THE TORSION INDEX OF A p-COMPACT GROUP

JAUME AGUADÉ

1. Introduction and statement of results

The torsion index of a compact connected Lie group was defined by Grothendieck in 1958 ([10]) and has been investigated by several authors ([14], [6], [15], etc.). Recently, the computation of the torsion indices of all simply connected compact Lie groups has been completed (see [16]). Since we are going to work at a single prime p, instead of the torsion index of a Lie group G we want to consider its p-primary part $t_p(G)$. We summarize the properties of $t_p(G)$ which are relevant to the present work in the following proposition ($\mathbb{Z}_p$ denotes the ring of p-adic integers).

Theorem 1.1. *Let p be a prime and let G be a compact connected Lie group with a maximal torus T and corresponding Weyl group W. The positive integer $t_p(G)$ has the following properties:*

(TI1) *If A is a finite abelian p-subgroup of G, then A has a subgroup of index dividing $t_p(G)$ which is contained in a conjugate of T.*

(TI2) *$t_p(G)$ kills the kernel and the cokernel of the homomorphism*

$$H^*(BG;\mathbb{Z}_p) \to H^*(BT;\mathbb{Z}_p)^W.$$

(TI3) *$H^*(G/T;\mathbb{Z}_p)$ is torsion free and concentrated in even degrees $\leq N = \dim(G) - \operatorname{rank}(G)$, with $H^N(G/T;\mathbb{Z}_p) \cong \mathbb{Z}_p$. Then, $t_p(G)$ is the order of the cokernel of $H^N(BT;\mathbb{Z}_p) \to H^N(G/T;\mathbb{Z}_p)$.*

(TI4) *If p is not a torsion prime for G, then $t_p(G) = 1$.*

Notice that the property (TI3) can be taken as a *definition* of the (p-primary) torsion index $t_p(G)$. The other properties are well known and can be found in [15] which provides proofs or references for all of them. Actually, the properties above are usually stated using $H^*(-;\mathbb{Z})$ and $t(G) = \prod_p t_p(G)$ instead of $H^*(-;\mathbb{Z}_p)$ and $t_p(G)$ but it is easy to see that both formulations are indeed equivalent. For property (TI2) one should notice that $H^*(BT;\mathbb{Z}_p)^W = H^*(BT;\mathbb{Z})^W \otimes \mathbb{Z}_p$. This follows from exactness of $- \otimes \mathbb{Z}_p$ and the fact that the elements invariant under W can be viewed as the kernel of the homomorphism $\oplus_{g\in W}(1-g)$.

The purpose of this paper is to extend the theorem above to connected p-compact groups ([8]) and to compute the torsion indices in all cases. We prove:

Theorem 1.2. *Let p be a prime and let X be a connected p-compact group with maximal torus T and corresponding Weyl group W. There is an integer $t_p(X)$ such that*

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Departament de Matemàtiques.
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UAB Universitat Autònoma de Barcelona

(1) *The properties (TI1), (TI2), (TI3), (TI4) in theorem 1.1 hold after replacing G with X.*
(2) *If X is exotic, then $t_p(X) = 1$ for p odd and $t_2(X) = 2$.*

Here we use the work *exotic* with the same meaning as in [1]: A p-compact group X is exotic if the associated pseudoreflection representation of the Weyl group of X over the p-adic field is irreducible and does not come from a reflection group over $\mathbb{Z}$.

Section 2 deals with the (easier) odd prime case and we show that if we define $t_p(X) = 1$ for any exotic X then properties (TI1), (TI2), (TI3), (TI4) hold true. The hardest part consist in computing the torsion index of the only exotic 2-compact group that we (following [12]) denote G_3 (other authors denote it as $DI(4)$). We need a comprehensive review of the cohomology of G_3 and BG_3 (section 3) and some computations on the cohomology of the exotic homogeneous space $G_3/\operatorname{Spin}(7)$ (section 4) before we can prove that $t_2(G_3) = 2$. Finally, we prove theorem 1.2 in section 6.

