ASYMPTOTIC PROFILE IN SELECTION-MUTATION EQUATIONS: GAUSS VERSUS CAUCHY DISTRIBUTIONS

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Abstract. In this paper, we study the asymptotic (large time) behavior of a selection-mutation-competition model for a population structured with respect to a phenotypic trait when the rate of mutation is very small. We assume that the reproduction is asexual, and that the mutations can be described by a linear integral operator. We are interested in the interplay between the time variable t and the rate ε of mutations. We show that depending on $\alpha > 0$, the limit $\varepsilon \to 0$ with $t = \varepsilon^{-\alpha}$ can lead to population number densities which are either Gaussian-like (when α is small) or Cauchy-like (when α is large).

1. Introduction

1.1. **Selection-mutation-competition models.** The phenotypic diversity of a species impacts its ability to evolve. In particular, the importance of the variance of the population along a phenotypic trait is illustrated by the *fundamental theorem of natural selection* [16], and the *breeder's equation* [23]: the evolution speed of a population along a one dimensional fitness gradient (or under artificial selection) is proportional to the variance of the initial population. Recently, the phenotypic variance of populations has also come to light as an important element to describe the evolutionary dynamics of ecosystems (where many interacting species are considered) [30, 4, 29].

Over the last decade, the issue of *Evolutionary Rescue* emerged as an important question [3, 10, 18] (see also the seminal work of Luria and Delbrück [24]), and led to a new interest in the phenotypic distribution of populations, beyond phenotypic variance. Evolutionary Rescue is concerned with a population living in an environment that changes suddenly. The population will survive either if some individuals in the population carry an unusual trait that turns out to be successful in the new environment, or if new mutants able to survive in the new environment appear before the population goes extinct (see [26] for a discussion on the relative effect of *de novo mutations* and *standing variance* in Evolutionary Rescue). In any case, the fate of the population will not be decided by the properties of the bulk of its density, but rather by the properties of the tail of the initial distribution of the individuals, close to the favourable traits for the new environment. A first example of such problems comes from emerging diseases [17]: Animal infections sometimes are able to infect humans. This phenomenon, called zoonose, is the source of many human epidemics: HIV, SARS, Ebola, MERS-CoV, etc. A zoonose may happen if a pathogen that reaches a human has the unusual property of being adapted to this new human host. A second example comes from the emergence of microbes resistant to an antimicrobial drug that is suddenly spread in the environment of the microbe. This second phenomenon can easily be tested experimentally [3, 27], and has major public health implications [9].

Most papers devoted to the genetic diversity of populations structured by a continuous phenotypic trait describe the properties of mutation-selection equilibria. It is however also interesting to describe the genetic diversity of populations that are not at equilibrium (*transient dynamics*): pathogen populations for instance are often in transient situations, either invading a new host, or being eliminated by the immune system. We refer to [19] for a review on transient dynamics in ecology. For asexual populations structured by a continuous phenotypic trait, several models exist, corresponding to different biological assumptions [13]. If the mutations are modeled by a diffusion, the steady populations (for a model close to (1), but where mutations are modelled by a Laplacian) are Gaussian distributions [21, 7]. Furthermore, [1, 12] have considered some transient dynamics for this model. In the model that we will consider (see (1)), the mutations are modelled by a non-local term. It was shown in [6] (see also [7]) that mutation-selection equilibria are then Cauchy profiles (under some assumptions), and this result has been extended to more general mutation kernels in [8], provided that the mutation rate is small enough. Finally, let us notice that the case of sexual populations is rather different, since recombinations by themselves can imply that a *mutation-recombination equilibrium* exists, even without selection. We refer to the infinitesimal model [5], and to [28] for some studies on the phenotypic distribution of sexual species in a context close to the one presented here for asexual populations.

In this article, we consider a population consisting of individuals structured by a quantitative phenotypic trait $x \in I$ (I open interval of $\mathbb{R}$ containing 0), and denote by $f := f(t,x) \geq 0$ its density. Here, the trait x is fully inherited by the offspring (if no mutation occurs), so that x is indeed rather a breeding value than a phenotypic trait (see [25]). We assume that individuals reproduce with a rate 1, and die at a rate

$$x^2 + \int_I f(t,y)\,\mathrm{d}y.$$

This means that individuals with trait $x = 0$ are those who are best adapted to their environment, and that the fitness decreases like a parabola around this optimal trait (this is expected in the surroundings of a trait of maximal fitness). It also means that the strength of the competition modeled by the logistic term is identical for all traits. When an individual of trait $x \in I$ gives birth, we assume that the offspring will have the trait x with probability $1-\varepsilon$, and a different trait x' with probability $\varepsilon \in (0,1)$. ε is then the probability that a mutation affects the phenotypic trait of the offspring. We can now define the growth rate of the population of trait x (that is the difference between the rate of *births without mutation*, minus the death rate) as

$$r_\varepsilon(t,x) = 1 - \varepsilon - x^2 - \int_I f(t,y)\,\mathrm{d}y.$$

When a mutation affects the trait of the offspring, we assume that the trait x' of the mutated offspring is drawn from a law over the set of phenotypes $I \subset \mathbb{R}$ with a density $\gamma := \gamma(x) \in L^1(I)$. The function γ then satisfies

$$\gamma(x) \geq 0, \qquad \int_I \gamma(x)\,dx = 1,$$

(we assume moreover in some of the mathematical statements that γ is bounded, C^1, with bounded derivative and that it is strictly positive on I). The main assumption here is that the law of the trait of a mutated offspring does not depend on the trait of its parent. This classical assumption, known as *house of cards* is not the most realistic, but it can be justified when the mutation rate is small [7] (see also [8]). All in all, we end up with the following equation:

$$\frac{\partial f_\varepsilon(t,x)}{\partial t} = r_\varepsilon(t,x)\, f_\varepsilon(t,x) + \varepsilon\,\gamma(x) \int_I f_\varepsilon(t,y)\,\mathrm{d}y. \tag{1}$$

This paper is devoted to the study of the asymptotic behaviour of the solutions of equation (1) when ε is small and t large and it is organized as follows. In the rest of Section 1 the main results are quoted, first in an informal way, and then as rigourous statements. Section 2 contains the proof of Proposition 1.1 and Theorem 1.2, and finally, in Section 3, Theorem 1.3 is proved.

1.2. **Asymptotic study of the model.** When we consider the solutions of (1), two particular profiles naturally appear:

- *A Cauchy profile:* For a given mutation rate $\varepsilon > 0$ small enough, one expects that $f_\varepsilon(t,x)$ will converge, as t goes to infinity, to the unique steady-state of (1), which is the following Cauchy profile

$$f_\varepsilon(\infty,x) := \frac{\varepsilon\,\gamma(x)\,\mathcal{I}_\varepsilon(\infty)}{\mathcal{I}_\varepsilon(\infty) - (1-\varepsilon) + x^2}, \tag{2}$$

  where $\mathcal{I}_\varepsilon(\infty)$ is such that $\int_I f_\varepsilon(\infty,x)\,dx = \mathcal{I}_\varepsilon(\infty)$. This steady-state of (1) is the so-called *mutation-selection equilibrium* of the *House of cards* model (1), which has been introduced in [6] (we also refer to [7] for a broader presentation of existing results).

- *A Gaussian profile:* If $\varepsilon = 0$, the solution of (1) can be written as

$$f_0(t,x) = f(0,x)\,e^{-\int_0^t \mathcal{I}_0(s)\,ds + t - t\,x^2}, \tag{3}$$

where $\mathcal{I}_0(t) := \int_I f_0(t,x)\,dx$, so that a Gaussian-like behavior (with respect to x) naturally appears in this case. Surprisingly, we are not aware of any reference to this property in the population genetics literature.

We will show that, as suggested by the above arguments, we can describe the phenotypic distribution of the population, that is $x \mapsto f_\varepsilon(t,x)$, when either $t \gg 1$ (large time for a given mutation rate $\varepsilon > 0$), or $0 \leq \varepsilon \ll 1$ (small mutation rate, for a given time interval $t \in [0,T]$). Before providing the precise statements of our results (see Subsection 1.3), we will briefly describe them here, and illustrate them with numerical simulations. The numerical simulations presented in Fig. 1 and Fig. 2 are obtained thanks to a finite difference scheme (explicit Runge-Kutta in time), and we illustrate our result with a single simulation of (1) with $\varepsilon = 10^{-2}$, $I = [-3/2, 3/2]$, $\gamma(x) = \frac{1}{40\pi} e^{\frac{-x^2}{20}}$ and $f_\varepsilon(0,x) = \Gamma_2(\varepsilon, x-1)$ (see the definition of Γ_2 in eq. (4) below). The initial condition corresponds to a population at the mutation-selection equilibrium which environment suddenly changes (the optimal trait originally in $x = 1$ moves to $x = 0$ at $t = 0$). This example is guided by the Evolutionary Rescue experiments described in Subsection 1.1, where the sudden change is obtained by the addition of e.g. salt or antibiotic to a bacterial culture.

We describe two phases of the dynamics of the population:

- *Large time: Cauchy profile.* We show that $f_\varepsilon(t,x)$ is asymptotically (when the mutation rate $\varepsilon > 0$ is small) close to

$$\Gamma_2(\varepsilon, x) = \frac{\varepsilon\,\gamma(0)}{\gamma(0)^2\pi^2\,\varepsilon^2 + x^2}, \tag{4}$$

  provided $t \gg \varepsilon^{-4}$. The population is then a time-independent Cauchy distribution for large times. This theoretical result is coherent with numerical results: we see in Fig. 1 that $f_\varepsilon(t,\cdot)$ is well described by $\Gamma_2(\varepsilon,\cdot)$, as soon as $t \geq 10^5$, which is confirmed by the value of $\|f_\varepsilon(t,\cdot) - \Gamma_2(\varepsilon,\cdot)\|_{L^1(I)}$ for $t \geq 10^5$ given by Fig. 2.

- *Short time: Gaussian profile.* We also show that $f_\varepsilon(t,x)$ is asymptotically (when the mutation rate $\varepsilon > 0$ is small) close to

$$\Gamma_1(t,\varepsilon,x) = \frac{f(0,x)\,\sqrt{t}}{f(0,0)\int_I e^{-x^2}\,dx} e^{-x^2\,t}, \tag{5}$$

  provided $1 \ll t \ll \varepsilon^{-2/3}$. The population has then a Gaussian-type distribution for short (but not too short) times. This theoretical result is coherent with numerical simulations: we see in Fig. 1 that $f_\varepsilon(t,\cdot)$ is well described by $\Gamma_1(t,\varepsilon,\cdot)$ for $t \in [10^2, 10^4]$, which is confirmed by the value of $\|f_\varepsilon(t,\cdot) - \Gamma_2(\varepsilon,\cdot)\|_{L^1(I)}$ for $t \in [10^2, 10^4]$ given by Fig. 2.

Another way to look at these results is to consider $t \geq 0$ and $\varepsilon > 0$ as two parameters, and to see the approximations presented above as approximations of $f_\varepsilon(t,\cdot)$ for some set of parameters: $f_\varepsilon(t,\cdot) \sim_{\varepsilon\to 0} \Gamma_2(\varepsilon,\cdot)$ for $(t,\varepsilon) \in \{(\tilde t,\tilde\varepsilon); \tilde t \gg \tilde\varepsilon^{-4}\}$, while $f_\varepsilon(t,\cdot) \sim_{\varepsilon\to 0} \Gamma_1(t,\varepsilon,\cdot)$ for $(t,\varepsilon) \in \{(\tilde t,\tilde\varepsilon); 1 \ll \tilde t \ll \tilde\varepsilon^{-2/3}\}$. We have represented these sets in Fig 3.

As described in the Subsection 1.1, the phenotypic distribution of species is involved in many ecological and epidemiological problematics. Our study is a general analysis of this problem and we do not have a particular application in mind. An interesting and (to our knowledge) new feature described by our study is that the tails of the trait distribution in a population can change drastically between "short times", that is $1 \ll t \ll \varepsilon^{-2/3}$ and "large times", that is $t \gg \varepsilon^{-4}$: the distribution is initially close to a Gaussian distribution, with small tails, and then converges to a thick tailed Cauchy distribution. This result could have significant consequences for *evolutionary rescue*: the tails of the distribution then play an important role. Quantifying the effect of this property of the tails of the distributions would however require further work, in particular on the impact of stochasticity (the number of pathogen is typically large, but finite). The plasticity of the pathogen (see [11]) may also play an important role.

1.3. **Main (rigorous) statements.** Here we state the two main theorems of the paper, together with a proposition asserting the well-posedness of the equation that we consider.

