# CONNECTIONS OF SEPARATRICES IN QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract. The goal of this article is to determine the maximum number of separatrix connections on phase portraits of quadratic differential systems. We prove that this number is exactly two when the singularities involved in the connections are topologically equivalent to elemental or semi-elemental singularities (except for two cases with three connections which imply the existence of centers). If the singularities are more degenerate then the number of connections is at least two, and we state the conjecture that this number is exactly two.

## 1. Introduction

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x, y), \qquad \frac{dy}{dt} = q(x, y), \tag{1}$$

where $p, q \in \mathbb{R}[x, y]$, i.e. $p$, $q$ are polynomials in $x$, $y$ over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

For about a century, the question of how many topologically different phase portraits of quadratic differential systems has been on the mind of many mathematicians. While the study of conic curves was completed two millennia ago, the study of $n$-dimensional quadrics was completed about two centuries ago, the study of $n$-dimensional linear differential systems was done for over one century, the problem of classifying 2-dimensional quadratic differential systems is still open. The global answer to this problem depends greatly on the solution to Hilbert's 16th problem, and until a definitive result on this last one, the problem of the number of quadratic phase portraits will persist. However one can reduce the problem to the case of phase portraits modulo limit cycles, that is, assuming every limit cycle and its interior collapsed to the focus inside.

If an answer for this problem is obtained, and if we had a bound $k$ ($H(2) \leq k$) for the number of limit cycles occurring in quadratic systems, then a bound for the number of phase portraits of quadratic differential systems could be obtained. But the plain multiplication of the number of phase portraits modulo limit cycles by the bound on the number of limit cycles is not yet a satisfying result because, on one side, not all phase portraits will admit the maximum of limit cycles. And on the other side, some portraits having several foci may have more that $k$ different phase portraits.

Great efforts were made to obtain phase portraits of quadratic systems. Using different approaches, many mathematicians have provided hundreds of phase portraits. Some looking for very degenerate systems or with a high level of instability. Others looking for the most generic cases. Still others starting from some specific property of the system like a center, a weak focus, or many others. The moment has arrived to try to collect all this work, organize it in a simple way, so that it becomes clear what has been done up to now, and to see what remains to do. Moreover, we will add here a new approach so as to cover a so far unexplored direction of study.

In [9] the authors gave a complete geometrical classification of singularities (finite and infinite) of quadratic differential systems. There are 1764 different such configurations (one less than it is mentioned in [9] due to



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a counting error) plus some conjectured empty. Since some of the geometric properties used to distinguish these configurations are not of topological nature, the number of topologically different configurations of singularities is much lower. This number is just 207 (the case (192) is declared empty because it is equivalent to (137)) and can be found in [8, 10].

We must recall now the concept of codimension. This concept was studied in [10] and defined for different equivalence relations applied to mathematical objects occurring in differential systems. Quadratic systems depend on 12 parameters but by means of affine changes of variables $x$ and $y$ and time rescaling one can write any quadratic system in just 5 parameters. This may induce us to think that quadratic systems could have levels of codimension from 0 (the most generic ones also known as the structurally stable ones) up to codimension 4. However, this is not true because we do not have a global chart for the moduli space for the quadratic differential systems modulo the group action. To be more explicit, it is not true that all quadratic differential systems can be written in a single normal form with 5 parameters. One needs to start with some initial condition, like for example a singular point of some specific kind, then we move the point to a fix position by using the group action and start reducing the number of parameters. Then these different starting hypotheses forces that we can not reduce the study to a single copy in $\mathbb{R}^5$ but to several copies of it in order to cover all parameter space, and some of them depart from already non-generic situations. In book [9, Chapter 6] the reader can find 28 normal forms for quadratic systems according to the number and multiplicity of the finite singularities. Most of them depend on 5 or 6 parameters, and some of them portray quite degenerate singularities, which justifies their bigger codimension. In [10] it is shown that the most degenerate topological configuration of singularities from [8] of a quadratic system has up to topological codimension 9 inside $\mathbb{R}^{12}$.

We are trying to classify the set of phase portraits of quadratic systems under the topological equivalence relation modulo limit cycles and this involves subtilities that needed a concept of codimension adapted for this purpose [10].

In [10] we give the codimension of each one of the 207 topological configurations of singularities from [8]. The simplest cases are those configurations in which all singularities (finite and infinite) are elemental ones (and hyperbolic), that is, saddles or anti-saddles (nodes or foci). These are the configurations of codimension 0. Then we have those which have exactly one saddle-node (finite or infinite which corresponds to the codimension 1). The codimension 2 configurations are also easy to grasp since they are those which either have 2 saddle-nodes, or a cusp singularity or an infinite nilpotent singularity of multiplicity 3. Configurations of singularities with higher codimension complete the rest of possibilities included the ones having an infinite number of singularities. But codimension of a phase portrait is not just the codimension of its configuration of singularities. Notice that by the reduction already done from geometrical configurations to topological ones, the triple semi-elemental nodes or saddles, or the quadruple semi-elemental saddle–node and others, have already been reduced to topologically equivalent singularities of lower multiplicity. Even infinite intricate singularities of multiplicity 7 whose neighborhoods are formed just by parabolic sectors are topologically equivalent to simple nodes of codimension 0.

The case of configurations with centers is trickier as explained in [10] and it becomes simpler to denote the phase portraits with centers according to the name they received in [41] instead of trying to assign a codimension to them.

Gathering all this information one produces Table 1 in which in the first column we list all the codimensions of the different topological configurations of singularities Now we consider the global codimension of the phase portrait and we do it modulo limit cycles. So, the codimension produced by weak foci or double limit cycles is not considered according to this criterium. We only add to the codimension provided by the singularities, the codimension provided by the number of separatrix connections. In every box we detail the known facts on every case. For example, in the cell $(0,0)$ we have the 44 structurally stable phase portraits found in [2]. In the cell $(0,1)$ we have the 61 phase portraits of the class (D), which have a stable configuration of singularities and a separatrix connection. In the cell $(1,1)$ we have the 141 phase portraits of class (A) (respectively (B);

| CCS | Global codimension of Phase Portraits | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **0** | 44 | 61 | $\geq 5$ | ? | ? | ? | ? | ? | ? | ? |
| **1** | x | 141 | $\geq 93$ | $\geq 0$ | ? | ? | ? | ? | ? | ? |
| **2** | x | x | $\geq 182$ | $\geq 134$ | $\geq 12$ | ? | ? | ? | ? | ? |
| **3** | x | x | x | $\geq 129$ | 68 | 4 | ? | ? | ? | ? |
| **4** | x | x | x | x | 54 | 11 | 0 | 0 | 0 | 0 |
| **5** | x | x | x | x | x | 31 | 1 | 0 | 0 | 0 |
| **6** | x | x | x | x | x | x | 19 | 0 | 0 | 0 |
| **7** | x | x | x | x | x | x | x | 8 | 0 | 0 |
| **8** | x | x | x | x | x | x | x | x | 3 | 0 |
| **9** | x | x | x | x | x | x | x | x | x | 1 |

Table 1. Number of phase portraits according codimension

(C)) which have a single finite saddle-node (respectively an infinite saddle-node of the type $\binom{0}{2}SN$; an infinite saddle-node of the type $\binom{1}{1}SN$). All together they produce the 202 phase portraits of structurally unstable phase portraits of codimension 1 found in [3] (the number has been reduced by 2 from the 204 claimed there after further studies which detected two errors).

At codimension of singularities 6 we find the phase portraits with a multiplicity 7 singularity plus some degenerate systems and some phase portraits with codimension of singularities lower than 6 but with some separatrix connections. Then, on the row 0 we have first the structurally stable systems already known and next come the cases where there is an instability produced by a single separatrix connection, and this number is also already known. Next there follow the cases where with the same sets of singularities, the phase portraits have either 2 or 3 or more connections of separatrices (if such numbers are possible). The boxes under the diagonal are all labeled with "x" since the codimension of the phase portrait cannot be lower than the codimension already provided by the singularities. Some boxes are indicated to have some known value which can be found with a thorough check of bibliography (without considering the conjectured impossible cases) and some boxes above the diagonal are filled with questions marks "?" since these are the cases that in this paper we want to prove are not realizable. We want to call the main diagonal of this table as Diagonal 0 since this is the diagonal where we will find the phase portraits with no separatrix connections. In this same way, we will call Diagonal 1 the one above Diagonal 0 and so on.

