F} \\ & \searrow^{f} \quad \downarrow & \downarrow \\ & BN \longrightarrow & B\pi_1 B\mathcal{F} \end{array}$$

is homotopy commutative. Therefore $\mathrm{Im}(\pi_1(\beta)) = \mathrm{Im}(f) \leq \mathrm{Im}(\pi_1(ev))$.

Now we are ready to complete the proof. By Step 2 there exists a saturated fusion system $(S', \mathcal{F}')$ and a BP-equivalence $f\colon B\mathcal{F}' \to B\mathcal{F}$ such that the homotopy cofibre of $ev\colon \bigvee_{[BP,B\mathcal{F}']_*} BP \to B\mathcal{F}'$ is 1-connected. Hence $CW_{BP}(B\mathcal{F}) \simeq CW_{BP}(B\mathcal{F}')$ and $\pi_1 CW_{BP}(B\mathcal{F}) \cong \pi_1 CW_{BP}(B\mathcal{F}')$ is a finite p-group by Step 1.

□

For the proof of Proposition 4.2 we need the next version of a result of N. Castellana, J.A. Crespo and J. Scherer.

Proposition 4.3 ([CCS07, Proposition 2.1]). *Let P be a finite p-group and let $F \to E \xrightarrow{\pi} BG$ be a fibration, where G is a discrete group. Let N be the normal subgroup of G generated by the image of homomorphisms $P \to G$ such that the induced pointed map $BP \to BG$ lifts to E. Then the pullback of the fibration along $BN \to BG$*

$$\begin{array}{ccccc} E' & \xrightarrow{f} & E & \xrightarrow{pr} & B(G/N) \\ \downarrow & & \downarrow \pi & & \| \\ BN & \to & BG & \xrightarrow{pr'} & B(G/N) \end{array}$$

induces a BP-equivalence $f\colon E' \to E$ on the total space level.

Proof. We want to show that f induces a BP-equivalence. The top fibration in the diagram yields a fibration

$$\mathrm{map}_*(BP, E') \xrightarrow{f_*} \mathrm{map}_*(BP, E)_{\{c\}} \xrightarrow{pr_*} \mathrm{map}_*(BP, B(G/N))_c,.$$

Since the base space is homotopically discrete, we only need to check that all components of $\mathrm{map}_*(BP, E)$ are sent by pr_* to $\mathrm{map}_*(BP, B(G/N))_c$. Thus consider a pointed map $h\colon BP \to E$. The composite $pr \circ h$ is homotopy equivalent to a map induced by a group homomorphism $\alpha = \pi_1(pr \circ h)\colon P \to G$ whose image is in N by construction. Therefore $pr \circ h \simeq pr' \circ \pi \circ h$ is null-homotopic. □

Remark 4.4. In the original version the authors consider $P = B\mathbb{Z}/p^m$ for $m > 1$ and $\bar{N}$ to be the (normal) subgroup generated by all elements $g \in G$ of order p^i for some $i \leq r$ such that the inclusion $f_g\colon B\langle g\rangle \to BG$ lifts to E, but this subgroup does not have the desired properties. Consider the fibration $B\mathbb{Z}/2 \xrightarrow{\iota} B\mathbb{Z}/4 \xrightarrow{pr} B\mathbb{Z}/2$, this fibration has no section and hence $\bar{N} = \{0\}$. Then $E' \simeq B\mathbb{Z}/2$ and $CW_{B\mathbb{Z}/4}(B\mathbb{Z}/2) \simeq B\mathbb{Z}/2 \not\simeq B\mathbb{Z}/4 = CW_{B\mathbb{Z}/4}(B\mathbb{Z}/4)$, contradicting the proposition. However, $N \cong \mathbb{Z}/2 \cong \langle g\rangle$, since $f_g = pr$. Hence $E' \simeq B\mathbb{Z}/4$ and $f : E' \to B\mathbb{Z}/4$ is an equivalence, in particular it is a $B\mathbb{Z}/4$-equivalence.

5 Cellular properties of the classifying space of a saturated fusion system

The goal of this section is to prove the main theorem of the paper. Given a finite p-group P, this result characterizes the property of being BP-cellular for classifying spaces of saturated fusion systems in terms of the fusion data.

Theorem 5.1. *Let $(S, \mathcal{F})$ be a saturated fusion system and let P be a finite p-group. Then $B\mathcal{F}$ is BP-cellular if and only if $S = Cl_{\mathcal{F}}(P)$.*

Corollary 5.2. *Let $(S, \mathcal{F})$ be a saturated fusion system and let P be a finite p-group.*

(a) The classifying space $B\mathcal{F}$ is BS-cellular.

(b) Let $(S, \mathcal{F}')$ be a saturated fusion system with $\mathcal{F} \subset \mathcal{F}'$. If $B\mathcal{F}$ is BP-cellular then $B\mathcal{F}'$ is also BP-cellular.

(c) Let A be a pointed connected space. If $Cl_{\mathcal{F}}((\pi_1 A)_{ab}) = S$, then $B\mathcal{F}$ is A-cellular.

(d) Let $\Omega_{p^m}(S)$ be the (normal) subgroup of S generated by its elements of order p^i, with $i \leq m$. Then $B\mathcal{F}$ is $B\mathbb{Z}/p^m$-cellular if and only if $S = Cl_{\mathcal{F}}(\Omega_{p^m}(S))$. In particular, there is a non-negative integer $m_0 \geq 0$ such that $B\mathcal{F}$ is $B\mathbb{Z}/p^m$-cellular for all $m \geq m_0$.

Proof. (a) Direct from Theorem 5.1 since $Cl_{\mathcal{F}}(S) = S$.

(b) It follows from the the inclusions $Cl_{\mathcal{F}}(P) \leq Cl_{\mathcal{F}'}(P) \leq S$.

(c) Notice that $SP^{\infty}A \simeq \prod_{i\geq 1} K(H_i(A;\mathbb{Z}), i)$ is A-cellular by [DF96, Corollary 4.A.2.1], so $B(\pi_1 A)_{ab} \simeq K(H_1(A;\mathbb{Z}), 1)$ is also A-cellular from [DF96, 2.D]. Then, $B\mathcal{F}$ is $B(\pi_1 A)_{ab}$-cellular using (a).

(d) We have $Cl_{\mathcal{F}}(\Omega_{p^m}(S)) \cong Cl_{\mathcal{F}}(\mathbb{Z}/p^m)$. Note that $B\Omega_{p^m}(S)$ is $B\mathbb{Z}/p^m$-cellular and that there exists an $m_0 \geq 0$ such that S is generated by elements of order a power of p less than p^{m_0}.

□

The strategy of proof for Theorem 5.1 goes by analyzing the fibre sequence given in [Cha96, Theorem 20.5]

$$CW_{BP}(B\mathcal{F}) \xrightarrow{c} B\mathcal{F} \xrightarrow{r} P_{\Sigma BP}C,$$

where C is the homotopy cofibre of the evaluation map $ev\colon \bigvee_{[BP,B\mathcal{F}]_*} BP \to B\mathcal{F}$ and r is the composite $B\mathcal{F} \to C \to P_{\Sigma BP}C$.