2. The odd prime case

The classification theorem for p-compact groups ([2]), tells us that any connected p-compact group X splits uniquely as a product $X \cong G_p^\wedge \times X_1$ where G is a compact connected Lie group and X_1 is a product of exotic p-compact groups. Notice that the splitting is as p-compact groups and not just as spaces. This splitting implies that it is enough to prove theorem 1.2 for each exotic p-compact group, since it is already known to be true for the (p-completions of) compact connected Lie groups. Let us discus this in some more detail. If theorem 1.2 holds for the p-compact groups X_1 and X_2, let $X = X_1 \times X_2$ and let us define $t_p(X) = t_p(X_1)\, t_p(X_2)$. We need to check that properties (TI1) to (TI4) hold for X if they hold for X_1 and X_2. (TI4) is trivial and (TI3) is straightforward. To prove (TI2) let us observe that the kernel of $\gamma : H^*(BX; \mathbb{Z}_p) \to H^*(BT; \mathbb{Z}_p)^W$ is equal to the torsion elements in $H^*(BX; \mathbb{Z}_p)$. If X is of Lie type, this is well known (cf. [9]). If X is exotic and $p = 2$ (i.e. $X = G_3$) then this is assertion 4 in [12] and if p is odd, this is proven in [1]. Then, it is clear that $t_p(X_1)\, t_p(X_2)$ kills the kernel of γ. It is obvious that $t_p(X_1)\, t_p(X_2)$ kills the cokernel of γ as well. Finally, (TI1) follows easily since we can use the theory of kernels of homomorphisms between p-compact groups which is developed in [8], section 7.

Let us assume now p odd and let X be an exotic p-compact group. These objects are very well understood. In particular, they satisfy the following properties (see [1]). Let T and W denote a maximal torus of X and the corresponding Weyl group, respectively. Then,

(1) X is simply connected and center free and $H^*(X; \mathbb{Z}_p)$ is torsion free.
(2) The natural map $BT \to BX$ induces an isomorphism
$$H^*(BX; \mathbb{Z}_p) \cong H^*(BT; \mathbb{Z}_p)^W.$$
In particular, $H^*(BX; \mathbb{Z}_p)$ is concentrated in even degrees.
(3) $H^*(X/T; \mathbb{Z}_p)$ is a free $\mathbb{Z}_p$-module concentrated in even degrees. Moreover (see [13], th. 7.5.1) $H^*(X/T; \mathbb{Z}_p) \otimes \mathbb{Q}$ is a Poincaré duality algebra with fundamental class in degree $\dim(X) - \operatorname{rank}(X)$. Actually, as a W-module, $H^*(X/T; \mathbb{Z}_p) \otimes \mathbb{Q}$ coincides with the regular representation of W.

We also need another property of p-compact groups (which holds also for $p = 2$) that follows from the work in [5].

(4) If X is any p-compact group such that $H^*(BX; \mathbb{F}_p)$ is concentrated in even degrees, then any finite abelian p-subgroup of X is conjugated to a subgroup of the maximal torus of X. In particular, this holds for any product of exotic p-compact groups for p odd.

Theorem 1.2 for p odd follows immediately from all these properties of p-compact groups. □

3. The 2-compact group G_3 and its maximal torus

In this section we recollect several properties of G_3 that we need in the forthcoming sections. We state these properties without proof because either they can be found in the papers [7], [12], [4], [11] or they follow from straightforward computations that are left to the reader.

As it is well known, G_3 is an exotic connected 2-compact group of rank three whose Weyl group W is the reflection group number 24 in the Shephard-Todd list of finite complex reflection groups. Its existence was stablished by Dwyer and Wilkerson in [7]. We remind that some authors call this 2-compact group $DI(4)$ but we follow the notation used in [12]. As an abstract group, W is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times GL_3(\mathbb{F}_2)$ and for a maximal torus T of G_3, there is a basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ of $H^2(BT; \mathbb{Z}_2)$ such that the action of W on $H^*(BT; \mathbb{Z}_2)$ is given by the pseudoreflections

$$s_1 = \begin{pmatrix} -1 & -\bar{\alpha} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

where $\alpha, \bar{\alpha} \in \mathbb{Z}_2$ are the roots of $x^2 - x + 2$ chosen in such a way that α is odd and $\bar{\alpha}$ is even.

