Intermittency for the Hyperbolic Anderson Model with rough noise in space

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Abstract

In this article, we consider the stochastic wave equation on the real line driven by a linear multiplicative Gaussian noise, which is white in time and whose spatial correlation corresponds to that of a fractional Brownian motion with Hurst index $H \in (\frac{1}{4}, \frac{1}{2})$. Initial data are assumed to be constant. First, we prove that this equation has a unique solution (in the Skorohod sense) and obtain an exponential upper bound for the p-th moment of the solution, for any $p \geq 2$. Condition $H > \frac{1}{4}$ turns out to be necessary for the existence of solution. Secondly, we show that this solution coincides with the one obtained by the authors in a recent publication, in which the solution is interpreted in the Itô sense. Finally, we prove that the solution of the equation in the Skorohod sense is weakly intermittent.

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1 Introduction

In this article, we continue the investigations from the recent article [3], by focusing on the stochastic wave equation with constant initial conditions and a linear term $\sigma(u) = \lambda u$ multiplying the noise:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) &= \dfrac{\partial^2 u}{\partial x^2}(t,x) + \lambda u(t,x)\dot{X}(t,x), \quad t>0,\, x\in\mathbb{R} \\ u(0,x) &= \eta, \quad x\in\mathbb{R} \\ \dfrac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R} \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}$ and $\eta \in \mathbb{R}$. To avoid trivial situations, we assume that $\lambda \neq 0$ and $\eta \neq 0$. This problem is known in the literature as the Hyperbolic Anderson Model, by analogy with its parabolic counterpart.

As in reference [3], we assume that the noise $\dot{X}$ is white in time and behaves in space like the formal derivative of a fractional Brownian motion with Hurst index $H \in (\frac{1}{4}, \frac{1}{2})$. This noise is given by a zero-mean Gaussian process $X = \{X(\varphi); \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$, defined on a complete probability space $(\Omega, \mathfrak{F}, P)$, with covariance:

$$E[X(\varphi)X(\psi)] = \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\mu(d\xi)dt, \quad \varphi,\psi \in \mathcal{D}(\mathbb{R}_+\times\mathbb{R}),$$

where $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ is the space of infinitely differentiable functions on $\mathbb{R}_+ \times \mathbb{R}$ with compact support, and $\mathcal{F}\varphi(t,\cdot)$ is the Fourier transform of the function $\varphi(t,\cdot)$, defined by:

$$\mathcal{F}\varphi(t,\cdot)(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\varphi(t,x)dx, \quad \xi \in \mathbb{R}.$$

We assume that the measure μ is given by $\mu(d\xi) = c_H|\xi|^{1-2H}d\xi$, with $\frac{1}{4} < H < \frac{1}{2}$ and

$$c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}.$$

Since the Fourier transform of μ in the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions on $\mathbb{R}$ is *not* a locally integrable function, the noise X is not of the same form as the one considered by Robert Dalang in his seminal article [9], and the stochastic integral with respect to X was constructed in [3] using different methods.

In the present article, the solution of equation (1) is defined using the divergence operator from Malliavin calculus, as opposed to the Itô-type stochastic integral used in article [3]. We say that the two solutions are interpreted in the Skorohod sense, respectively the Itô sense. Nevertheless, we will show that the two solutions coincide, by extending to the case of the noise X a classical result from Malliavin calculus which says that the Skorohod integral of a *measurable and adapted* process with respect to the Brownian motion coincides with its Itô integral (see Section 4 for details.)

In the first part of the paper, we show that equation (1) has a unique solution (in the Skorohod sense), whose moments of order $p \geq 2$ are bounded by an exponential function of t, up to some constants; see Theorem 3.4 for the precise statement. The proof of this result is based on Malliavin calculus techniques. More precisely, we write the Wiener chaos expansion of the solution $u(t,x)$ in the Skorohod sense, and we estimate the second moment of each multiple Wiener integral which appear in this expansion, following very closely the method used in [18] for the parabolic case.

This methodology has been used in the case of the stochastic heat equation in [14, 15, 16, 4], and also in the recent paper [18], in which the noise is the same as the one considered in the present article. In fact, article [18] contains a thorough analysis of the heat equation with a general diffusion coefficient $\sigma(u)$ multiplying the noise. On the other hand, the stochastic wave equation driven by a noise different than the one considered here (either smoother in space or fractional in time) was studied in references [11, 10, 1, 2]. Finally, the stochastic wave and heat equations with an affine diffusion coefficient $\sigma(u) = au + b$ and the same noise X as here have been studied in the recent article [3], using the classical method of Picard iterations, the solution being interpreted in the Itô sense. As explained above, it turns out that the solution to equation (1) in the Skorohod sense coincides with the one obtained in [3]. The advantage of the method based on the Wiener chaos expansion is that it allows us obtain estimates for the p-th moments of the solution, which lead to the weak intermittency property of the solution.

Recently, there has been a lot of interest in studying the intermittency property of solutions to stochastic partial differential equations, such as the stochastic heat and wave equations. For the former, we refer the reader to [5, 7, 13, 6, 17]. On the other hand, intermittency for the solution of the stochastic wave equation driven by a noise which is white in time and has a smoother space correlation than the one considered here was studied in [10, 8]. Finally, in [2], it was proved that the solutions to the stochastic wave and heat equations exhibit an intermittency-type property, even when the noise is fractional in time and has the same spatial correlation as in Dalang's article [9].

The notion of intermittency which will be considered here is *weak intermittency*, which is defined as follows. Recall that the *lower* and *upper Lyapunov exponents* of order $p \geq 2$ of the random field $u = \{u(t,x); t \geq 0, x \in \mathbb{R}\}$ which solves equation (1) are defined, respectively, by:

$$\underline{\gamma}(p) := \liminf_{t\to\infty} \frac{1}{t} \inf_{x\in\mathbb{R}} \log E|u(t,x)|^p \tag{2}$$

and

$$\overline{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}} \log E|u(t,x)|^p. \tag{3}$$

The goal of the present article is to prove that u is *weakly intermittent*, which means that

$$\underline{\gamma}(2) > 0 \quad \text{and} \quad \overline{\gamma}(p) < \infty \quad \text{for all } p \geq 2.$$

The physical interpretation of this property is that, as time becomes large, the paths of the random field u exhibit very high peaks concentrated on small spatial islands.

We observe that, in most of the references cited above, a weaker notion of *weak intermittence* has been considered, namely $\overline{\gamma}(2) > 0$ and $\overline{\gamma}(p) < +\infty$ for all $p \geq 2$.

As noticed in [1], the fact that $\overline{\gamma}(p) < \infty$ for all $p \geq 2$ is a direct consequence of the second moment estimate obtained for the n-th term appearing in the Wiener chaos expansion of $u(t,x)$. Condition $\underline{\gamma}(2) > 0$ turns out to be more delicate and is proved by showing that the second moment of the solution admits a lower exponential estimate. Though our noise is rougher than the one considered in some related references, e.g. [13], where the above-mentioned weaker notion of *weak intermittency* is considered, we succeeded to write a rather simplified proof of $\underline{\gamma}(2) > 0$ by taking advantage of the noise's roughness. More precisely, the fact that $H < \frac{1}{2}$ implies that the function $\xi \to \xi^{1-2H}$ is increasing on $\mathbb{R}_+$, which allowed us to find a suitable lower bound for $\underline{\gamma}(2)$; see the proof of Theorem 5.1 for details.

We notice that, though intermittency properties for the parabolic Anderson model driven by the noise X have already been studied in [18], the proof of our Theorem 5.1 also works in this latter case; see Remark 5.3.

Finally, we should mention that, as in [3] and [18], we were not able to eliminate of the restriction $H > \frac{1}{4}$, which is needed for the existence of the solution. This restriction might have a deeper meaning, but from the technical point of view, it comes from the requirement

$$\int_{\mathbb{R}} |\mathcal{F}G(t,\cdot)(\xi)|^2 |\xi|^{2(1-2H)} d\xi < \infty,$$

which arises naturally in our calculations. Here $G(t,x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$, $t > 0$ and $x \in \mathbb{R}$, is the fundamental solution of the wave equation in $\mathbb{R}$. Indeed, we prove that $H > \frac{1}{4}$ is a necessary condition for the existence of the solution to equation (1) with noise X with $0 < H < \frac{1}{2}$ (see Proposition 3.5).

The paper is organized as follows. In Section 2, we present the Malliavin calculus setting associated to our noise and introduce the tools needed in the sequel. In Section 3, we prove that equation (1) has a unique solution in the Skorohod sense. In Section 4, we compare the solution obtained in the previous section with the Itô-type solution studied in [3]. An important step here is to compare the Skorohod-type integral with respect to our noise with the Itô-type integral defined in the latter reference. This result is of independent interest and is proved in Appendix A. Finally, in Section 5, we prove that the solution to equation (1) is weakly intermittent, in the sense described above.

Along the paper we use the notation C for any positive real constant, independently of its value.

2 Preliminaries on Malliavin calculus

In order to give a meaning of solution to equation (1), we use an approach based on Malliavin calculus with respect to the isonormal Gaussian process determined by the noise $\dot{X}$. We describe briefly this procedure. We refer to [20] for more details.

Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, where

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := \int_0^\infty \int_{\mathbb{R}} \mathcal{F}\varphi(t,\cdot)(\xi)\overline{\mathcal{F}\psi(t,\cdot)(\xi)}\mu(d\xi)dt.$$

The map $\varphi \mapsto X(\varphi) \in L^2(\Omega)$ is an isometry which can be extended to $\mathcal{H}$. We denote this map by

$$X(\varphi) = \int_0^\infty \int_{\mathbb{R}} \varphi(t,x)X(dt,dx), \quad \varphi \in \mathcal{H}.$$

We say that $X(\varphi)$ is the Wiener integral of φ with respect to X. Then, $\{X(\varphi); \varphi \in \mathcal{H}\}$ defines an isonormal Gaussian process and we can develop the Malliavin calculus techniques based on it.