It is interesting to notice the different approaches that research has taken on this subject. Most of the classical works (Reyn, Jager, Vulpe/centers) has been done following the Diagonal 0 from down to up, starting with less generic situations and stopping when the number of parameters was too high to be able to allow a reasonable study. The recent works [2] and [3] on structural stability have taken a left-up to right-down orientation. They have looked for structurally stable quadratic systems (modulo limit cycles) and for structurally unstable systems of codimension 1 whether this codimension comes from coalescence of singularities, or from separatrix connections. Work on codimension 2 is in progress. Finally the recent work [8] based on the book [9] opens the direction from left down to right up for any row, that is, once all topologically distinct configurations of singularities are known, it becomes easier to find all possible phase portraits with those configurations and with no separatrix connections. Thus the Diagonal 0 would be complete so as to go to Diagonal 1.

In this paper we initiate work in a new direction so as to fill up all the unknown boxes, from upper-right to Diagonal 3. And the way to do it, is studying which are the possibilities to have several connections of separatrices.

The main result of this paper is that only the cases in three diagonals (0, 1 and 2) of Table 1 need to be studied for the .

In this paper we will also prove some incompatibilities between simultaneous presence of two specific types of separatrix connections in a phase portrait of a system in **QS**.

Since the main theorem (Theorem 1*) is based on several results and needs some notations later introduced, we have put it at the end of Section 2.

## 2. Our results

In this paper we look for connections of separatrices. We want to know what is the maximum number of connections for the family of quadratic systems.

We want to study the cases in which there are more that one separatrix connections. This goal must be split in two parts. We can first study all the cases in which all infinite singularities have at most one affine separatrix on one side of the local chart. This discards the cases with some specific infinite nilpotent or intricate singularities, more precisely, the configurations of infinite singularities 23–29 and 37–40 out of 46 ([9, Chapter 6, Figure 6.1]) and some with the infinite line filled up with singularities.

The second part includes these cases which being quite degenerate have lower numbers of their singularities and separatrices. An infinite singularity of a quadratic system can have at most two affine separatrices on the same side of a local chart (this can be deduced from the already mentioned Figure 6.1 of [9] or from diagrams in [8]).

In [3] the authors already started the study of phase portraits having a single connection of separatrices and stated that there are five types:

(a) Heteroclinic orbit between two finite singularities;
(b) Homoclinic orbit (involves two separatrices of the same finite singularity);
(c) Heteroclinic orbit between a finite and an infinite singularity;
(d) Heteroclinic orbit between an infinite singularity and its opposite;
(e) Heteroclinic orbit between an infinite singularity and another infinite singularity different from its opposite, in fact an adjacent one because if there is another singularity between them, the stabilities of the affine separatrices of the two infinite singularities that we want to connect, would be the same.

Each one of these singularities must have at least one separatrix. Generically they will be saddles. In fact, in [3] the authors talked about saddles instead of singularities having separatrices since their object of study was on codimension 1.

Originally the $(b)$ connection was considered with the singularity being in the affine plane, but now we can also think of "loops" with the singularity at infinity. This last case may only happen when an infinite singularity has two affine separatrices on the same side of the line at infinity. We point out that in [11] this type of connection was already presented and denoted as $(b)_\infty$.

So, it is trivial to see that if a phase portraits has 2 separatrix connections, the possibilities are:

$$\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,d),(c,e),(d,d),(d,e),(e,e)\}.$$

And if a phase portrait has 3 separatrix connections, the possibilities are:

$$\{(a,a,a),(a,a,b),(a,a,c),(a,a,d),(a,a,e),(a,b,b),(a,b,c),(a,b,d),\ (a,b,e),(a,c,c),(a,c,d),(a,c,e),$$
$$(a,d,d),(a,d,e),(a,e,e),(b,b,b),(b,b,c),(b,b,d),(b,b,e),(b,c,c),(b,c,d),\ (b,c,e),(b,d,d),(b,d,e),$$
$$(b,e,e),(c,c,c),(c,c,d),(c,c,e),(c,d,d),(c,d,e),(c,e,e),(d,d,d),(d,d,e),(e,e,e)\}.$$

However, we will see that all but one of the possibilities with 3 connection are not realizable, and neither are some with 2 connections.

Notice that the cases with 3 separatrix connections imply the existence of at least six separatrices, and the number of configurations of singularities which have such number of separatrices (or larger) is small.

For the case of finite nilpotent singularities, one can easily check [27] where all the systems with a nilpotent finite singularity are studied, in order to realize that there are no examples with more than two separatrix connections. For the two phase portraits found missing in [27] (see [11]) this statement also holds. The case of intricate finite singularities is even easier since there are only seven different phase portraits in the Poincaré disc.

On the other hand in the case of an infinite nilpotent or an intricate singularity, one can go through [11] to see that the phase portraits presented there have each at most two separatrix connections.

In order to discard many cases we will make an important use of the classification done by Zegeling [45] of the systems with 3 finite saddles plus one anti-saddle. This forces the existence of 3 pairs of infinite nodes. Even though the classification was done using old criteria and does not really specify the exact number of phase portraits, from Zegeling's work we see that his set of singularities produces the topological configurations (1) $s, s, s, a; N, N, N$ or (2) $s, s, s, c; N, N, N$ from [8]. The second case has 4 possible phase portraits which are from $Vul_8$ to $Vul_{11}$ (see [41]) and the first case has 13 possible phase portraits: 4 of them structurally stable (from $\mathbb{S}^2_{7,1}$ to $\mathbb{S}^2_{7,4}$ of [2]), 6 of codimension 1 (from $U^1_{D,15}$ to $U^1_{D,20}$ of [3]) and 3 of codimension 2, to know, pictures $b$, $c^a_1$ and $c^a_2$ of [45]. This set of phase portraits includes one case with three heteroclinic connections which sit on invariant straight lines passing through finite singularities. The existence of such trio of connections implies that the anti-saddle must be a center giving phase portrait $Vul_{10}$. And we will prove that there is just only one more possibility of an $(a, a, a)$ connection (with not all connections being straight lines) and this also forces the existence of a center.

In [10] a new notation for quadratic systems was introduced. This notation has already been used in [11] to give names to phase portraits with a nilpotent or intricate singularity at infinity, later used also in [13].

The correspondence between the notation of Zegeling and the two newer ones are:

(1) Phase portrait $d$ corresponds to structurally stable $\mathbb{S}^2_{7,1} \rightarrow QS1^{(0)}_1$;
(2) Phase portrait $e^d$ corresponds to structurally stable $\mathbb{S}^2_{7,2} \rightarrow QS1^{(0)}_2$;
(3) Phase portrait $e^c$ corresponds to structurally stable $\mathbb{S}^2_{7,3} \rightarrow QS1^{(0)}_3$;
(4) Phase portrait $e^b$ corresponds to structurally stable $\mathbb{S}^2_{7,4} \rightarrow QS1^{(0)}_4$;
(5) Phase portrait $c_3b$ corresponds to structurally unstable of codimension 1 $U^1_{D,15} \rightarrow QS1^{(1)}_1$;
(6) Phase portrait $c^b_1$ corresponds to structurally unstable of codimension 1 $U^1_{D,16} \rightarrow QS1^{(1)}_2$;
(7) Phase portrait $e^e$ corresponds to structurally unstable of codimension 1 $U^1_{D,17} \rightarrow QS1^{(1)}_3$;
(8) Phase portrait $c^b_2$ corresponds to structurally unstable of codimension 1 $U^1_{D,18} \rightarrow QS1^{(1)}_4$;
(9) Phase portrait $e^a$ can be completed with a stable focus inside the graphic and then it corresponds to the structurally unstable of codimension 1 $U^1_{D,19} \rightarrow QS1^{(1)}_5$, or with an unstable focus inside the graphic and then it corresponds to the structurally unstable of codimension 1 $U^1_{D,20} \rightarrow QS1^{(1)}_6$ or with a center and then it corresponds to $Vul_{11} \rightarrow QS2_1$;
(10) Phase portrait $b$ is a codimension 2 phase portrait of the not yet completed class (DD) but since the class of phase portraits with the topological configuration (1) $s, s, s, a; N, N, N$ has been completely studied, we may denote it by $QS1^{(2)}_1$;
(11) Phase portrait $c^a_1$ can be completed with a focus (by symmetry its stability does not matter) and then it is a codimension 2 phase portrait not yet classified in class (DD) (we can already denote it by $QS1^{(2)}_2$), or with a center and then it corresponds to $Vul_8 \rightarrow QS2_2$;

