L^p estimates for the maximal singular integral in terms of the singular integral

Anna Bosch-Camós, Joan Mateu and Joan Orobitg

Abstract

This paper continues the study, initiated in the works [MOV] and [MOPV], of the problem of controlling the maximal singular integral T^*f by the singular integral Tf. Here T is a smooth homogeneous Calderón-Zygmund singular integral operator of convolution type. We consider two forms of control, namely, in the weighted $L^p(\omega)$ norm and via pointwise estimates of T^*f by $M(Tf)$ or $M^2(Tf)$, where M is the Hardy-Littlewood maximal operator and $M^2 = M \circ M$ its iteration. The novelty with respect to the aforementioned works, lies in the fact that here p is different from 2 and the L^p space is weighted.

1 Introduction

Let T be a smooth homogeneous Calderón-Zygmund singular integral operator on $\mathbb{R}^n$ with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n} \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{1}$$

where Ω is a homogeneous function of degree 0 whose restriction to the unit sphere S^{n-1} is C^∞ and satisfies the cancellation property

$$\int_{|x|=1} \Omega(x)\, d\sigma(x) = 0,$$

σ being the normalized surface measure in S^{n-1}. Thus, Tf is the principal value convolution operator

$$Tf(x) = \text{p.v.} \int f(x-y)K(y)\, dy \equiv \lim_{\varepsilon \to 0} T^\varepsilon f(x), \tag{2}$$

where $T^\varepsilon f$ is the truncated operator at level ε defined by

$$T^\varepsilon f(x) = \int_{|x-y|>\varepsilon} f(x-y)K(y)\, dy.$$

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Departament de Matemàtiques.
http://www.uab.cat/matematiques
UAB Universitat Autònoma de Barcelona

For $f \in L^p$, $1 \le p < \infty$, the limit in (2) exits for almost all x. One says that the operator T is even (or odd) if the kernel (1) is even (or odd), that is, if $\Omega(-x) = \Omega(x)$, $x \in \mathbb{R}^n \setminus \{0\}$ (or $\Omega(-x) = -\Omega(x)$, $x \in \mathbb{R}^n \setminus \{0\}$). Let T^* be the maximal singular integral

$$T^* f(x) = \sup_{\varepsilon > 0} |T^\varepsilon f(x)|, \quad x \in \mathbb{R}^n.$$

In this paper we consider the problem of characterizing those smooth Calderón-Zygmund operators for which one can control $T^* f$ by Tf in the weighted L^p norm

$$\|T^* f\|_{L^p(\omega)} \le C \|Tf\|_{L^p(\omega)}, \quad f \in L^p(\omega), \text{ and } \omega \in A_p, \tag{3}$$

where A_p is the Muckenhoupt class of weights (see below for the definition). A stronger way of saying that T^* is controlled by T is the pointwise inequality

$$T^* f(x) \le C(M^s(Tf)(x)), \ x \in \mathbb{R}^n, \quad s \in \{1, 2\}, \tag{4}$$

where M denotes the Hardy-Littlewood maximal operator and $M^2 = M \circ M$ its iteration. For the case $p = 2$ and $\omega = 1$, the relationship between (3) and (4) has been studied in [MOV] for even kernels and in [MOPV] for odd kernels (see also [MV]). We will prove that, for any $1 < p < \infty$ and $\omega \in A_p$, the class of operators satisfying (3) coincides with the family of operators obtained for $p = 2$ and $\omega = 1$, thus giving an affirmative answer to Question 1 of [MOV, p. 1480]. Our main result states that for smooth Calderón-Zygmund operators, inequality (4) (with s depending on the parity of the kernel) is equivalent to (3) and also is equivalent to an algebraic condition involving the expansion of Ω in spherical harmonics.

Now we need to introduce some notation. The homogeneous function Ω, like any square-integrable function in S^{n-1} with zero integral, has an expansion in spherical harmonics of the form

$$\Omega(x) = \sum_{j=1}^{\infty} P_j(x), \quad x \in S^{n-1}, \tag{5}$$

where P_j is a homogeneous harmonic polynomial of degree j. For the case of even operators in the above sum we only have the even terms P_{2j} and for the odd case we only have the polynomials of odd degree P_{2j+1}. In any case, when Ω is infinitely differentiable on the unit sphere one has that, for each positive integer M,

$$\sum_{j=1}^{\infty} j^M \|P_j\|_\infty < \infty, \tag{6}$$

where the supremum norm is taken on S^{n-1}. When Ω is of the form

$$\Omega(x) = \frac{P(x)}{|x|^d}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

with P a homogeneous harmonic polynomial of degree $d \ge 1$, one says that T is a higher order Riesz transform. If the homogeneous polynomial P is not required to be harmonic, but has still zero integral on the unit sphere, then we call T a polynomial operator.

Let's recall the definition of Muckenhoupt weights. Let ω be a non negative locally integrable function, and $1 < p < \infty$. Then $\omega \in A_p$ if and only if there exits a constant C such that for all cubes $Q \subset \mathbb{R}^n$

$$\left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\right)^{p-1} \leq C.$$

The important fact worth noting is that Calderón-Zygmund operators and the Hardy-Littlewood maximal operator are bounded on $L^p(\omega)$, when $1 < p < \infty$ and ω belongs to A_p. See [Du, Chapter 7] or [Gr2, Chapter 9] to get more information on weights.

Now we state our result. We start with the case of even operators.

Theorem 1. *Let T be an even smooth homogeneous Calderón-Zygmund operator with kernel* (1)*. Then the following are equivalent:*

(a)

$$T^* f(x) \leq C M(Tf)(x), \quad x \in \mathbb{R}^n.$$

(b) If $p \in (1, \infty)$ and $\omega \in A_p$, then

$$||T^* f||_{L^p(\omega)} \leq C||Tf||_{L^p(\omega)}, \quad \textit{for all } f \in L^p(\omega).$$

(c) Assume that the expansion (5) *of Ω in spherical harmonics is*

$$\Omega(x) = \sum_{j=j_0}^{\infty} P_{2j}(x), \quad P_{2j_0} \neq 0\,.$$

Then, for each j there exists a homogeneous polynomial Q_{2j-2j_0} of degree $2j-2j_0$ such that $P_{2j} = P_{2j_0} Q_{2j-2j_0}$ and $\sum_{j=j_0}^{\infty} \gamma_{2j} Q_{2j-2j_0}(\xi) \neq 0$, $\xi \in S^{n-1}$. Here for a positive integer k we have set

$$\gamma_k = i^{-k} \pi^{\frac{n}{2}} \frac{\Gamma(\frac{k}{2})}{\Gamma(\frac{n+k}{2})}. \tag{7}$$

(d)

$$||T^* f||_{1,\infty} \leq C||Tf||_1, \quad \textit{for all } f \in H^1(\mathbb{R}^n).$$

Recall that $||g||_{1,\infty}$ denotes the weak L^1 norm of g and $H^1(\mathbb{R}^n)$ is the Hardy space. Calderón-Zygmund operators act on H^1. (For instance, see [Du, Chapter 6], [Gr2, Chapter 7] for more information on the Hardy space.)

To get the above result for odd kernels we will replace the Hardy-Littlewood maximal operator in *(a)* by its iteration.