The first goal is to compute the kernel of $r_p^{\wedge}$. In order to apply the theory of kernels, developed in Section 3, the target of the map needs to be a connected p-complete $\Sigma B\mathbb{Z}/p$-null space.

Since $\pi_1 B\mathcal{F}$ is a finite p-group, the same holds for $P_{\Sigma BP}C$ [Bou94, 2.9], therefore Bousfield-Kan p-completion of the previous homotopy fibration is a homotopy fibre sequence ([BK72, II.5.1])

$$CW_{BP}(B\mathcal{F})_p^{\wedge} \xrightarrow{c_p^{\wedge}} B\mathcal{F} \xrightarrow{r_p^{\wedge}} (P_{\Sigma BP}C)_p^{\wedge}.$$

Lemma 5.3. *If X is a 1-connected space and P is a finite p-group, then $(P_{\Sigma BP}X)_p^{\wedge}$ is $\Sigma B\mathbb{Z}/p$-null.*

Proof. We have the following weak homotopy equivalences

$$\operatorname{map}_*(\Sigma B\mathbb{Z}/p, (P_{\Sigma BP}X)_p^{\wedge}) \simeq \operatorname{map}_*(B\mathbb{Z}/p, \Omega(P_{\Sigma BP}X)_p^{\wedge}) \simeq \operatorname{map}_*(B\mathbb{Z}/p, (\Omega P_{\Sigma BP}X)_p^{\wedge})$$

where the last equivalence holds by [BK72, V.4.6 (ii)] since X is 1-connected (and so is $P_{\Sigma BP}X$ by [Bou94, 2.9]).

The commutation rules between nullification functors and loops in [DF96, 3.A.1] show that $\Omega P_{\Sigma BP}X \simeq P_{BP}(\Omega X)$. Finally, $\operatorname{map}_*(B\mathbb{Z}/p, (P_{BP}\Omega X)_p^{\wedge}) \simeq \operatorname{map}_*(B\mathbb{Z}/p, (P_{BP}\Omega X)) \simeq *$ where the first equivalence follows from Miller's theorem [Mil84, Thm 1.5] and the second from the fact that BP is $B\mathbb{Z}/p$-acyclic ([Dwy96, Lemma 6.13]). □

Next we describe a criteria for detecting when a map from an A-cellular space is null-homotopic which will be useful later.

Proposition 5.4. *Let X and Y be pointed connected spaces. Assume that X is A-cellular and Y is ΣA-null. Then a pointed map $f\colon X \to Y$ is null-homotopic if and only if for any map $g\colon A \to X$ the composite $f \circ g$ is null-homotopic.*

Proof. If f is null-homotopic then for any map $g\colon A \to X$ the composite $f \circ g$ is null-homotopic. A

Assume that for any map $g\colon A \to X$, the composite $f \circ g$ is null-homotopic. Let C be the homotopy cofibre of $ev\colon \bigvee_{[A,X]_*} A \to X$. Since $f \circ ev \simeq *$ by assumption, there is a map $\tilde{f}\colon C \to Y$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc}
\bigvee_{[A,X]_*} A & & \\
\downarrow{\scriptstyle ev} & \searrow^{\simeq *} & \\
X & \xrightarrow{\ f\ } & Y. \\
\downarrow & \nearrow_{\tilde{f}} & \\
C & &
\end{array}$$

Then $f \simeq *$ if and only if $\tilde{f} \simeq *$.

Since X is A-cellular, $P_{\Sigma A}C \simeq *$. Finally $\tilde{f} \simeq *$ because $\mathrm{map}_*(C, Y) \simeq \mathrm{map}_*(P_{\Sigma A}C, Y) \simeq \mathrm{map}_*(*, Y) \simeq *$, where the first equivalence follows from the fact that Y is ΣA-null. □

A key step in the proof of Theorem 5.1 is the following computation of the kernel of the map $r_p^\wedge$.

Proposition 5.5. *Let $(S, \mathcal{F})$ be a saturated fusion system. Then* $\ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$.

Proof. We will show that $\ker(r_p^\wedge) \leq Cl_{\mathcal{F}}(P)$ and that $f(P) \leq \ker(r_p^\wedge)$ for all $f\colon P \to S$, then since $\ker(r_p^\wedge)$ is strongly $\mathcal{F}$-closed, the definition of $Cl_{\mathcal{F}}(P)$ implies the equality.

By universal properties of cellularization and p-completion, any map $BP \to B\mathcal{F}$ lifts to $CW_{BP}(B\mathcal{F})_p^\wedge$, then $f(P) \leq \ker(r_p^\wedge)$ for any homomorphism $f\colon P \to S$.

According to Proposition 3.7, there exist $m \geq 0$ and a pointed map between classifying spaces $k\colon B\mathcal{F} \to B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge$ such that $\ker(k) = Cl_{\mathcal{F}}(P)$. Let $\iota\colon B\ker(r_p^\wedge) \to B\mathcal{F}$ be the composite $B\ker(r_p^\wedge) \to BS \to B\mathcal{F}$. It is enough to show that $k \circ \iota$ is nullhomotopic.

There is a lift $\tilde{\iota}\colon B\ker(r_p^\wedge) \to CW_{BP}(B\mathcal{F})$ such that following diagram is homotopy commutative

$$\begin{array}{ccccc}
 & & CW_{BP}(B\mathcal{F})_p^\wedge & & \\
 & \nearrow^{\tilde{\iota}} & \downarrow{\scriptstyle c} & \searrow^{k \circ c} & \\
B\ker(r_p^\wedge) & \xrightarrow{\ \iota\ } & B\mathcal{F} & \xrightarrow{\ k\ } & B(\Sigma_{|S/K|} \wr \Sigma_{p^m})_p^\wedge \\
 & \searrow_{* \simeq} & \downarrow{\scriptstyle r_p^\wedge} & & \\
 & & (P_{\Sigma BP}C)_p^\wedge. & &
\end{array}$$

We will show that $k \circ c$ is nullhomotopic by applying Proposition 5.4. Recall that any map $f\colon BP \to B\mathcal{F}$ is homotopic to $\Theta \circ B\rho\colon BP \to BS \to B\mathcal{F}$ where $\rho \in \mathrm{Hom}(P, S)$ (see [BLO03b, Theorem 4.4]). Then, the map k has the property that for any $f\colon BP \to B\mathcal{F}$, the composite $k \circ f$ is nullhomotopic.

In particular, if $c\colon CW_{BP}(B\mathcal{F}) \to B\mathcal{F}$ is the augmentation, for any $f\colon BP \to CW_{BP}(B\mathcal{F})$ we also have $(k \circ c) \circ f \simeq *$. Therefore $k \circ c \simeq *$ by Proposition 5.4 (since $B(\Sigma_{|S/K|} \wr \Sigma_{p^m})^\wedge_p$ is $\Sigma B\mathbb{Z}/p$-null and hence ΣBP-null by Remark 3.3). □

An immediate consequence of Proposition 5.5 is one implication in Theorem 5.1, that is, if $B\mathcal{F}$ is BP-cellular then $S = Cl_{\mathcal{F}}(P)$. In order to prove the prove the other implication, we will need a couple of technical results.