Recall that the square of the norm in $\mathcal{H}$ can also be written as follows:

$$\|\varphi\|_{\mathcal{H}}^2 = C_H \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(x) - \varphi(y)|^2 |x-y|^{2-2H}\, dxdydt,$$

where $C_H = H(1-2H)/2$.

Moreover, it can be seen that $\mathcal{H}$ is a space of functions in both variables t and x; see [19] for the proof of this fact, in the case when the noise is independent of t.

Let $\mathcal{G}$ be the σ-field generated by $\{X(\varphi); \varphi \in \mathcal{H}\}$. By Theorem 1.1.1 of [20], every random variable $F \in L^2(\Omega, \mathcal{G}, P)$ has the Wiener chaos expansion:

$$F = E(F) + \sum_{n\geq 1} F_n \quad \text{with} \quad F_n \in \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the n-th Wiener chaos space associated to X.

Each random variable $F \in \mathcal{H}_n$ can be represented as $F = I_n(f)$ for some $f \in \mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes n}$ is the n-th tensor product of $\mathcal{H}$ and $I_n : \mathcal{H}^{\otimes n} \to \mathcal{H}_n$ is the multiple Wiener integral with respect to X (see, e.g. [20, Prop. 1.1.4]). In our case, the norm in $\mathcal{H}^{\otimes n}$ is given by:

$$\|f\|_{\mathcal{H}^{\otimes n}}^2 = \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^n} |\mathcal{F}f(t_1,\cdot,\ldots,t_n,\cdot)(\xi_1,\ldots,\xi_n)|^2 \mu(d\xi_1)\ldots\mu(d\xi_n)dt_1\ldots dt_n. \tag{4}$$

For any $f \in \mathcal{H}^{\otimes n}$,

$$I_n(f) = I_n(\widetilde{f}) \tag{5}$$

and

$$E|I_n(f)|^2 = E|I_n(\widetilde{f})|^2 = n!\, \|\widetilde{f}\|_{\mathcal{H}^{\otimes n}}^2,$$

where $\widetilde{f}$ is the symmetrization of f in all n variables:

$$\widetilde{f}(t_1,x_1,\ldots,t_n,x_n) = \frac{1}{n!}\sum_{\rho\in S_n} f(t_{\rho(1)},x_{\rho(1)},\ldots,t_{\rho(n)},x_{\rho(n)}),$$

and we denote by S_n the set of all permutations of $\{1,\dots,n\}$. By the orthogonality of the Wiener chaos spaces, for any $f \in \mathcal{H}_n$ and $g \in \mathcal{H}_m$,

$$E[I_n(f)I_m(g)] = \begin{cases} n!\,\langle \widetilde{f}, \widetilde{g}\rangle_{\mathcal{H}^{\otimes n}} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{6}$$

The Wiener chaos expansion of $F \in L^2(\Omega, \mathcal{G}, P)$ becomes:

$$F = \sum_{n\geq 0} I_n(f_n) = \sum_{n\geq 0} I_n(\widetilde{f}_n),$$

where $f_n \in \mathcal{H}^{\otimes n}$ for $n \geq 1$, $f_0 = \widetilde{f}_0 = E(F)$ and $I_0 : \mathbb{R} \to \mathbb{R}$ is the identity map. Using again the orthogonality of the Wiener chaos spaces, we obtain that

$$E|F|^2 = \sum_{n\geq 0} E|I_n(f_n)|^2 = \sum_{n\geq 0} n!\,\|\widetilde{f}_n\|^2_{\mathcal{H}^{\otimes n}}.$$

Let $\mathcal{S}$ be the class of smooth random variables of the form

$$F = f(X(\varphi_1), \dots, X(\varphi_n)), \tag{7}$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $\varphi_i \in \mathcal{H}$, $n \geq 1$, and $C_b^\infty(\mathbb{R}^n)$ is the class of bounded C^∞-functions on $\mathbb{R}^n$, whose partial derivatives of all orders are bounded. The **Malliavin derivative** of F of the form (7) is an $\mathcal{H}$-valued random variable given by:

$$DF := \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\varphi_1), \dots, X(\varphi_n))\varphi_i.$$

We endow $\mathcal{S}$ with the norm $\|F\|_{\mathbb{D}^{1,2}} := (E|F|^2)^{1/2} + (E\|DF\|^2_{\mathcal{H}})^{1/2}$. The operator D can be extended to the space $\mathbb{D}^{1,2}$, the completion of $\mathcal{S}$ with respect to $\|\cdot\|_{\mathbb{D}^{1,2}}$.

The **divergence operator** δ is defined as the adjoint of the operator D. The domain of δ, denoted by Dom δ, is the set of $u \in L^2(\Omega; \mathcal{H})$ such that

$$|E\langle DF, u\rangle_{\mathcal{H}}| \leq c(E|F|^2)^{1/2}, \quad \forall F \in \mathbb{D}^{1,2},$$

where c is a constant depending on u. If $u \in$ Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathcal{H}}, \quad \forall F \in \mathbb{D}^{1,2}. \tag{8}$$

In particular, $E[\delta(u)] = 0$. If $u \in$ Dom δ, we use the notation

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x) X(\delta t, \delta x),$$

and we say that $\delta(u)$ is the **Skorohod integral** of u with respect to X.

The following result is the analogue of Proposition 1.3.7 of [20] for the noise X.

Proposition 2.1. *Let $u \in L^2(\Omega; \mathcal{H})$ such that for each $t > 0$ and $x \in \mathbb{R}$, $u(t,x) \in L^2(\Omega, \mathcal{G}, P)$ has the Wiener chaos expansion*

$$u(t,x) = \sum_{n\geq 0} I_n(f_n(\cdot, t, x)),$$

for some $f_n(\cdot, t, x) \in \mathcal{H}^{\otimes n}$. Then $u \in \operatorname{Dom} \delta$ if and only if the series $\sum_{n\geq 1} I_{n+1}(f_n)$ converges in $L^2(\Omega)$. In this case,

$$\delta(u) = \sum_{n\geq 0} I_{n+1}(f_n) = \sum_{n\geq 0} I_{n+1}(\widetilde{f}_n),$$

where $\widetilde{f}_n$ is the symmetrization of f_n in all $n+1$ variables.

3 Solution in the Skorohod sense

In this section, we define the concept of solution to equation (1) in the Skorohod sense, and we prove that this solution exists and is unique. As a by-product of this procedure, we obtain immediately an exponential upper bound for the p-th moment of the solution, which means that $\overline{\gamma}(p) < \infty$ for any $p \geq 2$. Moreover, we also prove that condition $H > \frac{1}{4}$ is necessary for the existence and uniqueness of solution to equation (1).

Let G be the Green function of the wave operator on $\mathbb{R}_+ \times \mathbb{R}$, i.e.

$$G(t,x) = \frac{1}{2} 1_{\{|x|<t\}}, \quad t > 0, x \in \mathbb{R}.$$

We consider the filtration

$$\mathcal{F}_t = \sigma(\{X(1_{[0,s]}\varphi); 0 \leq s \leq t, \varphi \in \mathcal{D}(\mathbb{R})\}) \vee \mathcal{N}, \quad t \geq 0,$$

where $\mathcal{N} = \{F \in \mathfrak{F}; P(F) = 0\}$.

Definition 3.1. We say that a process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a **solution of** (1) **(in the Skorohod sense)** if, for any $t \geq 0$ and $x \in \mathbb{R}$, $E|u(t,x)|^2 < \infty$ and

$$u(t,x) = \eta + \int_0^t \int_{\mathbb{R}} G(t-s, x-y) \lambda u(s,y) X(\delta s, \delta y), \tag{9}$$

i.e. the process $v^{(t,x)} = \{1_{[0,t]}(s) \lambda G(t-s, x-y) u(s,y); s \geq 0, y \in \mathbb{R}\}$ belongs to $\operatorname{Dom} \delta$ and $u(t,x) = \eta + \delta(v^{(t,x)})$.

To see when the solution exists, for any $t > 0$ and $x \in \mathbb{R}$, we define $f_0(t,x) = \eta$, and

$$\begin{aligned} &f_n(t_1, x_1, \ldots, t_n, x_n, t, x) \\ &\qquad = \lambda^n G(t-t_n, x-x_n) \ldots G(t_2-t_1, x_2-x_1)\, \eta 1_{\{0<t_1<\ldots<t_n<t\}}, \end{aligned} \tag{10}$$

for $n \geq 1$. We let $\widetilde{f}_0(t,x) = \eta$ and for $n \geq 1$, we let $\widetilde{f}_n(\cdot,t,x)$ be the symmetrization of $f_n(\cdot,t,x)$:

$$\begin{aligned}&\widetilde{f}_n(t_1,x_1,\ldots,t_n,t,x)\\&\quad=\eta\frac{\lambda^n}{n!}\sum_{\rho\in S_n}G(t-t_{\rho(n)},x-x_{\rho(n)})\ldots G(t_{\rho(2)}-t_{\rho(1)},x_{\rho(2)}-x_{\rho(1)})1_{\{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t\}}.\end{aligned} \tag{11}$$

The main result of the section (see Theorem 3.4 below) is based on the following proposition.

Proposition 3.2. *Equation* (1) *has a solution if and only if, for any $t > 0$ and $x \in \mathbb{R}$, it holds*

$$\sum_{n\geq 0} n!\,\|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} < \infty. \tag{12}$$

In this case, the solution $u = \{u(t,x); t \geq 0, x \in \mathbb{R}\}$ is unique and has the Wiener chaos expansion:

$$u(t,x) = \sum_{n\geq 0} I_n(f_n(\cdot,t,x)), \tag{13}$$

with kernels $f_n(\cdot,t,x), n \geq 1$ given by (10) *and $f_0(t,x) = \eta$. Moreover, for any $t \geq 0$ and $x \in \mathbb{R}$,*

$$E|u(t,x)|^2 = \sum_{n\geq 0} n!\,\|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}.$$

Proof. We use the same argument as on p. 302-303 of [15] for the Parabolic Anderson Model (with a noise different than here). Assume that a solution $u = \{u(t,x)\}$ to equation (1) exists. Since $u(t,x) \in L^2(\Omega,\mathcal{G},P)$, it has the Wiener chaos expansion (13) for some $f_n(\cdot,t,x) \in \mathcal{H}^{\otimes n}$ for $n \geq 1$ and $f_0(t,x) = \eta$.

