REGULARITY OF $\mathcal{C}^1$ AND LIPSCHITZ DOMAINS IN TERMS OF THE BEURLING TRANSFORM

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ABSTRACT. Let $\Omega \subset \mathbb{C}$ be a bounded $\mathcal{C}^1$ domain, or a Lipschitz domain "flat enough", and consider the Beurling transform of χ_Ω:

$$B\chi_\Omega(z) = \lim_{\varepsilon \to 0} \frac{-1}{\pi} \int_{w \in \Omega, |z-w| > \varepsilon} \frac{1}{(z-w)^2} \, dm(w).$$

Using a priori estimates, in this paper we solve the following free boundary problem: if $B\chi_\Omega$ belongs to the Sobolev space $W^{\alpha,p}(\Omega)$ for $0 < \alpha \leq 1$, $1 < p < \infty$ such that $\alpha p > 1$, then the outward unit normal N on $\partial\Omega$ is in the Besov space $B_{p,p}^{\alpha-1/p}(\partial\Omega)$. The converse statement, proved previously by Cruz and Tolsa, also holds. So we have

$$B(\chi_\Omega) \in W^{\alpha,p}(\Omega) \quad \Longleftrightarrow \quad N \in B_{p,p}^{\alpha-1/p}(\partial\Omega).$$

Together with recent results by Cruz, Mateu and Orobitg, from the preceding equivalence one infers that the Beurling transform is bounded in $W^{\alpha,p}(\Omega)$ if and only if the outward unit normal N belongs to $B_{p,p}^{\alpha-1/p}(\partial\Omega)$, assuming that $\alpha p > 2$.

1. INTRODUCTION

In this paper we show that the boundedness of the Beurling transform in the Sobolev spaces $W^{\alpha,p}(\Omega)$, with $0 < \alpha \leq 1$ and $1 < p < \infty$ such that $\alpha p > 1$, characterizes the Besov smoothness of the boundary $\partial\Omega$, whenever Ω is a $\mathcal{C}^1$ domain, or a Lipschitz domain "flat enough". This can be considered as a free boundary problem.

The Beurling transform of a function $f : \mathbb{C} \to \mathbb{C}$, with $f \in L^p$ for some $1 \leq p < \infty$, is defined by

$$Bf(z) = \lim_{\varepsilon \to 0} \frac{-1}{\pi} \int_{|z-w| > \varepsilon} \frac{f(w)}{(z-w)^2} \, dm(w).$$

It is known that this limit exist a.e. The Beurling transform plays an essential role in the theory of quasiconformal mappings in the plane, because it intertwines the ∂ and $\bar{\partial}$ derivatives. Indeed, in the sense of distributions, one has

$$B(\bar{\partial}f) = \partial f.$$

Let $\Omega \subset \mathbb{C}$ be a bounded domain (open and connected). We say that $\Omega \subset \mathbb{C}$ is a (δ, R)-Lipschitz domain if for each $z \in \partial\Omega$ there exists a Lipschitz function

$A : \mathbb{R} \to \mathbb{R}$ with slope $\|A'\|_\infty \leq \delta$ such that, after a suitable rotation,
$$\Omega \cap B(z, R) = \big\{(x, y) \in B(z, R) : y > A(x)\big\}.$$
If we do not care about the constants δ and R, then we just say that Ω is a Lipschitz domain. If in this definition we assume the function A to be of class $\mathcal{C}^1$, then we say that Ω is a $\mathcal{C}^1$ domain.

In [CT] it has been shown that for any Lipschitz domain Ω and $0 < \alpha \leq 1$ and $1 < p < \infty$ such that $\alpha p > 1$, if the outward unit normal is in the Besov space $B_{p,p}^{\alpha-1/p}(\partial\Omega)$, then $B(\chi_\Omega)$ belongs to the Sobolev space $W^{\alpha,p}(\Omega)$. More precisely, the following estimate has been proved:

$$(1.1) \qquad \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)} \leq c\, \|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)},$$

where N stands for the outward normal unitary vector on $\partial\Omega$, $\dot{W}^{\alpha,p}(\Omega)$ is a homogeneous sobolev space on Ω, and $\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)$ is a homogeneous Besov space on $\partial\Omega$. See the next section for the precise definition of Sobolev and Besov spaces, as well as their homogeneous versions. The constant c in (1.1) may depend on p and on the Lipschitz character of Ω, i.e. on δ and on $\mathcal{H}^1(\partial\Omega)/R$ (here $\mathcal{H}^1$ stands for the length or 1-dimensional Hausdorff measure). Observe that, by the L^p boundedness of the Beurling transform,
$$\|B(\chi_\Omega)\|_{W^{\alpha,p}(\Omega)} \leq c \big(m(\Omega)^{1/p} + \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}\big).$$
Thus (1.1) guaranties that $B(\chi_\Omega) \in W^{\alpha,p}(\Omega)$ whenever $N \in \dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)$.

Our main result is a (partial) converse to (1.1):

Theorem 1.1. *Let $\Omega \subset \mathbb{C}$ be a (δ, R)-Lipschitz domain and $0 < \alpha \leq 1$ and $1 < p < \infty$ such that $\alpha p > 1$. If $\delta = \delta(p) > 0$ is small enough, then*

$$(1.2) \qquad \|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)} \leq c\, \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)} + c\, \mathcal{H}^1(\partial\Omega)^{-\alpha+2/p}.,$$

where c depends on δ and p.

Some remarks are in order. Notice first that $\mathcal{C}^1$ domains are (δ, R)-Lipschitz domains for every $\delta > 0$ and an appropriate $R = R(\delta)$. So the theorem applies to all $\mathcal{C}^1$ domains. Then, by combining the results from [CT] with Theorem 1.1, one infers that, for a $\mathcal{C}^1$ domain Ω and α, p as above,

$$(1.3) \qquad B(\chi_\Omega) \in W^{\alpha,p}(\Omega) \quad \Longleftrightarrow \quad N \in B_{p,p}^{\alpha-1/p}(\partial\Omega).$$

Let us remark that the inequality
$$\|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)} \leq c\, \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}$$
fails in general. Indeed, when Ω is an open ball it turns out that $B(\chi_\Omega)$ vanishes identically on Ω. So $\|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)} = 0$, while $\|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)} \neq 0$, because N is not constant.

Recall that the Besov spaces $B_{p,p}^{\alpha-1/p}$ appear naturally in the context of Sobolev spaces. Indeed, a well known theorem of Gagliardo [Ga] asserts that the traces of

the functions from $W^{1,p}(\Omega)$ on $\partial\Omega$ coincide with the functions from $B_{p,p}^{1-1/p}(\partial\Omega)$, whenever Ω is a Lipschitz domain. An analogous result holds for $0 < \alpha < 1$. So, by this result theorem with (1.3), one deduces that $B(\chi_\Omega) \in W^{\alpha,p}(\Omega)$ if and only if N is the trace of some (vectorial) function from $W^{\alpha,p}(\Omega)$, which looks a rather surprising statement at first sight.

Our motivation for the characterization of those domains such that $B\chi_\Omega \in W^{\alpha,p}(\Omega)$ arises from the results of Cruz, Mateu and Orobitg in [CMO]. In this paper the authors study the smoothness of quasiconformal mappings when the Beltrami coefficient belongs to $W^{\alpha,p}(\Omega)$, for some fixed $1 < p < \infty$ and $0 < \alpha < 1$. As an important step in their arguments, they prove following kind of $T1$ theorem:

Theorem ([CMO]). *Let $\Omega \subset \mathbb{C}$ be a bounded $\mathcal{C}^{1+\varepsilon}$ domain, for some $\varepsilon > 0$, and let $0 < \alpha \leq 1$ and $1 < p < \infty$ be such that $\alpha p > 2$. Then, the Beurling transform is bounded in $W^{\alpha,p}(\Omega)$ if and only if $B(\chi_\Omega) \in W^{\alpha,p}(\Omega)$.*

As a corollary of the preceding result and Theorem 1.1 we obtain the following.

Corollary 1.2. *Let $\Omega \subset \mathbb{C}$ be a Lipschitz domain and let $0 < \alpha \leq 1$ and $1 < p < \infty$ be such that $\alpha p > 2$. Then, the Beurling transform is bounded in $W^{\alpha,p}(\Omega)$ if and only if the outward unit normal of Ω is in the Besov space $B_{p,p}^{\alpha-1/p}(\partial\Omega)$.*

Observe that the fact that $N \in \dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)$ implies that the local parameterizations of the boundary can be taken from $B_{p,p}^{1+\alpha-1/p}(\mathbb{R}) \subset \mathcal{C}^{1+\varepsilon}(\mathbb{R})$ because $\alpha p > 2$, and thus the theorem from Cruz-Mateu-Orobitg applies. For more details, see Lemma 3.1 below.

On the other hand, it is worth mentioning that the boundedness of the Beurling transform in the Lipschitz spaces $\mathrm{Lip}_\varepsilon(\Omega)$ for domains Ω of class $\mathcal{C}^{1+\varepsilon}$ has been studied in [MOV], [LV], and [De], because of the applications to quasiconformal mappings and PDE's.

We will prove Theorem 1.1 by reducing it to the case of the so called special Lipschitz domain, where Ω is the open set lying above a Lipschitz graph. That is, given a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$, one sets

$$(1.4) \qquad \Omega = \{(x,y) \in \mathbb{C} : y > A(x)\}.$$

In this situation, we will show the following:

Theorem 1.3. *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with compact support and consider the special Lipschitz domain Ω defined in (1.4). For $0 < \alpha \leq 1$ and $1 < p < \infty$ be such that $\alpha p > 1$, there exists $\delta = \delta(\alpha, p) > 0$ small enough such if $\|A'\|_\infty \leq \delta$, then*

$$(1.5) \qquad \|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}} \leq c \, \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)},$$

with c depending on α, p.

Above, $\dot{B}_{p,p}^{1+\alpha-1/p}$ and $\dot{W}^{\alpha,p}(\Omega)$ stand for homogeneous Besov and Sobolev spaces, on $\mathbb{R}$ and on Ω, respectively.

Let us remark that the converse inequality

$$(1.6) \qquad \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)} \leq c(\delta) \, \|A\|_{\dot{B}^{1+\alpha-1/p}_{p,p}}$$

also holds for special Lipschitz domains, without the smallness assumption on δ. This has been shown in [CT]. Notice, in particular, that (1.6) shows that $B\chi_\Omega$ is constant in Ω if this is a half plane. Of course, this can be proved without appealing to (1.6), in a much more elementary way. This property plays a key role in the arguments in [CT] and also in the ones of the present paper.

It is easy to check that

$$(1.7) \qquad \|A\|_{\dot{B}^{1+\alpha-1/p}_{p,p}} \approx \|N\|_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)},$$

where, as above, $N(z)$ stands for the outward unitary normal at z. So (1.5) is analogous to (1.2). For the detailed arguments, see Lemma 3.1 below.

We will prove Theorem 1.3 by means of a Fourier type estimate. To this end, we will need to approximate the Lipschitz graph by lines at many different scales. We will estimate the errors in the approximation in terms of the so called β_1 coefficients. Given an interval $I \subset \mathbb{R}$ and a function $f \in L^1_{loc}$, one sets

$$(1.8) \qquad \beta_1(f, I) = \inf_\rho \frac{1}{\ell(I)} \int_{3I} \frac{|f(x) - \rho(x)|}{\ell(I)} \, dx,$$

where the infimum is taken over all the affine functions $\rho : \mathbb{R} \to \mathbb{R}$. The coefficients β_1's (and other variants β_p, β_∞,...) appeared first in the works of Jones [Jo] and David and Semmes [DS1] on quantitative rectifiability. They have become a useful tool in problems which involve geometric measure theory and multi-scale analysis. See [DS2], [Lé], [MT], [To1], or [To2], for example, besides the aforementioned references. In the present paper we will use the β_1's to measure the Besov smoothness of the boundary of Lipschitz domains, by means of a characterization of Besov spaces in terms of β_1's due to Dorronsoro [Do].

The plan of the paper is the following. In Section 2, some preliminary notation and background is introduced. In particular, several characterizations of Besov spaces are described. In Section 3 we prove some auxiliary lemmas which will be used later. Theorem 1.3 is proved in Section 4, and then this is used in Section 5 to deduce Theorem 1.1. In the final Section 6 we show that Theorems 1.1 and 1.3 also hold replacing the $\dot{W}^{\alpha,p}$ seminorm of $B\chi_\Omega$ by the $\dot{B}^\alpha_{p,p}(\Omega)$ one, for $0 < \alpha < 1$.

2. Preliminaries

2.1. **Basic notation.** As usual, in the paper the letter 'c' stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as c_0, retain their values at different occurrences. The notation $A \lesssim B$ means that there is a fixed positive constant c such that $A \leq cB$. So $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

The notation $I(x, r)$ stands for an interval in $\mathbb{R}$ with center x and radius r.

2.2. **Dyadic and Whitney cubes.** By a cube in $\mathbb{R}^n$ we mean a cube with edges parallel to the axes. Most of the cubes in our paper will be dyadic cubes, which are assumed to be half open-closed. The collection of all dyadic cubes is denoted by $\mathcal{D}(\mathbb{R}^n)$. They are called intervals for $n = 1$ and squares for $n = 2$. The side length of a cube Q is written as $\ell(Q)$, and its center as z_Q. The lattice of dyadic cubes of side length 2^{-j} is denoted by $\mathcal{D}_j(\mathbb{R}^n)$. Also, given $a > 0$ and any cube Q, we denote by $a\,Q$ the cube concentric with Q with side length $a\,\ell(Q)$.

Recall that any open subset $\Omega \subset \mathbb{R}^n$ can be decomposed in the so called Whitney cubes, as follows:

$$\Omega = \bigcup_{k=1}^{\infty} Q_k,$$

where Q_k are disjoint dyadic cubes (the "Whitney cubes") such that for some constants $r > 20$ and $D_0 \geq 1$ the following holds,

- (i) $5Q_k \subset \Omega$.
- (ii) $rQ_k \cap \Omega^c \neq \varnothing$.
- (iii) For each cube Q_k, there are at most D_0 squares Q_j such that $5Q_k \cap 5Q_j \neq \varnothing$. Moreover, for such squares Q_k, Q_j, we have $\frac{1}{2}\ell(Q_k) \leq \ell(Q_j) \leq 2\,\ell(Q_k)$.

We will denote by $\mathcal{W}(\Omega)$ the family $\{Q_k\}_k$ of Whitney cubes of Ω.

If $\Omega \subset \mathbb{C}$ is a Lipschitz domain, then $\partial\Omega$ is a chord arc curve. Recall that a chord arc curve is just the bilipschitz image of a circumference. Then one can define a family $\mathcal{D}(\partial\Omega)$ of "dyadic" arcs which play the same role as the dyadic intervals in $\mathbb{R}$: for each $j \in \mathbb{Z}$ such that $2^{-j} \leq \mathcal{H}^1(\partial\Omega)$, $\mathcal{D}_j(\partial\Omega)$ is a partition of $\partial\Omega$ into pairwise disjoint arcs of length $\approx 2^{-j}$, and $\mathcal{D}(\partial\Omega) = \bigcup_j \mathcal{D}_j(\partial\Omega)$. As in the case of $\mathcal{D}(\mathbb{R}^n)$, two arcs from $\mathcal{D}(\partial\Omega)$ either are disjoint or one contains the other. The construction of $\mathcal{D}(\partial\Omega)$ easy: take an arc length parameterization $S^1(0, r_0) \to \partial\Omega$, consider a dyadic family of arcs of $S^1(0, r_0)$, and then let $\mathcal{D}(\partial\Omega)$ be the image of the dyadic arcs from $S^1(0, r_0)$.