Lemma 5.6. *Let $(S, \mathcal{F})$ be a saturated fusion system. If $S = Cl_{\mathcal{F}}(P)$, then homotopy cofibre C of the evaluation $ev\colon \bigvee_{[BP,B\mathcal{F}]_*} BP \to B\mathcal{F}$ is 1-connected.*

Proof. From the cofibration sequence and Seifert-Van Kampen theorem, we have that $\pi_1 C \cong \pi_1 B\mathcal{F}/N$, where N is the minimal normal subgroup of $\pi_1 B\mathcal{F}$ containing $\mathrm{Im}(\pi_1(ev))$. Given $f\colon BP \to B\mathcal{F}$, there is group homomorphism $g\colon P \to S$ such that $f \simeq \Theta \circ Bg$ (see [BLO03b, Theorem 4.4]). Let $\bar{N}$ be normal subgroup of S generated by all $g(P)$, where $g \in \mathrm{Hom}(P, S)$. Then the fundamental group can be described $\pi_1(C) \cong S/\bar{N}O^p_{\mathcal{F}}(S)$.

First, $\bar{N}O^p_{\mathcal{F}}(S)$ is strongly $\mathcal{F}$-closed by [DGPS11, Proposition A.9]. Morever it contains $g(P)$ for all $g \in \mathrm{Hom}(P, S)$. Therefore we have inclusions $Cl_{\mathcal{F}}(P) \leq \bar{N}O^p_{\mathcal{F}}(S) \leq S$. Since we are assuming $S = Cl_{\mathcal{F}}(P)$, the previous inclusions are all equalities and C is then 1-connected. □

Proposition 5.7. *Let Z be a $\Sigma^i BP$-null space where P is a finite p-group and $i \geq 0$. If $\pi_1 Z$ is a finite p-group then $Z^\wedge_p$ is also $\Sigma^i BP$-null.*

Proof. Since $\pi_1 Z$ is a finite p-group, there is a homotopy commutative diagram of fibre sequences

$$\begin{array}{ccccc}
F & \longrightarrow & Z\langle 1\rangle & \longrightarrow & Z^\wedge_p\langle 1\rangle \\
\| & & \downarrow & & \downarrow \\
F & \longrightarrow & Z & \longrightarrow & Z^\wedge_p \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & B\pi_1(Z) & \xrightarrow{\simeq} & B\pi_1(Z^\wedge_p)
\end{array}$$

From the proof of Proposition 3.1 in [DZ87], it is easy to see that $[\Sigma^i BP, Z^\wedge_p]_* \cong [\Sigma^i BP, Z]_* \cong *$. Then it is enough to check that $\mathrm{map}_*(\Sigma^i BP, \Omega Z^\wedge_p) \simeq *$. Consider the homotopy fibre sequence $\Omega Z^\wedge_p \to F \to Z$. Since Z is $\Sigma^i BP$-null, $\mathrm{map}_*(\Sigma^i BP, \Omega Z^\wedge_p) \simeq \mathrm{map}_*(\Sigma^i BP, F)$. This last mapping space is homotopy equivalent to $\mathrm{map}_*(\Sigma^i BP, F^\wedge_p)$ by Miller's theorem [Mil84, Thm 1.5] since F is nilpotent. Finally $F^\wedge_p \simeq *$. □

We are now ready to prove the main theorem of this paper.

Proof of Theorem 5.1. First assume that $B\mathcal{F}$ is BP-cellular. Then $P_{\Sigma BP}C$ is contractible and $r \simeq *$. This implies that $\ker(r^\wedge_p) = S$, but also $\ker(r^\wedge_p) = Cl_{\mathcal{F}}(P)$ by Proposition 5.5. Therefore $S = Cl_{\mathcal{F}}(P)$.

Now assume that $S = Cl_{\mathcal{F}}(P)$. Since C is 1-connected by Lemma 5.6 (and therefore $P_{\Sigma BP}C$ is so by [Bou94, 2.9]), consider the p-completed homotopy fibre sequence ([BK72, II.5.1])

$$CW_{BP}(B\mathcal{F})^\wedge_p \xrightarrow{c^\wedge_p} B\mathcal{F} \xrightarrow{r^\wedge_p} (P_{\Sigma BP}C)^\wedge_p .$$

We will show that $(P_{\Sigma BP}C)^\wedge_p$ is weakly contractible. For this we apply first the theory of kernels developed in Section 3 to $r^\wedge_p$ (see Lemma 5.3 and Lemma 5.6). By assumption and Proposition 5.5, $\ker(r^\wedge_p) = S$ and therefore $r^\wedge_p$ is null-homotopic by Theorem 3.6. Then there is a splitting $CW_{BP}(B\mathcal{F})^\wedge_p \simeq B\mathcal{F} \times \Omega(P_{\Sigma BP}C)^\wedge_p$.

Let $X = CW_{BP}(B\mathcal{F})^\wedge_p$ and $Y = \Omega(P_{\Sigma BP}C)^\wedge_p$ for simplicity. We will show that $P_{BP}(X)^\wedge_p \simeq *$ and that $P_{BP}(Y)^\wedge_p \simeq Y$. Since both nullification and p-completion functors commute with products, the previous splitting $X \simeq B\mathcal{F} \times Y$ shows that $Y \simeq *$. Since $(P_{\Sigma BP}C)^\wedge_p$ is 1-connected, then it follows that it is contractible.

First, we have that $P_{BP}(X)^\wedge_p \simeq P_{BP}(CW_{BP}(B\mathcal{F}))^\wedge_p \simeq *$ by checking the hypothesis of [CF15, Lemma 3.9]. One only needs to check that $P_{BP}(X)$ is p-good and that $P_{BP}(X)^\wedge_p$ is BP-null. Notice that $\pi_1 X$ is a finite p-group since C is 1-connected by Lemma 5.6. Therefore the space $P_{BP}(X)^\wedge_p$ is BP-null by Proposition 5.7 and the space $P_{BP}(X)$ is p-good by Proposition 4.2. Finally, recall that BP-cellular spaces are BP-acyclic ([DF96, 3.B.1]).

Now,

$$P_{BP}(Y)^\wedge_p \simeq P_{BP}(\Omega(P_{\Sigma BP}C)^\wedge_p)^\wedge_p \simeq (\Omega P_{\Sigma BP}((P_{\Sigma BP}C)^\wedge_p))^\wedge_p \simeq (\Omega(P_{\Sigma BP}C)^\wedge_p)^\wedge_p \simeq \Omega(P_{\Sigma BP}C)^\wedge_p = Y$$

where the second equivalence follows from commutation rules [DF96, 3.A.1], the third holds because $(P_{\Sigma BP}C)^\wedge_p$ is ΣBP-null by Millers's theorem [Mil84, Thm 1.5], and the forth is a commutation of taking loops and p-completion [BK72, V.4.6 (ii)].

