Algebraic-geometric and Diophantine aspects of Liouvillian integrability in the theory of polynomial vector fields

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Abstract

The goal of this article is to determine within the theory of polynomial differential systems the role of the geometry of their invariant algebraic curves and in particular the role of their multiplicities as well as of the multiplicities of singularities of the systems located on them, in their Liouvillian integrability. By analyzing in depth the family of Liouvillian integrable systems $QSL_{\geq 4}$ of quadratic differential systems with real coefficients and with invariant lines of total multiplicity at least 4 we determine this role. We also show that along with the geometry and in particular the multiplicities of the curves, also diophantian equations play an important role. Our theorems give specific reasons for why exactly the class splits in three non-intersecting subclasses with different kinds of integrating factors, depending on how the systems can be perturbed into generic ones, i.e. systems with all their invariant curves as well as their singularities simple (of multiplicity one). Analyzing the limiting process for integrating factors of the perturbed systems as the perturbation parameter ϵ tends to zero we get to understand why we end up with three distinct types of integrating factors.

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1 Introduction

We consider here real planar differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1.1}$$

where $p,\ q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials with real coefficients. We denote by $\tilde{D}$ the vector field

$$\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y} \tag{1.2}$$

associated to (1.1).

We call degree of a system (1) (or of a vector fields $\tilde{D}$), the integer $d = \max(\deg p, \deg q)$. We call *quadratic*, a system of the above form with d = 2. The problems we are considering in this work stem from the Darboux geometric theory of integrability [4], in terms of invariant algebraic curves of the systems. Darboux' work is for complex vector fields on the complex projective plane. His theory can be applied successfully for real systems (1.1) because for every real system (1.1) we can associate its complexfication, i.e. the system with the same coefficients but considered over the complex plane.

Definition 1.1. *An algebraic invariant curve of a system* (1.1) *is a set* $\{(x, y) \in \mathbb{C}^2 | f(x, y) = 0\}$ *where* $f(x, y) \in \mathbb{C}[x, y]$*, i.e. it is a polynomial in* x, y *with complex coefficients such that there exists* $K(x, y) \in \mathbb{C}[x, y]$ *called the cofactor of the curve* $f = 0$ *satisfying the following identity in* $\mathbb{C}[x, y]$*:*

$$f_x p(x, y) + f_y q(x, y) = f(x, y)K(x, y).$$

Note that if d is the degree of the system then the degree $deg(K)$ of K is at most $d - 1$.

Although the algebraic curve over $\mathbb{C}$ is a locus of points in $\mathbb{C}^2$ we sometimes just write the algebraic curve as $f(x, y)$ or just f. Because this is the locus of the solution points of $f = 0$ in $\mathbb{C}^2$ the polynomial $f(x, y)$ needs to be considered modulo multipliciation by a non-zero complex nmber.

Every real or complex system (1.1) can be extended to the real or complex foliation with singularities over the real, respectively complex foliation with singularities over the projective plane (see fr example [8]).

During the past twenty years the theory of Darboux has been extended so as to include the concept of multiplicity [3] of such curves and of exponential factors and this research area has become very active since. In particular, the theory of Darboux has been applied to obtain the topological classifications of families of real differential systems whose complexifications possess invariant algebraic curves over $\mathbb{C}$. For example the family $QSL_{\geq 4}$ of quadratic differential systems (1.1) with invariant lines of total multiplicity $M \geq 4$ has been topologically classified by first classifying geometrically this family. By *total multiplicity of invariant lines* of a system with a finite number of invariant lines we mean the sum of the multiplicities of all its invariant lines including the line at infinity.

In analogous way the geometric and topological classification of quadratic differential systems with invariant conics is in the process of being completed.

In [6] (1993) Christopher and Kooij gave sufficient algebraic conditions for obtaining an integrating factor of a polynomial differential system (1.1) that is the inverse of a polynomial. Recall the following definition:

Definition 1.2. *A function* $R(x, y)$*,* $R \neq 0$*, defined on an open set* $U \subset \mathbb{K}^2$*, with* $K = \mathbb{R}$ *or* $K = \mathbb{C}$ *is called an integrating factor for system* (1.1) *on* U *if it satisfies* $div(Rp, Rq) = 0$ *where* div *denote the divergence of a vector field and* R *does not vanish in* U*. The condition* $div(Rp, Rq) = 0$ *amounts to the following condition for* R*:*

$$\tilde{D}(R(x, y)) = -div(R(x, y)R. \tag{1.3}$$

In this article we are first concerned with integrating factors which are inverses of polynomials. A polynomial is called *an inverse integrating factor* of a system (1.1) (we denote it by PIIF) if its inverse is an integrating factor. A theorem of Christopher and Kooij gives sufficient conditions for existence of an inverse integrating factor for a system (1.1).

The hypotheses in this theorem were formulated in algebraic terms but their content is geometrical. This result intrigued and interested us because (i) it is a geometrical result which is very general, it is for any degree n of the systems and for any degrees of the curves, (ii) and at the same time it is very restrictive as it has abundant and even some redundant hypotheses on the relative position of the invariant algebraic curves involved in the expression of the polinomial inverse integrating factor.

In the hierarchy of integrating factors expressed in terms of polynomials defining invariant algebraic curves, the polynomial integrating factors and those that are inverses of polynomials are of course the simplest.

The theorem Christopher and Kooij is stated as follows:

Theorem 1.1. *[6] Consider a polynomial system (1.1) that has k algebraic solutions $C_i(x, y) = 0$ such that*
(a) all curves $C_i = 0$ are irreducible, non-singular and have no repeated factors in their highest order terms,
(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,
(c) no two curves have a common factor in their highest order terms,
(d) the sum of the degrees of the curves is $n + 1$, where n is the degree of system.
Then system (1.1) has an integrating factor $\mu(x, y) = \dfrac{1}{C_1C_2...C_k}$.

The theorem can be rewritten in more explicit geometrical terms as follows way as follows:

Theorem 1.1* *[10] Consider a polynomial system (1.1) that has k algebraic solutions $C_i(x, y) = 0$ such that*
(a) The curves $C_i = 0$ are irreducible, non-singular and their projective completions are transversal to the line at infinity (equivalently they intersect the line at infinity in $n_i = deg(\mathcal{C}_i)$ points where $deg(\mathcal{C}_i)$ is degree of the curve $\mathcal{C}_i$).
(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,
(c) no two curves intersect at a point on the line at infinity $Z = 0$,
(d) the sum of the degrees of the curves is $n + 1$, where n is the degree of system.
Then system (1.1) *has an integrating factor $\mu(x, y) = \dfrac{1}{C_1C_2...C_k}$.*

Occasionally we shall call anyone of these two formulations the (C-K) theorem.

In the hypotheses of this result we have many restrictions imposed on the configuration of invariant curves for producing this special type of an integrating factor. Are they necessary?

We observe that we have very restrictive hypotheses in the theorem that are on how the curves intersect. The four hypothesis are sufficient for obtaining a PIIF of the specific form. They are however not also necessary. Here below are some examples that prove this.

- Example 1 [12]: $dx/dt = x^2 - 1, dy/dt = y^2 - 1$

The following lines are invariant: $C_1 = x + 1, C_2 = x - 1, C_3 = y + 1, C_4 = y - 1$ and $\dfrac{1}{C_1C_2C_3C_4}$ is an integrating factor. We observe that this case is not covered by the (C-K) theorem. Indeed, the system does not satisfy condition (c) nor (d) because for example C_1 and C_2 intersect at infinity and the sum of the degrees of the curves is not $n + 1$ which in this case is 3.

We add here some additional information on this example: the system here has have an additional invariant line i.e $x - y = 0$ having a total of six distinct invariant lines, including the line at infinity.

Since the maximum total multiplicity of invariant lines for a quadratic system is 6 [1] we conclude that each line is of multiplicity 1 in this example. Clearly all singular points finite and infinite are also simple. So within the systems with six distinct invariant lines this system is generic in the sense that none of the singular points and none of the lines are of multiplicity $m > 1$.

- Example 2 [12]: $dx/dt = x^2, dy/dt = y^2$

The following lines are invariant: $C_1 = x$, $C_2 = y$, (C_1, C_2 are double lines), $C_3 = x - y$, $\dfrac{1}{C_1C_2C_3}$ is an integrating factor and the origin is a common point of the three lines and thus the first part of the condition (b) is not satisfied. Here we also have two other PIIFs which are x^2y^2 and $(x-y)^2$ (see the calculations in Section 3 for Systems 6.5.)

Observation 1.1. *In the above two Examples we see that we obtain polynomial inverse integrating factors which are the products of invariant lines but none of them are covered by the (C-K) theorem, the systems not satisfying all the hypotheses in the theorem.*

- Example 3 [14]: $dx/dt = gx^2 + (h-1)xy, dy/dt = (g-1)xy + hy^2$, $gh(g+h-1) \neq 0$.

This is a 2-parameter family of systems having the following invariant simple lines: $C_1 = x, C_2 = x - y, C_3 = y$ and $R(x, y) = \frac{1}{x(x-y)y}$ is the unique integrating factor that exists for all values of the two parameters. The systems do not satisfy conditions (b) as the three affine lines intersect at the origin. We point out that here unlike in the Example 2 the multiplicity of each line is one. However the origin is a singular point of the systems and it is of multiplicity 4.

Observation 1.2. *These three examples show that the restrictions in the hypotheses of the Theorem 1.1 on how the curves intersect in the finite plane or at infinity are not necessary for existence of a PIIF and even for a PIIF of the form given in (d). The preceding examples raise right away the following questions:*

Question 1. *How do we generalize the (C-K) theorem so as to also cover the above examples?.*

Question 2. *We observe that in the Example 2 two of the lines are double and we still have an inverse integrating factor and it is even of the sane type as the one in the (C-K) theorem, i.e it is the product of the three lines. Does multiplicity of invariant lines have a role in the production of inverse integrating factors and more generally in integrating factors and if it does then what is this role?*

Question 3. *We observe that the examples 2 and 3 are distinguished by the fact that while in Example 2 the lines x, y are double and the origin is simple, in the Example 3 the lines are simple and the origin is a multiple singular point of the systems of multiplicity 3. Do multiplicities of singularities of a system* (1.1) *located on the invariant curves also play a role in the production of integrating factors and if they do what is this role?*

In our next example we also see that the restriction to curves that are not tangent in the finite plane (formulated in the second part of (b) is also not necessary.

- Example 4: $\dot{x} = -y - x^2 - y^2, \dot{y} = x + xy$

This system has the invariant line $L : y + 1 = 0$ and the invariant ellipse $C : 6x^2 + 3y^2 + 2y - 1 = 0$. The line L is tangent to the ellipse at the point $(0, -1)$. So clearly the curves L and C do not satisfy the second part of the hypotheses (b) but thet satisfy all the other conditions in the theorem. However the product of the curves, i.e. $(y + 1)(6x^2 + 3y^2 + 2y - 1)$ is a PIIF for the system.

Question 4. *While the (C-K) theorem give us sufficient but not also necessary conditions for existence of a PIIF for a system* (1.1) *how do we characterize the systems that possess PIIFs by giving necessary and sufficient conditions for their existence? In other words how could we generalize Theorem 1.1 (Theorem 1.1*) by finding better hypotheses for obtaining a polynomial inverse integrating factor?*

Observation 1.3. *There is a striking difference in the forms of the PIIFs in these last two examples: In both examples 2 and 3 we have the same three invariant lines $x, y, x-y$ intersecting at the origin. But while in Example 3 the PIIF is $x(x-y)y$ and it is the unique PIIF that we have for all values of the parameters, in Example 2 we have besides $x(x-y)y$ two other PIIFs where the curves appear raised to the power 2, x^2y^2 and $(x-y)^2$. Is there a geometrical or Diophantian reason that explains this difference?*

Finding necessary and sufficient conditions for existence of PIIFs or for solving the problem of Poincaré [9] are very difficult questions because they are global problems. Indeed we are asked to determine out of all possible ordered couples $(S, \{C_i, i \leq k\})$ formed by a system (S) of the form (1.1) of any degrees n endowed with a finite number of invariant algebraic curves $C_i, i = 1, 2, .., k$, of *any degrees* n_i, what are the ordered couples $(S, \{C_i, i \leq k\})$ such that (S) has a PIIF expressed in terms of the curves C_i's? Poincaré's problem is global for similar reasons.

In his address at the ICM in Paris in 1900 Hilbert said:

"*In dealing with mathematical problems specialization plays, as I believe, a still more important role than generalization. Perhaps in most cases where we seek in vain the answer to a question, the cause of the failure lies in the fact that problems simpler and easier than the one in hand have been either not at all or incompletely solved.*"

A more recent similar remark made by Michel Talagrand is cited in News from the AMS in the announcement of Talagrand's Abel Prize:

"It helps to be humble and to start by understanding fully the simple situations."

We consider here only the particular case of quadratic systems, i.e. where in (1.1) we have $\max\{\deg(p), \deg(q)\} = 2$. We first try to solve this problem in the simplest of cases where the curves are just lines and the integrating factor will use only lines. For a non-degenerate quadratic system the total multiplicity of invariant lines, including the line at infinity is at most six.

In [11] the global geometric classification of quadratic systems with real coefficients and invariant lines of total multiplicity (see Section 2) greater than or equal to 5 was given in terms of the configurations of their invariant lines. The concept of *configuration of invariant lines* of a polynomial differential system was defined in [11].

Definition 1.3. *We call configuration of invariant lines of a real polynomial differential system* (1.1) *a finite set $\{f_i(x, y)\}$, $i = 1, 2, \ldots k$, $f_i(x, y) \in \mathbb{C}[x, y]$ of invariant lines of the complexification of the system equipped with their multiplicities and with all real singularities situated on the lines, each one equipped with its own multiplicity.*

Observation 1.4. *We observe that this is a mixed concept: on one side we have the algebraic geometric notion of algebraic curve and on the other side we also have dynamical notions such as the singularities of the differential systems located on algebraic curves that are invariant for the* *systems. The curves as well as the singularities of the systems located on them could be multiple and multiplicities of both the curves and of singularities located on them involve perturbations of the systems. For all these reasons we expect that the concept of configuration of invariant lines must be at the core of Liouvillian integrability.*

In our articles [11, 12] and [13, 14] our first goal was to obtain the topological classification of real quadratic systems with invariant lines of at least four total multiplicity, in other words to obtain the phase portraits of this class. Since only the real singularities appear on these phase portraits, we defined the concept of configuration of invariant lines in Definition 1.3 so as to include only the real singularities located on these lines. In fact when working on Liouvillian integrability the concept of configuration should include all singularities real or complex, finite and infinite located on the invariant lines or more generally invariant curves.

Definition 1.4. *We call configuration of invariant lines of a polynomial differential system* (1.1) *with coefficients in* $\mathbb{C}$ *a finite set* $\{f_i(x,y)\}$, $i=1,2,\ldots k$, $f_i(x,y)\in\mathbb{C}[x,y]]$ *of invariant lines of the system equipped with their multiplicities and with all their singularities situated on the curves, each one equipped with its own multiplicity.*

This concept generalizes of course in obvious way to the concept of *configuration of invariant algebraic curves* of a polynomial differential system over $\mathbb{C}$.

The idea for obtaining the phase portraits of $QSL_{\geq 4}$ was to first study the geometry of this family unsing the notion of configuration on invariant lines. This split the family QSL_4 into a number of much smaller families of systems, each one of them possessing the same configuration of invariant lines. It was then easier to handle the topological classification for each one of these much smaller subfamilies [11, 12]. As a biproduct of the geometrical classification the authors also obtained the Liouvillian integrability of the systems in this family. But the problem of why certain systems possess a polynomial inverse integrating factor or more general types of integrating factors was not considered in [11, 12].

Our initial goal was to find amswers to the questions raised above. The papers [11, 12] and also [14] provide us with a rich class of examples to explore in depth. In searching for answers to the questions raised before we encounter *arithmetic constraints* and we wanted to understand how much these constraints weigh versus the geometry of the configurations of lines in the production of polynomial inverse integrating factors.

Furthermore, we wanted to fully understand. Liovillian integrability as it splits according to various types of integrating factors in terms of on one side the algebro-geometric and dynamic mixed concept of configuration of invariant lines and on the other side of diophantine equations.

Definition 1.5. *We denote by* QSL_i *(respectively* $QSL_{\geq i}$*) the family of quadratic systems with real coefficients and invariant lines with total multiplicity* i *(respectively* $\geq i$*).*

The paper is organized as follows: section 2 includes basic concepts and known results on polynomial vector fields necessary for the next sections. In sections 3 and 4 we explore in depth the families QSL_i for $i\in\{6,5,4\}$. Our main results are stated in theorems 3.1, 3.2 and 4.i with $i\leq 4$.

2 Basics concepts and results

We consider here planar differential systems of the form (1.1) where p, q are polynomials in x, y with real coefficients. In order to make this work as self-contained as possible we give below some basic definitions and results necessary for this work.

Definition 2.1. *(Darboux) [4]) An algebraic solution of an equation* (1.1) *is an algebraic invariant curve $C(x,y) = 0$ (considered over $\mathbb{C}^2$),* deg $C \geq 1$*, with C an irreducible polynomial over $\mathbb{C}$.*

Definition 2.2. *A point $(x_0, y_0) \in \mathbb{C}^2$ is a singular point (or a singularity) of an algebraic curve $C(x,y) = 0$ if $C \in \mathbb{C}[x,y]$ and all its first partial derivatives vanish at (x_0, y_0). An algebraic curve is non-singular if it has no singularities.*

Definition 2.3. *Let U be an open subset of $\mathbb{K}^2$ with $K = \mathbb{R}$ or $K = \mathbb{C}$. A function $H : U \to K$ is a first integral of system* (1.1) *if it is constant on all solution curves $(x(t), y(t))$ of system* (1.1)*, i.e $H(x(t), y(t)) = C$, where $C \in \mathbb{K}$ is a constant, for all values of t for which the solution $(x(t), y(t))$ is defined on U.*
Note that if H is differentiable in U then H is a first integral on U if and only if $H_x P + H_y Q = 0$.

Definition 2.4. *If a system* (1.1) *has a first integral of the form $H(x,y) = f_1^{\lambda_1} ... f_p^{\lambda_p}$ where f_i are the invariant algebraic curves of system* (1.1) *and $\lambda_i \in \mathbb{C}$, then we say that system* (1.1) *is Darboux integrable and we call the function H a Darboux function.*

The problem of integrating a polynomial system by using its algebraic invariant curves over $\mathbb{C}$ was considered for the first time by Darboux in [4].

Theorem 2.1. *(Darboux[4]) (1978). Suppose that a polynomial system* (1.1) *over $\mathbb{C}$ has m invariant algebraic curves $f_i(x,y) = 0,\ i \leq m$, with $f_i \in \mathbb{C}[x,y]$ and with $m > n(n+1)/2$ where n is the degree of the system. Then we can find complex numbers $\lambda_1, ..., \lambda_m$ such that $f_1^{\lambda_1} ... f_m^{\lambda_m}$ is a first integral of the system.*

The proof of this theorem is immediate because having a first integral of this form means that $\tilde{D}(f_1^{\lambda_1} ... f_m^{\lambda_m}) = 0$ and this is equivalent to solving the equation $\sum \lambda_i K_i = 0$ for λ_i where the K_i's are the cofactors of the curves $f_i = 0$. Since $\deg(K_i) \leq n-1$ then the K_i's have $\dfrac{n(n+1)}{2}$ coefficients each and hence the equation $\sum \lambda_i K_i = 0$ is equivalent to solving a system of $\dfrac{n(n+1)}{2}$ linear homogeneous equations in m unknowns λ_i's. Since $m \geq \dfrac{n(n+1)}{2} + 1$ this system is solvable in λ_i's over the complex numbers.