Sometimes, the arcs from $\mathcal{D}(\partial\Omega)$ will be called dyadic "cubes", because of the analogy with $\mathcal{D}(\mathbb{R}^n)$.

If Ω is a special Lipschitz domain, that is, $\Omega = \{(x, y) \in \mathbb{C} : y > A(x)\}$, where $A : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, there exists an analogous family $\mathcal{D}(\partial\Omega)$. In this case, setting $T(x) = (x, A(x))$, one can take $\mathcal{D}(\partial\Omega) = T(\mathcal{D}(\mathbb{R}))$, for instance.

If Ω is either a Lipschitz or a special Lipschitz domain, to each $Q \in \mathcal{W}(\Omega)$ we assign a cube $\phi(Q) \in \mathcal{D}(\partial\Omega)$ such that $\phi(Q) \cap \rho Q \neq \varnothing$ and $\operatorname{diam}(\phi(Q)) \approx \ell(Q)$. So there exists some big constant M depending on the parameters of the Whitney decomposition and on the chord arc constant of $\partial\Omega$ such that

$$\phi(Q) \subset M\,Q, \qquad \text{and} \qquad Q \subset B(z, M\ell(\phi(Q))) \quad \text{for all } z \in \phi(Q).$$

From this fact, it easily follows that there exists some constant c_2 such that for every $Q \in \mathcal{W}(\Omega)$,

$$\#\{P \in \mathcal{D}(\partial\Omega) : P = \phi(Q)\} \leq c_2.$$

2.3. **Sobolev spaces.** Recall that for an open domain $\Omega \subset \mathbb{R}^n$, $1 \le p < \infty$, and a positive integer m, the Sobolev space $W^{m,p}(\Omega)$ consists of the functions $f \in L^1_{loc}(\Omega)$ such that

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} < \infty,$$

where $D^\alpha f$ is the α-th derivative of f, in the sense of distributions. The homogeneous Sobolev seminorm $\dot{W}^{m,p}$ is defined by

$$\|f\|_{\dot{W}^{m,p}(\Omega)} := \left(\sum_{|\alpha|=m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}.$$

For a non integer $0 < \alpha < 1$, one sets

$$(2.1) \qquad D^\alpha f(x) = \left(\int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}}\, dm(y) \right)^{\frac{1}{2}},$$

and then

$$\|f\|_{W^{\alpha,p}(\Omega)} = \left(\|f\|^p_{L^p(\Omega)} + \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}.$$

The homogeneous Sobolev seminorm $\dot{W}^{\alpha,p}(\Omega)$ equals

$$\|f\|_{\dot{W}^{\alpha,p}(\Omega)} = \|D^\alpha f\|_{L^p(\Omega)}.$$

2.4. **Besov spaces.** In this section we review some basic results concerning Besov spaces. We only consider the 1-dimensional case, and pay special attention to the homogeneous Besov spaces $\dot{B}^\alpha_{p,p}$, with $0 < \alpha < 1$.

Consider a radial $\mathcal{C}^\infty$ function $\eta : \mathbb{R} \to \mathbb{R}$ whose Fourier transform $\widehat{\eta}$ is supported in the annulus $A(0, 1/2, 3/2)$, such that setting $\eta_k(x) = \eta_{2^{-k}}(x) = 2^k \eta(2^k x)$,

$$(2.2) \qquad \sum_{k \in \mathbb{Z}} \widehat{\eta_k}(\xi) = 1 \qquad \text{for all } \xi \ne 0.$$

Then, for $f \in L^1_{loc}(\mathbb{R})$, $1 \le p, q < \infty$, and $\alpha > 0$, one defines the seminorm

$$\|f\|_{\dot{B}^\alpha_{p,q}} = \left(\sum_{k \in \mathbb{Z}} \|2^{k\alpha} \eta_k * f\|^q_p \right)^{1/q},$$

and the norm

$$\|f\|_{B^\alpha_{p,q}} = \|f\|_p + \|f\|_{\dot{B}^{2-1/p}_{p,p}}.$$

The homogeneous Besov space $\dot{B}^\alpha_{p,q} \equiv \dot{B}^\alpha_{p,q}(\mathbb{R})$ consists of the functions such that $\|f\|_{\dot{B}^\alpha_{p,q}} < \infty$, while the functions in the Besov space $B^\alpha_{p,q} \equiv B^\alpha_{p,q}(\mathbb{R})$ are those such that $\|f\|_{B^\alpha_{p,q}} < \infty$. If one chooses a function different from η which satisfies the same properties as η above, then one obtains an equivalent seminorm and norm, respectively. Let us remark that the seminorm $\| \cdot \|_{\dot{B}^\alpha_{p,q}}$ is a norm if one considers functions modulo polynomials.

For any function f and any $h > 0$, denote $\Delta_h(f)(x) = f(x+h) - f(x)$. For $1 \le p, q < \infty$ and $0 < \alpha < 1$, it turns out that

$$(2.3) \qquad \|f\|_{\dot{B}^\alpha_{p,q}}^p \approx \int_0^\infty \frac{\|\Delta_h(f)\|_q^p}{h^{\alpha p + 1}} \, dh,$$

assuming f to be compactly supported, say. Otherwise the comparability is true modulo polynomials, that is, above we should replace $\|\Delta_h(f)\|_q$ by

$$\inf_{p \text{ polynomial}} \|\Delta_h(f + p)\|_q.$$

See [Tr, p. 242], for instance. Analogous characterizations hold for Besov spaces with regularity $\alpha \ge 1$. In this case it is necessary to use differences of higher order.

Observe that, for $p = q$ and $0 < \alpha < 1$, one has

$$(2.4) \qquad \|f\|_{\dot{B}^\alpha_{p,p}}^p \approx \iint_0^\infty \frac{|\Delta_h(f)|^p}{h^{\alpha p + 1}} \, dh \, dx \approx \iint \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + 1}} \, dx \, dy,$$

for f with compact support. These results motivate the definition of the $\dot{B}^\alpha_{p,p}$-seminorm over a chord arc curve. If Γ is such a curve and $f \in L^1(\mathcal{H}^1 \lfloor \Gamma)$, then one defines

$$\|f\|_{\dot{B}^\alpha_{p,p}(\Gamma)}^p = \iint_{(x,y) \in \Gamma^2} \frac{|f(x) - f(y)|^p}{|x-y|^{\alpha p + 1}} \, d\mathcal{H}^1(x) \, d\mathcal{H}^1(y).$$

The same definition applies if Γ is a Lipschitz graph, say. If $\gamma : S^1(0, r) \to \Gamma$ or $\gamma : \mathbb{R} \to \Gamma$ is a bilipschitz parameterization of Γ (such as the arc length parameterization), clearly we have

$$\|f\|_{\dot{B}^\alpha_{p,q}(\Gamma)} \approx \|f \circ \gamma\|_{\dot{B}^\alpha_{p,q}(S^1(0,r))},$$

for f compactly supported.

Concerning the Besov spaces of regularity $1 < \alpha < 2$, let us remark that, for $f \in L^1_{loc}(\mathbb{R})$,

$$(2.5) \qquad \|f\|_{\dot{B}^\alpha_{p,q}}^p \approx \|f'\|_{\dot{B}^{\alpha-1}_{p,q}}^p,$$

where f' is the distributional derivative of f. Further we will use the following characterization in terms of the coefficients β_1 defined in (1.8), due to Dorronsoro [Do, Theorems 1 and 2]. For $0 < \alpha < 2$ and $1 \le p, q < \infty$, one has:

$$\|f\|_{\dot{B}^\alpha_{p,q}} \approx \left(\int_0^\infty \left(h^{-\alpha+1} \|\beta_1(f, I(\cdot, h))\|_p \right)^q \frac{dh}{h} \right)^{1/q}.$$

Again, this comparability should be understood modulo polynomials, unless f is compactly supported, say. In the case $p = q$, an equivalent statement is the following:

$$\|f\|_{\dot{B}^\alpha_{p,p}}^p \approx \left(\sum_{I \in \mathcal{D}(\mathbb{R})} \left(\frac{\beta_1(f, I)}{\ell(I)^{\alpha-1}} \right)^p \ell(I) \right)^{1/p}.$$

For other indices $\alpha \ge 2$, there are analogous results which involve approximation by polynomials of a fixed degree instead of affine functions. Let us remark that the

coefficients $\beta_1(f, I)$ are not introduced in [Do], and instead a different notation is used.

3. Auxiliary lemmas

3.1. **About Besov spaces.**

Lemma 3.1. *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with $\|A'\|_\infty \leq c_0$ and $\Gamma \subset \mathbb{C}$ its graph. Denote by $N_0(x)$ the unit normal of Γ at $(x, A(x))$ (whose vertical component is negative, say), which is defined a.e. Then,*

$$(3.1) \qquad |\Delta_h(A')(x)| \approx |\Delta_h N_0(x)|,$$

with constants depending on c_0. Thus, for $1 \leq p < \infty$ and $0 < \alpha < 1$,

$$(3.2) \qquad \|A\|_{\dot{B}^{\alpha+1}_{p,p}} \approx \|A'\|_{\dot{B}^{\alpha}_{p,p}} \approx \|N_0\|_{\dot{B}^{\alpha}_{p,p}},$$

with constants depending on α and p, and also on c_0 in the second estimate.

Above, we set

$$\|N_0\|_{\dot{B}^{\alpha}_{p,p}} := \|N_{0,1}\|_{\dot{B}^{\alpha}_{p,p}} + \|N_{0,2}\|_{\dot{B}^{\alpha}_{p,p}},$$

where $N_{0,i}$, $i = 1, 2$, are the components of N_0.

For the proof, see [CT].

Remark 3.2. As mentioned in Subsection 2.4, from the characterization of Besov spaces in terms of differences, it turns out that if $\gamma : \mathbb{R} \to \Gamma$ is an arc length parameterization of the Lipschitz graph Γ, and $N(z)$ stands for the unit normal at $z \in \Gamma$ (with a suitable orientation), then

$$\|N \circ \gamma\|_{\dot{B}^{\alpha-1/p}_{p,p}} \approx \|N_0\|_{\dot{B}^{\alpha-1/p}_{p,p}} \approx \|N\|_{\dot{B}^{\alpha-1/p}_{p,p}(\Gamma)},$$

for $0 < \alpha \leq 1$ and $1 < p < \infty$ such that $\alpha p > 1$.

Recall that for a Lipschitz domain Ω, whose boundary has an arc length parameterization $\gamma : S^1(0, r_0) \to \partial\Omega$ (with $2\pi r_0 = \mathcal{H}^1(\partial\Omega)$), if $N(z)$ stands for the outward unit normal at $z \in \partial\Omega$, we also have

$$\|N\|_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)} \approx \|N \circ \gamma\|_{\dot{B}^{\alpha-1/p}_{p,p}(S^1(0,r_0))}.$$

In (1.8) we defined the coefficients β_1 associated to a function f. Now we introduce an analogous notion replacing f by a chord arc curve Γ (which may be the boundary of a Lipschitz domain). Given $P \in \mathcal{D}(\Gamma)$, we set

$$(3.3) \qquad \beta_1(\Gamma, P) = \inf_L \frac{1}{\ell(P)} \int_{3P} \frac{\mathrm{dist}(x, L)}{\ell(P)} \, d\mathcal{H}^1(x),$$

where the infimum is taken over all the lines $L \subset \mathbb{C}$.

Next lemma is a direct consequence of the previous results and the characterization of homogeneous Besov spaces in terms of the β_1's from Dorronsoro. For the detailed proof, see [CT].

Lemma 3.3. *Let Ω be a Lipschitz domain. Suppose that the outward unit normal satisfies $N \in \dot{B}_{p,p}^{\alpha}(\partial\Omega)$, for some $1 \le p < \infty$, $0 < \alpha < 1$. Then,*

$$\sum_{P \in \mathcal{D}(\partial\Omega)} \left(\frac{\beta_1(\partial\Omega, P)}{\ell(P)^{\alpha}} \right)^{p} \ell(P) \lesssim \|N\|_{\dot{B}_{p,p}^{\alpha}(\partial\Omega)}^{p} + c\,\mathcal{H}^{1}(\partial\Omega)^{1-\alpha p}.$$

with c depending on $\mathcal{H}^1(\partial\Omega)/R$.

Lemma 3.4. *Consider functions $\varphi, f : \mathbb{R} \to \mathbb{R}$ and let $1 \le p < \infty$ and $0 < \alpha < 1$. We have*

$$\|f\|_{\dot{B}_{p,p}^{\alpha}}^{p} \lesssim \iint_{0}^{\infty} \frac{|\varphi\,\Delta_h(f)|^p}{h^{\alpha p+1}}\, dh\, dx + \|\varphi\|_{\dot{B}_{p,p}^{\alpha}}^{p} \|f\|_{\infty}^{p}.$$

Proof. Recall that

$$(3.4) \qquad \|\varphi f\|_{\dot{B}_{p,p}^{\alpha}}^{p} \approx \iint_{0}^{\infty} \frac{|\Delta_h(\varphi f)|^p}{h^{\alpha p+1}}\, dh\, dx.$$

To prove the lemma, we just use that

$$\Delta_h(\varphi\, f)(x) = \varphi(x)\,\Delta_h(f)(x) + f(x+h)\,\Delta_h(\varphi)(x).$$

Then, plugging this identity into (3.4) the lemma follows easily. $\qquad\square$

Lemma 3.5. *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipscthitz function supported on an interval I. For $1 \le p < \infty$ and $0 < \alpha < 1$, we have*

$$\|A\|_{\dot{B}_{p,p}^{\alpha}} \lesssim \|A'\|_{\infty}\, \ell(I)^{1-\alpha+1/p},$$

with a constant depending on α and p.

Proof. Denote $\ell = \ell(I)$. We set

$$(3.5)$$
$$\begin{aligned}
\|A\|_{\dot{B}_{p,p}^{\alpha}}^{p} &\approx \iint_{0}^{\infty} \frac{|\Delta_h(A)|^p}{h^{\alpha p}} \frac{dh}{h}\, dx \\
&= \iint_{0 < h \le \ell} \frac{|A(x+h) - A(x)|^p}{h^{\alpha p+1}}\, dh\, dx + \iint_{h > \ell} \frac{|A(x+h) - A(x)|^p}{h^{\alpha p+1}}\, dh\, dx.
\end{aligned}$$

For the first integral on the right side we use fact that the integrand vanishes unless $x \in 3I$ and the Lipschitz condition $|A(x+h) - A(x)| \le \|A'\|_{\infty} h$:

$$\begin{aligned}
\iint_{0<h\le\ell} \frac{|A(x+h) - A(x)|^p}{h^{\alpha p+1}}\, dh\, dx &\le \|A'\|_{\infty}^{p} \int_{x\in 3I} \int_{h \le \ell} \frac{h^p}{h^{\alpha p+1}}\, dh\, dx \\
&= 3\|A'\|_{\infty}^{p}\ell \int_{0<h\le\ell} h^{p(1-\alpha)-1}\, dh \approx \|A'\|_{\infty}^{p}\, \ell^{p+1-\alpha p}.
\end{aligned}$$

Concerning the last integral in (3.5), we use again the fact that A is supported on I and the estimate $|A(x+h) - A(x)| \leq \|A'\|_\infty \ell$:

$$\iint_{h>\ell} \frac{|A(x+h) - A(x)|^p}{h^{\alpha p+1}} \, dh \, dx$$

$$\leq \|A'\|_\infty^p \ell^p \left(\int_{x \in I} \int_{h>\ell} \frac{1}{h^{\alpha p+1}} \, dh \, dx + \int_{x \in I-h} \int_{h>\ell} \frac{1}{h^{\alpha p+1}} \, dh \, dx \right) \approx \|A'\|_\infty^p \ell^{p+1-\alpha p}.$$

$\square$

3.2. **About the Beurling transform of χ_Ω.** Let $\Omega \subset \mathbb{C}$ be an open set. If Ω has finite Lebesgue measure, then

$$(3.6) \qquad B\chi_\Omega(z) = \lim_{\varepsilon \to 0} \frac{-1}{\pi} \int_{|z-w|>\varepsilon} \frac{1}{(z-w)^2} \chi_\Omega(w) \, dm(w).$$

Otherwise, $B(\chi_\Omega)$ is a BMO function and, thus, it is defined modulo constants. Actually, a possible way to assign a precise value to $B(\chi_\Omega)(z)$ is the following:

$$(3.7) \qquad B\chi_\Omega(z) = \lim_{\varepsilon \to 0} \frac{-1}{\pi} \int_{|z-w|>\varepsilon} \left(\frac{1}{(z-w)^2} - \frac{1}{(z_0-w)^2} \right) \chi_\Omega(w) \, dm(w),$$

where z_0 is some fixed point, with $z_0 \notin \overline{\Omega}$, for example. It is easy to check that the preceding principal value integral exists for all $z \in \mathbb{C}$.