Summarizing, we have proved that $c\colon CW_{BP}(B\mathcal{F}) \to B\mathcal{F}$ is a mod p equivalence. Finally, using Proposition 4.1 we get that c is a weak equivalence since $CW_{BP}(B\mathcal{F})^\wedge_p \simeq B\mathcal{F}$ and $B\mathcal{F}_{\mathbb{Q}} \simeq *$. □

6 Examples

Let G be a finite group. The situation when G is generated by elements of order p is well studied by R. Flores and R. Foote in [FF11]. We start by giving a simple example where G is not generated by elements of order p^i.

Example 6.1. Let $G = \Sigma_3$, the permutation group of 3 elements. Σ_3 is generated by transpositions, i.e, by elements of order 2, but the Sylow 3-subgroup of Σ_3 is $S = \mathbb{Z}/3$. Therefore, BS is $B\mathbb{Z}/3^r$-cellular for all $r \geq 1$ and hence $(B\Sigma_3)^\wedge_3$ is $B\mathbb{Z}/3^r$-cellular for all $r \geq 1$ by Corollary 5.2.

Notice that $B\Sigma_3$ is not $B\mathbb{Z}/3^r$-cellular for any $r \geq 1$: applying $\mathrm{map}_*(B\mathbb{Z}/3^r, -)$ to the homotopy fibre sequence $B\mathbb{Z}/3 \xrightarrow{Bi} B\Sigma_3 \to B\mathbb{Z}/2$, we see that Bi is a $B\mathbb{Z}/3^r$-equivalence for any r, since $\mathrm{map}_*(B\mathbb{Z}/3^r, B\mathbb{Z}/2) \simeq *$. Then $CW_{B\mathbb{Z}/3^r}(B\Sigma_3) \simeq CW_{B\mathbb{Z}/3^r}(B\mathbb{Z}/3) \simeq B\mathbb{Z}/3$.

We are now interested in the study of $CW_{BP}(B\mathcal{F})$ when $S \neq Cl_{\mathcal{F}}(P)$. Our aim is to get tools to identify $P_{BP}(C)^\wedge_p$ in the p-completed Chacholsky's fibration describing $CW_{BP}(B\mathcal{F})$.

Proposition 6.2. *Let $(S, \mathcal{F})$ be a saturated fusion system, $\mathcal{L}$ be the associated centric linking system and let P be a finite p-group. Assume that there is a saturated fusion system $(S', \mathcal{F}')$, with associated linking system $\mathcal{L}'$, and a factorization (up to homotopy)*

$$\begin{array}{ccccc} |\mathcal{L}| & \xrightarrow{\eta} & B\mathcal{F} & = & B\mathcal{F} \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_p^\wedge}\downarrow & & \downarrow{\scriptstyle r_p^\wedge} \\ |\mathcal{L}'| & \xrightarrow{\eta} & B\mathcal{F}' & \xrightarrow{\tilde{r}} & (P_{\Sigma BP}C)_p^\wedge \end{array}$$

such that

(i) the map $\tilde{r}$ is injective, i.e., $\ker(\tilde{r}) = \{e\}$,

(ii) and or F, the homotopy fibre of $\pi \colon |\mathcal{L}| \to |\mathcal{L}'|$, or F_p, the homotopy fibre of $\pi_p^\wedge$, is $B\mathbb{Z}/p$-acyclic.

Then $\tilde{r}$ is a homotopy equivalence. In particular, $CW_{BP}(B\mathcal{F}) \simeq F_p$.

Proof. We will proceed as follows: first we will construct a map $\tilde{\pi}_p^\wedge \colon (P_{\Sigma BP}C)_p^\wedge \to B\mathcal{F}'$ under $B\mathcal{F}$, and then we will prove that $\tilde{\pi}_p^\wedge$ is a homotopy inverse of $\tilde{r}_p^\wedge$. This will give us that $CW_{BP}(B\mathcal{F})_p^\wedge \simeq F_p$. Finally we will show, in this situation, that $CW_{BP}(B\mathcal{F})$ is p-complete and hence $CW_{BP}(B\mathcal{F}) \simeq F_p$.

(a) *Construction of $\tilde{\pi}_p^\wedge \colon (P_{\Sigma BP}C)_p^\wedge \to B\mathcal{F}'$.* In order to apply Zabrodsky lemma [Dwy96, Proposition 3.4] to the following situation

$$\begin{array}{ccc} CW_{BP}(B\mathcal{F})_p^\wedge & & \\ \downarrow{\scriptstyle c_p^\wedge} & & \\ B\mathcal{F} & \xrightarrow{\pi_p^\wedge} & B\mathcal{F}' \\ \downarrow{\scriptstyle r_p^\wedge} & & \\ (P_{\Sigma BP}C)_p^\wedge & & \end{array}$$

we need to check that two assumptions are satisifed:

- $\pi_p^\wedge \circ c_p^\wedge \simeq *$: or equivalently, $\pi_p^\wedge \circ c \colon CW_{BP}(B\mathcal{F}) \to B\mathcal{F}'$ is null-homotopic since $B\mathcal{F}'$ is p-complete and $CW_{BP}(B\mathcal{F})$ is p-good.
  According to Proposition 5.4 it is sufficient to show that the composite $\pi_p^\wedge \circ c \circ f$ is null-homotopic for all $f \in \mathrm{map}_*(BP, CW_{BP}(B\mathcal{F}))$. But if we postcompose with $\tilde{r}$, we have $\tilde{r} \circ \pi_p^\wedge \circ c \circ f \simeq r_p^\wedge \circ c \circ f \simeq *$. There exists $\rho \in \mathrm{Hom}(P, S')$ such that $\pi_p^\wedge \circ c \circ f \simeq \Theta' \circ B\rho$, then $\rho(P) \le ker(\tilde{r}) = \{e\}$.
- $\mathrm{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega B\mathcal{F}') \simeq *$: since $B\mathcal{F}'$ and $CW_{BP}(B\mathcal{F})$ are p-good space we have $\mathrm{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, B\mathcal{F}') \simeq \mathrm{map}_*(CW_{BP}(B\mathcal{F}), B\mathcal{F}')$ and hence, taking loops,

  $$\mathrm{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega B\mathcal{F}') \simeq \mathrm{map}_*(CW_{BP}(B\mathcal{F}), \Omega B\mathcal{F}').$$

  The last mapping space is contractible since $\Omega B\mathcal{F}'$ is BP-null (p:mapping space) and $CW_{BP}(B\mathcal{F})$ is BP-acyclic ([DF96, 3.B.1]).