G_3 has $\mathrm{Spin}(7)$ as a 2-compact subgroup of maximal rank. This means that there is a map $\phi : B\,\mathrm{Spin}(7)^\wedge_2 \to BG_3$ whose homotopical fibre is $\mathbb{F}_2$-finite. It is natural to denote this fibre by $G_3/\mathrm{Spin}(7)$. The restriction of ϕ to a maximal torus of $\mathrm{Spin}(7)$ is a maximal torus of G_3.

There is a subgroup $V \subset \mathrm{Spin}(7)$ (explicitly described in [7]) which is an elementary abelian 2-group of rank four and such that the homomorphisms

$$H^*(BG_3; \mathbb{F}_2) \xrightarrow{\phi^*} H^*(B\,\mathrm{Spin}(7); \mathbb{F}_2) \xrightarrow{k^*} H^*(BV; \mathbb{F}_2) \cong \mathbb{F}_2[V^*]$$

are monomorphisms. Moreover, the image of $(\phi k)^*$ coincides with the rank four Dickson algebra which is the algebra of invariants of $H^*(BV; \mathbb{F}_2)$ under the action of the full linear group $GL(V^*)$, and the image of k^* coincides with the algebra of invariants $H^*(BV; \mathbb{F}_2)^H$ where $H \subset GL(V^*)$ can be described, in some appropriate basis of V^*, as the set of matrices with first row equal to $(1, 0, 0, 0)$. These algebras of invariants are well known (also as algebras over the Steenrod algebra) and we have isomorphisms (subscripts denote degrees)

$$H^*(BG_3; \mathbb{F}_2) \cong \mathbb{F}_2[c_8, c_{12}, c_{14}, c_{15}],$$

$$H^*(B\,\mathrm{Spin}(7); \mathbb{F}_2) \cong \mathbb{F}_2[d_4, d_6, d_7, d_8],$$

where the generators c_i and d_i can be explicitly described. In particular, we can see that ϕ^* is given by $\phi^*(c_8) = d_4^2 + d_8$, $\phi^*(c_{12}) = d_6^2 + d_4 d_8$, $\phi^*(c_{14}) = d_7^2 + d_6 d_8$, $\phi^*(c_{15}) = d_7 d_8$. Sq^1 vanishes on d_4, d_7, d_8, while $Sq^1(d_6) = d_7$.

As said before, a maximal torus T of Spin(7) is also a maximal torus of G_3. We have maps

$$BT_2^\wedge \xrightarrow{i} B\,\mathrm{Spin}(7)_2^\wedge \xrightarrow{\phi} BG_3$$

and we can see the Weyl group W_1 of Spin(7) as a subgroup of W, namely $W_1 = \langle s_1, s_2, s_1 s_3 s_2 s_1 s_2 s_3 s_1 \rangle$. It is known that the homomorphism

$$i^* : H^*(B\,\mathrm{Spin}(7); \mathbb{Z}_2) \to H^*(BT; \mathbb{Z}_2)^{W_1}$$

is surjective and its kernel coincides with the ideal of torsion elements. The integral invariants of W_1 are computed in [4]. They turn out to form a polynomial algebra on generators of degrees 4, 8, 12:

$$H^*(BT; \mathbb{Z}_2)^{W_1} \cong \mathbb{Z}_2[u_4, u_8, u_{12}].$$

Choosing an appropriate basis $\{x_1, x_2, A\}$ of $H^2(BT; \mathbb{Z}_2)$, these generators are

$$\begin{aligned} u_4 &= (1/2)(x_1^2 + x_2^2 + x_3^2) \\ u_8 &= (1/16)(x_1^4 + x_2^4 + x_3^4 - 2x_1^2 x_2^2 - 2x_1^2 x_3^2 - 2x_2^2 x_3^2) \\ u_{12} &= x_1^2 x_2^2 x_3^2 \end{aligned}$$

where we have used the notation $x_3 = 2A - x_1 - x_2$ and one can check that in spite of the denominators, these polynomials belong to $\mathbb{Z}_2[x_1, x_2, A]$.

