Continuity in the Hurst parameter of the law of the symmetric integral with respect to the fractional Brownian motion

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Abstract

We prove the convergence in law, in the space of continuous functions $\mathscr{C}([0,T])$, of the Russo-Vallois symmetric integral of a non-adapted process with respect to the fractional Brownian motion with Hurst parameter $H > 1/2$ to the Russo-Vallois symmetric integral with respect to the fractional Brownian motion with parameter H_0, when H tends to $H_0 \in [1/2, 1)$.

Keywords: Convergence in law; Fractional Brownian motion; Russo-Vallois symmetric integral.
MSC 2000: 60B12, 60J55, 60G15.

1 Introduction

In the last years, great efforts have been made to develop the stochastic integration with respect to more general integrators than a semimartingale. For instance, since 1993 Russo and Vallois [13] have developed a regularization procedure, whose philosophy is similar to the discretization. In that work, they have introduced forward (generalizing Itô), backward and symmetric (generalizing Stratonovich) stochastic integrals.

Our aim in the present paper is to study the continuity in the Hurst parameter of the law in $\mathscr{C}([0,T])$ of the Russo-Vallois symmetric integrals with respect to the fractional Brownian motion B^H when $H \in (1/2, 1)$.

In previous works, we have proved the continuity with respect to the Hurst parameter of the law of the functionals given by the multiple fractional integrals with $H \in (1/2, 1)$, the local time (for any value of H) and the first order integral with $H \in (0, 1/2)$, respectively (see [7], [8] and [9] for more details).

It is worth pointing out that this kind of results justifies the use of $B^{\hat{H}}$ as a model in applied situations where $\hat{H}$ is some estimation of the unknown true value of H.

We will prove in our main result (Theorem 3.13) that, under certain conditions, the law (in the space of continuous functions) of the Russo-Vallois symmetric integral with respect to the fractional Brownian motion with Hurst parameter $H > 1/2$, converges weakly to that of the corresponding integral with respect to B^{H_0} when $H \to H_0$, with $H_0 \in [\frac{1}{2}, 1)$.

Concretely, we will consider a family of stochastic processes $\{u^H,\ H \in V_0\}$ where V_0 is an interval which contains H_0 and each $u^H = \{u_t^H,\ t \in [0,T]\}$ is a continuous stochastic process satisfying certain conditions. Our purpose is to show that the family $\{X^H,\ H \in V_0\}$ of stochastic processes given by $X_t^H := \int_0^t u_s^H \, dB_s^H$ converges in law to X^{H_0}, in the space $\mathscr{C}([0,T])$, when $H \to H_0$.

For proving such convergence in law we will follow the usual procedure. First of all, we will check the tightness of the laws and then we will show the convergence of the finite-dimensional distributions. For the tightness we have to assume uniform bounds in H for the integrals of the moments of the processes u^H and their Malliavin derivatives. On the other hand for the convergence of the finite-dimensional distributions we must suppose some conditions (uniform in H) of regularity of the processes and their Malliavin derivatives. We also need to impose the following joint convergence

$$(u^H, B^H) \xrightarrow{\mathscr{L}} (u^{H_0}, B^{H_0}),$$

when $H \to H_0$ in $(\mathscr{C}([0,T]))^2$.

In the case $H_0 = \frac{1}{2}$, we have to assume some additional conditions of regularity of the Malliavin derivative that are very similar to those previously known as sufficient conditions for the existence of Stratonovich-type integral introduced by Nualart and Pardoux in [11].

We have organized the paper as follows. In Section 2, following [3], we give some preliminaries about the Russo-Vallois symmetric integral with respect to the fractional Brownian motion with Hurst parameter $H > 1/2$. We prove in Section 3 the main result about the convergence in law of the Russo-Vallois integral process. To illustrate this result we show a very simple application in Section 4. Finally, in Section 5 we give the proof of Proposition 2.3.

2 Preliminaries

Let $B^H = \{B_t^H,\ t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ defined on a probability space $(\Omega, \mathscr{F}, P)$, that is a centered Gaussian process with covariance function given by

$$R_H(s,t) = E[B_s^H B_t^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

From now on we only consider a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ although H_0 may be also equal to $\frac{1}{2}$.

Let $\mathscr{S}$ be the set of step functions on the interval $[0,T]$. Consider the Hilbert space $\mathscr{H}^H$ defined as the closure of the space of step functions $\mathscr{S}$ in $[0,T]$ with respect to the scalar product given by

$$\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle_{\mathscr{H}^H} = R_H(s,t).$$

This space contains elements that are not functions but they are distributions (see [6]). The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t$ provides an isometry between the Hilbert space $\mathscr{H}^H$ and the Gaussian space $\mathscr{H}_1(B^H)$ associated with B^H. We will denote this isometry by $\varphi \longrightarrow B^H(\varphi)$.

Let $|\mathscr{H}^H|$ be the linear space of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|^2_{|\mathscr{H}^H|} = \alpha_H \int_0^T \int_0^T |\varphi_r||\varphi_u||r-u|^{2H-2} dr du < +\infty.$$

It is not difficult to show that $|\mathscr{H}^H|$ is a Banach space with the norm $\|\cdot\|_{|\mathscr{H}^H|}$ and $\mathscr{S}$ is dense in it. In addition, it has been shown in [12] that this space equipped with the scalar product

$$\langle \varphi, \psi \rangle_{\mathscr{H}^H} = \alpha_H \int_0^T \int_0^T \varphi_r \psi_u |r-u|^{2H-2} dr du$$

is not complete but it is isometric to a subspace of $\mathscr{H}^H$. We can identify $|\mathscr{H}^H|$ with this subspace.

Since B^H is a Gaussian process we can develop a Malliavin calculus (or stochastic calculus of variations) with respect to it. The integration with respect to fractional Brownian motion has been tackled by different authors (see for instance [1, 5]) and even when the integrator is a more general Gaussian process (see [2] for more details).

Let $\mathcal{S}^H$ be the set of smooth and cylindrical random variables of the form

$$F = f(B^H(\varphi_1), \ldots, B^H(\varphi_n)),$$

where $n \geqslant 1$, $\varphi_i \in \mathscr{H}^H$ and $f \in \mathscr{C}_b^\infty(\mathbb{R}^n)$ (that is, f and its partial derivatives are bounded) . The derivative operator with respect B^H applied to a F of the above type is defined as the $\mathscr{H}^H$-valued random variable

$$D^H F \;=\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \ldots, B^H(\varphi_n))\varphi_i.$$

For $p \geqslant 1$, D^H is a closable operator from $L^p(\Omega)$ into $L^p(\Omega, \mathscr{H}^H)$.

For any $p \geqslant 1$, the Sobolev space $\mathbb{D}_H^{1,p}$ is the closure of $\mathcal{S}^H$ with respect the following norm

$$\|F\|_{H,1,p}^p = \|F\|_{L^p(\Omega)}^p + E\|D^H F\|_{L^p(\Omega, \mathscr{H}^H)}^p.$$

In a similar way, for a Hilbert space V, $\mathbb{D}_H^{1,p}(V)$ denotes the corresponding Sobolev space of V-valued random variables.

Another important operator of the Malliavin calculus is the divergence operator denoted by δ^H (also known as Skorohod integral). This operator is the adjoint of the derivative operator. We say that $u \in L^2(\Omega, \mathscr{H}^H)$ belongs to the domain of δ^H, denoted by $Dom\ \delta^H$, if there exists a constant C such that

$$E\big(\langle D^H F, u\rangle_{\mathscr{H}^H}\big) \leqslant C\|F\|_{L^2(\Omega)},$$

for any $F \in \mathcal{S}^H$. In this case $\delta^H(u)$ is defined by the following integration by parts formula:

$$E\big(F\delta^H(u)\big) = E\big(\langle D^H F, u\rangle_{\mathscr{H}^H}\big), \qquad F \in \mathbb{D}_H^{1,2}.$$

The Skorohod integral satisfies the following properties:

(a) For any $u \in \mathbb{D}_H^{1,2}(\mathscr{H}^H) \subset Dom\ \delta^H$ we have that

$$E(\delta^H(u))^2 = E\|u\|_{\mathscr{H}^H}^2 + E\langle D^H u,\ (D^H u)^*\rangle_{\mathscr{H}^H \otimes \mathscr{H}^H},$$

where $(D^H u)^*$ is the adjoint of $D^H u$ in the Hilbert space $\mathscr{H}^H \otimes \mathscr{H}^H$.