We start with the issue of well-posedness:

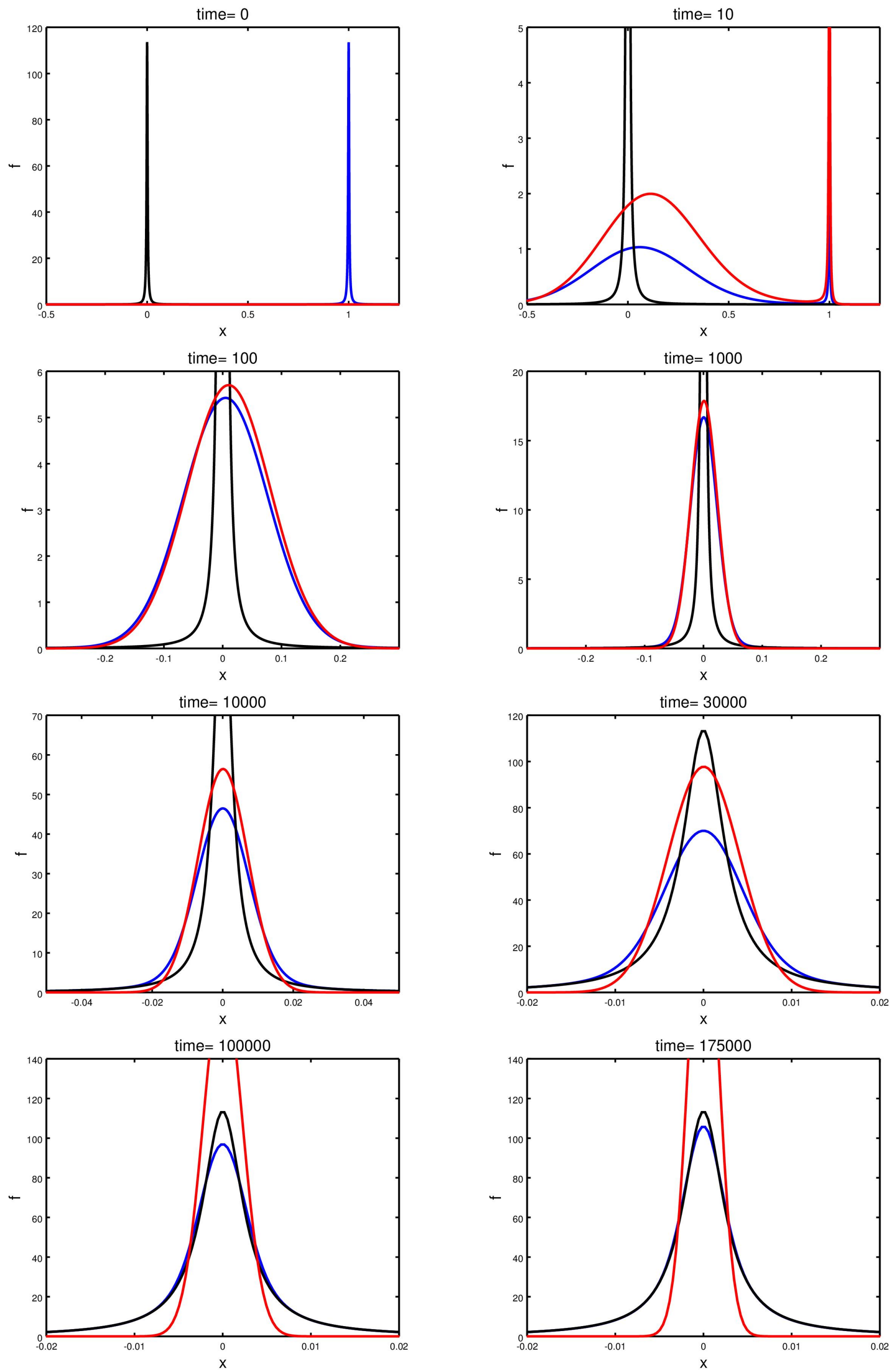

FIGURE 1. The different graphs correspond to different time points, from $t = 0$ to $t = 175\,000$, of the same simulation of (1) for $\varepsilon = 10^{-2}$ (see in the text for a complete description). In each of these plots, the blue (resp. red, black) line represents $x \mapsto f_\varepsilon(t,x)$ (resp. $x \mapsto \Gamma_1(t,\varepsilon,x)$, $x \mapsto \Gamma_2(\varepsilon,x)$). Note that in this figure, the scales of both axis change from one graph to the other, to accommodate with the dynamics of the solution $f(t,\cdot)$.

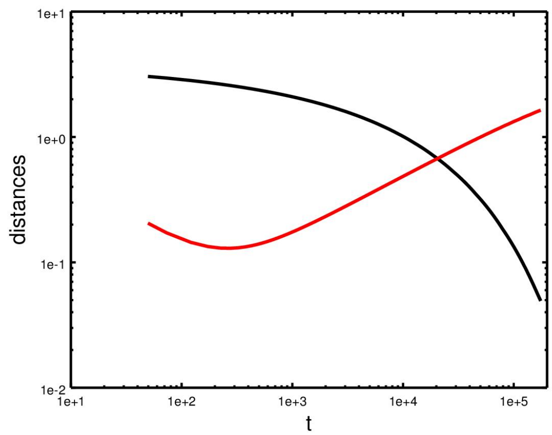

FIGURE 2. Simulation of (1) with $\varepsilon = 10^{-2}$ (see in the text for a complete description). The red line represents $\|f_\varepsilon(t,\cdot) - \Gamma_1(t,\varepsilon,\cdot)\|_{L^1(I)}$, while the black line represents $\|f_\varepsilon(t,\cdot) - \Gamma_2(\varepsilon,\cdot)\|_{L^1(I)}$.

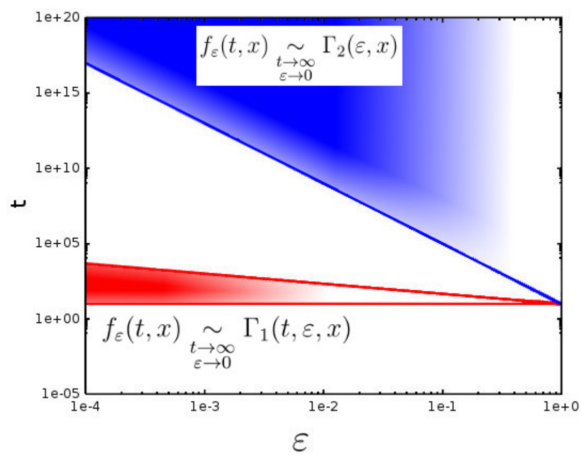

FIGURE 3. Representation of the set $\{(\tilde t,\tilde\varepsilon); \tilde t \gg \tilde\varepsilon^{-4}\}$ (in blue), where the approximation $f_\varepsilon(t,\cdot) \sim_{\varepsilon\to 0} \Gamma_2(\varepsilon,\cdot)$ holds provided that $\varepsilon > 0$ is small enough; and of the set $\{(\tilde t,\tilde\varepsilon); 1 \ll \tilde t \ll \tilde\varepsilon^{-2/3}\}$ (in red), where the approximation $f_\varepsilon(t,\cdot) \sim_{\varepsilon\to 0} \Gamma_1(t,\varepsilon,\cdot)$ holds provided that $\varepsilon > 0$ is small enough.

Proposition 1.1. *We consider the interval I ($I =]a,b[$, $-\infty \le a < b \le +\infty$). We assume that γ is continuous, bounded and strictly positive on I, and that $\int_I \gamma(x)\,dx = 1$. We consider an initial datum f_0 which is nonnegative, non identically 0, and integrable on I. Finally we take $\varepsilon \in]0,1/2[$.*

Then the initial value problem (1) (with $f(0,\cdot) = f_0$) admits a unique nonnegative (global in time) mild solution in $C(\mathbb{R}_+; L^1(I))$.

Then we turn to the two statements which are relative to the asymptotic behavior when $\varepsilon \to 0$ and $t \to +\infty$.

Theorem 1.2. *Under the assumptions of Proposition 1.1, and assuming moreover that γ has textcolorreda bounded derivative on I, there exist positive constants $K, \hat K$ and $\tilde K$ (independent of ε) such that the mild* *solution f of* (1) *satisfies (for $t > 0$)*

$$\left\| x \mapsto f(t,x) - \frac{\varepsilon\gamma(0)}{(\gamma(0)\pi\varepsilon)^2 + x^2} \right\|_{L^1(I)} \leq K\, \varepsilon^{\frac{-\hat{K}}{\varepsilon^2}}\, e^{-\hat{K}\varepsilon^2 t} + \tilde{K}\varepsilon \ln\left(\frac{1}{\varepsilon}\right).$$

Theorem 1.3. *Under the assumptions of Proposition 1.1, and assuming moreover that a and b are finite, $f_0 \in W^{1,\infty}(I)$, $f_0(0) > 0$ and $\int_I f_0(x)\,dx < 1$, there exists a constant $C > 0$ (independent of ε) such that the mild solution f of* (1) *satisfies (for $t > 4$)*

$$\left\| x \mapsto f(t,x) - \frac{f(0,x)\sqrt{t}e^{-x^2 t}}{f(0,0)\, \int_I e^{-y^2}\,dy} \right\|_{L^1(I)} \leq C\left(\frac{1}{\sqrt{t}} + \varepsilon\, t^{\frac{3}{2}}\right) e^{C\,\varepsilon\,t}. \tag{6}$$

A careful check of the proof shows that the constant C appearing in (6) *only depends on $|a-b|$, some upper bounds on $\|\gamma\|_{L^\infty}$ and $\|f_0\|_{W^{1,\infty}}$, and a lower bound on $f_0(0)$.*

Note first that, as previously stated in the introduction, if we consider times t which are large enough, for example $t = \varepsilon^{-4-\delta}$ with $\delta > 0$, then, since $(-\varepsilon^2 \ln\varepsilon - \varepsilon^{-2-\delta})$ is asymptotically equivalent to $-\varepsilon^{-2-\delta}$, Theorem 1.2 ensures that

$$\left\| x \mapsto f(\varepsilon^{-4-\delta},x) - \frac{\varepsilon\gamma(0)}{(\gamma(0)\pi\varepsilon)^2 + x^2} \right\|_{L^1(I)} = O_{\varepsilon\to 0}\left(\varepsilon\,\ln\left(\frac{1}{\varepsilon}\right)\right),$$

so that the Cauchy profile is indeed asymptotically reached for very large times.

The interpretation of Theorem 1.3 is a little more intricate, since it concerns intermediate times. For example, it implies the following description of the population's phenotypic diversity during transitory times, that is times t satisfying $1 \ll t \ll \varepsilon^{-\frac{2}{3}}$: There exists $C > 0$ such that for $\kappa > 0$ and $\varepsilon > 0$ small enough (in such a way that the interval $\left[\kappa^{-2}, \kappa^{\frac{2}{3}}\varepsilon^{-\frac{2}{3}}\right]$ is not empty, that is $\varepsilon < \kappa^4$),

$$\forall t \in \left[\kappa^{-2}, \kappa^{\frac{2}{3}}\varepsilon^{-\frac{2}{3}}\right], \qquad \left\| x \mapsto f(t,x) - \frac{f(0,x)\sqrt{t}e^{-x^2 t}}{f(0,0)\, \int_I e^{-y^2}\,dy} \right\|_{L^1(I)} \leq C\kappa.$$

For example, if $t = \varepsilon^{-1/2}$, then

$$\left\| x \mapsto f(t,x) - \frac{f(0,x)\sqrt{t}e^{-x^2 t}}{f(0,0)\, \int_I e^{-y^2}\,dy} \right\|_{L^1(I)} \leq C\,\varepsilon^{1/4} = C\,t^{-1/2}.$$

Note finally that these results hold for models which are slightly more general than equation (1). In fact, in both theorems one can assume that the competition term is a weighted population instead of the total population number. In Theorem 1.3, one could also assume that the mutation kernel depends on the parents trait.

2. Proof of Proposition 1.1 and Theorem 1.2

We start here the proofs of Proposition 1.1 and of Theorem 1.2. We recall that $I =]a,b[$, $-\infty \leq a < 0 < b \leq \infty$, and $\gamma := \gamma(x)$ is a bounded, continuous, strictly positive function, such that $\gamma(x) > 0$ and $\int_I \gamma(x)\,dx = 1$. For the proof of Theorem 1.2, we will also assume that it has a bounded derivative on I.

We begin with the study of the linear operator associated to eq. (1).

2.1. **Spectrum of the linear operator.** Let us define

$$(A_\varepsilon f)(x) := (1-\varepsilon)f(x) - x^2\, f(x) + \varepsilon\gamma(x) \int_I f(y)\, dy \tag{7}$$

as the operator corresponding to the linear part in eq. (1).

We begin with a basic lemma which enables to define the semigroup generated by this operator.

Lemma 2.1. *The linear operator A_ε, defined on $L^1(I)$ and with domain $D(A_\varepsilon) = \{f \in L^1(I) : \int_I x^2\, |f(x)|\, dx < \infty\}$, generates an irreducible positive C^0-semigroup (denoted from now on by $T_\varepsilon(t)$).*

Proof. The multiplication linear operator $(A^0_\varepsilon f)(x) := (1-\varepsilon)f(x) - x^2\, f(x)$ is the generator of a positive C^0-semigroup. Since γ is strictly positive, $A_\varepsilon - A^0_\varepsilon$ is a positive bounded perturbation whose only invariant closed ideals are 0 and the whole space $L^1(I)$. So $T_\varepsilon(t)$ is irreducible (see [14], Corollary 9.22). $\square$

Next, we present a proposition which gives information about the spectrum of A_ε.