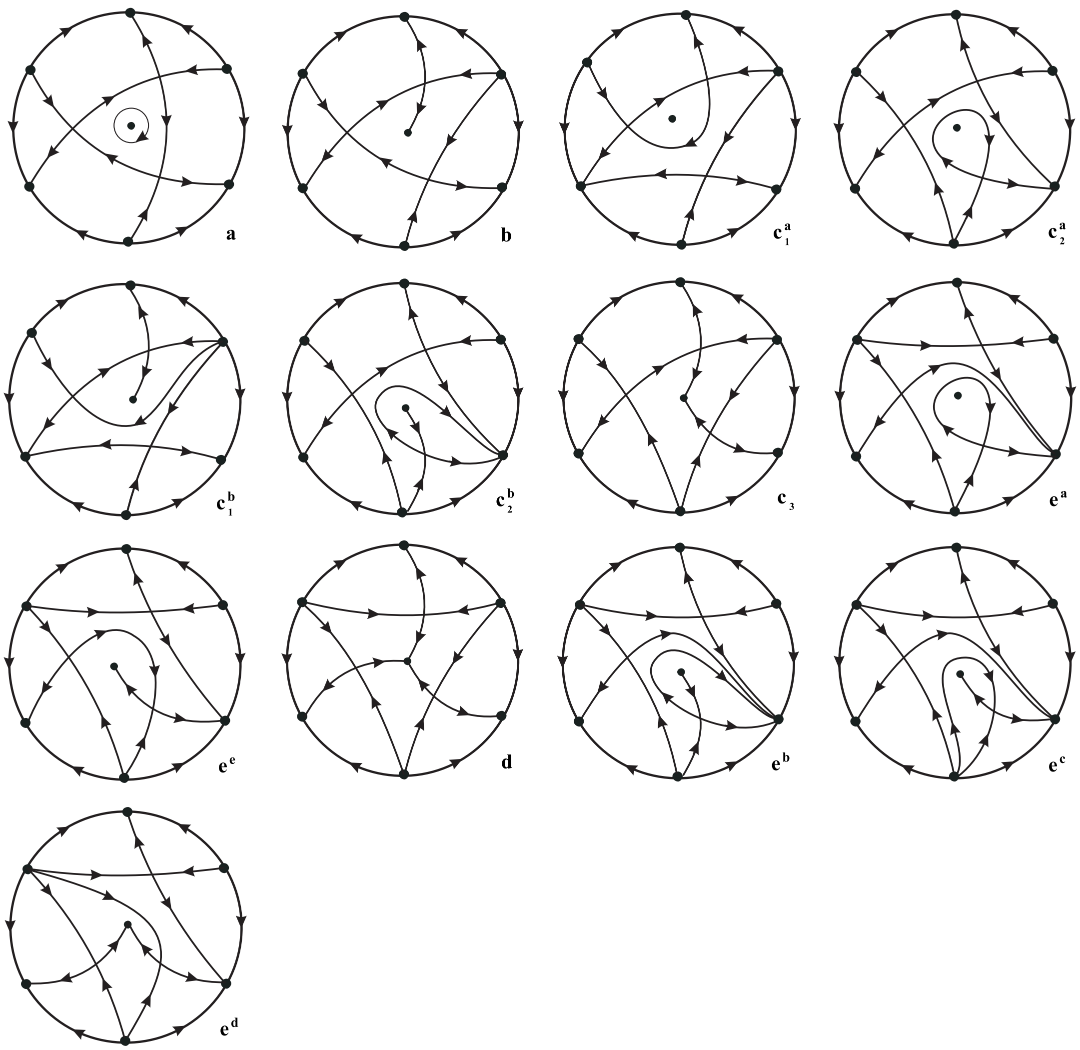



Figure 1. Phase portraits with three saddles and one anti-saddle.

(12) Phase portrait $c_2^a$ can be completed with a focus (by symmetry its stability does not matter) and then it is a codimension 2 phase portrait not yet classified in class (DD) (we can already denote it by $QS1_3^{(2)}$), or with a center and then it corresponds to $Vul_9 \rightarrow QS2_3$;

(13) Phase portrait $a$ always has a center and it corresponds to $Vul_{10} \rightarrow QS2_4$.

We add Figure 2, so as to complete the phase portraits with 3 saddles and one antisaddle that Zegeling left incomplete. That is, we change the phase portraits described in the above points 9), 11) and 12) by the complete ones with the definitive notation.

So, our first proposition will simply be:

**Proposition 1.** *If a quadratic system has 3 finite saddles and one anti-saddle, then its phase portrait corresponds to one of 17 possibilities described above and presented in* Figures 1 *and 2, 13 of them from class (1) with a focus or node and 4 from class (2) with a center.*

This completes the classification of the phase portraits determined by the configuration of singularities (1) $s, s, s, a; N, N, N$ and (2) $s, s, s, c; N, N, N$.

We observe that from 13 phase portraits corresponding to the topological configuration (1) we can obtain easily the phase portraits corresponding to the topological configuration (28) $s, s, sn; N, N, N$. More concretely we prove the next

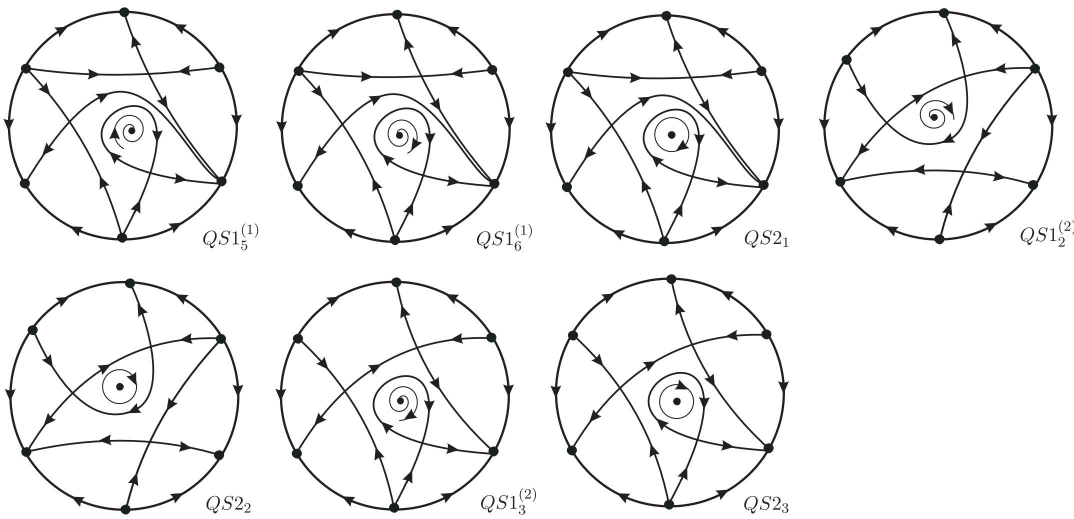



FIGURE 2. Phase portraits with three saddles and one anti-saddle.??

**Proposition 2.** *If a quadratic system has 2 finite saddles and one finite saddle-node, then its phase portrait corresponds to one of 11 phase portraits presented* FIGURE *3.*

*Proof.* We start by producing all the *potential* phase portraits of this class. By potential phase portrait we understand a phase portrait which is compatible with the corresponding global topological configuration of singularities. So we simply start from the 9 cases of FIGURE 1 on which the finite anti-saddle receives at least