The generators u_4, u_8 and u_{12} have a rather simple form as polynomials on x_1, x_2, A, but this basis of $H^2(BT; \mathbb{Z}_2)$ dos not coincide with the basis $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ that we have used to describe the action of W on $H^*(BT; \mathbb{Z}_2)$. The basis change is given by the matrix

$$\begin{pmatrix} 0 & -\bar{\alpha}/2 & -(1+\alpha)/2 \\ 1 & 0 & -(1+\alpha)/2 \\ 0 & \bar{\alpha} & \alpha \end{pmatrix} \in GL_3(\mathbb{Z}_2).$$

Using this matrix we can express the generators u_4, u_8, u_{12} as polynomials in $\epsilon_1, \epsilon_2, \epsilon_3$ and so we have an explicit description of the homomorphism

$$\mathbb{Z}_2[u_4, u_8, u_{12}] = H^*(B\,\mathrm{Spin}(7); \mathbb{Z}_2)/\mathrm{Torsion} \to H^*(BT; \mathbb{Z}_2) = \mathbb{Z}_2[\epsilon_1, \epsilon_2, \epsilon_3].$$

Finally, we want to use this to describe the homomorphism

$$\mathbb{F}_2[d_4, d_6, d_7, d_8] = H^*(B\,\mathrm{Spin}(7); \mathbb{F}_2) \xrightarrow{i^*} H^*(BT; \mathbb{F}_2) = \mathbb{F}_2[\epsilon_1, \epsilon_2, \epsilon_3].$$

In the Bockstein spectral sequence for $B\,\mathrm{Spin}(7)$ we have $E_2 = E_\infty = \mathbb{F}_2[\bar{d}_4, \bar{d}_8, \bar{d}_6^2]$ and the surjection

$$j : H^*(B\,\mathrm{Spin}(7); \mathbb{Z}_2)/\mathrm{Torsion} \to E_\infty$$

is given by $j(u_4) = \bar{d}_4$, $j(u_8) = \bar{d}_8$, $j(u_{12}) = \bar{d}_6^2$. From this it is straightforward to perform the computations that yield

$$\begin{aligned} i^*(d_4) &= \epsilon_1^2 + \epsilon_1\epsilon_2 + \epsilon_2^2 \\ i^*(d_6) &= Sq^2 i^*(d_4) = \epsilon_1^2\epsilon_2 + \epsilon_1\epsilon_2^2 \\ i^*(d_7) &= 0 \\ i^*(d_8) &= \epsilon_1\epsilon_2\epsilon_3(\epsilon_1+\epsilon_2+\epsilon_3) + \epsilon_3^2(\epsilon_1+\epsilon_2+\epsilon_3)^2. \end{aligned}$$

4. The exotic homogeneous space $G_3/\operatorname{Spin}(7)$

In this section we want to investigate the cohomology of the exotic homogeneous space $G_3/\operatorname{Spin}(7)$. The computations presented here are probably known to experts, but it may be worthwhile to work them out here in some detail.

Let us consider the fibration $G_3/\operatorname{Spin}(7) \xrightarrow{\rho} B\operatorname{Spin}(7)^{\wedge}_2 \to BG_3$ and let $V \subset \operatorname{Spin}(7)$ denote the elementary abelian 2-group of rang 4 considered in the preceding section. To simplify the notation, let us write $S = H^*(BV;\mathbb{F}_2)$. Then, we have $H^*(B\operatorname{Spin}(7);\mathbb{F}_2) = S^H$ and $H^*(BG_3;\mathbb{F}_2) = S^G$ for $G = GL_4(\mathbb{F}_2)$.