Proposition 2.2. *The linear operator A_ε has only one eigenvalue. It is a strictly dominant algebraically simple eigenvalue $\lambda_\varepsilon > 1-\varepsilon$ and a pole of the resolvent, with corresponding normalized positive eigenvector*

$$\psi_\varepsilon(x) = \frac{\varepsilon\gamma(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}.$$

Moreover, for ε small enough, $\lambda_\varepsilon < 1$.

The rest of the spectrum of the linear operator A_ε is equal to the interval $J = [\min(1-\varepsilon-a^2, 1-\varepsilon-b^2), 1-\varepsilon]$.

Proof. In the sequel, the norm $||\ ||$ is the L^1 norm on I. Let us first show that any λ belonging to the set $J = \mathrm{Range}(1-\varepsilon-x^2)$ belongs to the spectrum of A_ε. In order to do this, for $\lambda = 1-\varepsilon-x_0^2, x_0 \in \mathring{I}$, let us define $f_n(x) = \frac{n}{2}\left(\chi_{[x_0, x_0+\frac{1}{n}]}(x) - \chi_{[x_0-\frac{1}{n}, x_0]}(x)\right)$ for n such that $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}] \subset I$. We then have $\|f_n\| = 1$ and $\|(A_\varepsilon - \lambda Id)f_n\| = \frac{n}{2}\int_{x_0-\frac{1}{n}}^{x_0+\frac{1}{n}} |x^2 - x_0^2| dx \to 0$. So $(\min(1-\varepsilon-a^2, 1-\varepsilon-b^2), 1-\varepsilon]$ is contained in the spectrum of A_ε. The claim follows from the fact that the spectrum is a closed set.

On the other hand, notice that (for $x_0 \in I$), $1-\varepsilon-x_0^2$ is not an eigenvalue, since the potential corresponding eigenfunction $\frac{\gamma(x)}{x_0^2-x^2}$ is not an integrable function on I (remember that γ does not vanish).

Let us now compute the resolvent operator of A_ε, that is, let us try to solve the equation

$$A_\varepsilon f - \lambda f = g \in L^1(I). \tag{8}$$

For $\lambda \notin J$, defining $p := \int_I f(y)\, dy$, (8) gives

$$f(x) = \frac{\varepsilon\gamma(x)p - g(x)}{\lambda - (1-\varepsilon-x^2)}. \tag{9}$$

Integrating, we get

$$\left(1 - \varepsilon\int_I \frac{\gamma(x)}{\lambda-(1-\varepsilon-x^2)}\, dx\right) p = \int_I \frac{-g(x)}{\lambda-(1-\varepsilon-x^2)}\, dx, \tag{10}$$

and λ belongs to the resolvent set unless the factor of p on the left hand side vanishes. Therefore $\sigma(A) = J \cup \{\lambda \in \mathbb{C} : \varepsilon\int_I \frac{\gamma(x)}{\lambda-(1-\varepsilon-x^2)}\, dx = 1\}$.

Since for any real number $\lambda > 1-\varepsilon$, the function $F_\varepsilon(\lambda) := \varepsilon\int_I \frac{\gamma(x)}{\lambda-(1-\varepsilon-x^2)}\, dx$ is continuous, strictly decreasing, and satisfies $\lim_{\lambda\to 1-\varepsilon} F_\varepsilon(\lambda) = +\infty$ (recall that $\gamma(0) > 0$) and $\lim_{\lambda\to+\infty} F_\varepsilon(\lambda) = 0$, we see that there is a unique real solution of $F_\varepsilon(\lambda) = 1$ in $(1-\varepsilon, \infty)$. We denote it by λ_ε.

Taking $g(x) = 0$ in (8), we see that λ_ε is an eigenvalue with corresponding normalized strictly positive eigenvector

$$\psi_\varepsilon = \frac{\varepsilon\gamma(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}.$$

Taking $g = \psi_\varepsilon$ and $\lambda = \lambda_\varepsilon$ we see that the left hand side in (10) vanishes, whereas the right hand side is strictly negative, so that $A_\varepsilon f - \lambda_\varepsilon f = \psi_\varepsilon$ has no solution and hence λ_ε is algebraically simple.

Indeed, it also follows from (10) that the range of $A_\varepsilon - \lambda_\varepsilon\, Id$ coincides with the kernel of the linear form defined on $L^1(I)$ by the L^∞ function $\frac{1}{\lambda_\varepsilon-(1-\varepsilon)+x^2}$ (which is the eigenvector corresponding to the eigenvalue λ_ε of the adjoint operator A_ε^*) and hence it is a closed subspace of $L^1(I)$. Therefore, λ_ε is a pole of the resolvent (see Theorem A.3.3 of [14]). Furthermore, since

$$F_\varepsilon(1) = \varepsilon \int_I \frac{\gamma(x)}{\varepsilon + x^2}\, dx = \int_I \frac{\gamma(x)}{1+\left(\frac{x}{\sqrt{\varepsilon}}\right)^2}\, dx \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

we see that $F_\varepsilon(1) < 1$ for ε small enough, and hence $\lambda_\varepsilon < 1$.

Substituting λ by $a + bi$ in the characteristic equation

$$1 + \varepsilon \int_I \frac{\gamma(x)}{(1-\varepsilon - x^2 - \lambda)}\, dx = 0, \tag{11}$$

we have that the imaginary part is $-\varepsilon b \int_I \frac{\gamma(x)}{(1-\varepsilon-x^2-\lambda)}\, dx$. Since $\gamma(x) > 0$, there are no non real solutions of (11). $\square$

Remark 2.1. Note that $\lim_{\varepsilon\to 0} \lambda_\varepsilon = 1$.

We now write an expansion of the eigenvalue λ_ε.

Proposition 2.3. *Let us assume that γ has a bounded derivative on I. Let λ_ε be the dominant eigenvalue of the operator A_ε. Then*

$$\left|\lambda_\varepsilon - (1-\varepsilon) - \gamma(0)^2\pi^2\varepsilon^2\right| = O_{\varepsilon\to 0}\left(\varepsilon^3 \ln \frac{1}{\varepsilon}\right)$$

Proof. Let us consider the change of variable $x = \nu_\varepsilon z$ where $\nu_\varepsilon = \sqrt{\lambda_\varepsilon - (1-\varepsilon)}$. We have

$$1 = \varepsilon \int_a^b \frac{\gamma(x)}{(\lambda_\varepsilon - (1-\varepsilon - x^2))}\, dx = \varepsilon \int_{\frac{a}{\nu_\varepsilon}}^{\frac{b}{\nu_\varepsilon}} \frac{\gamma(\nu_\varepsilon z)}{\nu_\varepsilon^2 + (\nu_\varepsilon z)^2} \nu_\varepsilon\, dz = \frac{\varepsilon}{\nu_\varepsilon} \int_{\frac{a}{\nu_\varepsilon}}^{\frac{b}{\nu_\varepsilon}} \frac{\gamma(\nu_\varepsilon z)}{1+z^2}\, dz.$$

Then

$$\begin{aligned}
\left|\tfrac{\nu_\varepsilon}{\varepsilon} - \gamma(0)\pi\right| &= \left|\int_{\frac{a}{\nu_\varepsilon}}^{\frac{b}{\nu_\varepsilon}} \tfrac{\gamma(\nu_\varepsilon z)}{1+z^2}\, dz - \gamma(0)\pi\right| \\
&\leq \left|\int_{\frac{a}{\nu_\varepsilon}}^{\frac{b}{\nu_\varepsilon}} \tfrac{\gamma(\nu_\varepsilon z)}{1+z^2}\, dz - \int_{\mathbb{R}} \tfrac{\gamma(\nu_\varepsilon z)}{1+z^2}\, dz\right| + \left|\int_{\mathbb{R}} \tfrac{\gamma(\nu_\varepsilon z)-\gamma(0))}{1+z^2}\, dz\right| \\
&\leq 4\|\gamma\|_\infty \int_{\frac{B}{\nu_\varepsilon}}^{+\infty} \tfrac{dz}{1+z^2} + 2\|\gamma'\|_\infty \nu_\varepsilon \int_0^{\frac{A}{\nu_\varepsilon}} \tfrac{z}{1+z^2}\, dz,
\end{aligned}$$

where we have used

$$|\gamma(\nu_\varepsilon z) - \gamma(0)| \leq \min\left(\|\gamma\|_\infty, \|\gamma'\|_\infty \nu_\varepsilon |z|\right),$$

and have denoted $A := \frac{\|\gamma\|_\infty}{\|\gamma'\|_\infty}$ and $B := \min(|a|, b, A)$.
Since

$$4\|\gamma\|_\infty \int_{\frac{B}{\nu_\varepsilon}}^{+\infty} \frac{dz}{1+z^2} = 4\|\gamma\|_\infty \arctan\left(\frac{\nu_\varepsilon}{B}\right) \leq 4\|\gamma\|_\infty \frac{\nu_\varepsilon}{B},$$

and

$$2\|\gamma'\|_\infty \nu_\varepsilon \int_0^{\frac{A}{\nu_\varepsilon}} \frac{z}{1+z^2}\, dz = \|\gamma'\|_\infty \nu_\varepsilon \ln\left(1 + \frac{A^2}{\nu_\varepsilon^2}\right),$$

we obtain

$$\left|\nu_\varepsilon - \varepsilon\gamma(0)\pi\right| \leq \varepsilon\nu_\varepsilon \left(\frac{4\|\gamma\|_\infty}{B} + \|\gamma'\|_\infty \ln\left(1 + \frac{A^2}{\nu_\varepsilon^2}\right)\right), \tag{12}$$

which implies

$$\varepsilon\left(\gamma(0)\pi-\nu_\varepsilon\left(\frac{4\|\gamma\|_\infty}{B}+\|\gamma'\|_\infty\ln(1+\frac{A^2}{\nu_\varepsilon^2})\right)\right)$$
$$\leq \nu_\varepsilon \leq \varepsilon\left(\gamma(0)\pi+\nu_\varepsilon\left(\frac{4\|\gamma\|_\infty}{B}+\|\gamma'\|_\infty\ln\left(1+\frac{A^2}{\nu_\varepsilon^2}\right)\right)\right).$$

Since

$$\nu_\varepsilon\left(\frac{4\|\gamma\|_\infty}{B}+\|\gamma'\|_\infty\ln\left(1+\frac{A^2}{\nu_\varepsilon^2}\right)\right)\xrightarrow{\varepsilon\to 0}0,$$

we have

$$\frac{\gamma(0)\pi\varepsilon}{2}\leq\nu_\varepsilon\leq 2\,\gamma(0)\pi\varepsilon \tag{13}$$

for ε small enough.
Therefore, using (13) in (12) we get

$$\left|\nu_\varepsilon-\varepsilon\gamma(0)\pi\right|\leq\varepsilon^2 2\gamma(0)\pi\left(\frac{4\|\gamma\|_\infty}{B}+\|\gamma'\|_\infty\ln\left(1+\frac{4A^2}{\gamma(0)^2\pi^2\varepsilon^2}\right)\right)\leq C\varepsilon^2\ln\left(\frac{1}{\varepsilon}\right). \tag{14}$$

Finally, by (13) and (14),

$$|\lambda_\varepsilon-(1-\varepsilon)-\gamma(0)^2\pi^2\varepsilon^2|=|\nu_\varepsilon+\gamma(0)\pi\varepsilon|\,|\nu_\varepsilon-\gamma(0)\pi\varepsilon|\leq 3\gamma(0)\pi C\varepsilon^3\ln\left(\frac{1}{\varepsilon}\right).$$

□

2.2. **Proof of Proposition 1.1.** Let us define $\tilde{A}_\varepsilon=A_\varepsilon-\lambda_\varepsilon Id$ and let $\tilde{T}_\varepsilon(t)=e^{-\lambda_\varepsilon t}T_\varepsilon(t)$ be the semigroup generated by $\tilde{A}_\varepsilon$. We now rewrite equation (1) as

$$\frac{\partial f(t,x)}{\partial t}=\tilde{A}_\varepsilon f(t,x)+\left(\lambda_\varepsilon-\int_I f(t,y)\,dy\right)f(t,x). \tag{15}$$

We look for solutions of (15) (with positive initial condition $f_0\in L^1(I)$) which can be written as $f(t,x)=h(t)(\tilde{T}_\varepsilon(t)f_0)(x)$, with $h:=h(t)$ a real valued function of time such that $h(0)=1$. Substituting in (15), it follows that f is indeed a solution of eq. (1) if $h(t)$ satisfies the following initial value problem for an ordinary differential equation:

$$h'(t)=\left(\lambda_\varepsilon-h(t)\int_I\left(\tilde{T}_\varepsilon(t)f_0\right)(x)\,dx\right)h(t),\qquad h(0)=1. \tag{16}$$

Then h satisfies the integral equation

$$h(t)=1+\int_0^t\left(\lambda_\varepsilon-h(s)\int_I\left(\tilde{T}_\varepsilon f_0\right)(x)\,dx\right)h(s)\,ds,$$

from which the following identity follows

$$h(t)\tilde{T}_\varepsilon(t)f_0=\tilde{T}_\varepsilon(t)f_0+\int_0^t\tilde{T}_\varepsilon(t-s)\left(\lambda_\varepsilon-h(s)\int_I\left(\tilde{T}_\varepsilon(s)f_0\right)(x)dx\right)h(s)\tilde{T}_\varepsilon(s)f_0ds,$$

i.e., $f(t,x)$ is a solution of the variations of constants equation.

