On the isoperimetric and Hurwitz inequalities

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Abstract

In this paper we provide lower bounds for both the isoperimetric deficit $\Delta = L^2 - 4\pi F$, where F is the area enclosed by a convex curve of length L, and the Hurwitz's deficit $\pi |F_e| - \Delta$, F_e being the algebraic area enclosed by the evolute of the curve. These bounds involve some geometrical invariants related to the curve.

1 Introduction

Let K be a plane convex set of area F with boundary $C = \partial K$ of length L. As it is well known, the isoperimetric inequality states

$$F \le \frac{L^2}{4\pi},$$

with equality only for discs.

Hurwitz, in his paper about the use of Fourier series in some geometrical problems [3] (see also [2]), besides the isoperimetric inequality, proves the following inequality, which is a sort of reverse isoperimetric inequality,

$$\frac{L^2}{4\pi} \le F + \frac{1}{4}|F_e|,\tag{1}$$

where F_e is the algebraic area enclosed by the evolute of C. Equality holds when C is parallel to an astroid or a circle.

Recall that the *evolute* of a plane curve is the locus of its centers of curvature or, equivalently, the *envelope* of all the normals to this curve (i.e., the tangents to the evolute are the normals to the curve).

Introducing the isoperimetric deficit $\Delta = L^2 - 4\pi F$ the above two inequalities can be written as

$$0 \le \Delta \le \pi |F_e|.$$

In this note we provide lower bounds for both Δ and $\pi |F_e| - \Delta$. These bounds involve some geometrical quantities that we describe below.

The *pedal curve* of a plane curve C with respect to a fixed point O is the locus of points X so that the line OX is perpendicular to the tangent to C passing through X. The *Steiner* point of a plane convex set K, or the curvature centroid of K, is the center of mass of ∂K with respect to the density function that assigns to each point of ∂K its curvature.

Let A denote the area enclosed by the pedal curve of $C = \partial K$ with respect to the Steiner point of K. In Theorem 3.1 it is proved that

$$\Delta \ge 3\pi (A - F). \tag{2}$$

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So, the quantity $3\pi(A - F)$ is a lower estimate of the isoperimetric deficit. Since $A \ge F$, this inequality implies the isoperimetric one. Corollary 3.3 shows that equality holds in (2) for circles and curves which are parallel to an astroid.

The next estimate depends on the quantity Ω defined by

$$\Omega = \int \sin^3 \omega \, dP,\tag{3}$$

where ω is the angle between the tangent lines to C from P, and the integral is extended to the exterior of C. We say that ω is the viewing angle to C from P.

In Theorem 4.1 we prove that

$$\pi |F_e| - \Delta \ge \frac{40}{9} \left(\pi (A - F) - 8\Omega + 6L^2 \right).$$
(4)

Since the right hand side is non negative, this inequality implies Hurwitz's result (1). Corollary 4.3 shows that equality holds in (4) for convex bodies K which are the Minkowski sum of a disc or the interior of a curve parallel to an astroid plus the interior of an hyocyclid of three cusps.

Finally in Proposition 5.1 it is shown that for convex curves C parallel to an astroid the evolute of C is similar, with ratio 2, to this astroid.

2 Preliminaries

Support function

A straight line G in the plane is determined by the angle ϕ that the direction perpendicular to G makes with the positive x-axis and the distance $p = p(\phi)$ of G from the origin. The equation of G then takes the form

$$x\cos\phi + y\sin\phi - p = 0. \tag{5}$$

Equation (5), when $p = p(\phi)$ varies with ϕ , is the equation of a family of lines. If we assume that the 2π -periodic function $p(\phi)$ is differentiable, the envelope of the family is obtained from (5) and the derivative of its left-hand side, as follows:

$$-x\sin\phi + y\cos\phi - p' = 0, \quad p' = dp/d\phi. \tag{6}$$

From (5) and (6) we arrive at a parametric representation of the envelope of the lines (5):

$$x = p\cos\phi - p'\sin\phi, \qquad y = p\sin\phi + p'\cos\phi.$$

If the envelope is the boundary ∂K of a convex set K and the origin is an interior point of K, then $p(\phi)$ is called the *support function* of K (or the support function of the convex curve ∂K).