Observation 2.1. *We may encounter the case when an equation of the form $\sum \lambda_i K_i = 0$ has solutions in λ_i for fewer than $\dfrac{n(n+1)}{2}$ curves and since such an equation is equivalent to having a first integral of the form $f_1^{\lambda_1} ... f_m^{\lambda_m}$ the condition in the theorem of Darboux is only sufficient but also not necessary as the following example shows.*

Example: [10] Consider the system: $\begin{cases} \dot{x} = & 2x^2 + xy + 3 \\ \dot{y} = & 2y^2 + xy + 3 \end{cases}$

This system admits the invariant line $x - y$ and the invariant hyperbola $2 + xy$. Then $m = 2 < \dfrac{n(n+1)}{2} = 3$. However we still have here a Darboux first integral $H(x,y) = (x-y)^{\frac{-3}{2}}(2 + xy)$.

Poincaré was enthusiastic about the work of Darboux [4]. He wrote two papers inspired by this work and also proposed a problem now called the problem of Poincaré. He also proposed this topic for a prize. But except for Painlevé who won the prize and for Autonne who won an honorary mention, for a very long time the theory of Darboux attracted only very few people. The first significant result after Darboux' work is Jouanolou's theorem published 101 years later.

Theorem 2.2. *(Jouanolou (1979) [5]) Suppose that a polynomial system (1) has m invariant algebraic curves $f_i(x,y)=0,\ i\le m$, with $f_i\in\mathbb{C}[x,y]$ and with $m>n(n+1)/2+1$ where n is the degree of the system. Then the system has a first integral that is rational i.e. of the form $f(x,y)/g(x,y)$ where f,g are polynomials in x,y with complex coefficients.*

During the last decade of the XX-th century several papers appeared on the theory of Darboux [4] and additional concepts such as the notion of *exponential factor* was introduced. The area became very active in the first decades of the XXI-st century after the introduction of the concept of multiplicity of an invariant algebraic curve (see [3]) and work on families with multiple invariant curves appeared, for example see [11, 12] and [13, 14]. Adding these two concepts produced an extension of the theory of Darboux which is much more complete and powerful.

Definition 2.5. *Let $F(x,y)=e^{(G(x,y)/H(x,y))}$ with $G,H\in C[x,y]$ without non-constant common factors. We say that F is an exponential factor of system* (1.1) *if it satisfies the equality $F_xP+F_yQ=FL$ for some $L\in C[x,y]$ and F_x,F_y are the partial derivatives of F. The polynomial L is called the cofactor of the exponential factor F.*

We remark that an exponential factor could be a non-zero constant, with a zero cofactor.

Definition 2.6. *We say that a system* (1.1) *has a generalized Darboux first integral (respectively generalized Darboux integrating factor) if it admits a first integral (respectively integrating factor) of the form:*
$e^{G(x,y)}\prod_{i=1}^{s}C_i(x,y)^{\lambda_i}$, *where $G(x,y)\in\mathbb{C}(x,y)$ (G is a rational function over $\mathbb{C}$) and $C_i\in\mathbb{C}[x,y]$,* $\deg(\mathbb{C}_i)\ge 1,\ i=1,2,...,s,\ C_i$ *irreducible over $\mathbb{C}$ and $\lambda_i\in\mathbb{C}$.*

Remark 2.2. *In the current litterature in some papers a Darboux function is an expression of the form $e^{G(x,y)}\prod_{i=1}^{s}C_i(x,y)^{\lambda_i}$. We are interested here in the phenomena that produce these two kinds of integrating factors, i.e. the above one and $\prod_{i=1}^{s}C_i(x,y)^{\lambda_i}$ and for this reason we keep the distinction in their names.*

Theorem 2.3. *[2, 16] A planar polynomial differential system* (1.1) *has a Liouvillian first integral if and only if it has a generalized Darboux integrating factor.*

If a system (1) has a rational first integral $H(x,y)=\dfrac{f(x,y)}{g(x,y)}$ with $f,g\in\mathbb{C}[x,y]$, then the solution curves being located on its level curves $H(x,y)=C$ where C is a constant are then algebraic curves $f(x,y)-Cg(x,y)=0$. We call degree of the first integral H the number $\max(\deg(f),\deg(g))$. Then all the algebraic invariant curves of the system have a degree bounded by the degree of H.

Proposition 2.1. *[4] If a system* (1.1) *has an integrating factor (or first integral) of the form $F=\prod_{i=1}^{s}C_i(x,y)^{\lambda_i}$ with $C_i\in\mathbb{C}[x,y]$ then for all $i\in\{1,...,s\}$, $C_i=0$ is an algebraic invariant curve of (1).*

Definition 2.7. *[10] (1) We say that an invariant curve* $\mathcal{L} : f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ *for a system* (1.1) (S_0) *of degree* n *has multiplicity* m *if there exists a sequence of systems* (1.1) (S_k) *of degree* n *converging to* (S_0) *in the topology of* $S^{N-1}(\mathbb{R})$ *where* $N = (n+1)(n+2)$ *such that each* (S_k) *has* m *distinct invariant curves* $L_{1,k} : f_{1,k}(x, y) = 0, ..., L_{m,k} : f_{m,k}(x, y) = 0$ *over* $\mathbb{C}$, $\deg(f) = \deg(f_{i,k}) = r$, *converging to* $\mathcal{L}$ *as* $k \to \infty$, *in the topology of* $P_{R-1}(\mathbb{C})$, *with* $R = \dfrac{(r+1)(r+2)}{2}$ *and this does not occur for* $m + 1$.
(2) Consider a system (1.1) (S_0) *of degree* n. *We say that the line at infinity* $L_\infty : Z = 0$ *has multiplicity* m *if there exists a sequence of systems* (1.1) (S_k) *of degree* n *converging to* (S_0) *in the topology of* $S^{N-1}(\mathbb{R})$ *where* $N = (n+1)(n+2)$ *such that each* (S_k) *has* $m-1$ *distinct invariant lines* $L_{1,k} : f_{1,k}(x, y) = 0, ..., L_{m-1,k} : f_{m-1,k}(x, y) = 0$ *over* $\mathbb{C}$, *converging to the line at infinity* L_∞ *as* $k \to \infty$, *in the topology of* $S^{m-1}(\mathbb{R})$ *and this does not occur for* m.

This definition of multiplicity is similar with the definition of *strong geometric multiplicity* given in [3]. However the two definitions coincide in the case of invariant lines if we allow the curves $f_{i,k}(x, y)$ to belong to systems of bounded degrees instead of just quadratic.

Definition 2.8. *Suppose we have a polynomial differential system with a configuration of affine invariant curves* $f_1, \cdots, f_k$ *with multiplicities* m_i *and the line* $Z = 0$ *with multiplicity* m. *We call total curve of this configuration the curve* $F_1^{m_1}(x, y, z) \cdots F_k^{m_k}(x, y, z)Z^m = 0$, *where* F_i *is the homogeneization of* f_i.

Definition 2.9. *Let* S_0 *be a quadratic system. An unfolding of* S_0 *within the quadratic family of systems* (1.1) *is a family of quadratic systems* S_ε *depending continuously on a parameter* ε *such that when* ε *tends to zero then* S_ε *tends to* S_0 *in the topology of the sphere* $S^{11}(\mathbb{R})$ *where* n *is the degree of* S_0. *(Here* ε *is a real number).*

3 Polynomial inverse integrating factors in terms of lines for the family of systems $QSL_{\geq 5}$

The family $QSL_{\geq 5}$ was studied in [11, 12]. The main goal of these articles was to obtain the topological classification of the systems in this family. We recall below the notion of topological equivalence for real differential systems (1.1) not necessarily polynomial.

Definition 3.1. *Two differential systems* (1.1) S_1 *and* S_2 *are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of* S_1 *to the oriented phase curves of* S_2 *and preserving the orientation.*

In order to classify topologically the systems the authors first classified them geometrically by using the geometry of their invariant lines encoded in the concept of *configuration* of invariant lines of the systems [13].

Definition 3.2. *Consider apolynomial system* (1.1) *endowed with a finite number of algebraic invariant lines. By configuration of invariant lines of the system we mean a finite set of lines over* $\mathbb{C}$ *of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these curves, each one of these singularities endowed with its own multiplicity.*

In studying the integrability of systems in terms of algebraic invariant curves occasionally we will use fewer curves than the ones present in a given configuration and for this reason we give the following definition:

Definition 3.3. *We call sub-configuration of a given configuration $\mathcal{C}$ of invariant curves of a system* (1.1) *a configuration with all its invariant curves forming a subset of those of $\mathcal{C}$ endowed with the same multiplicities and the same real singularities with the same multiplicities as those in $\mathcal{C}$.*

The authors first classified geometrically the family $QSL_{\geq 5}$ in [11] i.e. with respect to their configurations of invariant lines. Then they used this classification for obtaining the finer topological classification of this family resulting in a total of 42 phase portraits [12].

Once having the configurations of invariant algebraic curves it was natural to apply the theory of Darboux and prove the Liouvillian integrability of this family by showing first that all the systems in this family have a PIIF and then by providing the generalized Darboux first integrals. The integrability result was simply a biproduct of the knowledge they obtained from the configurations of lines and the authors of [11, 12] did not go any further in exploring anyone of the questions raised in our Introduction. One of the goals of this article is to go deeper in this study so as to provide here answers for them.

3.1 The systems in QSL_6

The family QSL_6 has a total of 11 distinct configurations of invariant lines each one leading to a unique phase portrait. In the generic case of this family we have 5 distinct affine invariant lines so clearly by Jouanolou's theorem the system has a rational first integral. This also happens in the non-generic cases where we have at least one multiple invariant line of multiplicity greater than one.

With the purpose of bringing light to the open questions raised in the Introduction we first perform a full analysis of all the polynomial inverse integrating factors of the systems in the family $\mathbf{QSL_6}$. In Table 1 of [12] (see Appendix, Table I) only one PIIF was shown for each generic system in QSL_6. In fact we show that in the *generic* cases these systems actually have exactly 6 distinct PIIF's. This is due to the geometry of the systems which have six distinct invariant lines allowing for 6 distinct combinations of them in forming PIIF's.

Definition 3.4. *In the family of polynomial differential systems over $\mathbb{C}$ with invariant algebraic curves $f(x,y)=0$, $f(x,y)\in\mathbb{C}[x,y]$ we call generic a system whose invariant algebraic curves defined by irreducible polynomials $f(x,y)\in\mathbb{C}[x,y]$ are all simple and all the singularities of the system are also simple.*

In the cases where we have a multiple line of a system (1.1) in a family QSL_i we not so surprisingly have fewer PIIFs than in the generic case, as we have fewer distinct lines and hence fewer possibility to combine them for producing a PIIF. A more mathematical argument will be more convincing for fully explain this phenomenon. We first introduce the following:

Definition 3.5. *Let $\Omega\subset\mathbb{R}^2$ be an open subst and $(f_\varepsilon)_\varepsilon$ a family of C^1 functions on Ω. We say that $f_\varepsilon \longrightarrow f_0 \quad$ in $C^1_{\text{loc}}(\Omega)$ if, for every compact set $K\subset\Omega$, $f_\varepsilon\to f_0, \quad \partial_x f_\varepsilon\to\partial_x f_0, \quad \partial_y f_\varepsilon\to\partial_y f_0$ uniformly on K as $\varepsilon\to 0$.*

Proposition 3.1. *Consider the polynomial system* $(S_0): \dot{x} = P(x,y), \dot{y} = Q(x,y)$, *and a continuous polynomial perturbation* $(S_\varepsilon): \dot{x} = P_\varepsilon(x,y), \dot{y} = Q_\varepsilon(x,y)$, *where* $P_\varepsilon, Q_\varepsilon$ *are polynomials in* (x,y) *whose coefficients depend continuously on* ε, *and* $P_\varepsilon \to P,\ Q_\varepsilon \to Q \quad$ *as* $\varepsilon \to 0$.

Let $\Omega \subset \mathbb{R}^2$ *be an open set. Assume:*

1. *For each* ε, *there exists an integrating factor* $R_\varepsilon \in C^1(\Omega)$ *for* (S_ε), *i.e.*

$$D_\varepsilon(R_\varepsilon) = -R_\varepsilon \operatorname{div}(P_\varepsilon, Q_\varepsilon) \text{ on } \Omega, \text{ where } D_\varepsilon = P_\varepsilon \partial_x + Q_\varepsilon \partial_y.$$

2. $R_\varepsilon \to R_0$ *in* $C^1_{\text{loc}}(\Omega)$ *as* $\varepsilon \to 0$, *and* $R_0 \not\equiv 0$.

Then R_0 *is an integrating factor for* (S_0) *on* Ω, *that is,* $D(R_0) = -R_0 \operatorname{div}(P,Q)$ *on* Ω, *where* $D = P\partial_x + Q\partial_y$

Proof: For each ε, we have on Ω

$$P_\varepsilon\, \partial_x R_\varepsilon + Q_\varepsilon\, \partial_y R_\varepsilon = -R_\varepsilon \operatorname{div}(P_\varepsilon, Q_\varepsilon). \tag{3.1}$$

Write $P_\varepsilon(x,y) = \sum_{i,j} a_{ij}(\varepsilon) x^i y^j, Q_\varepsilon(x,y) = \sum_{i,j} b_{ij}(\varepsilon) x^i y^j$, where the coefficients a_{ij}, b_{ij} are continuous in ε. Then $\partial_x P_\varepsilon(x,y) = \sum_{i,j} i\, a_{ij}(\varepsilon) x^{i-1} y^j, \quad \partial_y Q_\varepsilon(x,y) = \sum_{i,j} j\, b_{ij}(\varepsilon) x^i y^{j-1}$.

Let $K \subset \Omega$ be compact. There exists $R > 0$ such that $|x| \leq R$, $|y| \leq R$ on K. Hence $\sup_K |\partial_x P_\varepsilon - \partial_x P| \leq \sum_{i,j} i\, |a_{ij}(\varepsilon) - a_{ij}(0)|\, R^{i+j-1} \xrightarrow[\varepsilon\to 0]{} 0$, then $\partial_x Q_\varepsilon \to \partial_x Q$ and similarly $\partial_y Q_\varepsilon \to \partial_y Q$ uniformly on K. Therefore

$\operatorname{div}(P_\varepsilon, Q_\varepsilon) = \partial_x P_\varepsilon + \partial_y Q_\varepsilon \longrightarrow \partial_x P + \partial_y Q = \operatorname{div}(P,Q)$ uniformly on every compact subset of Ω

On the other hand, $R_\varepsilon \to R_0$ in $C^1_{\text{loc}}(\Omega)$ implies that, for every compact $K \subset \Omega$,

$$R_\varepsilon \to R_0, \quad \partial_x R_\varepsilon \to \partial_x R_0, \quad \partial_y R_\varepsilon \to \partial_y R_0 \text{ uniformly on } K.$$

Consequently, on any compact $K \subset \Omega$,

$$P_\varepsilon\, \partial_x R_\varepsilon \to P\, \partial_x R_0, \qquad Q_\varepsilon\, \partial_y R_\varepsilon \to Q\, \partial_y R_0, \text{ and } R_\varepsilon \operatorname{div}(P_\varepsilon, Q_\varepsilon) \to R_0 \operatorname{div}(P,Q) \text{ uniformly on } K.$$

Passing to the limit in (3.1), we obtain on K $P\, \partial_x R_0 + Q\, \partial_y R_0 = -R_0 \operatorname{div}(P,Q) \quad$ on K.

Let $z_0 \in \Omega$. Since Ω is open, there exists a compact set $K \subset \Omega$ whose interior contains z_0. The above equality holds on K, hence in particular at z_0. As z_0 is arbitrary in Ω, we conclude that

$$P\, \partial_x R_0 + Q\, \partial_y R_0 = -R_0 \operatorname{div}(P,Q) \quad \text{on } \Omega, \text{that is,} D(R_0) = -R_0 \operatorname{div}(P,Q) \quad \text{on } \Omega.$$

Thus R_0 is an integrating factor of (S_0) on Ω.

∎

In the proof of the following proposition there intervene, apart from elements of algebraic-geometric nature, also elements of arithmetic nature. To stress this significant aspect we write the first part of the proof in great detail underscoring this fact.

Theorem 3.1. *We consider here the systems in the family* $\mathbf{QSL}_6$*.*
(i) There are 4 generic systems in this family and each one of them has exactly six polynomial inverse integrating factors given in the table below for the system 6.1 in [12] (our Example 1 in the Introduction):

System 6.1	*Invariant lines and their multiplicities* / *Respective cofactors*	*All Inverse integrating factor* R_i *which splits into linear factors*
$\begin{cases}\dot{x} = x^2 - 1 \\ \dot{y} = y^2 - 1\end{cases}$	$x \pm 1(1),\ y \pm 1(1),\ x - y(1)$ $x \mp 1, y \mp 1, x + y$	$R_1 = (x-y)^2$ $R_2 = (x-1)(y+1)(x-y)$ $R_3 = (x+1)(y-1)(x-y)$ $R_4 = (x-1)^2(y+1)^2$ $R_5 = (x+1)^2(y-1)^2$ $R_6 = (x+1)(x-1)(y+1)(y-1)$

The remaining generic cases 6.2, 6.3, 6.4 are completely analogous to 6.1 with the only difference that some of the lines and of the singular points are complex.
(ii) If a system in this family has an invariant line of multiplicity two (respectively 3) then the system has exactly 3 PIIFs (respectively exactly one).
(iii$_1$*) Every system* (S_0) *in this family possessing a multiple invariant line (i.e. is of the form 6.5,... 6.11) admits a polynomial perturbation into generic systems* $(S(\epsilon))$ *having 6 PIIFs for every sufficiently small* ϵ*.*
(iii$_2$*) Let* R_0 *be a PIIF of a system which has at least one multiple invariant line* L_0 *of multiplicity* $m \geq 1$*. Let* $(S(\epsilon))$ *be the perturbation of* (S_0) *in (iii*$_1$*) and suppose that* R_ϵ *is a PIIF of* $(S(\epsilon))$ *such that* $R_0 = lim_{\epsilon \to 0} R_\epsilon$*. Then* $d \leq d_\epsilon$ *where* d_ϵ *(respectively d) is the number of distinct invariant lines occurring in* R_ϵ *(respectivelyin* R_0*). Assuming that* L_0 *occurs in* R_0 *with the exponent* k *then* $k \leq m$*.*

Proof: i) Every system in QSL_6 is equivalent via the group action to one of the eleven systems indicated in Table 1 in [12] (our Table I in Appendix). For the first and second examples of the Introduction, both in QSL_6, we do the calculations in great detail in order to highlight the Diophantine as well as algebraic-geometric aspects encountered.
Let $\tilde{D}$ be the corresponding vector field of a system. Consider :
- System 6.1 : $\dot{x} = x^2 - 1,\ \dot{y} = y^2 - 1$
Here we have 6 invariant lines: $L_{1,2} = x \pm 1$, $L_{3,4} = y \pm 1$ and $L_6 = x - y$ with cofactors K_i. Suppose that $R = L_1^{\lambda_1} \dots L_6^{\lambda_6}$ is a Darboux integrating factor i, i.e.

$$\tilde{D}(R) = \operatorname{div}(P, Q)R \tag{3.2}$$

Direct calculations and using the fact that the lines are invariant then give

$$\lambda_1 K_1 R + \lambda_2 K_2 R + \lambda_3 K_3 R + \lambda_4 K_4 R + \lambda_5 K_5 R = (2x + 2y)R \tag{3.3}$$

and hence finding an integrating factors is equivalent to solving the equation in λ_i:

$$\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 + \lambda_4 K_4 + \lambda_5 K_5 = (2x + 2y), i.e. \tag{3.4}$$

$$\lambda_1(x-1) + \lambda_2(x+1) + \lambda_3(y-1) + \lambda_4(y+1) + \lambda_5(x+y) = 2x + 2y \tag{3.5}$$

and identifying coefficients we obtain:

$$\lambda_1 + \lambda_2 + \lambda_5 = 2; \lambda_3 + \lambda_4 + \lambda_5 = 2; \ -\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 = 0. \tag{3.6}$$

This is a linear non-homogeneous system of 3 equations in five unknowns with integer coefficients so the solutions will be in $\mathbb{Q}$ and using linear algebra we obtain the general solution:

$$\lambda_3 = \lambda_2; \lambda_4 = \lambda_1; \lambda_5 = 2 - (\lambda_1 + \lambda_2).$$

corresponding to arbitrary values of $\lambda_1, \lambda_2 \in \mathbb{Q}$. But we are interested in obtaining PIIF's hence we have to ask that all λ_i's be non-negative integers and in particular in the Diophantine equation

$$\lambda_5 = 2 - (\lambda_1 + \lambda_2)$$

Since the solutions should be in non-negative integers we could only have the following 6 solutions for (λ_1, λ_2):

$$(0,0), (0,1), (1,0), (2,0), (0,2), (1,1)$$

leading to the 6 PIIFs given in the Table above.