The computation of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is best worked out with the Eilenberg-Moore spectral sequence

$$\operatorname{Tor}_{H^*(BG_3;\mathbb{F}_2)}(H^*(B\operatorname{Spin}(7);\mathbb{F}_2),\mathbb{F}_2) \quad\Rightarrow\quad H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2).$$

Here the key observation is that $H^*(B\operatorname{Spin}(7);\mathbb{F}_2) = S^H$ is a free module over $H^*(BG_3;\mathbb{F}_2) = S^G$ because of the following classic argument. S is an integral extension of S^G, hence S^H is also an integral extension of S^G and, since S^H is a finitely generated algebra, we obtain that S^H is a finitely generated S^G-module. But both S^H and S^G are polynomial algebras and we can apply [3] V 5.5 or [13] 6.7.1 to conclude that S^H is S^G-free.

Hence the Eilenberg-Moore spectral sequence collapses to an isomorphism

$$H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2) \cong \mathbb{F}_2[\bar{d}_4,\bar{d}_6,\bar{d}_7]/(\bar{d}_6^{\,2}+\bar{d}_4^{\,3},\bar{d}_7^{\,2}+\bar{d}_4^{\,2}\bar{d}_6,\bar{d}_4^{\,2}\bar{d}_7)$$

where $\bar{d}_4$, $\bar{d}_6$, $\bar{d}_7$ are the images of $d_4, d_6, d_7 \in H^*(B\operatorname{Spin}(7);\mathbb{F}_2)$, respectively. It is rather easy to completely work out the algebra structure of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$. We obtain the following.

(1) The Poincaré series of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is

$$1+t^4+t^6+t^7+t^8+t^{10}+t^{11}+t^{12}+t^{13}+t^{14}+t^{16}+t^{17}+t^{18}+t^{20}+t^{24}$$

and the Euler characteristic is $7 = [H:W]$.

(2) An additive basis for $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is given by

$$\left\{\bar{d}_4^{\,i},\ i=0,\dots,6,\ \bar{d}_6,\ \bar{d}_7,\ \bar{d}_4\bar{d}_6,\ \bar{d}_4\bar{d}_7,\ \bar{d}_6\bar{d}_7,\ \bar{d}_4^{\,2}\bar{d}_6,\ \bar{d}_4\bar{d}_6\bar{d}_7,\ \bar{d}_4^{\,3}\bar{d}_6\right\}.$$

(3) $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ is a Poincaré duality algebra with top class $\bar{d}_4^{\,6}$ (see [13] 6.5).

(4) The Bockstein spectral sequence of $H^*(G_3/\operatorname{Spin}(7);\mathbb{F}_2)$ collapses after the second term, i.e. $H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2)$ has only torsion of order 2. We have

$$H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2)/\text{Torsion} \cong \mathbb{Z}_2[\bar{a}]/\bar{a}^7$$

and

$$H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2) \cong \mathbb{Z}_2[\bar{a},\bar{c}]/(\bar{a}^7,\, \bar{c}^3,\, \bar{a}^2\bar{c},\, 2\bar{c}).$$

In particular, the top class in $H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2)$ is $\bar{d_4}^6$ in dimension 24 and it is in the image of

$$\phi^* : H^*(B\operatorname{Spin}(7);\mathbb{Z}_2) \to H^*(G_3/\operatorname{Spin}(7);\mathbb{Z}_2).$$

5. The torsion index of G_3

To compute the torsion index of the 2-compact group G_3 we need a lemma on Poincaré duality in fibrations. I'm grateful to Aniceto Murillo for some helpful conversations on this subject. For this lemma we use the following notation. Let $\mathcal{O}$ denote the ring of integers or the ring of p-adic integers. Cohomology is taken with coefficients in $\mathcal{O}$ and we assume that all spaces are of finite type over $\mathcal{O}$. We say that $\eta \in H^n(X)$ is an *orientation class* if $H^i(X) = 0$ for $i > n$, $H^n(X) \cong \mathcal{O}$, and η is a generator of $H^n(X)$.