We fix $t > 0$ and $x \in \mathbb{R}$ and we write the Wiener chaos expansion for the variable $u(s,y)$ for $s \in [0,t]$ and $y \in \mathbb{R}$. We multiply this by the deterministic function $1_{[0,t]}(s)\lambda G(t-s-x-y)$. It follows that the process $v^{(t,x)}$ given in Definition 3.1 has the Wiener chaos expansion $v^{(t,x)}(s,y) = \sum_{n\geq 0} I_n(g_n^{(t,x)}(\cdot,s,y))$, with kernels:

$$g_n^{(t,x)}(\cdot,s,y) = 1_{[0,t]}(s)\lambda G(t-s,x-y)\widetilde{f}_n(\cdot,s,y), \tag{14}$$

being $\widetilde{f}_n(\cdot,s,y)$ the symmetrization of $f_n(\cdot,s,y)$ in the first n variables.

By Proposition 2.1, $v^{(t,x)} \in \mathrm{Dom}\,\delta$ if and only if $\sum_{n\geq 0} I_{n+1}(g_n^{(t,x)})$ converges in $L^2(\Omega)$. In this case, $\delta(v^{(t,x)}) = \sum_{n\geq 0} I_{n+1}(g_n^{(t,x)})$ and relation $u(t,x) = \eta + \delta(v^{(t,x)})$ becomes:

$$\sum_{n\geq 0} I_n(f_n(\cdot,t,x)) = \eta + \sum_{n\geq 0} I_{n+1}(g_n^{(t,x)}). \tag{15}$$

We denote by $\widetilde{g_n^{(t,x)}}$ the symmetrization of g in all $n+1$ variables, i.e.

$$\begin{aligned}
&\widetilde{g_n^{(t,x)}}(t_1,x_1,\ldots,t_n,x_n,s,y) \\
&\quad = \frac{1}{n+1}\Big[g_n^{(t,x)}(t_1,x_1,\ldots,t_n,x_n,s,y) \\
&\quad + \sum_{i=1}^{n} g_n^{(t,x)}(t_1,x_1,\ldots,t_{i-1},x_{i-1},s,y,t_{i+1},x_{i+1},\ldots,t_n,x_n,t_i,x_i)\Big]
\end{aligned}$$

Relation (15) can be written also as:

$$\eta + \sum_{n\geq 0} I_{n+1}(\widetilde{f}_{n+1}(\cdot,t,x)) = \eta + \sum_{n\geq 0} I_n(\widetilde{g_n^{(t,x)}}).$$

By the uniqueness of the Wiener chaos expansion with symmetric kernels (see Theorem 1.1.2 of [20]), we infer that for any $n\geq 0$,

$$\widetilde{f}_{n+1}(\cdot,t,x) = \widetilde{g_n^{(t,x)}},$$

that is

$$\widetilde{f}_{n+1}(t_1,x_1,\ldots,t_n,x_n,t_{n+1},x_{n+1},t,x) = \widetilde{g_n^{(t,x)}}(t_1,x_1,\ldots,t_n,x_n,t_{n+1},x_{n+1}).$$

This allows us to find $\widetilde{f}_n(\cdot,t,x)$ recursively:

$$\begin{aligned}
\widetilde{f}_1(t_1,x_1,t,x) &= \widetilde{g_0^{(t,x)}}(t_1,x_1) = 1_{[0,t]}(t_1)\lambda G(t-t_1,x-x_1)\eta \\
\widetilde{f}_2(t_1,x_1,t_2,x_2,t,x) &= \widetilde{g_1^{(t,x)}}(t_1,x_1,t_2,x_2) \\
&= \frac{1}{2}\left[g_1^{(t,x)}(t_1,x_1,t_2,x_2) + g_1^{(t,x)}(t_2,x_2,t_1,x_1)\right] \\
&= \frac{1}{2}[1_{[0,t]}(t_2)\lambda G(t-t_2,x-x_2)f_1(t_1,x_1,t_2,x_2) + \\
&\quad 1_{[0,t]}(t_1)\lambda G(t-t_1,x-x_1 f_1(t_2,x_2,t_1,x_1)] \\
&= \frac{\lambda^2}{2}[1_{[0,t]}(t_2)G(t-t_2,x-x_2)1_{[0,t_2]}(t_1)G(t_2-t_1,x_2-x_1)\eta + \\
&\quad 1_{[0,t]}(t_1)G(t-t_1,x-x_1)1_{[0,t_1]}(t_2)G(t_1-t_2,x_1-x_2)\eta],
\end{aligned}$$

and so on. This shows that the kernels $\widetilde{f}_n(\cdot,t,x)$ must be of the form (11).

The series $\sum_{n\geq 0} I_{n+1}(g_n^{(t,x)}) = \sum_{n\geq 0} I_{n+1}(f_{n+1}(\cdot,t,x))$ converges in $L^2(\Omega)$ if and only if (12) holds. In this case, the solution u exists and is unique, with the Wiener chaos expansion (13) with kernels $f_n(\cdot,t,x)$ given by (10). $\square$

We will also need the following technical result which follows from Lemma 4.5 of [17] using the change of variable $s_j = t - t_{n+1-j}$ for $j=1,\ldots,n$.

Lemma 3.3. *Let $T_n(t) = \{(t_1, \ldots, t_n); 0 < t_1 < \ldots < t_n < t\}$ for any $t > 0$ and $n \geq 1$. Then, for any $\beta_1, \ldots, \beta_n > -1$, we have:*

$$I_n(t, \beta_1, \ldots, \beta_n) := \int_{T_n(t)} \prod_{j=1}^{n} (t_{j+1} - t_j)^{\beta_j} dt_1 \ldots dt_n = \frac{\prod_{j=1}^{n} \Gamma(\beta_j + 1)}{\Gamma(|\beta| + n + 1)} t^{|\beta|+n},$$

where $|\beta| = \sum_{j=1}^{n} \beta_j$ and we denote $t_{n+1} = t$. Consequently, if there exist $M > \varepsilon > 0$ such that $\varepsilon \leq \beta_j + 1 \leq M$ for all $j = 1, \ldots, n$, then

$$I_n(t, \beta_1, \ldots, \beta_n) \leq \frac{C^n}{\Gamma(|\beta| + n + 1)} t^{|\beta|+n},$$

where $C = \sup_{x \in [\varepsilon, M]} \Gamma(x)$.

At this point we proceed to state and prove the main result of the section.

Theorem 3.4. *Equation (1) has a unique solution $u = \{u(t, x); t \geq 0, x \in \mathbb{R}\}$ which satisfies: for any $t \geq 0$, $x \in \mathbb{R}$ and $p \geq 2$,*

$$E|u(t, x)|^p \leq |\eta|^p C_1 \exp(C_2 |\lambda|^{2/(2H+1)} p^{(2H+2)/(2H+1)} t), \tag{16}$$

where C_1 and C_2 are some positive constants which depend on H.

Proof. We first show the existence and uniqueness of the solution. According to Proposition 3.2, the necessary and sufficient condition for the existence of the solution u is that the series (12) is convergent. Moreover, when it converges, this series is equal to $E|u(t, x)|^2$.

To evaluate this series, we proceed as on page 49 of [18] in the case of the Parabolic Anderson Model with the same noise as here, except that we have a simplified initial condition.

We fix $t > 0$ and $x \in \mathbb{R}$. We define the Fourier transform for any function $\varphi \in L^1(\mathbb{R}^n)$,

$$\mathcal{F}\varphi(\xi_1 \ldots, \xi_n) = \int_{\mathbb{R}^n} e^{-i \sum_{j=1}^{n} \xi_j x_j} \varphi(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

Let $f_n(\cdot, t, x)$ be the kernel given by (10). By direct calculation, we obtain that

$$\begin{aligned}
&\mathcal{F} f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)(\xi_1, \ldots, \xi_n) \\
&\quad = \eta \lambda^n e^{-i(\xi_1 + \ldots + \xi_n) x} \overline{\mathcal{F}G(t_2 - t_1, \cdot)(\xi_1)} \\
&\qquad \times \overline{\mathcal{F}G(t_3 - t_2, \cdot)(\xi_1 + \xi_2)} \ldots \overline{\mathcal{F}G(t - t_n, \cdot)(\xi_1 + \ldots + \xi_n)} 1_{\{0 < t_1 < \ldots < t_n < t\}}
\end{aligned}$$

and hence

$$\begin{aligned}
&\mathcal{F} \widetilde{f}_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)(\xi_1, \ldots, \xi_n) \\
&\quad = e^{-i(\xi_1 + \ldots + \xi_n) x} \frac{\eta \lambda^n}{n!} \sum_{\rho \in S_n} \overline{\mathcal{F}G(t_{\rho(2)} - t_{\rho(1)}, \cdot)(\xi_{\rho(1)})} \\
&\qquad \times \overline{\mathcal{F}G(t_{\rho(3)} - t_{\rho(2)}, \cdot)(\xi_{\rho(1)} + \xi_{\rho(2)})} \ldots \overline{\mathcal{F}G(t - t_{\rho(n)}, \cdot)(\xi_{\rho(1)} + \ldots + \xi_{\rho(n)})} \\
&\qquad \times 1_{\{0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t\}}
\end{aligned}$$