Since $dx = -(p + p'') \sin \phi \, d\phi$ and $dy = (p + p'') \cos \phi \, d\phi$ (we here assume that the function p is of class C^2), arclength measure on ∂K is given by

$$ds = \sqrt{dx^2 + dy^2} = |p + p''| \, d\phi \tag{7}$$

and the radius of curvature ρ by

$$\rho = \frac{ds}{d\phi} = |p + p''|.$$

It is well known (see for instance [4], page 3) that a necessary and sufficient condition for a periodic function p to be the support function of a convex set K is that p + p'' > 0. Finally, it follows from (7) that the length of a closed convex curve that has support function p of class C^2 is given by

$$L = \int_0^{2\pi} p \, d\phi. \tag{8}$$

The area of the convex set K is expressed in terms of the support function by

$$F = \frac{1}{2} \int_{\partial K} p ds = \frac{1}{2} \int_0^{2\pi} p(p + p'') d\phi = \frac{1}{2} \int_0^{2\pi} p^2 d\phi - \frac{1}{2} \int_0^{2\pi} p'^2 d\phi.$$
(9)

For any curve C given by $(x(\phi), y(\phi))$, convex or not, we will say that $p(\phi)$ is the generalized support function of C when

$$\begin{aligned} x(\phi) &= p(\phi)\cos(\phi) - p'(\phi)\sin(\phi), \\ y(\phi) &= p(\phi)\sin(\phi) + p'(\phi)\cos(\phi). \end{aligned}$$

Note that $p(\phi)$ is not necessarily a distance, as it happens when we define the support function of a convex set. In fact, $|p(\phi)|$ is the distance from the origin to the tangent to C at the point $(x(\phi), y(\phi))$.

It is easy to see that the generalized support function $p_e(\phi)$ of the evolute of $C = \partial K$ is $p_e(\phi) = -p'(\phi + \pi/2)$, where $p(\phi)$ is the support function of C, see [1]. Hence, assuming $p(\phi)$ is a C^3 -function, the algebraic area F_e enclosed by the evolute of C is given by

$$F_e = \frac{1}{2} \int_0^{2\pi} p'(p' + p''') \, d\phi = \frac{1}{2} \int_0^{2\pi} p'^2 \, d\phi - \frac{1}{2} \int_0^{2\pi} p''^2 \, d\phi.$$

Steiner point

The Steiner point of a convex set K of the Euclidean plane is defined by

$$S(K) = \frac{1}{\pi} \int_0^{2\pi} (\cos\phi, \sin\phi) p(\phi) d\phi,$$

where $p(\phi)$ is the support function of ∂K (see [2]).

Thus, if

$$p(\phi) = a_0 + \sum_{n \ge 1} a_n \cos n\phi + b_n \sin n\phi,$$

is the Fourier series of the 2π -periodic function $p(\phi)$, the Steiner point is

$$S(K) = (a_1, b_1).$$

The Stiener point of K is also known as the curvature centroid of K because under appropriate smothness conditions it is the center of mass of ∂K with respect to the density function that assigns to each point of ∂K its curvature.

The relation between the support function $p(\phi)$ of a convex set K and the support function $q(\phi)$ of the same convex set but with respect to a new reference with origin at the point (a, b), and axes parallel to the previous x and y-axes, is given by

$$q(\phi) = p(\phi) - a\cos\phi - b\sin\phi.$$

Hence, taking the Steiner point as a new origin, we have

$$q(\phi) = a_0 + \sum_{n \ge 2} a_n \cos n\phi + b_n \sin n\phi$$

Pedal curve

If the curve C is given in cartesian coordinates as the envelope of the lines $x \cos \phi + y \sin \phi - p(\phi) = 0$, then the pedal curve $\mathcal{P} = \mathcal{P}(\phi)$ of C with respect to the origin, is given by

$$\mathcal{P}(\phi) = (p(\phi)\cos\phi, p(\phi)\sin\phi),$$

or, in polar coordinates, by $r = p(\phi)$.

In particular, if C is closed, the area enclosed by \mathcal{P} is

$$A = \frac{1}{2} \int_0^{2\pi} p^2 \, d\phi. \tag{10}$$

If F is the area enclosed by C, we obviously have $A \ge F$ with equality if and only if C is a circle.