(b) For any F in $\mathbb{D}_H^{1,2}$ and any $u \in Dom\ \delta^H$ such that Fu and $F\delta^H(u) - \langle D^H F,\ u\rangle_{\mathscr{H}^H}$ are square integrable, we have that $Fu \in Dom\ \delta^H$ and

$$\delta^H(Fu) = F\delta^H(u) - \langle D^H F,\ u\rangle_{\mathscr{H}^H}. \tag{1}$$

We consider also $|\mathscr{H}^H| \otimes |\mathscr{H}^H|$ the space of measurable functions φ on $[0,T]^2$ such that

$$\|\varphi\|^2_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|} = \alpha_H \int_{[0,T]^4} |\varphi_{r,\theta}||\varphi_{u,\eta}||r-u|^{2H-2}|\theta-\eta|^{2H-2} dr du d\theta d\eta < +\infty.$$

The space $|\mathscr{H}^H| \otimes |\mathscr{H}^H|$ is a Banach space with the norm $\|\cdot\|_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|}$. As well, this space equipped with the scalar product

$$\langle \varphi, \psi \rangle_{|\mathscr{H}^H\otimes|\mathscr{H}^H|} = \alpha_H^2 \int_{[0,T]^4} \varphi_{r,\theta}\psi_{u,\eta}|r-u|^{2H-2}|\theta-\eta|^{2H-2} dr du d\theta d\eta$$

is isometric to a subspace of $\mathscr{H}^H \otimes \mathscr{H}^H$. We identify $|\mathscr{H}^H| \otimes |\mathscr{H}^H|$ with this subspace.

For any $p \geqslant 1$, denote by $\mathbb{D}^{1,p}(|\mathscr{H}^H|)$ the subspace of $\mathbb{D}^{1,p}_H(\mathscr{H}^H)$ formed by the elements u such that $u \in |\mathscr{H}^H|$ a.s., $D^H u \in |\mathscr{H}^H| \otimes |\mathscr{H}^H|$ a.s. and

$$E\|u\|^p_{|\mathscr{H}^H|} + E\|D^H u\|^p_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|} < +\infty.$$

The Sobolev space $\mathbb{D}^{1,2}_H(|\mathscr{H}^H|) \subset \mathbb{D}^{1,2}_H(\mathscr{H}^H)$ is included in the domain of the Skorohod integral δ^H and from the above properties we have

$$E(\delta^H(u))^2 \leqslant E\|u\|^2_{|\mathscr{H}^H|} + E\|D^H u\|^2_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|}.$$

For any $p > 1$, the divergence of a process $u \in \mathbb{D}^{1,p}_H(|\mathscr{H}^H|)$ satisfies the following inequality of norms

$$E(|\delta^H(u)^p|) \leqslant C_p \left(E(\|u\|^p_{\mathscr{H}^H}) + E(\|D^H u\|^p_{\mathscr{H}^H\otimes\mathscr{H}^H}) \right). \tag{2}$$

We give now the definition of the symmetric stochastic integral introduced by Russo and Vallois in [13] in the particular case in which the integrator process is a fractional Brownian motion.

Definition 2.1. *Let $u = \{u_t,\ t \in [0,T]\}$ be a stochastic processes with integrable trajectories. The symmetric integral of the process u with respect to the fractional Brownian motion B^H is defined as the limit in probability when ε tends to zero of*

$$\frac{1}{2\varepsilon}\int_0^T u_s (B^H_{(s+\varepsilon)\wedge T} - B^H_{(s-\varepsilon)\vee 0}) ds,$$

provided that this limit exists and it is denoted by $\int_0^T u_t dB^H_t$.

The following proposition gives sufficient conditions for the existence of the symmetric integral in the case $H > 1/2$ and provides a representation of this integral in terms of the Skorohod integral and a trace term (see Proposition 3 of [3]).

Proposition 2.2. *Let $u = \{u_t,\ t \in [0,T]\}$ be a stochastic process in the space $\mathbb{D}_H^{1,2}(|\mathscr{H}^H|)$. Suppose also that*

$$\int_0^T \int_0^T |D_s^H u_t||t-s|^{2H-2} ds dt < +\infty, \quad a.s. \tag{3}$$

Then the symmetric integral exists and we have

$$\int_0^T u_t dB_t^H = \delta^H(u) + \alpha_H \int_0^T \int_0^T D_s^H u_t |t-s|^{2H-2} ds dt,$$

where $\alpha_H = H(2H-1)$.

A sufficient condition for (3) is

$$\int_0^T \left(\int_s^T |D_s^H u_t|^p dt \right)^{1/p} ds < \infty,$$

for some $p > \frac{1}{2H-1}$.

Under the assumptions of the above proposition, the indefinite symmetric integral $\int_0^t u_s dB_s^H = \int_0^T u_s 1_{[0,t]}(s) dB_s^H$ exists and for any $t \in [0,T]$ can be decomposed as

$$\int_0^t u_s dB_s^H = \delta^H(u\mathbf{1}_{[0,t]}) + \alpha_H \int_0^t \int_0^T D_r^H u_s |s-r|^{2H-2} dr ds.$$

From now on we use the following notation for the indefinite Skorohod integral

$$\int_0^t u_s \delta^H B_s^H = \delta^H(u\mathbf{1}_{[0,t]}).$$

In the standard Brownian motion case, Nualart and Pardoux defined a Stratonovich-type integral by means of a limit of Riemann sums (see Section 7 in [11]). In Theorem 3.1.1. of [10] the following sufficient condition for the existence of this integral for a process $u \in \mathbb{D}^{1,2}(L^2([0,T]))$ appears: If there exist $D^{1|2,+}u$ and $D^{1|2,-}u$ elements of $L^1([0,T] \times \Omega)$ satisfying that

$$\lim_{\delta \to 0} \int_0^T \sup_{0<y<\delta} E|D_r^{1|2,-} u_r - D_r^{1|2} u_{(r-y)\vee 0}| dr = 0, \tag{4}$$

(resp.

$$\lim_{\delta \to 0} \int_0^T \sup_{-\delta<y<0} E|D_r^{1|2,+} u_r - D_r^{1|2} u_{(r-y)\wedge T}| dr = 0,) \tag{5}$$

then, the Stratonovich-type integral of u exists.

Furthermore, in this case one can define the trace as

$$\nabla_t^{1|2} u_t = D_t^{1|2,-} u_t + D_t^{1|2,+} u_t.$$

In the following proposition we will see that these conditions also imply the existence of the Russo-Vallois symmetric integral. As well, this integral admits a decomposition in terms of the Skorohod integral and the trace.

Proposition 2.3. *Let $u \in \mathbb{D}^{1,2}(L^2([0,T]))$ be a stochastic process that satisfy conditions (4) and (5). Then there exists the Russo-Vallois symmetric integral of the process u with respect to the standard Brownian motion and, in addition, it admits the following representation*

$$\int_0^t u_s dB_s^{1|2} = \delta^{1|2}(u\mathbf{1}_{[0,t]}) + \int_0^t \frac{1}{2}\nabla_r^{1|2} u_r dr. \tag{6}$$

In order to lighten this section of preliminaries, the proof of this result is transferred to Section 5.

We will finish this section with the following lemma that provides an upper bound independent of H for the norms $\|\cdot\|_{|\mathscr{H}^H|}$ and $\|\cdot\|_{|\mathscr{H}^H|\times|\mathscr{H}^H|}$. See Lemma 3.1 and Lemma 4.1 of [7] for its proof.