Lemma 5.1. *Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a fibration of 1-connected spaces and assume that $\eta^F \in H^m(F)$ and $\eta^B \in H^n(B)$ are orientation classes. Assume $\alpha \in H^m(E)$ is such that $j^*(\alpha) = \lambda\eta^F$ for some $\lambda \neq 0$. Then there is an orientation class η^E for E such that $\alpha \cdot \pi^*(\eta^B) = \lambda\eta^E$.*

Proof. This follows easily from the cohomology spectral sequence of the fibration $F \xrightarrow{j} E \xrightarrow{\pi} B$. First of all, it is clear that $H^i(E) = 0$ for $i > n+m$ while $H^{n+m}(E) = E_\infty^{n,m} = E_2^{n,m} \cong \mathcal{O}$. Recall that the cohomology spectral sequence is multiplicative in the sense that (up to some signs which would not play any role here) the product in E_2 induced by the products in $H^*(B)$ and $H^*(F)$ yields a product in each E_r, $2 \leq r \leq \infty$, in such a way that the product in E_∞ is compatible with the product in $H^*(E)$.

At the E_2 level we have that $\eta^E := \eta^F {\cdot} \eta^B$ is a generator of $E_2^{n,m} = E_\infty^{n,m} = H^{n+m}(E)$. The hypothesis $j^*(\alpha) = \lambda\eta^F$, $\lambda \neq 0$ implies that α has filtration zero in $H^m(E)$ and its image in $E_\infty^{0,m}$ is $\lambda\eta^F$. Then, $\lambda\eta^E = (\lambda\eta^F)\cdot[\eta^B]$ holds in E_∞ where $[\eta^B]$ denotes the image of η^B in $E_\infty^{n,0}$. Since $E_\infty^{i,m+n-i} = 0$ for $i \neq n$, we deduce $\lambda\eta^E = \alpha\cdot\pi^*(\eta^B)$ as desired. □

Now we can proceed to the computation of the torsion index of G_3 or, to be more precise, to the computation of the order of the cokernel of $k^* : H^{42}(BT;\mathbb{Z}_2) \to H^{42}(G_3/T;\mathbb{Z}_2)$. We consider the diagram

$$\begin{array}{ccccc} (\operatorname{Spin}(7)/T)^\wedge_2 & \xrightarrow{j} & G_3/T & \xrightarrow{\pi} & G_3/\operatorname{Spin}(7) \\ & & \Big\downarrow k & & \Big\downarrow \phi \\ & & (BT)^\wedge_2 & \xrightarrow{i} & (B\operatorname{Spin}(7))^\wedge_2 \end{array}$$

$\operatorname{Spin}(7)/T$ is a compact orientable differentiable manifold of dimension 18 and we can choose an orientation class $\eta \in H^{18}(\operatorname{Spin}(7)/T;\mathbb{Z}_2)$. The torsion indices of the Lie groups $\operatorname{Spin}(n)$ have been computed by Totaro for all values of n ([15]) and it turns out that the torsion index of Spin(7) is equal to 2. This means that there is $\omega \in H^*(BT;\mathbb{Z})$ such that $f^*(\omega) = 2\eta$ for the natural map $f : \mathrm{Spin}(7)/T \to BT$.

The computations in the preceding section show that there is an orientation class $\rho \in H^{24}(G_3/\mathrm{Spin}(7);\mathbb{Z}_2)$ which is in the image of ϕ^*. Let $\rho = \phi^*(\gamma)$. We can apply now the lemma above to the fibration $\mathrm{Spin}(7)/T \to G_3/T \to G_3/\mathrm{Spin}(7)$ with $\alpha = k^*(\omega)$ and deduce that there is an orientation class $\theta \in H^{42}(G_3/T;\mathbb{Z}_2)$ such that $k^*(\omega \cdot i^*(\gamma)) = 2\,\theta$. This implies that the torsion index of G_3 divides 2.