The other three generic cases 6.2, 6.3 and 6.4 are done in a completely analogous manner with the distinction that some invariant lines could be complex.

(ii) Consider now the non-generic cases starting with:

- System 6.5: $\dot{x} = x^2, \ \ \dot{y} = y^2$

The invariant lines with their associated multiplicities are: $x(2), y(2), x - y(1)$ and their respective cofactors are: $x, y, x + y$. The divergence of the system is $2x + 2y$. We search for a PIIF of the form $x^{\lambda_1} y^{\lambda_2} (x - y)^{\lambda_3}$. Hence the λ_i's must satisfy the equation:

$$\lambda_1 x + \lambda_2 y + \lambda_3 (x - y) = 2x + 2y$$

and identifying coefficients we obtain:

$$\lambda_1 + \lambda_3 = 2, \lambda_2 + \lambda_3 = 2 \tag{3.7}$$

and hence we have the Diophantine equations:

$$\lambda_1 = \lambda_2 = 2 - \lambda_3 \tag{3.8}$$

where λ_i's must be non-negative integers, in particular we have $0 \leq \lambda_3 \leq 2$ and therefore $\lambda_3 \in \{0, 1, 2\}$ leading to just three PIIF's: $(x-y)^2, xy(x-y), x^2y^2$. We thus proved (ii).

(iii_j) We consider the following perturbation of this system:

$$(S_\varepsilon) : \dot{x} = x^2 - \varepsilon^2, \ \ \dot{y} = y^2 - \varepsilon^2$$

This quadratic perturbation of our system has the following affine invariant lines: $l_1^\varepsilon = x + \varepsilon, \ l_2^\varepsilon = x - \varepsilon, \ l_3^\varepsilon = y + \varepsilon, \ l_4^\varepsilon = y - \varepsilon$ and $l_5 = x - y$. The invariant lines x and y of the non-generic system (6.5) are both of multiplicity 2. Each double line produces in the perturbation (S_ε) two distinct lines: x produces the lines $l_1^\varepsilon = x + \varepsilon, \ l_2^\varepsilon = x - \varepsilon$ both of which tending to x and similarly the line y produces the lines $l_3^\varepsilon = y + \varepsilon, \ l_4^\varepsilon = y - \varepsilon$ tending to y. So it is clear that the PIIF is $R_6(\epsilon) = l_1^\epsilon l_2^\epsilon l_3^\epsilon l_4^\epsilon$

and that it tends to x^2y^2. While calculations gave us this PIIF before, here we have the reason why the calculations led to this PIIF, namely the multiplicity 2 of the affine lines x,y and we see that this exponent of the line in the PIIF is exactly the number of lines that coalesce in the limit for anyone of the PIIFs of the perturbation so this exponent k is less than or equal to the multiplicity of the line and in this particular case they are equal.

For all the remaining non-generic systems possessing invariant lines of multiplicity two, i.e. from 6.6 to 6.9 we have similar proofs.

- System 6.10: $\dot{x} = x^2, \ \ \dot{y} = 1$
In this case we only have exactly two invariant lines both triple: x and the line at infinity.

$$(S_\varepsilon) : \dot{x} = (1-\varepsilon)^2x^2 - \varepsilon^2, \dot{y} = (2\varepsilon^2y + 1)(2\varepsilon y + 1)$$

is a quadratic polynomial perturbation of the system 6.10 and it has the following invariant lines: $l_1^\varepsilon = (1-\varepsilon)x + \varepsilon,\ l_2^\varepsilon = (1-\varepsilon)x - \varepsilon,\ l_3^\varepsilon = 2\varepsilon y + 1,\ l_4^\varepsilon = 2\varepsilon^2 y + 1$ and $l_5 = (\varepsilon-1)^2x - 4\varepsilon^3y - \varepsilon(\varepsilon+1)$ and the line at infinity. Hence for all values of ϵ it belongs to the family $\mathbf{QSL}_6$. The three lines l_1^ε, l_2^ε and l_5^ε all tend to the triple line x while l_3^ε and l_4^ε with the projective completions $L_3^\epsilon = 2\epsilon Y + Z$ and $L_4^\epsilon = 2\epsilon^2 Y + Z$ tend to the line at infinity $Z = 0$. The perturbation (S_ε) has an equivalent configuration to the one of the generic system (6.1). For $\epsilon \neq 0$ the systems (S_ε) have six PIIF's: $R_1^\varepsilon = (l_5^\varepsilon)^2,\ R_2^\varepsilon = l_2^\varepsilon l_3^\varepsilon l_5^\varepsilon,\ R_3^\varepsilon = l_1^\varepsilon l_4^\varepsilon l_5^\varepsilon,\ R_4^\varepsilon = (l_2^\varepsilon)^2(l_3^\varepsilon)^2,\ R_5^\varepsilon = (l_1^\varepsilon)^2(l_4^\varepsilon)^2$ and $R_6^\varepsilon = l_1^\varepsilon l_2^\varepsilon l_3^\varepsilon l_4^\varepsilon$.
$\lim\limits_{\varepsilon\to 0}\{R_i^\varepsilon\} = \{x^2\}$ for $i = 1, 2, \dots, 6$. Using a previous Proposition we obtain that $R := x^2$ is a polynomial inverse integrating factors for the system (6.10) which could also be calculated directly using the Darboux theory.

We see that here the exponent of this line x in the PIIF is the number of affine lines that appear in the PIIF of a perturbation of the system into generic systems and that coalesce in taking the limit when the perturbation parameter tends to zero. Clearly this number is less than or equal to the multiplicity of the line.
The remaining case 6.11 is treated similarly. ∎

Observation 3.1. *(a) We underscore here the significant role played by Diophantine equations in obtaining this Theorem. The abundance of 6 PIIF for the generic cases of QSL_6 is due to the occurring Diophantine equations.*
(b) We observe that from the 6 PIIFs obtained in the generic cases in Theorem 1.1, only two of them, R_2 and R_3, are of (C-K) type.
We observe that none of the other three generic systems 6.2, 6.3, 6.4. have configurations of (C-K) type. However they also have PIIFs but they cannot be obtained from the (C-K) theorem. This brings us back to our Question 4: How do we generalize the (C-K) theorem?

Observation 3.2. *(a) Considering now the non-generic cases 6.i, $5 \leq i \leq 11$ we observe that it is the multiplicity of the lines as well as their unfoldings within generic systems that play a role in the presence of PIIFs. In essence it is because the non-generic systems can be perturbed into generic ones possessing a PIIF, the non-generic ones inherit this property. In QSL_6 the non-generic cases are just the ones possessing a multiple line and it is the presence of multiplicity that provides the existence of the perturbation. We shall later see, for more general cases that the multiplicity of the*

singularities of the systems also play a role in the production of PIIFs.
(b) In the case of QSL_6 the exponents appearing in the PIIFs of the non-generic cases are explained geometrically by the way the lines in the unfoldings behave with respect to the process of taking the limit, for example some may go to infinity and hence they will not appear in the resulting PIIF while others coalesce resulting in an exponent of a factor being greater than one.

3.2 The family of systems in QSL_5

In [12] the authors used the geometrical classification of QSL_5 in terms of configurations of invariant lines to obtain the topological classification of this family leading to 31 distinct phase portraits. As a biproduct of the geometrical classification (see [11]) the authors obtained the proof that all systems in this family possess a PIIF and a generalized Darboux first integral.

All the systems in QSL_5 have a PIIF but unlike the case of QSL_6 where all systems have a rational first integral, in QSL_5 this family has in general a generalized Darboux first integral.

This is a richer family than QSL_6 which splits into 30 distinct cases according to the normal forms and distinct configurations of invariant lines modulo the group action of affine transformations and time rescaling. Eleven of these cases are one-parameter families and the remaining ones do not depend on a parameter. We now perform a full analysis of the family QSL_5.

Theorem 3.2. *(i) In QSL_5 there are exactly six families with generic systems,none of them of (C-K) type, with configurations leading to a unique PIIF in all cases except for 5.1 which for some values of the parameter g has two PIIFs.*
In only two of the six cases, 5.1 and 5.2 we have subconfigurations with exactly three invariant lines in (C-K) position. The four other generic cases have no combinations of three out of four affine invariant lines that are in (C-K) positions, all having at least two parallel lines. We have however a unique PIIF for all these four other generic systems which is the product of the four affine invariant lines.

(ii) Every system (S_0) in QSL_5 possessing an invariant line L of multiplicity $m > 1$ has a polynomial unfolding (S_ϵ) within the generic systems of the family QSL_5 and therefore it has at least one PIIF namely $R_0 = lim_{\epsilon\to 0}(R_\epsilon)$ where R_ϵ is a PIIF of (S_ϵ).
In passing to the limit two things could occur:
(a) k of the linear factors in a PIIF of the perturbation would coalesce into L yielding a PIIF R with L raised to the power $k \le m$.
(b) a PIIF R_ϵ may loose a factor L_ϵ when L_ϵ tends to a constant as ϵ tends to zero, i.e. L_ϵ tends to the line at infinity.

(iii) Apart from the PIIF obtained for all values of the parameter g in the normal forms in [12], due to Diophantine constraints the systems 5.1, 5.8 and 5.14 have an additional PIIF expressed in terms of invariant lines whenever

$$g \in \{\dots, -1/4, -1/3, -1/2, 1/2, 1/3, 1/4, \dots\}.$$

Proof: (i) **- Systems 5.1:** $\dot{x} = (x+1)(gx+1),\ \dot{y} = (g-1)xy + y^2$ with $g(g^2-1) \ne 0$.
The invariant lines are: $x+1$, $gx+1$, $x-y+1$ and y are all simple with respective co-factors: $K_1 = gx+1$, $K_2 = g(x+1)$, $K_3 = gx+y+1$, $K_4 = (g-1)x+y$. We have $div(p,q) = (3g-1)x+2y+1+g$.

We need to solve in λ_i's the equation $\sum \lambda_i K_i = div(p, q)$. This leads to a system of three linear non-homogeneous equations in four unknowns whose solution is:

$$\lambda_1 = 0, \quad \lambda_2 = 1 - 1/g + \lambda_4/g, \quad \lambda_3 = 2 - \lambda_4$$

where λ_4 is arbitrary. We search for a PIIF hence all λ_i's must be non-negative integers. In this case $\lambda_4 \leq 2$ and hence $\lambda_4 = 0, 1, 2$.

For $\lambda_4 = 0$ we have $\lambda_2 = 1 - 1/g \in \mathbb{Z}_{\geq 0}$ and $\lambda_3 = 2$. hence $1/g \in \mathbb{Z}$ i.e. hence $g = 1/m$ where $m \neq 0$ and we also have $1/g \leq 1$ hence $m \leq 1$. In view of our assumption $g(g^2 - 1) \neq 0$ we get that $m \neq \pm 1$. Hence $m < -1$ and so we obtain the PIIF=$(gx + 1)^{1-m}(x - y + 1)^2 = (gx + 1)^{1-1/g}(x - y + 1)^2$ for $g \in \{\ldots, -1/4, -1/3, -1/2\}$.

If $\lambda_4 = 2$ in an analogous way we get in this case the PIIF $(gx + 1)^{1+1/g}(x - y + 1)^2$ for $g \in \{1/2, 1/3, 1/4, \ldots\}$.

In the case $\lambda_4 = 1$ we get the PIIF=$(gx + 1)(x - y + 1)y$ and this is the case of the three lines in the (C-K) position.

-Systems 5.2: They are treated analogously to 5.1 with the only diference being that 5.2 has two complex conjugate invariant lines instead of having all lines real and the equation $\sum \lambda_i K_i = div(p, q)$ leads to a non-homogeneous system of linear equations whose solution is

$$\lambda_1 = 1 + 2i/g - 2i\lambda_3/g, \quad \lambda_2 = 0, \quad \lambda_4 = 2 - \lambda_3.$$

Here the λ_i's must be non-negative integers. Hence $\lambda_3 = 0, 1, 2$. For $\lambda_3 = 0$ we get $\lambda_4 = 2$, $\lambda_2 = 0$ and $\lambda_1 = 1 + 2i/g$. Since we considered $g \in \mathbb{R}$ this is impossible. Consider now $\lambda_3 = 1$. Then $\lambda_2 = 0$ and $\lambda_4 = 1$. In this case the PIIF is $R = (x + 2)(y + i(x - 2) + g)(y - i(x - 2) + g)$ and it is the PIIF given by three lines in (C-K) general position. Had we also included the possibility that $g \in \mathbb{C}$ then we would have had an analogous result as the one for the previous family 5.1.

The generic **Systems 5.i:** with $i \in \{3, 4, 5, 6\}$ behave similarly to the generic cases 6.i with $i \in \{1, 2, 3, 4\}$ as their configurations corespond to subconfigurations of these systems 6.i. They all have PIIFs which are the product of the four affine invariant lines, two by two parallel and hence excluded in the (C-K) theorem because of its triple intersection of lines at points at infnity.

(ii) We first make the following observation:

Observation 3.3. *We observe that the systems in the family 5.8 can be unfolded in systems with configuration 5.1 by unfolding the multiple line into two parallel ones and those of 5.14 by unfolding the double line into two lines intersecting in the finite plane. Because of these unfoldings we expect to have analogous results to those of 5.1 also for 5.8 and 5.14.*

We first prove the existence of exactly two PIIFs for two families 5.8 and 5.14. They are the only ones with exactly two PIIFs.

-Systems 5.8: The normal form of systems 5.8 is:

$$dx/dt = gx^2, \quad dy/dt = (g - 1)xy + y^2$$

where g is a real parameter such that $g(g^2 - 1) \neq 0$. The invariant lines here are: $L_1 = x$ which is double, $L_2 = y$ and $L_3 = x - y$ which are both simple. Their corresponding cofactors are: $K_1 = gx$,

$K_2 = (g-1)x + y$, $K_3 = bx + y$. We need to solve the equation $\sum \lambda_i K_i = div(p,q)$ where the divergence here is $div(p,q) = (3g-1)x + 2y$. Identifying coefficients we obtain the system

$$g\lambda_1 + (g-1)\lambda_2 + g\lambda_3 = 3g-1, \quad \lambda_2 + \lambda_3 = 2$$

with the solutions:

$$\lambda_1 = \frac{g+1-\lambda_3}{g}, \quad \lambda_2 = 2 - \lambda_3.$$

We need that λ_1's be in $\mathbb{Z}_{\geq 0}$. Hence $\lambda_3 \in \{0,1,2\}$. Consider first $\lambda_3 = 0$. Then $\lambda_1 = 1 + 1/g$ and it must belong to $\mathbb{Z}_{\geq 0}$. Hence $1/g = n \in \mathbb{Z}$ and $1/g \geq -1$ and since $g(g^2-1) \neq 0$ we must have $1/g = n > 1$ and $g = 1/n \in \{1/2, 1/3, 1/4, ...\}$. In this case the PIIF is $x^{1+1/g}y^2 = x^{1+n}y^2$ where $n > 1$.
Let $\lambda_3 = 1$. Then $\lambda_1 = \lambda_2 = 1$ and the PIIF is here $xy(x-y)$.
Let now $\lambda_3 = 2$. Then $\lambda_1 = 1 - 1/g \in \mathbb{Z}_{\geq 0}$ and $\lambda_2 = 0$. Hence $1/g \leq 1$ and $1/g = n \in \mathbb{Z}_{\leq -2}$. In this case for $g \in \{..., -1/4, -1/3, -1/2\}$ the PIIF is $x^{1-1/g}(x-y)^2$.

We observe that the proof for 5.8 above is entirely similar to the proof for systems 5.1. The reason is that 5.8 can be perturbed into a system with a configuration equivalent to the generic one in 5.1 and passing to the limit of the two PIIF's for the perturbation we get those for 5.8 as we show here below.

We split the double vertical line into two parallel invariant lines in the following perturbation of systems 5.8:

$$(S_\epsilon(g)): \; dx/dt = (x+\epsilon)(gx+\epsilon), \quad dy/dt = (g-1)xy + y^2$$

The invariant lines here are: $L_1^\epsilon = x + \epsilon$, $L_2^\epsilon = gx + \epsilon$, $L_3^\epsilon = x - y + \epsilon$ and $L_4^\epsilon = y$. All singularities in $(S_\epsilon(g))$ are simple and only 6 of them are lying on the lines. The 4-tuple singular point at the origin has split into 4 simple real singularities, three of them determining the triangle which has been formed. Clearly this is an equivalent configuration to the one for the generic systems 5.1 and since this configuration produces exactly two PIIFs, one given by the triangle via the (C-K) result: $R_1^\epsilon = L_2^\epsilon L_3^\epsilon L_4^\epsilon$, and the other obtained only for some special values of g. More precisely if $g = 1/n$ with $n \in \mathbb{Z}_{<-1}$ we have $R_2^\epsilon = (L_2^\epsilon)^{1-1/g}(L_3^\epsilon)^2$ and in case $n \in \mathbb{Z}_{>1}$ we have $R_2^\epsilon = (L_2^\epsilon)^{1+1/g}(L_3^\epsilon)^2$. Passing to the limit when $\epsilon \to 0$ we obtain the two PIIF's of 5.8.

-Systems 5.14: The normal form of these systems is:

$$dx/dt = (1+x)(gx+1), \quad dy/dt = (g-1)xy$$

where g is a real parameter such that $g(g^2-1) \neq 0$. The invariant lines are here the double line $x+1$ and the two simple lines y, $gx+1$. For these systems the same type of arguments as in the preceding case occur as we can easily ocnstruct a perturbations in which the double line splits in two lines resulting in a configuration equivalent to 5.1. The resulting PIIFs for this perturbation explain geometrically one of the two PIIFs while the Diophantine arguments explain the presence of a second PIIF for special values of the parameter g.

The three families of systèms 5.1, 5.8 and 5.14 behave similarly because the two families 5.8 and 5.14, both possessing a unique double affine line can be perturbed into generic systems with configurations equivalent to those of 5.1.

-System 5.7: The normal form of this systems is:

$$dx/dt = 1 + x, \quad dy/dt = -xy + y^2$$

The invariant lines are: y, $x+1$, $x-y+1$ and their respective cofactors are: $y-x$, 1, $1+y$. The divergence of the systems is: $1-x+2y$. Solving the corresponding equation in λ_i's, i.e. $\lambda_1(y-x) + \lambda_2 + \lambda_3(1_y) = 1-x+2y$ we obtain $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = 0$. Therefore $y(x-y+1)$ is the unique FIIP.