Moreover the following results hold:

Lemma 3.6. *Let $\Omega \subset \mathbb{C}$ be an open set. The function $B(\chi_\Omega)$ is analytic in $\mathbb{C} \setminus \partial\Omega$ and moreover, for every $z \in \mathbb{C} \setminus \partial\Omega$,*

$$(3.8) \qquad \partial B(\chi_\Omega)(z) = \frac{2}{\pi} \int_{|z-w|>\varepsilon} \frac{1}{(z-w)^3} \chi_\Omega(w) \, dm(w),$$

for $0 < \varepsilon < \mathrm{dist}(z, \partial\Omega)$.

When Ω has infinite measure, saying that $B(\chi_\Omega)$ is analytic in $\mathbb{C} \setminus \partial\Omega$ means that the function defined in (3.7) is analytic for each choice of z_0. Notice that, in any case, the derivative $\partial B(\chi_\Omega)$ is independent of z_0.

Lemma 3.7. *Let $\Pi \subset \mathbb{C}$ be a half plane. Then $\partial B(\chi_\Pi) = 0$ in $\mathbb{C} \setminus \partial\Pi$. Equivalently, for all $z \notin \partial\Pi$ and $0 < \varepsilon < \mathrm{dist}(z, \partial\Pi)$, we have*

$$\int_{|z-w|>\varepsilon} \frac{1}{(z-w)^3} \chi_\Pi(w) \, dm(w) = 0.$$

For the proofs of the preceding two lemmas, see [CT], for example. By using very similar arguments, for any $z \in \mathbb{C} \setminus \partial\Omega$, one gets

$$(3.9) \qquad \partial^2 B(\chi_\Omega)(z) = \frac{-6}{\pi} \int_{|z-w|>\varepsilon} \frac{1}{(z-w)^4} \chi_\Omega(w) \, dm(w),$$

for $0 < \varepsilon < \mathrm{dist}(z, \partial\Omega)$. The details are left for the reader.

Lemma 3.8. *Let Ω be either a Lipschitz or a special Lipschitz domain. For all $w \in \mathbb{C} \setminus \partial\Omega$ and all ε with $0 < \varepsilon < \mathrm{dist}(w, \partial\Omega)$, we have*

$$(3.10) \qquad \partial B(\chi_\Omega)(w) = \frac{i}{2\pi} \int_{\partial\Omega} \frac{1}{(w-z)^2} \, d\bar{z}.$$

Proof. In sense of distributions, we have

$$\partial B(\chi_\Omega) = \frac{-1}{\pi} \partial \left(\mathrm{p.v.} \frac{1}{z^2} * \chi_\Omega \right).$$

Suppose that first that Ω is bounded. Then we have

$$(3.11) \qquad \mathrm{p.v.} \frac{1}{z^2} * \chi_\Omega = \mathrm{p.v.} \frac{1}{z^2} * \partial\chi_\Omega,$$

It turns out that, in the sense of distributions,

$$\partial\chi_\Omega = -\frac{i}{2} \, d\bar{z} \lfloor \partial\Omega.$$

Indeed, given $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$,

$$\langle \partial\chi_\Omega, \, \varphi \rangle = - \int_\Omega \partial\varphi \, dm = -\frac{i}{2} \int_{\partial\Omega} \varphi \, d\bar{z},$$

which proves our claim.

So we deduce that

$$\partial B(\chi_\Omega) = \frac{i}{2\pi} \, \mathrm{p.v.} \frac{1}{z^2} * d\bar{z} \lfloor \partial\Omega,$$

in the sense of distributions. From the first identity in (3.10), it is clear $\partial B(\chi_\Omega)$ is analytic in $\mathbb{C} \setminus \partial\Omega$. So the identity above holds pointwise in $\mathbb{C} \setminus \partial\Omega$.

For a special a Lipschitz domain we have to be a little careful, because both $\mathrm{p.v.} \dfrac{1}{z^2}$ and χ_Ω are distributions with non compact support, and so the identity (3.11) is not clear.

Consider the upper half plane $\Pi = \{(x,y) : y > 0\}$. Since $B(\chi_\Pi)$ is constant in $\mathbb{C} \setminus \mathbb{R}$, $\partial B(\chi_\Pi) = 0$ in $\mathbb{C} \setminus \mathbb{R}$, and so

$$\partial B(\chi_\Omega) = \partial B(\chi_\Omega - \chi_\Pi) = \frac{-1}{\pi} \partial \left(\mathrm{p.v.} \frac{1}{z^2} * (\chi_\Omega - \chi_\Pi) \right) \quad \text{in } \mathbb{C} \setminus \mathbb{R}.$$

Now, observe that $\chi_\Omega - \chi_\Pi$ has compact support, because the Lipschitz function A has compact support, and thus

$$\left(\mathrm{p.v.} \frac{1}{z^2} * (\chi_\Omega - \chi_\Pi) \right) = \mathrm{p.v.} \frac{1}{z^2} * \partial(\chi_\Omega - \chi_\Pi).$$

In the sense of distributions,

$$\partial\chi_\Omega = -\frac{i}{2} \, d\bar{z} \lfloor \partial\Omega \quad \text{and} \quad \partial\chi_\Pi = -\frac{i}{2} \, d\bar{z} \lfloor \partial\Pi.$$

As a consequence,

$$\partial B(\chi_\Omega - \chi_\Pi) = \frac{i}{2\pi} \, \mathrm{p.v.} \frac{1}{z^2} * (d\bar{z} \lfloor \Omega - d\bar{z} \lfloor \partial\Pi).$$

Therefore,

$$\partial B(\chi_\Omega) = \frac{i}{2\pi} \, \text{p.v.} \, \frac{1}{z^2} * (d\bar{z}\lfloor\Omega - d\bar{z}\lfloor\partial\Pi) \quad \text{in } \mathbb{C} \setminus \mathbb{R}.$$

Taking into account that $d\bar{z}\lfloor\partial\Pi = dz\lfloor\partial\Pi$ and, by Cauchy's formula,

$$\int_{\partial\Pi} \frac{1}{(w-z)^2} \, dz = 0 \quad \text{for all } w \in \mathbb{C} \setminus \mathbb{R},$$

we deduce that

$$\partial B(\chi_\Omega)(w) = \frac{i}{2\pi} \int_{\partial\Omega} \frac{1}{(w-z)^2} \, d\bar{z} \quad \text{for all } w \in \mathbb{C} \setminus (\mathbb{R} \cup \partial\Omega).$$

Since $\partial B(\chi_\Omega)$ is analytic in $\mathbb{C} \setminus \partial\Omega$, the identity above holds for all $w \in \mathbb{C} \setminus \partial\Omega$. $\quad\square$

The following result is a straightforward consequence of the preceding lemma.

Lemma 3.9. *Let* $\Omega = \{(x,y) \in \mathbb{C} : y = A(x)\}$ *be a special Lipschitz domain (with* A *Lipschitz and compactly supported). Then we have*

$$\partial B(\chi_\Omega)(w) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{A'(x)}{(x + iA(x) - w)^2} \, dx, \qquad \text{for all } w \notin \partial\Omega.$$

Proof. By Cauchy's formula, it follows that

$$\int_{\partial\Omega} \frac{1}{(w-z)^2} \, dz = 0.$$

From this fact and the preceding lemma, we infer that

$$(3.12) \qquad\qquad \partial B(\chi_\Omega)(w) = \frac{i}{2\pi} \int_{\partial\Omega} \frac{1}{(z-w)^2} \, (d\bar{z} - dz).$$

The boundary $\partial\Omega$ can be parameterized by $\{x + iA(x) : x \in \mathbb{R}\}$. Then we have $dz = (1 + iA'(x)) \, dx$ and thus $d\bar{z} - dz = -2iA'(x) \, dx$. Plugging this identity into (3.12), one concludes the lemma. $\quad\square$

Lemma 3.10. *Let* $\Omega \subset \mathbb{C}$ *be either a Lipschitz or a special Lipschitz domain. Then, for all* $z \in Q$ *with* $Q \in \mathcal{W}(\Omega)$, *Then,*

$$(3.13) \qquad\qquad \left|\partial B\chi_\Omega(z)\right| \lesssim \sum_{R \in \mathcal{D}(\partial\Omega):R \supset \phi(Q)} \frac{\beta_1(R)}{\ell(R)} + \frac{1}{\text{diam}(\Omega)},$$

and

$$(3.14) \qquad\qquad \left|\partial^2 B\chi_\Omega(z)\right| \lesssim \sum_{R \in \mathcal{D}(\partial\Omega):R \supset \phi(Q)} \frac{\beta_1(R)}{\ell(R)^2} + \frac{1}{\text{diam}(\Omega)^2}.$$

Proof. The estimate (3.13) has been proved in [CT] (see equation (5.2) there). The proof of the inequality (3.14) is very similar. For completeness, we sketch the arguments. We may assume that $\beta_1(c_4\phi(Q)) \leq \varepsilon_0$, with $\varepsilon_0 > 0$ small enough, for some fixed absolute constant $c_4 > 10$, say. Indeed, from (3.9) it turns out

that $\left|\partial^2 B\chi_\Omega(z)\right| \leq c/\ell(Q)^2$, by choosing $\varepsilon = \ell(Q)$ there, and so (3.14) holds if $\beta_1(c_4\phi(Q)) > \varepsilon_0$, with some constant depending on ε_0.

So suppose that $\beta_1(c_4\phi(Q)) \leq \varepsilon_0$, with ε_0 very small. In this case, L_Q is very close to $\partial\Omega$ near $\phi(Q)$, and then one infers that $\mathrm{dist}(z, L_Q) \approx \ell(Q)$. Denote by Π_Q the half plane whose boundary is L_Q and contains z. Take $0 < \varepsilon < \mathrm{dist}(z, \partial\Omega)$. Since $(\frac{1}{z^4}\chi_{B(0,\varepsilon)^c}) * \chi_{\Pi_Q}$ vanishes on $\Pi_Q \ni z$ (because $\partial^2 B\chi_\Omega(z) = 0$), we have

$$\left|\partial^2 B\chi_\Omega(z)\right| = \left|\left(\frac{6\pi}{z^4}\chi_{B(0,\varepsilon)^c}\right) * (\chi_\Omega - \chi_{\Pi_Q})(z)\right| \leq \frac{6\pi}{|z|^3} * \chi_{\Omega\Delta\Pi_Q}(z).$$

For each $n \geq 0$, let B_n be a ball centered at $w' \in \phi(Q)$ with

$$\mathrm{diam}(B_n) = 2^n\mathrm{diam}(\phi(Q)) \approx 2^n\ell(Q),$$

and set also $B_{-1} = \varnothing$. For some n_0 such that $\mathrm{diam}(\Omega) \approx \mathrm{diam}(B_{n_0})$, similarly to [CT], we have

$$\frac{6\pi}{|z|^4} * \chi_{\Omega\Delta\Pi_Q}(z) = \sum_{n=0}^{n_0} \frac{6\pi}{|z|^4} * \chi_{B_n\cap(\Omega\Delta\Pi_Q)}(z) + \frac{6\pi}{|z|^4} * \chi_{B_N^c\cap(\Omega\Delta\Pi_Q)}(z)$$

$$\leq c\sum_{n=0}^{n_0} \frac{1}{\ell(2^nQ)^4} m(B_n \cap (\Omega\Delta\Pi_Q)) + \frac{c}{\mathrm{diam}(\Omega)^2}.$$

By Lemma 4.3 from [CT], we have

$$m(B_n \cap (\Omega\Delta\Pi_Q)) \leq c \sum_{P\in\mathcal{D}(\partial\Omega):\phi(Q)\subset P\subset R} \beta_1(P)\mathrm{diam}(R)^2,$$

where $R \in \mathcal{D}(\partial\Omega)$ is some cube containing $\phi(Q)$ such that $\ell(R) \approx \mathrm{diam}(B_n)$. Then we obtain

$$(3.15) \qquad \frac{6\pi}{|z|^4} * \chi_{\Omega\Delta\Pi_Q}(z) \leq c \sum_{\substack{R\in\mathcal{D}(\partial\Omega):\\ R\supset\phi(Q)}} \frac{1}{\ell(R)^4} \sum_{\substack{P\in\mathcal{D}(\partial\Omega):\\ \phi(Q)\subset P\subset R}} \beta_1(P)\ell(R)^2 + \frac{c}{\mathrm{diam}(\Omega)^2}$$

$$= c \sum_{\substack{P\in\mathcal{D}(\partial\Omega):\\ P\supset\phi(Q)}} \beta_1(P) \sum_{\substack{R\in\mathcal{D}(\partial\Omega):\\ R\supset P}} \frac{1}{\ell(R)^2} + \frac{c}{\mathrm{diam}(\Omega)^2}$$

$$\leq c \sum_{\substack{P\in\mathcal{D}(\partial\Omega):\\ P\supset\phi(Q)}} \frac{\beta_1(P)}{\ell(P)^2} + \frac{c}{\mathrm{diam}(\Omega)^2},$$

which proves (3.14). $\qquad\qquad\square$

Since $B\chi_\Omega$ is analytic in Ω, it is clear that

$$\|B\chi_\Omega\|_{\dot{W}^{1,p}(\Omega)}^p \approx \int_\Omega |\partial B\chi_\Omega|^p\, dm.$$

On the other hand, for $0 < \alpha < 1$, we set $\|B\chi_\Omega\|_{\dot{W}^{\alpha,p}(\Omega)} = \|D^\alpha B\chi_\Omega\|_{L^p(\Omega)}$, with $D^\alpha f$ defined in (2.1). The following lemma relates $D^\alpha B\chi_\Omega$ to $\partial B\chi_\Omega$, and it will play a key role for the proof of Theorems 1.1 and 1.3 in the case $0 < \alpha < 1$.