Using (4), we obtain that:

$$
\begin{aligned}
&\|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}\\
&= \eta^2\lambda^{2n}\int_{(0,t)^n}\int_{\mathbb{R}^n}|\mathcal{F}\widetilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot)(\xi_1,\ldots,\xi_n)|^2\mu(d\xi_1)\ldots\mu(d\xi_n)dt_1\ldots dt_n\\
&= \eta^2\lambda^{2n}\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t}\int_{\mathbb{R}^n}|\mathcal{F}\widetilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot)(\xi_1,\ldots,\xi_n)|^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu(d\xi_1)\ldots\mu(d\xi_n)dt_1\ldots dt_n\\
&= \frac{\eta^2\lambda^{2n}}{(n!)^2}\sum_{\rho\in S_n}\int_{0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t}\int_{\mathbb{R}^n}|\mathcal{F}G(t_{\rho(2)}-t_{\rho(1)},\cdot)(\xi_{\rho(1)})|^2\\
&\quad\times|\mathcal{F}G(t_{\rho(3)}-t_{\rho(2)},\cdot)(\xi_{\rho(1)}+\xi_{\rho(2)})|^2\ldots|\mathcal{F}G(t-t_{\rho(n)},\cdot)(\xi_{\rho(1)}+\ldots\xi_{\rho(n)})|^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu(d\xi_{\rho(1)})\ldots\mu(d\xi_{\rho(n)})dt_{\rho(1)}\ldots dt_{\rho(n)}\\
&= \frac{\eta^2\lambda^{2n}}{n!}\int_{0<t_1'<\ldots<t_n'<t}\int_{\mathbb{R}^n}|\mathcal{F}G(t_2'-t_1',\cdot)(\xi_1')|^2|\mathcal{F}G(t_3'-t_2',\cdot)(\xi_1'+\xi_2')|^2\ldots\\
&\quad\times|\mathcal{F}G(t-t_n',\cdot)(\xi_1'+\ldots+\xi_n')|^2\mu(d\xi_1')\ldots\mu(d\xi_n')dt_1'\ldots dt_n',
\end{aligned}
$$

where for the last equality we used the change of variable $\xi_j'=\xi_{\rho(j)}$ and $t_j'=t_{\rho(j)}$.

Recall that $T_n(t)=\{(t_1,\ldots,t_n);0<t_1<\ldots<t_n<t\}$. Hence

$$
\begin{aligned}
&n!\,\|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}\\
&= \eta^2\lambda^{2n}c_H^n\int_{T_n(t)}\int_{\mathbb{R}^n}|\mathcal{F}G(t_2-t_1,\cdot)(\xi_1)|^2|\mathcal{F}G(t_3-t_2,\cdot)(\xi_1+\xi_2)|^2\ldots\\
&\qquad\qquad |\mathcal{F}G(t-t_n,\cdot)(\xi_1+\ldots+\xi_n)|^2|\xi_1|^{1-2H}\ldots|\xi_n|^{1-2H}dt_1\ldots dt_n \qquad (17)\\
&= \eta^2\lambda^{2n}c_H^n\int_{T_n(t)}\int_{\mathbb{R}^n}|\mathcal{F}G(t_2-t_1,\cdot)(\eta_1)|^2|\mathcal{F}G(t_3-t_2,\cdot)(\eta_2)|^2\ldots|\mathcal{F}G(t-t_n,\cdot)(\eta_n)|^2\\
&\qquad\qquad |\eta_1|^{1-2H}|\eta_2-\eta_1|^{1-2H}\ldots|\eta_n-\eta_{n-1}|^{1-2H}d\eta_1\ldots d\eta_n dt_1\ldots dt_n,
\end{aligned}
$$

where for the last equality we used the change of variable $\eta_j=\xi_1+\ldots+\xi_j$ for $j=1,\ldots,n$.

Using the inequality $(a+b)^p\leq a^b+b^p$ for $p\in(0,1)$ and $a,b>0$, we have:

$$
|\eta_j-\eta_{j-1}|^{1-2H}\leq(|\eta_{j-1}|+|\eta_j|)^{1-2H}\leq|\eta_{j-1}|^{1-2H}+|\eta_j|^{1-2H}.
$$

We use the following fact: for any finite set S and positive numbers $(a_j)_{j\in S}$ and $(b_j)_{j\in S}$,

$$
\prod_{j\in S}(a_j+b_j)=\sum_{I\subset S}\left(\prod_{j\in I}a_j\right)\left(\prod_{J\in S\setminus I}b_j\right). \qquad (18)
$$

Hence,

$$\begin{aligned}
\prod_{j=2}^{n} |\eta_j - \eta_{j-1}|^{1-2H} &\le \prod_{j=2}^{n} (|\eta_{j-1}|^{1-2H} + |\eta_j|^{1-2H}) \\
&= \sum_{I \subset \{2,\ldots,n\}} \left(\prod_{j \in I} |\eta_{j-1}|^{1-2H} \right) \left(\prod_{j \in \{2,\ldots,n\} \setminus I} |\eta_j|^{1-2H} \right) \\
&= \sum_{I \subset \{2,\ldots,n\}} \left(\prod_{j \in I-1} |\eta_j|^{1-2H} \right) \left(\prod_{j \in \{2,\ldots,n\} \setminus I} |\eta_j|^{1-2H} \right),
\end{aligned}$$

where $I-1 = \{j-1; j \in I\}$. Note that the last sum can be written as

$$\sum_{\alpha \in D_n} \prod_{j=1}^{n} |\eta_j|^{\alpha_j},$$

where D_n is a set of cardinality 2^{n-1} consisting of multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with the following properties:

$$|\alpha| = \sum_{j=1}^{n} \alpha_j = (n-1)(1-2H),$$

$$\alpha_1 \in \{0, 1-2H\} \quad \text{and} \quad \alpha_j \in \{0, 1-2H, 2(1-2H)\} \text{ for } j = 2, \ldots, n.$$

Hence

$$\prod_{j=2}^{n} |\eta_j - \eta_{j-1}|^{1-2H} \le \sum_{\alpha \in D_n} \prod_{j=1}^{n} |\eta_j|^{\alpha_j}. \tag{19}$$

Using the notation $t_{n+1} = t$, we obtain

$$\begin{aligned}
&n! \, \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \\
&\quad \le \eta^2 \lambda^{2n} c_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} \prod_{j=1}^{n} |\mathcal{F}G(t_{j+1} - t_j, \cdot)(\eta_j)|^2 |\eta_1|^{1-2H} \\
&\qquad \times \sum_{\alpha \in D_n} \prod_{j=1}^{n} |\eta_j|^{\alpha_j} d\eta_1 \ldots d\eta_n dt_1 \ldots dt_n \\
&\quad = \eta^2 \lambda^{2n} c_H^n \sum_{\alpha \in D_n} \int_{T_n(t)} \left(\int_{\mathbb{R}} |\mathcal{F}G(t_2 - t_1, \cdot)(\eta_1)|^2 |\eta_1|^{1-2H+\alpha_1} d\eta_1 \right) \\
&\qquad \times \prod_{j=2}^{n} \left(\int_{\mathbb{R}} |\mathcal{F}G(t_{j+1} - t_j, \cdot)(\eta_j)|^2 |\eta_j|^{\alpha_j} d\eta_j \right) dt_1 \ldots dt_n.
\end{aligned}$$

Note that for any $\alpha \in (-1, 1)$ and $t > 0$, we have: (see relation (3.3) in [3])

$$\int_{\mathbb{R}} |\mathcal{F}G(t, \cdot)(\xi)|^2 |\xi|^{\alpha} d\xi = C_\alpha t^{1-\alpha}, \tag{20}$$

where $C_\alpha > 0$ is a constant which depends on α. We use this relation for $1 - 2H + \alpha_1$ and for α_j with $j = 2, \ldots, n$. In order to have $2(1 - 2H) < 1$ we need to assume that $H > 1/4$.

It follows that:

$$n! \, \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \leq \eta^2 \lambda^{2n} C^n \sum_{\alpha \in D_n} \int_{T_n(t)} (t_2 - t_1)^{2H - \alpha_1} \prod_{j=2}^{n} (t_{j+1} - t_j)^{1 - \alpha_j} dt_1 \ldots dt_n,$$

where $C > 0$ is constant depending on H. At this point, we apply Lemma 3.3 with $\beta_1 = 2H - \alpha_1$ and $\beta_j = 1 - \alpha_j$ for all $j = 2, \ldots, n$. We note that $\beta_j \in [0, 1]$ for all $j = 1, \ldots, n$ and hence, we can take $\varepsilon = 1$ and $M = 2$ in Lemma 3.3. Then

$$|\beta| = \sum_{j=1}^{n} \beta_j = 2H + (n - 1) - |\alpha| = 2Hn.$$

We obtain that

$$n! \, \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \leq \eta^2 \frac{\lambda^{2n} C^n t^{n(2H+1)}}{\Gamma(n(2H+1)+1)} \leq \eta^2 \frac{\lambda^{2n} C^n t^{n(2H+1)}}{(n!)^{2H+1}}, \tag{21}$$

where $C > 0$ is a constant which depends on H. For the last inequality we used the fact that $\Gamma(an + 1) \geq C(n!)^a$ for all $n \geq 1$ and for any $a > 1$ (see e.g. relation (68) of [2]). Using Lemma A.1 of [2], we infer that

$$\sum_{n \geq 0} n! \, \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \leq \eta^2 \sum_{n \geq 0} \frac{\lambda^{2n} C^n t^{n(2H+1)}}{(n!)^{2H+1}} \leq \eta^2 C_1 \exp(|\lambda|^{2/(2H+1)} C_2 t),$$

for some positive constants C_1 and C_2 which depend on H. This proves that the solution exists, is unique and satisfies (16) for $p = 2$.