The calculation above lead us to a PIIF for this system and we want to see a geometric or dynamic reason for this.

We first observe that the line at infinity is a double line and it has two double points. This means that we can split the double line in two lines but what we show here below is that this can be done within the class QSL_5

For this we consider the following petrurbation into generic systems 5.1:

$$(S_\epsilon) : dx/dt = (x+1)(\epsilon x + 1), \quad dy/dt = (\epsilon - 1)xy + y^2$$

The invariant lines of this systems are: y, $x+1$, $y-x$, $\epsilon x + 1$, $y-x-1$. Associated with these lines we have the corresponding cofactors $K_1 = (\epsilon-1)x + y$; $K_2 = \epsilon x + 1$; $K_3 = \epsilon(x+1)$, $K_4 = \epsilon x + y + 1$. The resulting equation to be solved is $\Sigma_i K_i = div(S_\epsilon)$ where div denotes the divergence of the perturbation $div = (3\epsilon - 1)x + 2y + \epsilon + 1$. Mathematica gives

$$\lambda_2 = 0, \quad \lambda_3 = 1 - 1/\epsilon + \lambda_1/\epsilon, \quad \lambda_4 = 2 - \lambda_1.$$

Hence we have a Darboux integrating factor of the form:

$$R_\epsilon(\lambda_1) = y^{\lambda_1}(\epsilon x + 1)^{1-1/\epsilon+\lambda_1/\epsilon}(y-x-1)^{2-\lambda_1}.$$

To obtain a PIIF we need to have all λ_i's non-negative integers and hence $\lambda_1 \in \{0,1,2\}$ but. only $\lambda_1 = 1$ produces a PIIF which is $R_\epsilon(1) = y(\epsilon x + 1)(y-x-1)$. We see here that the three lines producing this PIIF are in general position as expressed in the (C-K) theorem. Furthermore $lim_{\epsilon \to 0}(R_\epsilon) = y(x-y+1)$. In passing to the limit one of the three lines disappearaed at infinity and we are only left with two factors. We see that we can have a PIIF in case we have a multiple line (the line at infinity here) possessing two double singular points of the system and see here how this PIIF originates in the generic configuration 5.1.

The remaining systems are treated in the same way. The point (iii) was treated in the development of the proof for (i) and (ii). ∎

We saw that in $QSL_{\geq 5}$ all generic systems have at least a point where three distinct invariant lines intersect. More precisely the following proposition holds.

Proposition 3.2. *Consider a quadratic differential system in $QSL_{\geq 5}$ or $QSL^{\mathbb{C}}_{\geq 5}$ and suppose that this system is generic. Then either*
1) this system has exactly two finite and two infinite singular points through which pass exactly three invariant lines in which case the system belongs to QSL_6 or
2) this system has exactly two singular points (either both infinite or one finite and the other one infinite) through which pass exactly three invariant lines in which case the system belongs to QSL_5.

The presence of singular points through which pass three invariant lines, for systems in $QSL_{\geq 5}$ does not encroach on having polynomial inverse integrating factors for all the system in this family.

4 In depth analysis of Liouvillian integrability of the family QSL_4

We recall that Liouvillian integrability is characterized in terms of the type of the integrating factors the systems could have (see Theorem 2.3). We therefore look at the types of integrating factors present in the family QSL_4. Unlike the systems in $QSL_{\geq 5}$ whose integrating factors are all inverses of polynomial functions, QSL_4 presents all the spectrum of integrating factors leading to Liouvillian integrability, i.e. inverses of polynomials, Darboux and generalized Darboux.
Furthermore, while all generic systems in $QSL_{\geq 5}$ possess PIIFs, in QSL_4 they could possess, apart from PIIFs also DIFs.

The family QSL_4 is thus a very good testing ground for examining the engin that splits the family and more generally Liouvillian integrable polynomial differential systems into these three subfamilies corresponding to the three types of integrating factors.

A simple answer is given by the solutions of the system of linear equations with rational coefficients constructed by identifying the coefficients in the polynomial equation:

$$\sum \lambda_i K_i = -div(p,q), \quad 1 \leq i \leq m$$

in the unknowns λ_i where K_i is the cofactor of an invariant line f_i for $1 \leq i \leq m$ and (p,q) appears in one the 46 normal forms of QSL_4 modulo the group action (see [14]). This is a system of linear non-homogeneous equations over the rationals. If we can solve the system then we obtain the exponents of a DIF. If in addition we can solve it over $\mathbb{Z}_{\geq 0}$ then we obtain an integrating factor whose inverse is polynomial. In case we have no such solutions and we used all the present invariant lines then we try to find exponential factors and we need to solve the system $\sum \lambda_i K_i = -div(p,q), \ 1 < i \leq k$ where K_i for $m < i \leq k$ are cofactors of invariant lines and of exponential factors and the rest proceeds as before.

This answers the question we asked in the first paragraph via an algebraic and diophantine argument that leaves us however not completely satisfied. Indeed, while the systems are endowed with an algebraic-geometric structure encoded in the configuration of their invariant lines, in the above arguments we do not see the role of theis geometric structure in separating integrating factors in these three groups.

We therefore pursue our analysis in more depth with the aim of unravelling the contribution of the geometry of the systems and possibly also specific analytic contributions in the production of their integrating factors. This more complex engine turns out to be their types of unfoldings into generic systems, and most importantly the multiplicities of the invariant lines as well as the multiplicities of singularities of the systems. In short everything that the concept of configuration of invariant curves of a system includes. We perform this in depth analysis on the Liouvillian integrable family QSL_4.

In view of Theorem 6 of [14] the space QSL_4 is formed by 46 distinct families of systems modulo the action of the affine group and time rescaling and all these systems possess DIFs or GDIFs. Because these normal forms are essential for this work and will very often intervene in the this article we give in the Appendix a slightly modified version of the Table 1 in [14] (called Table I in Appendix) containing all these 46 normal forms. We will also need often the configurations corresponding to these normal form which are contained in Diagram 1 of [14]. For this reason we also give this Diagram 1 in the Appendix with indications on how to use it.

We call *semi-generic* a system whose invariant lines are all simple and there is at least one singular point of multiplicity higher than one.

Modulo the action of the group of affine transformations and time rescaling the family QSL_4 is composed of the following distinct subspaces:

- generic systems, all appearing in six 2-parameter families;
- semi-generic systems, they form the content of nine families, four of them depending on two parameters and five on one parameter;
- non-generic and also non-semi-generic systems; these are those having at least one multiple invariant line and they are the most numerous ones: 31 of the 46 families are filled up with them.

Our next four theorems analyse in depth the Liouvillian integrability of the systems in this family based on their types of integrating factors starting with the polinomial inverse integrating factors (PIIF), then the Darboux ones (DIF) and generalized Darboux integrating factors (GDIF).

These four theorems make explicit for each one of these cases the algebro-geometric structure involving the contribution of multiplicities of the lines and of singularities and also the analytical and diophantine properties that distinguish them.

The results in the next four theorem involve numerous perturbations of nongeneric systems belonging to one of the 46 normal forms modulo the group action for QSL_4, into generic ones. This led to numerous calculations of the integrating factors of these perturbations, all expressed in terms of invariant lines and when passing to their limit as the perturbation parameter ε tends to zero lead to integrating factors where the invariant curves are raised to some exponents or to exponential factors. Some calculations are similar to those encountered in the previous section and except for calculations of limits leading to exponential factors in the resulting GDIF's, they simply consist in solvings linear equations over the rationals. We present the results in eight Tables organizing each group of systems by their specific geometric features, ivolving whenever needed the multiplicities of the lines.

Theorem 4.1. *(A) The systems in QSL_4 with a PIIF: Out of the 46 families of systems of QSL_4, modulo the group action, distinct according to their configurations of invariant lines, 21 of them have a unique PIIF for all values of the parameters. These are characterized by possessing polynomial perturbations in $\varepsilon \in \mathbb{R}$, into generic systems in QSL_4 of (C-K) type. We split them below according to their geometric and analytic features.*
(A_1) Generic systems with a PIIF: There are only two families 4.1, 4.2 *out of the 21, which are generic and they are also the only families with configurations of (C-K) type. Each system in these families has a unique PIIF which is the product of the three distinct affine invariant lines.*

(A_2) Semi-generic systems with a PIIF (Table 1): There are only four semi-generic families with PIIF: 4.5, 4.8, 4.12, 4.15. *All these systems possess configurations not of (C-K) type and have a*

unique multiple singularity of multiplicity four.
A semi-generic system (S_0) *has a PIIF R if and only if there exists a polynomial perturbation* (S_ε), $\varepsilon \in \mathbb{R}$ *of* (S_0) *into generic systems of (C-K) type 4.1 or 4.2 with three distinct invariant lines* $L_i(\varepsilon)$, *linear in* ε, *leading to a PIIF* $R(\varepsilon)$, *product of the three lines such that* $lim_{\varepsilon\to 0}(R(\varepsilon)) = R$ *(see Table 1 below).*
The multiple singular point of a system 4.5 or 4.8 results from the coalescence (in passing to the limit as $\varepsilon \to 0$*) of the three affine singularities in the 4.1 (respectively 4.2) perturbation of 4.5 (respectively of 4.8), producing an affine singular point of multiplicity four. Analogously every system in the two families 4.12 or 4.15 has a unique multiple infinite singular point of multiplicity four resulting from the coalescence (in passing to the limit as* $\varepsilon \to 0$*) of an affine singular point with two infinite ones.*
(A_3) *Systems possessing a multiple invariant line Tables 2 and 3): All the remaining 15 out of the 21 families with PIIF have at least one multiple line and they admit small polynomial perturbations into one of the two generic systems of (C-K) type 4.1, 4.2 or into four semi-generic families listed above in* (A_2) *with a PIIF* $R(\varepsilon)$ *product of the three perturbed distinct lines producing a PIIF* $R_0 = R(0)$ *as* ε *tends to zero (see Tables 2 and 3). If a line appears in* R_0 *with the exponent* m *then* m *is the multiplicity of the line. If the multiple line is at infinity with multiplicity* m *then* R_0 *will only feature* r *affine lines with* $r = 4 - m$.

Proof: We prove each one of the statments based on [14] and the first three of the eight Tables appearing further below.

(A_1) This point follows easily from inspecting the Table 1 and Diagram 1 in [14] or here in the Appendix and by applying the (C-K) theorem or by calculations using the method of Darboux.

(A_2) The fact that $4.i$ for $i \in \{5, 8, 12, 15\}$ are semi-generic systems according to our definition can be seen on page 1 of the Diagram 1 in [14]. The perturbations of the systems into generic ones with (C-K) configurations for proving this point are indicated in our Table 1.

(A_3) The perturbations for proving this third point are shown in Table 3 of [13]. The proof of the point (A_3) is based on our calculations by applying the method of Darboux and they lead to the results exhibited in the Tables 2 and 3 below. The necessity of the condition is clear because the perturbation deforms a generic system of (C-K) type into one with at least one multiple line. If this line is affine, it splits into affine lines and its multiplicity appears as exponent of the line in the PIIF. If a line goes to infinity leading to a multiple line at infinity we clearly will have fewer lines appearing in the PIIF. ∎

Observation 4.1. *We observe that in the case* (A_3) *of this theorem all systems possess at least one invariant line of multiplicity greater than one. There was no need for an exponential factor to construct an integrating factor which is of the simplest kind: it is the inverse of a PIIF. But if we have an exponential factor necessarily this is attached to a multiple curve (see [3]).*

Theorem 4.2. *(B) The systems with a DIF distinct from the ones mentioned above:*
(B_1) *Generic systems with a DIF: There are four remaining families of generic systems:* 4.3, 4.6, 4.9, 4.13. *They are 2-parameter families. Their DIFs are given in Table 1 in [14] and here in the Appendix in Table I.*
The families 4.3 and 4.6 have a unique DIF that cannot be the inverse of a PIIF for any values of the two parameters.

Table 1: Perturbations of semi-generic systems in QSL_4 with PIIF into generic systems of (C-K) type

Perturbed systems	Invariant lines	$PIIF\ \mathcal{R}_i^\varepsilon$; $\mathcal{R}_i = \lim_{\varepsilon\to 0}\mathcal{R}_i^\varepsilon$
(4.5_ε) $((4.1))$ $\begin{cases}\dot x=(x+\varepsilon)(gx-y+hy),\\ \dot y=\big(y-\frac{g}{h}\varepsilon\big)\big((g-1)x+hy\big)\end{cases}$	$L_1=x+\varepsilon,\ L_2=y-\frac{g}{h}\varepsilon,$ $L_3=hx-hy+\varepsilon(g+h-1)$	$\mathcal{R}_5^\varepsilon=L_1L_2L_3$; $\mathcal{R}_5=hx(x-y)y$
(4.8_ε) $((4.2))$ $\begin{cases}\dot x=gx^2+(h+1)xy,\\ \dot y=-x^2+gxy+hy^2+\\ \qquad\frac{\varepsilon u}{2gh}x+\varepsilon y+\frac{\varepsilon^2 v}{4g^2h},\\ u=1+g^2-h^2,\\ v=g^2+(h+1)^2\end{cases}$	$L_1=x,$ $L_2=2gh(x-iy)-\varepsilon(h+1+ig),$ $L_3=2gh(x+iy)-\varepsilon(h+1-ig)$	$\mathcal{R}_8^\varepsilon=L_1L_2L_3$; $\mathcal{R}_8=4g^2h^2x(x^2+y^2)$
(4.12_ε) $((4.1))$ $\begin{cases}\dot x=(x+h-1)(x+h+1+\varepsilon uy),\\ \dot y=y\big[(1-g)x+\varepsilon(g-1)(h-1)y\big],\\ u=1+g-h+gh\end{cases}$	$L_1=x+h-1,\ L_3=y,$ $L_2=x+2\varepsilon y+1+h$	$\mathcal{R}_{12}^\varepsilon=L_1L_2L_3$; $\mathcal{R}_{12}=\big[(x+h)^2-1\big]y$
(4.15_ε) $((4.2))$ $\begin{cases}\dot x=1+h^2+2hx+x^2+\\ \qquad\varepsilon y+[g\varepsilon^2y^2]/w,\\ \dot y=(1-g)xy+\frac{2\varepsilon}{w}gh(g-1)y^2,\\ w=(1+g)^2+(g-1)^2h^2,\ g\neq-1\end{cases}$	$L_{1,2}=gh^2+ghx\mp 2igh\mp igx$ $-g-h^2-hx-\varepsilon y\mp ix-1,$ $L_3=y$	$\mathcal{R}_{15}^\varepsilon=L_1L_2L_3$; $\mathcal{R}_{15}=y\big[(x+h)^2+1\big]$
(4.15_ε) $((4.2))$ $\begin{cases}\dot x=1+h^2+2hx+x^2+\varepsilon y+\\ \qquad 2\varepsilon xy-\frac{\varepsilon^2}{4h^2}(5-4h+4h^2)y^2,\\ \dot y=2xy+[\varepsilon y^2]/h\quad(for\ g=-1)\end{cases}$	$L_{1,2}=\pm 2ih+2h^2+2hx+$ $(1\mp 2i)\varepsilon y-2h\varepsilon y,\ L_3=y$	$\mathcal{R}_{15}^\varepsilon=L_1L_2L_3$; $\mathcal{R}_{15}=y\big[(x+h)^2+1\big]$

The families 4.9 and 4.13 have a unique DIF which for some particular values of the two parameters is the inverse of a PIIF due to diophantine properties.

(B_2) The semi-generic systems with a DIF: These are the five one-parameter families 4.4, 4.7, 4.10, 4.11, 4.14*. They admit perturbations, polynomial in ε, into generic systems belonging to the four generic families with DIF mentioned above, producing a DIF for the systems when the perturbation parameter ε tends to zero (see Table 4).*

(B_3) The systems with an affine multiple invariant line and possessing a DIF are 4.30 *and* 4.43 *(see Table 5 below) and those with the multiple line at infinity are:* 4.28, 4.32, 4.37 *(Table 6 below). With the exception of 4.30 the other four types of systems admit polynomial perturbations in the perturbation parameter ε into generic systems with a DIF $R(\varepsilon)$ verifying two conditions: (i) the three affine lines in $R(\varepsilon)$ have coefficients belonging to $\mathbb{Z}[g,\varepsilon]$ which are linear or quadratic in ε and (ii) their exponents in $R(\varepsilon)$ are linear in ε or do not depend on ε.*
The systems in the family 4.30 admit a polynomial perturbation into generic systems with a DIF: $R(\varepsilon)$ such that the affine lines in the perturbation and their exponents in $R(\varepsilon)$ have coefficients of the form $F+G\sqrt{v}$ where F does not depend on ε or is linear in ε and it is a rational function in the variables and the parameter, and G is a rational function in the parameter of the family and in

Table 2: Systems in QSL_4 with PIIF and a multiple affine line

Perturbed systems	Invariant lines	$PIIF\ \mathcal{R}_i^{\varepsilon}$
	Corresponding cofactors	$\mathcal{R}_i = \lim_{\varepsilon\to 0} \mathcal{R}_i^{\varepsilon}$
Inverse integrating factors of all systems of type $\{(2,1),(1,2)\}$ with the affine line double which possess perturbations into generic or semi-generic systems with PIIF		
$(4.20_\varepsilon) \equiv (IV.20_\varepsilon)$ $\big((4.5)\big)$ [4, Table 3]	$L_1 = x + \varepsilon,\ L_2 = x + \varepsilon y, L_3 = y$	$\mathcal{R}_{20}^{\varepsilon} = L_1L_2L_3$
	$K_1 = gx + \varepsilon y + y, K_2 = gx + y,$ $K_3 = gx - x + y$	$\mathcal{R}_{20} = x^2y$
$(4.24_\varepsilon) \equiv (IV.24_\varepsilon)$ $\big((4.12)\big)$ [4, Table 3]	$L_1 = x + 1 + \varepsilon,\ L_2 = x + 1 - \varepsilon,\ L_3 = y$	$\mathcal{R}_{20}^{\varepsilon} = L_1L_2L_3$
	$K_1 = x + 1 - \varepsilon,\quad K_2 = x + 1 + \varepsilon$ $K_3 = (1 - g)x$	$\mathcal{R}_{24} = (x+1)^2y$
$(4.25_\varepsilon) \equiv (IV.25_\varepsilon)$ $\big((4.1)\big)$ [4, Table 3]	$L_1 = x,\ L_2 = x + \varepsilon y + \varepsilon,\ L_3 = y$	$\mathcal{R}_{25}^{\varepsilon} = L_1L_2L_3$
	$K_1 = g\varepsilon + gx + \varepsilon y + y,\ K_2 = gx + y,$ $K_3 = 1 - x + gx + y$	$\mathcal{R}_{25} = x^2y$
$(4.31_\varepsilon) \equiv (IV.31_\varepsilon)$ $\big((4.12)\big)$ [4, Table 3]	$L_{1,2} : -g\varepsilon + (1+\varepsilon)x + (1+\varepsilon)x^2 = 0,$ $L_3 = x + \varepsilon y + \varepsilon + 1$	$\mathcal{R}_{25}^{\varepsilon} = L_1L_2L_3$
	$K_{1,2} : -g\varepsilon + (1+\varepsilon)x + (1+\varepsilon)x^2 = 0,$ $K_3 = x$	$\mathcal{R}_{31} = x(x+1)^2$
Inverse integrating factors of systems with PIIF and of type $\{(3,1),(1,1)\}$ with the affine line triple which possess perturbations into generic or semi-generic systems		
$(4.42_\varepsilon) \equiv (IV.42_\varepsilon)$ $\big((4.5)\big)$ [4, Table 3]	$L_1 = x,\ L_2 = 2x + \varepsilon y,\ L_3 = x - 2\varepsilon y$	$\mathcal{R}_{42}^{\varepsilon} = L_1L_2L_3$
	$K_1 = (2gy + \varepsilon x)/2,\ K_2 = y(g + \varepsilon^2),$ $K_3 = (2gy + 2\varepsilon^2 y + 5\varepsilon x)/2$	$\mathcal{R}_{42} = 2x^3$
$(4.44_\varepsilon) \equiv (IV.44_\varepsilon)$ $\big((4.12)\big)$ [4, Table 3]	$L_{1,2} : (1+\varepsilon)x^2 + \varepsilon(1+\varepsilon)x - g\varepsilon = 0,$ $L_3 = x + \varepsilon y + \varepsilon(1+\varepsilon)$	$\mathcal{R}_{44}^{\varepsilon} = L_1L_2L_3$
	$K_{1,2} : (1+\varepsilon)x^2 + \varepsilon(1+\varepsilon)x - g\varepsilon = 0,\ K_3 = x$	$\mathcal{R}_{44} = x^3$
$(4.45_\varepsilon) \equiv (IV.45_\varepsilon)$ $\big((4.1)\big)$ [4, Table 3]	$L_1 = x,\ L_2 = x + \varepsilon y,\ L_3 = x + 2\varepsilon y - 2\varepsilon^2/g$	$\mathcal{R}_{45}^{\varepsilon} = L_1L_2L_3$
	$K_1 = (g^2y + g\varepsilon x - 2g\varepsilon + 2\varepsilon^3)/g,$ $K_2 = (2\varepsilon^2 - g)(\varepsilon - gy)/g,\ K_3 = gy - 2\varepsilon^2 y - \varepsilon x$	$\mathcal{R}_{45} = x^3$

ε that tends to a finite limit when ε tends to zero. Furthermore v is a polynomial in the parameter of the family and in ε, linear in ε and hence $R(\varepsilon)$ has a finite limit which is a DIF when ε tends to zero.