Next, we prove that the torsion index of G_3 cannot be equal to 1. It is enough to prove that the homomorphism $H^{42}(BT;\mathbb{F}_2) \to H^{42}(G_3/T;\mathbb{F}_2)$ is equal to zero. Let us consider the $\mathbb{F}_2$-spectral sequence of the fibration $G_3 \to G_3/T \to BT_2^\wedge$. We have that

$$H^*(G_3;\mathbb{F}_2) \cong \mathbb{F}_2[x_7]/x_7^2 \otimes E(x_{11},x_{13})$$
$$Sq^4(x_7) = x_{11},\ Sq^2(x_{11}) = x_{13},\ Sq^1(y_{13}) = x_7^2.$$

Hence, the generators $x_7, x_{11}, x_{13}, x_7^2$ are transgressive to $c_8, c_{12}, c_{14}, 0$ respectively. Here we denote by c_8, c_{12}, c_{14} the images in $H^*(BT;\mathbb{F}_2)$ of the generators $c_8, c_{12}, c_{14} \in H^*(BG_3;\mathbb{F}_2)$. Recall that in section 3 we have computed these elements as explicit polynomials in some basis $\{\epsilon_1,\epsilon_2,\epsilon_3\}$ of $H^2(BT;\mathbb{F}_2)$.

In the E_2-term of the spectral sequence of $G_3 \to G_3/T \to BT_2^\wedge$, let us consider the row containing x_7^2. All elements in this row are permanent cycles and the only boundaries are the elements of the form $x_7^2 q$ with q in the ideal of $\mathbb{F}_2[\epsilon_1,\epsilon_2,\epsilon_3]$ generated by c_8, c_{12}, c_{14}. If we compute the quotient algebra $\mathbb{F}_2[\epsilon_1,\epsilon_2,\epsilon_3]/(c_8,c_{12},c_{14})$ (using any choice of a computer algebra software) we see that it is a graded algebra with Poincaré series equal to

$$1 + 3t^2 + 6t^4 + 10t^6 + 14t^8 + 18t^{10} + 21t^{12} + 22t^{14} + 21t^{16} + 18t^{18} + 14t^{20} + 10t^{22} + 6t^{24} + 3t^{26} + t^{28}.$$

and so in particular there is an element $q \in H^{28}(BT;\mathbb{F}_2)$ which does not belong to the ideal (c_8,c_{12},c_{14}). Hence, the element $x_7^2 q$ in the E_2-term of the spectral sequence survives to a non trivial element in $H^{42}(G_3/T;\mathbb{F}_2)$ which cannot be in the image of $H^*(BT;\mathbb{F}_2)$. This finishes the proof of

Theorem 5.2. *The cokernel of $H^{42}(BT;\mathbb{Z}_2) \to H^{42}(G_3/T;\mathbb{Z}_2)$ has order two.*

6. Proof of theorem 1.2

In section 2 we saw that it is enough to prove theorem 1.2 for each exotic p-compact group and we also saw that theorem 1.2 is true for all odd primes. Since it is known ([2]) that the only exotic 2-compact group is G_3, the only thing that remains to be proved is that G_3 satisfies the properties (TI1) to (TI4) with $t_2(G_3) = 2$.

(TI4) is void and (TI3) is just theorem 5.2 plus some facts about G_3/T which were proven in [2]. In [12] it is proven that the torsion elements in $H^*(BG_3;\mathbb{Z}_2)$ are of order two and the homomorphism $H^*(BG_3;\mathbb{Z}_2) \to H^*(BT;\mathbb{Z}_2)^W$ is surjective. This implies immediately that (TI2) holds. It is well known that G_3 has up to conjugation only one maximal elementary abelian 2-subgroup. It has rank four and is the group that we have denoted by V in section 3. Since V has a subgroup of rank three which injects into a maximal torus of G_3, we have (TI1) and the proof is complete. □

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès, Spain.

E-mail address: aguade@mat.uab.cat