To obtain (16) for $p \geq 2$ arbitrary, we proceed as in the proof of Proposition 5.1 of [2]. We denote by $\| \cdot \|_p$ the $L^p(\Omega)$-norm. Using Minkowski's inequality, the equivalence of the $\| \cdot \|_p$-norms on a *fixed* Wiener chaos space $\mathcal{H}_n$ and (21), we have

$$\begin{aligned} \|u(t, x)\|_p &\leq \sum_{n \geq 0} \|I_n(f_n(\cdot, t, x))\|_p \leq \sum_{n \geq 0} (p-1)^{n/2} \|I_n(f_n(\cdot, t, x))\|_2 \\ &= \sum_{n \geq 0} (p-1)^{n/2} \left(n! \|\widetilde{f}_n(\cdot, t, x))\|^2_{\mathcal{H}^{\otimes n}} \right)^{1/2} \\ &\leq |\eta| \sum_{n \geq 0} (p-1)^{n/2} \frac{|\lambda|^n C^{n/2} t^{n(H+1/2)}}{(n!)^{H+1/2}}. \end{aligned}$$

Using again Lemma A.1 of [2], we obtain:

$$\|u(t, x)\|_p \leq |\eta| C_1 \exp(C_2 |\lambda|^{2/(2H+1)} p^{1/(2H+1)} t).$$

Relation (16) follows taking power p. $\square$

Finally, the optimality of the condition $H > \frac{1}{4}$ is addressed in the following proposition.

Proposition 3.5. *Assume that $H \in (0, \frac{1}{2})$ and that equation* (1) *admits a unique solution in the Skorohod sense. Then $H > \frac{1}{4}$.*

Proof. By Proposition 3.2, equation (1) has a solution if and only if condition (12) holds. Note that, for any $n \geq 1$, in (17) we proved that

$$\begin{aligned} & n! \, \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \\ & \quad = \eta^2 \lambda^{2n} c_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} |\mathcal{F}G(t_2 - t_1, \cdot)(\eta_1)|^2 |\mathcal{F}G(t_3 - t_2, \cdot)(\eta_2)|^2 \dots |\mathcal{F}G(t - t_n, \cdot)(\eta_n)|^2 \\ & \qquad\qquad \times |\eta_1|^{1-2H} |\eta_2 - \eta_1|^{1-2H} \dots |\eta_n - \eta_{n-1}|^{1-2H} d\eta_1 \dots d\eta_n dt_1 \dots dt_n, \end{aligned}$$

where we recall that $T_n(t) = \{(t_1, \dots, t_n); 0 < t_1 < \dots < t_n < t\}$. Assume that $n \geq 2$ and note that if $a, b \in \mathbb{R}$ have opposite signs, then $|a - b| = |a| + |b|$. Based on this simple observation, we estimate from below the integral with respect to $(\eta_1, \eta_2) \in \mathbb{R}^2$ by the integral on the set $(\eta_1, \eta_2) \in \mathbb{R}_+ \times \mathbb{R}_-$ and use that, for any $(\eta_1, \eta_2) \in \mathbb{R}_+ \times \mathbb{R}_-$, we have

$$|\eta_2 - \eta_1| = |\eta_1| + |\eta_2| \geq |\eta_1|.$$

Hence

$$|\eta_2 - \eta_1|^{1-2H} \geq |\eta_1|^{1-2H}$$

since the function $\xi \mapsto \xi^{1-2H}$ is increasing on $\mathbb{R}$ (because $1 - 2H > 0$). Then, by Fubini theorem,

$$\begin{aligned} n! \, \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} \geq \eta^2 \lambda^{2n} c_H^n & \int_{T_n(t)} \left(\int_{\mathbb{R}_+} |\mathcal{F}G(t_2 - t_1, \cdot)(\eta_1)|^2 |\eta_1|^{2-4H} \, d\eta_1 \right) \\ & \times \int_{\mathbb{R}_-} \int_{\mathbb{R}^{n-2}} |\mathcal{F}G(t_3 - t_2, \cdot)(\eta_2)|^2 \dots |\mathcal{F}G(t - t_n, \cdot)(\eta_n)|^2 \\ & \times |\eta_3 - \eta_2|^{1-2H} \cdots |\eta_n - \eta_{n-1}|^{1-2H} d\eta_2 \dots d\eta_n dt_1 \dots dt_n. \end{aligned}$$

For all $r > 0$, the function $\mathcal{F}G(r, \cdot)(\eta_1) = \frac{\sin(r|\eta_1|)}{|\eta_1|}$ is symmetric, which implies

$$\int_{\mathbb{R}_+} |\mathcal{F}G(r, \cdot)(\eta_1)|^2 |\eta_1|^{2-4H} \, d\eta_1 = \frac{1}{2} \int_{\mathbb{R}} |\mathcal{F}G(r, \cdot)(\eta_1)|^2 |\eta_1|^{2-4H} \, d\eta_1,$$

and the latter integral is convergent if and only if $H \in (\frac{1}{4}, \frac{3}{4})$ (see [3, Eq. (3.3)]). This concludes the proof. □

4 Solution in the Itô sense

In this section, we introduce the concept of solution of equation (1) in the Itô sense, as defined in [3], and we show that this solution coincides with the solution in the Skorohod sense defined in the Section 3.

We recall from Section 2.2 of [3] the construction of the stochastic integral with respect to the noise X. We say that g is an *elementary process* on $\mathbb{R}_+ \times \mathbb{R}$ if it is of the form

$$g(\omega, t, x) = Y(\omega) 1_{(a,b]}(t) 1_{(v,w]}(x), \tag{22}$$

where $0 \leq a < b \leq T$, Y is a $\mathbb{R}$-valued bounded $\mathcal{F}_a$-measurable random variable, and $v, w \in \mathbb{R}$ with $v < w$. The integral of a process g of the form (22) with respect to X is defined by:

$$(g \cdot X)_t := \int_0^t \int_{\mathbb{R}} g(s,x) X(ds,dx) = Y(X_{t\wedge b}((v,w]) - X_{t\wedge a}((v,w])).$$

This definition is extended to the set $\mathcal{E}_r$ of all linear combinations of elementary processes. For any process $g \in \mathcal{E}_r$ and for any $T > 0$, we have

$$E\left|\int_0^T \int_{\mathbb{R}} g(t,x) X(dt,dx)\right|^2 = E\int_0^T \int_{\mathbb{R}} |\mathcal{F}g(t,\cdot)(\xi)|^2 \mu(d\xi) dt =: \|g\|_{0,T}^2.$$

We fix $T > 0$. Then the map $g \mapsto g \cdot X$ is an isometry which is extended to the completion $\mathcal{P}_0^T$ of $\mathcal{E}_r$ with respect to the norm $\|g\|_{0,T}^2$. For any $g \in \mathcal{P}_0^T$, we say that

$$(g \cdot X)_t = \int_0^t \int_{\mathbb{R}} g(s,x) X(ds,dx), \quad t \in [0,T]$$

is the **Itô integral** of g with respect to X.

Following [3], we have the following definition.

Definition 4.1. We say that a random field $u = \{u(t,x);\, t \in [0,T] \times \mathbb{R}\}$ is a **solution of** (1) **(in the Itô sense)** if, for any $(t,x) \in [0,T] \times \mathbb{R}$,

$$u(t,x) = \eta + \int_0^t \int_{\mathbb{R}} G(t-s, x-y) \lambda u(s,y) X(ds,dy) \quad \text{a.s.} \tag{23}$$

By Theorem 1.1 of [3], we know that (1) has a unique solution in the Itô sense, which is obtained as the limit of the sequence $(u_n)_{n\geq 0}$ of Picard iterations, given by: $u_0(t,x) = \eta$ and

$$u_{n+1}(t,x) = \eta + \int_0^t \int_{\mathbb{R}} G(t-s, x-y) \lambda u_n(s,y) X(ds,dy), \quad n \geq 0. \tag{24}$$

The following result is the analogue of Proposition 1.3.11 of [20] for the noise X; see also Theorem 2.9 of [12]. Its proof is included in Appendix A, for the sake of completeness.

Theorem 4.2. *Let $u = \{u(t,x); t \geq 0,\, x \in \mathbb{R}\}$ be a process such that u restricted to $[0,T]$ belongs to $\mathcal{P}_0^T$ for any $T > 0$. Then for any $t > 0$, $u\,\mathbf{1}_{[0,t]} \in$ Dom δ and its Skorohod integral coincides with the Itô integral, that is*

$$\int_0^\infty \int_{\mathbb{R}} u(s,x) \mathbf{1}_{[0,t]}(s) X(\delta s, \delta x) = \int_0^t \int_{\mathbb{R}} u(s,x) X(ds,dx).$$

The following result is an immediate consequence of Theorem 4.2.

Theorem 4.3. *The solution of equation* (1) *in the Itô sense coincides with the solution of* (1) *in the Skorohod sense. Moreover, the n-th Picard iteration is given by a predictable modification of*

$$u_n(t,x) = \sum_{k=0}^{n} I_k(f_k(\cdot,t,x)), \tag{25}$$

where $f_k(\cdot,t,x)$ is defined by (10) *for $k \geq 1$ and $f_0(t,x) = \eta$.*

Proof. Let u be the solution of (1) in the Itô sense. Fix $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and consider the process $v^{(t,x)}(s,y) = 1_{[0,t]}(s)\lambda G(t-s,x-u)u(s,y)$ for $s > 0$ and $y \in \mathbb{R}$. Then $v^{(t,x)}$ restricted to $[0,T]$ belongs to $\mathcal{P}_0^T$ for any $T > 0$. By Theorem 4.2, $v^{(t,x)} \in \mathrm{Dom}\,\delta$ and its Skorohod integral coincides with its Itô integral, i.e. the Skorohod integral in (9) coincides with the Itô integral in (23). Hence, u is the solution of (1) in the Skorohod sense.