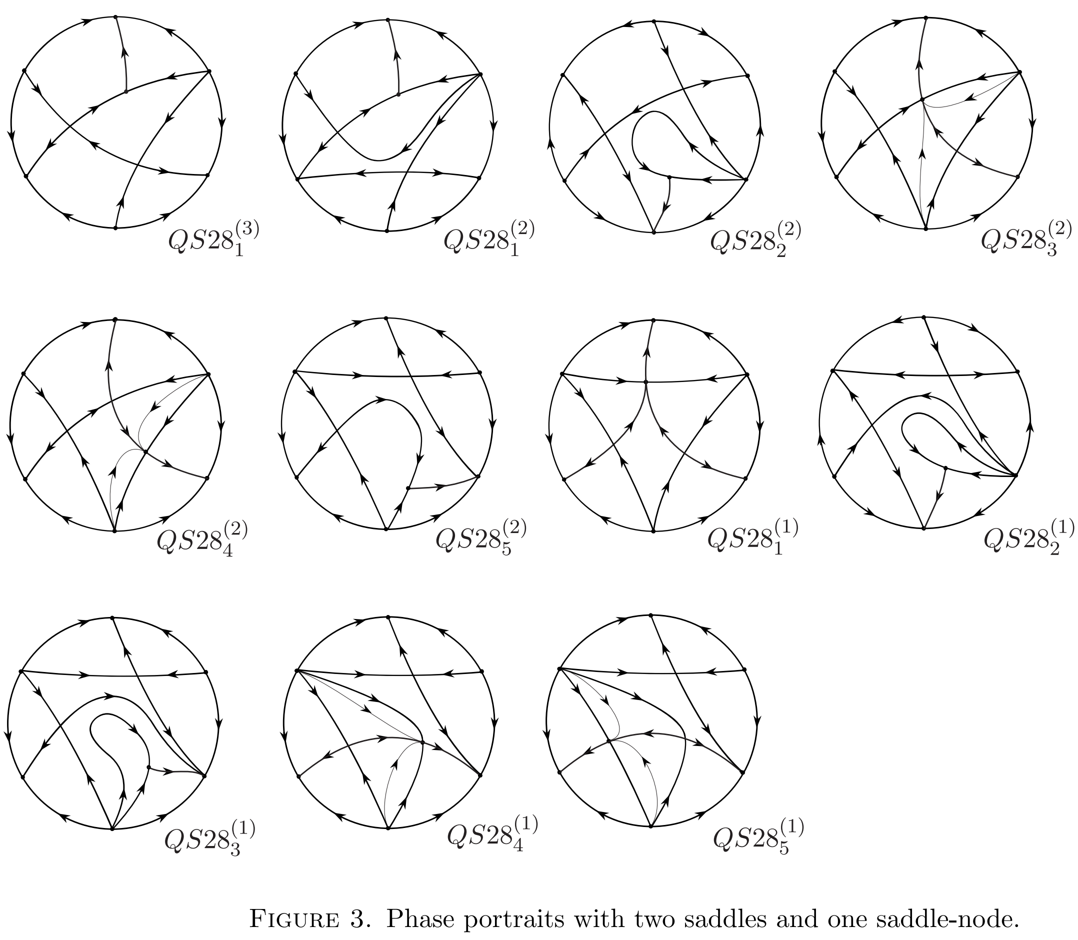



FIGURE 3. Phase portraits with two saddles and one saddle-node.

one separatrix and produce the coalescence of the finite anti-saddle with a saddle. Some of the cases offer just one possibility, others offer two different possibilities, one offers three possibilities which are all the same by symmetry. All of them are presented in FIGURE 3. Then it remains to provide an example for each of them or prove that some could not be realizable. However in this case, all are realizable and we provide the examples in the web page ?????. □

Other important results found in many references are:

**Lemma 1** ([44])**.** *Any straight line has at most two contact points (taking into account the multiplicity of the points) with orbits of a quadratic system unless the straight line itself is a union of trajectories. In case there are two finite saddles on that line, the infinite singularities must be nodes.*

**Lemma 2.** *Inside a limit cycle of a quadratic system, there is always a unique singularity which must be a focus. Inside a graphic it must al most a unique singularity which can be a focus or a center.*

We will also need Lemma 3.20 from [3], but since there may be a misunderstanding in its formulation, we reformulate it here more clearly.

**Lemma 3.** *Let $X$ be a quadratic system. Let $P$ be an elemental or semi-elemental infinite singular point having separatrices. If $P$ is a saddle (or a saddle–node with first eigenvalue equal to zero), the eigenvector directed to the affine plane, determines the direction of a straight line and both affine separatrices are at the same side of this line considered from the point of view of the local chart at infinity. From the point of view of the Poincaré disc, both separatrices are on opposite sides.*

Now we may start producing results that will restrict the possibilities for having some combinations of separatrix connections.

**Proposition 3.** *If a quadratic system has a separatrix connection $(d)$ from an infinite singularity to the opposite singularity, then the connection is an invariant straight line without singularities and the system has real finite singular points with total multiplicity at most 2. Moreover, the system has either two complex finite singularities, or two finite singularities have coalesced with the infinite singularity on the $(d)$ connection.*

*Proof.* The fact that a connection of separatrices of type (d) must be an invariant straight line is a known fact already proven in many papers (see for example [2, Corollary 4.6]) and it cannot have finite singularities on it by simple definition of what $(d)$ means.

Then, such a system can be written by means of an affine change of variables into:

$$x' = x(a + bx + cy), \qquad y' = q_2(x, y), \tag{2}$$

and this system clearly can have singularities outside the axis $x = 0$ of total multiplicity at most 2. Moreover, the other two singularities must be on the line $x = 0$. Thus, either they have complex $y$ coordinate, or they have gone both to infinite line on the direction $x = 0$ coalescing with the infinite singularity. □

**Proposition 4.** *A quadratic system cannot have two loops having the same saddle singularity.*

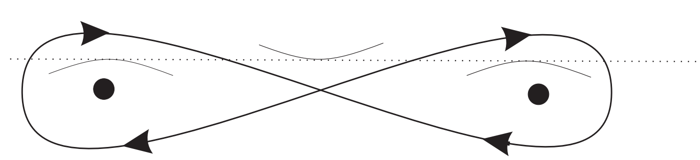


FIGURE 4. Impossibility of two loops in a same saddle

*Proof.* Clearly one can draw a line crossing both loops (no need to touch any singularity), and this forces immediately 3 contact points (see FIGURE 4). □

**Proposition 5.** *If a polynomial differential system has a multiple singularity where one of its local separatrices turns to be a separatrix connection with an elemental saddle, then there exists a bifurcation of this system that produces a saddle point from the multiple point keeping the separatrix connection.*

*Proof.* An isolated multiple singularity (finite or infinite) may always split in elemental singularities by means of an adequate perturbation. If there is a separatrix connection, this means that there must be at least one hyperbolic sector in the neighborhood of the singularity, and this implies that a saddle could bifurcate from the multiple singularity.

After that, a rotated vector field can be used as a complementary change to the perturbation so as to restore the separatrix connection (if needed).

This argument works if the multiple singularity is finite. It also works when the singularity is infinite and the saddle that must keep the connection becomes finite, because then a rotated vector field does not move the finite singularities and the connection may easily be recovered.

But when the multiple singularity is infinite, and the only saddle that may split from it is an infinite one, then the separatrix connection must end on an infinite saddle (the simplest of this situation is a $\binom{0}{2}SN$), then the argument that a rotated vector field will recover the separatrix connection is not convincing. The reason is that a rotated vector field does affect the position and even the existence of the real infinite singularities. It could perfectly happen that by means of the rotated vector field, the recently split singularities would join again (and even become complex) before one can guarantee that the separatrix connection has been recovered. In some cases, one may be lucky and the rotated vector field will separate the infinite singularities while getting closer to recovering the separatrix connection. But in other cases the previous situation occurs and the infinite points get closer. However this problem has been solved in [1] where the author needed to prove the existence of several phase portraits with a finite saddle-node and a separatrix connection. The algorithm is the following: Except in some cases in which the separatrix connection takes place along an algebraic invariant curve (and then one can make the perturbation while maintaining the connection), in the remaining cases, the conditions for the connection cannot be determined algebraically. Then one does not know for sure which is the exact parameter value $\alpha^*$ for which the connection is achieved. One starts with a parameter $\alpha_1$ such that $|\alpha_1 - \alpha^*| << 1$ which means that we do not have in fact the separatrix connection, but we are very close to having it, and depending on the sign of $\alpha_1 - \alpha^*$ the relative position of the separatrices will be different. Then one makes the perturbation to split the infinite singular point but maintaining the relative position of the separatrices unchanged. Next one needs to check in which sense the vector field must be rotated so that the separatrices get closer and produce the connection. Next one checks if this sense of rotation moves the recently obtained infinite singularities closer or not. If they enlarge the distance between them, we will obtain the desired separatrix connection and we are done. But if they get closer, all we must do is to start again with a different parameter value $\alpha_2$ so that $(\alpha_1 - \alpha^*)(\alpha_2 - \alpha^*) < 0$. Now the relative position of the separatrices close to produce the connection will be the opposite than the one before. Then doing again the perturbation to split the infinite singularity, now the sense of rotation of the rotated vector field needed to produce the separatrix connection will at the same time enlarge the distance between the recently separated singularities and we will obtain the desired separatrix connection with elemental singularities.