Theorem 2. *Let T be an odd smooth homogeneous Calderón-Zygmund operator with kernel* (1). *Then the following are equivalent:*

(a)

$$T^*f(x) \le CM^2(Tf)(x), \quad x \in \mathbb{R}^n.$$

(b) If $p \in (1, \infty)$ and $\omega \in A_p$ then

$$||T^*f||_{L^p(\omega)} \le C||Tf||_{L^p(\omega)}, \quad \textit{for all } f \in L^p(\omega).$$

(c) Assume that the expansion (5) *of Ω in spherical harmonics is*

$$\Omega(x) = \sum_{j=j_0}^{\infty} P_{2j+1}(x), \quad P_{2j_0+1} \neq 0\,.$$

Then, for each j there exists a homogeneous polynomial Q_{2j-2j_0} of degree $2j-2j_0$ such that $P_{2j+1} = P_{2j_0+1}Q_{2j-2j_0}$ and $\sum_{j=j_0}^{\infty} \gamma_{2j+1} Q_{2j-2j_0}(\xi) \neq 0$, $\xi \in S^{n-1}$, with γ_{2j+1} as in (7).

Clearly, both in Theorem 1 as in Theorem 2, the condition *(a)* implies *(b)* is a consequence of the boundedness of the Hardy-Littlewood maximal operator on weighted L^p spaces. The proof of *(c)* implies *(a)* in Theorem 1 is proved in [MOV] and the same implication in Theorem 2 is proved in [MOPV]. So the only task to be done is to show that *(b)* implies *(c)* in both theorems (and $(d) \Rightarrow (c)$ in Theorem 1). One of the crucial points in the proof of the implication $(b) \Rightarrow (c)$ for the case $p = 2$ and $\omega = 1$ in [MOV] and [MOPV] is to use Plancherel Theorem to get a pointwise inequality to work with it. For $p \neq 2$ we will get the corresponding pointwise inequality using properties of the Fourier transform of the kernels as L^p multipliers.

In Section 2 we introduce L^p Fourier multipliers and some tools to control their norm (see Lemma 1). Section 3 is devoted to the proof of $(b) \Rightarrow (c)$, for polynomial operators. The general case is discussed in Section 4.

As usual, the letter C will denote a constant, which may be different at each occurrence and which is independent of the relevant variables under consideration.

2 Multipliers

Recall that, given $1 \le p < \infty$, one denotes by $\mathcal{M}_p(\mathbb{R}^n)$ the space of all bounded functions m on $\mathbb{R}^n$ such that the operator

$$T_m(f) = (\hat{f}\, m)^\vee, \quad f \in \mathcal{S},$$

is bounded on $L^p(\mathbb{R}^n)$ (or is initially defined in a dense subspace of $L^p(\mathbb{R}^n)$ and has a bounded extension on the whole space). As usual, $\mathcal{S}$ denotes the space of Schwartz functions, $\hat{f}$ is the Fourier transform of f and $f^\vee$ the inverse Fourier transform.

The norm of m in $\mathcal{M}_p(\mathbb{R}^n)$ is defined as the norm of the bounded linear operator $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Elements of the space $\mathcal{M}_p(\mathbb{R}^n)$ are called L^p (Fourier) multipliers. Similarly, we speak of $L^p(\omega)$ multipliers. It is well known that $\mathcal{M}_2$, the set of all L^2 multipliers, is L^∞ and that $\mathcal{M}_1(\mathbb{R}^n)$ is the set of Fourier transforms of finite Borel measures on $\mathbb{R}^n$. The basic theory on multipliers may be found for example in the monographs [Du], [Gr1].

Let $0 \leq \phi \leq 1$ be an smooth function such that $\phi(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$, and $\phi(\xi) = 0$ if $|\xi| \geq 1$. Given $\xi_0 \in \mathbb{R}^n$, we define $\phi_\delta(\xi) = \phi(\frac{\xi-\xi_0}{\delta})$. Consider $m \in L^\infty$ such that m is continuous in some neighbourhood of ξ_0 with $m(\xi_0) = 0$. It is clear, by Plancherel Theorem, that the norm of $m\phi_\delta$ in $\mathcal{M}_2$ approaches zero when $\delta \to 0$. We ask if the same result holds when m is an L^p multiplier. Adding some regularity to m we get a positive answer.

Lemma 1. *Let $\xi_0 \in \mathbb{R}^n$, $0 < \delta \leq \delta_0$ and $m \in \mathcal{M}_p \cap \mathcal{C}^n(B(\xi_0, \delta_0))$ with $m(\xi_0) = 0$. Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $0 \leq \phi \leq 1$ such that $\phi(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$, and $\phi(\xi) = 0$ if $|\xi| \geq 1$. Set $\phi_\delta(\xi) = \phi(\frac{\xi-\xi_0}{\delta})$ and let $T_{m\phi_\delta}$ be the operator with multiplier $m\phi_\delta$.*

1. *If $\omega \in A_p$, $1 < p < \infty$, then $||T_{m\phi_\delta}||_{L^p(\omega)\to L^p(\omega)} \longrightarrow 0$, when $\delta \to 0$.*
2. *$||T_{m\phi_\delta}||_{L^1\to L^{1,\infty}} \longrightarrow 0$, when $\delta \to 0$.*
3. *$||T_{m\phi_\delta}||_{H^1\to L^1} \longrightarrow 0$, when $\delta \to 0$.*

To prove Lemma 1, we use the next theorem due to Kurtz and Wheeden. Following [KW], we say that a function m belongs to the class $M(s,l)$ if

$$m_{s,l} := \sup_{R>0} \left(R^{s|\alpha|-n} \int_{R<|x|<2R} |D^\alpha m(x)|^s \, dx \right)^{1/s} < +\infty, \text{ for all } |\alpha| \leq l, \tag{8}$$

where s is a real number greater or equal to 1, l a positive integer and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multiindex of nonnegative integers.

Theorem 3 ([KW, p. 344]). *Let $1 < s \leq 2$ and $m \in M(s,n)$.*

1. *If $1 < p < \infty$ and $\omega \in A_p$, then there exists a constant C, independent of f, such that*
$$||T_m f||_{L^p(\omega)} \leq C||f||_{L^p(\omega)}.$$
2. *There exists a constant C, independent of f and λ, such that*
$$|\{x \in \mathbb{R}^n : |T_m f(x)| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1}, \quad \lambda > 0.$$
3. *There exists a constant C, independent of f, such that*
$$||T_m f||_{L^1} \leq C||f||_{H^1}.$$

Analyzing the proof we check that, in all cases, the constant C, which appears in the statements *1*, *2* and *3* of the previous Theorem, depends linearly on the constant $m_{s,n}$ defined at (8). We also remark that when $\omega = 1$ the proof can be adapted to the case $H^1 \to L^1$, so we get statement *3* which is not explicitly written in [KW].

Proof of Lemma 1. Using Theorem 3 we only need to prove that the multiplier $m\phi_\delta$ is in $M(s,n)$ for some $1 < s \le 2$, and the constant $m_{s,n}$ tends to 0 if δ tends to 0.