For the second statement, let $(u_n)_{n\geq 0}$ be the sequence defined by (25). It can be proved that each u_n is $L^2(\Omega)$-continuous, and hence it has a predictable modification. We work with this modification (denoted also by u_n). We write (25) for $u_n(s,y)$, and we multiply this relation by $1_{[0,t]}(s)\lambda G(t-s,x-y)$. We obtain:

$$\begin{aligned} 1_{[0,t]}(s)G(t-s,x-y)u_n(s,y) &= \sum_{k=0}^{n} I_k(1_{[0,t]}(s)\lambda G(t-s,x-y)f_k(\cdot,s,y)) \\ &= \sum_{k=0}^{n} I_k(f_{k+1}(\cdot,s,y,t,x)). \end{aligned}$$

By Proposition 2.1, the process $1_{[0,t]}(\cdot)\lambda G(t-\cdot,x-\cdot)u_n$ is Skorohod integrable and its Skorohod integral is given by

$$\int_0^t \int_{\mathbb{R}} G(t-s,x-y)\lambda u_n(s,y)X(\delta s,\delta y) = \sum_{k=0}^{n} I_{k+1}(f_{k+1}(\cdot,t,x)) = \eta - u_{n+1}(t,x).$$

Now, we use this equality to show by induction that the sequence $(u_n)_{n\geq 0}$ satisfies exactly the recurrence relation (24). Indeed, since u_0 is deterministic, one clearly obtains that relation (24) holds for $n = 0$. In [3, Thm. 1.1], we proved that any Picard iterate defined through (24) is well-defined. Therefore, the restriction of the process $1_{[0,t]}(\cdot)\lambda G(t-\cdot,x-\cdot)u_1$ to $[0,T]$ belongs to $\mathcal{P}_0^T$ for any $T > 0$, and hence, by Theorem 4.2, its Skorohod integral coincides with the Itô integral. This shows that relation (24) holds for $n = 1$. The same argument can be used in the general induction step. □

5 Intermittency

In this section, we prove that the solution u to equation (1) is weakly intermittent. Recall the definitions (2) and (3) of the lower and upper Lyapunov exponents of u of order p, respectively. We will prove the following result.

Theorem 5.1. *The random field u is weakly intermittent, i.e.*

$$\underline{\gamma}(2) > 0 \quad \textit{and} \quad \overline{\gamma}(p) < \infty \quad \textit{for all } p \geq 2.$$

Proof. For the upper bound, we observe that the estimate (16) clearly implies that $\overline{\gamma}(p) < \infty$ for all $p \geq 2$. So it remains to prove the lower bound, i.e. $\underline{\gamma}(2) > 0$. For this, we will obtain an exponential lower bound for the second order moment of the solution u. Surprisingly, we will see that our noise's roughness (i.e. the fact that $H < \frac{1}{2}$) works in our advantage.

By Proposition 3.2, we know that, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$E|u(t,x)|^2 = \sum_{n=0}^{\infty} n! \, \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}},$$

where the kernels $\widetilde{f}_n$ are given in (11). On the other hand, equation (17) in the proof of Theorem 3.4 yields

$$\begin{aligned} & n! \, \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\ & = \eta^2 \lambda^{2n} c_H^n \int_{T_n(t)} \int_{\mathbb{R}^n} |\mathcal{F}G(t_2-t_1,\cdot)(\eta_1)|^2 |\mathcal{F}G(t_3-t_2,\cdot)(\eta_2)|^2 \dots |\mathcal{F}G(t-t_n,\cdot)(\eta_n)|^2 \\ & \qquad \times |\eta_1|^{1-2H} |\eta_2-\eta_1|^{1-2H} \dots |\eta_n-\eta_{n-1}|^{1-2H} d\eta_1 \dots d\eta_n dt_1 \dots dt_n, \end{aligned}$$

In order to bound the above term from below, we are going to use the same idea as in the proof of Proposition 3.5. Namely, we consider the set $A = \{(\eta_1,\dots,\eta_n); \eta_1 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_-, \dots, \eta_{n-1} \in \mathbb{R}_+, \eta_n \in \mathbb{R}_-\}$ if n is even and $A = \{(\eta_1,\dots,\eta_n); \eta_1 \in \mathbb{R}_+, \eta_2 \in \mathbb{R}_-, \dots, \eta_{n-1} \in \mathbb{R}_-, \eta_n \in \mathbb{R}_+\}$ if n is odd. For any $(\eta_1,\dots,\eta_n) \in A$, we have

$$|\eta_j - \eta_{j-1}| = |\eta_j| + |\eta_{j-1}| \geq |\eta_j|, \quad \text{for all } j = 2,\dots,n,$$

and hence

$$|\eta_j - \eta_{j-1}|^{1-2H} \geq |\eta_j|^{1-2H}, \quad \text{for all } j = 2,\dots,n,$$

since the function $\xi \mapsto \xi^{1-2H}$ is increasing on $\mathbb{R}_+$ (because $1-2H > 0$). Then

$$\begin{aligned} & n! \, \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\ & \geq \eta^2 \lambda^{2n} c_H^n \int_{T_n(t)} \int_{A} |\mathcal{F}G(t_2-t_1,\cdot)(\eta_1)|^2 |\mathcal{F}G(t_3-t_2,\cdot)(\eta_2)|^2 \dots |\mathcal{F}G(t-t_n,\cdot)(\eta_n)|^2 \\ & \qquad \times |\eta_1|^{1-2H} |\eta_2|^{1-2H} \dots |\eta_n|^{1-2H} \, d\eta_1 \dots d\eta_n dt_1 \dots dt_n \\ & = \eta^2 \lambda^{2n} C^n \int_{T_n(t)} \prod_{j=1}^{n} \int_{\mathbb{R}} |\mathcal{F}G(t_{j+1}-t_j,\cdot)(\eta)|^2 |\eta|^{1-2H} \, d\eta dt_1 \dots dt_n, \end{aligned}$$

where $t_{n+1} = t$, and we have taken into account that, for all $r > 0$,

$$\int_{\mathbb{R}_+} |\mathcal{F}G(r,\cdot)(\eta)|^2 |\eta|^{1-2H} d\eta = \int_{\mathbb{R}_-} |\mathcal{F}G(r,\cdot)(\eta)|^2 |\eta|^{1-2H} d\eta = \frac{1}{2} \int_{\mathbb{R}} |\mathcal{F}G(r,\cdot)(\eta)|^2 |\eta|^{1-2H} d\eta.$$

Then , since $\int_{\mathbb{R}} |\mathcal{F}G(r,\cdot)(\eta)|^2 |\eta|^{1-2H} d\eta = C\, r^{2H}$ (see, e.g. [3, Eq. (3.3)]), we can infer that

$$
\begin{aligned}
n!\, \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} &\geq \eta^2 \lambda^{2n} C^n \int_{T_n(t)} \prod_{j=1}^n (t_{j+1}-t_j)^{2H}\, dt_1 \dots dt_n \\
&= \eta^2 \lambda^{2n} C^n \frac{\Gamma(2H+1)^n}{\Gamma(2Hn+n+1)}\, t^{(2H+1)n} \\
&= \eta^2 \lambda^{2n} C^n \frac{t^{(2H+1)n}}{\Gamma((2H+1)n+1)}.
\end{aligned}
$$

Here we have applied Lemma 3.3 with $\beta_j = 2H$ for all j.

At this point, we apply the following fact, which can be verified by applying Stirling's Formula:

$$
\frac{\Gamma(an+1)}{a^{an} n^{\frac{1-a}{2}} (n!)^a} \leq C, \quad \text{for all } \ n \geq 0 \ \text{ and } \ a \in \mathbb{R}_+.
$$

We apply this inequality with $a = 2H+1$. Hence

$$
\begin{aligned}
n!\, \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} &\geq \eta^2 \lambda^{2n} C^n \frac{t^{(2H+1)n}}{(n!)^{2H+1} (2H+1)^{(2H+1)n} n^{-H}} \\
&= \eta^2 \lambda^{2n} C^n \, \frac{n^H t^{(2H+1)n}}{(n!)^{2H+1}} \\
&\geq \eta^2 \lambda^{2n} C^n \, \frac{t^{(2H+1)n}}{(n!)^{2H+1}},
\end{aligned}
$$

because $n^H \geq 1$ for all $n \geq 1$.

Therefore

$$
\begin{aligned}
\sum_{n=0}^\infty n!\, \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} &= \eta^2 + \sum_{n=1}^\infty n!\, \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\
&\geq \eta^2 \sum_{n=0}^\infty \lambda^{2n} C^n \, \frac{t^{(2H+1)n}}{(n!)^{2H+1}} \\
&\geq C \exp(C \lambda^{\frac{2}{2H+1}}\, t),
\end{aligned}
$$

where the last inequality follows by Lemma 5.2 below. This concludes the proof. $\square$

In the proof of the next lemma, we will apply the following inequality, which can be easily checked by induction: for any sequence $(a_n)_{n\geq 0}$ of positive real numbers and for any $p \geq 1$, it holds

$$
\left(\sum_{n=0}^\infty a_n \right)^p \leq 2^{p-1} \sum_{n=0}^\infty \left(2^{p-1}\right)^n a_n^p. \tag{26}
$$

Lemma 5.2. *For any $p>0$ and $x>0$,*

$$\sum_{n\geq 0}\frac{x^n}{(n!)^p}\geq c_1\exp(c_2x^{1/p}),$$

where c_1 and c_2 are some positive constants which depend on p.