On the other hand, the nonlinear part of the right hand side of (15) is a locally Lipschitz function of $f\in L^1(I)$. From this uniqueness follows, whereas global existence and nonnegativity are clear from (16).

2.3. **Asymptotic behavior of the nonlinear equation.** Let us start this subsection with a proposition ensuring in a quantitative way the convergence of the solution towards a nontrivial steady state:

Proposition 2.4. *Under the assumptions of Theorem 1.2, for $\varepsilon > 0$ small enough, and any $\rho_\varepsilon < (\gamma(0)\pi\varepsilon)^2$, there exists a constant $C_\varepsilon > 0$ (depending on f_0 and ε) such that*

$$\|f(t,\cdot) - \lambda_\varepsilon\, \psi_\varepsilon\|_{L^1(I)} \le C_\varepsilon\, e^{-\rho_\varepsilon\, t}.$$

Furthermore, taking $\rho_\varepsilon = \frac{\alpha_\varepsilon}{2} = \frac{\lambda_\varepsilon - (1-\varepsilon)}{2}$, the following more explicit (in terms of dependence w.r.t ε) estimate holds

$$\|f(.,t) - \lambda_\varepsilon\, \psi_\varepsilon\|_{L^1(I)} \le K\, \varepsilon^{\frac{-\hat{K}}{\varepsilon^2}}\, e^{\frac{-\alpha_\varepsilon t}{2}},$$

where $K, \hat{K} > 0$ depend on f_0 but not on ε.

The rest of this subsection will be devoted to proving Proposition 2.4 taking advantage of the special form of the solution, $f(t) = h(t)T_\varepsilon(t)f_0$, as written in the proof of Proposition 1.1. In order to do so, we will also use some lemmas (Lemmas 2.5-2.10 which are stated below).

Let us start with one in which some properties of the spectrum of $\tilde{A}_\varepsilon = A_\varepsilon - \lambda_\varepsilon Id$ are used to study the asymptotic behavior of the semigroup $\tilde{T}_\varepsilon(t)$ generated by $\tilde{A}_\varepsilon$.

Lemma 2.5. a) *The essential growth bound of the semigroup generated by $\tilde{A}_\varepsilon$ is $\omega_{ess}(\tilde{T}_\varepsilon) = 1 - \varepsilon - \lambda_\varepsilon$.*
b) *The growth bound of the semigroup generated by $\tilde{A}_\varepsilon$ is $\omega_0(\tilde{T}_\varepsilon) = 0$.*

Proof. a) $\tilde{A}_\varepsilon$ is a compact (one rank) perturbation of $\tilde{A}^0_\varepsilon f := (1 - \varepsilon - x^2 - \lambda_\varepsilon)f$. Then $\omega_{ess}\left(\tilde{T}_\varepsilon\right) = \omega_{ess}\left(\tilde{T}^0_\varepsilon\right)$ where $\tilde{T}^0_\varepsilon(t)$ is the semigroup generated by $\tilde{A}^0_\varepsilon$ (see [2]).

Since $\tilde{A}^0_\varepsilon$ is a multiplication operator, $\omega_{ess}\left(\tilde{T}^0_\varepsilon\right) = 1 - \varepsilon - \lambda_\varepsilon$ and the result follows.

b) By Proposition 2.2, the spectral bound of $\tilde{A}_\varepsilon$ is 0 and the spectral mapping theorem holds for any positive C^0-semigroup on L^1 (see [14]).

□

Let us now write, for a positive non identically zero f_0, $\left(\tilde{T}_\varepsilon(t)\right) f_0(x) = c_{f_0}\psi_\varepsilon(x) + v(t,x)$ where $\psi_\varepsilon(x)$ is the eigenvector corresponding to the eigenvalue 0 of $\tilde{A}_\varepsilon$, and $c_{f_0}\psi_\varepsilon(x)$ is the spectral projection of f_0 on the kernel of $\tilde{A}_\varepsilon$ (Note that $c_{f_0} > 0$ since f_0 is positive and $\tilde{A}_\varepsilon$ is the generator of an irreducible positive semigroup). We also define $\varphi(t) := \int_I v(t,x)\,dx$. The following lemma gives the asymptotic behavior of c_{f_0}:

Lemma 2.6. *Let us assume that f_0 is a positive integrable function on I. Then there exist positive constants K_1, K_2 (independent of ε but depending on f_0) such that $K_1\,\varepsilon^2 \le c_{f_0} \le K_2$. Moreover, $\lim_{\varepsilon\to 0} c_{f_0} = 0$.*

Proof. Recall that $c_{f_0} = \langle \psi^*_\varepsilon, f_0\rangle$ where ψ^*_ε is the eigenvector of the adjoint operator A^*_ε corresponding to the eigenvalue λ_ε, normalized such that $\langle \psi^*_\varepsilon, \psi_\varepsilon\rangle = 1$. Since

$$\psi^*_\varepsilon = \frac{\left(\varepsilon \int_I \frac{\gamma(x)}{(\lambda_\varepsilon - (1-\varepsilon-x^2))^2}\,dx\right)^{-1}}{\lambda_\varepsilon - (1 - \varepsilon - x^2)},$$

we see that

$$c_{f_0} = \frac{\int_I \frac{f_0(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}\,dx}{\varepsilon \int_I \frac{\gamma(x)}{(\lambda_\varepsilon-(1-\varepsilon-x^2))^2}\,dx}.$$

Let us start by bounding the denominator from above. Using that, by Proposition 2.3, for ε small enough, $\lambda_\varepsilon - (1-\varepsilon) \ge \frac{(\gamma(0)\pi\varepsilon)^2}{2}$, we obtain the bound

$$\begin{aligned} \varepsilon \int_I \frac{\gamma(x)}{(\lambda_\varepsilon-(1-\varepsilon-x^2))^2}\,dx &\le \varepsilon \sup_x \gamma(x) \int_{\mathbb{R}} \frac{1}{\left(\frac{(\gamma(0)\pi\varepsilon)^2}{2}+x^2\right)^2}\,dx \\ &= \sup_x \gamma(x)\, \frac{\sqrt{2}}{\gamma(0)^3(\pi\varepsilon)^2} =: \frac{K_0}{\varepsilon^2}. \end{aligned} \tag{17}$$

Similarly, since for ε small enough, $\lambda_\varepsilon - (1-\varepsilon) \le 2(\gamma(0)\pi\varepsilon)^2$, we have

$$\varepsilon \int_I \frac{\gamma(x)}{\left(\lambda_\varepsilon - (1-\varepsilon-x^2)\right)^2}\,dx \ge \varepsilon \min_{[-x_0,x_0]} \gamma(x) \int_{-x_0}^{x_0} \frac{dx}{\left(2(\gamma(0)\pi\varepsilon)^2 + x^2\right)^2} \ge \frac{K_3}{\varepsilon^2}, \tag{18}$$

where x_0 is any positive value in I. For the numerator we have, on the one hand,

$$\varepsilon^2 \int_I \frac{f_0(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}\,dx \le \int_I \frac{\varepsilon^2}{\frac{(\gamma(0)\pi\varepsilon)^2}{2} + x^2} f_0(x)\,dx, \tag{19}$$

where the right hand side tends to 0 when ε goes to 0 by an easy application of the Lebesgue dominated convergence theorem (note that the integrand is bounded above by $\frac{2}{(\gamma(0)\pi)^2} f_0(x)$).

On the other hand, notice that there exists an interval $J \subset I$ which does not contain 0 such that $\int_J f_0(x)\,dx > 0$. Then, since

$$\int_I \frac{f_0(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}\,dx \ge \int_J \frac{f_0(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}\,dx$$

and

$$\lim_{\varepsilon\to 0} \int_J \frac{f_0(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}\,dx = \int_J \frac{f_0(x)}{x^2}\,dx > 0,$$

there exists a constant $K_4 > 0$ such that

$$\int_I \frac{f_0(x)}{\lambda_\varepsilon - (1-\varepsilon-x^2)}\,dx > K_4. \tag{20}$$

By (18) and(19),

$$c_{f_0} = \frac{\varepsilon^2 \int_I \frac{f_0(x)}{\lambda_\varepsilon-(1-\varepsilon-x^2)}\,dx}{\varepsilon^3 \int_I \frac{\gamma(x)}{\lambda_\varepsilon-(1-\varepsilon-x^2)}\,dx} \le \frac{\varepsilon^2 \int_I \frac{f_0(x)}{\lambda_\varepsilon-(1-\varepsilon-x^2)}\,dx}{K_3} \xrightarrow{\varepsilon\to 0} 0$$

and by (17) and (20), and ε small enough,

$$c_{f_0} \ge \frac{K_4}{\frac{K_0}{\varepsilon^2}} =: K_1\varepsilon^2.$$

This completes the proof. □

Remark 2.2. If $f_0(x)$ is bounded below by a positive number c in a neighbourhood $(-\delta,\delta)$ of 0, then the lower estimate can be improved using that

$$\int_{-\delta}^{\delta} \frac{\varepsilon}{k^2\varepsilon^2 + x^2}\,dx = \frac{2}{k}\arctan\left(\frac{\delta}{k\varepsilon}\right) \xrightarrow{\varepsilon\to 0^+} \frac{\pi}{k}.$$

Indeed, for ε small enough

$$\begin{aligned} \varepsilon \int_I \tfrac{f_0(x)}{\lambda_\varepsilon-(1-\varepsilon-x^2)}\,dx &\ge \varepsilon \int_I \tfrac{f_0(x)}{2(\gamma(0)\pi\varepsilon)^2+x^2}\,dx \\ &\ge c\int_{-\delta}^{\delta} \tfrac{\varepsilon}{(\sqrt{2}\gamma(0)\pi)^2\varepsilon^2+x^2}\,dx \xrightarrow{\varepsilon\to 0^+} \tfrac{c}{\sqrt{2}\gamma(0)}. \end{aligned}$$

So in this case, for ε small enough,

$$c_{f_0} \ge \frac{\frac{c}{\sqrt{2}\gamma(0)\varepsilon}}{\frac{K_0}{\varepsilon^2}} =: K\varepsilon$$

for some constant K independent of ε.

The next two lemmas enable to estimate $\varphi(t)$ (defined immediately before the statement of Lemma 2.6). In the first one, the dependence w.r.t. ε is not explicit.

Lemma 2.7. *For ε small enough and any $\rho_\varepsilon < (\gamma(0)\pi\varepsilon)^2$ there exists $K_\varepsilon > 0$ such that $|\varphi(t)| \le \|v(t,\cdot)\| \le K_\varepsilon\, e^{-\rho_\varepsilon t}\,\|f_0\|$.*

Proof. Since, by Lemma 2.5, $\omega_{ess}(\tilde{A}_\varepsilon) < \omega_0(\tilde{A}_\varepsilon)$, we can apply Theorem 9.11 in [14], and get the estimate

$$\|v(t,\cdot)\| = \|\tilde{T}_\varepsilon(t)f_0 - c_{f_0}\psi_\varepsilon\| \le K_\varepsilon e^{-\eta t}\|f_0\| \quad \forall \eta < \lambda_\varepsilon - (1-\varepsilon).$$

Proposition 2.3 gives then the statement. $\square$

We now give an estimate of the dependence of K_ε on ε, provided that ρ_ε is chosen far enough from its limit value. More precisely, we choose $\rho_\varepsilon = \frac{\lambda_\varepsilon - (1-\varepsilon)}{2} =: \frac{\alpha_\varepsilon}{2}$.

Lemma 2.8. *For ε small enough, there exists a constant K independent of ε and of f_0 such that*

$$\left\|\tilde{T}_\varepsilon(t)f_0 - c_{f_0}\,\psi_\varepsilon\right\| \le K\,\varepsilon^{-4}\,e^{\frac{-\alpha_\varepsilon}{2}\,t}\,\|f_0\|.$$

Proof. Since the proof of this result is quite technical, we delay it to the end of this section (subsection 2.5). $\square$

We now turn to the study of the scalar function $h(t)$. Notice that (16) can be written as

$$h'(t) = \Big(\lambda_\varepsilon - (c_{f_0} + \varphi(t))\,h(t)\Big)\,h(t), \qquad h(0) = 1. \tag{21}$$

The next two lemmas are devoted to the analysis of the asymptotic behavior of $h(t)$. In the first one, the dependence w.r.t. ε of the constants is not explicit.