Assume that $\xi_0 \neq 0$ and that $\delta < \delta_0$ is small enough. For $|\alpha| \le n$, using Leibniz rule one has

$$\begin{aligned}
&\sup_{R>0}\left(R^{s|\alpha|-n}\int_{R<|\xi|<2R}|D^\alpha(m\phi_\delta)(\xi)|^s\,d\xi\right)^{1/s}\\
&= \sup_{R>0}\left(R^{s|\alpha|-n}\int_{\{R<|\xi|<2R\}\cap B(\xi_0,\delta)}|D^\alpha(m\phi_\delta)(\xi)|^s\,d\xi\right)^{1/s}\\
&\le C|\xi_0|^{|\alpha|-\frac{n}{s}}\left(\int_{B(\xi_0,\delta)}|D^\alpha(m\phi_\delta)(\xi)|^s\,d\xi\right)^{1/s}\\
&\le C|\xi_0|^{|\alpha|-\frac{n}{s}}\left(\sum_{\beta_i\le\alpha_i,\,1\le i\le n}\binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2}\cdots\binom{\alpha_n}{\beta_n}\int_{B(\xi_0,\delta)}|D^{\alpha-\beta}(m)(\xi)D^\beta(\phi_\delta)(\xi)|^s\,d\xi\right)^{1/s}.
\end{aligned}$$

Now we will get a bound for each term in the above sum. In order to get it, we consider different cases. In all the cases we will use that for any multiindex α we have $|D^\alpha\phi_\delta(\xi)| \lesssim \frac{1}{\delta^{|\alpha|}}$ and that the modulus of continuity of m, denoted by $\omega(m,\xi_0,\delta)$, satisfies $\omega(m,\xi_0,\delta) \le C\delta$.

Case 1. $|\alpha| = n$.

For $\beta = \alpha$ one has that

$$\begin{aligned}
\int_{B(\xi_0,\delta)}|D^{\alpha-\beta}(m)(\xi)D^\beta(\phi_\delta)(\xi)|^s\,d\xi &= \int_{B(\xi_0,\delta)}|m(\xi)|^s|D^\alpha(\phi_\delta)(\xi)|^s\,d\xi\\
&\le C\frac{1}{\delta^{ns}}|\omega(m,\xi_0,\delta)|^s\delta^n\\
&\le C\delta^{s+n-ns}
\end{aligned}$$

and this term tends to 0 as δ tends to 0 taking $1 < s < \frac{n}{n-1}$. For the remaining terms, that is $\alpha \neq \beta$, we have

$$\begin{aligned}
\int_{B(\xi_0,\delta)}|D^{\alpha-\beta}(m)(\xi)D^\beta(\phi_\delta)(\xi)|^s d\xi &\le C\frac{1}{\delta^{|\beta|s}}\delta^n\\
&= C\delta^{n-s|\beta|}\\
&\le C\delta^{s+n-ns},
\end{aligned}$$

where the derivatives of m are bounded by a constant, and the last inequality holds when δ is small enough. So, if $1 < s < \frac{n}{n-1}$, this term goes to 0 as δ goes to 0.

Case 2. $|\alpha| = k < n$.

For $|\beta| = |\alpha|$, using the boundedness of the modulus of continuity of m we have

$$\begin{aligned}\int_{B(\xi_0,\delta)} |D^{\alpha-\beta}(m)(\xi)D^{\beta}(\phi_\delta)(\xi)|^s\, d\xi &= \int_{B(\xi_0,\delta)} |m(\xi)|^s |D^{\alpha}(\phi_\delta)(\xi)|^s d\xi \\ &\leq C\frac{1}{\delta^{ks}}|\omega(m,\xi_0,\delta)|^s\delta^n \\ &= C\delta^{s+n-ks} \\ &\leq C\delta^{s+n-ns}\end{aligned}$$

and this term, again, goes to 0 as δ goes to 0, whenever $1 < s < \frac{n}{n-1}$.

Finally, if $|\beta| < |\alpha|$, one gets the same bound

$$\begin{aligned}\int_{B(\xi_0,\delta)} |D^{\alpha-\beta}(m)(\xi)D^{\beta}(\phi_\delta)(\xi)|^s d\xi &\leq C\frac{1}{\delta^{|\beta|s}}\delta^n \\ &= C\delta^{n-s|\beta|} \\ &\leq C\delta^{s+n-ns}.\end{aligned}$$

When $\xi_0 = 0$ one has

$$\begin{aligned}&\sup_{R>0}\left(R^{s|\alpha|-n}\int_{R<|\xi|<2R}|D^{\alpha}(m\phi_\delta)(\xi)|^s d\xi\right)^{1/s} \\ &= \sup_{\delta\geq R>0}\left(R^{s|\alpha|-n}\int_{R<|\xi|<2R}|D^{\alpha}(m\phi_\delta)(\xi)|^s d\xi\right)^{1/s}.\end{aligned}$$

Observe that for $|\alpha| > 0$, $D^{\alpha}\phi_\delta$ lives on $\{\delta/2 \leq |\xi| \leq \delta\}$. Then, similar calculations complete the proof. $\square$

To prove the first case of Lemma 1 there is another argument due to J. Duoandikoetxea. We thank him for providing us the following lemma. In fact, it is only necessary to assume that the multiplier m is continuous.

Lemma 2. *Let $\xi_0 \in \mathbb{R}^n$, $0 < \delta \leq \delta_0$, $1 < q < 2$ and $m \in \mathcal{M}_q \cap \mathcal{C}(B(\xi_0,\delta_0))$ with $m(\xi_0) = 0$. Set $\phi_\delta(\xi)$ as above and let $T_{m\phi_\delta}$ be the operator with multiplier $m\phi_\delta$.*

(a) For any $p \in (q,2)$ we have

$$\|T_{m\phi_\delta}\|_{L^p\to L^p} \longrightarrow 0, \quad \textit{when} \;\; \delta \to 0.$$

(b) Let $\omega \in A_p$ with $p \in (q,2)$ and let $s > 1$ such that $\omega^s \in A_p$. If m is an $L^p(\omega^s)$ multiplier, then

$$\|T_{m\phi_\delta}\|_{L^p(\omega)\to L^p(\omega)} \longrightarrow 0, \quad \textit{when} \;\; \delta \to 0.$$

Remark 1. Clearly, a similar result holds when $2 < p < q$.

Proof. We first observe that $||T_{m\phi_\delta}||_{L^2\to L^2} = \|m\phi_\delta\|_\infty = \varepsilon(\delta)$ and $\varepsilon(\delta) \to 0$ as $\delta \to 0$ since m is continuous in ξ_0. On the other hand, $\|m\phi_\delta\|_{\mathcal{M}_q} \le \|\phi_\delta^\vee\|_{L^1}\|m\|_{\mathcal{M}_q} = C\|m\|_{\mathcal{M}_q}$, where C is a constant independent of δ. That is, for all $\delta > 0$

$$\|T_{m\phi_\delta} f\|_q \le M\|f\|_q.$$

Then, applying the Riesz-Thorin theorem (e.g. [Gr1, p. 34]), for any $p \in (q, 2)$ ($\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$) we have

$$\|T_{m\phi_\delta} f\|_p \le M^{1-\theta}\varepsilon(\delta)^\theta\|f\|_p = \varepsilon_1(\delta)\|f\|_p, \tag{9}$$

where $\varepsilon_1(\delta) \to 0$ as $\delta \to 0$ and (a) is proved. For proving (b), since $\omega^s \in A_p$ and ϕ_δ is a cutoff smooth function, note that

$$\|T_{m\phi_\delta} f\|_{L^p(\omega^s)} \le C\|f\|_{L^p(\omega^s)}, \tag{10}$$

where one can check that C is a constant independent of δ. Finally, from (9) and (10), applying the interpolation theorem with change of measure of Stein-Weiss (e.g. [BeL, p. 115]), we get

$$\|T_{m\phi_\delta} f\|_{L^p(\omega)} \le C^{1/s}\varepsilon_1(\delta)^{1-1/s}\|f\|_{L^p(\omega)}$$

as desired. □

3 The polynomial case

As we remarked in the Introduction, to have a complete proof of Theorems 1 and 2 only remains to prove that *(b)* implies *(c)* (and *(d)* implies *(c)* in Theorem 1). Our procedure to get the above implications follows essentially the arguments used in [MOV] and [MOPV]. The main difficulty to overcome is that for $p \neq 2$, we cannot apply Plancherel Theorem and we replace it by a Fourier multiplier argument.