Lemma 2.4. *For any $f \in L^2([0,T])$ and any $H \in \left(\frac{1}{2}, 1\right)$ we have that*

$$\|f\|^2_{|\mathscr{H}^H|} \leqslant C_T \|f\|^2_{L^2([0,T])}.$$

Moreover, for any $g \in L^2([0,T]^2)$ and any $H \in \left(\frac{1}{2}, 1\right)$ we have that

$$\|g\|^2_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|} \leqslant C_T \|g\|^2_{L^2([0,T]^2)}.$$

3 Convergence in law of the Russo-Vallois symmetric integrals with respect to B^H

Fixed $H_0 \geqslant \frac{1}{2}$, we consider the family of stochastic processes $\{X^H,\ H \in V_0\}$ defined by the Russo-Vallois symmetric integrals with respect to the fractional Brownian motion

$$X^H := \left\{ X_t^H = \int_0^t u_s^H dB_s^H,\ t \in [0,T] \right\}, \tag{7}$$

where V_0 is an interval given by

$$V_0 = \begin{cases} [\frac{1}{2}, H_2], & \text{if } H_0 = \frac{1}{2}, \text{ with } \frac{1}{2} < H_2 < 1; \\ [H_1, H_2], & \text{if } H_0 > \frac{1}{2}, \text{ with } \frac{1}{2} < H_1 < H_0 < H_2 < 1. \end{cases}$$

Suppose that the stochastic processes u^H satisfy the following block of hypotheses:

Block A :

There exists $p > 2$ such that

(A1)

$$\int_0^T \sup_{H \in V_0} E|u_s^H|^p ds < +\infty.$$

(A2)

$$\sup_{H\in V_0}\int_0^T \sup_{x\in[0,T]} E|D_r^{H} u_x^{H}|^p dr = K_D < +\infty.$$

Remark 3.1. *The above conditions imply the existence of the Russo-Vallois symmetric integral for the process u^H, when $H>\frac{1}{2}$. By Proposition 2.2, this integral exists if $u^H \in \mathbb{D}_H^{1,2}(|\mathscr{H}^H|)$ and*

$$E\left(\int_0^T\int_0^T |D_s^H u_t^H||t-s|^{2H-2} ds dt\right) < +\infty.$$

In fact, for $H>\frac{1}{2}$ condition (A1) implies that $u^H \in \mathbb{D}_H^{1,2}(L^2([0,T])) \subset \mathbb{D}_H^{1,2}(|\mathscr{H}^H|)$ and by (A2) we have that

$$\begin{aligned} E\left(\int_0^T\int_0^T |D_s^H u_t^H||t-s|^{2H-2} ds dt\right) &\leqslant \int_0^T \sup_{t\in[0,T]} E|D_s^H u_t^H| \left(\int_0^T |t-s|^{2H-2} dt\right) ds \\ &\leqslant \frac{C_T}{2H-1}\int_0^T \sup_{t\in[0,T]} E|D_s^H u_t^H| ds < +\infty. \end{aligned}$$

In addition, Block A of hypotheses assures the existence of a continuous version of the Russo-Vallois symmetric integral. This can be easily seen by using the following result (see Theorem 5, [3]):

Theorem 3.2. *Let $u^H = \{u_t^H,\ t\in[0,T]\}$ be a stochastic process of $\mathbb{D}_H^{1,p}(|\mathscr{H}^H|)$, where $pH>1$ and assume that*

$$\int_0^T |Eu_r^H|^p dr + \int_0^T E\left(\int_0^T |D_s^H u_r^H|^{1/H} ds\right)^{pH} dr < +\infty. \tag{8}$$

Then the integral $X^H = \{\int_0^t u_s^H dB_s^H,\ t\in[0,T]\}$ has a version with continuous trajectories. Moreover for all $\gamma < H-\frac{1}{p}$ there exists a random constant C_γ a.s. finite such that

$$|X_t^H - X_s^H| \leqslant C_\gamma |t-s|^\gamma.$$

In fact, the process u^H satisfies the hypotheses of Theorem 3.2. By Condition (A1) we have that the term

$$\int_0^T |Eu_r^H|^p dr < +\infty.$$

On the other hand, if $p>2$ and $H>\frac{1}{2}$, $pH>1$ and applying Hölder's inequality and Condition (A2) we obtain

$$\int_0^T E\left(\int_0^T |D_s^H u_r^H|^{\frac{1}{H}} ds\right)^{pH} dr \leqslant T^{pH-1}\int_0^T\int_0^T E|D_s^H u_r^H|^p ds dr < +\infty.$$

We will prove the convergence in law of the family of processes $\{X^H,\ H \in V_0\}$ showing firstly the tightness of the family of laws and then, the convergence of the finite-dimensional distributions.

3.1 Tightness

In the next proposition, we will prove the tightness of the family of the laws of stochastic processes $\{X^H,\ H \in V_0\}$ in $\mathscr{C}([0,T])$.

Proposition 3.3. *Let $\{X^H\}_{H\in V_0}$ be the family of stochastic processes defined in (7) where the processes $\{u^H\}_{H\in V_0}$ satisfy Block A of hypotheses. Then, the family of laws of $\{X^H\}_{H\in V_0}$ in the space of continuous functions $\mathscr{C}([0,T])$ is tight.*

Proof. For any $t \in [0,T]$ we write

$$X_t^H = \int_0^t u_s^H \delta^H B_s^H + \alpha_H \int_0^t \int_0^T D_r^H u_s^H |s-r|^{2H-2} dr ds.$$

Since $X_0^H = 0$, using Billingsley criterion (see Theorem 12.3, [4]) it suffices to check that for any $s < t$

$$E|X_t^H - X_s^H|^\beta \leqslant C(F(t)-F(s))^{1+\alpha} \tag{9}$$

where $\alpha,\ \beta,\ C > 0$ are positive constants and F is a continuous and increasing function.
In this case, for any $s < t$ we have that

$$E|X_t^H - X_s^H|^p \leqslant 2^{p-1}\left(E\left|\int_s^t u_r^H \delta^H B_r^H\right|^p + \alpha_H^p E\left|\int_s^t \int_0^T D_r^H u_x^H |x-r|^{2H-2} dr dx\right|^p\right). \tag{10}$$

Taking into account that the Skorohod integral satisfies the L^p-estimates given in (2) we get

$$E\left|\int_s^t u_r^H \delta^H B_r^H\right|^p \leqslant C_p(E\|u^H \mathbf{1}_{[s,t]}\|^p_{|\mathscr{H}^H|} + E\|D^H u^H \mathbf{1}_{[s,t]}\|^p_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|}).$$

We shall now estimate each term of the right-hand side. By applying Lemma 2.4 and Hölder's inequality we have that

$$\begin{aligned} E\|u^H \mathbf{1}_{[s,t]}\|^p_{|\mathscr{H}^H|} &\leqslant C_T E\|u^H \mathbf{1}_{[s,t]}\|^p_{L^2([0,T])} \\ &\leqslant C_T\ (t-s)^{\frac{p}{2}-1} \int_s^t E|u_r^H|^p dr. \end{aligned}$$

From the following inequality

$$cd \leqslant \frac{1}{p'}\ c^{p'} + \frac{1}{q'}\ d^{q'}, \forall\ c,\ d > 0 \quad \textit{and} \quad \forall\ p',\ q' > 1 \quad \textit{such that}\ \frac{1}{p'} + \frac{1}{q'} = 1, \tag{11}$$

we obtain that

$$E\|u^H \mathbf{1}_{[s,t]}\|^p_{|\mathscr{H}^H|} \leqslant C_T\left(\frac{1}{p'}(t-s)^{(\frac{p}{2}-1)p'} + \frac{1}{q'}\left(\int_s^t E|u_r^H|^p dr\right)^{q'}\right).$$

Similarly, we can also estimate the moment of order p of the derivative

$$
\begin{aligned}
E\|D^{H} u^{H} \mathbf{1}_{[s,t]}\|^p_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|} &\leqslant C_T \; E\|D^{H} u^{H} \mathbf{1}_{[s,t]}\|^p_{L^2([0,T]^2)} \\
&\leqslant C_T T^{\frac{p}{2}-1} E\left(\int_s^t \left(\int_0^T |D_r^{H} u_x^{H}|^2 dr\right)^{p/2} dx\right) \\
&\leqslant C_T T^{\frac{p}{2}-1}(t-s)^{\frac{p}{2}-1} \int_s^t \int_0^T E|D_r^{H} u_x^{H}|^p dr dx.
\end{aligned}
$$

From (11) we deduce that

$$
E\|D^{H} u^{H} \mathbf{1}_{[s,t]}\|^p_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|} \leqslant C_{T,p} \; \left(\frac{1}{p'} \; (t-s)^{(\frac{p}{2}-1)p'} + \frac{1}{q'}\left(\int_s^t \int_0^T E|D_r^{H} u_x^{H}|^p dr dx\right)^{q'}\right).
$$

Hence, we obtain

$$
\begin{aligned}
E\left|\int_s^t u_r^{H} \delta^H B_r^{H}\right|^p \leqslant C_{T,\,p} &\left(\frac{1}{p'} \; (t-s)^{(\frac{p}{2}-1)p'} + \frac{1}{q'}\left(\int_s^t \sup_{H\in V_0} E|u_r^{H}|^p dr\right)^{q'}\right. \\
&\left. + \frac{1}{q'}\left(\int_s^t \sup_{H\in V_0} \int_0^T E|D_r^{H} u_x^{H}|^p dr dx\right)^{q'}\right).
\end{aligned}
$$