Lemma 2.9. *For $\varepsilon > 0$ small enough and any $\rho_\varepsilon < (\gamma(0)\pi\varepsilon)^2$, there exists a positive constant $\hat{C}_\varepsilon > 0$ such that $\left|h(t) - \frac{\lambda_\varepsilon}{c_{f_0}}\right| \le \hat{C}_\varepsilon\,e^{-\rho_\varepsilon\,t}$.*

Proof. The solution of (21) is explicitly given by

$$h(t) = \frac{e^{\lambda_\varepsilon t}}{1 + \int_0^t (c_{f_0} + \varphi(s))\,e^{\lambda_\varepsilon s}\,ds} = \frac{1}{e^{-\lambda_\varepsilon t} + \frac{c_{f_0}}{\lambda_\varepsilon}\,(1 - e^{-\lambda_\varepsilon t}) + e^{-\lambda_\varepsilon t}\int_0^t \varphi(s)\,e^{\lambda_\varepsilon s}\,ds}.$$

Then

$$\begin{aligned}
\left|h(t) - \tfrac{\lambda_\varepsilon}{c_{f_0}}\right| &= \left|\frac{1}{e^{-\lambda_\varepsilon t} + \frac{c_{f_0}}{\lambda_\varepsilon}(1-e^{-\lambda_\varepsilon t}) + e^{-\lambda_\varepsilon t}\int_0^t \varphi(s)e^{\lambda_\varepsilon s}\,ds} - \frac{1}{\frac{c_{f_0}}{\lambda_\varepsilon}}\right| \\
&= \frac{\frac{\lambda_\varepsilon}{c_{f_0}}\left|e^{-\lambda_\varepsilon t}\left(1 - \frac{c_{f_0}}{\lambda_\varepsilon}\right) + e^{-\lambda_\varepsilon t}\int_0^t \varphi(s)e^{\lambda_\varepsilon s}\,ds\right|}{e^{-\lambda_\varepsilon t} + e^{-\lambda_\varepsilon t}\int_0^t (c_{f_0} + \varphi(s))e^{\lambda_\varepsilon s}\,ds} \\
&\le \hat{C}_\varepsilon e^{-\rho_\varepsilon t},
\end{aligned}$$

where for the last inequality we have used that the denominator is a positive continuous function bounded below (it takes the value 1 for $t = 0$ and its limit is $\frac{c_{f_0}}{\lambda_\varepsilon}$ when t goes to infinity). We also used the following estimate for the numerator: since, by Lemma 2.7, $|\varphi(s)| \le K_\varepsilon\,e^{-\rho_\varepsilon s}\|f_0\|$, then

$$\begin{aligned}
\left|e^{-\lambda_\varepsilon t}\big(1 - \tfrac{c_{f_0}}{\lambda_\varepsilon}\big) + e^{-\lambda_\varepsilon t}\int_0^t \varphi(s)\,e^{\lambda_\varepsilon s}\,ds\right| &\le e^{-\lambda_\varepsilon t}\left(\left|1 - \tfrac{c_{f_0}}{\lambda_\varepsilon}\right| - \tfrac{K_\varepsilon\,\|f_0\|}{\lambda_\varepsilon - \rho_\varepsilon}\right) + \tfrac{K_\varepsilon\,\|f_0\|}{\lambda_\varepsilon - \rho_\varepsilon}\,e^{-\rho_\varepsilon t} \\
&\le 2K_\varepsilon\,e^{-\rho_\varepsilon t}\|f_0\|.
\end{aligned}$$

$\square$

In order to give an estimate of the dependence of $\hat{C}_\varepsilon$ w.r.t. ε, we need to bound the denominator more precisely and to take a value of ρ_ε separated of its limit value. As in Lemma 2.8, we choose $\rho_\varepsilon = \frac{\lambda_\varepsilon - (1-\varepsilon)}{2} =: \frac{\alpha_\varepsilon}{2}$.

Lemma 2.10. *For $\varepsilon > 0$ small enough, there exist constants K_7 and K_8 (independent of ε) such that*

$$\left|h(t) - \frac{\lambda_\varepsilon}{c_{f_0}}\right| \le K_8\,\varepsilon^{\frac{-K_7}{\varepsilon^2}}\,e^{-\frac{\alpha_\varepsilon t}{2}}.$$

Proof. Using Lemma 2.7 and the fact that the second term is positive we see that

$$
\begin{aligned}
e^{-\lambda_\varepsilon t} + e^{-\lambda_\varepsilon t} \int_0^t (c_{f_0} + \varphi(s))\, e^{\lambda_\varepsilon s}\, ds \quad &\geq \quad e^{-\lambda_\varepsilon t} + \max\left(0, c_{f_0}(1 - e^{-\lambda_\varepsilon t}) - K_\varepsilon e^{-\rho_\varepsilon t}\right) \\
&\geq e^{-\lambda_\varepsilon t_\varepsilon}
\end{aligned}
\tag{22}
$$

for any t_ε such that

$$
c_{f_0}\,(1 - e^{-\lambda_\varepsilon t_\varepsilon}) - K_\varepsilon\, e^{-\rho_\varepsilon t_\varepsilon} \geq e^{-\lambda_\varepsilon t_\varepsilon}. \tag{23}
$$

(Notice that the left hand side in (23) is an increasing function of t_ε). This indeed happens if $K_\varepsilon\, e^{-\rho_\varepsilon t_\varepsilon} \leq \frac{c_{f_0}}{2}$ and $(1 + c_{f_0})\, e^{-\lambda_\varepsilon t_\varepsilon} \leq \frac{c_{f_0}}{2}$. Since the second condition is weaker than the first one for ε small enough, (23) holds whenever t_ε is such that $e^{-\rho_\varepsilon t_\varepsilon} \leq \frac{c_{f_0}}{2K_\varepsilon}$, i.e., $e^{-\lambda_\varepsilon t_\varepsilon} \leq \left(\frac{c_{f_0}}{2K_\varepsilon}\right)^{\frac{\lambda_\varepsilon}{\rho_\varepsilon}}$ and $\varepsilon > 0$ is sufficiently small. So, $\left(\frac{c_{f_0}}{2K_\varepsilon}\right)^{\frac{\lambda_\varepsilon}{\rho_\varepsilon}}$ is also a lower bound in (22), and we finally have

$$
\left| e^{-\lambda_\varepsilon t} + e^{-\lambda_\varepsilon t} \int_0^t (c_{f_0} + \varphi(s)) e^{-\lambda_\varepsilon s} ds \right| \geq \left(\frac{c_{f_0}}{2K_\varepsilon}\right)^{\frac{\lambda_\varepsilon}{\rho_\varepsilon}}.
$$

Using the bound on the numerator given in the proof of Lemma 2.9, the previous estimate, and using also Lemma 2.8, Lemma 2.6 and Proposition 2.3, we obtain

$$
\begin{aligned}
\left| h(t) - \tfrac{\lambda_\varepsilon}{c_{f_0}} \right| \quad &\leq \quad \frac{2K_\varepsilon\, e^{-\rho_\varepsilon t} \|f_0\|}{\left(\frac{c_{f_0}}{2K_\varepsilon}\right)^{\frac{\lambda_\varepsilon}{\rho_\varepsilon}}} \\
&\leq \quad \frac{2K_5\, \varepsilon^{-4}\, e^{-\frac{\alpha_\varepsilon t}{2}} \|f_0\|}{\left(\frac{K_1 \varepsilon^2}{2K_5\, \varepsilon^{-4}}\right)^{K_6\, \varepsilon^{-2}}} \\
&= \quad 2K_5 \left(\frac{2K_5}{K_1}\right)^{-\frac{K_6}{\varepsilon^2}} \varepsilon^{-4 - \frac{6K_6}{\varepsilon^2}}\, e^{-\frac{\alpha_\varepsilon t}{2}} \|f_0\| \\
&\leq \quad K_8\, \varepsilon^{\frac{-K_7}{\varepsilon^2}}\, e^{-\frac{\alpha_\varepsilon t}{2}}.
\end{aligned}
\tag{24}
$$

□

Finally, a standard application of the triangular inequality and Lemmas 2.6, 2.7 and 2.9 give

$$
\begin{aligned}
\|f(t,\cdot) - \lambda_\varepsilon \psi_\varepsilon(x)\| \quad &\leq \quad \left| h(t) - \tfrac{\lambda_\varepsilon}{c_{f_0}} \right| \left\| \tilde{T}_\varepsilon(t) f_0 \right\| + \tfrac{\lambda_\varepsilon}{c_{f_0}} \left\| \tilde{T}_\varepsilon(t) f_0 - c_{f_0}\, \psi_\varepsilon(x) \right\| \\
&\leq \quad \hat{C}_\varepsilon\, e^{-\rho_\varepsilon t}\, (K_2 + K_\varepsilon\, e^{-\rho_\varepsilon t}\, ||f_0||) + \tfrac{1}{K_1 \varepsilon^2} K_\varepsilon e^{-\rho_\varepsilon t} \\
&\leq \quad C_\varepsilon\, e^{-\rho_\varepsilon t}.
\end{aligned}
\tag{25}
$$

Using Lemmas 2.8 and 2.10 in the second inequality of (25), the last statement of Proposition 2.4 follows.

2.4. **Proof of Theorem 1.2.** By the triangular inequality,

$$
\begin{aligned}
\left\| f(t,\cdot) - \tfrac{\varepsilon \gamma(0)}{(\gamma(0)\pi\varepsilon)^2 + \cdot^2} \right\|_{L^1(I)} \quad &\leq \quad \|f(t,\cdot) - \lambda_\varepsilon \psi_\varepsilon\|_{L^1(I)} \\
&+ \quad \left\| \lambda_\varepsilon \psi_\varepsilon - \tfrac{\varepsilon \gamma(0)}{(\gamma(0)\pi\varepsilon)^2 + \cdot^2} \right\|_{L^1(I)}
\end{aligned}
.
$$

Hence by Proposition 2.3 and Proposition 2.4, we only need to estimate the last term, for which we have

$$\begin{aligned}
&\left\|\frac{\lambda_\varepsilon \varepsilon\gamma}{\lambda_\varepsilon-(1-\varepsilon)+\cdot^2}-\frac{\varepsilon\gamma(0)}{(\gamma(0)\pi\varepsilon)^2+\cdot^2}\right\|_{L^1(I)}\\
&\quad\le \left\|\frac{\varepsilon(\lambda_\varepsilon-1)\gamma}{\lambda_\varepsilon-(1-\varepsilon)+\cdot^2}\right\|_{L^1(I)}+\left\|\frac{\varepsilon\gamma}{\lambda_\varepsilon-(1-\varepsilon)+\cdot^2}-\frac{\varepsilon\gamma}{(\gamma(0)\pi\varepsilon)^2+\cdot^2}\right\|_{L^1(I)}\\
&\qquad+\left\|\frac{\varepsilon(\gamma-\gamma(0))}{(\gamma(0)\pi\varepsilon)^2+\cdot^2}\right\|_{L^1(I)}.
\end{aligned}$$

Let us bound the three terms. For the first one we have, by Proposition 2.3,

$$\begin{aligned}
\left\|\tfrac{\varepsilon(\lambda_\varepsilon-1)\gamma}{\lambda_\varepsilon-(1-\varepsilon)+\cdot^2}\right\|_{L^1(I)} &\le (\lambda_\varepsilon-1)\|\gamma\|_\infty\int_{\mathbb{R}}\tfrac{\varepsilon\,dx}{\frac{(\gamma(0)\pi\varepsilon)^2}{2}+x^2}\\
&= (\lambda_\varepsilon-1)\|\gamma\|_\infty\tfrac{\sqrt{2}}{\gamma(0)}=O(\varepsilon).
\end{aligned}$$

For the second one, by Proposition 2.3 and (17),

$$\begin{aligned}
&\left\|\frac{\varepsilon\gamma}{\lambda_\varepsilon-(1-\varepsilon)+\cdot^2}-\frac{\varepsilon\gamma}{(\gamma(0)\pi\varepsilon)^2+\cdot^2}\right\|_{L^1(I)}\\
&\quad\le |(\gamma(0)\pi\varepsilon)^2-(\lambda_\varepsilon-(1-\varepsilon))|\varepsilon\|\gamma\|_\infty\int_{\mathbb{R}}\frac{dx}{\left(\frac{(\gamma(0)\pi\varepsilon)^2}{2}+x^2\right)^2}\\
&\quad=\left|(\gamma(0)\pi\varepsilon)^2-(\lambda_\varepsilon-(1-\varepsilon))\right|\frac{K_0}{\varepsilon^2}=O\left(\varepsilon\ln\frac{1}{\varepsilon}\right).
\end{aligned}$$

For the third one, similarly to the proof of Proposition 2.3, denoting by $A:=\frac{\|\gamma\|_\infty}{\|\gamma'\|_\infty}$,

$$\begin{aligned}
\left\|\frac{\varepsilon(\gamma-\gamma(0))}{(\gamma(0)\pi\varepsilon)^2+\cdot^2}\right\|_{L^1(I)} &\le 2\varepsilon\int_0^A\frac{\|\gamma'\|_\infty x}{(\gamma(0)\pi\varepsilon)^2+x^2}\,dx+2\varepsilon\int_A^{+\infty}\frac{\|\gamma\|_\infty}{(\gamma(0)\pi\varepsilon)^2+x^2}\,dx\\
&= \varepsilon\|\gamma'\|_\infty\ln\left(1+\frac{A^2}{(\gamma(0)\pi\varepsilon)^2}\right)+2\frac{\|\gamma\|_\infty}{\gamma(0)\pi}\arctan\left(\frac{\gamma(0)\pi\varepsilon}{A}\right)\\
&= O\left(\varepsilon\ln\frac{1}{\varepsilon}\right).
\end{aligned}$$