Lemma 3.11. *Let $0 < \alpha < 1$ and $1 < p < \infty$ be such that $\alpha p > 1$. For $0 < \theta \le 1$, if Ω is either a Lipschitz domain or a special Lipschitz domain, we have*
(3.16)
$$\|B(\chi_\Omega)\|^p_{\dot{W}^{\alpha,p}(\Omega)} \gtrsim \theta^{p-\alpha p} \int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z,\partial\Omega)^{p-\alpha p}\, dm(z) - c_3\, \theta^{2p-\alpha p} \|N\|^p_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)},$$
where the constant c depends on p and α.

Notice that the integral on the right side is multiplied by $\theta^{p-\alpha p}$, while $\|N\|^p_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)}$ by $\theta^{2p-\alpha p}$. The fact that for $\theta \ll 1$ we have $\theta^{2p-\alpha p} \ll \theta^{p-\alpha p}$ will be important for the proof of Theorems 1.1 and 1.3.

Proof. For $x \in Q \in \mathcal{W}(\Omega)$, we have

(3.17) $$D^\alpha f(x)^2 = \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{2+2\alpha}}\, dm(y) \ge \int_{|y-x| \le \theta\ell(Q)} \frac{|f(x) - f(y)|^2}{|x-y|^{2+2\alpha}}\, dm(y).$$

Let f be analytic in Ω, such as $B\chi_\Omega$. For $x \in Q \in \mathcal{W}(\Omega)$ and $y \in \Omega$ such that
$$|x - y| \le \theta\ell(Q),$$
we have
$$|f(y) - f(x) - f'(x)(y-x)| \le \frac{1}{2} \sup_{w \in 3Q} |f''(w)|\, |x-y|^2.$$

Thus,

(3.18) $$2|f(y) - f(x)|^2 \ge |f'(x)(y-x)|^2 - \sup_{w \in 3Q} |f''(w)|^2\, |x-y|^4.$$

Plugging this estimate into (3.17) yields
$$\begin{aligned}
D^\alpha f(x)^2 \ge{}& \int_{|y-x| \le \theta\ell(Q)} \frac{1}{|x-y|^{2\alpha}}\, |f'(x)|^2\, dm(y) \\
&- \int_{|y-x| \le \theta\ell(Q)} |x-y|^{2-2\alpha} \sup_{w \in 3Q} |f''(w)|^2\, dm(y) \\
\gtrsim{}& \theta^{2-2\alpha}\ell(Q)^{2-2\alpha}\, |f'(x)|^2 - c\, \theta^{4-2\alpha}\ell(Q)^{4-2\alpha} \sup_{w \in 3Q} |f''(w)|^2.
\end{aligned}$$

Therefore, since $\ell(Q) \approx \operatorname{dist}(x,\partial\Omega)$,
$$\begin{aligned}
\|D^\alpha f\|^p_{L^p(\Omega)} \gtrsim{}& \theta^{p-\alpha p} \int_\Omega |f'(x)|^p \operatorname{dist}(x,\partial\Omega)^{p-\alpha p}\, dm(x) \\
&- c\, \theta^{2p-\alpha p} \sum_{Q \in \mathcal{W}(\Omega)} \ell(Q)^{2+2p-\alpha p} \sup_{w \in 3Q} |f''(w)|^p.
\end{aligned}$$

Hence, to prove (3.16) it is enough to show that, for $f = B\chi_\Omega$,

$$(3.19) \qquad S := \sum_{Q \in \mathcal{W}(\Omega)} \ell(Q)^{2+2p-\alpha p} \sup_{w \in 3Q} |f''(w)|^p \lesssim \|N\|^p_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)}.$$

Now, from Lemma 3.10, it turns out that for all $w \in 3Q$, with $Q \in \mathcal{W}(\Omega)$

$$\left|\partial^2 B\chi_\Omega(w)\right| \lesssim \sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \frac{\beta_1(R)}{\ell(R)^2} + \frac{1}{\text{diam}(\Omega)^2}.$$

Then we infer the term S in (3.19) satisfies

$$S \lesssim \sum_{Q \in \mathcal{W}(\Omega)} \ell(Q)^{2+2p-\alpha p} \left(\sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \frac{\beta_1(R)}{\ell(R)^2} \right)^p + \frac{m(\Omega)}{\text{diam}(\Omega)^{\alpha p}}.$$

The last term on the right side is bounded by $\text{diam}(\Omega)^{2-\alpha p}$. For the first one we use Cauchy-Schwarz, and then we get

$$\left(\sum_{R \in \mathcal{D}(\partial\Omega): RR \supset \phi(Q)} \frac{\beta_1(R)}{\ell(R)^2} \right)^p \leq \left(\sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \frac{\beta_1(R)^p}{\ell(R)^{2p-1/2}} \right) \left(\sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \frac{1}{\ell(R)^{p'/(2p)}} \right)^{p/p'}$$

$$\lesssim \sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \frac{\beta_1(R)^p}{\ell(R)^{2p-1/2}} \frac{1}{\ell(\phi(Q))^{1/2}}.$$

Thus,

$$\sum_{Q \in \mathcal{W}(\Omega)} \ell(Q)^{2+2p-\alpha p} \left(\sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \frac{\beta_1(R)}{\ell(R)^2} \right)^p$$

$$\lesssim \sum_{Q \in \mathcal{W}(\Omega)} \sum_{P \in \mathcal{D}(\partial\Omega): P \supset \phi(Q)} \frac{\beta_1(P)^p}{\ell(P)^{2p-1/2}} \ell(\phi(Q))^{3/2+2p-\alpha p}$$

$$= \sum_{P \in \mathcal{D}(\partial\Omega)} \frac{\beta_1(P)^p}{\ell(P)^{2p-1/2}} \sum_{Q \in \mathcal{W}(\Omega): \phi(Q) \subset P} \ell(\phi(Q))^{3/2+2p-\alpha p}.$$

Notice that

$$\sum_{Q \in \mathcal{W}(\Omega): \phi(Q) \subset P} \ell(\phi(Q))^{3/2+2p-\alpha p} \lesssim \sum_{\widetilde{Q} \in \mathcal{D}(\partial\Omega): \widetilde{Q} \subset P} \ell(\widetilde{Q})^{3/2+2p-\alpha p} \lesssim \ell(P)^{3/2+2p-\alpha p},$$

because $3/2 + 2p - \alpha p > 1$. Hence,

$$\sum_{Q \in \mathcal{W}(\Omega)} \ell(Q)^{2+2p-\alpha p} \left(\sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \frac{\beta_1(R)}{\ell(R)^2} \right)^p \lesssim \sum_{P \in \mathcal{D}(\partial\Omega)} \beta_1(P)^p \ell(P)^{2-\alpha p}.$$

Therefore,

$$S \lesssim \sum_{P \in \mathcal{D}(\partial\Omega)} \left(\frac{\beta_1(P)}{\ell(P)^{\alpha+1/p}} \right)^p \ell(P) + \text{diam}(\Omega)^{2-\alpha p} \approx \|N\|^p_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)},$$

by Lemma 3.3, as wished.

The arguments for special Lipschitz domains are analogous, and even easier. Roughly speaking, the only difference is that the terms above which involve $\operatorname{diam}(\Omega)$ do not appear. $\qquad\square$

4. The main lemma and the proof of Theorem 1.3

The main result of this section is the following.

Lemma 4.1 (Main Lemma). *Let $\Omega \subset \mathbb{C}$ be a special δ-Lipschitz domain. Let $1 < p < \infty$ and $0 < \alpha \le 1$ be such that $\alpha p > 1$. If δ is small enough, then*

$$(4.1) \qquad \int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z, \partial\Omega)^{p-\alpha p}\, dm(z) \gtrsim \|N\|^p_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)}.$$

Before worrying about the proof of the preceding result we show that this yields Theorem 1.3 as an easy consequence.

Proof of Theorem 1.3. In the case $\alpha = 1$, it is clear that

$$\int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z, \partial\Omega)^{p-\alpha p}\, dm(z) \approx \|B\chi_\Omega\|^p_{\dot{W}^{1,p}(\Omega)},$$

and thus the theorem is a straightforward consequence of (4.1). For $0 < \alpha < 1$, we need to use Lemma 3.11 too. Indeed, if θ is chosen small enough, from the Main Lemma, we will have

$$\theta^{p-\alpha p} \int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z, \partial\Omega)^{p-\alpha p}\, dm(z) \ge 2c_3\, \theta^{2p-\alpha p} \|N\|^p_{B^{\alpha-1/p}_{p,p}(\partial\Omega)},$$

where c_3 is the constant appearing in (3.16). Then Lemma 3.11 tells us that

$$(4.2) \qquad \|B(\chi_\Omega)\|^p_{\dot{W}^{\alpha,p}(\Omega)} \gtrsim \theta^{p-\alpha p} \int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z, \partial\Omega)^{p-\alpha p}\, dm(z).$$

Together with the Main Lemma again this implies that

$$\|B(\chi_\Omega)\|^p_{\dot{W}^{\alpha,p}(\Omega)} \gtrsim \|N\|^p_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)}.$$

$\qquad\square$

Notice also that from the Main Lemma, the inequality (4.2), and the fact that

$$\|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)} \lesssim \|N\|_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)},$$

proved in [CT], we deduce the following.

Corollary 4.2. *Let $\Omega \subset \mathbb{C}$ be a special δ-Lipschitz domain. Let $1 < p < \infty$ and $0 < \alpha < 1$ be such that $\alpha p > 1$. If δ is small enough, then*

$$\|B(\chi_\Omega)\|^p_{\dot{W}^{\alpha,p}(\Omega)} \approx \int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z, \partial\Omega)^{p-\alpha p}\, dm(z) \approx \|N\|^p_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)}.$$

The remaining of this section is devoted to the proof of the Main Lemma. First some remarks abut notation and terminology: recall that in Subsection 2.2 to each square $Q \in \mathcal{W}(\Omega)$ we assigned a cube $\phi(Q) \in \mathcal{D}(\partial\Omega)$ with diameter and distance to Q both comparable to $\ell(Q)$. In the case of special Lipschitz domains the following precise definition of $\phi(Q)$ is very convenient. Given a square $Q = (a, b] \times (c, d] \in \mathcal{W}(\Omega)$, we consider the arc

$$\phi(Q) = \{(x, A(x)) : a < x \leq b\}.$$

In particular, notice that $\phi(Q) \in \mathcal{D}(\partial\Omega)$. Observe also that $\ell(Q) = \mathcal{H}^1(\phi(Q))$ and that

$$\text{dist}(Q, \phi(Q)) \approx \ell(Q).$$

We denote

$$\ell(\phi(Q)) := \ell(Q).$$

Moreover, given $a > 1$ and $P \in \mathcal{D}(\Omega)$ of the form

$$P = \{(x, A(x)) : x \in I\},$$

for some interval $I \subset \mathbb{R}$, we let aP be the following arc from $\partial\Omega$:

$$P = \{(x, A(x)) : x \in aI\}.$$

Lemma 4.3. *Let Ω be as in Theorem 1.3. Consider a square $Q \in \mathcal{W}(\Omega)$ and denote by L_Q a line that minimizes $\beta_1(\phi(Q))$. Let $y = g_Q(x)$ the affine map defining L_Q. Then, for any $w \in 3Q$ we have*

$$\left| \text{Im} \left(\partial B \chi_\Omega(w) - \frac{1}{\pi} \int_{\mathbb{R}} \frac{A'(x)}{(x + i \, g_Q(x) - w)^2} \, dx \right) \right| \leq c \, \delta \sum_{P \in \mathcal{D}(\partial\Omega) : P \supset \phi(Q)} \frac{\beta_1(P)}{\ell(P)} \, .$$

Proof. Let w, Q, L_Q, g_Q be as in the statement above. By the preceding lemma,

$$(4.3) \qquad \text{Im} \left(\partial B \chi_\Omega(w) - \frac{1}{\pi} \int_{\mathbb{R}} \frac{A'(x)}{(x + i \, g_Q(x) - w)^2} \, dx \right)$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \text{Im} \left(\frac{1}{(x + iA(x) - w)^2} - \frac{1}{(x + i \, g_Q(x) - w)^2} \right) A'(x) \, dx.$$

Denoting $w = a + ib$, we have

(4.4)

$$\text{Im} \left(\frac{1}{(x + iA(x) - w)^2} - \frac{1}{(x + i \, g_Q(x) - w)^2} \right)$$

$$= -2(x - a) \left(\frac{A(x) - b}{\left((x - a)^2 + (A(x) - b)^2 \right)^2} - \frac{g_Q(x) - b}{\left((x - a)^2 + (g_Q(x) - b)^2 \right)^2} \right).$$

We write the expression on the right inside the big parentheses as follows:

$$\frac{\big(A(x) - b\big) - \big(g_Q(x) - b\big)}{\big((x-a)^2 + (A(x) - b)^2\big)^2}$$

$$+ \big(g_Q(x) - b\big) \left(\frac{1}{\big((x-a)^2 + (A(x) - b)^2\big)^2} - \frac{1}{\big((x-a)^2 + (g_Q(x) - b)^2\big)^2} \right)$$

$$=: T_1 + T_2.$$

The first term equals

$$(4.5) \qquad\qquad\qquad T_1 = \frac{A(x) - g_Q(x)}{|x + iA(x) - w|^4}.$$

Concerning T_2, we have

$$\frac{1}{\big((x-a)^2 + (A(x) - b)^2\big)^2} - \frac{1}{\big((x-a)^2 + (g_Q(x) - b)^2\big)^2}$$

$$= \frac{\big[2(x-a)^2 + (g_Q(x) - b)^2 + (A(x) - b)^2\big]\big[(g_Q(x) - b)^2 - (A(x) - b)^2\big]}{\big((x-a)^2 + (A(x) - b)^2\big)^2\big((x-a)^2 + (g_Q(x) - b)^2\big)^2}$$

$$= \frac{\big[2(x-a)^2 + (g_Q(x) - b)^2 + (A(x) - b)^2\big]\big[g_Q(x) + A(x) - 2b\big]\big[g_Q(x) - A(x)\big]}{\big((x-a)^2 + (A(x) - b)^2\big)^2\big((x-a)^2 + (g_Q(x) - b)^2\big)^2}.$$

Notice now that

$$(x-a)^2 + (g_Q(x) - b)^2 \approx (x-a)^2 + (A(x) - b)^2.$$

This follows easily from the fact that A is a Lipschitz graph with small slope and so we can assume that the slope of g_Q is small and bounded independently of Q. Then, from the last calculation, we obtain

$$|T_2| \lesssim \frac{\big|A(x) - g_Q(x)\big|}{\big((x-a)^2 + (A(x) - b)^2\big)^2} = \frac{\big|A(x) - g_Q(x)\big|}{|x + iA(x) - w|^4}.$$

From (4.3), (4.4), (4.5), the last estimate, and the fact that $\|A'\|_\infty \leq \delta$, we deduce that

(4.6)
$$\left| \mathrm{Im}\left(\partial B\chi_\Omega(w) - \int_{\mathbb{R}} \frac{A'(x)}{(x + i\, g_Q(x) - w)^2}\, dx \right) \right| \lesssim \int_{\mathbb{R}} \frac{|x - a|\, |A(x) - g_Q(x)|}{|x + iA(x) - w|^4}\, |A'(x)|\, dx$$

$$\lesssim \delta \int_{\mathbb{R}} \frac{|A(x) - g_Q(x)|}{|x + iA(x) - w|^3}\, dx.$$

Now we wish to estimate the last integral in (4.6). To this end, we set

$$(4.7) \qquad \int_{\mathbb{R}} \frac{|A(x) - g_Q(x)|}{|x + iA(x) - w|^3}\, dx \lesssim \sum_{k \geq 0} \int_{|x-a| \leq 2^k \ell(Q)} \frac{\mathrm{dist}(x, L_Q)}{(2^k \ell(Q))^3}\, dx.$$