(b) $\tilde{r}$ *and* $\tilde{\pi}_p^\wedge$ *are homotopy inverse.* First note that both $Id_{(P_{\Sigma BP}C)_p^\wedge}$ and $\tilde{r} \circ \tilde{\pi}$ factor $r_p^\wedge$ since $Id_{(P_{\Sigma BP}C)_p^\wedge} \circ r_p^\wedge \simeq \tilde{r} \circ \pi \simeq \tilde{r} \circ \tilde{\pi} \circ r_p^\wedge$. In order to show that $\tilde{r} \circ \tilde{\pi}_p^\wedge \simeq Id_{(P_{\Sigma BP}C)_p^\wedge}$, we will apply Zabrodsky lemma to the fibration $CW_{BP}(B\mathcal{F})_p^\wedge \to B\mathcal{F} \to (P_{\Sigma BP}C)_p^\wedge$ and the composite map $\tilde{r}_p^\wedge \circ \pi_p^\wedge \colon B\mathcal{F} \to (P_{\Sigma BP}C)_p^\wedge$. Then uniqueness up to homotopy of the factorization will give the desired equivalence.

We check that $\mathrm{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq *$ and $\tilde{r}_p^\wedge \circ \tilde{\pi}_p^\wedge \circ c_p^\wedge \simeq *$. First, by Proposition 5.7, $(P_{\Sigma BP}C)_p^\wedge$ is ΣBP-null, then $\Omega(P_{\Sigma BP}C)_p^\wedge$ is BP-null ([DF96, 3.A.1]). Now, since $CW_{BP}(B\mathcal{F})_p^\wedge$ and $\Omega(P_{\Sigma BP}C)_p^\wedge$ are p-good and $CW_{BP}(B\mathcal{F})$ is BP-acyclic, we have

$$\mathrm{map}_*(CW_{BP}(B\mathcal{F})_p^\wedge, \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq \mathrm{map}_*(CW_{BP}(B\mathcal{F}), \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq *$$

Finally, $\tilde{r}_p^\wedge \circ \tilde{\pi}_p^\wedge \circ c_p^\wedge \simeq r_p^\wedge \circ c_p^\wedge \simeq *$.

It remains to prove that $\tilde{\pi}_p^\wedge \circ \tilde{r}_p^\wedge \simeq Id_{B\mathcal{F}'}$. First note that $Id_{B\mathcal{F}'} \circ \pi \simeq \tilde{\pi} \circ r_p^\wedge \simeq \tilde{\pi} \circ \tilde{r} \circ \pi$, then both $Id_{B\mathcal{F}'}$ and $\tilde{\pi} \circ \tilde{r}$ factor π. In order to show that $Id_{B\mathcal{F}'} \simeq \tilde{\pi} \circ \tilde{r}$, we will apply again Zabrodsky's lemma to the fibration $F_p \to B\mathcal{F} \to B\mathcal{F}'$ (resp. $F \to |\mathcal{L}| \to |\mathcal{L}'|$, depending on whether F or F_p is $B\mathbb{Z}/p$-acyclic) and the composite $\tilde{\pi} \circ r_p^\wedge$ (resp. $\tilde{\pi} \circ r_p^\wedge \circ \eta$).

If $i \colon F_p \to B\mathcal{F}$, then $r_p^\wedge \circ i \simeq \tilde{r} \circ \pi \circ i \simeq *$. Then since $\Omega B\mathcal{F}'$ is $B\mathbb{Z}/p$-null and F_p is $B\mathbb{Z}/p$-acyclic (resp. F is $B\mathbb{Z}/p$-acyclic), $\mathrm{map}_*(F_p, \Omega B\mathcal{F}') \simeq *$ (resp. $\mathrm{map}_*(F, \Omega B\mathcal{F}') \simeq *$).

(c) $CW_{BP}(B\mathcal{F})$ *is p-complete.* By Proposition 4.1 we obtain the homotopy fibration

$$CW_{BP}(B\mathcal{F}) \to F_p \to (F_p)_{\mathbb{Q}},$$

where $(F_p)_{\mathbb{Q}} \simeq *$, since $\pi_i(F_p)$ are finite groups for $i \geq 1$.

□

Remark 6.3. Proposition 6.2 also holds without assuming the existence of π and only considering the factoritzation on p-completed classifying spaces if we know that the homotopy fiber F_p is $B\mathbb{Z}/p$-acyclic.

Corollary 6.4. *Let $(S, \mathcal{F})$ be a saturated fusion system and let P be a finite p-group. If $O^p_{\mathcal{F}}(S) \lhd Cl_{\mathcal{F}}(P)$, then $CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F}'$, where $B\mathcal{F}'$ is the connected cover of $B\mathcal{F}$ with $\pi_1 B\mathcal{F}' \cong Cl_{\mathcal{F}}(P)/O^p_{\mathcal{F}}(S)$.*

Proof. The connected cover $B\mathcal{F}'$ is the classifying space of a saturated fusion system over $Cl_{\mathcal{F}}(P)$ by [BCG+07, Theorem A]. So $B\mathcal{F}'$ is BP-cellular by Corollary 5.2. We get the following commutative diagram of homotopy fibrations

$$\begin{array}{ccccc}
B\tilde{\mathcal{F}} & = & B\tilde{\mathcal{F}} & & \\
\downarrow & & \downarrow & & \\
B\mathcal{F}' & \xrightarrow{f} & B\mathcal{F} & \longrightarrow & B(S/Cl_{\mathcal{F}}(P)) \\
\downarrow & & \downarrow & & \| \\
B(Cl_{\mathcal{F}}(P)/O^p_{\mathcal{F}}(S)) & \longrightarrow & B(S/O^p_{\mathcal{F}}(S)) & \longrightarrow & B(S/Cl_{\mathcal{F}}(P))
\end{array}$$

In order to apply Proposition 6.2, we have to find a factorization of $r_p^\wedge \colon B\mathcal{F} \to (P_{\Sigma BP}C)_p^\wedge$ by a map $\tilde{r}\colon B(S/Cl_{\mathcal{F}}(P)) \to (P_{\Sigma BP}C)_p^\wedge$ with trivial kernel, where we can assume that C is 1-connected by step 2 in the proof of Proposition 4.2.

By Lemma 5.3 and Remark 3.3, $\Omega(P_{\Sigma BP}C)_p^\wedge$ is BP-null and $B\mathcal{F}'$ is BP-cellular space (hence BP acyclic), then $\mathrm{map}_*(B\mathcal{F}, \Omega(P_{\Sigma BP}C)_p^\wedge) \simeq *$. Moreover, $r_p^\wedge \circ f \simeq *$, because $r_p^\wedge \circ f|_{BCl_{\mathcal{F}}(P)} \simeq *$ since $\ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$. We are in the conditions of applying Zabrodsky lemma (see [Dwy96, Proposition 3.4]) which shows that there is a map $\tilde{r}\colon B(S/Cl_{\mathcal{F}}(P)) \to (P_{\Sigma BP}C)_p^\wedge$ such that the diagram

$$\begin{array}{ccc} B\mathcal{F}' & & \\ \downarrow{\scriptstyle f} & & \\ B\mathcal{F} & =\!=\!=\!= & B\mathcal{F} \\ \downarrow & & \downarrow{\scriptstyle r_p^\wedge} \\ B(S/Cl_{\mathcal{F}}(P)) & \xrightarrow{\tilde{r}} & (P_{\Sigma BP}C)_p^\wedge \end{array}$$

commutes up to homotopy.