2.5. **Proof of Lemma 2.8.** Let us consider the linear initial value problem

$$\left\{\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \tilde{A}_\varepsilon u(t,x)=(a_\varepsilon(x)-\lambda_\varepsilon)\,u(t,x)+\varepsilon\gamma(x)\,\int_I u(t,y)\,dy,\\ u(0,x) &= u_0(x),\end{aligned}\right. \tag{26}$$

where $a_\varepsilon(x):=1-\varepsilon-x^2$. Let us recall that $s(\tilde{A}_\varepsilon)=0$ and $\varepsilon\int_I\frac{\gamma(x)}{\lambda_\varepsilon-a_\varepsilon(x)}\,dx=1$ (see Proposition 2.2). Applying the Laplace transform with respect to t to the previous equation, we obtain the identity

$$\mu\,\mathcal{L}[u](\mu,x)-u_0(x)=(a_\varepsilon(x)-\lambda_\varepsilon)\,\mathcal{L}[u](\mu,x)+\varepsilon\,\gamma(x)\int_I\mathcal{L}[u](\mu,y)\;dy,$$

that is

$$\mathcal{L}[u](\mu,x)=\frac{u_0(x)}{\mu+\lambda_\varepsilon-a_\varepsilon(x)}+\frac{\varepsilon\,\gamma(x)}{\mu+\lambda_\varepsilon-a_\varepsilon(x)}\int_I\mathcal{L}[u](\mu,y)\,dy. \tag{27}$$

Integrating (with respect to x), we obtain

$$\int_I\mathcal{L}[u](\mu,x)\,dx=\frac{\int_I\frac{u_0(x)}{\mu+\lambda_\varepsilon-a_\varepsilon(x)}\,dx}{1-\int_I\frac{\varepsilon\gamma(x)}{\mu+\lambda_\varepsilon-a_\varepsilon(x)}\,dx}=\frac{\int_I\frac{u_0(x)}{\mu+\lambda_\varepsilon-a_\varepsilon(x)}\,dx}{\varepsilon\mu\int_I\frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))(\mu+\lambda_\varepsilon-a_\varepsilon(x))}\,dx},$$

where we have used, for the second equality, $\varepsilon\int_I \frac{\gamma(x)}{\lambda_\varepsilon - a_\varepsilon(x)} = 1$. Substituting in (27), we get

$$\mathcal{L}[u](\mu,x) = \frac{u_0(x)}{\mu+\lambda_\varepsilon - a_\varepsilon(x)} + \frac{\int_I \frac{u_0(x)}{\mu+\lambda_\varepsilon - a_\varepsilon(x)}\,dx}{\mu\int_I \frac{\gamma(x)}{(\lambda_\varepsilon - a_\varepsilon(x))(\mu+\lambda_\varepsilon - a_\varepsilon(x))}\,dx}\,\frac{\gamma(x)}{(\mu+\lambda_\varepsilon - a_\varepsilon(x))}. \tag{28}$$

This Laplace transform is analytic for Re $\mu > 0$ (note that $\lambda_\varepsilon - a_\varepsilon(x)$ is positive and tends to zero when ε tends to zero). Then, for $s > 0$, we know, by the inversion theorem, that

$$u(t,x) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} \mathcal{L}[u](\mu,x)\,e^{\mu t}\,d\mu.$$

Using the theorem of residues, we can shift the integration path to the left in order to obtain, for any $s' \in (1-\varepsilon-\lambda_\varepsilon, 0)$,

$$u(t,x) = \mathrm{Res}_{\mu=0}\Big(\mathcal{L}[u](\mu,x)e^{\mu t}\Big) + \frac{1}{2\pi i}\int_{s'-i\infty}^{s'+i\infty}\mathcal{L}[u](\mu,x)e^{\mu\, t}\,d\mu,$$

where

$$\begin{aligned}
\mathrm{Res}_{\mu=0}\Big(\mathcal{L}[u](\mu,x)e^{\mu t}\Big) &= \lim_{\mu\to 0}\mu\mathcal{L}[u](\mu,x) \\
&= \lim_{\mu\to 0}\left(\frac{\mu u_0(x)}{\mu+\lambda_\varepsilon - a_\varepsilon(x)} + \frac{\int_I \frac{u_0(x)}{\mu+\lambda_\varepsilon - a_\varepsilon(x)}\,dx}{\int_I \frac{\gamma(x)}{(\lambda_\varepsilon - a_\varepsilon(x))(\mu+\lambda_\varepsilon - a_\varepsilon(x))}\,dx}\,\frac{\gamma(x)}{\mu+\lambda_\varepsilon - a_\varepsilon(x)}\right) \\
&= \frac{\langle u_0,\psi_\varepsilon^*\rangle}{\langle \psi_\varepsilon,\psi_\varepsilon^*\rangle}\,\psi_\varepsilon(x) = c_{u_0}\,\psi_\varepsilon(x)
\end{aligned}$$

(let us recall that $\psi_\varepsilon(x) = \frac{\varepsilon\gamma(x)}{\lambda_\varepsilon - a_\varepsilon(x)}$ and $\psi_\varepsilon^*(x) = \frac{\left(\varepsilon\int_I \frac{\gamma(x)\,dx}{(\lambda_\varepsilon-(1-\varepsilon-x^2))^2}\right)^{-1}}{\lambda_\varepsilon-(1-\varepsilon-x^2)}$).

Thus, we obtain that, for $s' \in (1-\varepsilon-\lambda_\varepsilon, 0)$,

$$u(t,x) = c_{u_0}\psi_\varepsilon(x) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{L}[u](s'+i\tau,x)\,e^{(s'+i\tau)\,t}\,d\tau. \tag{29}$$

We now define $g_\varepsilon(\mu) := \frac{\int_I \frac{u_0(x)\,dx}{\mu+\lambda_\varepsilon - a_\varepsilon(x)}}{\mu\int_I \frac{\gamma(x)\,dx}{(\lambda_\varepsilon - a_\varepsilon(x))(\mu+\lambda_\varepsilon - a_\varepsilon(x))}}$, so that we can write

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathcal{L}[u](s'+i\tau,x)e^{(s'+i\tau)t}\,d\tau &= \frac{1}{2\pi}u_0(x)e^{s't}\int_{-\infty}^{\infty}\frac{e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau \\
&+ \frac{1}{2\pi}\gamma(x)e^{s't}\int_{-\infty}^{\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau \\
&= e^{-(\lambda_\varepsilon - a_\varepsilon(x))t}u_0(x) \\
&+ \frac{1}{2\pi}\gamma(x)e^{s't}\int_{-\infty}^{\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau,
\end{aligned} \tag{30}$$

where we used the estimate $s'+\lambda_\varepsilon - a_\varepsilon(x) > 0$ and the identity $\int_{-\infty}^{\infty}\frac{e^{i\tau t}}{\alpha+i\tau}\,d\tau = 2\pi e^{-\alpha t}$ (for $\alpha > 0$).

We now would like to find a bound for $\left\|\frac{1}{2\pi}\gamma(x)e^{s't}\int_{-\infty}^{\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau\right\|_\infty$.

We see that

$$\begin{aligned}
\left\|\gamma(x)\,e^{s't}\int_{-\infty}^{\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau\right\|_\infty &\leq e^{s't}\|\gamma\|_\infty \sup_x\left|\int_{-\infty}^{\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau\right| \\
&= e^{s't}\sup_x\left|\int_{-\infty}^{\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau\right|
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
\left|\int_{-\infty}^{\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau}\,d\tau\right| &\leq \int_{-\infty}^{\infty}\frac{|g_\varepsilon(s'+i\tau)|}{|s'+\lambda_\varepsilon - a_\varepsilon(x)+i\tau|}\,d\tau \\
&\leq \int_{-\infty}^{\infty}\frac{|g_\varepsilon(s'+i\tau)|}{|s'+\lambda_\varepsilon-(1-\varepsilon)+i\tau|}\,d\tau
\end{aligned} \tag{32}$$

since $|s'+\lambda_\varepsilon-a_\varepsilon(x)+i\tau|\geq|s'+\lambda_\varepsilon-(1-\varepsilon)+i\tau|$.

Let us then find an upper bound for $g_\varepsilon(s'+i\tau)$. For the numerator of $g_\varepsilon(s'+i\tau)$ we can estimate

$$\left|\int_I \frac{u_0(x)}{s'+i\tau+\lambda_\varepsilon-a_\varepsilon(x)}\,dx\right|\leq\frac{\|u_0\|_1}{|s'+i\tau+\lambda_\varepsilon-(1-\varepsilon)|}.$$

We now find a lower bound for the denominator of $g_\varepsilon(s'+i\tau)$. We use the elementary estimate $|z|\geq\max(|\mathrm{Re}z|,|\mathrm{Im}z|)$ and we start with the real part.

$$\begin{aligned}
\left|\mathrm{Re}\int_I \frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))(s'+i\tau+\lambda_\varepsilon-a_\varepsilon(x))}\,dx\right| &= \left|\int_I \frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))}\frac{s'+\lambda_\varepsilon-a_\varepsilon(x)}{|s'+i\tau+\lambda_\varepsilon-a_\varepsilon(x)|^2}\,dx\right|\\
&= \int_I \frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))}\frac{s'+\lambda_\varepsilon-a_\varepsilon(x)}{|s'+i\tau+\lambda_\varepsilon-a_\varepsilon(x)|^2}\,dx\\
&= \int_I \frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))\left(s'+\lambda_\varepsilon-a_\varepsilon(x)+\frac{\tau^2}{s'+\lambda_\varepsilon-a_\varepsilon(x)}\right)}\,dx\\
&\geq \int_I \frac{\gamma(x)}{(\lambda_{\varepsilon_0}-(1-\varepsilon_0)+x^2)\left(\lambda_{\varepsilon_0}-(1-\varepsilon_0)+x^2+\frac{\tau^2}{x^2}\right)}\,dx\\
&= \int_I \frac{x^2\gamma(x)}{(\lambda_{\varepsilon_0}-(1-\varepsilon_0)+x^2)((\lambda_{\varepsilon_0}-(1-\varepsilon_0)+x^2)x^2+\tau^2)}\,dx\\
&=: F(\tau),
\end{aligned}$$

where in the last inequality we used the estimates $s'+\lambda_\varepsilon-a_\varepsilon(x)<\lambda_\varepsilon-a_\varepsilon(x)$, $s'+\lambda_\varepsilon-(1-\varepsilon)>0$. We also used that, since $\lambda_\varepsilon-(1-\varepsilon)$ is strictly positive and tends to zero when ε goes to zero, there exists ε_0 such that $\forall\varepsilon<\varepsilon_0$ we have $\lambda_{\varepsilon_0}-(1-\varepsilon_0)>\lambda_\varepsilon-(1-\varepsilon)$.
In a similar way, for the imaginary part,

$$\begin{aligned}
\left|\mathrm{Im}\int_I \frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))(s'+i\tau+\lambda_\varepsilon-a_\varepsilon(x))}\,dx\right| &= \left|\int_I \frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))}\frac{-\tau}{(s'+\lambda_\varepsilon-a_\varepsilon(x))^2+\tau^2}\,dx\right|\\
&= |\tau|\int_I \frac{\gamma(x)}{(\lambda_\varepsilon-a_\varepsilon(x))\left((s'+\lambda_\varepsilon-a_\varepsilon(x))^2+\tau^2\right)}\,dx\\
&\geq |\tau|\int_I \frac{\gamma(x)}{(\lambda_{\varepsilon_0}-(1-\varepsilon_0)+x^2)\left((\lambda_{\varepsilon_0}-(1-\varepsilon_0)+x^2)^2+\tau^2\right)}\,dx\\
&=: G(\tau).
\end{aligned}$$

Defining $H(\tau):=\max(F(\tau),G(\tau))$ we see that

$$|g_\varepsilon(s'+i\tau)|\leq\frac{\frac{\|u_0\|_1}{|s'+i\tau+\lambda_\varepsilon-(1-\varepsilon)|}}{|s'+i\tau|H(\tau)},\tag{33}$$

and then, using (31), (32) and (33)

$$\begin{aligned}
&\left\|\gamma(x)e^{s't}\int_{-\infty}^{+\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon-a_\varepsilon(x)+i\tau}\,d\tau\right\|_\infty\\
&\leq e^{s't}\int_{-\infty}^{+\infty}\frac{d\tau}{\sqrt{s'^2+\tau^2}|s'+i\tau+\lambda_\varepsilon-(1-\varepsilon)|^2H(\tau)}\|u_0\|_1.
\end{aligned}$$