Consider $R \in \mathcal{D}(\partial\Omega)$ such that $R \supset \phi(Q)$. Let L_R be a line that minimizes $\beta_1(R)$. Then, as shown in [CT],

$$\operatorname{dist}_H(L_Q \cap B(a, 5\ell(R)), \, L_R \cap B(a, 5\ell(R))) \leq c \sum_{P \in \mathcal{D}(\partial\Omega): Q \subset P \subset R} \beta_1(P) \, \ell(R),$$

where dist_H stands for the Hausdorff distance. As a consequence, for $x \in 3R$,

$$\operatorname{dist}(x, L_Q) = \operatorname{dist}(x, L_Q \cap B(a, 5\ell(R))) \lesssim \operatorname{dist}(x, L_R) + \sum_{P \in \mathcal{D}(\partial\Omega): Q \subset P \subset R} \beta_1(P) \, \ell(R).$$

Plugging this estimate into (4.7), we get

$$
\begin{aligned}
\int_{\mathbb{R}} \frac{|A(x) - g_Q(x)|}{|x + iA(x) - w|^3} \, dx &\lesssim \sum_{R \in \mathcal{D}(\partial\Omega): R \supset \phi(Q)} \int_{p_1(3R)} \frac{\operatorname{dist}(x, L_Q)}{\ell(R)^3} \, dx \\
&\lesssim \sum_{\substack{R \in \mathcal{D}(\partial\Omega): \\ R \supset \phi(Q)}} \int_{p_1(3R)} \frac{\operatorname{dist}(x, L_R)}{\ell(R)^3} \, dx + \sum_{\substack{R \in \mathcal{D}(\partial\Omega): \\ R \supset \phi(Q)}} \sum_{\substack{P \in \mathcal{D}(\partial\Omega): \\ \phi(Q) \subset P \subset R}} \frac{\beta_1(P)}{\ell(R)} \\
&\lesssim \sum_{\substack{R \in \mathcal{D}(\partial\Omega): \\ R \supset \phi(Q)}} \sum_{\substack{P \in \mathcal{D}(\partial\Omega): \\ \phi(Q) \subset P \subset R}} \frac{\beta_1(P)}{\ell(R)} \lesssim \sum_{\substack{P \in \mathcal{D}(\partial\Omega): \\ P \supset \phi(Q)}} \frac{\beta_1(P)}{\ell(P)}.
\end{aligned}
$$

Together with (4.6), this proves the lemma. $\qquad\square$

Lemma 4.4. *Let Ω be as in Theorem 1.3 and g_Q as in Lemma 4.3. We have*

$$(4.8) \quad \sum_{Q \in \mathcal{W}(\Omega)} \int_{3Q} \left| \operatorname{Im} \left(\partial B\chi_\Omega(w) - \frac{1}{\pi} \int_{\mathbb{R}} \frac{A'(x)}{(x + i\, g_Q(x) - w)^2} \, dx \right) \right|^p \ell(Q)^{p - \alpha p} \, dm(w)$$

$$\leq c\, \delta^p \, \|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p.$$

Proof. From Lemma 4.3 we infer that

$$\sum_{Q \in \mathcal{W}(\Omega)} \int_{3Q} \left| \operatorname{Im} \left(\partial B\chi_\Omega(w) - \frac{1}{\pi} \int_{\mathbb{R}} \frac{A'(x)}{(x + i\, g_Q(x) - w)^2} \, dx \right) \right|^p \ell(Q)^{p - \alpha p} \, dm(w)$$

$$\lesssim c\, \delta^p \sum_{Q \in \mathcal{W}(\Omega)} \left(\sum_{P \in \mathcal{D}(\partial\Omega): P \supset \phi(Q)} \frac{\beta_1(P)}{\ell(P)} \right)^p \ell(Q)^{2 + p - \alpha p}.$$

By Cauchy-Schwartz we have

$$
\begin{aligned}
\left(\sum_{P \in \mathcal{D}(\partial\Omega): P \supset \phi(Q)} \frac{\beta_1(P)}{\ell(P)} \right)^p &\leq \left(\sum_{P \in \mathcal{D}(\partial\Omega): P \supset \phi(Q)} \frac{\beta_1(P)^p}{\ell(P)^{p - \frac{1}{2}}} \right) \left(\sum_{P \in \mathcal{D}(\partial\Omega): P \supset \phi(Q)} \frac{1}{\ell(P)^{\frac{p'}{2p}}} \right)^{\frac{p}{p'}} \\
&\leq c \sum_{P \in \mathcal{D}(\partial\Omega): P \supset \phi(Q)} \frac{\beta_1(P)^p}{\ell(P)^{p - \frac{1}{2}}} \frac{1}{\ell(\phi(Q))^{1/2}}.
\end{aligned}
$$

Thus,

$$\sum_{Q\in\mathcal{W}(\Omega)}\left(\sum_{\substack{P\in\mathcal{D}(\partial\Omega):\\ P\supset\phi(Q)}}\frac{\beta_1(P)}{\ell(P)}\right)^p\ell(Q)^{p-\alpha p}\lesssim\sum_{Q\in\mathcal{W}(\Omega)}\sum_{\substack{P\in\mathcal{D}(\partial\Omega):\\ P\supset\phi(Q)}}\frac{\beta_1(P)^p}{\ell(P)^{p-\frac{1}{2}}}\,\ell(\phi(Q))^{3/2+p-\alpha p}$$

$$=\sum_{P\in\mathcal{D}(\partial\Omega)}\frac{\beta_1(P)^p}{\ell(P)^{p-\frac{1}{2}}}\sum_{\substack{Q\in\mathcal{W}(\Omega):\\ \phi(Q)\subset P}}\ell(\phi(Q))^{3/2+p-\alpha p}.$$

Notice that

$$\sum_{Q\in\mathcal{W}(\Omega):\phi(Q)\subset P}\ell(\phi(Q))^{3/2+p-\alpha p}\le c\sum_{\widetilde{Q}\in\mathcal{D}(\partial\Omega):\widetilde{Q}\subset P}\ell(\widetilde{Q})^{3/2+p-\alpha p}\le c\,\ell(P)^{3/2+p-\alpha p},$$

and so, the left side of (4.8) is bounded above by

$$c\,\delta^p\sum_{P\in\mathcal{D}(\partial\Omega)}\frac{\beta_1(P)^p}{\ell(P)^{\alpha p-2}}.$$

Observe now that the last sum can be written as

$$\sum_{P\in\mathcal{D}(\partial\Omega)}\frac{\beta_1(P)^p}{\ell(P)^{\alpha p-2}}=\sum_{P\in\mathcal{D}(\partial\Omega)}\left(\frac{\beta_1(P)}{\ell(P)^{\alpha-\frac{1}{p}}}\right)^p\ell(P).$$

By [Do, Theorems 1 and 2], this is comparable to $\|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p\approx\|N\|_{\dot{B}_{p,p}^{\alpha-1/p}}^p$, and so we are done. $\qquad\square$

As a corollary of the preceding lemma, using the finite overlapping of the squares $\{3Q\}_{Q\in\mathcal{W}(\Omega)}$, we deduce that

(4.9)
$$\int_\Omega|\partial B\chi_\Omega(w)|^p\mathrm{dist}(w,\partial\Omega)^{p-\alpha p}\,dm(w)\gtrsim\sum_{Q\in\mathcal{W}(\Omega)}\int_{3Q}|\mathrm{Im}(\partial B\chi_\Omega)|^p\ell(Q)^{p-\alpha p}\,dm(w)$$

$$\gtrsim\sum_{Q\in\mathcal{W}(\Omega)}\int_{3Q}\left|\int_\mathbb{R}\frac{A'(x)}{(x+i\,g_Q(x)-w)^2}\,dx\right|^p\ell(Q)^{p-\alpha p}\,dm(w)$$

$$-c\sum_{Q\in\mathcal{W}(\Omega)}\int_{3Q}\left|\mathrm{Im}\left(\partial B\chi_\Omega(w)-\int_\mathbb{R}\frac{A'(x)}{(x+i\,g_Q(x)-w)^2}\,dx\right)\right|^p\ell(Q)^{p-\alpha p}\,dm(w)$$

$$\ge\sum_{Q\in\mathcal{W}(\Omega)}\int_{3Q}\left|\int_\mathbb{R}\frac{A'(x)}{(x+i\,g_Q(x)-w)^2}\,dx\right|^p\ell(Q)^{p-\alpha p}\,dm(w)-c\,\delta^p\,\|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p.$$

We will prove below that

$$\sum_{Q\in\mathcal{W}(\Omega)}\int_{3Q}\left|\int_\mathbb{R}\mathrm{Im}\left(\frac{A'(x)}{(x+i\,g_Q(x)-w)^2}\right)\,dx\right|^p\ell(Q)^{p-\alpha p}\,dm(w)\gtrsim\|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p.$$

Together with (4.9), this will yield the Main Lemma 4.1, by taking δ small enough.

For a given $w = a + bi \in \Omega$ and $Q \in \mathcal{W}(\Omega)$ with $3Q \ni w$, we have

$$(4.10) \qquad \mathrm{Im}\left(\frac{1}{(x + i\, g_Q(x) - w)^2}\right) = \frac{-2(x - a)(g_Q(x) - b)}{\left((x - a)^2 + (g_Q(x) - b)^2\right)^2}.$$

Now we denote

$$s_Q(w) = b - g_Q(a).$$

Observe that $s_Q(w) \approx \mathrm{dist}(w, L_Q) \approx \mathrm{dist}(w, \partial\Omega)$. Now, for some number $|\theta_Q| \lesssim \delta$,

$$g_Q(x) = \theta_Q(x - a) + g_Q(a) = \theta_Q(x - a) + b - s_Q(w).$$

Writing θ instead of θ_Q and s instead of $s_Q(w)$ to simplify notation, the expression in (4.10) equals

$$(4.11) \qquad \frac{2(x - a)(s - \theta(x - a))}{\left((x - a)^2 + (\theta(x - a) - s)^2\right)^2} = \frac{2}{(1 + \theta^2)^2}\, \frac{s(x - a) - \theta(x - a)^2}{\left[\left(x - a - \frac{\theta s}{1+\theta^2}\right)^2 + \left(\frac{s}{1+\theta^2}\right)^2\right]^2}$$

Notice that, with the change of variables $y = x - a$, $t = s/(1 + \theta^2)$, the denominator in the last fraction can be written as $\left[(y - \theta\, t)^2 + t^2\right]^2$, while the numerator equals

$$(1 + \theta^2)\, t\, y - \theta\, y^2 = (t - \theta^2 t)(y - \theta\, t) + \theta\left[t^2 - (y - \theta\, t)^2\right].$$

So the last fraction on the right side of (4.11) equals

$$(4.12) \qquad \frac{(t - \theta^2 t)(y - \theta\, t)}{\left[(y - \theta\, t)^2 + t^2\right]^2} + \theta\, \frac{t^2 - (y - \theta\, t)^2}{\left[(y - \theta\, t)^2 + t^2\right]^2}.$$

From (4.10), (4.11), and (4.12), using that $|\theta| \leq \|A'\|_\infty \leq \delta \ll 1$ and recalling that θ depends on $Q \ni w$, but not on x, we obtain

$$\left|\int_{\mathbb{R}} \mathrm{Im}\left(\frac{A'(x)}{(x + i\, g_Q(x) - w)^2}\right) dx\right| \geq \left|\int_{\mathbb{R}} \frac{\frac{s}{1+\theta^2}\left(x - a - \frac{\theta s}{1+\theta^2}\right)}{\left[\left(x - a - \frac{\theta s}{1+\theta^2}\right)^2 + \left(\frac{s}{1+\theta^2}\right)^2\right]^2} A'(x)\, dx\right|$$

$$- c\,\delta \left|\int_{\mathbb{R}} \frac{\left(\frac{s}{1+\theta^2}\right)^2 - \left(x - a - \frac{\theta s}{1+\theta^2}\right)^2}{\left[\left(x - a - \frac{\theta s}{1+\theta^2}\right)^2 + \left(\frac{s}{1+\theta^2}\right)^2\right]^2} A'(x)\, dx\right|$$

$$=: I_1(w, Q) - c\,\delta\, I_2(w, Q).$$

Therefore,

(4.13)

$$\sum_{Q \in \mathcal{W}(\Omega)} \int_{3Q} \left|\int_{\mathbb{R}} \mathrm{Im}\left(\frac{A'(x)}{(x + i\, g_Q(x) - w)^2}\right) dx\right|^p \ell(Q)^{p - \alpha p}\, dm(w)$$

$$\gtrsim \sum_{Q \in \mathcal{W}(\Omega)} \int_{3Q} I_1(w, Q)^p \ell(Q)^{p - \alpha p}\, dm(w) - c\delta^p \sum_{Q \in \mathcal{W}(\Omega)} \int_{3Q} I_2(w, Q)^p \ell(Q)^{p - \alpha p}\, dm(w).$$

Lemma 4.5. *Under the assumptions and notation above,*

$$\sum_{Q\in\mathcal{W}(\Omega)}\int_{3Q}I_1(w,Q)^p\ell(Q)^{p-\alpha p}\,dm(w)\gtrsim\int_0^\infty\|t^{\frac{1}{p}-1-\alpha}\,\psi_t*A\|_p^p\,\frac{dt}{t},$$

where

$$\psi(x)=\frac{3x^2-1}{(x^2+1)^3}$$

and $\psi_t(x)=t^{-1}\psi(t^{-1}x)$. Moreover,

(4.14)
$$\int_0^\infty\|t^{\frac{1}{p}-1-\alpha}\,\psi_t*A\|_p^p\,\frac{dt}{t}\gtrsim\|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p.$$

Proof. Fix a square $Q\in\mathcal{W}(\Omega)$, and set $Q=(a_1,a_2]\times(b_1,b_2]$. Recall that, for $w=a+ib\in 3Q$,

$$I_1(w,Q)=\left|\int_{\mathbb{R}}\frac{\frac{s}{1+\theta^2}\left(x-a-\frac{\theta s}{1+\theta^2}\right)}{\left[\left(x-a-\frac{\theta s}{1+\theta^2}\right)^2+\left(\frac{s}{1+\theta^2}\right)^2\right]^2}\,A'(x)\,dx\right|,$$

where $s\equiv s_Q(w)=b-g_Q(a)$, and $\theta\equiv\theta_Q$ is the slope of the affine line defined by g_Q, which approximates $\partial\Omega\cap p_1(3Q)$. For $t>0$, consider the kernel

$$K_t(x)=\frac{t\,x}{[x^2+t^2]^2}.$$

Observe that

$$I_1(w,Q)=\left|K_{\frac{s}{1+\theta^2}}*A'\left(a+\frac{\theta s}{1+\theta^2}\right)\right|.$$

We have

$$\int_{3Q}I_1(w,Q)^p\,dm(w)=\int_{a_1-\ell(Q)}^{a_2+\ell(Q)}\int_{t_1-\ell(Q)}^{t_2+\ell(Q)}\left|K_{\frac{s}{1+\theta^2}}*A'\left(x+\frac{\theta s}{1+\theta^2}\right)\right|^p\,dt\,dx.$$