Now, we will study the second term of (10). For this, we will first apply Hölder's inequality and then, Fubini's theorem

$$
\begin{aligned}
&\alpha_H^p E\left|\int_s^t \int_0^T D_r^{H} u_x^{H} |x-r|^{2H-2} dr dx\right|^p \\
&\leqslant \alpha_H^p \left(\int_s^t \int_0^T E|D_r^{H} u_x^{H}|^p |x-r|^{2H-2} dr dx\right)\left(\int_s^t \int_0^T |x-r|^{2H-2} dr dx\right)^{p-1} \\
&\leqslant C_T (t-s)^{p-1} \alpha_H \int_s^t \int_0^T \sup_{x\in[0,T]} E|D_r^{H} u_x^{H}|^p |x-r|^{2H-2} dr dx \\
&= C_T (t-s)^{p-1} \alpha_H \int_0^T \sup_{x\in[0,T]} E|D_r^{H} u_x^{H}|^p \left(\int_s^t |x-r|^{2H-2} dx\right) dr \\
&\leqslant C_T (t-s)^{p-1} \sup_{H\in V_0} \int_0^T \sup_{x\in[0,T]} E|D_r^{H} u_x^{H}|^p dr \\
&\leqslant C_T K_D (t-s)^{p-1}.
\end{aligned}
$$

Summarazing, for any $s < t$ we have that

$$
\begin{aligned}
E|X_t^H - X_s^H|^p &\leqslant C_{p,T} \left(\frac{1}{p'}(t-s)^{(\frac{p}{2}-1)p'} + \frac{1}{q'}\left(\int_s^t \sup_{H\in V_0} E|u_r^H|^p dr \right)^{q'} \right. \\
&\quad \left. + \frac{1}{q'}\left(\int_s^t \sup_{H\in V_0} \int_0^T E|D_r^H u_x^H|^p dr dx \right)^{q'} \right) + C_T K_D (t-s)^{p-1} \\
&\leqslant C_{p,T} \left(\frac{1}{p'}(t-s)^{(\frac{p}{2}-1)p' \wedge (p-1)} + \frac{1}{q'}\left(\int_s^t \sup_{H\in V_0} E|u_r^H|^p dr \right)^{q'} \right. \\
&\quad \left. + \frac{1}{q'}\left(\int_s^t \sup_{H\in V_0} \int_0^T E|D_r^H u_x^H|^p dr dx \right)^{q'} \right). \qquad (12)
\end{aligned}
$$

Define

$$
F(x) = x + \int_0^x \sup_{H\in V_0} E|u_r^H|^p dr + \int_0^x \sup_{H\in V_0} \int_0^T E|D_r^H u_y^H|^p dr dy.
$$

F is a continuous and increasing function and by (12) we have that

$$
E|X_t^H - X_s^H|^p \leqslant C_{p,p',T} (F(t) - F(s))^{(\frac{p}{2}-1)p' \wedge (p-1) \wedge q'}.
$$

Finally, for any $p > 2$ there exist p' and q' such that $\frac{1}{p'} + \frac{1}{q'} = 1$ and such that the exponent $(\frac{p}{2}-1)p' \wedge (p-1) \wedge q'$ is greater than 1. So, we have proved (9). $\square$

3.2 Convergence of the finite-dimensional distributions

For any $\varepsilon > 0$, we define the stochastic process

$$
X^{H,\varepsilon} = \left\{ X_t^{H,\varepsilon} := \int_0^t u_s^{H,\varepsilon} dB_s^H,\ t \in [0,T] \right\}, \qquad (13)
$$

where $u_s^{H,\varepsilon}$ is an approximating process of u^H defined by

$$
u_t^{H,\varepsilon} = \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} u_s^H ds. \qquad (14)
$$

By convention we will assume that the process u vanishes outside the interval $[0,T]$.
Actually, we can write

$$
u_s^{H,\varepsilon} = (u^H * \varphi_\varepsilon)(s) \qquad (15)
$$

where $\varphi_\varepsilon(s) = \frac{1}{\varepsilon}\varphi\left(\frac{s}{\varepsilon}\right)$ denotes the approximation of the identity when $\varepsilon \to 0$ given by the function $\varphi(s) = \frac{1}{2}\mathbf{1}_{[-1,1]}(s)$.

Remark 3.4. *It is not difficult to check that the Russo-Vallois symmetric integral of the process $u^{H,\varepsilon}$ given in (14) exists because $u^{H,\varepsilon}$ is absolute continuous. Moreover, its integral is given by the integration by parts formula (see [13]).*

To prove the convergence in law of the finite-dimensional distributions we shall distinguish the following cases in terms of the parameter H_0: $H_0 > \frac{1}{2}$ and $H_0 = \frac{1}{2}$.

3.2.1 Case $H_0 > \frac{1}{2}$

Recall that in this case the interval V_0 has the form $V_0 = [H_1, H_2]$ with $\frac{1}{2} < H_1 < H_0 < H_2 < 1$. We will suppose that the family of processes $\{u^{H}\}_{H \in V_0}$ also satisfies the following hypotheses:

Block B :

(B1)

$$\lim_{\delta \to 0} \sup_{H \in V_0} \sup_{|y|<\delta} \int_0^T E|u_s^{H} - u_{s-y}^{H}|^2 ds = 0.$$

(B2) There exists $p \geqslant 2$ satisfying also $p > \frac{1}{2H_1 - 1}$ for which

$$\lim_{\delta \to 0} \sup_{H \in V_0} \sup_{|y|<\delta} \int_0^T \int_0^T E|D_r^{H} u_s^{H} - D_r^{H} u_{s-y}^{H}|^p dr ds = 0.$$

Next, we state the following technical result that we will need early.

Proposition 3.5. *Fix $H_0 > \frac{1}{2}$. Let X^{H} and $X^{H,\varepsilon}$ be the families of stochastic processes defined in (7) and (13). Suppose that $\{u^{H}\}_{H \in V_0}$ satisfies Block A and Block B of hypotheses. Then,*

$$\lim_{\varepsilon \to 0} \sup_{H \in V_0} E|X_t^{H} - X_t^{H,\,\varepsilon}| = 0. \tag{16}$$

Proof. We have that

$$\begin{aligned} E|X_t^{H} - X_t^{H,\,\varepsilon}| &= E\Big|\int_0^t (u_s^{H} - u_s^{H,\varepsilon}) dB_s^{H}\Big| \\ &\leqslant E\Big|\int_0^t (u_s^{H} - u_s^{H,\varepsilon}) \delta^{H} B_s^{H}\Big| + E\Big|\alpha_{H} \int_0^t \int_0^T (D_r^{H} u_s^{H} - D_r^{H} u_s^{H,\varepsilon}) |s-r|^{2H-2} dr ds\Big| \end{aligned}$$

To prove (16) we will see that each term of the right-hand side tends to zero, uniformly in H, when $\varepsilon \to 0$.
From the L^p-estimates for the Skorohod integral (see (2)) we can write

$$E\Big|\int_0^t (u_s^{H} - u_s^{H,\varepsilon}) \delta^{H} B_s^{H}\Big| \leqslant C_{T,p} \left(E\|u^{H} - u^{H,\varepsilon}\|^2_{|\mathscr{H}^{H}|} + E\|D^{H} u^{H} - D^{H} u^{H,\varepsilon}\|^2_{|\mathscr{H}^{H}| \otimes |\mathscr{H}^{H}|} \right)^{1/2}.$$

By Lemma 2.4 we have

$$E\|u^{H} - u^{H,\varepsilon}\|^2_{|\mathscr{H}^{H}|} \leqslant C_{T} E\|u^{H} - u^{H,\varepsilon}\|^2_{L^2([0,T])}$$

and

$$E\|D^{H} u^{H} - D^{H} u^{H,\varepsilon}\|^2_{|\mathscr{H}^{H}| \otimes |\mathscr{H}^{H}|} \leqslant C_{T} E\|D^{H} u^{H} - D^{H} u^{H,\varepsilon}\|^2_{L^2([0,T]^2)}.$$

It can be seen that the terms $E\|u^{H} - u^{H,\varepsilon}\|^2_{L^2([0,T])}$ and $E\|D^{H} u^{H} - D^{H} u^{H,\varepsilon}\|^2_{L^2([0,T]^2)}$ converge to zero, uniformly in H, when $\varepsilon \to 0$.