Proof: (B_1) The first statement follows from Table 1 of [14]. The second statement results by simply showing that solving the equation $\sum \lambda_i K_i = div(p,q)$ in λ_i's leads to a unique solution in this case. Then trying to seek non-negative integer solutions we end up with diophantine equations that have a solution for particular values of the parameters. This kind of calculation is entirely analogous to the ones performed for systems 5.1 and 5.8 in the preceding section.

Table 3: Systems in QSL_4 with PIIF and the multiple line at infinity

Perturbed systems	Invariant lines / Corresponding cofactors	$PIIF\ \mathcal{R}_i^\varepsilon$ / $\mathcal{R}_i = \lim_{\varepsilon\to 0}\mathcal{R}_i^\varepsilon$
Inverse integrating factors of all systems of type $\{(2,1),(1,2)\}$ with double line at infinity which possess perturbations into generic or semi-generic systems with PIIF		
$(4.18_\varepsilon) \equiv (IV.18_\varepsilon)$ $((4.1))$ [4, Table 3]	$L_1 = \varepsilon x + 1,\ L_3 = y,$ $L_2 = y - x(1-g\varepsilon) - (g+1)(g\varepsilon+1)$	$\mathcal{R}_{18}^\varepsilon = L_1L_2L_3$
	$K_1 = (g^2+g+gx+y), K_2 = g\varepsilon x + g + \varepsilon y + y,$ $K_3 = (y-x) - \varepsilon(g(g+1)(\varepsilon+1)+1-y)$	$\mathcal{R}_{18} = y(y-x-g-1)$
$(4.19_\varepsilon) \equiv (IV.19_\varepsilon)$ $((4.12))$ [4, Table 3]	$L_1 = x+g,\ L_2 = \varepsilon x + 1,\ L_3 = y$	$\mathcal{R}_{19}^\varepsilon = L_1L_2L_3$
	$K_1 = \varepsilon x + 1,\ K_2 = \varepsilon(x+g),\ K_3 = -x$	$\mathcal{R}_{19} = y(x+g)$
$(4.27_\varepsilon) \equiv (IV.27_\varepsilon)$ $((4.2))$ [4, Table 3]	$L_1 = \varepsilon x + 1, L_{2,3} = (1-2\varepsilon)(x \pm iy) - 1 \pm ig$	$\mathcal{R}_{27}^\varepsilon = L_1L_2L_3$
	$K_1 = 2\varepsilon(1-2\varepsilon)(gx+y),$ $K_{2,3} = (1-2\varepsilon)(2g\varepsilon x + g + 2\varepsilon y \mp ix - y \mp i)$	$\mathcal{R}_{27} = (x-1)^2+(y+g)^2$
$(4.29_\varepsilon) \equiv (IV.29_\varepsilon)$ $((4.12))$ [4, Table 3]	$L_{1,2} = x \pm 1,\quad L_3 = \varepsilon y + 1$	$\mathcal{R}_{29}^\varepsilon = L_1L_2L_3$
	$K_{1,2} = x \mp 1, K_3 = \varepsilon(x+g)$	$\mathcal{R}_{29} = x^2 - 1$
$(4.33_\varepsilon) \equiv (IV.33_\varepsilon)$ $((4.15))$ [4, Table 3]	$L_{1,2} = x \pm i,\quad L_3 = \varepsilon y + 1$	$\mathcal{R}_{33}^\varepsilon = L_1L_2L_3$
	$K_{1,2} = x \mp i, K_3 = \varepsilon(x+g)$	$\mathcal{R}_{33} = x^2 + 1$
Inverse integrating factors of all systems of type $\{(3,1),(1,1)\}$ with triple line at infinity which possess perturbations into generic or semi-generic systems with PIIF		
$(4.35_\varepsilon) \equiv (IV.35_\varepsilon)$ $((4.1))$ [4, Table 3]	$L_1 = \varepsilon x + 1,\ L_2 = \varepsilon^2(x+\varepsilon y) + \varepsilon + 1,\ L_3 = y$	$\mathcal{R}_{35}^\varepsilon = L_1L_2L_3$
	$K_1 = \varepsilon(\varepsilon+1)(y+g)+g\varepsilon^3 x, K_2 = \varepsilon^2(g+g\varepsilon x+y),$ $K_3 = g\varepsilon^2 + (g\varepsilon^3-1)x + \varepsilon^2 y$	$\mathcal{R}_{35} = y$
$(4.36_\varepsilon) \equiv (IV.36_\varepsilon)$ $((4.12))$ [4, Table 3]	$L_{1,2} = \varepsilon x \pm 1,\ L_3 = y$	$\mathcal{R}_{36}^\varepsilon = L_1L_2L_3$
	$K_{1,2} = -g\varepsilon(\varepsilon x \mp 1),\ K_3 = -x$	$\mathcal{R}_{36} = -y$
$(4.38_\varepsilon) \equiv (IV.38_\varepsilon)$ $((4.9))$ [4, Table 3]	$L_1 = x,\ L_2 = \varepsilon(2-g\varepsilon^2)x+1,\ L_3 = \varepsilon x + \varepsilon^2 y + 1$	$\mathcal{R}_{38}^\varepsilon = L_1^{1-2g\varepsilon^2} L_3^2 \times L_2^{\frac{1+3g\varepsilon^2-2g^2\varepsilon^4}{2-g\varepsilon^2}}$
	$K_1 = \varepsilon(2-g\varepsilon^2)x + 1,\ K_2 = \varepsilon(2-g\varepsilon^2)x,$ $K_3 = 1 - x + gx + y$	$\mathcal{R}_{38} = x$

(B_2) The perturbations for proving this point are presented in Table 4 along with the invariant lines and cofactors of the perturbed systems as well as the resulting DIF depending on ε and its limit when ε tends to zero. The results shown in this Table 4 prove the point (B_2).

(B_3) The perturbations of systems 4.30 and 4.43 as well as those for 4.28, 4.32, 4.37 proving this point are shown in Table 3 of [13]. The invariant lines and cofactors of the perturbed systems of 4.30 and 4.43 and their DIF's are shown in Table 5 along with their limits when ε tends to zero. The invariant lines and cofactors of the perturbed systems of 4.28, 4.32, 4.37 and their DIF's are

Table 4: Perturbations of semi-generic systems in QSL_4 into generic systems not of (C-K) type with DIF (C-K) configuration.

Perturbed systems	Invariant lines / Corresponding cofactors	$DIF\ \ \mathcal{R}_i^\varepsilon$ / $\mathcal{R}_i = \lim_{\varepsilon\to 0} \mathcal{R}_i^\varepsilon$
(4.4_ε) $((4.3))$ $\begin{cases} \dot{x} = (x+\varepsilon)(1+gx+\varepsilon y - y), \\ \dot{y} = (y+\varepsilon)(1+(g-1)x+\varepsilon y) \end{cases}$	$L_1 = x+\varepsilon,\ L_2 = x-y,$ $L_3 = y+\varepsilon$	$\mathcal{R}_4^\varepsilon = L_1^{\varepsilon-1} L_2^{-(g+\varepsilon)} L_3^{g-1}$
	$K_1 = 1+gx+(\varepsilon-1)y,$ $K_2 = gx+\varepsilon y+1+\varepsilon,$ $K_3 = (g-1)x+\varepsilon y+1$	$\mathcal{R}_4 = x^{-1}(x-y)^{-g}y^{g-1}$
(4.7_ε) $((4.6))$ $\begin{cases} \dot{x} = gx^2+(\varepsilon+1)xy, \\ \dot{y} = -1+gx+(\varepsilon-1)y-x^2 \\ \quad +gxy+\varepsilon y^2 \end{cases}$	$L_1 = x,\ L_2 = ix+y+1,$ $L_3 = -ix+y+1$	$\mathcal{R}_7^\varepsilon = L_1^{-(1+\varepsilon)} L_2^{(ig+\varepsilon-1)/2} \times$ $L_3^{i(g+i\varepsilon-i)/2}$
	$K_1 = gx+y+\varepsilon y,$ $K_2 = (g+i)x+\varepsilon y-1,$ $K_3 = (g-i)x+\varepsilon y-1$	$\mathcal{R}_7 = x^{-1}(1+ix+y)^{(ig-1)/2} \times$ $(1-ix+y)^{-(1+ig)/2}$
(4.10_ε) $((4.9))$ $\begin{cases} \dot{x} = (x+\varepsilon)^2-1, \\ \dot{y} = (y+g)(y+2gx-g) \end{cases}$	$L_1 = x+\varepsilon+1,\ L_3 = y+g,$ $L_2 = x+\varepsilon-1$	$\mathcal{R}_{10}^\varepsilon = L_1^{2g+g\varepsilon-1} L_2^{-(1+g\varepsilon)} L_3^{-2}$
	$K_1 = x+\varepsilon-1,$ $K_2 = x+\varepsilon+1,$ $K_3 = y+2gx-g$	$\mathcal{R}_{10} = (x+1)^{2g-1}(x-1)^{-1} \times$ $(y+g)^{-2}$
(4.11_ε) $((4.9))$ $\begin{cases} \dot{x} = (x+g)^2-1, \\ \dot{y} = (y+\varepsilon)(y+(1-\varepsilon)x-\varepsilon) \end{cases}$	$L_1 = x+g+1,\ L_3 = y+\varepsilon,$ $L_2 = x+g-1$	$\mathcal{R}_{11}^\varepsilon = L_1^{(g+\varepsilon-g\varepsilon-1)/2} L_3^{-2} \times$ $L_2^{(g\varepsilon-1-g-3\varepsilon)/2}$
	$K_1 = x+g-1,$ $K_2 = x+g+1,$ $K_3 = y+(1-\varepsilon)x-\varepsilon$	$\mathcal{R}_{11} = (x+g+1)^{(g-1)/2} y^{-2} \times$ $(x+g-1)^{-(1+g)/2}$
(4.14_ε) $((4.13))$ $\begin{cases} \dot{x} = (x+g)^2+1, \\ \dot{y} = (y+\varepsilon)(y+(1-\varepsilon)x-\varepsilon) \end{cases}$	$L_1 = x+g+i,\ L_3 = y+\varepsilon,$ $L_2 = x+g-i$	$\mathcal{R}_{14}^\varepsilon = L_1^{i(g\varepsilon-g-(2-i)\varepsilon+i)/2} \times$ $L_3^{-2}\, L_2^{i(g\varepsilon-g-(2+i)\varepsilon-i)/2}$
	$K_1 = x+g-i,$ $K_2 = x+g+i,$ $K_3 = y+(1-\varepsilon)x-\varepsilon$	$\mathcal{R}_{14} = (x+g+i)^{-(ig+1)/2} y^{-2} \times$ $(x+g-i)^{(ig-1)/2}$

shown in Table 6 along with their limits when ε tends to zero. Under these conditions in passing to the limit when ε tends to zero we have a finite limit for the exponents in $R(\varepsilon)$ and hence we obtain a DIF. ∎

It remains to consider the systems in QSL_4 which possess a GDIF which cannot be a DIF and to determine what exactly forces the appearance of an exponential factor in their integrating factors. From Table 1 of [14] (Table I of the Appendix) we see that there are 11 canonical forms for systems possessing a GDIF modulo the group of affine transformations anf time rescaling. All these systems have at least one multiple invariant line of multiplicity $m > 1$ and they turn out to be distinguished from all the preceding ones by the the way they can be perturbed into generic systems with a DIF which cannot be a PIIF. These perturbations are such that they lead to integrating factors $R(\varepsilon)$

Table 5: Systems in QSL_4 with DIF and a multiple affine line

Perturbed systems	Invariant lines / Corresponding cofactors	$DIF\ \mathcal{R}_i^\varepsilon$ / $\mathcal{R}_i = \lim_{\varepsilon\to 0}\mathcal{R}_i^\varepsilon$
Inverse integrating factors of all systems with DIF of type $\{(2,1),(1,2)\}$ with a double affine line which possess perturbations into generic or semi-generic systems with DIF		
$(4.30_\varepsilon)\equiv(IV.30_\varepsilon)$ $\big((4.9)\big)$ [4, Table 3]	$L_1 = x+(g+1+\sqrt{v})/(2g)$, $L_1 = x+(g+1-\sqrt{v})/(2g)$, $L_3 = x+1+\varepsilon y,\ v=(g-1)^2+4g\varepsilon$	$\mathcal{R}_{30}^\varepsilon = L_1^{c_1}L_2^{c_2}L_3^{-2}$, $c_{1,2} = \dfrac{(1-g)}{2g} \pm \dfrac{(v-4g\varepsilon)\sqrt{v}}{2gv}$
	$K_1 = gL_2,\ K_2 = gL_1$, $K_3 = gx-\varepsilon y+1$	$\mathcal{R}_{30} = (x+1)^{-2}(gx+1)^{\frac{1-g}{g}}$
Inverse integrating factors of systems of type $\{(3,1),(1,1)\}$ with the affine line triple which possess perturbations into generic or semi-generic systems with PIIF		
$(4.43_\varepsilon)\equiv(IV.43_\varepsilon)$ $\big((4.9)\big)$ [4, Table 3]	$L_{1,2} = x\pm\varepsilon,\ L_3 = x+g\varepsilon^2 y$	$\mathcal{R}_{43}^\varepsilon = L_1^{\frac{1-g}{2g}}L_2^{\frac{1-g}{2g}}L_3^{-2}$
	$K_{1,2} = g(x\mp\varepsilon), K_3 = g(x-\varepsilon^2 y)$	$\mathcal{R}_{43} = x^{\frac{1-3g}{g}}$

Table 6: Systems in QSL_4 with DIF and the line at infinity multiple

Perturbed systems	Invariant lines / Corresponding cofactors	$DIF\ \mathcal{R}_i^\varepsilon$ / $\mathcal{R}_i = \lim_{\varepsilon\to 0}\mathcal{R}_i^\varepsilon$
Integrating factors of all systems with DIF of type $\{(2,1),(1,2)\}$ with double line at infinity which possess perturbations into generic or semi-generic systems with DIF		
$(4.28_\varepsilon)\equiv(IV.28_\varepsilon)$ $\big((4.9)\big)$ [4, Table 3]	$L_{1,2} = x\pm 1,\ L_3 = 1+\varepsilon y$	$\mathcal{R}_{28}^\varepsilon = L_1^{(\varepsilon+g-2)/2}L_2^{(\varepsilon-g-2)/2}L_3^{-2}$
	$K_1 = x-1,\ K_2 = x+1$, $K_3 = \varepsilon(x+gy)$	$\mathcal{R}_{28} = (x+1)^{(g-2)/2}(x-1)^{-(g+2)/2}$
$(4.32_\varepsilon)\equiv(IV.32_\varepsilon)$ $\big((4.13)\big)$ [4, Table 3]	$L_{1,2} = x\pm i,\ L_3 = 1+\varepsilon y$	$\mathcal{R}_{32}^\varepsilon = L_1^{(\varepsilon-gi-2)/2}L_2^{(\varepsilon+gi-2)/2}L_3^{-2}$
	$K_1 = x-i,\ K_2 = x+i$, $K_3 = \varepsilon(x+gy)$	$\mathcal{R}_{32} = (x+i)^{-(ig+2)/2}(x-i)^{(gi-2)/2}$
Integrating factors of systems with DIF of type $\{(3,1),(1,1)\}$ with triple line at infinity which possess perturbations into generic or semi-generic systems with DIF		
$(4.37_\varepsilon)\equiv(IV.37_\varepsilon)$ $\big((4.9)\big)$ [4, Table 3]	$L_1 = x,\ L_2 = 1+2\varepsilon x$, $L_3 = \varepsilon x+\varepsilon^2 y+1$	$\mathcal{R}_{37}^\varepsilon = L_1^{-(1+g)}L_2^{(g-1)/2}L_3^{-2}$
	$K_1 = 1+2\varepsilon x,\ K_2 = 2\varepsilon x$, $K_3 = \varepsilon(g\varepsilon y+x)$	$\mathcal{R}_{37} = x^{-(g+1)}$

which in the passage to the limit as ε tends to sero, forces the appearance of an exponential factor into the integrating factor $R(\varepsilon)$ which becomes a GDIF.
We split the discussion in two cases: one in which the multiple line/lines is/are affine and the case when the multiple line at infinity is multiple (Table 8).