3 A lower bound for the isoperimetric deficit

We proceed now to provide a lower bound for the isoperimetric deficit.

Theorem 3.1 Let K be a convex set of area F with boundary $C = \partial K$ of class C^2 and length L. Let A be the area enclosed by the pedal curve of C with respect to the Steiner point S(K). Then

$$\Delta \ge 3\pi (A - F),\tag{11}$$

where $\Delta = L^2 - 4\pi F$ is the isoperimetric deficit.

Proof. Let $p(\phi)$ be the support function of C, with respect to an orthonormal reference with origin in the Steiner point, and axes parallel to the x and y-axes.

We know that the Fourier series of $p(\phi)$, is

$$p(\phi) = a_0 + \sum_{n \ge 2} a_n \cos n\phi + b_n \sin n\phi.$$

By Parseval's identity we have

$$\frac{1}{2\pi} \int_0^{2\pi} p^2 \, d\phi = a_0^2 + \frac{1}{2} \sum_{n \ge 2} (a_n^2 + b_n^2),\tag{12}$$

and similar expressions for p' and p''. Concretely we have

$$\int_{0}^{2\pi} p^{\prime 2} d\phi = \pi \sum_{n \ge 2} n^{2} (a_{n}^{2} + b_{n}^{2}), \qquad \int_{0}^{2\pi} p^{\prime \prime 2} d\phi = \pi \sum_{n \ge 2} n^{4} (a_{n}^{2} + b_{n}^{2}).$$
(13)

Hence, the isoperimetric deficit $\Delta = L^2 - 4\pi F$, according to (8) and (9), is given by

$$\begin{aligned} \Delta &= \left(\int_0^{2\pi} p \, d\phi\right)^2 - 2\pi \int_0^{2\pi} p^2 \, d\phi + 2\pi \int_0^{2\pi} p'^2 \, d\phi \\ &= 2\pi^2 \sum_{n \ge 2} (n^2 - 1)(a_n^2 + b_n^2) \ge \frac{3\pi^2}{2} \sum_{n \ge 2} n^2 (a_n^2 + b_n^2) = \frac{3\pi}{2} \int_0^{2\pi} p'^2 \, d\phi. \end{aligned}$$

But it follows from (9) and (10) that

$$\frac{1}{2} \int_0^{2\pi} p'^2 \, d\phi = \frac{1}{2} \int_0^{2\pi} p^2 \, d\phi - F = A - F,$$

and hence

$$\Delta \ge 3\pi (A - F). \qquad \Box$$

The above proof shows that $\Delta = 0$ if and only if $p(\phi) = a_0$, that is, when C is a circle.

Now we study the case of equality in (11). It is clear from the proof that $\Delta = 3\pi(A - F)$ if and only if

$$p(\phi) = a_0 + a_2 \cos 2\phi + b_2 \sin 2\phi$$

In order to characterize the curves with this support function we recall that the parametric equations of the *astroid* (a 4-cusped hypocycloid) are

$$\begin{aligned} x(\phi) &= 2a\sin^3(\phi), \\ y(\phi) &= 2a\cos^3(\phi), \end{aligned}$$

for some constant $a \in \mathbb{R}^+$, with $0 \le \phi \le 2\pi$. From this it is easy to see that the generalized support function $p(\phi)$ of the astroid is $p(\phi) = a \sin(2\phi)$, where ϕ is the angle between the normal $(-y'(\phi), x'(\phi))$ and the positive x-axis.

This implies that the curves with generalized support function given by

$$q(\phi) = b + p(\phi) = b + a\sin(2\phi),$$

where $b \in \mathbb{R}$, are parallel to an astroid. The distance between these curves and the astroid is |b|.

We have the following result.

Proposition 3.2 Let

$$p(\phi) = a_0 + a_2 \cos(2\phi) + b_2 \sin(2\phi)$$

be the support function of a closed convex curve C of length L, with $a_2^2 + b_2^2 \neq 0$. Then the interior parallel curve to C at distance $L/2\pi$ is an astroid.