□

**Remark 1.** *The above proposition allows us to restrict our coming proofs to systems where one of the singularities involved in the separatrix connections is an elemental saddle, except for the cases in which more than one separatrix connection involves the same singularity.*

Some other results related to singularities and indices of polynomial differential systems are summarized in the next proposition:

**Proposition 6.** *The following statements hold for a polynomial differential system $X$ of degree $n$ (some of the results are even true for analytic systems):*

- *a) $X$ has at most $n^2$ finite simple singularities (or finite singularities of total multiplicity $n^2$);*
- *b) $X$ has at most $n+1$ couples of infinite simple singularities;*
- *c) Counting just one representative of each couple of infinite singularities, $X$ has singularities of total multiplicity at most $n^2+n+1$;*
- *d) A simple saddle has index $-1$;*
- *e) A simple anti-saddle (focus, node, center) has index $+1$;*
- *f) On the Poincaré sphere, that is, counting twice all finite singularities, and all infinite singularities, the sum of all indices of singularities of $X$ is the Euler Characteristic of the surface, that is $+2$;*
- *g) On the Poincaré disk, that is, counting once all finite singularities, and one representative of each couple of infinite singularities, the sum of all indices of singularities of $X$ is $+1$;*
- *h) Due to g), $X$ cannot have more than $(n^2+n)/2$ saddles (for quadratic systems, the maximum is then $3$);*
- *i) Furthermore, $X$ cannot have more than $(n^2+n)/2$ singularities having local separatrices;*
- *j) A singularity has index whose absolute value is lower than or equal to the square root of its multiplicity. Together with statement c), this says that any singularity of $X$ has index between $-n$ and $n$;*
- *k) A singularity has index of the same parity as its multiplicity;*
- *l) $X$ can have at most $n$ pairs of infinite saddles, and also, at most $n$ pairs of infinite singularities having separatrices;*
- *m) Unless $X$ is degenerate, there cannot be more than $n$ finite singularities on a straight line, and if there are $n$, the line is invariant for the system;*
- *n) If we describe the flow on the line at infinity at any regular point of $X$ as clockwise or counterclockwise, the flow on two opposite regular points is the same (respectively opposite) if $n$ is odd (respectively if $n$ is even).*

Since all statements (except for $l$) are well known results, the only statement that needs a proof is $l$).

*Proof.* Assume that a system $X$ of degree $n$ has exactly $n+1$ infinite elemental saddles. So, the infinity brings a negative $-n-1$ index. The finite part must have total index $n+2$. The system must have finite singularities (real or complex) of total multiplicity $n^2$. If all finite singularities have coalesced into a single singularity, the maximum index would be $n$ by statement $j$) and any perturbation of that singularity will have total finite index equal to $n$. Thus, the required index $n+2$ is not achievable.

For the second part of the statement, if we had a combination of multiple infinite singularities (for example saddles-nodes formed by coalescence of a finite node and an infinite saddle or vice versa) so that we have $n+1$ pairs of infinite singularities and all of them would have at least one separatrix, by means of a perturbation we could eject all the nodes from infinity and leave $n+1$ saddles at infinity. □

The previous proposition works for systems of any degree, and we will apply it here to quadratic systems.

2.1. **Case of singularities topologically equivalent to elemental or semi-elemental singularities.** We will consider first the case in which all the singularities involved in a separatrix connection are finite elemental saddles, or saddles-nodes, or infinite singularities which have at most one affine separatrix on each side of the infinite line. Since we will need to keep talking about these singularities, let us call any of them for abbreviating as *AS-saddle*. In the following statements of results of this subsection, we subsume that the singularities involved in the separatrix connections are topologically equivalent to elemental or semi-elemental singularities.

**Proposition 7.** *If a phase portrait of a quadratic system has the configuration of separatrices* $(c)$ *and this is a straight semi-line, then the number of singularities (counted with multiplicity) in the other closed semi-line (that is, including the finite singularity and the infinite one) must be exactly* 3 *and there cannot be an AS-saddle in this open semi-line.*

*Proof.* By Proposition 5 we may assume that the two singularities forming the separatrix connection are elemental saddles. This already forces the existence of another singularity in the other semi-line. But this singularity cannot be another saddle since this would produce too many contact points along a parallel line to the invariant line as can be seen in Figure 5(a). In fact, if we do not force the existence of the second singularity below the finite saddle, we see that the vector field restricted to the two parallel dotted lines in Figure 5(b) below the finite singularity must be directed inside the region delimited by these lines. This implies that either there must be an attractive node, or the finite saddle is in fact a saddle-node, or the infinite singularity is in fact a $\binom{1}{1}SN$.

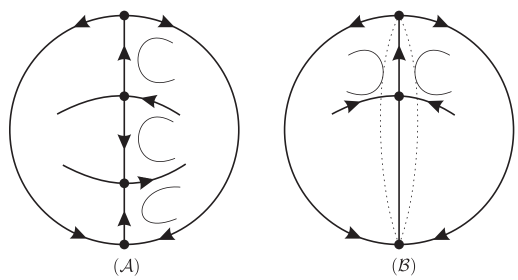



Figure 5. Proof of proposition 7

□

Now we start proving the non-existence of some couples of separatrix connections.

**Proposition 8.** *The configurations of separatrices* $(a, d)$,$(c, d)$, $(d, d)$ *and* $(d, e)$ *are not realizable.*

*Proof.* Clearly $(d, d)$ in not realizable since such separatrix connections would have different slopes and they would intersect at a finite singularity and thus we would no longer have a connection of type $(d)$. Remember that we only consider AS-saddles, and thus they cannot be intricate singularities with more that one separatrix on the same side of the infinite line.

Configuration $(a, d)$ implies the existence of two finite AS-saddles and since two is the maximum total multiplicity of finite singularities (by Proposition 3) they must be elemental saddles, so the index at infinity must be $+3$. Since we already have an AS-saddle at infinity, this is impossible.

The configuration $(c, d)$ is produced with two couples of infinite AS-saddles (one couple for the (d) connection and one AS-saddle for the (c)) together with a finite AS-saddle. Then, the infinity has two couples of AS-saddles and the finite part has one AS-saddle plus at most another singularity which if it exists must be an anti-saddle. So, the total index in the finite part is either $-1$ or 0. The other two finite singularities, be they complex or coalesced with an infinite singularity they produce the neutral index 0. Then, the three infinite singularities must produce total index at least $+1$ if the second finite singularity exists, or $+2$ if that second finite singularity has coalesced with an infinite one. But all these cases are impossible with two AS-saddles at infinity.

The configuration $(d, e)$ uses three infinite singularities with separatrices, and this is impossible by Proposition 6 $l$).

□

**Proposition 9.** *The configuration of separatrices* $(a, e)$ *is not realizable.*

*Proof.* Configuration $(e)$ needs 2 pairs of AS-saddles at infinity and configuration $(a)$ needs of two finite AS-saddles. This cannot happen in quadratic systems by Proposition 6(i). □

The rest of couples of separatrix connections are realizable as it can be checked in bibliography. We consider now triplets of connections. By extrapolation from the impossible couples, the next corollary follows immediately:

**Corollary 1.** *The configurations of separatrices* $(a, a, d)$, $(a, b, d)$, $(a, c, d)$, $(a, d, d)$, $(a, d, e)$, $(b, c, d)$, $(b, d, d)$, $(b, d, e)$, $(c, c, d)$, $(c, d, d)$, $(c, d, e)$, $(d, d, d)$, $(d, d, e)$, $(d, e, e)$, $(a, a, e)$, $(a, b, e)$, $(a, c, e)$ *and* $(a, e, e)$ *are not realizable.*

**Proposition 10.** *If a phase portrait has the configuration of separatrices* $(b, c)$*, then the saddle producing the loop* $(b)$ *is distinct from one producing the connection* $(c)$*.*

*Proof.* Assume the contrary. We have at least an infinite AS-saddle plus a finite saddle producing a loop and a focus inside it and both are connected with a single separatrix. Take the straight line passing through the infinite AS-saddle and the focus. This cannot be an invariant straight line, so, it already has the maximum of two contact points. Then part of the flow must look like Figure 6($\mathcal{A}$). But there is a third unavoidable contact point on the line, in the segment from the infinite AS-saddle producing the connection and the loop. Notice that if the finite AS-saddle is a saddle-node, the result is the same, since its three separatrices are not affected by the nodal sector. And even if the infinite AS-saddle is a $\binom{0}{2}SN$, then the flow must arrive to the infinite saddle-node tangent to the line at infinity (see [9]), and thus the extra contact point appears the same (see Figure 6($\mathcal{B}$)). □