Table 7: Systems in QSL_4 with GDIF and a multiple affine line

Perturbed systems	Invariant lines / Corresponding cofactors	$DIF\ \mathcal{R}_i^\varepsilon$ / $\mathcal{R}_i = \lim_{\varepsilon\to 0} \mathcal{R}_i^\varepsilon$
Integrating factors of all systems of type $\{(2,1),(1,2)\}$ with the double line affine which possess perturbations into generic or semi-generic systems with DIF		
$(4.21_\varepsilon) \equiv (IV.21_\varepsilon)$ $\big((4.4)\big)$ [4, Table 3]	$L_1 = x, L_2 = x+\varepsilon y+\varepsilon,\ L_3 = y+1,$	$\mathcal{R}_{21}^\varepsilon = L_1^{-(\varepsilon+1)/\varepsilon} L_2^{(1-g\varepsilon)/\varepsilon} L_3^{g-1}$
	$K_1 = gx+\varepsilon y+\varepsilon+y,\ K_2 = gx+y,$ $K_3 = -x+gx+y$	$\mathcal{R}_{21} = x^{-(g+1)} e^{\frac{y+1}{x}} y^{g-1}$
$(4.22_\varepsilon) \equiv (IV.22_\varepsilon)$ $\big((4.9)\big)$ [4, Table 3]	$L_{1,2} = x \pm \varepsilon, L_3 = y+1$	$\mathcal{R}_{22}^\varepsilon = L_1^{\frac{2-g\varepsilon-\varepsilon}{2g\varepsilon}} L_2^{-\frac{g\varepsilon+\varepsilon+2}{2g\varepsilon}} L_3^{-2}$
	$K_{1,2} = g(x \mp \varepsilon),\ K_3 = y-1-x+gx$	$\mathcal{R}_{22} = x^{-\frac{g+1}{g}} e^{\frac{2}{gx}} (y+1)^{-2}$
$(4.23_\varepsilon) \equiv (IV.23_\varepsilon)$ $\big((4.11)\big)$ [4, Table 3]	$L_1 = x,\ L_{2,3} = y+1 \pm \varepsilon$	$\mathcal{R}_{23}^\varepsilon = L_1^{-2} L_2^{\frac{1-\varepsilon}{2\varepsilon}} L_3^{-\frac{1+\varepsilon}{2\varepsilon}}$
	$K_1 = x+y,\ K_{2,3} = y+1 \mp \varepsilon$	$\mathcal{R}_{23} = x^{-2}(y+1)^{-1} e^{\frac{1}{y+1}}$
$(4.26_\varepsilon) \equiv (IV.26_\varepsilon)$ $\big((4.3)\big)$ [4, Table 3]	$L_1 = x,\ L_2 = x+\varepsilon y+\varepsilon,\ L_3 = y+1$	$\mathcal{R}_{26}^\varepsilon = L_1^{-(1+\varepsilon)/\varepsilon} L_2^{1/\varepsilon} L_3^{-1}$
	$K_1 = \varepsilon+y+\varepsilon y,\ K_2 = y,\ K_3 = y-x$	$\mathcal{R}_{26} = x^{-1} e^{(y+1)/x} (y+1)^{-1}$
Integrating factors of all systems with GDIF of type $\{(2,2)\}$ with one affine and one infinite double lines which possess perturbations into generic or semi-generic systems with DIF		
$(4.39_\varepsilon) \equiv (IV.39_\varepsilon)$ $\big((4.9)\big)$ [4, Table 3]	$L_{1,2} = x \pm \varepsilon,\ L_3 = 1+\varepsilon y$	$\mathcal{R}_{39}^\varepsilon = L_1^{\frac{1+\varepsilon^2-2\varepsilon}{2\varepsilon}} L_2^{\frac{\varepsilon^2-2\varepsilon-1}{2\varepsilon}} L_3^{-2}$
	$K_{1,2} = x \mp \varepsilon,\ K_3 = \varepsilon(x+gy)$	$\mathcal{R}_{39} = x^{-2} e^{1/x}$
$(4.40_\varepsilon) \equiv (IV.40_\varepsilon)$ $\big((4.9)\big)$ [4, Table 3]	$L_1 = x+(1+\varepsilon-u)/(2\varepsilon),$ $L_2 = x+\varepsilon y+1,$ $L_3 = \varepsilon x+(1+\varepsilon+u)/2$	$\mathcal{R}_{40}^\varepsilon = L_1^{c_1} L_2^{-2} L_3^{c_2},$ $c_{1,2} = \frac{1-\varepsilon}{2\varepsilon} \mp \frac{(1-\varepsilon)^2}{2\varepsilon u^2}\sqrt{u},$ $u = 5\varepsilon^2-2\varepsilon+1$
	$K_1 = L_2,\ K_2 = \varepsilon L_1,$ $K_3 = \varepsilon x-\varepsilon y+1$	$\mathcal{R}_{40} = (x+1)^{-2} e^x$
Integrating factors of all systems with GDIF of type $\{(3,1),(1,1)\}$ with the triple line affine which possess perturbations into generic or semi-generic systems with DIF		
$(4.41_\varepsilon) \equiv (IV.41_\varepsilon)$ $\big((4.3)\big)$ [4, Table 3]	$L_1 = x,\ L_2 = x+\frac{1}{2}g\varepsilon y+\frac{1}{2}\varepsilon,$ $L_3 = x-2g\varepsilon y-2\varepsilon$	$\mathcal{R}_{41}^\varepsilon = L_1^{\frac{1}{g\varepsilon^2}} L_2^{-\frac{4(g\varepsilon^2+1)}{5g\varepsilon^2}} L_3^{-\frac{6g\varepsilon^2+1}{5g\varepsilon^2}}$
	$K_1 = g(\varepsilon x+2y)/2, K_2 = g(g\varepsilon^2 y+\varepsilon^2+y)$ $K_3 = g(2g\varepsilon^2 y+2\varepsilon^2+5\varepsilon x+2y)/2$	$\mathcal{R}_{41} = x^{-2} e^{g(y+g)^2/(2x^2)}$

Theorem 4.3. *(C) The systems in QSL_4 with a GDIF possessing a multiple affine invariant line. Each one of these systems can be perturbed into generic ones (not of (C-K) type) having a DIF $R(\varepsilon)$ tending to a GDIF when ε tends to zero. The expressions of perturbations depend on the multiplicities of invariant lines appearing in their configurations. We group them in Table 7 according to their multiplicities and to the number of invariant multiple lines.*

$(C_{a,(2,1)})$: *Only one multiple affine invariant line which is double:*
The exponents in $R(\varepsilon)$ of the lines that split from the double line are of the form $\frac{1}{\varepsilon}(c_1\varepsilon + c_2)$, $c_{1,2}$ constants and $c_2 \neq 0$ and they appear in Table 7.
$(C_{a,(2,2)})$: *Two lines of multiplicity 2 each, one affine and the other one at infinity (Table 7):*
In this case we have only two normal forms and they are without parameters: 4.39 and 4.40. Two of the exponents in $R(\varepsilon)$ are of the form $\frac{1}{\varepsilon}F(\varepsilon)$ where $F(\varepsilon)$ belongs to a quadratic extension of the ring $\mathbb{Q}(\varepsilon)$ which tends to a finite nonzero limit when ε tends to zero and the thirs one is a constant.
$(C_{a,(3,1)})$: *A single multiple affine line of multiplicity 3. This occurs in only one canonical form 4.41 depending on one parameter.*
The exponents in $R(\varepsilon)$ are of the form $\frac{1}{\varepsilon^2}(c_1\varepsilon^2 + c_2)$, $c_{1,2}$ constants and $c_2 \neq 0$.

Proof: Assume that a system in QSL_4 possesses a GDIF. Then modulo the group action it must be, one of the 11 families with GDIF from the 46 families in QSL_4 appearing in [14]. They are: 4.16, 4.17, 4.21, 4.22, 4.23, 4.26, 4.34, 4.39, 4.40, 4.41, 4.46. Since the system has only one affine invariant line which is double and the line at infinity is simple then according to the configurations exhibited in Diagram 1 of [14] (here in the Appendix) it must be one of the seven systems 4.21, 4.22, 4.23, 4.26, 4.39, 4.40, 4.41. A simple calculation using the method of Darboux gives us the DIF $R(\varepsilon)$ which we see in Table 7 and also by the same method, the GDIF R of the unperturbed system previously listed in Table 1 of [14]. We know that $lim_{\varepsilon\to 0}(R(\varepsilon)) = R$. But by directly calculate this limit we see how exactly the exponential factor in GDIFs are produced. We do this for the system 4.21 with a double affine line. The integrating factor of the perturbation which appear for 4.21 in Table 7 is:

$$R_{21}(\varepsilon) = x^{-\frac{\varepsilon+1}{\varepsilon}}(x+\varepsilon y+\varepsilon)^{\frac{1-g\varepsilon}{\varepsilon}}(y+1)^{g-1}$$

This gives:

$$R_{21}(\varepsilon) = x^{-\frac{\varepsilon+1}{\varepsilon}}x^{\frac{1-g\varepsilon}{\varepsilon}}((1+\varepsilon(\frac{y+1}{x})^{\frac{1}{\varepsilon}})^{1-g\varepsilon} = x^{-(g+1)}((1+\varepsilon(\frac{y+1}{x})^{\frac{1}{\varepsilon}})^{1-g\varepsilon}(y+1)^{g-1}$$

And hence we have:

$$lim_{\varepsilon\to 0}(R_{21}(\varepsilon) = x^{-(g+1)}e^{\frac{y+1}{x}}(y+1)^{g-1}.$$

In similar ways calculations of the limits of $R(\varepsilon)$ for the other perturbations of systems with a GDIF and a multiple invariant affine line show analytically how the exponential factor is generated. We have the results in Table 7.

Theorem 4.4. *(D) The systems in QSL_4 with a GDIF and with the line at infinity multple.*
Each one of these systems admits a perturbation into generic systems in QSL_4 which are not of (C-K) type, having a DIF $R(\varepsilon)$ which tends to a GDIF when the perturbation parameter tends to zero. The expressions of perturbations depend on the multiplicity of the line at infinity and we group the results into Table 8 accordingly.
$(D_{\infty,2})$: *The line at infinity is double and there are two simple affine invariant lines. There are only two normal forms for these systems: 4.16 and 4.17.*
The perturbations of 4.16 and 4.17 are within 4.9 and 4.10 respectively [13]. The line at infinity splits into itself and an affine invariant line called L_2, in both linear in ε in both normal forms 4.9 and 4.10 and its exponents in the two $R(\varepsilon)$ are of the form $\frac{1}{\varepsilon}F(\varepsilon, g)$ where $F(\varepsilon, g) \in \mathbb{Q}(\varepsilon, g)$ for 4.17 and 4.16 and $F(\varepsilon, g)$ tends to a finite limit when $\varepsilon \to 0$.

$(D_{\infty,3})$: *The line at infinity is triple and we have a single affine invariant line which is simple. There is only one normal form for this case: 4.34. The perturbation of 4.34 is within systems with normal form 4.9. In the perturbation the line at infinity splits into two distinct invariant affine lines linear in* ε *and their exponents in* $R(\varepsilon)$ *are of the form* $\frac{1}{\varepsilon^2}(c_1\varepsilon^2 + c_2)$, $c_{1,2}$ *constants and* $c_i \neq 0$.
$(D_{\infty,4})$: *the line at infinity is the only invariant line and it is of multiplicity 4.*
There is only one normal form for these systems in [14] which is 4.46 and the systems with this form admit a polynomial perturbation within systems with normal form 4.13. In the perturbation the line at infinity splits into itself and three affine invariant lines with coefficients in $\mathbb{Q}[\sqrt{4-\varepsilon}]$ *and their exponents in* $R(\varepsilon)$ *are of the form.* $F_1(\varepsilon) + iF_2(\varepsilon)\sqrt{\varepsilon(4-\varepsilon)}$, $F_{1,2} \in \mathbb{Q}(\varepsilon)$, $F_{1,2} \neq 0$ *and they both have a finite limit when* ε *tends to zero.*

Table 8: Systems in QSL_4 with GDIF and the line at infinity multiple

Perturbed systems	Invariant lines / Corresponding cofactors	DIF $\mathcal{R}_i^\varepsilon$ / $\mathcal{R}_i = \lim_{\varepsilon\to 0}\mathcal{R}_i^\varepsilon$
Integrating factors of all systems with GDIF of type $\{(2,1),(1,2)\}$ *with the double line at infinity which possess perturbations into generic or semi-generic systems with DIF*		
$(4.16_\varepsilon) \equiv (IV.16_\varepsilon)$ $((4.9))$ [4, Table 3]	$L_1 = x+g,\ L_2 = \varepsilon x + 1,\ L_3 = y$	$\mathcal{R}_{16}^\varepsilon = L_1^{\frac{g\varepsilon+g-1}{1-g\varepsilon}} L_2^{\frac{g\varepsilon^2-\varepsilon-1}{\varepsilon(1-g\varepsilon)}} L_3^{-2}$
	$K_1 = \varepsilon x + 1, K_2 = \varepsilon(x+g),$ $K_3 = y - x$	$\mathcal{R}_{16} = e^{-x}(x+g)^{g-1}y^{-2}$
$(4.17_\varepsilon) \equiv (IV.17_\varepsilon)$ $((4.10))$ [4, Table 3]	$L_1 = x,\ L_2 = \varepsilon x + 1,\ L_3 = y$	$\mathcal{R}_{17}^\varepsilon = L_1^{-1} L_2^{-(\varepsilon+1)/\varepsilon} L_3^{-2}$
	$K_1 = \varepsilon x + 1,\ K_2 = \varepsilon x,\ K_3 = y - x$	$\mathcal{R}_{17} = x^{-1}e^{-x}y^{-2}$
Integrating factors of systems with GDIF of type $\{(3,1),(1,1)\}$ *with the triple line at infinity which possess perturbations into generic or semi-generic systems with DIF*		
$(4.34_\varepsilon) \equiv (IV.34_\varepsilon)$ $((4.9))$ [4, Table 3]	$L_{1,2} = \varepsilon x \pm 1,\ L_3 = y$	$\mathcal{R}_{34}^\varepsilon = L_1^{\frac{1-2g\varepsilon^2}{2g\varepsilon^2}} L_2^{\frac{1-2g\varepsilon^2}{2g\varepsilon^2}} L_3^{-2}$
	$K_{1,2} = \pm g\varepsilon(\varepsilon x \mp 1), K_3 = y - x$	$\mathcal{R}_{34} = y^{-2}e^{-\frac{x^2}{2g}}$
Integrating factors of systems with GDIF of type $\{(4,1)\}$ *with the line at infinity of multiplicty 4 which possess perturbations into generic or semi-generic systems with DIF*		
$(4.46_\varepsilon) \equiv (IV.46_\varepsilon)$ $((4.13))$ [4, Table 3]	$L_1 = 2\sqrt{\varepsilon}x + i\sqrt{4-\varepsilon} + \sqrt{\varepsilon},$ $L_2 = -2\sqrt{\varepsilon}x + i\sqrt{4-\varepsilon} - \sqrt{\varepsilon},$ $L_3 = \varepsilon x + \varepsilon^2 y + 1$	$\mathcal{R}_{46}^\varepsilon = L_1^{c_1} L_2^{c_3} L_3^{-2},$ $c_{1,2} = \frac{\varepsilon-3}{2} \mp i\frac{\varepsilon^2-5\varepsilon+2}{2u}\sqrt{u},$ $u = \varepsilon(4-\varepsilon)$
	$K_1 = -\frac{1}{2}\sqrt{\varepsilon}L_2,\ K_2 = \frac{1}{2}\sqrt{\varepsilon}L_1,$ $K_3 = \varepsilon(1+\varepsilon y - \varepsilon^2 y)$	$\mathcal{R}_{46} = e^{-x}$

Proof: $(D_{\infty,2})$: Assume that the system possesses a GDIF with the line at infinity double. Then modulo the group action the system must be one of the remaining 4 families with GDIF i.e. 4.16, 4.17, 4.34 and 4.46 in Table 1 of [14]. From Diagram 1 in [14] we see that among the four systems only 4.16 and 4.17 are with an affine double line. The two corresponding perturbations are given in Table 1 of [13] and appear here in Table 8. The DIFs $R(\varepsilon)$ of the perturbations are calculated using the theory of Darboux just like those exhibited previously for QSL_6 and QSL_5 and they appear here in Table 8. Clearly the GDIFs can be calculated directly by using the theory of Darboux. But they can also be calculated by taking its limit as ε tends to zero. This second way of obtaining R elucidates analytically how the exponential factor appears and we do this calculation here. We consider the DIF $R(\varepsilon)$ for a system 4.16 and we denote it in Table 8 by $R_{16}^{\varepsilon} = L_1^{\frac{g\varepsilon+g-1}{1-g\varepsilon}} L_2^{\frac{g\varepsilon^2-\varepsilon-1}{\varepsilon(1-g\varepsilon)}} L_3^{-2}$, Since the lines L_1 and L_2 are independent of ε and their exponents in R_{16}^{ε} tend respectively to $g-1$ and -2, we only need to calculate the limit of the second factor appearing in $R(\varepsilon)$ i.e.

$$lim_{\varepsilon\to 0}(\varepsilon x+1)^{\frac{g\varepsilon^2-\varepsilon-1}{\varepsilon(1-g\varepsilon)}} = lim_{\varepsilon\to 0}[(\varepsilon x+1)^{\frac{1}{\varepsilon}}]^{\frac{g\varepsilon^2-\varepsilon-1}{1-g\varepsilon}} = e^{-x}.$$

A simple analogous calculation can be done for 4.17.
($D_{\infty,3}$): The systems in QSL_4 that have a triple line at infinity can be brought modulo the group action to a single normal form 4.34. Each system in this family admits a polynomial perturbation into systems with normal form 4.9 [13]. The triple line at infinity splits into itself and two affine lines $L_{1,2} = 1 \pm \varepsilon x$. Calculations using the method of Darboux lead to an integrating factor which is a DIFs $R(\varepsilon)$ appearing here in Table 8 as $R_{34}^{\varepsilon} = L_1^{\frac{1-2g\varepsilon}{2g\varepsilon^2}} L_2^{\frac{1-2g\varepsilon}{2g\varepsilon^2}} L_3^{-2}$. The GDIF R of a system in 4.34 was calculated using the method of Darboux in [14]. It can also be calculated easily by taking the limit of $R(\varepsilon)$ as ε tends to zero obtaining R_{17} in Table 8. Below we do this calculation.

$$lim_{\varepsilon\to 0}(R_{34}^{\varepsilon}) = (1+\varepsilon x)^{\frac{1-2g\varepsilon}{2g\varepsilon^2}}(1-\varepsilon x)^{\frac{1-2g\varepsilon}{2g\varepsilon^2}} = [(1+\varepsilon^2(-x^2))^{\frac{1}{\varepsilon^2}}]^{\frac{1-2g\varepsilon}{2g}} = e^{-\frac{x^2}{2g}}$$

($D_{\infty,4}$): The systems in QSL_4 with the line at infinity of multiplicity 4 occur only with the cofiguration Config 4.46 in Diagram 1 of [14] and their normal form modulo the group action appears as 4.46 in Table 1 in [14]. Any system with this configuration admits a polynomial perturbation within systems with configuration and normal form 4.13 and its expression is given in Table 1 of [13]. The three affine lines that split from the quadruple line at infinity in the perturbed systems are linear in both $\sqrt{\varepsilon}$ and $\sqrt{4-\varepsilon}$ and they appear in the 2nd column of Table 8 while the DIF $R(\varepsilon)$ of the perturbed system appears in Table 8 denoted by R_{46}^{ε}. It was computed algebraically by the method of Darboux [4] and it appears in Table 8 together with the GDIF R_{46} of systems 4.46. It can be computed by the extended method of Darboux using also exponential factors and it can also be computed directly by taking the limit of R_{46}^{ε} as ε tends to zero but the calculation here is a bit more complicated than in the preceding cases.