We begin with the proof of *(b)* implies *(c)* in Theorem 1 for the case $\omega = 1$. Then we show how to adapt this proof to the case with weights, to the case of odd operators and to the case of weak L^1. Thus, we assume that T is an even polynomial operator with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P_2(x)}{|x|^{2+n}} + \frac{P_4(x)}{|x|^{4+n}} + \cdots + \frac{P_{2N}(x)}{|x|^{2N+n}}, \quad x \neq 0,$$

where P_{2j} is a homogeneous harmonic polynomial of degree $2j$. Each term has the multiplier (see [St, p. 73])

$$\left(\text{p.v.}\ \frac{P_{2j}(x)}{|x|^{2j+n}}\right)^\wedge(\xi) = \gamma_{2j}\frac{P_{2j}(\xi)}{|\xi|^{2j+n}}.$$

Then,

$$\widehat{\text{p.v. } K}(\xi) = \frac{Q(\xi)}{|\xi|^{2N}}, \qquad \xi \neq 0,$$

where Q is the homogeneous polynomial of degree $2N$ defined by

$$Q(x) = \gamma_2 P_2(x)|x|^{2N-2} + \cdots + \gamma_{2j} P_{2j}(x)|x|^{2n-2j} + \cdots + \gamma_{2N} P_{2N}(x).$$

We want to obtain a convenient expression for the function $K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}$, the kernel K off the unit ball B (see (12)). To find it, we need a simple technical lemma which we state without proof.

Lemma 3 ([MOV, p. 1435])**.** *Assume that φ is a radial function of the form*

$$\varphi(x) = \varphi_1(|x|)\chi_B(x) + \varphi_2(|x|)\chi_{\mathbb{R}^n\setminus\overline{B}}(x),$$

where φ_1 is continuously differentiable on $[0,1)$ and φ_2 on $(1,\infty)$. Let L be a second order linear differential operator with constant coefficients. Then the distribution $L\varphi$ satisfies

$$L\varphi = L\varphi(x)\chi_B(x) + L\varphi(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x),$$

provided φ_1, φ_1', φ_2 and φ_2' extend continuously to the point 1 and the two conditions

$$\varphi_1(1) = \varphi_2(1), \quad \varphi_1'(1) = \varphi_2'(1)$$

are satisfied.

Consider the differential operator $Q(\partial)$ defined by the polynomial $Q(x)$ above and let E be the standard fundamental solution of the N-th power Δ^N of the Laplacian. Then $Q(\partial)E = \text{p.v. } K(x)$, which may be verified by taking the Fourier transform of both sides. The concrete expression of $E(x) = |x|^{2N-n}(a(n,N) + b(n,N)\log|x|^2)$ (e.g. [MOV, p. 1464]) is not important now, just note that it is a radial function. Consider the function

$$\varphi(x) = E(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) + (A_0 + A_1|x|^2 + \cdots + A_{2N-1}|x|^{4N-2})\chi_B(x),$$

where B is the open ball of radius 1 centered at origin and the constants $A_0, A_1, \ldots,$ A_{2N-1} are chosen as follows. Since $\varphi(x)$ is radial, the same is true for $\Delta^j\varphi$ if j is a positive integer. Thus, in order to apply N times Lemma 3, one needs $2N$ conditions, which (uniquely) determine $A_0, A_1, \ldots, A_{2N-1}$. Therefore, for some constants $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}$,

$$\Delta^N\varphi = (\alpha_0 + \alpha_1|x|^2 + \cdots \alpha_{N-1}|x|^{2(N-1)})\chi_B(x) = b(x), \tag{11}$$

where the last identity is the definition of b. Let's remark that b is a bounded function supported in the unit ball and it only depends on N and not on the kernel K. Since

$$\varphi = E * \Delta^N\varphi,$$

taking derivatives of both sides we obtain

$$Q(\partial)\varphi = Q(\partial)E * \Delta^N \varphi = \text{p.v. } K(x) * b = T(b).$$

On the other hand, applying Lemma 3,

$$Q(\partial)\varphi = K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) + Q(\partial)(A_0 + A_1|x|^2 + \cdots + A_{2N-1}|x|^{4N-2})(x)\chi_B(x).$$

We write

$$S(x) := -Q(\partial)(A_0 + A_1|x|^2 + \cdots + A_{2N-1}|x|^{4N-2})(x),$$

and we get

$$K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T(b)(x) + S(x)\chi_B(x). \tag{12}$$

Let's remark that S will be null when Q is a harmonic polynomial (see [MOV, p. 1437]). Consequently

$$T^1 f = T(b) * f + S\chi_B * f.$$

Our assumption is the L^p estimate between T^* and T. Since the truncated operator T^1 at level 1 is obviously dominated by T^*, we have

$$\begin{aligned} \|S\chi_B * f\|_p &\le \|T^1 f\|_p + \|Tb * f\|_p \\ &\le \|T^* f\|_p + \|b * Tf\|_p \\ &\le C\|Tf\|_p + \|b\|_1\|Tf\|_p \\ &= C\|Tf\|_p, \end{aligned} \tag{13}$$

that is, for any $f \in L^p$

$$\|S\chi_B * f\|_p \le C\|\text{p.v. } K * f\|_p. \tag{14}$$

If $p = 2$, we can use Plancherel and this L^2 inequality translates into a pointwise inequality between the Fourier multipliers:

$$|\widehat{S\chi_B}(\xi)| \le C|\widehat{\text{p.v. } K}(\xi)| = \frac{Q(\xi)}{|\xi|^{2N}}, \quad \xi \ne 0. \tag{15}$$

If $p \ne 2$ we must resort to Fourier multipliers to get (15). We observe that the multipliers we are dealing with, $\widehat{S\chi_B}$ and $\widehat{\text{p.v. } K}$, are in $\mathcal{C}^\infty \setminus \{0\}$ and in $\mathcal{M}_p$. Let $\xi_0 \ne 0$, we write

$$\begin{aligned} &\widehat{S\chi_B}(\xi) = \widehat{S\chi_B}(\xi)(\xi_0) + E_1(\xi) \quad &&\text{with} \quad E_1(\xi) = \widehat{S\chi_B}(\xi) - \widehat{S\chi_B}(\xi_0) \\ &\widehat{\text{p.v. } K}(\xi) = \widehat{\text{p.v. } K}(\xi_0) + E_2(\xi) \quad &&\text{with} \quad E_2(\xi) = \widehat{\text{p.v. } K}(\xi) - \widehat{\text{p.v. } K}(\xi_0) \end{aligned}$$

and so

$$\|\text{p.v. } K * f\|_p \le |\widehat{\text{p.v. } K}(\xi_0)|\, \|f\|_p + \|T_{E_2} f\|_p, \tag{16}$$

$$\|S\chi_B * f\|_p \ge |\widehat{S\chi_B}(\xi_0)|\, \|f\|_p - \|T_{E_1} f\|_p, \tag{17}$$

where T_{E_i} denotes the operator with multiplier E_i ($i = 1, 2$). Using (17), (14) and (16) consecutively, we get

$$\begin{aligned}|\widehat{S\chi_B}(\xi_0)|\,\|f\|_p - \|T_{E_1}f\|_p &\le \|S\chi_B * f\|_p \\ &\le C\|\text{p.v. } K * f\|_p \\ &\le C(|\widehat{\text{p.v. } K}(\xi_0)|\,\|f\|_p + \|T_{E_2}f\|_p)\end{aligned}$$

and therefore

$$|\widehat{S\chi_B}(\xi_0)| \le C\left(|\widehat{\text{p.v. } K}(\xi_0)| + \frac{||T_{E_2}f||_p}{||f||_p} + \frac{||T_{E_1}f||_p}{||f||_p}\right), \quad \xi_0 \ne 0. \tag{18}$$