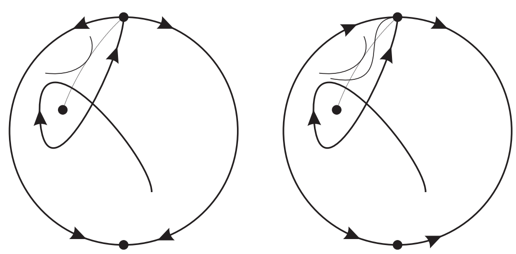



FIGURE 6. Proof of proposition 10

**Proposition 11.** *Assume a phase portrait has the configuration of separatrices* $(c, c)$*. Then if it involves two finite AS-saddles, it cannot have a third separatrix connection, and the only possible phase portrait is given in Figure 7($\mathcal{C}$) or the corresponding ones with any of the nodes coalescing with one of the possible saddles. If the configuration of separatrices involves only one finite AS-saddle, it must form the* $(c)$ *connections with two non-opposite and moreover, adjacent infinite AS-saddles.*

*Proof.* The connections cannot involve two finite AS-saddles and two non opposite infinite AS-saddles since in that case the total number of singularities with separatrices would be greater than or equal to four (see Proposition 6 $(i)$). It can neither be one finite AS-saddle sending two local separatrices with different stabilities to a pair of opposite infinite AS-saddles since a straight line passing through these singularities would have too many contact points (see FIGURE 7($\mathcal{A}$) ).

Assume that there are two finite AS-saddles and a pair of opposite AS-saddles forming the connections. By Proposition 7, a connection $(c)$ cannot be part of a straight line, otherwise, the other connection would not be compatible.

So we draw a AS-finite saddle having a $(c)$ connection with an infinite AS-saddle, as a curved line. If we take the straight line $L$ joining these two singularities, this line cannot have more contact points, and so there

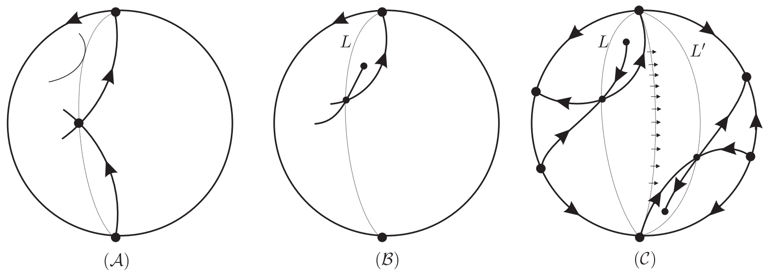



FIGURE 7. Proof of proposition 11

must be a separatrix of the AS-finite saddle (and a finite anti-saddle) in the region defined by $L$ and the connecting separatrix (see FIGURE 7($\mathcal{B}$)). There is also the possibility that the finite anti-saddle has coalesced with the saddle, so we have a saddle-node whose eigenvector related with the parabolic sector enters into the region mentioned above.

With two AS-finite saddles and one pair of infinite AS-saddles which generically have index $-1$, the total index forces the existence of singular points of total index $+4$, that is, generically four anti-saddles, two finite and two infinite (if any one of the AS-saddle is not an elementary saddle, it must have coalesced with some of the other mentioned singularities and the index argument holds). We have located already one of the finite anti-saddles. The second connection $(c)$ must take place on the same semi-plane (defined by $L$) of the first connection since the eigenvector of the infinite AS-saddle associated to these separatrices is the same. So, taking the line $L'$ passing through the singularities involved in the second connection, we also have located the second finite anti-saddle using the same argument as above for the first anti-saddle. Since we cannot put more finite singularities, the four remaining finite separatrices can only produce generically phase portrait in FIGURE 7($\mathcal{C}$) (with the possibility of one infinite node coalescing with infinite AS-saddle forming a saddle–node). And this phase portrait cannot have another separatrix connection.

The other possibility remaining is that only one finite saddle and two adjacent infinite AS-saddles are involved in the two connections. We will later check the compatibility of this (c,c) connection with other connections. □

**Theorem 1.** *There are no phase portraits with three separatrix connections except for the configuration of connections* $(a, a, a)$.

*Proof.* We have already discarded some of the possibilities in Corollary 1. Now we will discard the rest.

Configuration $(b, b)$ (two loops) is realizable, but we will now see that no other connection may occur in this case.

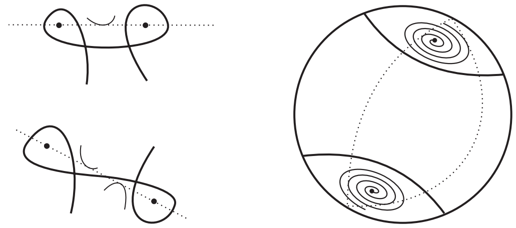
Clearly we cannot have $(b, b, b)$ (three loops) since this implies three saddles and three nodes.

In order to obtain $(a, b, b)$ with just four singularities, the saddles forming the two loops must also produce the $(a)$ connection. Then, taking the line that joins the two anti-saddles we will obtain more contact point than allowed (see FIGURE 8($\mathcal{A}$)).

In order to obtain $(b, b, c)$ with just four singularities, one of the saddles forming the loop must also produce the $(c)$ connection. This is already forbidden by Proposition 10.

The configuration $(b, b, d)$ is excluded by Proposition 3 since having $(d)$ implies no more than 2 finite singularities, and the two loops need four.

And finally the configuration $(b, b, e)$ is not allowed since the connection $(e)$ forms a graphic with the line at infinity that needs a focus inside, plus the two loops, this makes 5 singularities.



($\mathcal{A}$) ($\mathcal{B}$)

FIGURE 8. Proof of Theorem 1

The configuration of separatrices $(a, a, b)$ is not realizable since two connections between two pairs of finite saddles plus a loop need a minimum of three saddles plus the focus inside the loop. Then we apply Proposition 1 and we see that none of the phase portraits given in FIGURE 1 have all these three connections.

The configuration of separatrices $(e, e, e)$ is not realizable since it implies the existence of three pairs of AS-saddles at infinity.

The configurations of separatrices $(b, e, e)$ and $(c, e, e)$ imply the existence of 2 pairs of AS-saddles at infinity by the $(e, e)$ which form graphics and thus, two finite foci (or centers) exist one in each graphic. Therefore, we must have a third infinite singularity which is a node by Proposition 6(k). Now take any infinite point on one of these graphics and the opposite one and draw a straight line from one to the other. Consider also all the parallel lines to this one. Any line of this type must have a contact point inside each one of the graphics, and the set of straight lines produced, cover all plane. So, there cannot be other finite singularities, and thus, there cannot be other separatrix connections of the classes $(a)$, $(b)$ or $(c)$ since they all need at least one finite singularity (see FIGURE 8($\mathcal{B}$)).

In a similar way, neither the configuration $(c, c, e)$ is possible. Assume it is, then we must have again two graphics, one formed by the $(e)$ connection and another formed by the $(c, c)$ connections (see Proposition 11). The infinite singular points of one graphic are opposite to the infinite singularities of the the other graphic because we cannot have all three infinite singularities having separatrices. Then take a point from the infinite arc of one of the graphics, and draw the line passing through the finite AS-saddle up to the opposite point. This line has three contact points, one inside each graphic plus the finite singularity, and we arrive at a contradiction.

Consider the configuration of separatrices $(b, c, e)$. The configuration of separatrices $(c, e)$ implies the existence of two pairs of AS-saddles at infinity plus a finite AS-saddle to produce $(c)$. If we want to add a connection (b), this adds a saddle plus a focus inside the loop. Remember that the saddle forming the loop cannot be involved in the (c) connection by Proposition 10. Then the AS-saddle forming the (c) connection is a simple saddle. Both finite anti-saddles are inside graphics, therefore the only source and sink for the orbits outside the graphics are the two opposite infinite nodes (or the corresponding parabolic sectors of an infinite $\binom{0}{2}SN$). Then there are 5 finite separatrices plus one infinite (three attractive and three repelling ones) which must go to the infinite nodes (or saddle-nodes) and this is impossible, see FIGURE 9. So, configuration $(b, c, e)$ is not realizable.