Proof. We make the change of variable $u = \phi - \phi_0 + \frac{\pi}{4}$, where

$$\tan 2\phi_0 = \frac{b_2}{a_2}.$$

We obtain

$$p(u) = a_0 + a\sin 2u$$

where $a = \sqrt{a_2^2 + b_2^2}$. Hence the given curve is parallel to an astroid at distance $|a_0|$. By the condition of convexity, p + p'' > 0, and a_0 is positive. Since $L = \int_0^{2\pi} p(\phi) d\phi = 2\pi a_0$, the proposition is proved. \Box

Corollary 3.3 Equality in Theorem 3.1 holds if and only if C is a circle or a curve parallel to an astroid.

Proof. We have seen that equality holds when

$$p(\phi) = a_0 + a_2 \cos 2\phi + b_2 \sin 2\phi$$

If $a_2 = b_2 = 0$, $p(\phi) = a_0$ is the support function of a circle. If $a_2^2 + b_2^2 \neq 0$, the result follows directly from Proposition 3.2. \Box

Remark 3.4 As it is well known (see for instance [4], page 8) the area F_r enclosed by the interior parallel at distance r to a closed curve is given by

$$F_r = F - Lr + \pi r^2$$

where L and F are respectively the length and the area corresponding to the given curve. In particular, if $r = L/2\pi$, we get

$$F_{L/2\pi} = F - \frac{L^2}{4\pi},$$

or, equivalently

$$F_{L/2\pi} = -\frac{\Delta}{4\pi}.$$

This means that the isoperimetric inequality $\Delta \geq 0$ is equivalent to $F_{L/2\pi} \leq 0$, a fact that suggests a more geometric proof of the isoperimetric inequality, by showing that in the process of collapsing, the curve reverses orientation. Moreover, $F_{L/2\pi} = 0$ holds only for a circle.

Remark 3.5 Combining Theorem 3.1 with Hurwitz's inequality (1) we have the estimate

$$A - F \le \frac{1}{3} |F_e|,\tag{14}$$

with equality for curves parallel to an astroid or circles.

4 A lower bound for the Hurwitz's deficit

We proceed now to find a lower bound for the Hurwit's deficit $\pi |F_e| - \Delta$.

If

$$p(\phi) = a_0 + \sum_{n \ge 1} a_n \cos n\phi + b_n \sin n\phi,$$

is the Fourier series of the support function of a convex curve C, it is easy to see that the quantities $a_n^2 + b_n^2$, for $n \ge 2$, are invariants of this curve under the group of movements of the plane. We shall use the following geometrical interpretation of $a_2^2 + b_2^2$, due to Hurwitz, [3]:

$$\Omega := \int \sin^3 \omega \, dP = \frac{3}{4}L^2 + \frac{\pi^2}{4}(a_2^2 + b_2^2),\tag{15}$$

where ω is the viewing angle of C from P, and the integral is extended to the exterior of C.

Theorem 4.1 Let K be a convex set of area F with boundary $C = \partial K$ of class C^3 and length L, and let A be the area enclosed by the pedal curve of C with respect to the Steiner point S(K). Let F_e be the algebraic area enclosed by the evolute of C and denote $\Delta = L^2 - 4\pi F$ the isoperimetric deficit. Then

$$\pi |F_e| - \Delta \ge \frac{40}{9} \left(\pi (A - F) - 8\Omega + 6L^2 \right)$$
(16)

where Ω is defined in (3).

Proof. Recall that the area of the evolute in terms of the support function is given by

$$|F_e| = \frac{1}{2} \int_0^{2\pi} p''^2 \, d\phi - \frac{1}{2} \int_0^{2\pi} p'^2 \, d\phi.$$

If $p(\phi)$ is taken with respect to the Steiner point S(K) then, using equalities (8), (9), (12) and (13), we have

$$\begin{aligned} \pi |F_e| - \Delta &= -\left(\int_0^{2\pi} p \, d\phi\right)^2 + 2\pi \int_0^{2\pi} p^2 \, d\phi - 2\pi \int_0^{2\pi} p'^2 \, d\phi + \frac{\pi}{2} \int_0^{2\pi} p''^2 \, d\phi \\ &- \frac{\pi}{2} \int_0^{2\pi} p'^2 \, d\phi. \\ &= 2\pi^2 \sum_{n \ge 2} (1 - n^2) (a_n^2 + b_n^2) + \frac{\pi^2}{2} \sum_{n \ge 2} (n^4 - n^2) (a_n^2 + b_n^2) \\ &= 2\pi^2 \sum_{n \ge 2} (1 - n^2 + \frac{n^4 - n^2}{4}) (a_n^2 + b_n^2) = \frac{\pi^2}{2} \sum_{n \ge 3} (n^4 - 5n^2 + 4) (a_n^2 + b_n^2) \end{aligned}$$