Observe now that, assuming δ small enough, for each $x\in p_1(3Q)$ we have

$$\int_{t_1-\ell(Q)}^{t_2+\ell(Q)}\left|K_{\frac{s}{1+\theta^2}}*A'\left(x+\frac{\theta s}{1+\theta^2}\right)\right|^p\,dt=\int_{t_1-\ell(Q)}^{t_2+\ell(Q)}\left|K_{\frac{t-g_Q(x)}{1+\theta^2}}*A'\left(x+\frac{\theta s}{1+\theta^2}\right)\right|^p\,dt$$

$$\geq\int_{t_1-\frac{1}{2}\ell(Q)}^{t_2+\frac{1}{2}\ell(Q)}\left|K_{t-A(a_1)}*A'\left(x+\frac{\theta s}{1+\theta^2}\right)\right|^p\,dt.$$

Also, it follows easily that

$$\int_{a_1-\ell(Q)}^{a_2+\ell(Q)} \int_{t_1-\frac{1}{2}\ell(Q)}^{t_2+\frac{1}{2}\ell(Q)} \left| K_{t-A(a_1)} * A'\left(x+\frac{\theta s}{1+\theta^2}\right)\right|^p dt\, dx$$

$$\geq \int_{a_1}^{a_2} \int_{t_1-\frac{1}{2}\ell(Q)}^{t_2+\frac{1}{2}\ell(Q)} \left| K_{t-A(a_1)} * A'\left(x\right)\right|^p dt\, dx$$

$$\geq \int_{a_1}^{a_2} \int_{t_1}^{t_2} \left| K_{t-A(x)} * A'\left(x\right)\right|^p dt\, dx.$$

As a consequence,

(4.15)

$$\sum_{Q\in\mathcal{W}(\Omega)} \int_{3Q} I_1(w,Q)^p \ell(Q)^{p-\alpha p}\, dm(w) \gtrsim \sum_{Q\in\mathcal{W}(\Omega)} \iint_{(x,t)\in Q} \left| K_{t-A(x)} * A'\left(x\right)\right|^p t^{p-\alpha p}\, dt\, dx$$

$$= \iint_{(x,t)\in\Omega} \left| K_{t-A(x)} * A'\left(x\right)\right|^p t^{p-\alpha p}\, dt\, dx$$

$$= \int_{x\in\mathbb{R}} \int_0^\infty \left| K_t * A'\left(x\right)\right|^p t^{p-\alpha p}\, dt\, dx.$$

Notice now that $K_t * A' = (K_t)' * A$. It is easy to check that $(K_t)' = -t^{-2}\,\psi_t$, and then the first inequality claimed in the lemma follows just writing

$$\left| K_t * A'\left(x\right)\right|^p t^{p-\alpha p} = \left| t^{\frac{1}{p}-1-\alpha}\,\psi_t * A\left(x\right)\right|^p \frac{1}{t}.$$

To prove (4.14), first we calculate the Fourier transform of ψ_t. Notice that $K_t = c\,(P_t)'$, where P_t is the Poisson kernel and c is some absolute constant. So,

(4.16) $$\widehat{\psi_t}(\xi) = c\,t^2\xi^2\,e^{-2\pi\, t\,|\xi|}.$$

Consider a radial $\mathcal{C}^\infty$ function η whose Fourier transform is supported in the annulus $A(0, 1/2, 3/2)$, and setting $\eta_{(k)}(x) = \eta_{2^{-k}}(x) = 2^k\,\eta(2^k\,x)$,

(4.17) $$\sum_{k\in\mathbb{Z}} \widehat{\eta_{(k)}}(\xi) = 1 \qquad \text{for all } \xi \neq 0.$$

Then we have

$$\|A\|^p_{\dot{B}^{1+\alpha-1/p}_{p,p}} \approx \sum_{k\in\mathbb{Z}} \|2^{k(1+\alpha-\frac{1}{p})}\eta_{(k)} * A\|^p_p.$$

Notice now that there exists some Schwartz function τ such that $\eta = \psi * \tau$. Indeed, we only have to take

$$\widehat{\tau}(\xi) = \frac{\widehat{\eta}(\xi)}{c\,\xi^2\,e^{-2\pi\,|\xi|}},$$

so that $\widehat{\tau} \in \mathcal{C}^\infty_c$. Similarly, for any $s \in [1,2]$, we take some Schwartz function τ^s such that

$$\eta = \psi_s * \tau^s.$$

Then, for every $k \in \mathbb{Z}$ and every $s \in [1, 2]$, we have

$$\eta_{(k)} * A = \psi_{s2^{-k}} * \tau_{2^{-k}}^s * A,$$

where $\tau_{2^{-k}}^s(x) = 2^k \tau^s(2^k x)$. Thus,

$$\|\eta_{(k)} * A\|_p \le \|\tau_{2^{-k}}^s\|_1 \|\psi_{s2^{-k}} * A\|_p \le c \|\psi_{s2^{-k}} * A\|_p,$$

where we took into account that $\|\tau_{2^{-k}}^s\|_1 = \|\tau^s\|_1 \le c$ for some constant independent of s. Then we deduce

$$\|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p \lesssim \sum_{k \in \mathbb{Z}} \|2^{k(1+\alpha-\frac{1}{p})} \psi_{s2^{-k}} * A\|_p^p \qquad \text{for all } s \in [1, 2].$$

As a consequence, by Fubini and a change of variables,

$$\|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p \lesssim \sum_{k \in \mathbb{Z}} \int_1^2 \|2^{k(1+\alpha-\frac{1}{p})} \psi_{s2^{-k}} * A\|_p^p \frac{ds}{s}$$

$$= \sum_{k \in \mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} \|2^{k(1+\alpha-\frac{1}{p})} \psi_t * A\|_p^p \frac{dt}{t} \approx \int_0^\infty \|t^{\frac{1}{p}-1-\alpha} \psi_t * A\|_p^p \frac{dt}{t}.$$

This proves the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.6. *Under the assumptions and notation above,*

$$\sum_{Q \in \mathcal{W}(\Omega)} \int_{3Q} I_2(w, Q)^p \ell(Q)^{p-\alpha p} \, dm(w) \lesssim \int_0^\infty \|t^{\frac{1}{p}-1-\alpha} \psi_t * A\|_p^p \frac{dt}{t},$$

where ψ_t is as in Lemma 4.5.

Proof. The arguments are quite similar to the ones for Lemma 4.5. Consider $Q \in \mathcal{W}(\Omega)$, and set $Q = (a_1, a_2] \times (b_1, b_2]$. Recall that, for $w = a + ib \in 3Q$,

$$I_2(w, Q) = \left| \int_{\mathbb{R}} \frac{\left(\frac{s}{1+\theta^2}\right)^2 - \left(x - a - \frac{\theta s}{1+\theta^2}\right)^2}{\left[\left(x - a - \frac{\theta s}{1+\theta^2}\right)^2 + \left(\frac{s}{1+\theta^2}\right)^2\right]^2} A'(x) \, dx \right|,$$

where $s \equiv s_Q(w) = b - g_Q(a)$, and $\theta \equiv \theta_Q$ is the slope of the affine line defined by g_Q. For $t > 0$, consider the kernel

$$J_t(x) = \frac{t^2 - x^2}{[x^2 + t^2]^2}.$$

Now we have

$$I_2(w, Q) = \left| J_{\frac{s}{1+\theta^2}} * A' \left(a + \frac{\theta s}{1+\theta^2}\right) \right|,$$

and so

$$\int_{3Q} I_2(w, Q)^p \, dm(w) = \int_{a_1-\ell(Q)}^{a_2+\ell(Q)} \int_{t_1-\ell(Q)}^{t_2+\ell(Q)} \left| J_{\frac{s}{1+\theta^2}} * A' \left(x + \frac{\theta s}{1+\theta^2}\right) \right|^p dt \, dx.$$

Then, assuming δ small enough, for each $x \in p_1(3Q)$ we have

$$\int_{t_1-\ell(Q)}^{t_2+\ell(Q)} \left| J_{\frac{s}{1+\theta^2}} * A'\left(x + \frac{\theta s}{1+\theta^2}\right) \right|^p dt = \int_{t_1-\ell(Q)}^{t_2+\ell(Q)} \left| J_{\frac{t-g_Q(x)}{1+\theta^2}} * A'\left(x + \frac{\theta s}{1+\theta^2}\right) \right|^p dt$$

$$\leq \int_{t_1-\frac{3}{2}\ell(Q)}^{t_2+\frac{3}{2}\ell(Q)} \left| J_{t-A(a_1)} * A'\left(x + \frac{\theta s}{1+\theta^2}\right) \right|^p dt.$$

Also, it follows easily that

$$\int_{a_1-\ell(Q)}^{a_2+\ell(Q)} \int_{t_1-\frac{3}{2}\ell(Q)}^{t_2+\frac{3}{2}\ell(Q)} \left| J_{t-A(a_1)} * A'\left(x + \frac{\theta s}{1+\theta^2}\right) \right|^p dt\, dx$$

$$\leq \int_{a_1-2\ell(Q)}^{a_2+2\ell(Q)} \int_{t_1-\frac{3}{2}\ell(Q)}^{t_2+\frac{3}{2}\ell(Q)} \left| J_{t-A(a_1)} * A'(x) \right|^p dt\, dx$$

$$\leq \int_{a_1-2\ell(Q)}^{a_2+2\ell(Q)} \int_{t_1-2\ell(Q)}^{t_2+2\ell(Q)} \left| J_{t-A(x)} * A'(x) \right|^p dt\, dx.$$

As a consequence,

(4.18)
$$\sum_{Q\in\mathcal{W}(\Omega)} \int_{3Q} I_2(w,Q)^p \ell(Q)^{p-\alpha p}\, dm(w) \leq \sum_{Q\in\mathcal{W}(\Omega)} \iint_{(x,t)\in 5Q} \left| J_{t-A(x)} * A'(x) \right|^p t^{p-\alpha p}\, dt\, dx$$

$$\approx \iint_{(x,t)\in\Omega} \left| J_{t-A(x)} * A'(x) \right|^p t^{p-\alpha p}\, dt\, dx$$

$$= \int_{x\in\mathbb{R}} \int_0^\infty \left| J_t * A'(x) \right|^p t^{p-\alpha p}\, dt\, dx.$$

We denote $\varphi_t = t^2(J_t)'$, so that $J_t * A' = (J_t)' * A = t^{-2}\varphi_t * A$. Moreover, it turns out that $\varphi_t(x) = t^{-1}\varphi(t^{-1}x)$, where $\varphi \equiv \varphi_1$. Then we have

(4.19)
$$\int_{x\in\mathbb{R}} \int_0^\infty \left| J_t * A'(x) \right|^p t^{p-\alpha p}\, dt\, dx = \int_0^\infty \left\| t^{\frac{1}{p}-1-\alpha} \varphi_t * A \right\|_p^p \frac{dt}{t}.$$

To calculate the Fourier transform of φ, notice that $J_t(x) = c\,(Q_t)'(x)$, where Q_t is the conjugated Poisson kernel and c is some absolute constant. So,

$$\widehat{J_t}(\xi) = c\,\xi\,\mathrm{sgn}(\xi)e^{-2\pi t|\xi|} = c\,|\xi|\,e^{-2\pi t|\xi|},$$

and

$$\widehat{\varphi_t}(\xi) = c\,t^2\xi\,|\xi|\,e^{-2\pi t|\xi|}.$$

So, recalling (4.16), it turns out that $\widehat{\varphi_t}(\xi) = c\,\mathrm{sgn}(\xi)\,\widehat{\psi_t}(\xi)$. That is, φ_t is the Hilbert transform of ψ_t, modulo a constant factor. Thus, denoting by H the Hilbert transform,

$$\|\varphi_t * A\|_p = c\,\|H(\psi_t * A)\|_p \leq c\,\|\psi_t * A\|_p,$$

and so, by (4.19),

$$\int_{x\in\mathbb{R}}\int_0^\infty \left|J_t * A'(x)\right|^p t^{p-\alpha p}\, dt\, dx \le c \int_0^\infty \|t^{\frac{1}{p}-1-\alpha}\, \psi_t * A\|_p^p\, \frac{dt}{t},$$

and the lemma follows. □

Proof of the Main Lemma 4.1. By (4.13) and Lemmas 4.5 and 4.6, assuming δ small enough, we get

$$\sum_{Q\in\mathcal{W}(\Omega)} \int_{3Q} \left|\int_{\mathbb{R}} \frac{A'(x)}{(x + i\, g_Q(x) - w)^2}\, dx\right|^p \ell(Q)^{p-\alpha p}\, dm(w)$$

$$\gtrsim (1 - c\,\delta^p) \int_0^\infty \|t^{\frac{1}{p}-1-\alpha}\, \psi_t * A\|_p^p\, \frac{dt}{t} \gtrsim \|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p.$$

Together with (4.9), this implies that

$$\int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z,\partial\Omega)^{p-\alpha p} dm(z) \gtrsim \|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p - c\,\delta^p \|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p \gtrsim \|A\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p,$$

for δ small enough again. □

5. The proof of Theorem 1.1

Let $\Omega \subset \mathbb{C}$ be a bounded domain which is (δ, R)-Lipschitz. We have to show that

$$(5.1) \qquad \|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)}^p \lesssim \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}^p + \mathcal{H}^1(\partial\Omega)^{2-\alpha p}.$$

To this end we will prove:

Lemma 5.1. *Let $\Omega \subset \mathbb{C}$ be a (δ, R)-Lipschitz domain. Let $1 < p < \infty$ and $0 < \alpha \le 1$ be such that $\alpha p > 1$. If δ is small enough, then*

$$(5.2) \qquad \|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)}^p \lesssim \int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z,\partial\Omega)^{p-\alpha p}\, dm(z) + \mathcal{H}^1(\partial\Omega)^{2-\alpha p}.$$

Let us show first that this result yields Theorem 1.1 as an easy consequence.

Proof of Theorem 1.1. In the case $\alpha = 1$, it is clear that (5.1) follows from (5.2). So assume that $0 < \alpha < 1$. In this case, if θ is chosen small enough, from Lemma 5.1, we have

$$\theta^{p-\alpha p}\left(\mathcal{H}^1(\partial\Omega)^{2-\alpha p} + \int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z,\partial\Omega)^{p-\alpha p} dm(z)\right) \ge 2c_3\, \theta^{2p-\alpha p}\|N\|_{B_{p,p}^{\alpha-1/p}(\partial\Omega)}^p,$$

where c_3 appears in (3.16). Then, by Lemma 3.11,

$$(5.3)$$

$$\theta^{p-\alpha p}\, \mathcal{H}^1(\partial\Omega)^{2-\alpha p} + \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}^p \gtrsim \theta^{p-\alpha p}\int_\Omega |\partial B\chi_\Omega(z)|^p \operatorname{dist}(z,\partial\Omega)^{p-\alpha p}\, dm(z).$$

Together with Lemma 5.1 again this gives (5.1). □

Suppose that Ω is simply connected. Consider and arc length parameterization of $\partial\Omega$ given by $\gamma : S^1(0, r_0) \to \partial\Omega$, where $2\pi r_0 = \mathcal{H}^1(\partial\Omega)$. Recall that, for a function $f : S^1(0, r_0) \to \mathbb{R}$ and $0 < \alpha < 1$ and $1 < p < \infty$,

$$\|f\|_{\dot{B}^\alpha_{p,p}(S^1(0,r_0))}^p = \iint_{(s,t) \in S^1(0,r_0) \times S^1(0,r_0)} \frac{|f(s) - f(t)|^p}{|s-t|^{\alpha p + 1}} \, ds \, dt.$$

Then, taking into account that $\|N \circ \gamma\|_{\dot{B}^{\alpha-1/p}_{p,p}(S^1(0,r_0))} \approx \|N\|_{\dot{B}^{\alpha-1/p}_{p,p}(\partial\Omega)}$, (5.2) is equivalent to

$$(5.4) \quad \iint_{(s,t) \in S^1(0,r_0) \times S^1(0,r_0)} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s-t|^{\alpha p}} \, ds \, dt$$

$$\lesssim \int_\Omega |\partial B \chi_\Omega(z)|^p \operatorname{dist}(z, \partial\Omega)^{p-\alpha p} \, dm(z) + \mathcal{H}^1(\partial\Omega)^{2-\alpha p}.$$

We will use the following notation: given $a > 1$ and a small arc $I \subset S^1(0, r_0)$, we denote by aI the arc of $S^1(0, r_0)$ with the same mid point as I and length $\ell(aI) = a\,\ell(I)$.