Let $[x] \in \ker(\tilde{r}) \leq S/Cl_{\mathcal{F}}(P)$, where $x \in S$. Since the above diagram commutes, $r_p^\wedge|_{B\langle x\rangle} \simeq \tilde{r}|_{B\langle [x]\rangle} \simeq *$, then $x \in \ker(r_p^\wedge) = Cl_{\mathcal{F}}(P)$ and hence $[x] = e$. Finally, Proposition 6.2 gives us the equivalence $CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F}'$. □

Corollary 6.5. *Let $(S, \mathcal{F})$ be a fusion system and let P be a finite p-group. If $Cl_{\mathcal{F}}(P) \lhd \mathcal{F}$, then $CW_{BP}(B\mathcal{F})$ is homotopy equivalent to the homotopy fibre of $B\mathcal{F} \to B(\mathcal{F}/Cl_{\mathcal{F}}(P))$.*

Proof. Let $K = Cl_{\mathcal{F}}(P)$. Since K is normal in $\mathcal{F}$, there is a saturated fusion system $(S/K, \mathcal{F}/K)$, and a map defined between the nerve of the associated linking systems $\pi\colon |\mathcal{L}| \to |\mathcal{L}/K|$ whose homotopy fibre is BK, and an injective factorization of $r_p^\wedge$ by $B\mathcal{F}/K$ from Proposition 3.8. Then the result follows from Proposition 6.2. □

Example 6.6. Let G be a finite group and let S be a p-Sylow subgroup of G. Assume that $N_G(S)$ controls fusion in G. Then $BN_G(S)_p^\wedge \simeq BG_p^\wedge$ and S is normal in $\mathcal{F}_S(N_G(S))$. On account of Corollary 6.5, for all finite p-group P, $CW_{BP}(BG_p^\wedge)$ is equivalent to the homotopy fibre of $BN_G(S)_p^\wedge \to B(N_G(S)/Cl_{\mathcal{F}_S(N_G(S))}(P))_p^\wedge$.

An example is given by $G = \mathbb{Z}/p^n \wr \mathbb{Z}/q = (\mathbb{Z}/p^n)^q \rtimes \mathbb{Z}/q$, when $p \neq q$ and $n \geq 1$. The Sylow p-subgroup of G is $S = (\mathbb{Z}/p^n)^q$ and $Cl_{\mathcal{F}_S(G)}(\mathbb{Z}/p^r) = (\mathbb{Z}/p^r)^q$ is abelian and hence normal in $\mathcal{F}_S(G)$ by Corollary 3.9. Therefore $BG_p^\wedge$ is $B\mathbb{Z}/p^r$-cellular if and only if $r \geq n$ by Theorem 5.1. Then $CW_{B\mathbb{Z}/p^r}(BG_p^\wedge)$ is equivalent to the homotopy fibre of $BG_p^\wedge \to B(G/(\mathbb{Z}/p^r)^q)_p^\wedge$ by Corollary 6.5.

Other explicit examples appear in [FS07, Example 5.2]. The authors proved that the normalizer of the Sylow of the Suzuki group $\mathrm{Sz}(2^n)$, with n an odd integer at least 3, is $N_{\mathrm{Sz}(2^n)}(S) = S \rtimes \mathbb{Z}/(2^n - 1)$ and it controls fusion in $\mathrm{Sz}(2^n)$. In this case, S is $B\mathbb{Z}/2^m$-cellular for all $m \geq 2$ and hence $B\mathrm{Sz}(2^n)_2^\wedge$ is so. Moreover, $Cl_{\mathcal{F}_S(\mathrm{Sz}(2^n))}(\mathbb{Z}/2) \cong (\mathbb{Z}/2)^n$ and hence $CW_{B\mathbb{Z}/2}(B\mathrm{Sz}(2^n)_2^\wedge)$ is equivalent to the homotoy fibre of $BN_{\mathrm{Sz}(2^n)}(S)_2^\wedge \to B(N_{\mathrm{Sz}(2^n)}(S)/(\mathbb{Z}/2)^n)_2^\wedge$.

7 The cellularization of the classifying spaces of a family of exotic fusion systems at $p = 3$

Let $S = B(3, r; 0, \gamma, 0)$, for $r \geq 4$ and $\gamma = 0, 1, 2$, be the family of finite 3-groups of order 3^r (see [DRV07, Theorem A.2, Proposition A.9]) generated by $\{s, s_1, \dots, s_{r-1}\}$ where

- $s_i = [s_{i-1}, s]$ for all $i \in \{2, \dots, r-1\}$,
- $[s_1, s_i] = 1$ for all $i \in \{2, \dots, r-1\}$,
- $s_1^3 s_2^3 s_3 = s_{r-1}^{\gamma}$,
- $s_i^3 s_{i+1}^3 s_{i+2} = 1$ for all $i \in \{2, \dots, r-1\}$,
- $s^3 = 1$.

The center of S is $\langle s_{r-1} \rangle$. The normal subgroup $\gamma_1 = \langle s_1, \dots, s_{r-1} \rangle$ of S is of index 3 and the corresponding group extension is split. There are group isomorphisms

$$B(3, r; 0, \gamma, 0) = \langle s_1, s_2 \rangle \rtimes \langle s \rangle = \begin{cases} (\mathbb{Z}/3^m \times \mathbb{Z}/3^m) \rtimes \mathbb{Z}/3 & \text{, if } r = 2m+1, \\ (\mathbb{Z}/3^m \times \mathbb{Z}/3^{m+1}) \rtimes \mathbb{Z}/3 & \text{, if } r = 2m. \end{cases}$$

In [DRV07, Theorem 5.10], the authors construct families of exotic 3-local finite groups $\mathcal{F}$ whose Sylow 3-subgroup is $S = B(3, r; 0, \gamma, 0)$.

Proposition 7.1. *Let $\mathcal{F}$ be an exotic fusion system over $B(3, r; 0, \gamma, 0)$ such that $\mathcal{F}$ has at least one $\mathcal{F}$-Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then*

(i) If $\gamma = 0$, then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$-cellular for all $l \geq 1$.

(ii) Assume $\gamma \neq 0$. Then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$-cellular if and only if $l \geq 2$. If $l = 1$, $Cl_{\mathcal{F}}(\mathbb{Z}/3) = \langle s, s_2 \rangle$.

Proof. By Theorem 5.1 we are reduced to the computation of $Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$. Let $N := \langle s_2, s \rangle$ which is a proper normal subgroup of S. This subgroup N contains the center of S, $Z(S) = \langle s_{r-1} \rangle \cong \mathbb{Z}/3$ by [DRV07, Lemma A.10]. Moreover, the description of $\mathcal{F}$ and the computation of automorphisms of S in [DRV07, Theorem 5.10,Lemma A.14] show that N is strongly $\mathcal{F}$-closed.