FIGURE 9. Figure $(b, c, e)$

There are different topological ways to obtain configuration $(b, c, c)$ but we are going to prove that all them are not realizable. Assume first that each connection is produced by a different saddle so, there are 3 finite saddles. Then, we must have at least one pair of infinite AS-saddles, and this is not possible. So, we must reduce the number of finite saddles involved. We know by Proposition 10 that the saddle forming the loop cannot be involved in the connection $(c)$, so both connections $(c)$ must be produced by the same saddle, and there is a second finite saddle forming the loop. By Proposition 11, this must form a graphic with a finite saddle and two adjacent infinite AS-saddles which rises the total number of singularities with separatrices to four.

Configuration $(c, c, c)$ needs two couples of AS-saddles at infinity (there cannot be three). So, the maximum number of finite saddles is 1 (we are generically in family 12 of [2]). Then, the same finite saddle must produce all three connections and two of them must go to opposite infinite singularities. But this is not possible by Proposition 7.

Consider configuration $(a, c, c)$. From Proposition 11, the two (c) connections are formed either by two different finite AS-saddles, and then, there is no other possible connection, or are formed by a single finite AS-saddle and two adjacent infinite AS-saddles. In the second case, we already have 3 singularities with separatrices, and thus we cannot add an (a) connection.

Configuration $(a, a, c)$ implies the existence of at least two finite AS-saddles and one infinite AS-saddle, and this is the maximum number of singularities with separatrices one may have. So, both $(a)$ must be produced by the two finite saddles (thus generating a graphic having an invariant straight line and a focus or center inside) and one free separatrix of these saddles must form the $(c)$ connection with an infinite AS-saddle. The connection $(c)$ cannot be on an invariant straight line since the only possibility offered by Proposition 7 cannot be applied here. Assume that the infinite AS-saddle is a saddle. Now take the line passing through this infinite saddle and the focus inside the graphic and a new contact point is unavoidable (see FIGURE 10($\mathcal{A}$)). Assume now that the infinite AS-saddle is saddle-node. If it is a $\binom{1}{1}SN$ then both hyperbolic sectors must be separated by the separatrix connection and we arrive at the same contradiction given in FIGURE 10($\mathcal{A}$).

Suppose now that the infinite AS-saddle is a $\binom{0}{2}SN$. We claim that the hyperbolic sector must be on the left of the separatrix connection. Indeed, if it is on the right side, then between the infinite node $[1 : 0 : 0]$ and the saddle-node $[0 : 1 : 0]$ it is forced to be an infinite node and we get a contradiction, obtaining infinite singularities of total multipicity $\geq 4$ (see FIGURE 10($\mathcal{B}$)). So our claim is proved, i.e. the hyperbolic sector is on the left of the separatrix connection and we get the same contradiction as in the case when the infinite AS-saddle is a saddle.

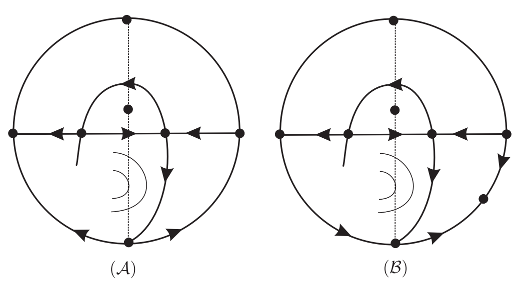


FIGURE 10

Consider now configuration $(a, b, c)$. The connection $(a)$ implies the existence of two finite AS-saddles and $(c)$ implies the existence of an infinite one, totaling the maximum allowable number of AS-saddles. So, we draw the loop, and one of the free separatrices of the AS-saddle must make the $(a)$ connection with the other finite AS-saddle. Let us start by locating the focus inside the loop on coordinates $(-1, 0)$ and the second AS-saddle on $(1, 0)$. Then the saddle forming the loop may be situated in any half-plane, and we may assume it is on $y < 0$ due to the reflection with respect to $y = 0$. We draw the loop and we also extend the right free separatrix up to the second saddle to produce the (a) connection. This separatrix cannot be tangent to $y = 0$ because that would produce a double contact point. Thus the other separatrix with the same eigenvector, stays in the half-plane $y > 0$. Then we need a second anti-saddle in the region delimited by the axis $y = 0$, and the (a) separatrix connection, otherwise we would have another contact point on $y = 0$ between the singularities. Then, the second pair of separatrices of the AS-saddle at $(1, 0)$ must be as in FIGURE 11($\mathcal{A}$). We have already assigned a direction to the separatrices. So we have already controlled all finite singularities. The second saddle and the second anti-saddle could form a saddle-node but this does not affect the argument. Now we must have an infinite saddle and two infinite nodes (or a $\binom{0}{2}SN$ plus a node). So we have two separatrices from the second saddle in the half-plane $y > 0$, and one of them must form the connection (c). Recall that the same saddle cannot be involved in a (b) and (c) connection by Proposition 10.

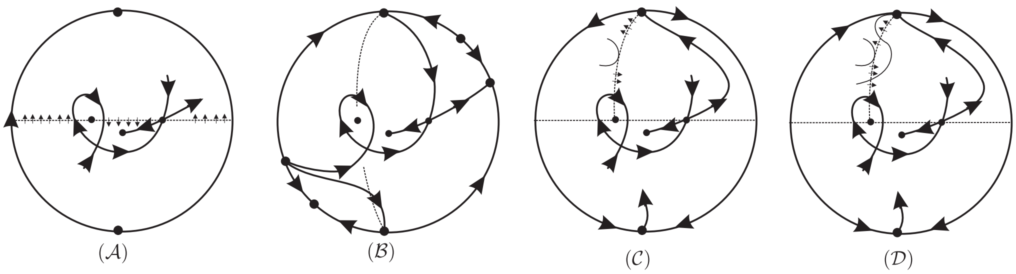



FIGURE 11

Assume first that the stable separatrix forms the (c) connection, then the other infinite nodes must be in the first quadrant and the phase portrait is the one given in FIGURE 11($\mathcal{B}$). Draw now a straight line from the infinite AS-saddle to its opposite passing through the focus inside the loop. When this line leaves the loop on the lower half-plane either does it through the saddle, left of it or right of it. In any case, this straight line produces 3 contact points.

Assume now that the unstable separatrix forms the (c) connection. If the infinite AS-saddle is a saddle then we directly obtain a contradiction using the same line as before since a contact point must appear between the infinite saddle and the loop (see FIGURE 11($\mathcal{C}$)). Suppose that the infinite AS-saddle is a $\binom{0}{2}SN$. Then we get the same contradiction because the orbits must arrive at the parabolic sector of this saddle-node tangent to infinite line (see FIGURE 11($\mathcal{D}$)).

Altogether we have proved the impossibility of the configuration (a,b,c).

It is trivial to see that the configuration (b,c,d) is impossible by combining the statement of Propositions 3 and 10.

So, since all cases have been discarded, the theorem is proved. □

**Theorem 2.** *The connections $(a, a, a)$ can produce just two phase portraits which are $Vul_{10}$ and $Vul_3$ from* [41] *(see* FIGURE *12).*

*Proof.* It is clear that three connections of type $(a)$ can be done by three straight lines with three saddles, or by a straight line and a conic (ellipse) as shown by the already known examples. We must simply prove that there is no other way to do it, and moreover, that the anti-saddles inside the graphics are always centers.

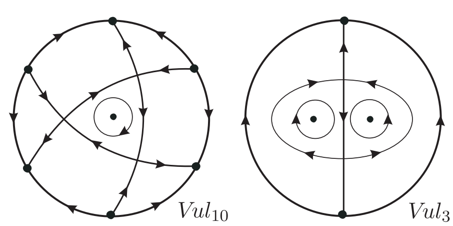



FIGURE 12. The connections $(a, a, a)$.

The case of three saddles is already done in [45] where all systems with three finite saddles and one anti-saddle are studied. There it is proved that having the three connections, the only possible phase portrait is $Vul_{10}$.

If we only make use of two finite saddles, then the connections must form two graphics. Moreover the connection in the middle must be a straight line since an heteroclinic finite graphic of a quadratic system must always be convex.

Assume that the two separatrix connections different from the straight line going from one saddle to the other form a closed set similar to the ellipse of $Vul_3$ but not necessarily being an algebraic curve. Assume also that both singularities inside the graphics are strong foci (it is not possible a focus and a center by [41]). The foci will then have some fixed stability which will not change under small perturbations of the coefficients.