Now, choosing appropriate functions in (18) we obtain the pointwise inequality. Let $\phi_\delta(\xi) = \phi(\frac{\xi-\xi_0}{\delta})$ as in Lemma 1 and define $g_\delta \in \mathcal{S}(\mathbb{R}^n)$ by $\widehat{g_\delta}(\xi) = \phi_\delta(\xi)$. Then $T_{E_j}g_\delta = T_{E_j}(g_{2\delta} * g_\delta) = T_{E_j\phi_{2\delta}}(g_\delta)$, because $\phi_{2\delta} = 1$ on the support of ϕ_δ. Changing f by g_δ in (18) we have

$$\begin{aligned}|\widehat{S\chi_B}(\xi_0)| &\le C\left(|\widehat{\text{p.v. } K}(\xi_0)| + \frac{\|T_{E_2\phi_{2\delta}}g_\delta\|_p}{\|g_\delta\|_p} + \frac{\|T_{E_1\phi_{2\delta}}g_\delta\|_p}{\|g_\delta\|_p}\right) \\ &\le C\left(|\widehat{\text{p.v. } K}(\xi_0)| + \|T_{E_2\phi_{2\delta}}\|_{L^p\to L^p} + \|T_{E_1\phi_{2\delta}}\|_{L^p\to L^p}\right).\end{aligned}$$

Applying Lemma 1 to the multipliers E_j we prove that the two last terms tend to zero as δ tends to zero. So, for $\omega = 1$, we get (15) and from here we would follow the arguments in [MOV, p. 1457].

For the weighted case we must be careful with the inequalities in (13). In general, the inequality $\|f * F\|_{L^p(\omega)} \le C\|f\|_1\|F\|_{L^p(\omega)}$ is not satisfied. That is, we can not control $\|b * Tf\|_{L^p(\omega)}$ by a constant times $\|b\|_1\|Tf\|_{L^p(\omega)}$. However, in the even case b is a bounded function supported in the unit ball and so

$$|(b * Tf)(x)| = \left|\int_{|x-y|<1} b(x-y)Tf(y)\,dy\right| \le CM(Tf)(x).$$

Moreover

$$\|b * Tf\|_{L^p(\omega)} \le C\|Tf\|_{L^p(\omega)},$$

because $\omega \in A_p$. So, $\|S\chi_B * f\|_{L^p(\omega)} \le C\|\text{p.v. } K * f\|_{L^p(\omega)}$ and proceeding as above, we would get (15).

The proof of *(b)* implies *(c)* in Theorem 2 can be handled in much the same way. The only significant difference, because now the polynomial is odd, lies on the function b in (12), which is not supported in the unit ball but it is a BMO function satisfying the decay $|b(x)| \le C|x|^{-n-1}$ if $|x| > 2$ (see [MOPV, section 4]). In any case, $b \in L^1$ and the set of inequalities (13) remains valid for the case $\omega = 1$.

On the other hand, for any ω in the Muckenhoupt class we write, arguing as in [MOPV, p. 3675],

$$\begin{aligned}|(b*Tf)(x)| &= \left|\int_{|x-y|<2}(b(x-y)-b_{B(0,2)})Tf(y)\,dy\right|\\ &\quad+|b_{B(0,2)}|\int_{|x-y|<2}|Tf(y)|\,dy+\int_{|x-y|>2}|b(x-y)|\,|Tf(y)|\,dy\\ &= I+II+III,\end{aligned}$$

where $b_{B(0,2)}=|B(0,2)|^{-1}\int_{B(0,2)}b$. To estimate the local term I we use the generalized Hölder's inequality and the pointwise equivalence $M_{L(\log L)}f(x)\simeq M^2f(x)$ ([P]) to get

$$|I|\le C\|b\|_{\mathrm{BMO}}\|Tf\|_{L(\log L),B(x,2)}\le CM^2(Tf)(x).$$

Notice that $b_{B(0,2)}$ is a dimensional constant. Hence

$$|II|\le CM(Tf)(x).$$

Finally, from the decay of b we obtain

$$|III|\le C\int_{|x-y|>2}\frac{|Tf(y)|}{|x-y|^{n+1}}\,dy\le CM(Tf)(x),$$

by using a standard argument which consists in estimating the integral on the annuli $\{2^k\le|x-y|<2^{k+1}\}$. Therefore

$$|(b*Tf)(x)|\le CM^2(Tf)(x). \tag{19}$$

So, we obtain

$$\|b*Tf\|_{L^p(\omega)}\le C\|Tf\|_{L^p(\omega)},$$

because $\omega\in A_p$. Then, $\|S\chi_B*f\|_{L^p(\omega)}\le C\|\mathrm{p.v.}\ K*f\|_{L^p(\omega)}$ and we get (15).

It remains to prove that *(d)* implies *(c)* in Theorem 1. To get this implication we need to precise some properties of the functions g_δ that we explain below. First of all, note that $g_\delta(x)=e^{ix\xi_0}\delta^n g(\delta x)$ where $\hat g=\phi$. So it is clear that the norms $\|g_\delta\|_1=\|g\|_1$ and $\|g_\delta\|_{1,\infty}=\|g\|_{1,\infty}$ do not depend on the parameter $\delta>0$. When $\delta<|\xi_0|$, since $\int g_\delta(x)\,dx=\phi_\delta(0)=0$ and $g_\delta\in\mathcal{S}(\mathbb{R}^n)$, we have that $g_\delta\in H^1$. But, some computations are required to check that $\|g_\delta\|_{H^1}\le C$ with constant C independent of δ.