We observe now that, for $n \ge 3$, we have $n^4 - 5n^2 + 4 \ge \frac{40}{9}n^2$, with equality only for n = 3. Therefore

$$\pi |F_e| - \Delta \ge \frac{20\pi^2}{9} \sum_{n \ge 3} n^2 (a_n^2 + b_n^2) = \frac{20\pi^2}{9} \sum_{n \ge 2} n^2 (a_n^2 + b_n^2) - \frac{20\pi^2}{9} 4(a_2^2 + b_2^2) \ge 0,$$

which can be written as

$$\begin{aligned} \pi |F_e| - \Delta &\geq \frac{40\pi}{9} \left(\frac{1}{2} \int_0^{2\pi} p'^2 \, d\phi - \frac{1}{2} \int_0^{2\pi} p^2 \, d\phi + \frac{1}{2} \int_0^{2\pi} p^2 \, d\phi \right) - \frac{80\pi^2}{9} (a_2^2 + b_2^2) \\ &= \frac{40\pi}{9} (A - F) - \frac{80\pi^2}{9} (a_2^2 + b_2^2) \geq 0. \end{aligned}$$

Replacing $a_2^2 + b_2^2$ by its value from (15) one gets the desired result. \Box

The proof above shows that

$$\pi(A-F) - 8\Omega + 6L^2 \ge 0,$$

and so inequality (16) implies Hurwitz's inequality $\Delta \leq \pi |F_e|$. Moreover $\Delta = \pi |F_e|$ holds if and only if the support function $p(\phi)$ is of the form

$$p(\phi) = a_0 + a_2 \cos(2\phi) + b_2 \sin(2\phi),$$

and so, by Proposition 3.2, C must be parallel to an astroid or a circle.

Now we will analyze when equality holds in (16). For this we recall that the parametric equations of an *hypocycloid of three cusps*, with respect to a suitable orthogonal system, are

$$\begin{aligned} x(t) &= -2a\cos t - a\cos 2t \\ y(t) &= -2a\sin t + a\sin 2t \end{aligned}$$

with $a \in \mathbb{R}, t \in [0, 2\pi]$.

The relationship between the parameter t and the angle $\phi(t)$ between the normal vector (-y'(t), x'(t)) and the positive x-axis is

$$\phi(t) = \alpha(t) - \frac{\pi}{2},$$

where $\alpha(t)$ denotes the angle between the tangent vector (x'(t), y't) and the positive x-axis. Hence

$$\tan \phi(t) = -\cot \alpha(t) = \frac{\sin t + \sin 2t}{\cos t - \cos 2t} = \cot \frac{t}{2}.$$

Thus

$$t = \pi - 2\phi(t).$$

On the other hand, the generalized support function $p(\phi)$ of the hypocycloid must verify

$$\left(\begin{array}{c} x(\phi) \\ y(\phi) \end{array}\right) = \left(\begin{array}{c} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{array}\right) \left(\begin{array}{c} p(\phi) \\ p'(\phi) \end{array}\right),$$

 \mathbf{SO}

$$\begin{pmatrix} p(\phi) \\ p'(\phi) \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} -2a\cos(\pi - 2\phi) - a\cos 2(\pi - 2\phi) \\ -2a\sin(\pi - 2\phi) + a\sin 2(\pi - 2\phi) \end{pmatrix}$$

Then

$$p(\phi) = a\cos(3\phi).$$

Proposition 4.2 Let

$$p(\phi) = a_3 \cos 3\phi + b_3 \sin 3\phi$$

be the generalized support function of a closed curve C, with $a_3^2 + b_3^2 \neq 0$. Then C is a hypocycloid of three cusps.