∎

Appendix

In this section we give a slightly modified version of Table 1 of [14] which contains the canonical forms from (4.1) to (4.46) of systems in QSL_4 modulo the group action. The Table 1 also contains the results of the first calculations of integrating factors for this family. In addition we give here the Diagram 1 from [14] which has all the configurations for QSL_4. The reader is advised to only look at the configurations and ignore the various geometric invariants of the left sides of the pages

Table I

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$ / First integral $\mathcal{F}_i$
(4.1) $\begin{cases}\dot{x} = gx + gx^2 + (h-1)xy,\\ \dot{y} = -hy + (g-1)xy + hy^2,\end{cases}$ $(g,h) \in \mathbb{R}^2$, $gh(g+h-1) \neq 0$, $(g-1)(h-1)(g+h) \neq 0$	$x\,(1)$, $y\,(1)$, $x - y + 1\,(1)$	$\mathcal{R}_1 = xy(x-y+1)$
	$g(x+1)+y(h-1)$, $x(g-1)+h(y-1)$, $gx+hy$	$\mathcal{F}_1 = x^h y^g (1+x-y)^{1-g-h}$
(4.2) $\begin{cases}\dot{x} = gx^2 + (h+1)xy,\\ \dot{y} = h[g^2+(h+1)^2]+2ghy-x^2\\ \quad +(g^2+1-h^2)x+gxy+hy^2,\end{cases}$ $(g,h) \in \mathbb{R}^2$, $h(h+1) \neq 0$, $g^2+(h-1)^2 \neq 0$	$x\,(1)$, $\pm i(y+g)+$ $x-h-1\,(1)$	$\mathcal{R}_2 = x\big[(y+g)^2+$ $(x-h-1)^2\big]$
	$gx+(h+1)y$, $\mp i(x+h+h^2)+$ $g(x+h)+hy$	$\mathcal{F}_2 = x^{-2h}(\alpha_+)^{\beta_-}(\alpha_-)^{\beta_+}$, $\alpha_\pm = x-h-1 \pm i(y+g)$, $\beta_\pm = (x+1) \pm ig$
(4.3) $\begin{cases}\dot{x} = x + gx^2 + (h-1)xy,\\ \dot{y} = y + (g-1)xy + hy^2,\end{cases}$ $(g,h) \in \mathbb{R}^2$, $gh(g+h-1) \neq 0$, $(g-1)(h-1)(g+h) \neq 0$	$x\,(1)$, $y\,(1)$, $x-y\,(1)$	$\mathcal{R}_3 = x^{1-h}y^{1-g} \times$ $(x-y)^{g+h}$
	$gx+1+y(h-1)$, $x(g-1)+hy+1$, $gx+hy+1$	
(4.4) $\begin{cases}\dot{x} = x + gx^2 - xy,\\ \dot{y} = y + (g-1)xy,\end{cases}$ $g \in \mathbb{R}$, $g(g-1) \neq 0$	$x\,(1)$, $x-y\,(1)$, $y\,(1)$	$\mathcal{R}_4 = xy^{1-g}(x-y)^g$
	$gx+1-y$, $gx+1$ $x(g-1)+1$	
(4.5) $\begin{cases}\dot{x} = gx^2 + (h-1)xy,\\ \dot{y} = (g-1)xy + hy^2,\end{cases}$ $(g,h) \in \mathbb{R}^2$, $gh(g+h-1) \neq 0$, $(g-1)(h-1)(g+h) \neq 0$	$x\,(1)$, $x-y\,(1)$, $y\,(1)$	$\mathcal{R}_5 = xy(x-y)$
	$gx+y(h-1)$, $gx+hy$, $x(g-1)+hy$	$\mathcal{F}_5 = x^h y^g (y-x)^{1-g-h}$
(4.6) $\begin{cases}\dot{x} = gx^2 + (h+1)xy,\\ \dot{y} = -1 + gx + (h-1)y\\ \quad -x^2+gxy+hy^2,\ (g,h) \in \mathbb{R}^2,\end{cases}$ $h(h+1)\big[g^2+(h-1)^2\big] \neq 0$	$x\,(1)$, $\mathcal{I}_\pm = x\pm$ $i(y+1)\,(1)$	$\mathcal{R}_6 = \mathcal{I}_+^{(1-h-ig)/2} \times$ $\mathcal{I}_-^{(1-h+ig)/2} x^{h+1}$
	$gx+(h+1)y$, $\pm ix+1+gx+hy$	
(4.7) $\begin{cases}\dot{x} = gx^2 + xy, \quad g \in \mathbb{R},\\ \dot{y} = -1 + gx - y - x^2 + gxy\end{cases}$	$x\,(1)$, $\mathcal{I}'_\pm = x\pm$ $i(y+1)\,(1)$	$\mathcal{R}_7 = \mathcal{I}'^{\,(1-ig)/2}_+ \times$ $\mathcal{I}'^{\,(1+ig)/2}_- \, x$
	$gx+y$, $\pm ix+1+gx$	
(4.8) $\begin{cases}\dot{x} = gx^2+(h+1)xy,\ (g,h) \in \mathbb{R}^2,\\ \dot{y} = -x^2 + gxy + hy^2,\end{cases}$ $h(h+1)\big[g^2+(h-1)^2\big] \neq 0$	$x\,(1)$, $x \pm iy\,(1)$	$\mathcal{R}_8 = x(x^2+y^2)$
	$gx+(h+1)y$, $\pm ix+gx+hy$	$\mathcal{F}_8 = x^{-2h}(x+iy)^{h+1-ig}$ $\times(x-iy)^{h+1+ig}$

as they are not necessary for this work. They served their purpose in [14] as they ensured that the 46 configurations presented in [14] are indeed distinct but these invariants are redundant here.

We point out that in Diagram 1 of [14] there are 46 configurations of invariant lines. However, it

Table I (*continued*)

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$
(4.9) $\begin{cases}\dot{x}=x^2-1,\ (g,h)\in\mathbb{R}^2,\\ \dot{y}=(y+h)[y+(1-g)x-h],\end{cases}$ $g(g-1)[(g\pm1)^2-4h^2]\neq0$	$y+h\,(1)$, $\mathcal{I}''_{\pm}=x\pm1\,(1)$ / $x(1-g)+y-h$, $x\mp1$	$\mathcal{R}_9=(y+h)^2\times$ $\mathcal{I}''^{(g+1-2h)/2}_{+}\times$ $\mathcal{I}''^{(g+1+2h)/2}_{-}$
(4.10) $\begin{cases}\dot{x}=x^2-1,\quad g\in\mathbb{R},\\ \dot{y}=(y+g)(y+2gx-g),\end{cases}$ $g(2g-1)\neq0$	$y+g\,(1)$, $x\pm1\,(1)$ / $2gx+y-g$, $x\mp1$	$\mathcal{R}_{10}=(x+1)^{1-2g}\times$ $(y+h)^2(x-1)$
(4.11) $\begin{cases}\dot{x}=(x+g)^2-1,\quad g\in\mathbb{R},\\ \dot{y}=y(x+y),\qquad g\neq\pm1\end{cases}$	$y\,(1)$, $\mathcal{I}'''_{\pm}=x+$ $+g\pm1\,(1)$ / $x+y$, $x+g\mp1$	$\mathcal{R}_{11}=\mathcal{I}'''^{\,(1-g)/2}_{+}\times$ $\mathcal{I}'''^{\,(1+g)/2}_{-}y^2$
(4.12) $\begin{cases}\dot{x}=(x+h)^2-1,\ (g,h)\in\mathbb{R}^2,\\ \dot{y}=(1-g)xy,\ g(g-1)\neq0,\end{cases}$ $(h^2-1)\big[\ h^2(g-1)^2-(g+1)^2\big]\neq0$	$y\,(1)$, $x+h\pm1\,(1)$ / $x+y$, $x+h\mp1$	$\mathcal{R}_{12}=(x+h+1)\times$ $(x+h-1)y$ / $\mathcal{F}_{12}=y^{2/(g-1)}(x+h+1)^{1+h}$ $\times(x+h-1)^{1-h}$
(4.13) $\begin{cases}\dot{x}=x^2+1,\quad (g,h)\in\mathbb{R}^2,\\ \dot{y}=(y+h)[y+(1-g)x-h],\end{cases}$ $g(g-1)\big[(g+1)^2+h^2\big]\neq0$	$y+h\,(1)$, $x\pm i\,(1)$ / $x(1-g)+y-h$, $x\mp i$	$\mathcal{R}_{13}=(y+h)^2\times$ $(x+i)^{(1+g+2ih)/2}\times$ $(x-i)^{(1+g-2ih)/2}$
(4.14) $\begin{cases}\dot{x}=(x+g)^2+1,\\ \dot{y}=y(x+y),\quad g\in\mathbb{R}\end{cases}$	$y\,(1)$, $x+g\pm i\,(1)$ / $x+y$, $x+g\mp i$	$\mathcal{R}_{14}=y^2\times$ $(x+g+i)^{(1+ig)/2}\times$ $(x+g-i)^{(1-ig)/2}$
(4.15) $\begin{cases}\dot{x}=(x+h)^2+1,\\ \dot{y}=(1-g)xy,\quad (g,h)\in\mathbb{R}^2,\end{cases}$ $g(g-1)\big[(g+1)^2+h^2\big]\neq0$	$y\,(1)$, $x+h\pm i$, (1) / x, $x+h\mp i$	$\mathcal{R}_{15}=y\ \big[(x+h)^2+1\big]$ / $\mathcal{F}_{15}=y^{2/(g-1)}(x+h+i)^{1-ih}$ $\times(x+h-i)^{1+ih}$
(4.16) $\begin{cases}\dot{x}=g+x,\qquad g\in\mathbb{R},\\ \dot{y}=y(y-x),\ \ g(g-1)\neq0\end{cases}$	$x+g\,(1)$, $y\,(1)$ / 1, $y-x$	$\mathcal{R}_{16}=e^xy^2(x+g)^{1-g}$
(4.17) $\begin{cases}\dot{x}=x,\\ \dot{y}=y(y-x)\end{cases}$	$x\,(1)$, $y\,(1)$ / 1, $y-x$	$\mathcal{R}_{17}=x\,e^xy^2$
(4.18) $\begin{cases}\dot{x}=g(g+1)+gx+y,\ \ g\in\mathbb{R},\\ \dot{y}=y(y-x),\ \ g(g+1)\neq0\end{cases}$	$y\,(1)$, $x-y+g+1\,(1)$ / $y-x$, $y+g$	$\mathcal{R}_{18}=y(x-y+g+1)$ / $\mathcal{F}_{18}=e^xy^g(x-y+g+1)^{-g-1}$

was shown in [15] that two configurations were omitted in [14]. These are denoted by *Config. 4.9a* (see [15, Remark 6.2]) and *Config. 4.11a* (see [15, Remark 6.6]). These two configurations are pictured in Figure 1.

We note that for both configurations *Config. 4.9* and *Config. 4.9a* we have the same canonical form and the same expression for the DIF depending on two parameters g, h.

Table I (*continued*)

Orbit representative	Invariant lines and their multiplicities	Inverse integrating factor $\mathcal{R}_i$
	Respective cofactors	
(4.19) $\begin{cases}\dot{x}=g+x, & g\in\mathbb{R},\\ \dot{y}=-xy, & g(g-1)\neq 0\end{cases}$	$x+g\,(1),\quad y\,(1)$	$\mathcal{R}_{19}=y(x+g)$
	$1,\quad -x$	$\mathcal{F}_{19}=e^{x}y(x+g)^{-g}$
(4.20) $\begin{cases}\dot{x}=x(gx+y), & g\in\mathbb{R},\\ \dot{y}=(g-1)xy+y^2, & g(g-1)\neq 0\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{20}=x^2y$
	$gx+y,\ x(g-1)+y$	$\mathcal{F}_{20}=x^{1-g}\,y^{g}\,e^{y/x}$
(4.21) $\begin{cases}\dot{x}=x(gx+y),\ g(g-1)\neq 0,\\ \dot{y}=(y+1)(gx-x+y),\ g\in\mathbb{R}\end{cases}$	$x\,(2),\quad y+1\,(1)$	$\mathcal{R}_{21}=x^{g+1}\times$ $e^{-(gx+y+1)/x}\times$ $(y+1)^{1-g}$
	$gx+y,\ x(g-1)+y$	
(4.22) $\begin{cases}\dot{x}=gx^2,\ g\in\mathbb{R},\ g(g-1)\neq 0,\\ \dot{y}=(y+1)[y+(g-1)x-1]\end{cases}$	$x\,(2),\quad y+1\,(1)$	$\mathcal{R}_{22}=x^{(g+1)/g}\times$ $(y+1)^2e^{-2/(gx)}$
	$x,\quad x(g-1)+y-1$	
(4.23) $\begin{cases}\dot{x}=x^2+xy,\\ \dot{y}=(y+1)^2\end{cases}$	$x\,(1),\quad y+1\,(2)$	$\mathcal{R}_{23}=x^2(y+1)\times$ $e^{-1/(y+1)}$
	$x+y,\ y+1$	
(4.24) $\begin{cases}\dot{x}=(x+1)^2, & g\in\mathbb{R},\\ \dot{y}=(1-g)xy, & g(g-1)\neq 0\end{cases}$	$x+1\,(2),\quad y\,(1)$	$\mathcal{R}_{24}=(x+1)^2\,y$
	$x+1,\quad x$	$\mathcal{F}_{24}=(x+1)^{g-1}\,y\times$ $e^{(g-1)/(x+1)}$
(4.25) $\begin{cases}\dot{x}=gx^2+xy, & g(g-1)\neq 0,\\ \dot{y}=y+(g-1)xy+y^2, & g\in\mathbb{R}\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{25}=x^2\,y$
	$gx+y,\ x(g-1)+y+1$	$\mathcal{F}_{25}=x^{1-g}\,y^{g}\,e^{(y+1)/x}$
(4.26) $\begin{cases}\dot{x}=xy,\\ \dot{y}=(y+1)(y-x)\end{cases}$	$x\,(2),\quad y\,(1)$	$\mathcal{R}_{26}=x(y+1)\times$ $e^{-(y+1)/x}$
	$y,\quad y-x$	
(4.27) $\begin{cases}\dot{x}=2gx+2y, & g\in\mathbb{R},\\ \dot{y}=g^2+1-x^2-y^2\end{cases}$	$y+g\pm i(x-1)\,(1)$	$\mathcal{R}_{27}=(x-1)^2+$ $(y+g)^2$
	$g-y\pm i(x+1)$	$\mathcal{F}_{27}=e^{x}(\alpha_-)^{1-ig}(\alpha_+)^{1+ig}$, $\alpha_\pm=y+g\pm i(x-1)$
(4.28) $\begin{cases}\dot{x}=x^2-1, & g\in\mathbb{R},\\ \dot{y}=x+gy, & g(g^2-4)\neq 0\end{cases}$	$x+1\,(1),\ x-1\,(1)$	$\mathcal{R}_{28}=(x-1)^{1+g/2}\times$ $(x+1)^{1-g/2}$
	$x-1,\ x+1$	
(4.29) $\begin{cases}\dot{x}=x^2-1, & g\in\mathbb{R},\\ \dot{y}=g+x, & g\neq\pm 1\end{cases}$	$x+1\,(1),\ x-1\,(1)$	$\mathcal{R}_{29}=x^2-1$
	$x-1,\ x+1$	$\mathcal{F}_{29}=e^{2y}\,(x-1)^{-1-g}$ $\times(x+1)^{-1+g}$
(4.30) $\begin{cases}\dot{x}=(x+1)(gx+1),\ g\in\mathbb{R},\\ \dot{y}=1+(g-1)xy,\ g(g^2-1)\neq 0\end{cases}$	$x+1\,(2),\ gx+1\,(1)$	$\mathcal{R}_{30}=(x+1)^2\times$ $(gx+1)^{(g-1)/g}$
	$gx+1,\ x+1$	
(4.31) $\begin{cases}\dot{x}=x(x+1),\ g\in\mathbb{R},\\ \dot{y}=g-x^2+xy,\ g(g+1)\neq 0\end{cases}$	$x+1\,(2),\ x\,(1)$	$\mathcal{R}_{31}=(x+1)^2\times$ $(gx+1)^{(g-1)/g}$
	$x,\ x+1$	$\mathcal{F}_{31}=x^{-g}(x+1)^{1+g}\,e^{\frac{y-g+1}{x+1}}$

Table I (*continued*)

Orbit representative	Invariant lines and their multiplicities	Respective cofactors	Inverse integrating factor $\mathcal{R}_i$
(4.32) $\begin{cases}\dot{x}=x^2+1, & g\in\mathbb{R},\\ \dot{y}=x+gy, & g\neq 0\end{cases}$	$x\pm i\,(1)$	$x\mp i$	$\mathcal{R}_{32}=(x+i)^{1+ig/2}\times (x-i)^{1-ig/2}$
(4.33) $\begin{cases}\dot{x}=x^2+1, & g\in\mathbb{R},\\ \dot{y}=g+x\end{cases}$	$x\pm i\,(1)$	$x\mp i$	$\mathcal{R}_{33}=x^2+1$ $\mathcal{F}_{33}=e^{-2y}(x-i)^{1-ig}\times(x+i)^{1+ig}$
(4.34) $\begin{cases}\dot{x}=g, & g\in\{-1,1\},\\ \dot{y}=y(y-x)\end{cases}$	$y\,(1)$	$y-x$	$\mathcal{R}_{34}=y^2\,e^{x^2/(2g)}$
(4.35) $\begin{cases}\dot{x}=g+y,\\ \dot{y}=xy, & g\in\{-1,1\}\end{cases}$	$y\,(1)$	x	$\mathcal{R}_{35}=y$ $\mathcal{F}_{35}=y^{-2g}e^{x^2-2y}$
(4.36) $\begin{cases}\dot{x}=g,\\ \dot{y}=xy, & g\in\{-1,1\}\end{cases}$	$y\,(1)$	x	$\mathcal{R}_{36}=y$ $\mathcal{F}_{36}=y\,e^{-x^2/(2g)}$
(4.37) $\begin{cases}\dot{x}=x, & g(g^2-1)\neq 0\\ \dot{y}=gy-x^2, & g\in\mathbb{R}\end{cases}$	$x\,(1)$	1	$\mathcal{R}_{37}=x^{g+1}$ $\mathcal{F}_{37}=x^{-g}(x^2+2y-gy)\ (g\neq 2)$, $\widetilde{\mathcal{F}}_{37}=ye^{y/x^2}\ (g=2)$
(4.38) $\begin{cases}\dot{x}=x,\\ \dot{y}=g-x^2, & 0\neq g\in\mathbb{R}\end{cases}$	$x\,(1)$	1	$\mathcal{R}_{38}=x$ $\mathcal{F}_{38}=x^{-2g}e^{x^2+2y}$
(4.39) $\dot{x}=x^2,\quad \dot{y}=x+y$	$x\,(2)$	x	$\mathcal{R}_{39}=x^2e^{-1/x}$
(4.40) $\dot{x}=1+x,\quad \dot{y}=1-xy$	$x+1\,(2)$	1	$\mathcal{R}_{40}=(x+1)^2e^{-x}$
(4.41) $\begin{cases}\dot{x}=gxy, & g\in\{-1,1\}\\ \dot{y}=y-x^2+gy^2\end{cases}$	$x\,(3)$	y	$\mathcal{R}_{41}=x^2e^{-g(y+g)^2/(2x^2)}$
(4.42) $\begin{cases}\dot{x}=gxy, & g\in\{-1,1\}\\ \dot{y}=-x^2+gy^2\end{cases}$	$x\,(3)$	y	$\mathcal{R}_{42}=x^3$ $\mathcal{F}_{42}=x^{2/g}\,e^{y^2/x^2}$
(4.43) $\begin{cases}\dot{x}=gx^2, & g(g^2-1)\neq 0\\ \dot{y}=1+(g-1)xy, & g\in\mathbb{R}\end{cases}$	$x\,(3)$	x	$\mathcal{R}_{43}=x^{2g}\alpha^{1-g}\ (g\neq 1/2)$; $\widetilde{\mathcal{R}}_{43}=x^3e^{-xy}\ (g=1/2)$, $\alpha=1+(2g-1)xy$ $\mathcal{F}_{43}=x^{1-2g}\alpha^g\ (g\neq 1/2)$, $\widetilde{\mathcal{F}}_{43}=x^{-2}\,e^{xy}\ (g=1/2)$

On the other hand considering [15, Lemma 6.1] it follows that the configuration *Config. 4.9a* is distinguished from *Config. 4.9* by the sign of the invariant polynomial H_{16} defined in [15]. More exactly, if $H_{16}<0$ we obtain *Config. 4.9a* whereas for if $H_{16}>0$ we get *Config. 4.9*. For canonical systems (4.9) from Table I we have the expression of H_{16} and its sign are given by the formulas:

$$H_{16}=180\big[4h^2-(g-1)^2\big]\big[4h^2-(g+1)^2\big]^2 \ \Rightarrow\ \text{sign}\,(H_{16})=\text{sign}\,(4h^2-(g-1)^2).$$

Table I (*continued*)

Orbit representative	Invariant lines and their multiplicities / Respective cofactors	Inverse integrating factor $\mathcal{R}_i$
44) $\begin{cases}\dot{x} = x^2, \quad g \in \{-1, 1\}\\ \dot{y} = g - x^2 + xy\end{cases}$	$x\,(3)$	$\mathcal{R}_{44} = x^3$
	x	$\mathcal{F}_{44} = x^2\, e^{(g+2xy)/x^2}$
45) $\begin{cases}\dot{x} = gxy, \quad g \in \{-1, 1\}\\ \dot{y} = x - x^2 + gy^2\end{cases}$	$x\,(3)$	$\mathcal{R}_{45} = x^3$
	y	$\mathcal{F}_{45} = x^2\, e^{(2x+gy^2)/x^2}$
46) $\dot{x} = 1, \quad \dot{y} = y - x^2$	–	$\mathcal{R}_{46} = e^x$
	–	$\mathcal{F}_{46} = e^{-x}\big[(x+1)^2 - y + 1\big]$

For both configurations *Config. 4.11* and *Config. 4.11a* we have the same canonical form for systems which are semi-generic and the same terturbation into generic systems. Hence we have the same expression for the DIFs of the perturbation depending on two parameters g, h leading to the same DIF.