Lemma 4. *When $0<\delta<|\xi_0|$, $\|g_\delta\|_{H^1}\le C$ with constant C independent of δ.*

Proof. We have $g_\delta(x)=e^{ix\xi_0}\delta^n g(\delta x)$ with $g\in\mathcal{S}(\mathbb{R}^n)$ and $\int g_\delta=0$. Set $F_0^\delta(x)=\chi_{B(0,\delta^{-1})}(x)$ and, for $j\ge 1$, $F_j^\delta(x)=\chi_{B(0,2^j\delta^{-1})}(x)-\chi_{B(0,2^{j-1}\delta^{-1})}(x)$. Note that

$\sum_{j=0}^{\infty} F_j^\delta(x) \equiv 1$. Consider the atomic decomposition of g_δ

$$g_\delta(x) = \sum_{j=0}^{\infty}(g_\delta(x) - c_j^\delta)F_j^\delta(x) + \sum_{j=0}^{\infty}[(c_j^\delta + d_j^\delta)F_j^\delta(x) - d_{j+1}^\delta F_{j+1}^\delta(x)]$$
$$:= \sum_{j=0}^{\infty} a_j^\delta(x) \ + \ \sum_{j=0}^{\infty} A_j^\delta(x),$$

where $c_j^\delta = \dfrac{\int g_\delta F_j^\delta}{\int F_j^\delta}$, $d_0^\delta = 0$ and $d_{j+1}^\delta = \dfrac{\int g_\delta(F_0^\delta + \cdots + F_j^\delta)}{\int F_{j+1}^\delta}$, so that $\int a_j^\delta(x)\,dx = \int A_j^\delta(x)\,dx = 0$. Note that a_j^δ is supported in the ball $B(0, 2^j\delta^{-1})$ and A_j^δ is supported in $B(0, 2^{j+1}\delta^{-1})$.

Since $g \in \mathcal{S}(\mathbb{R}^n)$ we have $(1 + |z|^{n+1})|g(z)| \leq C$. Then

$$|g_\delta(x)F_j^\delta(x)| = \delta^n |g(\delta x)| F_j^\delta(x) \leq \delta^n \sup_{|z|\sim 2^j} |g(z)| \leq C\left(\frac{\delta}{2^j}\right)^n 2^{-j} = \frac{C2^{-j}}{|B(0, 2^j\delta^{-1})|}$$

and therefore

$$|c_j^\delta| = \left|\frac{\int g_\delta F_j^\delta}{\int F_j^\delta}\right| \leq \frac{C2^{-j}}{|B(0, 2^j\delta^{-1})|}.$$

On the other hand, $\int g_\delta(F_0^\delta + \cdots + F_j^\delta) = \int_{|x|\geq 2^j\delta^{-1}} g_\delta(x)\,dx$, because $\int g_\delta = 0$, and so

$$d_{j+1}^\delta = \frac{\int_{|x|\geq 2^j\delta^{-1}} g_\delta(x)\,dx}{\int F_{j+1}^\delta} \leq \frac{\int_{|z|\geq 2^j} |g(z)|\,dz}{|B(0, 2^{j+1}\delta^{-1})|} \leq \frac{C2^{-j}}{|B(0, 2^{j+1}\delta^{-1})|}.$$

Consequently

$$\|a_j^\delta\|_{H^1} \leq \frac{C}{2^j} \qquad \text{and} \qquad \|A_j^\delta\|_{H^1} \leq \frac{C}{2^j}.$$

Therefore, for all $\delta \in (0, |\xi_0|)$, $\|g_\delta\|_{H^1} \leq C$ as we claimed. □

Finally, for functions f in H^1, and again using (12), we have

$$\begin{aligned}\|S\chi_B * f\|_{1,\infty} &\leq 2(\|T^1 f\|_{1,\infty} + \|Tb * f\|_{1,\infty}) \\ &\leq C(\|T^* f\|_{1,\infty} + \|b * Tf\|_1) \\ &\leq C\|Tf\|_1 + \|b\|_1\|Tf\|_1) \\ &= C\|Tf\|_1 = C\|\text{p.v. } K * f\|_1.\end{aligned}$$

Taking $\xi_0 \neq 0$ and using the same notation as before, we have

$$||\text{p.v. } K * f||_1 \leq |\widehat{\text{p.v. } K}(\xi_0)|\,||f||_1 + ||T_{E_2} f||_1,$$
$$||S\chi_B * f||_{1,\infty} \geq \frac{1}{2}|\widehat{S\chi_B}(\xi_0)|\,\|f\|_{1,\infty} - ||T_{E_1} f||_{1,\infty}$$

and consequently

$$|\widehat{S\chi_B}(\xi_0)| \le C\left(|\widehat{\text{p.v. } K}(\xi_0)|\frac{\|f\|_1}{\|f\|_{1,\infty}} + \frac{\|T_{E_2}f\|_1}{\|f\|_{1,\infty}} + \frac{\|T_{E_1}f\|_{1,\infty}}{\|f\|_{1,\infty}}\right), \quad \xi_0 \neq 0.$$

Replacing f by g_δ and using the properties of g_δ (that is, $\|g_\delta\|_1 = \|g\|_1$, $\|g_\delta\|_{1,\infty} = \|g\|_{1,\infty}$ and Lemma 4) we obtain

$$\begin{aligned}
|\widehat{S\chi_B}(\xi_0)| &\le C\left(|\widehat{\text{p.v. } K}(\xi_0)|\frac{\|g_\delta\|_1}{\|g_\delta\|_{1,\infty}} + \frac{\|T_{E_2\phi_{2\delta}}g_\delta\|_1}{\|g_\delta\|_{1,\infty}} + \frac{\|T_{E_1\phi_{2\delta}}g_\delta\|_{1,\infty}}{\|g_\delta\|_{1,\infty}}\right)\\
&\le C\left(|\widehat{\text{p.v. } K}(\xi_0)|\frac{\|g\|_1}{\|g\|_{1,\infty}} + \frac{\|T_{E_2\phi_{2\delta}}\|_{H^1\to L^1}\|g_\delta\|_{H^1}}{\|g_\delta\|_{1,\infty}} + \frac{\|T_{E_1\phi_{2\delta}}\|_{L^1\to L^{1,\infty}}\|g_\delta\|_1}{\|g_\delta\|_{1,\infty}}\right)\\
&\le C\left(|\widehat{\text{p.v. } K}(\xi_0)| + \|T_{E_2\phi_{2\delta}}\|_{H^1\to L^1} + \|T_{E_1\phi_{2\delta}}\|_{L^1\to L^{1,\infty}}\right)
\end{aligned}$$

and therefore, applying Lemma 1 on the right hand side of this inequality, we get

$$|\widehat{S\chi_B}(\xi_0)| \le C|\widehat{\text{p.v. } K}(\xi_0)| \quad \xi_0 \neq 0$$

as desired.

4 The general case

In our procedure for the polynomial case, the function b has been crucial. It provides a convenient way to express the function $K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}$, where K is the kernel of the operator T. As we mentioned before, b only depends on the degree of the homogeneous polynomial and on the space $\mathbb{R}^n$. In the even case $2N$ (see (11)), $b = b_{2N}$ is the restriction to the unit ball of some polynomial of degree $2N-2$. In the odd case $2N+1$, b_{2N+1} is a BMO function with certain decay at infinity. Until now, we did not need to pay attention to the size of the parameters appearing in the definition of b because the degree of the polynomial (either $2N$ or $2N+1$) was fixed. In this section we require a control of the L^1, L^∞ or BMO norms of b, as well as its decay at infinity. We summarize all we need in next lemma.

Lemma 5. *There exists a constant C depending only on n such that*

(i) $|\widehat{b_{2N}}(\xi)| \le C$ *and* $|\widehat{b_{2N+1}}(\xi)| \le C, \quad \xi \in \mathbb{R}^n$.

(ii) $\|b_{2N}\|_{L^\infty(B)} \le C(2N)^{2n+2}$ *and* $\|\nabla b_{2N}\|_{L^\infty(B)} \le C(2N)^{2n+4}$.