Now, since F and G are strictly positive continuous functions, $F(0)>0$ and $\tau G(\tau)$ tends to a positive limit when τ goes to ∞, there exists a constant $C>0$ (independent of ε) such that $H(\tau)\geq\frac{C}{1+\tau}$. Choosing $s'=-\frac{\alpha_\varepsilon}{2}$, where $\alpha_\varepsilon=\lambda_\varepsilon-(1-\varepsilon)$, we can write

$$\begin{aligned}
\left\|x\mapsto\gamma(x)\,e^{s't}\int_{-\infty}^{+\infty}\frac{g_\varepsilon(s'+i\tau)e^{i\tau t}}{s'+\lambda_\varepsilon-a_\varepsilon(x)+i\tau}\,d\tau\right\|_\infty &\leq \frac{e^{-\alpha_\varepsilon t}}{C}\int_0^{+\infty}\frac{2(1+\tau)}{\left((\frac{\alpha_\varepsilon}{2})^2+\tau^2\right)^{\frac{3}{2}}}\,d\tau\|u_0\|_1\\
&= \frac{e^{-\frac{\alpha_\varepsilon t}{2}}}{C}\left(\frac{8}{\alpha_\varepsilon^2}+\frac{4}{\alpha_\varepsilon}\right)\|u_0\|_1.
\end{aligned}$$

Finally, going back to (29) and using (30), we end up with

$$\|u(t,\cdot)-c_{u_0}\psi_\varepsilon\| \leq \left(1+\frac{1}{\pi C}\left(\frac{4}{\alpha_\varepsilon^2}+\frac{2}{\alpha_\varepsilon}\right)\right) e^{-\frac{\alpha_\varepsilon t}{2}}\,\|u_0\|_1 \leq K_5\,\varepsilon^{-4} e^{-\frac{\alpha_\varepsilon t}{2}}\,\|u_0\|_1.$$

3. Proof of Theorem 1.3

We start here the proof of Theorem 1.3. From now on, C will designate a strictly positive constant depending only on some upper bounds on $\|\gamma\|_{L^\infty}$, $\|f_0\|_{W^{1,\infty}}$, on a lower bound on f_0, and on $|b-a|$ (assumed to be finite in this theorem).

Thanks to the variation of the constant formula, the solution f of (1) satisfies:

$$\begin{aligned}
f(t,x) &= f(0,x)\,e^{(1-\varepsilon-x^2)\,t-\int_0^t\int_I f(s,y)\,dy\,ds}\\
&\quad+\varepsilon\int_0^t\left(\gamma(x)\int_I f(s,y)\,dy\right)e^{(1-\varepsilon-x^2)(t-s)-\int_s^t\int_I f(\sigma,y)\,dy\,d\sigma}\,ds\\
&= f(0,x)\,e^{(1-\varepsilon-x^2)\,t-\int_0^t\mathcal{I}(s)\,ds}\\
&\quad+\varepsilon\int_0^t\gamma(x)\,\mathcal{I}(s)e^{(1-\varepsilon-x^2)(t-s)-\int_s^t\mathcal{I}(\sigma)\,d\sigma}\,ds,
\end{aligned}\tag{34}$$

where

$$\mathcal{I}(t) := \int_I f(t,y)\,dy.$$

Obtaining a precise estimate on $t\mapsto e^{(1-\varepsilon)(t-s)-\int_s^t\mathcal{I}(\sigma)\,d\sigma}$ is the cornerstone of the proof of Theorem 1.3.

3.1. **Preliminary estimates.** If we sum (34) along $x\in\mathbb{R}$, we get, for $t\geq 0$:

$$\begin{aligned}
\mathcal{I}(t) &= \left(\int_I f(0,x)\,e^{-x^2t}\,dx\right)e^{(1-\varepsilon)t-\int_0^t\mathcal{I}(s)\,ds}\\
&\quad+\varepsilon\int_0^t\left(\int_I\int_I\gamma(x)f(s,y)e^{-x^2(t-s)}\,dx\,dy\right)e^{(1-\varepsilon)(t-s)-\int_s^t\mathcal{I}(\sigma)\,d\sigma}\,ds\\
&= \frac{z_1(t)}{\sqrt{t}}e^{(1-\varepsilon)t-\int_0^t\mathcal{I}(s)\,ds}+\varepsilon\int_0^t\frac{z_2(s,t-s)}{\sqrt{t-s}}e^{(1-\varepsilon)(t-s)-\int_s^t\mathcal{I}(\sigma)\,d\sigma}\,ds,
\end{aligned}\tag{35}$$

where

$$z_1(t) := \sqrt{t}\int_I f(0,x)\,e^{-x^2t}\,dx,\quad z_2(\sigma,\tau)=\sqrt{\tau}\int_I\int_I\gamma(x)\,f(\sigma,y)\,e^{-x^2\tau}\,dx\,dy.$$

If we integrate our equation w.r.t. x, we get

$$\begin{aligned}
\frac{\partial\mathcal{I}}{\partial t}(t) &= \mathcal{I}(t)\,(1-\varepsilon-\mathcal{I}(t))-\int_I x^2f(t,x)\,dx+\varepsilon\int_I\int_I\gamma(x)f(t,y)\,dx\,dy\\
&\leq \mathcal{I}(t)\,(1-\varepsilon-\mathcal{I}(t))+\varepsilon\,\mathcal{I}(t)\\
&\leq \mathcal{I}(t)\,(1-\mathcal{I}(t))\,,
\end{aligned}$$

which implies, since $\mathcal{I}(0)\leq 1$, that

$$0\leq\mathcal{I}(t)\leq 1.\tag{36}$$

Thanks to (35), (36) and the nonnegativity of z_1, z_2, one gets

$$\frac{z_1(t)}{\sqrt{t}}e^{(1-\varepsilon)t-\int_0^t\mathcal{I}(s)\,ds}\leq C,\tag{37}$$

while for some constants C, $C'>0$,

$$\begin{aligned}
z_1(t) &= \int_I f\left(0,\frac{x}{\sqrt{t}}\right)e^{-x^2}\,dx\\
&\geq \frac{1}{C}\int_{-C'}^{C'} f\left(0,\frac{x}{\sqrt{t}}\right)\,dx\geq\frac{1}{C}\quad,
\end{aligned}$$

for $t \geq 1$. Note that here we used a lower bound on $f(0,\cdot)$ around $x = 0$ (we have assumed that $f(0,0) > 0$ and that $f(0,\cdot)$ is continuous). Thanks to this lower bound, (37) becomes

$$e^{(1-\varepsilon)t - \int_0^t \mathcal{I}(s)\,ds} \leq C\sqrt{t}. \tag{38}$$

Thanks to (38) and (36), we can estimate the second term of (35) as follows:

$$\begin{aligned} w(t) &:= \varepsilon \int_0^t \frac{z_2(s,t-s)}{\sqrt{t-s}} e^{(1-\varepsilon)\,t - \int_0^t \mathcal{I}(\sigma)\,d\sigma} e^{\varepsilon s + \int_0^s (\mathcal{I}(\sigma)-1)\,d\sigma}\,ds \\ &\leq C\,\varepsilon\sqrt{t}\,\|z_2\|_{L^\infty} \int_0^t \frac{e^{C\varepsilon s}}{\sqrt{t-s}}\,ds \\ &\leq C\,\varepsilon\sqrt{t}\,\|z_2\|_{L^\infty}\, e^{C\varepsilon t} \int_0^t \frac{e^{-C\varepsilon s}}{\sqrt{s}}\,ds \leq C\,\varepsilon t\,\|z_2\|_{L^\infty}\, e^{C\varepsilon t}. \end{aligned} \tag{39}$$

In order to estimate $\|z_2\|_{L^\infty}$, we proceed as follows:

$$\begin{aligned} z_2(s,\tau) &= \int_I \int_I \gamma\left(\frac{x}{\sqrt{\tau}}\right) f(s,y)\, e^{-x^2}\,dx\,dy \\ &\leq C\,\mathcal{I}(s) \int_I e^{-x^2}\,dx \leq C. \end{aligned}$$

This estimate combined with (39) implies that $w(t)$ satisfies

$$0 \leq w(t) \leq C\,\varepsilon\,t\,e^{C\,\varepsilon\,t}. \tag{40}$$

Since $f(0,\cdot) \in W^{1,\infty}(I)$, we can estimate (for $t > 0$)

$$\begin{aligned} z_1(t) &= \int_I \left(f(0,0) + \int_0^{\frac{x}{\sqrt{t}}} \frac{\partial f}{\partial x}(0,z)\,dz \right) e^{-x^2}\,dx \\ &= f(0,0) \int_I e^{-x^2}\,dx + \lambda(t), \end{aligned} \tag{41}$$

where

$$\begin{aligned} |\lambda(t)| &\leq \int_I \left| \int_0^{\frac{x}{\sqrt{t}}} \frac{\partial f}{\partial x}(0,z)\,dz \right| e^{-x^2}\,dx \leq \frac{C}{\sqrt{t}} \int_I |x| e^{-x^2}\,dx \\ &\leq \frac{C}{\sqrt{t}}. \end{aligned} \tag{42}$$

3.2. **Estimate for $e^{(1-\varepsilon)t - \int_0^t \mathcal{I}(s)\,ds}$.** Thanks to (35) (and the definition of λ and w: see (41) and (39) respectively), we see that, for $t \geq 1$,

$$\mathcal{I}(t) = \frac{f(0,0)\int_I e^{-x^2}\,dx + \lambda(t)}{\sqrt{t}} e^{(1-\varepsilon)t - \int_0^t \mathcal{I}(s)\,ds} + w(t), \tag{43}$$

so that

$$\begin{aligned} e^{\int_0^t \mathcal{I}(s)\,ds} &= e^{\int_0^1 \mathcal{I}(s)\,ds} + \int_1^t \frac{d}{ds}\left(e^{\int_0^s \mathcal{I}(\sigma)\,d\sigma}\right)(s)\,ds \\ &= e^{\int_0^1 \mathcal{I}(s)\,ds} + \int_1^t \frac{f(0,0)\int_I e^{-x^2}\,dx}{\sqrt{s}} e^{(1-\varepsilon)s}\,ds \\ &\quad + \int_1^t \frac{\lambda(s)}{\sqrt{s}} e^{(1-\varepsilon)s}\,ds + \int_1^t w(s) e^{\int_0^s \mathcal{I}(\sigma)\,d\sigma}\,ds. \end{aligned} \tag{44}$$

We will now estimate each of the terms on the right hand side of (44). We start by estimating the third term on the right hand side, thanks to (42) and an integration by parts:

$$
\begin{aligned}
\left|\int_1^t \frac{\lambda(s)}{\sqrt{s}} e^{(1-\varepsilon)s}\, ds\right| &\leq C\int_1^t \frac{e^{(1-\varepsilon)s}}{s}\, ds \\
&\leq C\left[\frac{e^{(1-\varepsilon)t}}{(1-\varepsilon)t} + \int_1^t \frac{e^{(1-\varepsilon)s}}{(1-\varepsilon)s^2}\, ds\right] \\
&\leq \frac{C}{1-\varepsilon}\left[\frac{e^{(1-\varepsilon)t}}{t} + t\max_{s\in[1,t]} \frac{e^{(1-\varepsilon)s}}{s^2}\right] \\
&\leq \frac{2C}{(1-\varepsilon)t} e^{(1-\varepsilon)t}, \qquad (45)
\end{aligned}
$$

provided that $t > 4$ (this last assumption ensures, remembering that $0 < \varepsilon < 1/2$, that $\max_{s\in[1,t]} \frac{e^{(1-\varepsilon)s}}{s^2} = \frac{e^{(1-\varepsilon)t}}{t^2}$).

We now estimate the second term on the right hand side of (44), using an integration by parts:

$$
\begin{aligned}
&\int_1^t \frac{f(0,0)\int_I e^{-x^2}\, dx}{\sqrt{s}} e^{(1-\varepsilon)s}\, ds \\
&\quad = f(0,0)\left(\int_I e^{-x^2}\, dx\right)\left(\frac{e^{(1-\varepsilon)t}}{(1-\varepsilon)\sqrt{t}} - \frac{e^{1-\varepsilon}}{1-\varepsilon} + \int_1^t \frac{e^{(1-\varepsilon)s}}{2(1-\varepsilon)s^{3/2}}\, ds\right),
\end{aligned}
$$

and then, applying an estimate similar to the one used to obtain (45), we get, provided that $t > 4$,

$$
0 \leq \int_1^t \frac{e^{(1-\varepsilon)s}}{2\,(1-\varepsilon)s^{3/2}}\, ds \leq \int_1^t \frac{e^{(1-\varepsilon)\, s}}{2\,(1-\varepsilon)\, s}\, ds \leq \frac{1}{(1-\varepsilon)^2\, t} e^{(1-\varepsilon)\, t}. \qquad (46)
$$

Finally, we estimate the last term of the right hand side of (44), thanks to estimates (40) and (36), and for $t \geq 1$:

$$
\begin{aligned}
0 \leq \int_1^t w(s) e^{\int_0^s \mathcal{I}(\sigma)\, d\sigma}\, ds &\leq \int_1^t |w(s)| e^{\|\mathcal{I}\|_{L^\infty(\mathbb{R}_+)} s}\, ds \\
&\leq C\,\varepsilon \int_1^t s\, e^{C\varepsilon s}\, e^s\, ds \\
&\leq C\,\varepsilon \frac{t\, e^{(1+C\,\varepsilon)\, t}}{1+C\,\varepsilon}, \qquad (47)
\end{aligned}
$$

where we have used $s \leq t$ to obtain the last inequality.