Thus the system may be transformed by means of an affine change so that the invariant straight line is $x = 0$ and the saddles stay at $(0, \pm 1)$. So, the normal form is (with an additional time rescaling):

$$\frac{dx}{dt} = x(c + gx + ky), \qquad \frac{dy}{dt} = -1 + ex + lx^2 + mxy + y^2, \tag{3}$$

with $k < 0$ and $c < |k|$ and considering the assumption above we arrive at Figure 13($\mathcal{A}$)). Now perturb the system (3) into the following one:

$$\frac{dx}{dt} = x(c + gx + ky) + \alpha x + \beta x^2, \qquad \frac{dy}{dt} = -1 + ex + lx^2 + mxy + y^2. \tag{4}$$

This perturbation will not break the invariant straight line. Four different perturbations $\alpha > 0 = \beta$, $\alpha < 0 = \beta$, $\alpha = 0 < \beta$ and $\alpha = 0 > \beta$ will break the other two separatrix connections in four different ways, leaving the foci with the same stability if the perturbations are small enough (see Figure 13 from ($\mathcal{B}$) to ($\mathcal{E}$)). Then, in one of the perturbations we would get no limit cycle, in other two we would have a limit cycle around one or the other focus, and in another perturbation we would have a limit cycle around each focus (case ($\mathcal{C}$)). But this contradicts a known result which says that quadratic systems with an invariant straight line can have at most one limit cycle [23]. So, the assumption that we could have foci inside the graphics was false. □

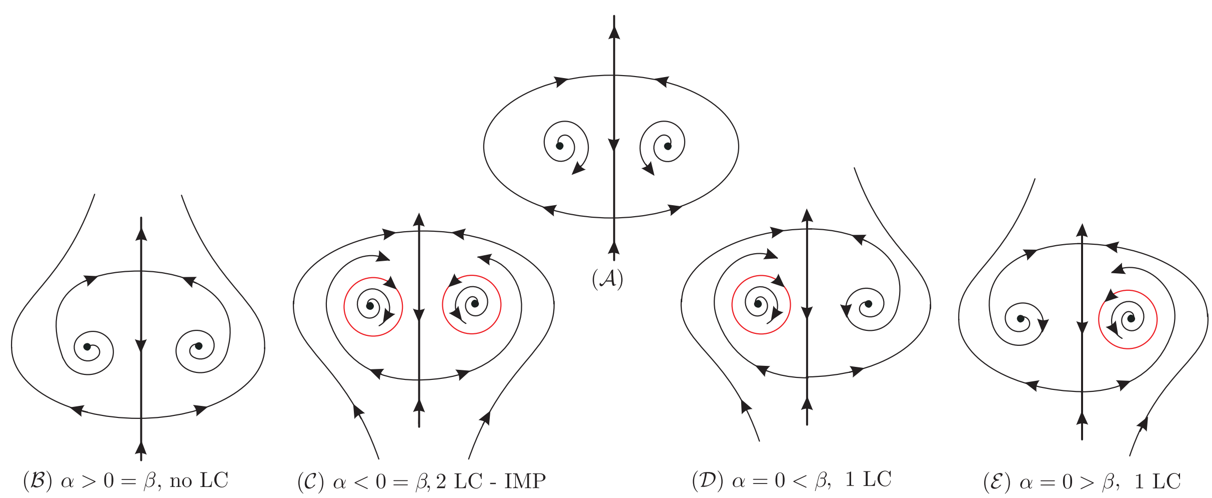



Figure 13. The proof of Theorem 2.

2.2. **Case of singularities not topologically equivalent to elemental or semi-elemental singularities.** Up to here we have proved all the statements above for phase portraits having singularities which are topologically equivalent to elemental or semi-elemental singularities (finite or infinite). That is, even if a singularity is nilpotent or intricate but topologically equivalent to elemental or semi-elemental one, the statements hold. Now we need to prove the statements related to the impossibility of those couples or triplets of connections for the remaining singularities. But this would be a very long task with many subcases to be considered. There is a better way for doing this. There are several papers in which a classification of phase portraits possessing nilpotent or intricate singularities (finite and infinite) is given. More precisely, in [22] finite intricate singularities are investigated; in [27] quadratic systems with a finite nilpotent singularity are considered; in [11] the phase portraits with infinite nilpotent or intricate singularities are given. This last paper relies on [12] for some of the classes of quadratic systems considered. From a detailed check of all the phase portraits contained in these papers it follows that none of them has three separatrix connections. Neither do they have any of the couples of connections which we have proved in Subsection 2.1 to be impossible for elemental or semi-elemental singularities.

However the papers [27] and [12] are based on bifurcations diagrams in the parameter spaces of some normal forms. This kind of study cannot guarantee that absolutely every phase portrait that could be generated by those normal forms, is going to be found. These works are always subject to the possible presence of "islands" in the parameter space. These islands could be of two types: $(i)$ on the shore of the island there is a double limit cycle which splits in two limit cycles inside the island; $(ii)$ on the shore of the island there is some separatrix connection and a different phase portrait in the interior of the island. The possible existence of these islands has been already mentioned in several papers using the same technique, but up to now no such islands have been found. Moreover, the maximum number of separatrix connections that can be found in the bifurcation diagrams of [27] and [12] is one.

Concerning our results presented here this means that the new statement of Theorem 1 extended to nilpotent or intricate singularities would have to include the expression "modulo existence of islands". So we rewrite Theorem 1 so as to cover all these possibilities:

**Theorem** $1^*$ *If all the singularities are elemental or semi-elemental, then there are no phase portraits with three separatrix connections except for the configuration of connections* $(a, a, a)$. *If there is at least one nilpotent or intricate singularity, the statement is subject to possible existence of islands.*

We believe that there do not exist such islands which could affect the above statement and thus we formulate the following conjecture:

**Conjecture 1.** *The maximum number of separatrix connections that a phase portrait of a quadratic differential system may have, is two, except for phase portraits* $Vul_{10}$ *and* $Vul_3$ *(where it is three).*

## Concluding comment

Summing up, we have shown that Diagonal 3 from Table 1 (and all higher diagonals) have no phase portraits. Hence, all effort to find and classify all quadratic phase portraits modulo limit cycles, must be done in Diagonals from 0 to 2. Most of the boxes are already completed and many others are relatively easy to complete. The most challenging box to be studied is codimension 0 (structurally stable) configurations of singularities, and global codimension of phase portraits 2 (two separatrix connections). In this article we have reduced from 15 to 10 the number of subfamilies with two separatrix connections that need to be studied. So, an update to Table 1 leads to Table 2.

## Acknowledgments

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The third and the fourth authors are very thankful for the hospitality provided by the Departament de Matemàtiques during their visits of Universitat Autònoma de Barcelona.

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[1] J. C. Artés, *S*tructurally unstable quadratic vector fields of codimension two: families possessing one finite saddle-node and a separatrix connection, Preprint, 2023.

| **CCS** | Global codimension of Phase Portraits | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **0** | 44 | 61 | $\geq 5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| **1** | x | 141 | $\geq 93$ | $\geq 0$ | 0 | 0 | 0 | 0 | 0 | 0 |
| **2** | x | x | $\geq 182$ | $\geq 134$ | $\geq 12$ | 0 | 0 | 0 | 0 | 0 |
| **3** | x | x | x | $\geq 129$ | 68 | 4 | 0 | 0 | 0 | 0 |
| **4** | x | x | x | x | 54 | 11 | 0 | 0 | 0 | 0 |
| **5** | x | x | x | x | x | 31 | 1 | 0 | 0 | 0 |
| **6** | x | x | x | x | x | x | 19 | 0 | 0 | 0 |
| **7** | x | x | x | x | x | x | x | 8 | 0 | 0 |
| **8** | x | x | x | x | x | x | x | x | 3 | 0 |
| **9** | x | x | x | x | x | x | x | x | x | 1 |

Table 2. Improvement of Table 1 after Theorem 2*

[2] J. C. Artés, R. Kooij and J. Llibre, *Structurally stable quadratic vector fields*, Memoires Amer. Math. Soc., 134(639), 1998. 108 pp.

[3] J. C. Artés, J. Llibre and A. C. Rezende, *Structurally unstable quadratic vector fields of codimension one*. Universitext, Birkhäuser, New York–Berlin, 2018, 283 pp.

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