By [15, Lemma 6.4] for systems (4.11) from Table I we obtain *Config. 4.11* if $H_4 < 0$ and *Config. 4.11a* if $H_4 > 0$. The expressions of H_4 and of its sign for systems (4.11) are:

$$H_4 = 48(g^2 - 1) \quad \Rightarrow \quad \text{sign}\,(H_4) = \text{sign}\,(g^2 - 1).$$

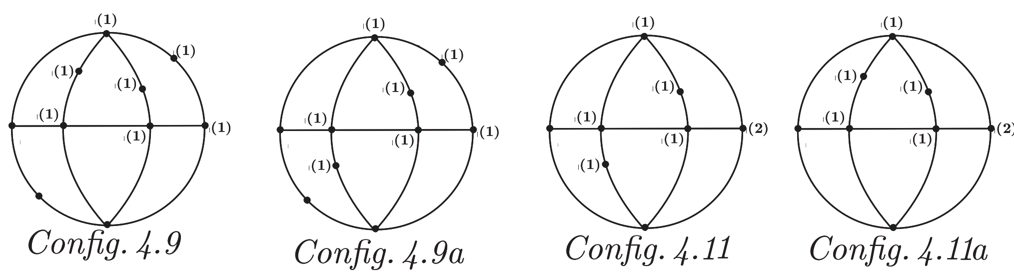
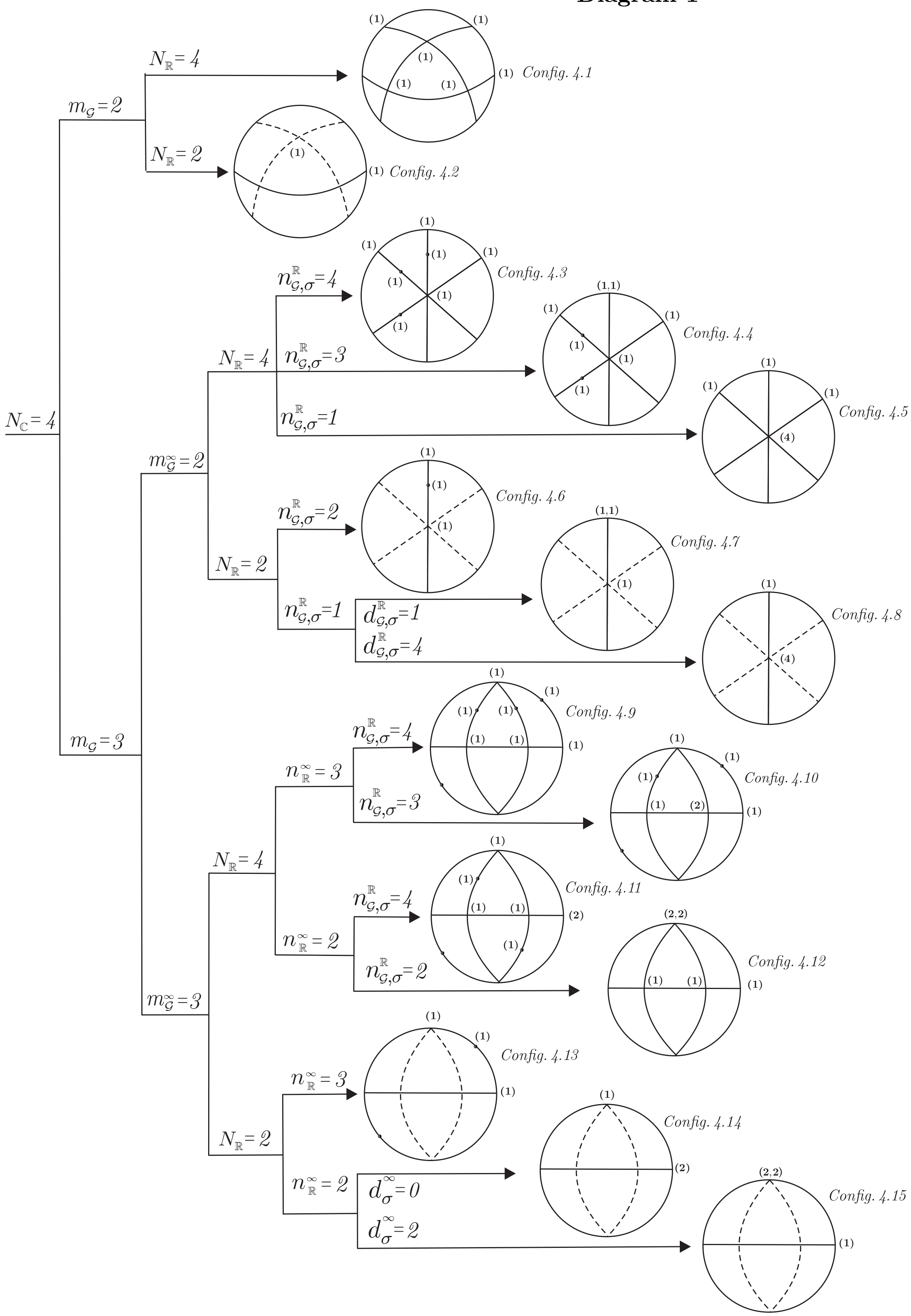


Figure 1: Configurations *Config. 4.9*, *4.9a*; *4.11*, *4.11a*.

In the Diagram 1 that follows we draw in boldface the lines that are multiple and we write next to the line its multiplicity without any parenthesis. We write the multiplicities of singularities with parenthesis, i.e. if an affine singularity s has the multiplicity 4 then we write (4) next to the singularity. If a singular point at infinity is simple then its multiplicity is denoted by (1). The multiple points at infinity can split into points in the finite plane and also points at infinity. If a singular point at infinity is not simple then its multiplicity is denoted by (i, j) to give the two kinds of multiplicities whose sum is the multiplicity of the singular point.

Diagram 1



$N_{\mathbb{C}}=4$
$m_{\mathcal{G}}=2$
$N_{\mathbb{R}}=4$
Config. 4.1
$N_{\mathbb{R}}=2$
Config. 4.2
$m_{\mathcal{G}}^{\infty}=2$
$N_{\mathbb{R}}=4$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.3
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=3$
Config. 4.4
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
Config. 4.5
$N_{\mathbb{R}}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=2$
Config. 4.6
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=1$
Config. 4.7
$d_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.8
$m_{\mathcal{G}}=3$
$m_{\mathcal{G}}^{\infty}=3$
$N_{\mathbb{R}}=4$
$n_{\mathbb{R}}^{\infty}=3$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.9
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=3$
Config. 4.10
$n_{\mathbb{R}}^{\infty}=2$
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=4$
Config. 4.11
$n_{\mathcal{G},\sigma}^{\mathbb{R}}=2$
Config. 4.12
$N_{\mathbb{R}}=2$
$n_{\mathbb{R}}^{\infty}=3$
Config. 4.13
$n_{\mathbb{R}}^{\infty}=2$
$d_{\sigma}^{\infty}=0$
Config. 4.14
$d_{\sigma}^{\infty}=2$
Config. 4.15

Diagram 1 *(continued)*

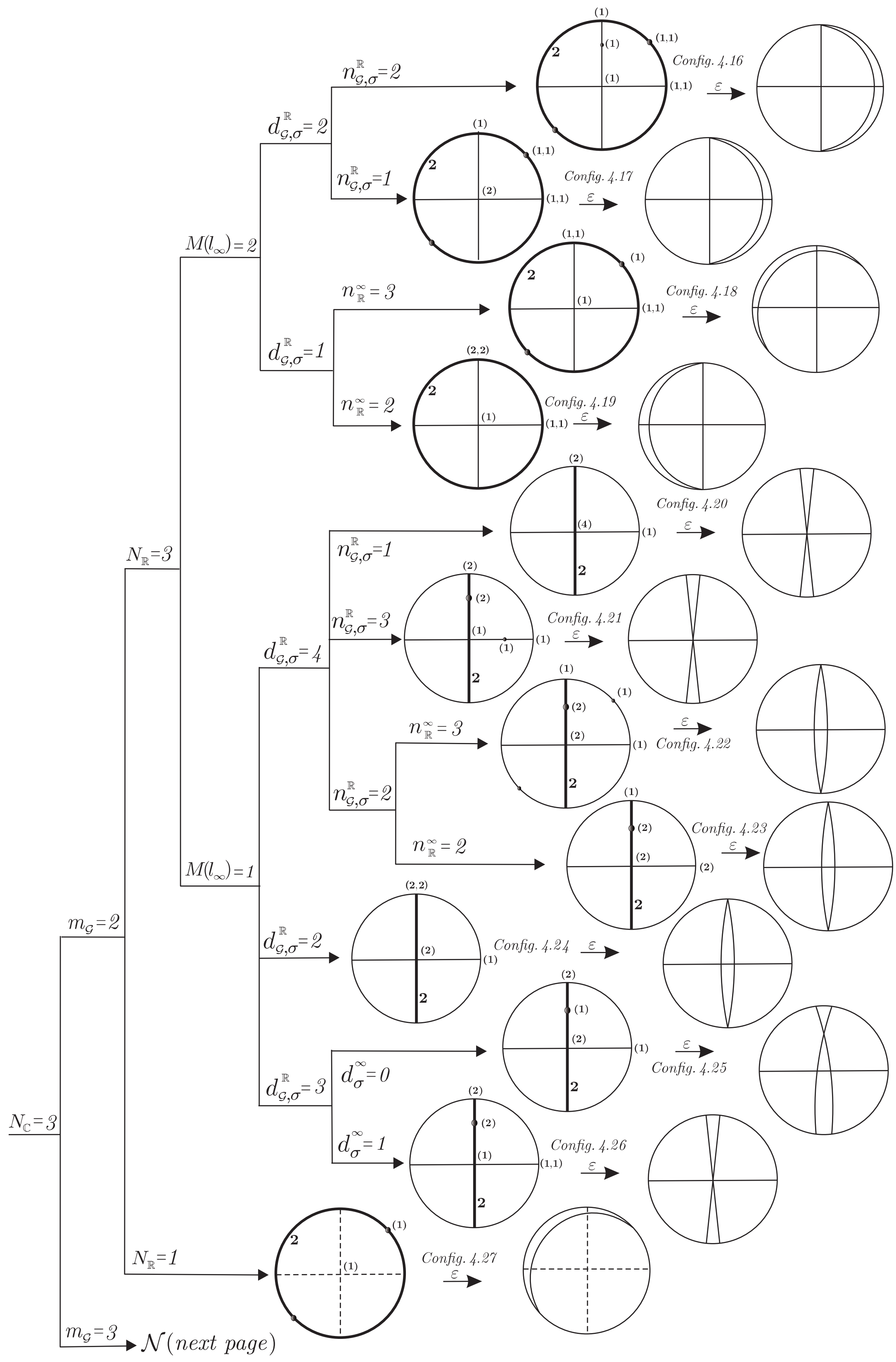


Diagram 1 *(continued)*

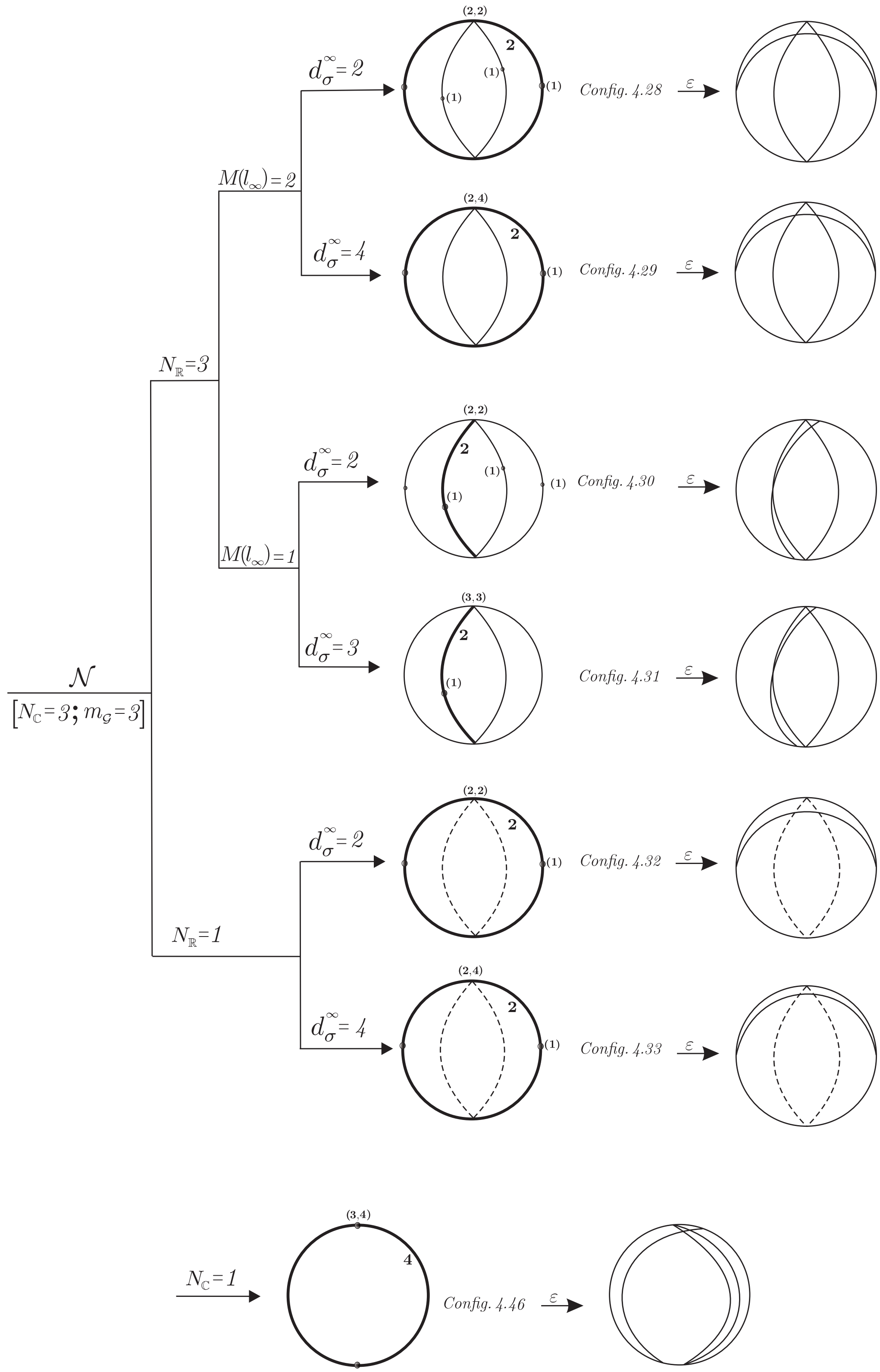
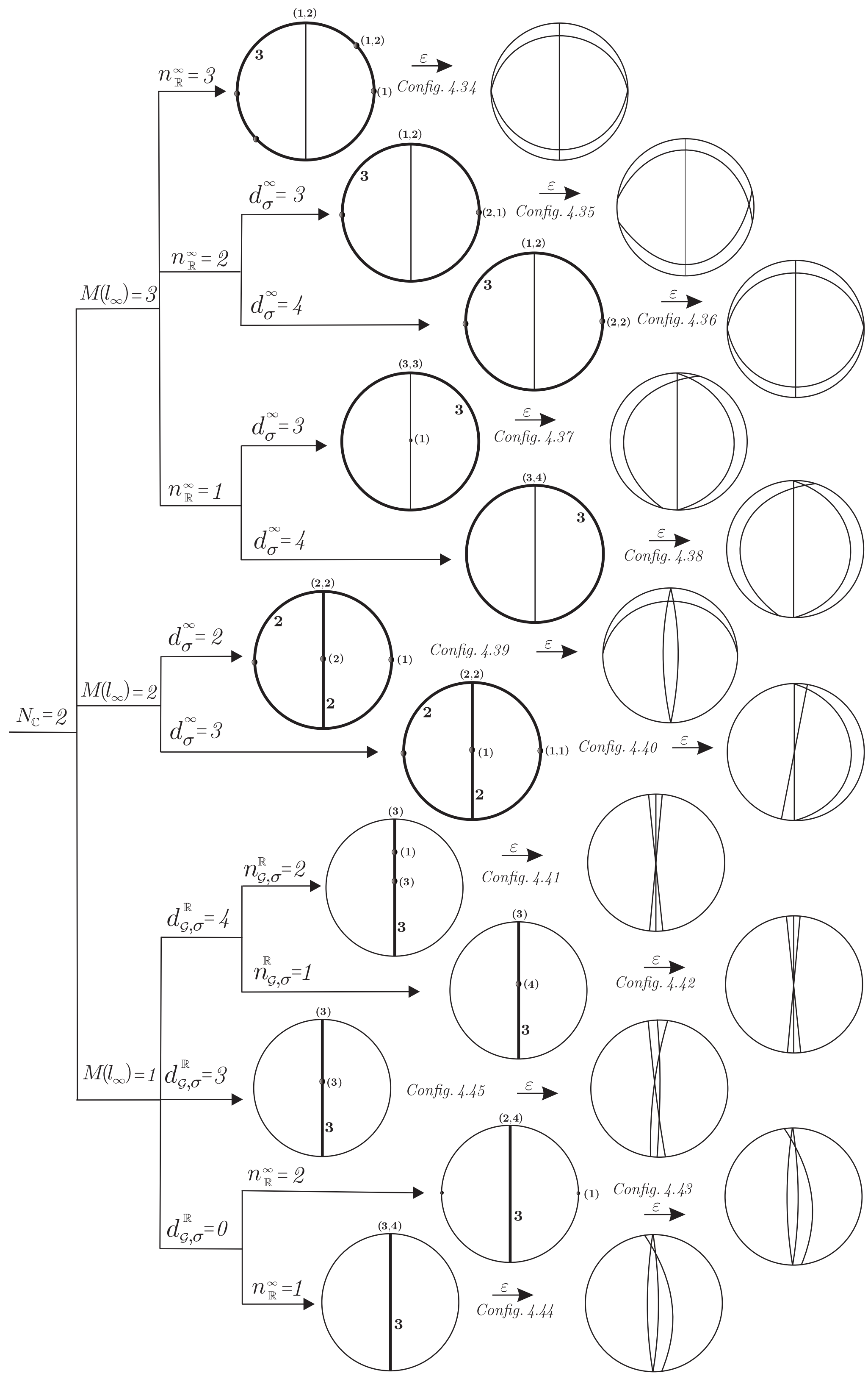


Diagram 1 *(continued)*



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