(iii) $\|b_{2N+1}\|_{\text{BMO}} \le C(2N+1)^{2n}$ *and* $\|b_{2N+1}\|_{L^2} \le C(2N+1)^{2n}$.

(iv) If $|x| > 2$ *then* $|b_{2N+1}(x)| \le C(2N+1)^{2n}|x|^{-n-1}$.

Proof. Parts *(i)*, *(ii)* and *(iii)* are proved in [MOV, Lemma 8] and [MOPV, Lemma 5]. It only remains to prove *(iv)*.

Recall that σ denotes the normalized surface measure in S^{n-1}, and let $h_1, \dots, h_d$ be an orthonormal basis of the subspace of $L^2(d\sigma)$ consisting of all homogeneous harmonic polynomials of degree $2N+1$. As it is well known, $d \simeq (2N+1)^{n-2}$. As in the proof of Lemma 6 in [MOV] we have $h_1^2 + \cdots + h_d^2 = d$, on S^{n-1}. Set

$$H_j(x) = \frac{1}{\gamma_{2N+1}\sqrt{d}}\, h_j(x), \quad x \in \mathbb{R}^n\,,$$

and let S_j be the higher order Riesz transform with kernel $K_j(x) = H_j(x)/|x|^{2N+1+n}$. The Fourier multiplier of S_j^2 is

$$\frac{1}{d}\,\frac{h_j(\xi)^2}{|\xi|^{4N+2}}, \quad 0 \neq \xi \in \mathbb{R}^n\,,$$

and thus

$$\sum_{j=1}^{d} S_j^2 = \text{Identity}\,. \tag{20}$$

We use again (12), but now the second term at the right hand side vanishes because each h_j is harmonic (see [MOV], p. 1437). We get

$$K_j(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = S_j(b_{2N+1})(x), \quad x \in \mathbb{R}^n, \quad 1 \le j \le d\,,$$

and so by (20)

$$b_{2N+1} = \sum_{j=1}^{d} S_j\left(K_j(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}}(x)\right)\,. \tag{21}$$

Therefore we set

$$\begin{aligned}\sum_{j=1}^{d} S_j\left(K_j(x)\,\chi_{\mathbb{R}^n \setminus \overline{B}}(x)\right) &= \sum_{j=1}^{d} S_j * S_j - \sum_{j=1}^{d} S_j\left(K_j(x)\,\chi_B(x)\right)\\ &= \delta_0 - \sum_{j=1}^{d} S_j\left(K_j(x)\,\chi_B(x)\right),\end{aligned}$$

where δ_0 is the Dirac delta at the origin. If $|x| > 2$, then

$$\begin{aligned}S_j(K_j(y)\,\chi_B(y))(x) &= \lim_{\varepsilon \to 0} \int_{\varepsilon < |y| < 1} K_j(x-y) K_j(y)\, dy\\ &= \lim_{\varepsilon \to 0} \int_{\varepsilon < |y| < 1} (K_j(x-y) - K_j(x)) K_j(y)\, dy.\end{aligned}$$

In this situation,

$$|K_j(x-y)-K_j(x)| \leq C\frac{|y|}{|x|^{n+1}}\left(\|H_j\|_\infty(2N+1)+\|\nabla H_j\|_\infty\right),$$

hence

$$|S_j(K_j(y)\,\chi_B(y))(x)| \leq C\frac{\|H_j\|_\infty(2N+1)+\|\nabla H_j\|_\infty}{|x|^{n+1}}\int_{|y|<1}\frac{\|H_j\|_\infty}{|y|^{n-1}}\,dy$$

where the supremum norms are taken on S^{n-1}. Clearly

$$\|H_j\|_\infty = \frac{1}{\gamma_{2N+1}}\left\|\frac{h_j}{\sqrt{d}}\right\|_\infty \leq \frac{1}{\gamma_{2N+1}} \simeq (2N+1)^{n/2}.$$

For the estimate of the gradient of H_j we use the inequality [St, p. 276]

$$\|\nabla H_j\|_\infty \leq C\,(2N+1)^{n/2+1}\,\|H_j\|_2,$$

where the L^2 norm is taken with respect to $d\sigma$. Since the h_j are an orthonormal system,

$$\|H_j\|_2 = \frac{1}{\sqrt{d}\,\gamma_{2N+1}} \simeq \frac{(2N+1)^{n/2}}{(2N+1)^{(n-2)/2}} \simeq 2N+1.$$

Gathering the above inequalities we get, when $|x|>2$,

$$|S_j(K_j(y)\,\chi_B(y))(x)| \leq C\frac{(2N+1)^{n+2}}{|x|^{n+1}}$$

and finally

$$|b_{2N+1}(x)| \leq Cd\frac{(2N+1)^{n+2}}{|x|^{n+1}} \leq C\frac{(2N+1)^{2n}}{|x|^{n+1}},$$

as claimed. □

Now, the kernel of the operator $Tf = \text{p.v. } K*f$ is of the type $K(x)=\frac{\Omega(x)}{|x|^n}$ being Ω a $C^\infty(S^{n-1})$ homogeneous function of degree 0, with vanishing integral on the sphere. Then, $\Omega(x)=\sum_{j\geq 1}^\infty \frac{P_{2j}(x)}{|x|^{2j}}$ with P_{2j} homogeneous harmonic polynomials of degree $2j$ when T is an even operator, and $\Omega(x)=\sum_{j\geq 0}^\infty \frac{P_{2j+1}(x)}{|x|^{2j+1}}$ with P_{2j+1} homogeneous harmonic polynomials of degree $2j+1$ when T is an odd operator. The strategy consists in passing to the polynomial case by looking at a partial sum of the series above. Set, for each $N\geq 1$, $K_N(x)=\frac{\Omega_N(x)}{|x|^n}$, where $\Omega_N(x)=\sum_{j=1}^N \frac{P_{2j}(x)}{|x|^{2j}}$ (or $\Omega_N(x)=\sum_{j=0}^N \frac{P_{2j+1}(x)}{|x|^{2j+1}}$ in the odd case), and let T_N be the operator with kernel K_N.

We begin by considering *(b)* implies *(c)* in Theorem 1 when $\omega=1$, that is, T is even and our hypothesis is $\|T^*f\|_p \leq C\|Tf\|_p$, $f\in L^p(\mathbb{R}^n)$. In this setting, the difficulty is that there is no obvious way of obtaining the inequality

$$\|T_N^*f\|_p \leq C\|T_Nf\|_p, \quad f\in L^p(\mathbb{R}^n). \tag{22}$$

Instead, we try to get (22) with $\|T_N f\|_p$ replaced by $\|Tf\|_p$ in the right hand side plus an additional term which becomes small as N tends to ∞. We start by writing

$$\begin{aligned}\|T_N^1 f\|_p &\le \|T^1 f\|_p + \|T^1 f - T_N^1 f\|_p \\ &\le C\|Tf\|_p + \|\sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c} * f\|_p.\end{aligned} \tag{23}$$

By (12), and since every P_{2j} is harmonic, there exists a bounded function b_{2j} supported on B such that

$$\frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c}(x) = \text{p.v. } \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j}.$$