Combining these estimates, estimate (44) becomes:

$$
e^{\int_0^t \mathcal{I}(s)\, ds - (1-\varepsilon)t} = \frac{f(0,0)\int_I e^{-x^2}\, dx}{(1-\varepsilon)\sqrt{t}} + \mu(t), \qquad (48)
$$

or

$$
e^{(1-\varepsilon)t - \int_0^t \mathcal{I}(s)\, ds} = \left(\frac{f(0,0)\int_I e^{-x^2}\, dx}{(1-\varepsilon)\sqrt{t}} + \mu(t)\right)^{-1}, \qquad (49)
$$

where, thanks to (44), (45), (46) and (47), for $t > 4$,

$$
-\frac{C}{t} \leq \mu(t) \leq C\left(\frac{1}{t} + \varepsilon t e^{C\varepsilon t}\right). \qquad (50)
$$

3.3. **Estimate for** $\left| e^{(1-\varepsilon)t-\int_0^t \mathcal{I}(s)\,ds} - \frac{\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx}\right|$**.** Thanks to (49),

$$\begin{aligned}
&\left| e^{(1-\varepsilon)t-\int_0^t \mathcal{I}(s)\,ds} - \frac{\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx}\right| \\
&= \left| \left(\frac{f(0,0)\,\int_I e^{-x^2}\,dx}{(1-\varepsilon)\sqrt{t}} + \mu(t)\right)^{-1} - \frac{\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx}\right| \qquad (51)\\
&= \frac{\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx} \left| \left(\frac{1}{1-\varepsilon} + \frac{\mu(t)\,\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx}\right)^{-1} - 1\right|. \qquad (52)
\end{aligned}$$

We notice that thanks to estimate (50),

$$f(0,0)\,\int_I e^{-x^2}\,dx + (1-\varepsilon)\mu(t)\sqrt{t} \geq \frac{f(0,0)\,\int_I e^{-x^2}\,dx}{2}, \qquad (53)$$

as soon as $t > 4$. Under the same assumption, we directly get from (50) that

$$|\mu(t)| \leq C\left(\frac{1}{t} + \varepsilon\, t\, e^{C\,\varepsilon\, t}\right). \qquad (54)$$

Using the bounds (53) and (54), we can show that as soon as $t > 4$,

$$\begin{aligned}
\left| \left(\frac{1}{1-\varepsilon} + \frac{\mu(t)\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx}\right)^{-1} - 1\right| &= \left| \frac{-\varepsilon f(0,0)\,\int_I e^{-x^2}\,dx - (1-\varepsilon)\mu(t)\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx + (1-\varepsilon)\mu(t)\sqrt{t}}\right| \\
&\leq C\left(\frac{1}{\sqrt{t}} + \varepsilon t^{\frac{3}{2}}\, e^{C\,\varepsilon\, t}\right), \qquad (55)
\end{aligned}$$

so that identity (52) leads to the bound (for $t > 4$)

$$\left| e^{(1-\varepsilon)t-\int_0^t \mathcal{I}(s)\,ds} - \frac{\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx}\right| \leq C\left(1 + \varepsilon\, t^2\, e^{C\,\varepsilon\, t}\right). \qquad (56)$$

Notice also, as this is going to be useful further on, that for $s > 4$, thanks to (48) and (54),

$$\begin{aligned}
\left| e^{\int_0^s \mathcal{I}(\sigma)\,d\sigma-(1-\varepsilon)s} - \frac{f(0,0)\,\int_I e^{-x^2}\,dx}{\sqrt{s}}\right| &= \left| \mu(s) + \varepsilon \frac{f(0,0)\,\int_I e^{-x^2}\,dx}{(1-\varepsilon)\sqrt{s}}\right| \\
&\leq C\left(\frac{1}{s} + \varepsilon\, s\, e^{C\,\varepsilon\, s}\right). \qquad (57)
\end{aligned}$$

3.4. **Conclusion of the proof of Theorem 1.3.** In this last part of the proof, we systematically consider times $t > 4$. We estimate

$$\begin{aligned}
&\left\| x \mapsto f(t,x) - \frac{f(0,x)\,\sqrt{t}\, e^{-x^2 t}}{f(0,0)\,\int_I e^{-y^2}\,dy}\right\|_{L^1(I)} \\
&\leq \left\| x \mapsto f(t,x) - \frac{f(0,x)\, e^{-x^2 t}}{\frac{f(0,0)\,\int_I e^{-y^2}\,dy}{(1-\varepsilon)\,\sqrt{t}} + \mu(t)}\right\|_{L^1(I)} \\
&\quad + \left\| x \mapsto \frac{f(0,x)\, e^{-x^2 t}}{\frac{f(0,0)\,\int_I e^{-y^2}\,dy}{(1-\varepsilon)\,\sqrt{t}} + \mu(t)} - \frac{f(0,x)\,\sqrt{t}\, e^{-x^2 t}}{f(0,0)\,\int_I e^{-y^2}\,dy}\right\|_{L^1(I)}. \qquad (58)
\end{aligned}$$

Let us start by estimating the second term of the right hand side of (58), thanks to estimate (55):

$$\begin{aligned}
&\left\| x \mapsto \frac{f(0,x)e^{-x^2 t}}{\frac{f(0,0)\,\int_I e^{-y^2}\,dy}{(1-\varepsilon)\sqrt{t}} + \mu(t)} - \frac{f(0,x)\sqrt{t}e^{-x^2 t}}{f(0,0)\,\int_I e^{-y^2}\,dy} \right\|_{L^1(I)} \\
&\quad \leq \left\| x \mapsto \frac{f(0,x)\sqrt{t}e^{-x^2 t}}{f(0,0)\,\int_I e^{-y^2}\,dy} \left| \left(\frac{1}{1-\varepsilon} + \frac{\mu(t)\sqrt{t}}{f(0,0)\,\int_I e^{-y^2}\,dy} \right)^{-1} - 1 \right| \right\|_{L^1(I)} \\
&\quad \leq \frac{||f(0,\cdot)||_\infty \sqrt{t}}{f(0,0)\,\int_I e^{-y^2}\,dy} \left| \left(\frac{1}{1-\varepsilon} + \frac{\mu(t)\sqrt{t}}{f(0,0)\,\int_I e^{-y^2}\,dy} \right)^{-1} - 1 \right| \int_I e^{-x^2 t}\,dx \\
&\quad \leq C\left(\frac{1}{\sqrt{t}} + \varepsilon t^{\frac{3}{2}}\, e^{C\,\varepsilon\,t} \right). \qquad (59)
\end{aligned}$$

We now rewrite the first term of the right hand side of (58), using formula (34) and (49):

$$\begin{aligned}
&\left\| x \mapsto f(t,x) - \frac{f(0,x)e^{-x^2 t}}{\frac{f(0,0)\,\int_I e^{-y^2}\,dy}{(1-\varepsilon)\sqrt{t}} + \mu(t)} \right\|_{L^1(I)} \\
&\quad = \left\| x \mapsto \varepsilon \int_0^t \left(\int_I \gamma(x) f(s,y)\,dy \right) e^{(1-\varepsilon-x^2)(t-s) - \int_s^t \mathcal{I}(\sigma)\,d\sigma}\,ds \right\|_{L^1(I)} \\
&\quad \leq C\varepsilon \int_I \int_0^t \mathcal{I}(s) e^{-x^2(t-s)} e^{(1-\varepsilon)t - \int_0^t \mathcal{I}(\sigma)\,d\sigma} e^{\int_0^s \mathcal{I}(\sigma)\,d\sigma - (1-\varepsilon)s}\,ds\,dx,
\end{aligned}$$

and then, thanks to (36), (56) and (57),

$$\begin{aligned}
&\left\| x \mapsto f(t,x) - \frac{f(0,x)e^{-x^2 t}}{\frac{f(0,0)\,\int_I e^{-y^2}\,dy}{(1-\varepsilon)\sqrt{t}} + \mu(t)} \right\|_{L^1(I)} \\
&\quad \leq C\,\varepsilon \int_0^4 \left(\int_I e^{-x^2(t-s)}\,dx \right) \left(\frac{\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx} + 1 + \varepsilon\,t^2\,e^{C\,\varepsilon\,t} \right) ds \\
&\qquad + C\varepsilon \int_4^t \left(\int_I e^{-x^2(t-s)}\,dx \right) \left(\frac{f(0,0)\,\int_I e^{-x^2}\,dx}{\sqrt{s}} + \frac{1}{s} + \varepsilon\,s\,e^{C\,\varepsilon\,s} \right) \\
&\qquad\qquad \left(\frac{\sqrt{t}}{f(0,0)\,\int_I e^{-x^2}\,dx} + 1 + \varepsilon\,t^2\,e^{C\,\varepsilon\,t} \right) ds \\
&\quad \leq C\,\varepsilon\,\frac{1}{\sqrt{t}} \left(\sqrt{t} + 1 + \varepsilon\,t^2\,e^{C\,\varepsilon\,t} \right) \\
&\qquad + C\varepsilon \int_4^t \frac{1}{\sqrt{t-s}} \left(\frac{1}{\sqrt{s}} + \varepsilon\,s\,e^{C\,\varepsilon\,s} \right) \left(\sqrt{t} + 1 + \varepsilon\,t^2\,e^{C\,\varepsilon\,t} \right) ds.
\end{aligned}$$

We estimate

$$\int_4^t \frac{s e^{C\varepsilon s}}{\sqrt{t-s}}\,ds \leq t e^{C\varepsilon t} \int_4^t \frac{ds}{\sqrt{t-s}} \leq C t^{\frac{3}{2}} e^{C\varepsilon t},$$

and then

$$
\begin{aligned}
&\left\| x \mapsto f(t,x) - \frac{f(0,x)e^{-x^2 t}}{\frac{f(0,0)\int_I e^{-y^2}\,dy}{(1-\varepsilon)\sqrt{t}} + \mu(t)} \right\|_{L^1(I)} \\
&\quad \le C\varepsilon \left(1 + \frac{1}{\sqrt{t}} + \varepsilon\, t^{\frac{3}{2}}\, e^{C\,\varepsilon\, t}\right) \\
&\qquad + C\,\varepsilon\,\left(1 + \varepsilon t^{\frac{3}{2}} e^{C\varepsilon t}\right)\,\left(\sqrt{t} + 1 + \varepsilon\, t^2\, e^{C\,\varepsilon\, t}\right) \\
&\quad \le C\left(\varepsilon + \varepsilon\sqrt{t} + \frac{\varepsilon}{\sqrt{t}} + \left(\varepsilon t^{\frac{3}{2}} + \varepsilon^2 t^2 + \varepsilon^2 t^{\frac{3}{2}} + \varepsilon^3 t^{\frac{7}{2}}\right) e^{C\varepsilon t}\right) \\
&\quad \le C\left(\varepsilon\sqrt{t} + \varepsilon t^{\frac{3}{2}} e^{C\varepsilon t}\right), \qquad (60)
\end{aligned}
$$

where we have used the fact that $\varepsilon t \le C e^{C\varepsilon t}$. Thanks to (59) and (60), (58) becomes:

$$
\begin{aligned}
\left\| x \mapsto f(t,x) - \frac{f(0,x)\sqrt{t}e^{-x^2 t}}{f(0,0)\int_I e^{-y^2}\,dy} \right\|_{L^1(I)} &\le C\left(\frac{1}{\sqrt{t}} + \varepsilon\,\sqrt{t} + \varepsilon\, t^{\frac{3}{2}}\, e^{C\,\varepsilon\, t}\right). \\
&\le \left(\frac{1}{\sqrt{t}} + \varepsilon\, t^{\frac{3}{2}}\right) e^{C\,\varepsilon\, t}.
\end{aligned}
$$

Theorem 1.3 follows from this estimate.

Note that as stated in the comments of the theorem, if we assume that $t \in \left[\frac{1}{\kappa^2}, \kappa^{\frac{2}{3}}\varepsilon^{-\frac{2}{3}}\right]$, then (6) becomes

$$
\left\| x \mapsto f(t,x) - \frac{f(0,x)\sqrt{t}e^{-x^2 t}}{f(0,0)\int_I e^{-y^2}\,dy} \right\|_{L^1(I)} \quad \le \quad C\left(\kappa + \kappa\, e^{C\,\kappa^{\frac{2}{3}}\varepsilon^{\frac{1}{3}}}\right),
$$

and if furthermore $\varepsilon \le \kappa \le 1$, then

$$
\left\| x \mapsto f(t,x) - \frac{f(0,x)\,\sqrt{t}\,e^{-x^2 t}}{f(0,0)\int_I e^{-y^2}\,dy} \right\|_{L^1(I)} \quad \le \quad C\,\kappa.
$$

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