Proof. We make the change of variable given by $u = \phi - \frac{\psi_0}{3}$, where

$$\tan\psi_0 = \frac{b_3}{a_3}.$$

Then

$$p(u) = a\cos(3u),$$

where $a = \frac{a_3}{\cos \psi_0}$, and the proposition follows. \Box

Corollary 4.3 Equality in Theorem 4.1 holds if and only if K is the Minkowski sum of two bodies K_1 and K_2 ,

$$K_1 + K_2 = \{x + y; x \in K_1, y \in K_2\},\$$

where K_1 is a disk or the interior of a curve parallel to an astroid, and K_2 is the interior of an hypocycloid of three cusps.

Proof. It is clear from the proof of Theorem 4.1 that equality holds if and only if the support function is of the form

 $p(\phi) = a_0 + a_2 \cos 2\phi + b_2 \sin 2\phi + a_3 \cos 3\phi + b_3 \sin 3\phi.$

If we put $p_1(\phi) = a_0 + a_2 \cos 2\phi + b_2 \sin 2\phi$ and $p_2(\phi) = a_3 \cos 3\phi + b_3 \sin 3\phi$, we have $p(\phi) = p_1(\phi) + p_2(\phi)$ and so K is the Minkowski's sum of the bodies K_1 and K_2 with generalized support functions $p_1(\phi)$ and $p_2(\phi)$ respectively.

We know, by Proposition 3.2, that K_1 is the interior of a curve parallel to an astroid or a circle and, by Proposition 4.2, K_2 is the interior of a hypocycloid. \Box

Remark 4.4 Combining Theorems 3.1 and 4.1 one gets the inequality

$$A - F \le \frac{9}{67} |F_e| + \frac{80}{67\pi} (4\Omega - 3L^2)$$

stronger than (14), with equality for curves described in Corollary 4.3.

Remark 4.5 Another interesting relation between the enclosed area, the length and the viewing angle ω of a curve C from a point P, also due to Hurwitz (cf. [3]), is

$$\int (\omega - \sin \omega) dP = \frac{L^2}{2} - \pi F,$$

where the integral is extended at the exterior of C.

Since $6(\omega - \sin \omega) \ge \sin^3 \omega$ for $0 \le \omega \le \pi$, we get the noteworthy inequalities:

$$\Omega \le 3L^2 - 6\pi F,$$

and by (15),

$$a_2^2 + b_2^2 \le \frac{9}{\pi^2}L^2 - \frac{24}{\pi}F,$$

which gives a bound for the invariant $a_2^2 + b_2^2$.

5 Parallel curves to an astroid and evolutes

We have seen the role played by the convex curves considered in Proposition 3.2. For such a curve C we show now the relationship between the astroid parallel to C and the evolute of C.

Proposition 5.1 Let

$$p(\phi) = a_0 + a_2 \cos(2\phi) + b_2 \sin(2\phi)$$

be the support function of a closed convex curve C of length L. Then the evolute of C and the interior parallel curve to C at distance $L/2\pi$, are similar with ratio 2.

Proof. We shall see that there is a similarity, composition of a rotation with an homothecy, applying the parallel curve on the evolute. We may assume, by Proposition 3.2, $p(\phi) = a_0 + c\sin(2\phi)$. The generalized support function of the parallel curve to C at distance $L/2\pi = a_0$ is $q(t) = c\sin(2\phi)$ and the corresponding one to the evolute of C is

$$p_e(\phi) = -p'(\phi + \frac{\pi}{2}) = 2c\cos(2\phi).$$

The generalized support function of the rotated $3\pi/4$ parallel curve is

$$\tilde{p}(\phi) = q(\phi - \frac{3\pi}{4}) = c\cos(2\phi)$$

Hence this rotated curve is homothetic, with ratio 2, to the evolute. \Box

In particular, the area of the evolute of such a curve is four times the area of the parallel curve at distance $L/2\pi$. Moreover, since Hurwitz's inequality is equivalent to

$$4|F_{L/2\pi}| - |F_e| \le 0,$$

the curves for which the area of the evolute is four times the area of the parallel curve at distance $L/2\pi$, are exactly the curves parallel to an astroid or circles.

Next figure shows the convex curve C with support function $p(\phi) = 5 + \sin(2\phi)$, its parallel interior curve at distance $L/2\pi = 5$, and the evolute of C.



Parallel curve (black) and evolute (red).

References

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