By Lemma 5 *(ii)* , we have that $\|b_{2j}\|_{L^1} \le C\|b_{2j}\|_{L^\infty(B)} \le C(2j)^{2n+2}$, and thus

$$\begin{aligned}\|\sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}} \chi_{\overline{B}^c} * f\|_p &= \|\sum_{j>N} \text{p.v. } \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j} * f\|_p \\ &\le \sum_{j>N} \|\frac{P_{2j}(x)}{|x|^{2j+n}}\|_{L^p \to L^p} \|b_{2j} * f\|_p \\ &\le \sum_{j>N} \|\frac{P_{2j}(x)}{|x|^{2j+n}}\|_{L^p \to L^p} \|b_{2j}\|_1 \|f\|_p \\ &\le C\|f\|_p \sum_{j>N} \|\frac{P_{2j}(x)}{|x|^{2j+n}}\|_{L^p \to L^p} (2j)^{2n+2} \\ &\le C\|f\|_p \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty)(2j)^{2n+2}.\end{aligned} \tag{24}$$

The last inequality follows from a well-known estimate for Calderón-Zygmund operators (e.g. [Gr1, Theorem 4.3.3]). On the other hand,

$$K_N(x)\chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T_N(b_{2N})(x) + S_N(x)\chi_B(x)$$

and then

$$T_N^1 f = \text{p.v. } K_N * b_{2N} * f + S_N \chi_B * f.$$

So, for each $f \in L^p(\mathbb{R}^n)$, using (23) and (24), we have the L^p inequality

$$\begin{aligned}&\|S_N \chi_B * f\|_p \le \|T_N^1 f\|_p + \|\text{p.v. } K_N * b_{2N} * f\|_p \\ &\le C\left(\|Tf\|_p + \|f\|_p \sum_{j>N} (\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty)(2j)^{2n+2} + \|\text{p.v. } K_N * b_{2N} * f\|_p \right).\end{aligned}$$

We emphasize that the corresponding multipliers $\widehat{S_N \chi_B}$, $\widehat{\text{p.v. } K}$ and $\text{p.v. } \widehat{K_N * b_{2N}} = \widehat{\text{p.v. } K_N}\widehat{b_{2N}}$ are in $\mathcal{C}^\infty \setminus \{0\}$ and in $\mathcal{M}_p$. Therefore, proceeding as in the polynomial case, and applying Lemma 1 we obtain the pointwise estimate for $\xi \neq 0$

$$|\widehat{S_N\chi_B}(\xi)| \leq C\left(|\widehat{\text{p.v. } K}(\xi)| + |(\widehat{\text{p.v. } K_N} \cdot \widehat{b_{2N}})(\xi)| + \sum_{j>N}(\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty)(2j)^{2n+2}\right)$$

$$\leq C\left(|\widehat{\text{p.v. } K}(\xi)| + |\widehat{\text{p.v. } K_N}(\xi)| + \sum_{j>N}(\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty)(2j)^{2n+2}\right),$$

where in the last step we have used Lemma 5 *(i)*, that is, $|\widehat{b_{2N}}(\xi)| \leq C$, for $\xi \in \mathbb{R}^n$.

The idea is now to take limits, as N goes to ∞, in the preceding inequality. By the definition of K_N and (6), the term on the right-hand side converges to $C|\widehat{\text{p.v. } K}(\xi)|$. The next task is to clarify how the left-hand side converges, but at this point we proceed as in [MOV, p. 1463] and we get the desired result.

This argument, which has been explained for the even case and $\omega = 1$, is also valid for the other cases, after taking into account the particular details listed below.

To get *(b)* implies *(c)* in Theorem 1 for any $\omega \in A_p$, we would use

$$\|b_{2j} * f\|_{L^p(\omega)} \leq C\|b_{2j}\|_{L^\infty(B)}\|Mf\|_{L^p(\omega)} \leq C(2j)^{2n+2}\|f\|_{L^p(\omega)}$$

to obtain the inequality analogous to (24).

In order to obtain *(d)* implies *(c)* in Theorem 1, note that if $c_j > 0$ and $\sum_{j=1}^\infty c_j = 1$, then $\|\sum g_j\|_{1,\infty} \leq \sum c_j^{-1}\|g_j\|_{1,\infty}$. We have

$$\begin{aligned}
\|\sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}}\chi_{\overline{B}^c} * f\|_{1,\infty} &= \|\sum_{j>N} \text{p.v. } \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j} * f\|_{1,\infty} \\
&\leq \sum_{j>N} j^2\|\text{p.v. } \frac{P_{2j}(x)}{|x|^{2j+n}} * b_{2j} * f\|_{1,\infty} \\
&\leq \sum_{j>N} j^2\|\frac{P_{2j}(x)}{|x|^{2j+n}}\|_{L^1\to L^{1,\infty}}\|b_{2j} * f\|_1 \\
&\leq \sum_{j>N} j^2\|\frac{P_{2j}(x)}{|x|^{2j+n}}\|_{L^1\to L^{1,\infty}}\|b_{2j}\|_1\|f\|_1 \\
&\leq C\|f\|_1\sum_{j>N} j^2\|\frac{P_{2j}(x)}{|x|^{2j+n}}\|_{L^1\to L^{1,\infty}}(2j)^{2n+2} \\
&\leq C\|f\|_1\sum_{j>N}(\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty)(2j)^{2n+4},
\end{aligned}$$

and therefore, for all functions $f \in H^1(\mathbb{R}^n)$,

$$\begin{aligned}\|S_N\chi_B * f\|_{1,\infty} &\le 2(\|T_N^1 f\|_{1,\infty} + \|\text{p.v. } K_N * b_{2N} * f\|_{1,\infty}) \\ &\le 4(\|T^1 f\|_{1,\infty} + \|\sum_{j>N} \frac{P_{2j}(x)}{|x|^{2j+n}}\chi_{\overline{B}^c} * f\|_{1,\infty}) + 2\|\text{p.v. } K_N * b_{2N} * f\|_{1,\infty}) \\ &\le C(\|Tf\|_1 + \|f\|_1 \sum_{j>N}(\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty)(2j)^{2n+4} \\ &\qquad + \|\text{p.v. } K_N * b_{2N} * f\|_{1,\infty}).\end{aligned}$$

Again, using Lemma 1, Lemma 4 and Lemma 5, we obtain, for $\xi \neq 0$,

$$|\widehat{S_N\chi_B}(\xi)| \le C\left(|\widehat{\text{p.v. } K}(\xi)| + |\widehat{\text{p.v. } K_N}(\xi)| + \sum_{j>N}(\|P_{2j}\|_\infty + \|\nabla P_{2j}\|_\infty)(2j)^{2n+4}\right)$$

as desired.

The implication $(b) \Rightarrow (c)$ in Theorem 2 can be adapted as follows. T is odd and the functions b_{2j+1} are in BMO. By Lemma 5, we have $\|\widehat{b_{2j+1}}\|_\infty \le C$, $\|b_{2j+1}\|_{\text{BMO}} \le C(2j+1)^{2n}$ and $\|b_{2j+1}\|_2 \le C(2j+1)^{2n}$. Moreover, $|b_{2j+1}(x)| \le C(2j+1)^{2n}|x|^{-n-1}$ if $|x| > 2$. Then, proceeding in the same way as in the proof of (19), we get

$$\|b_{2j+1} * f\|_{L^p(\omega)} \le C(2j+1)^{2n}\|f\|_{L^p(\omega)}$$

and so, the inequality analogous to (24) follows.

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Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia

E-mail: abosch@mat.uab.cat
E-mail: mateu@mat.uab.cat
E-mail: orobitg@mat.uab.cat