Proof. Case 1: $p<1$. We use the fact that $(\sum_{n\geq 0}a_n)^p\leq \sum_{n\geq 0}a_n^p$, for any positive real numbers $(a_n)_{n\geq 0}$. Hence,

$$\sum_{n\geq 0}\frac{x^n}{(n!)^p}=\sum_{n\geq 0}\left(\frac{x^{n/p}}{n!}\right)^p\geq\left(\sum_{n\geq 0}\frac{x^{n/p}}{n!}\right)^p=\exp(p\,x^{1/p}).$$

Case 2: $p\geq 1$. Using (26), we obtain:

$$\begin{aligned}\sum_{n\geq 0}\frac{x^n}{(n!)^p} &= \sum_{n\geq 0}2^{(p-1)n}\left(\frac{2^{-(p-1)n/p}x^{n/p}}{n!}\right)^p\\ &\geq 2^{1-p}\left(\sum_{n\geq 0}\frac{2^{-(p-1)n/p}x^{n/p}}{n!}\right)^p\\ &= 2^{1-p}\exp(p\,2^{-(p-1)/p}x^{1/p}).\end{aligned}$$

□

Remark 5.3. It is worth mentioning that the above prove also works for the case of the stochastic heat equation, i.e. the parabolic Anderson model, which would let us recover the same intermittency result of [18]. The minor modifications only involve the formula for the Fourier transform of the underlying fundamental solution. Indeed, in this case we have

$$G(t,x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{|x|^2}{2t}},\ t>0,\ x\in\mathbb{R}\qquad\text{and}\qquad \mathcal{F}G(t,\cdot)(\xi)=e^{-t|\xi|^2/2},\ \xi\in\mathbb{R}.$$

A Proof of Theorem 4.2

The proof of this theorem is based on two auxiliary lemmas. First of all, we observe that for any Borel set A of $\mathbb{R}_+$, any Borel set $B\subset A^c$ and any $f,g\in\mathcal{H}$, the random variables $I_1(f\mathbf{1}_A)$ and $I_1(g\mathbf{1}_B)$ are independent. Indeed, the random vector $\big(I_1(f\mathbf{1}_A),\,I_1(g\mathbf{1}_B)\big)$ is Gaussian and

$$\begin{aligned}E[I_1(f\mathbf{1}_A)I_1(g\mathbf{1}_B)] &= \langle f\mathbf{1}_A,g\mathbf{1}_B\rangle_{\mathcal{H}}\\ &= \int_0^\infty\int_{\mathbb{R}}\mathcal{F}f(t,\cdot)(\xi)\mathbf{1}_A(t)\overline{\mathcal{F}g(t,\cdot)(\xi)}\mathbf{1}_B(t)\mu(d\xi)dt=0.\end{aligned}$$

On the other hand, for any $A \in \mathcal{B}(\mathbb{R}_+)$, we define the σ-field

$$\mathcal{F}_A = \sigma\{X(1_C\varphi);\, C \in \mathcal{B}_0\,, C \subset A,\, \varphi \in \mathcal{D}(\mathbb{R})\} \vee \mathcal{N},$$

where $\mathcal{N}$ are the null sets of $\mathfrak{F}$ and $\mathcal{B}_0$ are the bounded Borel sets of $\mathbb{R}_+$.

The following result is the analogue of Lemma 1.2.5 of [20] for the noise X.

Lemma A.1. *Let $F \in L^2(\Omega)$ with Wiener chaos decomposition $F = \sum_{n=0}^{\infty} I_n(f_n)$ for some symmetric functions $f_n \in \mathcal{H}^{\otimes n}$, and let $A \in \mathcal{B}(\mathbb{R}_+)$. Then*

$$E[F|\mathcal{F}_A] = \sum_{n=0}^{\infty} I_n(f_n \mathbf{1}_A^{\otimes n}).$$

Proof. Since the conditional expectation is $L^2(\Omega)$-continuous and linear, it suffices to consider $F = I_n(f^{\otimes n})$, with $f \in \mathcal{H}$ and $\|f\|_{\mathcal{H}} = 1$. We denote by $f^{\otimes n}$ the function defined by: $f^{\otimes n}(t_1, x_1, \ldots, t_n, x_n) = f(t_1, x_1) \ldots f(t_n, x_n)$.

We will proceed by induction on n. For $n = 1$, we have

$$E[I_1(f)|\mathcal{F}_A] = E[I_1(f\mathbf{1}_A) + I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] = I_1(f\mathbf{1}_A),$$

since $I_1(f\mathbf{1}_A)$ is $\mathcal{F}_A$-measurable and $I_1(f\mathbf{1}_{A^c})$ is independent of $\mathcal{F}_A$.

Suppose that the assertion is true up to some $n \geq 2$ and let us check its validity for $n+1$. We will use the following recurrent formula for the multiple Itô-Wiener integrals (see the proof of Proposition 1.1.4 of [20]):

$$I_{n+1}(g^{\otimes(n+1)}) = I_n(g^{\otimes n}) I_1(g) - n\, \|g\|_{\mathcal{H}}^2\, I_{n-1}(g^{\otimes(n-1)}), \tag{27}$$

for any $g \in \mathcal{H}$. Applying (27) to $g = f$, we have

$$\begin{aligned} E[I_{n+1}(f^{\otimes(n+1)})|\mathcal{F}_A] &= E[I_n(f^{\otimes n})I_1(f)|\mathcal{F}_A] - nE[I_{n-1}(f^{\otimes(n-1)})|\mathcal{F}_A] \\ &= E[I_n(f^{\otimes n})I_1(f)|\mathcal{F}_A] - nI_{n-1}(f^{\otimes(n-1)}\mathbf{1}_A^{\otimes(n-1)}), \end{aligned} \tag{28}$$

where we have applied the induction hypothesis to I_{n-1}. We write

$$E[I_n(f^{\otimes n})I_1(f)|\mathcal{F}_A] = E[I_n(f^{\otimes n})I_1(f\mathbf{1}_A)|\mathcal{F}_A] + E[I_n(f^{\otimes n})I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A]. \tag{29}$$

Using the fact that $I_1(f\mathbf{1}_A)$ is $\mathcal{F}_A$-measurable and the induction hypothesis, we have

$$E[I_n(f^{\otimes n})I_1(f\mathbf{1}_A)|\mathcal{F}_A] = I_1(f\mathbf{1}_A)I_n(f^{\otimes n}\mathbf{1}_A^{\otimes n}), \tag{30}$$

To deal with $E[I_n(f^{\otimes n})I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A]$, we write

$$I_n(f^{\otimes n}) = I_n(f^{\otimes n}\mathbf{1}_A^{\otimes n}) + I_n(f^{\otimes n}(1 - \mathbf{1}_A^{\otimes n})).$$

Note that

$$E[I_n(f^{\otimes n}\mathbf{1}_A^{\otimes n})I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] = I_n(f^{\otimes n}\mathbf{1}_A^{\otimes n})E[I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] = 0,$$

since $I_n(f^{\otimes n}\mathbf{1}_A^{\otimes n})$ is $\mathcal{F}_A$-measurable and $I_1(f\mathbf{1}_{A^c})$ is independent of $\mathcal{F}_A$ with zero mean. Hence,

$$E[I_n(f^{\otimes n})I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] = E[I_n(f^{\otimes n}(1-\mathbf{1}_A^{\otimes n}))I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A]. \tag{31}$$

We claim that:

$$I_n(f^{\otimes n}(1-\mathbf{1}_A^{\otimes n})) = \sum_{k=1}^{n}\binom{n}{k} I_k(f^{\otimes k}\mathbf{1}_{A^c}^{\otimes k})I_{n-k}(f^{\otimes(n-k)}\mathbf{1}_A^{\otimes(n-k)}). \tag{32}$$

To see this, we note that by (18), we have

$$\begin{aligned} f^{\otimes n}(t_1,x_1,\ldots,t_n,x_n) &= \prod_{j=1}^{n}[f(t_j,x_j)\mathbf{1}_{A^c}(t_j)+f(t_j,x_j)\mathbf{1}_A(t_j)] \\ &= \sum_{J\subset\{1,\ldots,n\}}\left[\prod_{j\in J} f(t_j,x_j)\mathbf{1}_{A^c}(t_j)\right]\cdot\left[\prod_{j\in J^c} f(t_j,x_j)\mathbf{1}_A(t_j)\right]. \end{aligned}$$

Subtracting from this $(f^{\otimes n}1_A^{\otimes n})(t_1,x_1,\ldots,t_n,x_n) = \prod_{j=1}^{n} f(t_j,x_j)\mathbf{1}_A(t_j)$, we are left with the sum over all subsets $J\subset\{1,\ldots,n\}$, except $J=\emptyset$. Relation (32) follows by integrating with respect to $X(dt_1,dx_1)\ldots X(dt_n,dx_n)$ and denoting $\mathrm{card}(J)=k$.

We multiply (32) by $I_1(f\mathbf{1}_{A^c})$ and we take the conditional expectation with respect to $\mathcal{F}_A$. We obtain:

$$\begin{aligned} &E[I_n(f^{\otimes n}(1-\mathbf{1}_A^{\otimes n}))I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] \\ &\quad = \sum_{k=1}^{n}\binom{n}{k}E[I_k(f^{\otimes k}\mathbf{1}_{A^c}^{\otimes k})I_{n-k}(f^{\otimes(n-k)}\mathbf{1}_A^{\otimes(n-k)})I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A]. \end{aligned}$$

We evaluate separately each term in the sum on the right-hand of the previous relation. The term corresponding to $k=n$ is

$$E[I_n(f\mathbf{1}_{A^c}^{\otimes n}))I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] = 0,$$

because $I_n(f\mathbf{1}_{A^c}^{\otimes n})I_1(f\mathbf{1}_{A^c})$ is independent of $\mathcal{F}_A$ and its expectation is equal to 0 (due to (6), since $n\geq 2$). The term corresponding to $k=2,\ldots,n-1$ is

$$\binom{n}{k}E[I_k(f^{\otimes k}\mathbf{1}_{A^c}^{\otimes k})I_{n-k}(f^{\otimes(n-k)}\mathbf{1}_A^{\otimes(n-k)})I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] = 0,$$

because $I_{n-k}(f^{\otimes(n-k)}\mathbf{1}_A^{\otimes(n-k)})$ is $\mathcal{F}_A$-measurable and $I_1(f\mathbf{1}_{A^c})I_k(f^{\otimes k}\mathbf{1}_{A^c}^{\otimes k})$ is independent of $\mathcal{F}_A$ with null expectation. The term corresponding to $k=1$ is

$$n\,E[I_{n-1}(f^{\otimes(n-1)}\mathbf{1}_A^{\otimes(n-1)})\big(I_1(f\mathbf{1}_{A^c})\big)^2|\mathcal{F}_A] = n\,\|f\mathbf{1}_{A^c}\|_{\mathcal{H}}^2\,I_{n-1}(f^{\otimes(n-1)}\mathbf{1}_A^{\otimes(n-1)}).$$