First $s \in Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$ for all $l \geq 1$. If $Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$ is a proper normal subgroup of S, again by [DRV07, Lemma A.10], $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = N$ with $[N : S] = 3$, and

$$N \cong \begin{cases} 3^{1+2}_{+} & \text{, if } r = 4, \\ B(3, r-1; 0, 0, 0) & \text{, if } r > 4. \end{cases}$$

We have then inclusions $N \subset Cl_{\mathcal{F}}(\mathbb{Z}/3^l) \subset S$. And $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = S$ if and only if there exists $x \in S \setminus N$ such that $x^{3^l} = 1$.

Let $x = s^i s_1^j s_2^k \in S$, where $i = 0, 1, 2$. Since the index $[Cl_{\mathcal{F}}(\mathbb{Z}/3^l) : S] \in \{1, 3\}$, $x \in Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$ if and only if $x^2 \in Cl_{\mathcal{F}}(\mathbb{Z}/3^l)$. But if $x = s^2 s_1^j s_2^k$, then $x^2 = s^4 s_1^{j'} s_2^{k'} = s s_1^{j'} s_2^{k'}$, therefore we can assume $i = 0, 1$. Note that $s_i = [s_{i-1}, s] \in \langle s, s_2 \rangle \subset N$ for all $i \in \{3, \ldots, r-1\}$ and $s_1^3 = s_{r-1}^{\gamma} s_3^{-1} s_2^{-3} \in N$. If $j \equiv 0 \mod 3$ then $x \in N$.

If $i = 0$ then $x = s_1^j s_2^k$ and $[s_1, s_2] = 1$. Then $o(x) = 3$ if and only if $o(s_1^j) = 3$ and $o(s_2^k) = 3$. In particular, $3j | 3^m$, that is, $3|j$. But then $j \equiv 0 \mod 3$ and $x \in N$.

If $i = 1$ and $x = ss_1^j s_2^k$, one can compute $x^3 = (s_{r-1})^{\gamma i}$ using relations $s_1^a s = ss_1^a s_2^a$, $s_2^b = ss_2^b s_3^b$, $s_3^c s = ss_3^c s_4^c$, $s_1^3 s_2^3 s_3 = s_{r-1}^{\gamma}$ and $s_2^3 s_3^3 s_4 = 1$ with $a, b, c > 0$ ([DRV07, Proposition A.9]). We have that $x^{3^l} = (s_{r-1})^{\gamma i 3^{l-1}}$ and $x^{3^l} = 1$ if and only if $\gamma i 3^{l-1} \equiv 0 \mod 3$. That is, $x^9 = 1$ always, and $x^3 = 1$ if $\gamma = 0$ or $i \equiv 0 \mod 3$, but in this last case $x \in N$.

Summarizing, if $\gamma = 0$ then $x = ss_1 \in S \setminus N$ is of order 3 and then $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = S$ for all $l \geq 1$. If $\gamma \neq 0$ then $x = ss_1 \in S \setminus N$ is of order 9 and then $Cl_{\mathcal{F}}(\mathbb{Z}/3^l) = S$ for all $l \geq 2$. Finally if $\gamma \neq 0$, $Cl_{\mathcal{F}}(\mathbb{Z}/3) = N$. □

We will finish this section by describing $CW_{B\mathbb{Z}/3}(B\mathcal{F})$ when $\mathcal{F}$ is an exotic fusion system over $B(3, r; 0, \gamma, 0)$ with $\gamma \neq 0$ such that $\mathcal{F}$ has at least one $\mathcal{F}$-Alperin rank two elementary abelian 3-subgroup given in [DRV07].

Lemma 7.2. *Let $\mathcal{F}$ an exotic fusion system under the hypothesis of Proposition 7.1, $B\mathcal{F}$ is 1-connected.*

Proof. By Proposition 2.5, the fundamental group of $B\mathcal{F}$ is $\pi_1(B\mathcal{F}) \cong S/O^p_{\mathcal{F}}(S)$, where $O^p_{\mathcal{F}}(S) := \langle [Q, O^p(\mathrm{Aut}_{\mathcal{F}}(Q))] \mid Q \leq S \rangle$. The subgroup $O^p_{\mathcal{F}}(S)$ is a strongly $\mathcal{F}$-closed subgroup of S, and the arguments in the proof in [DRV07, page 1751] show that it must contain $N = \langle s, s_2 \rangle < S$. We will show that $O^p_{\mathcal{F}}(S) = S$ by proving that $s_1 \in O^p_{\mathcal{F}}(S)$. Checking tables in [DRV07, Theorem 5.10,Lemma A.14], we see that the automorphisms of order two η and/or ω are group elements in $\mathrm{Aut}_{\mathcal{F}}(S)$. By the description given there $\eta(s_1) = s_1 s_2^{f''}$ and $\omega(s_1) = s_1^{-1} s_2^{f''}$. Then $s_1^{-1}\eta(s_1) = s_1^{-2} s_2^{f''}$ or $s_1^{-1}\omega(s_1) = s_1^{-2} s_2^{f''}$ are elements of $O^p_{\mathcal{F}}(S)$, since $s_2, s_1^3 \in N \subset O^p_{\mathcal{F}}(S)$ then $s_1 \in O^p_{\mathcal{F}}(S)$. □

Proposition 7.3. *Let $\mathcal{F}$ be an exotic fusion system satisfying the hypothesis of Proposition 7.1 with $\gamma \neq 0$. Let $N = \langle s, s_2 \rangle < S$, then there exists a unique map (up to homotopy) $f \colon B\mathcal{F} \to (B\Sigma_3)^\wedge_3$ whose kernel is N.*

Proof. The proof of Proposition 3.7 shows that the quotient morphism $S \to S/N \cong \mathbb{Z}/3$ gives a fusion preserving homomorphism $\rho \colon S \to \Sigma_3$. We want to show that this morphism extends to a map $f : B\mathcal{F} \to (B\Sigma_3)^\wedge_3$.

By Proposition 2.9, $B\mathcal{F} \simeq (\mathrm{hocolim}_{O^c(\mathcal{F})} \tilde{B}P)^\wedge_3$, where $\tilde{B}P \simeq BP$ for $P \in \mathcal{F}^c$. The fusion preserving property of ρ shows that $B\rho \in \lim_{O(\mathcal{F})}[BP, (B\Sigma_3)^\wedge_3]$.

The obstructions for rigidifying the homotopy commutative diagram in the category of spaces lie in $\lim^{i+1}_{O^c(\mathcal{F})} \pi_i(\mathrm{map}(BP, (B\Sigma_3)^\wedge_3)_{\Theta' \circ B\rho|_P})$, for $i \geq 1$ (see [Woj87]). Note that since the 3-Sylow subgroup of Σ_3 is abelian, we have $\pi_1(\mathrm{map}(BP, (B\Sigma_3)^\wedge_3)_{\Theta' \circ B\rho|_P})$ is abelian being a quocient of $C_{\mathbb{Z}/3}(\rho(P))$. In fact, it will be trivial or $\mathbb{Z}/3$ (Proposition 2.6).