The main step for the proof of Lemma 5.1 consists of next lemma.

Lemma 5.2. *Suppose that Ω is simply connected. Under the assumptions and notation above, consider an arc $I \subset S^1(0, r_0)$ with $\ell(I) \leq R/4$ and denote by s_1, s_2 the end points of $2I$. Let $a = \delta^{1/2}\,\ell(I)$. Then, for $0 < \delta \ll 1$ small enough we have*

$$(5.5)$$

$$\iint_{\substack{s \in I \\ t \in 1.1I}} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s-t|^{\alpha p}} \, ds \, dt \lesssim \int_{\Omega \cap B(z_I, 4\ell(I))} |\partial B \chi_\Omega(z)|^p \operatorname{dist}(z, \partial\Omega)^{p-\alpha p} \, dm(z)$$

$$+ \sum_{i=1}^{2} \int_{|s-s_i| \leq 7a} \int_{t \in S^1(0,r_0)} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s-t|^{\alpha p}} \, dt \, ds + c(\delta)\, \ell(I)^{2-\alpha p}.$$

Proof. Denote by s_I the mid point of I and set $z_I = \gamma(s_I)$. Let $A : \mathbb{R} \to \mathbb{R}$ be the Lipschitz function whose graph Γ coincides with $\partial\Omega$ on $B(z_I, R)$, so that $\Omega \cap B(z_I, R)$ lies above Γ, after a suitable rotation. Notice that $\gamma(6I) \subset B(z_I, R)$, since $\ell(I) = \mathcal{H}^1(\gamma(I)) \leq R/4$. Let $J \subset \mathbb{R}$ be the interval such that $\{(x, A(x)) : x \in J\} = \gamma(I)$, so that $\{(x, A(x)) : x \in 5J\} \subset \gamma(6I) \subset B(z_I, R)$. Observe that

$$\gamma(1.1I) \subset \{(x, A(x)) : x \in 1.1J\}.$$

It also immediate to check that $\ell(I) \approx \ell(J)$. Moreover, translating Γ slightly if necessary, we may assume that one of the endpoints of $\gamma(I)$ lies on the horizontal coordinate axis. Notice that then, since $\|A'\|_\infty \leq \delta$, by the mean value theorem, it turns out that

$$(5.6) \qquad\qquad |A(x)| \lesssim \delta\,\ell(J) \qquad \text{for all } x \in 5J.$$

Moreover, we will assume that A is defined in the whole of $\mathbb{R}$ and that $\|A\|_\infty \lesssim \delta\ell(J)$ and $\|A'\|_\infty \lesssim \delta$.

Denote $z_i = \gamma(s_i)$, for $i = 1, 2$ (recall that s_1, s_2 are the end points of $2I$). Also, let x_i be such that $z_i = (x_i, A(x_i))$. Assume that $x_1 < x_2$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function which equals 1 on $[x_1, x_2]$ and vanishes on $\mathbb{R} \setminus [x_1 - a, x_2 + a]$, with $\|\varphi'\|_\infty \lesssim 1/a$ (recall that $a = \delta^{1/2}\ell(I) \approx \delta^{1/2}\ell(J)$). Observe that, since we are assuming δ to be very small,

$$[x_1 - a, x_2 + a] \subset 3J$$

We consider the auxiliary Lipschitz function $\widetilde{A} = \varphi A$ and its graph $\widetilde{\Gamma}$. Let

$$\widetilde{\Omega} = \{(x, y) \in \mathbb{C} : y > \widetilde{A}(x)\},$$

and denote by $\widetilde{N}(x)$ the outward unit normal at $(x, \widetilde{A}(x)) \in \widetilde{\Gamma}$. By Corollary 4.2, we have

$$\|\widetilde{N}\|_{\dot{B}^{\alpha-1/p}_{p,p}} \approx \int_{\widetilde{\Omega}} |\partial B\chi_{\widetilde{\Omega}}(z)|^p \operatorname{dist}(z, \partial\widetilde{\Omega})^{p-\alpha p}\, dm(z).$$

Indeed, using (5.6),

$$\|\widetilde{A}'\|_\infty \leq \|A'\|_\infty + \|\varphi'\|_\infty \|\chi_{3J}A\|_\infty \leq \delta + \frac{c}{a}\delta\,\ell(J) \lesssim \delta + \delta^{1/2} \approx \delta^{1/2},$$

and thus the assumption on the small slope of the Lipschitz function in Theorem 1.3 holds for δ small enough.

On the other hand, since $\widetilde{N}$ coincides with N on $\gamma(1.1I)$,

$$\|\widetilde{N}\|^p_{\dot{B}^{\alpha-1/p}_{p,p}} \approx \iint_{(x,y)\in\mathbb{R}^2} \frac{|\widetilde{N}(x) - \widetilde{N}(y)|^p}{|x-y|^{\alpha p}}\, dx\, dy \gtrsim \iint_{\substack{s\in I \\ t\in 1.1I}} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s-t|^{\alpha p}}\, ds\, dt.$$

Therefore, to prove the lemma it is enough to show that $\int_{\widetilde{\Omega}} |\partial B\chi_{\widetilde{\Omega}}|^p \operatorname{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p} dm$ is bounded above by the right side of (5.5).

Consider the rectangle

$$V = [x_1 - 2a,\, x_2 + 2a] \times [-a, a].$$

To estimate $\|\partial B(\chi_{\widetilde{\Omega}})\|^p_{L^p(\widetilde{\Omega})}$, we write

(5.7)
$$\int_{\widetilde{\Omega}} |\partial B\chi_{\widetilde{\Omega}}|^p \operatorname{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p}\, dm \leq \int_{\widetilde{\Omega}\setminus V} |\partial B\chi_{\widetilde{\Omega}}|^p \operatorname{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p}\, dm$$

$$+ \int_{\widetilde{\Omega}\cap V\setminus(B(x_1,4a)\cup B(x_2,4a))} |\partial B(\chi_{\widetilde{\Omega}})|^p \operatorname{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p}\, dm$$

$$+ \int_{\widetilde{\Omega}\cap(B(x_1,4a)\cup B(x_2,4a))} |\partial B(\chi_{\widetilde{\Omega}})|^p \operatorname{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p}\, dm.$$

Let us deal with the first integral on the right side. To this end, consider the upper half plane $\Pi = \{(x, y) \in \mathbb{C} : y > 0\}$. Recall that $\partial B\chi_\Pi(z) = 0$ for $z \in \Pi$. Therefore,

for $z \in \widetilde{\Omega} \setminus V \subset \Pi$, using the first identity in (3.10),

$$|\partial B(\chi_{\widetilde{\Omega}})(z)| = |\partial B(\chi_{\widetilde{\Omega}})(z) - \partial B(\chi_\Pi)(z)|$$

$$\leq \int_{\Pi \Delta \widetilde{\Omega}} \frac{1}{|z-w|^3} \, dm(w) \leq \frac{m(\Pi \Delta \widetilde{\Omega})}{\mathrm{dist}(z, \Pi \Delta \widetilde{\Omega})^3} \lesssim \frac{\delta \, \ell(J)^2}{\mathrm{dist}(z, \Pi \Delta \widetilde{\Omega})^3}.$$

It is easy to check that if $z \notin V$, then $\mathrm{dist}(z, \Pi \Delta \widetilde{\Omega}) \gtrsim a$. Using also the fact that $\mathrm{dist}(z, \Pi \Delta \widetilde{\Omega}) \approx |z - z_I|$ for $|z - z_I| \geq 4\ell(J)$, we obtain

(5.8)
$$\int_{\widetilde{\Omega} \setminus V} |\partial B(\chi_{\widetilde{\Omega}})|^p \mathrm{dist}(\cdot, \partial \widetilde{\Omega})^{p-\alpha p} \, dm \lesssim \int_{B(z_I, 4\ell(J))} \frac{\delta^p \, \ell(J)^{2p}}{a^{3p}} \, \ell(J)^{p-\alpha p} \, dm(z)$$

$$+ \int_{\mathbb{C} \setminus B(z_I, 4\ell(J))} \frac{\delta^p \, \ell(J)^{2p}}{|z - z_I|^{3p}} |z - z_I|^{p-\alpha p} \, dm(z)$$

$$\lesssim \frac{\delta^p \, \ell(J)^{3p+2-\alpha p}}{a^{3p}} + \frac{\delta^p \, \ell(J)^{2p}}{\ell(J)^{2p+\alpha p-2}} \approx \delta^{-p/2} \, \ell(I)^{2-\alpha p}.$$

Let us turn our attention to the second integral on the right side of (5.7) now. In this case, using that

$$\widetilde{\Omega} \cap V \setminus (B(x_1, 4a) \cup B(x_2, 4a)) = \Omega \cap V \setminus (B(x_1, 4a) \cup B(x_2, 4a)),$$

we write

$$\int_{\widetilde{\Omega} \cap V \setminus (B(x_1, 4a) \cup B(x_2, 4a))} |\partial B(\chi_{\widetilde{\Omega}})|^p \mathrm{dist}(\cdot, \partial \widetilde{\Omega})^{p-\alpha p} \, dm$$

$$\lesssim \int_{\Omega \cap B(z_I, 4\ell(I))} |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot, \partial \widetilde{\Omega})^{p-\alpha p} \, dm$$

$$+ \int_{V \setminus (B(x_1, 4a) \cup B(x_2, 4a))} |\partial B(\chi_\Omega) - \partial B(\chi_{\widetilde{\Omega}})|^p \mathrm{dist}(\cdot, \partial \widetilde{\Omega})^{p-\alpha p} \, dm.$$

We estimate the last integral arguing as above. Observe that for $z \in V \setminus (B(x_1, 4a) \cup B(x_2, 4a))$, we have $\mathrm{dist}(z, \Omega \Delta \widetilde{\Omega}) \gtrsim a$. Then we have

$$|\partial B(\chi_\Omega)(z) - \partial B(\chi_{\widetilde{\Omega}})(z)| \leq \int_{\Omega \Delta \widetilde{\Omega}} \frac{1}{|z - w|^3} \, dm(w)$$

$$\leq \int_{|z-w| \geq c^{-1}a} \frac{1}{|z - w|^3} \, dm(w) \lesssim \frac{1}{a}.$$

As a consequence, since $\mathrm{dist}(z, \partial \widetilde{\Omega}) \lesssim a$ for $z \in V$,

$$\int_{V \setminus (B(x_1, 4a) \cup B(x_2, 4a))} |\partial B(\chi_\Omega) - \partial B(\chi_{\widetilde{\Omega}})|^p \, \mathrm{dist}(\cdot, \partial \widetilde{\Omega})^{p-\alpha p} \, dm$$

$$\lesssim \int_{V \setminus (B(x_1, 4a) \cup B(x_2, 4a))} \frac{a^{p-\alpha p}}{a^p} \, dm \lesssim \frac{a \, \ell(J)}{a^{\alpha p}} \approx \delta^{(1-\alpha p)/2} \, \ell(I)^{2-\alpha p},$$

and thus, taking into account also that $\mathrm{dist}(z, \partial\widetilde{\Omega}) = \mathrm{dist}(z, \partial\Omega)$ in the domain of integration,

$$(5.9) \quad \int_{V \setminus (B(x_1, 4a) \cup B(x_2, 4a))} |\partial B(\chi_{\widetilde{\Omega}})|^p \mathrm{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p} \, dm$$

$$\lesssim \int_{V \setminus (B(x_1, 4a) \cup B(x_2, 4a))} |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot, \partial\Omega)^{p-\alpha p} \, dm + \delta^{(1-\alpha p)/2} \, \ell(I)^{2-\alpha p}$$

$$\leq \int_{\Omega \cap B(z_I, 4\ell(I))} |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot, \partial\Omega)^{p-\alpha p} \, dm + \delta^{(1-\alpha p)/2} \, \ell(I)^{2-\alpha p}.$$

It remains to estimate the last integral in (5.7). First we deal with the integral on $B(x_1, 4a)$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function such that $0 \leq \psi \leq 1$ which equals 1 in $[x_1 - 5a, x_1 + 5a]$ and vanishes in $\mathbb{R} \setminus [x_1 - 6a, x_1 + 6a]$, with $\|\psi'\|_\infty \leq c/a$. Denote $A_0 = \psi \, \widetilde{A} = \psi \, \varphi \, A$ and set

$$\Omega_0 = \{(x, y) \in \mathbb{C} : y > A_0(x)\}.$$

Then we have

(5.10)
$$\int_{\widetilde{\Omega} \cap B(x_1, 4a)} |\partial B(\chi_{\widetilde{\Omega}})|^p \, \mathrm{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p} \, dm \lesssim \int_{\widetilde{\Omega} \cap B(x_1, 4a)} |\partial B(\chi_{\Omega_0})|^p \mathrm{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p} \, dm$$

$$+ \int_{\widetilde{\Omega} \cap B(x_1, 4a)} |\partial B(\chi_{\Omega_0}) - \partial B(\chi_{\widetilde{\Omega}})|^p \mathrm{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p} \, dm.$$

Since A_0 coincides with $\widetilde{A}$ in $[x_1 - 5a, x_1 + 5a]$, it turns out that, for $z \in B(x_1, 4a)$, $\mathrm{dist}(z, \Omega_0 \Delta \widetilde{\Omega}) \gtrsim a$, and thus

$$|\partial B(\chi_{\Omega_0})(z) - \partial B(\chi_{\widetilde{\Omega}})(z)| \leq \int_{\Omega_0 \Delta \widetilde{\Omega}} \frac{1}{|z - w|^3} \, dm(w) \leq \int_{|z-w| \gtrsim a} \frac{1}{|z - w|^3} \, dm(w) \lesssim \frac{1}{a}.$$

Therefore, using that $\mathrm{dist}(\cdot, \partial\widetilde{\Omega}) \lesssim a$ on $B(x_1, 4a)$,

(5.11)
$$\int_{B(x_1, 4a)} |\partial B(\chi_{\Omega_0}) - \partial B(\chi_{\widetilde{\Omega}})|^p \mathrm{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p} \, dm \lesssim \frac{a^{p-\alpha p} \, a^2}{a^p} \approx \delta^{(2-\alpha p)/2} \, \ell(I)^{2-\alpha p}.$$

For the first integral on the right side of (5.10) we use the fact that $\widetilde{\Omega} \cap B(x_1, 4a) = \Omega_0 \cap B(x_1, 4a)$, and moreover that $\mathrm{dist}(\cdot, \partial\widetilde{\Omega}) = \mathrm{dist}(\cdot, \partial\Omega_0)$ in $\widetilde{\Omega} \cap B(x_1, 4a)$. Then, by Corollary 4.2 applied to Ω_0, we get