In summary,

$$E[I_n(f^{\otimes n}(1-\mathbf{1}_A^{\otimes n}))I_1(f\mathbf{1}_{A^c})|\mathcal{F}_A] = n\,\|f\mathbf{1}_{A^c}\|_{\mathcal{H}}^2\,I_{n-1}(f^{\otimes(n-1)}\mathbf{1}_A^{\otimes(n-1)}). \tag{33}$$

Combining (28), (29), (30), (31) and (33), we obtain:

$$\begin{aligned}
&E[I_{n+1}(f^{\otimes(n+1)})|\mathcal{F}_A] \\
&\quad = I_1(f\mathbf{1}_A)I_n(f^{\otimes n}\mathbf{1}_A^{\otimes n}) + n\,\|f\mathbf{1}_{A^c}\|_{\mathcal{H}}^2\, I_{n-1}(f^{\otimes(n-1)}\mathbf{1}_A^{\otimes(n-1)}) - nI_{n-1}(f^{\otimes(n-1)}\mathbf{1}_A^{\otimes(n-1)}) \\
&\quad = I_1(f\mathbf{1}_A)I_n(f^{\otimes n}\mathbf{1}_A^{\otimes n}) - n\,\|f\mathbf{1}_A\|_{\mathcal{H}}^2\, I_{n-1}(f^{\otimes(n-1)}\mathbf{1}_A^{\otimes(n-1)}) \\
&\quad = I_{n+1}(f^{\otimes(n+1)}\mathbf{1}_A^{\otimes(n+1)}),
\end{aligned}$$

where for the second equality we used the fact that $f\mathbf{1}_A$ and $f\mathbf{1}_{A^c}$ are orthogonal in $\mathcal{H}$ with $\|f\mathbf{1}_A\|_{\mathcal{H}}^2+\|f\mathbf{1}_{A^c}\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{H}}^2 = 1$, and for the last equality we used (27) with $g = f\mathbf{1}_A$. This concludes the proof. $\square$

We recall that a random variable $F \in L^2(\Omega)$ with the Wiener chaos expansion $F = \sum_{n\geq 0} I_n(f_n)$ for some symmetric functions $f_n \in \mathcal{H}^{\otimes n}$ is Malliavin differentiable (i.e. $F \in \mathbb{D}^{1,2}$) if and only if

$$\sum_{n\geq 1} nn!\|f_n\|_{\mathcal{H}^{\otimes n}}^2 < \infty \tag{34}$$

(see, e.g., Proposition 1.2.2 of [20]). In this case,

$$D_{t,x}F = \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(\cdot,t,x)) \tag{35}$$

(see Exercise 1.2.5 of [20] for the white noise case). We also remind that DF takes values in $\mathcal{H}$, which is a space of functions. The next result is the analogue of Proposition 1.2.8 of [20] for the noise X; see also Proposition 3.12 of [12].

Lemma A.2. *Let $F \in \mathbb{D}^{1,2}$ and $A \in \mathcal{B}(\mathbb{R}_+)$. Then $E[F|\mathcal{F}_A] \in \mathbb{D}^{1,2}$ and*

$$D(E[F|\mathcal{F}_A]) = \mathbf{1}_A E[DF|\mathcal{F}_A].$$

Proof. Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ be the Wiener chaos expansion of F, for some symmetric functions $f_n \in \mathcal{H}^{\otimes n}$. By Lemma A.1, $E[F|\mathcal{F}_A] = \sum_{n=0}^{\infty} I_n(f_n\mathbf{1}_A^{\otimes n})$. The fact that $E[F|\mathcal{F}_A] \in \mathbb{D}^{1,2}$ follows from the criterion for Malliavin differentiability stated above, since

$$\sum_{n=1}^{\infty} n\, n!\|f_n\mathbf{1}_A^{\otimes n}\|_{\mathcal{H}^{\otimes n}}^2 \leq \sum_{n=1}^{\infty} n\, n!\|f_n\|_{\mathcal{H}^{\otimes n}}^2 < \infty.$$

Moreover,

$$D_{t,x}(E[F|\mathcal{F}_A]) = \sum_{n\geq 1} nI_{n-1}(f_n(\cdot,t,x)1_A^{\otimes(n-1)}1_A(t)) = 1_A(t)\sum_{n\geq 1} nI_{n-1}(f_n(\cdot,t,x)1_A^{\otimes(n-1)}).$$

On the other hand, using (35) and the fact that the conditional expectation is linear and continuous, we have

$$E[D_{t,x}F|\mathcal{F}_A] = \sum_{n=1}^{\infty} n\, E[I_{n-1}(f_n(\cdot,t,x))|\mathcal{F}_A] = \sum_{n=1}^{\infty} n\, I_{n-1}(f_n(\cdot,t,x)\mathbf{1}_A^{\otimes(n-1)}),$$

where the last equality follows form Lemma A.1. $\square$

We recall the following results.

Proposition A.3 (Proposition 1.3.3 of [20]). *Let $F \in \mathbb{D}^{1,2}$ and $u \in Dom(\delta)$ such that $F\,u \in L^2(\Omega; \mathcal{H})$. Then $Fu \in Dom(\delta)$ and the following equality holds true:*

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathcal{H}},$$

provided that the right-hand side belongs to $L^2(\Omega)$.

Proposition A.4 (Proposition 1.3.6 of [20]). *Let $u \in L^2(\Omega; \mathcal{H})$ and $(u_n)_{n\geq 1} \subset \mathrm{Dom}\ \delta$ such that $E\|u_n - u\|_{\mathcal{H}}^2 \to 0$. Suppose that there exists a random variable $G \in L^2(\Omega)$ such that*

$$E(\delta(u_n)F) \to E(GF) \quad \textit{for all } F \in \mathcal{S},$$

where $\mathcal{S}$ is the class of smooth random variables of form (7). Then $u \in \mathrm{Dom}\ \delta$ and $\delta(u) = G$.

Now we are in position to prove Theorem 4.2.

Proof of Theorem 4.2. Case 1. u is a linear combination of elementary processes.

By linearity, it is enough to assume that $u = g$, where g is an elementary process of form (22). The Itô-type integral of g is given by

$$(g \cdot X)_t := Y[X_{t\wedge b}\big((v,w]\big) - X_{t\wedge a}\big((v,w]\big)].$$

Fix $t > 0$. We will prove that $g\mathbf{1}_{[0,t]} \in \mathrm{Dom}(\delta)$ and

$$\delta(g\mathbf{1}_{[0,t]}) = (g \cdot X)_t.$$

Without loss of generality, we assume that $Y \in \mathbb{D}^{1,2}$. (To see this, note that since $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$, there exists a sequence $(Y_n)_{n\geq 1} \subset \mathbb{D}^{1,2}$ such that $E|Y_n - Y|^2 \to 0$. Define $g_n(t,x) = Y_n 1_{(a,b]}(t) 1_{(v,w]}(x)$. Then $g_n 1_{[0,t]} \in \mathrm{Dom}\ \delta$ and $\delta(g_n 1_{[0,t]}) = (g_n \cdot X)_t$ for any $n \geq 1$. We apply Proposition A.4 with $u = g1_{[0,t]}$ and $u_n = g_n 1_{[0,t]}$.)

Observe that $(g\mathbf{1}_{[0,t]})(s,x) = Y\,\mathbf{1}_{(a\wedge t, b\wedge t]}(s)\mathbf{1}_{(v,w]}(x)$. Using Proposition A.3 with $u = \mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]} \in \mathrm{Dom}(\delta)$ and $F = Y$, we obtain that $g\mathbf{1}_{[0,t]} \in \mathrm{Dom}(\delta)$ and

$$\delta(g\mathbf{1}_{[0,t]}) = Y\delta(\mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]}) - \langle DY, \mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]}\rangle_{\mathcal{H}} = Y\,I_1(\mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]}),$$

due to the fact that

$$\begin{aligned}\langle DY, \mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]}\rangle_{\mathcal{H}} &= \langle D\big(E[Y|\mathcal{F}_a]\big), \mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]}\rangle_{\mathcal{H}} \\ &= \langle \mathbf{1}_{[0,a]}E[DY|\mathcal{F}_a], \mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]}\rangle_{\mathcal{H}} = 0,\end{aligned}$$

where for the second equality we used Lemma A.2 with $A = [0,a]$ and the $\mathcal{F}_a = \mathcal{F}_{[0,a]}$. The result follows using that

$$I_1(\mathbf{1}_{(a\wedge t, b\wedge t]}\mathbf{1}_{(v,w]}) = X_{t\wedge b}\big((v,w]\big) - X_{t\wedge a}\big((v,w]\big).$$

Case 2. General case. Let $t > 0$ be arbitrary. Fix $T \geq t$. Since $u1_{[0,T]} \in \mathcal{P}_0^T$, there exists a sequence $(u_n)_{n\geq 1}$ of simple processes defined on $[0,T] \times \mathbb{R}$ such that

$$E\|u_n - u\mathbf{1}_{[0,T]}\|_{\mathcal{H}}^2 \to 0.$$

By *Case 1*, $u_n\mathbf{1}_{[0,t]} \in \text{Dom}\,\delta$ and $\delta(u_n\mathbf{1}_{[0,t]}) = (u_n \cdot X)_t$. By the definition of the Itô integral, $(u_n \cdot X)_t \to (u \cdot X)_t$ in $L^2(\Omega)$. By applying Proposition A.4, we infer that $u\mathbf{1}_{[0,t]} \in \text{Dom}\,\delta$ and $\delta(u\mathbf{1}_{[0,t]}) = (u \cdot X)_t$.

□

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