We will show that for any $F \colon O(\mathcal{F}) \to \mathbb{Z}_{(p)} - Mod$, $\lim^i_{O(\mathcal{F})} F = 0$ for $i > 1$. From [BLO03b, Proposition 3.2, Corollary 3.3], we are reduced to show that derived limits of atomic functors have the same vanishing property. Note that from [DRV07, Theorem

5.10,Lemma A.14], the relevant outomorphism groups $\mathrm{Out}_{\mathcal{F}}(P)$ are $SL_2(\mathbb{F}_3)$ or $GL_2(\mathbb{F}_3)$. In both cases the 3-Sylow subgrup is of order 3, and then [JMO92, Proposition 6.2(i)] implies the result.

The obstructions to uniqueness lie in $\lim^i_{\mathcal{O}^c(\mathcal{F})} \pi_i(\mathrm{map}(BP, (B\Sigma_3)^\wedge_3)_{\Theta' \circ B\rho|_P})$, for $i \geq 1$ (see [Woj87]). By the previous paragrah we have to look at the first derived functor of atomic functors with value $\mathbb{Z}/3$. But since $\mathrm{Aut}(\mathbb{Z}/3) \cong \mathbb{Z}/2$, by [JMO92, Proposition 6.1(ii)] any element of order 3 will act trivially on $\mathbb{Z}/3$. Finally note that if a map $g\colon B\mathcal{F} \to (B\Sigma_3)^\wedge_3$ has kernel N, its restriction $g|_{BS}\colon BS \to (B\Sigma_3)^\wedge_3$ has to be homotopic to $B\rho$. □

Proposition 7.4. *Let $\mathcal{F}$ be an exotic fusion system over $B(3, r; 0, \gamma, 0)$ with $\gamma \neq 0$ such that $\mathcal{F}$ has at least one $\mathcal{F}$-Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then there exists a map $f\colon B\mathcal{F} \to (B\Sigma_3)^\wedge_p$ such that $CW_{B\mathbb{Z}/3}(B\mathcal{F})$ is the homotopy fiber of f.*

Proof. Let f be the map constructed in Proposition 7.3 with $\ker(f) = Cl_{\mathcal{F}}(S)$. Precisely because of this, $f \circ ev \simeq *$ where $ev\colon \bigvee_{[B\mathbb{Z}/3, B\mathcal{F}]_*} B\mathbb{Z}/3 \to B\mathcal{F}$. Then f factors through the cofibre C of ev and, since $(B\Sigma_3)^\wedge_3$ is $B\mathbb{Z}/3$-null, we obtain a factorization of f, $f'\colon P_{\Sigma B\mathbb{Z}/3}(C) \to (B\Sigma_3)^\wedge_3$ such that the following diagram is homotopy commutative

$$\begin{array}{ccc} B\mathcal{F} & = & B\mathcal{F} \\ \downarrow & & \downarrow{\scriptstyle r_3^\wedge} \\ P_{B\mathbb{Z}/3}(C) & \xrightarrow{f'} & (B\Sigma_3)^\wedge_3. \end{array}$$

The strategy if to construct a homotopy inverse of f', $\bar{\Theta}\colon (B\Sigma_3)^\wedge_3 \to P_{B\mathbb{Z}/3}(C)^\wedge_3$, up to 3-completion, which fits in the previous diagram up to homotopy.

Since Σ_3 has an abelian normal 3-Sylow subgroup $\mathbb{Z}/3$, we have that $(B\mathbb{Z}/3)_{h\mathbb{Z}/2} \to B\Sigma_3$ is an equivalence. Consider the fibre sequence $BN \to BS \to B\mathbb{Z}/3$ and the map $r_3^\wedge|_{BS}\colon BS \to P_{B\mathbb{Z}/3}(C)^\wedge_3$, by Zabrodsky's lemma [Dwy96, Proposition 3.4], $r_3^\wedge|_{BS}$ factors (uniquely up to homotopy) via $\Theta'\colon B\mathbb{Z}/3 \to P_{B\mathbb{Z}/3}(C)^\wedge_3$. In order to get $\bar{\Theta}$, we only need check that Θ' is $\mathbb{Z}/2$-equivariant up to homotopy. For any $\mathcal{F}$ in the hypothesis of the proposition, note that there is an element in $\omega' \in \mathrm{Out}_{\mathcal{F}}(S)$ which project to $\omega \in \mathrm{Out}_{\Sigma_3}(\mathbb{Z}/3)$ (they are called η or ω in the tables [DRV07, Theorem 5.10]). Since Θ' is unique up to homotopy factoring $r_3^\wedge \circ \Theta$, and $\Theta \circ \omega' \simeq \Theta$, it follows $\omega \circ \Theta' \simeq \Theta'$.

Next we check that $\bar{\Theta}$ is a homotopy inverse to f. First consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} CW_{B\mathbb{Z}/3}(B\mathcal{F}) & & & & \\ \downarrow & & & & \\ B\mathcal{F} & = & B\mathcal{F} & = & B\mathcal{F} \\ \downarrow{\scriptstyle r} & & \downarrow & & \downarrow{\scriptstyle r_3^\wedge} \\ P_{\Sigma B\mathbb{Z}/3}(C) & \xrightarrow{f'} & (B\Sigma_3)^\wedge_3 & \xrightarrow{\bar{\Theta}} & P_{\Sigma B\mathbb{Z}/3}(C)^\wedge_3. \end{array}$$

Since 3-completion on the bottom line also gives a homotopy commutative diagram, unicity on Zabrodsky's lemma (see [Dwy96, Proposition 3.4]) shows that $\bar{\Theta} \circ f'$ is 3-completion. So $\bar{\Theta} \circ (f')^\wedge_3 \simeq id$.

Now consider the following homotopy commutative diagram

$$
\begin{array}{ccccc}
BK & \longrightarrow & F & & \\
\downarrow & & \downarrow & & \\
BS & \xrightarrow{\Theta} & B\mathcal{F} & = & B\mathcal{F} \\
\downarrow & & \downarrow & & \downarrow f \\
B\mathbb{Z}/3 & \xrightarrow{\iota} & (B\Sigma_3)^\wedge_3 & \xrightarrow{(f')^\wedge_3 \circ \bar{\Theta}} & (B\Sigma_3)^\wedge_3 .
\end{array}
$$

The map $(f')^\wedge_3 \circ \bar{\Theta}$ is determined by its restriction to the Sylow 3-subgroup $\mathbb{Z}/3$ by Proposition 7.3. Again $\iota \circ (f')^\wedge_3 \circ \bar{\Theta}$ and ι give homotopy commutative diagrams when placed in the bottom line, then unicity on Zabrodsky's lemma (see [Dwy96, Proposition 3.4]) shows that they are homotopic. □

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