In fact, let us study the first term. By expression (15) and making an easy change of variables we get

$$
\begin{aligned}
|u_s^H - (u^H * \varphi_\varepsilon)(s)| &= \left| u_s^H - \int_{\mathbb{R}} u_x^H \varphi_\varepsilon(s-x)dx \right| \\
&\leqslant \int_{\mathbb{R}} \varphi_\varepsilon(y) |u_s^H - u_{s-y}^H| dy.
\end{aligned}
$$

Applying Cauchy-Schwarz's inequality, Fubini's theorem and using that φ_ε defines a probability measure, we obtain that

$$
\begin{aligned}
E\|u^H - u^{H,\,\varepsilon}\|^2_{L^2([0,T])} &\leqslant E\left(\int_0^T \left| \int_{\mathbb{R}} \varphi_\varepsilon(y) |u_s^H - u_{s-y}^H| dy \right|^2 ds \right) \\
&\leqslant E\left(\int_0^T \int_{\mathbb{R}} |u_s^H - u_{s-y}^H|^2 \varphi_\varepsilon(y) dy ds \right) \\
&= \int_{\mathbb{R}} \varphi_\varepsilon(y) \left(\int_0^T E|u_s^H - u_{s-y}^H|^2 ds \right) dy \\
&\leqslant \sup_{H \in V_0} \sup_{|y|<\varepsilon} \int_0^T E|u_s^H - u_{s-y}^H|^2 ds.
\end{aligned}
$$

Then condition (B1) implies the desired convergence to zero of $\sup_{H\in V_0} E\|u^H - u^{H,\,\varepsilon}\|^2_{L^2([0,T])}$. In a similar way, using condition (B2) (it suffices to take $p=2$), one can show that

$$
E\|D^H u^H - D^H u^{H,\varepsilon}\|^2_{|\mathscr{H}^H|\otimes|\mathscr{H}^H|} \stackrel{\varepsilon\to 0}{\longrightarrow} 0,
$$

uniformly in H.
It remains to check the convergence to zero of the following term

$$
E\left| \alpha_H \int_0^t \int_0^T (D_r^H u_s^H - D_r^H u_s^{H,\varepsilon}) |s-r|^{2H-2} dr ds \right|. \tag{17}
$$

Remember that the parameter $H \in V_0 = [H_1, H_2]$ with $\frac{1}{2} < H_1 < H_0 < H_2$.
By Fubini's theorem we have that

$$
\begin{aligned}
&E\left| \alpha_H \int_0^T \int_0^T (D_r^H u_s^H - D_r^H u_s^{H,\varepsilon}) |s-r|^{2H-2} dr ds \right| \\
&\qquad \leqslant \alpha_H \int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_s^{H,\varepsilon}| |s-r|^{2H-2} dr ds \\
&\qquad \leqslant \alpha_H \int_0^T \int_0^T \left(\int_{\mathbb{R}} E|D_r^H u_s^H - D_r^H u_{s-y}^H| \varphi_\varepsilon(y) dy \right) |s-r|^{2H-2} dr ds \\
&\qquad = \alpha_H \int_{\mathbb{R}} \left(\int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H| |s-r|^{2H-2} dr ds \right) \varphi_\varepsilon(y) dy.
\end{aligned} \tag{18}
$$

If $H \geqslant H_1$, there exists a constant $C_{T,H_1} > 0$ such that

$$|s-r|^{2H-2} \leqslant C_{T,H_1} |s-r|^{2H_1-2}$$

and thus, we have that

$$\int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H||s-r|^{2H-2} drds \leqslant C_{T,H_1} \int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H||s-r|^{2H_1-2} drds.$$

Applying Hölder's inequality with the particular value of $p > \frac{1}{2H_1-1}$ which has appeared in condition $(B2)$ and its conjugate $q = \frac{p}{p-1} > 1$ which satisfies that $q(2-2H_1) < 1$, we have

$$\int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H||s-r|^{2H_1-2} drds$$
$$\leqslant \left(\int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H|^p drds \right)^{1/p} \left(\int_0^T \int_0^T |s-r|^{(2H_1-2)q} drds \right)^{1/q}.$$

From condition (B2) we can deduce that

$$\alpha_H C_{T,H_1} \left(\int_0^T \int_0^T |s-r|^{(2H_1-2)q} drds \right)^{1/q} \left(\sup_{H \in V_0} \sup_{|y|<\varepsilon} \int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H|^p drds \right)^{1/p} \tag{19}$$

is an upper bound of (18) and taking into account that

$$\alpha_H \int_0^T \int_0^T |s-r|^{(2H_1-2)q} drds < \widetilde{C}_{T,H_1},$$

expression (19) converges to zero uniformly in $H \in V_0$, when $\varepsilon \to 0$.

□

Remark 3.6. *Lets note that in this proof the only step where we use that $H_0 > \frac{1}{2}$ is when we study the term (17). Thus, if $H_0 = \frac{1}{2}$ we must to pay special attention to this term.*

3.2.2 Case $H_0 = \frac{1}{2}$

In this case $V_0 = [H_0, H_2] \subset [\frac{1}{2}, 1)$. We will also assume that the family of stochastic processes $\{u^H\}_{H \in V_0}$ satisfies the following condition:

Condition C :

(C) There exist $D^{H,-} u^H$ and $D^{H,+} u^H$ elements of $L^1([0,T] \times \Omega)$ satisfying

$$\lim_{\delta \to 0} \sup_{H \in V_0} \int_0^T \sup_{0<y<\delta} E|D_r^{H,-} u_r^H - D_r^H u_{r-y}^H| dr = 0, \tag{20}$$

(resp.

$$\lim_{\delta \to 0} \sup_{H \in V_0} \int_0^T \sup_{-\delta<y<0} E|D_r^{H,+} u_r^H - D_r^H u_{r-y}^H| dr = 0.) \tag{21}$$

If the u^H satisfy Condition C then we can define a kind of trace (inspired in the case $H = 1/2$) of u^H as

$$\nabla_t^H u_t^H = D_t^{H,+} u_t^H + D_t^{H,-} u_t^H.$$

Remark 3.7. *Note that if $H_0 = \frac{1}{2}$, due to conditions (C) the process u^{H_0} is Stratonovich integrable because it satisfies conditions (4) and (5) (see Proposition 2.3).*
Furthermore, conditions (C) and (A2) imply that

$$\sup_{H\in V_0} \int_0^T E(|D_r^{H,+} u_r^H|)dr < +\infty \tag{22}$$

(resp.

$$\sup_{H\in V_0} \int_0^T E(|D_r^{H,-} u_r^H|)dr < +\infty.) \tag{23}$$

The following proposition is similar to Proposition 3.5 but only refers to the standard Brownian motion.

Proposition 3.8. *Let $B^{1|2}$ a standard Brownian motion. Suppose that the stochastic process $u^{1|2} = \{u_t^{1|2},\ t \in [0,T]\}$ satisfies conditions (4) and (5). Then*

$$\lim_{\varepsilon\to 0} E\left|\int_0^t u_s^{1|2,\varepsilon} dB_s^{1|2} - \int_0^t u_s^{1|2} dB_s^{1|2}\right| = 0. \tag{24}$$

Proof. For all $\varepsilon > 0$ the Stratonovich integral of $u^{1|2,\varepsilon}$ can be expressed as

$$\int_0^t u_s^{1|2,\varepsilon} dB_s^{1|2} = \delta^{1|2}((u^{1|2}\mathbf{1}_{[0,t]})^\varepsilon) + \frac{1}{2\varepsilon}\int_0^t \int_0^T D_r^{1|2} u_s^{1|2} \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r)drds, \tag{25}$$

because the trace of $(u^{1|2}\mathbf{1}_{[0,t]})^\varepsilon$ gives

$$\nabla_r^{1|2}\big((u^{1|2}\mathbf{1}_{[0,t]})^\varepsilon\big)_r = \frac{1}{\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} D_r^{1|2} u_s^{1|2}\mathbf{1}_{[0,t]}(s)ds.$$

In a similar way, for $u^{1|2}$ we have

$$\int_0^t u_s^{1|2} dB_s^{1|2} = \delta^{1|2}(u^{1|2}\mathbf{1}_{[0,t]}) + \int_0^t \frac{1}{2}\nabla_r^{1|2} u_r^{1|2} dr. \tag{26}$$

From expression (25) and using some arguments of the proof of Proposition 2.3, we can obtain the convergence in $L^2(\Omega)$ of the first summand of (25) to the first summand of (26) and also, the convergence in $L^1(\Omega)$ of the second summand of (25) to the second summand of (26). This concludes the proof.