(5.12)
$$\int_{\widetilde{\Omega} \cap B(x_1, 4a)} |\partial B(\chi_{\Omega_0})|^p \mathrm{dist}(\cdot, \partial\widetilde{\Omega})^{p-\alpha p} \, dm \leq \int_{\Omega_0} |\partial B(\chi_{\Omega_0})|^p \mathrm{dist}(\cdot, \partial\Omega_0)^{p-\alpha p} \, dm$$

$$\lesssim \|A_0\|_{\dot{B}_{p,p}^{1+\alpha-1/p}}^p.$$

We have

(5.13) $\qquad \|A_0\|_{\dot{B}_{p,p}^{1+\alpha-1/p}} \approx \|A_0'\|_{\dot{B}_{p,p}^{\alpha-1/p}} \leq \|\varphi\psi A'\|_{\dot{B}_{p,p}^{\alpha-1/p}} + \|(\varphi\psi)'A\|_{\dot{B}_{p,p}^{\alpha-1/p}}.$

From Lemma 3.4, we deduce that

$$\|\varphi\psi A'\|_{\dot{B}_{p,p}^{\alpha-1/p}}^p \lesssim \iint |\varphi\psi\Delta_h A'(x)|^p \frac{dh}{h^{\alpha p}}\,dx + \|\varphi\psi\|_{\dot{B}_{p,p}^{\alpha-1/p}}^p \|A'\|_\infty^p.$$

It is easy to check that $\|\varphi\psi\|_{B_{p,p}^{1-1/p}}^p \lesssim a^{2-\alpha p}$. Indeed, this a straightforward consequence of Lemma 3.5: since $\varphi\psi$ is supported on an interval with length $\lesssim a$ and $\|(\varphi\psi)'\|_\infty \lesssim 1/a$,

$$\|\varphi\psi\|_{\dot{B}_{p,p}^{\alpha-1/p}} \lesssim a^{1-\alpha+2/p}\,\|(\varphi\psi)'\|_\infty \lesssim a^{-\alpha+2/p},$$

and so our last claim holds. Then we obtain

(5.14) $\qquad \|\varphi\psi A'\|_{\dot{B}_{p,p}^{\alpha-1/p}}^p \lesssim \iint |\chi_{[x_1-6a,x_1+6a]}\,\Delta_h A'(x)|^p\,\frac{dh}{h^{\alpha p}}\,dx + \delta^p\,a^{2-\alpha p}.$

We split the integral on the right side above as follows:

$$\int_{|x-x_1|\leq 6a}\int_{|h|\leq \ell(I)/2}|\Delta_h A'(x)|^p\,\frac{dh}{h^{\alpha p}}\,dx + \int_{|x-x_1|\leq 6a}\int_{|h|>\ell(I)/2}|\Delta_h A'(x)|^p\,\frac{dh}{h^{\alpha p}}\,dx = I_1+I_2.$$

For I_2 we use the rough estimate $|\Delta_h A'(x)| \leq 2\delta$, and then, using that $\alpha p > 1$, we get

$$I_2 \leq 2^p\delta^p\int_{|x-x_1|\leq 6a}\int_{|h|>\ell(I)/2}\frac{dh}{h^{\alpha p}}\,dx \lesssim \delta^p a\,\ell(I)^{1-\alpha p} = c(\delta)\ell(I)^{2-\alpha p}.$$

For I_1, recall that $|A'(x)-A'(y)| \approx |N(x,A(x))-N(y,A(y))|$ for $x,y \in 5J$, by (3.1). Thus we get

$$\begin{aligned}I_1 &= \int_{|x-x_1|\leq 6a}\int_{|x-y|\leq \ell(I)/2}\frac{|A'(x)-A'(y)|^p}{|x-y|^{\alpha p}}\,dy\,dx\\ &\lesssim \int_{|s-s_1|\leq 7a}\int_{t\in S^1(0,r_0)}\frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,dt\,ds.\end{aligned}$$

Therefore, from (5.14) we derive
(5.15)

$$\|\varphi\psi A'\|_{\dot{B}_{p,p}^{\alpha-1/p}}^p \lesssim \int_{|s-s_1|\leq 7a}\int_{t\in S^1(0,r_0)}\frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,dt\,ds + c(\delta)\,\ell(I)^{2-\alpha p}.$$

To estimate $\|(\varphi\psi)'A\|_{\dot{B}_{p,p}^{\alpha-1/p}}$ we use that $(\varphi\psi)'A$ is a Lipschitz function supported on $[x_1-6a, x_1+6a]$ satisfying

$$\|[(\varphi\psi)'A]'\|_\infty \leq \|(\varphi\psi)''A\|_\infty + \|(\varphi\psi)'A'\|_\infty \lesssim \frac{\delta\,\ell(I)}{a^2} + \frac{\delta}{a} \lesssim \frac{1}{\ell(I)}.$$

Then Lemma 3.5 tells us that

(5.16) $\qquad \|(\varphi\psi)'A\|_{\dot{B}_{p,p}^{\alpha-1/p}} \lesssim a^{1-\alpha+2/p}\,\|[(\varphi\psi)'A]'\|_\infty \lesssim c(\delta)\,\ell(I)^{-\alpha+2/p}.$

From (5.10), (5.11), (5.12), (5.13), (5.15), and (5.16), we get

(5.17)
$$\int_{\widetilde{\Omega}\cap B(x_1,4a)} |\partial B(\chi_{\widetilde{\Omega}})|^p \mathrm{dist}(\cdot,\partial\widetilde{\Omega})\,dm$$
$$\lesssim \int_{|s-s_1|\le 7a}\int_{t\in S^1(0,r_0)} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,dt\,ds + c(\delta)\ell(I)^{2-\alpha p}.$$

An analogous inequality holds the integral over $B(x_2,4a)$.

Plugging the estimates obtained in (5.8), (5.9) and (5.17) into (5.7), we get

$$\|\partial B(\chi_{\widetilde{\Omega}})\,\mathrm{dist}(\cdot,\partial\widetilde{\Omega})^{1-\alpha}\|_{L^p(\widetilde{\Omega})}^p \lesssim \int_{\Omega\cap B(z_I,4\ell(I))} |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-\alpha p}\,dm$$
$$+\sum_{i=1}^{2}\int_{|s-s_i|\le 7a}\int_{t\in S^1(0,r_0)} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,dt\,ds + c(\delta)\,\ell(I)^{2-\alpha p},$$

which proves the lemma.   □

Proof of Lemma 5.1. Suppose first that Ω is simply connected, and let $\gamma\colon S^1(0,r_0)\to \partial\Omega$ be an arc length parameterization of $\partial\Omega$. We will prove (5.4).

Let $I\subset S^1(0,r_0)$ be an arc $\ell(I)\le R/4$. We set

$$\iint_{\substack{s\in I\\ t\in S^1(0,r_0)}} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,ds\,dt = \iint_{\substack{s\in I\\ t\in 1.1I}} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,ds\,dt$$
$$+\iint_{\substack{s\in I\\ t\in S^1(0,r_0)\setminus 1.1I}} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,ds\,dt.$$

To estimate the last integral we use the fact that $|s-t|\gtrsim \ell(I)$ in the domain of integration. Indeed, for $s\in I_j$ we have

$$\int_{t\in S^1(0,r_0)\setminus 1.1I} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,dt \le \int_{t\in S^1(0,r_0)\setminus 1.1I} \frac{2^p}{|s-t|^{\alpha p}}\,dt \lesssim \frac{1}{\ell(I)^{\alpha p-1}}.$$

From the last estimate and Lemma 5.2 we obtain

(5.18)
$$\iint_{\substack{s\in I\\ t\in S^1(0,r_0)}} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,ds\,dt \lesssim \int_{\Omega\cap B(z_I,4\ell(I))} |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-\alpha p}\,dm$$
$$+\sum_{i=1}^{2}\int_{|s-s_i|\le 7a}\int_{t\in S^1(0,r_0)} \frac{|N(\gamma(s))-N(\gamma(t))|^p}{|s-t|^{\alpha p}}\,dt\,ds + c(\delta)\,\ell(I)^{2-\alpha p},$$

where s_1, s_2 are the end points of $2I$ and $a=\delta^{1/2}\,\ell(I)$.

Given $u \in S^1(0, r_0)$, we denote by I_u the arc of $S^1(0, r_0)$ with length $R/4$ whose mid point is u. Now we average the inequality (5.18) over all the intervals I_u, $u \in S^1(0, r_0)$. By Fubini, we have

$$\int_{u \in S^1(0,r_0)} \iint_{\substack{s \in I_u \\ t \in S^1(0,r_0)}} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s - t|^{\alpha p}} \, ds \, dt \, du$$

$$= \frac{R}{4} \iint_{\substack{s \in S^1(0,r_0) \\ t \in S^1(0,r_0)}} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s - t|^{\alpha p}} \, ds \, dt = \frac{R}{4} \, \|N \circ \gamma\|^p_{\dot{B}^{\alpha - 1/p}_{p,p}}.$$

Now we turn to the right side of (5.18). Concerning the first integral, we have

$$\int_{u \in S^1(0,r_0)} \int_{\Omega \cap B(z_{I_u}, 4\ell(I))} |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot, \partial\Omega)^{p - \alpha p} dm \, du$$

$$= \int_\Omega \left(\int_{u : |\gamma(u) - w| \leq 4\ell(I)} du \right) |\partial B(\chi_\Omega)(w)|^p \mathrm{dist}(w, \partial\Omega)^{p - \alpha p} \, dm(w).$$

Taking into account that Ω is a Lipschitz domain, it follows that, for each $w \in \mathbb{C}$,

$$\mathcal{H}^1\{u : |\gamma(u) - w| \leq R\} \leq c\, R.$$

Thus,

$$\int_{u \in S^1(0,r_0)} \int_{\Omega \cap B(z_{I_u}, R)} |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot, \partial\Omega)^{p - \alpha p} \, dm \, du$$

$$\lesssim R \int_\Omega |\partial B(\chi_\Omega)|^p \mathrm{dist}(\cdot, \partial\Omega)^{p - \alpha p} \, dm.$$

Now we consider the second term on the right side of (5.18), for $i = 1$, say:

$$\int_{u \in S^1(0,r_0)} \int_{|s - s_i| \leq 7a} \int_{t \in S^1(0,r_0)} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s - t|^{\alpha p}} \, dt \, ds \, du$$

$$= 14a \int_{s \in S^1(0,r_0)} \int_{t \in S^1(0,r_0)} \frac{|N(\gamma(s)) - N(\gamma(t))|^p}{|s - t|^{\alpha p}} \, dt \, ds$$

$$\approx \delta^{1/2} R \, \|N \circ \gamma\|^p_{\dot{B}^{\alpha - 1/p}_{p,p}(S^1(0,r_0))}.$$

Clearly, the integral of the last term of (5.18) over $S^1(0, r_0)$ equals $c(\delta) R^{2 - \alpha p} r_0$. Then finally we deduce that

$$\frac{R}{4} \, \|N \circ \gamma\|^p_{\dot{B}^{\alpha - 1/p}_{p,p}(S^1(0,r_0))} \lesssim R \, \|\partial B(\chi_\Omega) \mathrm{dist}(\cdot, \partial\Omega)^{1 - \alpha}\|^p_{L^p(\Omega)}$$

$$+ \delta^{1/2} R \, \|N \circ \gamma\|^p_{\dot{B}^{\alpha - 1/p}_{p,p}(S^1(0,r_0))} + c(\delta) R^{2 - \alpha p} r_0.$$

Thus, if δ is taken small enough, then

$$\|N \circ \gamma\|^p_{\dot{B}^{\alpha - 1/p}_{p,p}(S^1(0,r_0))} \lesssim \|\partial B(\chi_\Omega) \mathrm{dist}(\cdot, \partial\Omega)^{1 - \alpha}\|^p_{L^p(\Omega)} + c(\delta) R^{1 - \alpha p} r_0,$$

and thus the theorem follows (in the case where Ω is simply connected).

If Ω is not simply connected, then $\partial\Omega$ has a finite number of components (because it is a Lipschitz domain). Arguing as above, one can show that (5.4) holds for the arc length parameterization γ of each component, and then we are done. $\qquad\square$

6. THE BEURLING TRANSFORM IN $B_{p,p}^\alpha(\Omega)$

Recall that for a Lipschitz or special Lipschitz domain and $f \in L_{loc}^1(\Omega)$, one sets

$$\|f\|_{\dot{B}_{p,p}^\alpha(\Omega)}^p = \iint \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 1}}\, dx\, dy.$$

In [CT] it was shown that, for $0 < \alpha < 1$, the estimate (1.1) is also valid with $\|B(\chi_\Omega)\|_{\dot{B}_{p,p}^\alpha(\Omega)}$ replacing $\|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}$. That is,

(6.1) $$\|B(\chi_\Omega)\|_{\dot{B}_{p,p}^\alpha(\Omega)} \leq c\, \|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)}.$$

One can also check that Theorems 1.1 and 1.3 also hold with $\|B(\chi_\Omega)\|_{\dot{B}_{p,p}^\alpha(\Omega)}$ instead of $\|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}$. So we have:

Theorem 6.1. *Let $0 < \alpha < 1$ and $1 < p < \infty$ such that $\alpha p > 1$. Let $\Omega \subset \mathbb{C}$ be either a (δ, R)-Lipschitz domain or a δ-Lipschitz domain, and assume that δ is small enough. If Ω is a Lipschitz domain, then*

$$\|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)} \leq c\, \|B(\chi_\Omega)\|_{\dot{B}_{p,p}^\alpha(\Omega)} + \mathcal{H}^1(\partial\Omega)^{-\alpha+2/p},$$

and if it is a special Lipschitz domain,

$$\|N\|_{\dot{B}_{p,p}^{\alpha-1/p}(\partial\Omega)} \leq c\, \|B(\chi_\Omega)\|_{\dot{B}_{p,p}^\alpha(\Omega)}.$$

Observe that from (6.1), Theorem 6.1 and the analogous results involving $\dot{W}^{\alpha,p}(\Omega)$ one deduces that, under the assumptions of Theorem 6.1,

$$B(\chi_\Omega) \in \dot{B}_{p,p}^\alpha(\Omega) \quad \Longleftrightarrow \quad B(\chi_\Omega) \in \dot{W}^{\alpha,p}(\Omega).$$

Moreover, if Ω is a special Lipschitz domain,

$$\|B(\chi_\Omega)\|_{\dot{B}_{p,p}^\alpha(\Omega)} \approx \|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}.$$

The proof of Theorem 6.1 follows by arguments very similar to the ones for Theorems 1.1 and 1.3. One can check that the key Lemma 3.11 also holds replacing $\|B(\chi_\Omega)\|_{\dot{W}^{\alpha,p}(\Omega)}$ by $\|B(\chi_\Omega)\|_{\dot{B}_{p,p}^\alpha(\Omega)}$. Indeed, one writes

$$\|B\chi_\Omega\|_{\dot{B}_{p,p}^\alpha(\Omega)}^p \geq \iint_{|x-y|\leq\ell(Q_x)} \frac{|B\chi_\Omega(x) - B\chi_\Omega(y)|^p}{|x - y|^{\alpha p + 1}}\, dx\, dy,$$

where Q_x is the square from $\mathcal{W}(\Omega)$ that contains x. Then, one uses the estimate (3.18) and then argues as in the proof of the lemma for the $\dot{W}^{\alpha,p}(\Omega)$ norm. Applying this new version of Lemma 3.11 together with the Main Lemma 4.1 and Lemma 5.1, Theorem 6.1 follows.

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