□

For the proof of the next proposition we will make use of the following approximation of the identity when $H \downarrow \frac{1}{2}$

$$\psi_H(s) = c_H |s|^{2H-2} \mathbf{1}_{[-T,T]}(s) \tag{27}$$

where

$$c_H = \frac{2H-1}{2T^{2H-1}}. \tag{28}$$

Observe that when $H_0 = \frac{1}{2}$ condition (B2) does not make sense. Instead of this condition we will assume

(B2')

$$\lim_{\delta \to 0} \sup_{H \in V_0} \sup_{|y|<\delta} \int_0^T \int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H|^2 drds = 0.$$

Proposition 3.9. *Suppose that $\{u^H\}_{H \in V_0}$ satisfies the blocks A and B (with (B2') replacing (B2)) of hypotheses and Condition C. Then, given $\rho > 0$ there exist $\varepsilon > 0$ and $\eta > 0$ such that*

$$\sup_{H \in (\frac{1}{2}, \frac{1}{2}+\eta)} E|X_t^H - X_t^{H,\varepsilon}| < \rho \tag{29}$$

and

$$E|X_t^{1|2} - X_t^{1|2,\varepsilon}| < \rho. \tag{30}$$

Proof. To prove this result, we will see that, at least for a small enough $\varepsilon > 0$, each term of the following majorization tends to zero, when $H \to \frac{1}{2}$:

$$E|X_t^H - X_t^{H,\varepsilon}| \leqslant E\Big|\int_0^t (u_s^H - u_s^{H,\varepsilon})\delta^H B_s^H\Big| + E\Big|\alpha_H \int_0^t \int_0^T (D_r^H u_s^H - D_r^H u_s^{H,\varepsilon})|s-r|^{2H-2} drds\Big| \tag{31}$$

In the proof of Proposition 3.5 we have seen that the first term tends to zero, uniformly in H, when $\varepsilon \to 0$.
Recall that (see for instance Remark 3.6) the main term that we have to treat for the case $H_0 = \frac{1}{2}$ is the second term:

$$E\Big|\alpha_H \int_0^t \int_0^T (D_r^H u_s^H - D_r^H u_s^{H,\varepsilon})|s-r|^{2H-2} drds\Big|.$$

We can write

$$E\left|\alpha_H \int_0^T\int_0^T (D_r^H u_s^H - D_r^H u_s^{H,\varepsilon})|r-s|^{2H-2} drds\right| \leqslant A_1(H) + A_2(H) + A_3(H,\varepsilon) + A_4(H,\varepsilon) + A_5(H,\varepsilon),$$

where

$$A_1(H) = E\left|\alpha_H \int_0^T\int_0^T D_r^H u_s^H |r-s|^{2H-2} drds - c_H \int_0^T\int_0^T D_r^H u_s^H |r-s|^{2H-2} drds\right|,$$

$$A_2(H) = E\left|c_H \int_0^T\int_0^T D_r^H u_s^H |r-s|^{2H-2} drds - \int_0^T \frac{1}{2}\nabla_r^H u_r^H dr\right|,$$

$$A_3(H,\varepsilon) = E\left|\int_0^T \frac{1}{2}\nabla_r^H u_r^H dr - \int_0^T \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} D_r^H u_x^H dx\right) dr\right|,$$

$$A_4(H,\varepsilon) = E\left|\int_0^T \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} D_r^H u_x^H dx\right) dr - c_H \int_0^T\int_0^T D_r^H u_s^{H,\varepsilon} |r-s|^{2H-2} drds\right|,$$

$$A_5(H,\varepsilon) = E\left|c_H \int_0^T\int_0^T D_r^H u_s^{H,\varepsilon} |r-s|^{2H-2} drds - \alpha_H \int_0^T\int_0^T D_r^H u_s^{H,\varepsilon} |r-s|^{2H-2} drds\right|,$$

with c_H defined in (28).
Notice that the term $\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} D_r^H u_x^H dx$ which appears in $A_3(H,\varepsilon)$ and $A_4(H,\varepsilon)$ plays the role of the trace $\nabla^H u^{H,\varepsilon}$.
Next, we will see that each one of these terms becomes small when H is close to $\frac{1}{2}$ at least for some $\varepsilon > 0$:

- For the first term we have

$$\begin{aligned} A_1(H) &= |\alpha_H - c_H| E\left|\int_0^T\int_0^T D_r^H u_s^H |r-s|^{2H-2} drds\right| \\ &\leqslant |\alpha_H - c_H| \int_0^T \sup_{s\in[0,T]} E|D_r^H u_s^H| \left(\int_0^T |r-s|^{2H-2} ds\right) dr \\ &\leqslant |\alpha_H - c_H| \frac{2T^{2H-1}}{2H-1} K_D^{1/p} \\ &= |2HT^{2H-1} - 1| K_D^{1/p}, \end{aligned}$$

  where we have used condition (A2). On the other hand, the term $|2HT^{2H-1} - 1|$ tends to zero as $H \to 1/2$.Thus, $A_1(H)$ converges to zero as $H \to \frac{1}{2}$.

- We can bound the term $A_5(H,\varepsilon)$ in the following way

$$\begin{aligned} A_5(H,\varepsilon) &= |\alpha_H - c_H| E\left|\int_0^T\int_0^T \left(\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} D_r^H u_x^H dx\right) |r-s|^{2H-2} drds\right| \\ &\leqslant |\alpha_H - c_H| \frac{1}{2\varepsilon}\int_0^T\int_0^T \left(\int_{s-\varepsilon}^{s+\varepsilon} \sup_{x\in[0,T]} E|D_r^H u_x^H| dx\right) |r-s|^{2H-2} drds \\ &\leqslant |\alpha_H - c_H| \int_0^T \sup_{x\in[0,T]} E|D_r^H u_x^H| \left(\int_0^T |r-s|^{2H-2} ds\right) dr \\ &\leqslant |2HT^{2H-1} - 1| K_D^{1/p}. \end{aligned}$$

  Therefore, $A_5(H,\varepsilon)$ becomes small when H is close to $\frac{1}{2}$, for any $\varepsilon > 0$.

- Now we will study the term $A_2(H)$. Using the approximation of the identity defined in (27), this term can be expressed as

$$E\left|\int_0^T \left(\frac{1}{2}\nabla_r^H u_r^H - (D_r^H u_{\bullet}^H * \psi_H)(r)\right) dr\right|$$

with

$$\frac{1}{2}\nabla_r^H u_r^H - (D_r^H u_{\bullet}^H * \psi_H)(r) = \int_{-T}^T \psi_H(y)\left(\frac{1}{2}\nabla_r^H u_r^H - D_r^H u_{r-y}^H\right) dy.$$

Using Fubini's theorem we obtain the following equality

$$E\left|\int_0^T \left(\frac{1}{2}\nabla_r^H u_r^H - (D_r^H u_{\bullet}^H * \psi_H)(r)\right) dr\right| = E\left|\int_{-T}^T \psi_H(y)\left(\int_0^T (\frac{1}{2}\nabla_r^H u_r^H - D_r^H u_{r-y}^H) dr\right) dy\right|.$$

For any $\delta \in (0,T)$ we have that

$$\begin{aligned} A_2(H) \leqslant\ & E\left|\int_{|y|\leqslant\delta} \psi_H(y)\left(\int_0^T \frac{1}{2}\nabla_r^H u_r^H - D_r^H u_{r-y}^H dr\right) dy\right| \\ & + E\left|\int_{|y|>\delta} \psi_H(y)\left(\int_0^T \frac{1}{2}\nabla_r^H u_r^H - D_r^H u_{r-y}^H dr\right) dy\right|. \end{aligned} \tag{32}$$

We will insert the following term

$$D_r^{H,-} u_r^H \mathbf{1}_{\{0<y\leqslant\delta\}} + D_r^{H,+} u_r^H \mathbf{1}_{\{-\delta\leqslant y<0\}}$$

into the first summand of (32). Then, we have that

$$\begin{aligned} & E\Big|\int_{|y|\leqslant\delta} \psi_H(y)\left(\int_0^T \left(\frac{1}{2}\nabla_r^H u_r^H - D_r^H u_{r-y}^H\right) dr\right) dy\Big| \\ & \leqslant E\Big|\int_{|y|\leqslant\delta} \psi_H(y)\left(\int_0^T \left(\frac{1}{2}\nabla_r^H u_r^H - (D_r^{H,-} u_r^H \mathbf{1}_{\{0<y\leqslant\delta\}} + D_r^{H,+} u_r^H \mathbf{1}_{\{-\delta\leqslant y<0\}})\right) dr\right) dy\Big| \\ & + E\Big|\int_{|y|\leqslant\delta} \psi_H(y)\left(\int_0^T \left((D_r^{H,-} u_r^H \mathbf{1}_{\{0<y\leqslant\delta\}} + D_r^{H,+} u_r^H \mathbf{1}_{\{-\delta\leqslant y<0\}}) - D_r^H u_{r-y}^H\right) dr\right) dy\Big|. \end{aligned} \tag{33}$$

By applying Fubini's theorem and the symmetry of the function ψ_H we obtain that the first summand is equal to zero. In fact, we have

$$
\begin{aligned}
&\Big| \int_{|y|\leqslant \delta} \psi_H(y) \left(\int_0^T \left(\frac{1}{2} \nabla_r^H u_r^H - (D_r^{H,-} u_r^H \mathbf{1}_{\{0<y\leqslant \delta\}} + D_r^{H,+} u_r^H \mathbf{1}_{\{-\delta\leqslant y<0\}}) \right) dr \right) dy \Big| \\
&= \Big| \int_0^T \left(\frac{1}{2} \nabla_r^H u_r^H \left(\int_{|y|\leqslant \delta} \psi_H(y) dy \right) \right. \\
&\qquad\qquad \left. - D_r^{H,-} u_r^H \left(\int_0^\delta \psi_H(y) dy \right) - D_r^{H,+} u_r^H \left(\int_{-\delta}^0 \psi_H(y) dy \right) \right) dr \Big| \\
&= \Big| \int_0^T \left(D_r^{H,-} u_r^H \left(\frac{1}{2} \int_{|y|\leqslant \delta} \psi_H(y) dy - \int_0^\delta \psi_H(y) dy \right) \right. \\
&\qquad\qquad \left. + D_r^{H,+} u_r^H \left(\frac{1}{2} \int_{|y|\leqslant \delta} \psi_H(y) dy - \int_{-\delta}^0 \psi_H(y) dy \right) \right) dr \Big| = 0.
\end{aligned}
$$

The second summand of (33) can be bounded by the following sum whose terms tend to zero when δ goes to zero

$$
\begin{aligned}
&E \Big| \int_{|y|\leqslant \delta} \psi_H(y) \int_0^T (D_r^{H,-} u_r^H - D_r^H u_{r-y}^H) \mathbf{1}_{\{0<y\leqslant \delta\}} dr dy \Big| \\
&+ E \Big| \int_{|y|\leqslant \delta} \psi_H(y) \int_0^T (D_r^{H,+} u_r^H - D_r^H u_{r-y}^H) \mathbf{1}_{\{-\delta\leqslant y<0\}} dr dy \Big|.
\end{aligned} \tag{34}
$$

In fact, taking suprema in the first summand over H and y we have

$$
\begin{aligned}
&E \Big| \int_{|y|\leqslant \delta} \psi_H(y) \int_0^T (D_r^{H,-} u_r^H - D_r^H u_{r-y}^H) \mathbf{1}_{\{0<y\leqslant \delta\}} dr dy \Big| \\
&\qquad \leqslant \left(\sup_{H\in V_0} \sup_{0<y\leqslant \delta} \int_0^T E |D_r^{H,-} u_r^H - D_r^H u_{r-y}^H| dr \right) \int_{-T}^T \psi_H(y) dy.
\end{aligned}
$$

Now using condition (20) we have that we can choose δ such that the term

$$
\sup_{H\in V_0} \sup_{0<y\leqslant \delta} \int_0^T E |D_r^{H,-} u_r^H - D_r^H u_{r-y}^H| dr
$$

becomes sufficiently small. By similar arguments to those used above one can see an analogous result for the second summand of (34).

We next detail the steps of the treatment of the second term of (32). On one hand we have that

$$
\begin{aligned}
&E \Big| \int_{|y|>\delta} \psi_H(y) \left(\int_0^T \frac{1}{2} \nabla_r^H u_r^H - D_r^H u_{r-y}^H dr \right) dy \Big| \\
&\qquad \leqslant \int_{|y|>\delta} \psi_H(y) \left(\int_0^T E \left| \frac{1}{2} \nabla_r^H u_r^H - D_r^H u_{r-y}^H \right| dr \right) dy.
\end{aligned}
$$

From condition (A2) and inequalities (22) and (23) we have that

$$\sup_{H\in(\frac{1}{2},\frac{1}{2}+\eta)}\sup_{y\in[0,T]}\int_0^T E\left|\frac{1}{2}\nabla_r^H u_r^H - D_r^H u_{r-y}^H\right| dr < +\infty.$$

Moreover, using that ψ_H is an approximation of the identity when $H \downarrow \frac{1}{2}$, for the δ chosen in the treatment of (34) we have that:

$$\int_{|y|>\delta}\psi_H(y)dy \overset{H\downarrow\frac{1}{2}}{\longrightarrow} 0.$$

Thus, we deduce that

$$E\Big|\int_{|y|>\delta}\psi_H(y)\left(\int_0^T \frac{1}{2}\nabla_r^H u_r^H - D_r^H u_{r-y}^H dr\right)\Big|$$

becomes small when H is close to $\frac{1}{2}$.

- The term $A_3(H,\varepsilon)$ can be bounded as follows

$$\begin{aligned}
A_3(H,\varepsilon) &\leqslant \frac{1}{2\varepsilon}\int_0^T\left(\int_r^{r+\varepsilon} E|D_r^{H,+}u_r^H - D_r^H u_x^H|\,dx\right)dr \\
&\qquad\qquad + \frac{1}{2\varepsilon}\int_0^T\left(\int_{r-\varepsilon}^r E|D_r^{H,-}u_r^H - D_r^H u_x^H|\,dx\right)dr \\
&\leqslant \frac{1}{2}\sup_{H\in V_0}\int_0^T \sup_{x\in[r,r+\varepsilon]} E|D_r^{H,+}u_r^H - D_r^H u_x^H|\,dr \\
&\qquad\qquad + \frac{1}{2}\sup_{H\in V_0}\int_0^T \sup_{x\in[r-\varepsilon,r]} E|D_r^{H,-}u_r^H - D_r^H u_x^H|\,dr.
\end{aligned}$$

  Using condition (20) we have that given $\rho > 0$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ we have

$$\sup_{H\in V_0} A_3(H,\varepsilon) < \frac{\rho}{2}.$$

- We proceed now to finish the proof with the treatment of the term $A_4(H,\varepsilon)$.

  By Proposition 3.5, there exists ε_1 such that the first summand of (31) is lesser than ρ for any $0 < \varepsilon < \varepsilon_1$. On the other hand, by Proposition 3.8, there exists ε_2 such that for any $0 < \varepsilon < \varepsilon_2$ inequality (30) is also satisfied. From now on, fix an $\varepsilon > 0$ such that $\varepsilon < \varepsilon_2 \wedge \varepsilon_1 \wedge \varepsilon_0$.

Similar arguments to those used above give

$$\begin{aligned}A_4(H,\varepsilon) &= E\left|\int_{-T}^{T}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon}D_r^{H}u_x^{H}\,dx - D_r^{H}u_{r-y}^{H,\varepsilon}\right)dr\right)dy\right|\\&\leqslant E\left|\int_{|y|\leqslant\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon}D_r^{H}u_x^{H}\,dx - \frac{1}{2\varepsilon}\int_{r-y-\varepsilon}^{r-y+\varepsilon}D_r^{H}u_x^{H}dx\right)dr\right)dy\right|\\&\quad + E\left|\int_{|y|>\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon}D_r^{H}u_x^{H}\,dx - \frac{1}{2\varepsilon}\int_{r-y-\varepsilon}^{r-y+\varepsilon}D_r^{H}u_x^{H}dx\right)dr\right)dy\right|.\end{aligned} \tag{35}$$

The first summand of (35) can be bounded by

$$\begin{aligned}&E\left|\int_{|y|\leqslant\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{(r-\varepsilon)\wedge(r-y-\varepsilon)}^{(r-\varepsilon)\vee(r-y-\varepsilon)}D_r^{H}u_x^{H}dx\right)dr\right)dy\right|\\&+E\left|\int_{|y|\leqslant\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{(r+\varepsilon)\wedge(r-y+\varepsilon)}^{(r+\varepsilon)\vee(r-y+\varepsilon)}D_r^{H}u_x^{H}dx\right)dr\right)dy\right|.\end{aligned}$$

So, applying Fubini's theorem we have that

$$\begin{aligned}&E\left|\int_{|y|\leqslant\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{(r-\varepsilon)\wedge(r-y-\varepsilon)}^{(r-\varepsilon)\vee(r-y-\varepsilon)}D_r^{H}u_x^{H}dx\right)dr\right)dy\right|\\&\leqslant\int_{|y|\leqslant\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{(r-\varepsilon)\wedge(r-y-\varepsilon)}^{(r-\varepsilon)\vee(r-y-\varepsilon)}\sup_{x\in[0,T]}E|D_r^{H}u_x^{H}|dx\right)dr\right)dy\\&\leqslant\int_0^T\left(\sup_{x\in[0,T]}E|D_r^{H}u_x^{H}|\left(\int_{|y|\leqslant\varepsilon}\frac{c_H|y|^{2H-1}}{2\varepsilon}dy\right)dr\right)\\&=\frac{(2H-1)\varepsilon^{2H-1}}{8HT^{2H-1}}\int_0^T\sup_{x\in[0,T]}E|D_r^{H}u_x^{H}|dr.\end{aligned}$$

For $H\in\left(\frac{1}{2},\frac{1}{2}+\eta\right)$ we have that this term becomes small if $\eta>0$ is small enough. We treat now the second summand of (35). This term can be bounded by

$$\begin{aligned}&\int_{|y|>\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon}E|D_r^{H}u_x^{H}|\,dx\right)dr\right)dy\\&+\int_{|y|>\varepsilon}\psi_{H}(y)\left(\int_0^T\left(\frac{1}{2\varepsilon}\int_{r-y-\varepsilon}^{r-y+\varepsilon}E|D_r^{H}u_x^{H}|dx\right)dr\right)dy.\end{aligned}$$

Taking supremum on x we obtain the same upper bound for each of these terms:

$$\int_{|y|>\varepsilon}\psi_{H}(y)\left(\int_0^T\sup_{x\in[0,T]}E|D_r^{H}u_x^{H}|\,dr\right)dy. \tag{36}$$

Thus, condition (A2) and the following computation

$$\int_{|y|>\varepsilon} \psi_H(y)dy = 1 - \left(\frac{\varepsilon}{T}\right)^{2H-1}$$

imply that (36) tends to zero when $H \to \frac{1}{2}$. This concludes the proof.

□

In order to prove the convergence of the finite-dimensional distributions of $\{X^H\}_{H\in V_0}$ we will use that $X^{H,\varepsilon}$ is a continuous functional of (u^H, B^H). Indeed, first of all, applying the integration by parts formula (see for instance [13]) we have that

$$\begin{aligned} X_t^{H,\varepsilon} = \int_0^t u_s^{H,\varepsilon} dB_s^H &= u_t^{H,\varepsilon} B_t^H - u_0^{H,\varepsilon} B_0^H - \int_0^t (u_s^{H,\varepsilon})' B_s^H ds \\ &= \left(\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} u_s^H ds\right) B_t^H - \int_0^t \left(\frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} u_r^H dr\right)' B_s^H ds. \end{aligned} \tag{37}$$

Suppose that $0<\varepsilon<\frac{T}{3}$. Taking into account the convention that u is null out of the interval $[0,T]$, the term $\int_0^t (u_s^{H,\varepsilon})' B_s^H ds$ has a different expression depending on the value of t:

(i) If $0 \leqslant t < \varepsilon$ then

$$\int_0^t (u_s^{H,\varepsilon})' B_s^H ds = \int_0^t \frac{u_{s+\varepsilon}}{2\varepsilon} B_s^H ds.$$

(ii) If $\varepsilon \leqslant t < T-\varepsilon$ then

$$\int_0^t (u_s^{H,\varepsilon})' B_s^H ds = \int_0^\varepsilon \frac{u_{s+\varepsilon}}{2\varepsilon} B_s^H ds + \int_\varepsilon^t \frac{u_{s+\varepsilon}-u_{s-\varepsilon}}{2\varepsilon} B_s^H ds.$$

(iii) If $T-\varepsilon \leqslant t < T$ then

$$\int_0^t (u_s^{H,\varepsilon})' B_s^H ds = \int_0^\varepsilon \frac{u_{s+\varepsilon}}{2\varepsilon} B_s^H ds + \int_\varepsilon^{T-\varepsilon} \frac{u_{s+\varepsilon}-u_{s-\varepsilon}}{2\varepsilon} B_s^H ds + \int_{T-\varepsilon}^t \frac{0-u_{s-\varepsilon}}{2\varepsilon} B_s^H ds.$$

We next see that the functional of u^H and B^H involved in (37) is continuous.

Lemma 3.10. *Fixed* $0<\varepsilon<\frac{T}{3}$ *we define the map*

$$\begin{array}{rccl} \Psi^\varepsilon : & (\mathscr{C}([0,T]))^2 & \longrightarrow & \mathscr{C}([0,T]) \\ & (x,y) & \mapsto & \Psi^\varepsilon(x,y)(t) := \Psi_1^\varepsilon(x)(t)y(t) - \int_0^t y(s)\Psi_2^\varepsilon(x(s))ds, \end{array}$$

with

$$\Psi_1^\varepsilon(x)(t) := \frac{1}{2\varepsilon}\int_{(t-\varepsilon)\vee 0}^{(t+\varepsilon)\wedge T} x(s)ds,$$

and

$$\Psi_2^\varepsilon(x)(t) := \begin{cases} \frac{x(t+\varepsilon)}{2\varepsilon},\ if\ 0 \leqslant t < \varepsilon, \\ \\ \frac{x(t+\varepsilon)-x(t-\varepsilon)}{2\varepsilon},\ if\ \varepsilon \leqslant t < T-\varepsilon, \\ \\ \frac{-x(t-\varepsilon)}{2\varepsilon},\ if\ T-\varepsilon \leqslant t < T. \end{cases}$$

Then Ψ^ε is continuous.

Proof. We have that

$$\Psi^\varepsilon(x,y)(t) = \begin{cases} \Psi_1^\varepsilon(x)(t)y(t) - \int_0^t y(s)\ \frac{x(s+\varepsilon)}{2\varepsilon}\ ds,\ 0 \leqslant t < \varepsilon, \\ \\ \Psi_1^\varepsilon(x)(t)y(t) - \int_0^\varepsilon y(s)\ \frac{x(s+\varepsilon)}{2\varepsilon}\ ds - \int_\varepsilon^t y(s)\ \frac{x(s+\varepsilon)-x(s-\varepsilon)}{2\varepsilon}\ ds,\ \varepsilon \leqslant t < T-\varepsilon, \\ \\ \Psi_1^\varepsilon(x)(t)y(t) - \int_0^\varepsilon y(s)\ \frac{x(s+\varepsilon)}{2\varepsilon}\ ds - \int_\varepsilon^{T-\varepsilon} y(s)\ \frac{x(s+\varepsilon)-x(s-\varepsilon)}{2\varepsilon}\ ds \\ \qquad\qquad - \int_{T-\varepsilon}^t y(s)\ \frac{-x(s-\varepsilon)}{2\varepsilon}\ ds,\ T-\varepsilon \leqslant t < T. \end{cases}$$

Notice that the functional Ψ^ε really takes values in $\mathscr{C}([0,T])$ and it can be easily checked that is a continuous map from $\mathscr{C}([0,T])^2$ into $\mathscr{C}([0,T])$. $\square$

In the following lemma we will prove the convergence in law of $\{X^{H,\varepsilon}\}_{H\in V_0}$ to $X^{H_0,\varepsilon}$ in $\mathscr{C}([0,T])$, when $H \to H_0$, for all $0 < \varepsilon < T/3$.

Lemma 3.11. *Let $\{u^H\}_{H\in V_0}$ be a family of processes with continuous trajectories such that*

$$(u^H, B^H) \xrightarrow{\mathscr{L}} (u^{H_0}, B^{H_0}),$$

in $(\mathscr{C}([0,T]))^2$ when $H \to H_0$. Then for all $H_0 \in \left[\frac{1}{2}, 1\right)$ the family of processes $\{X^{H,\varepsilon}\}_{H\in V_0}$ defined in (13) converges in law to $X^{H_0,\varepsilon}$ in $\mathscr{C}([0,T])$ when $H \to H_0$.

Proof. From expression (37) and Lemma 3.11 we have that $X^{H,\varepsilon} = \Psi^\varepsilon(u^H, B^H)$ is a continuous functional of (u^H, B^H). So, using that

$$(u^H, B^H) \xrightarrow{\mathscr{L}} (u^{H_0}, B^{H_0}),$$

when $H \to H_0$ in $(\mathscr{C}([0,T]))^2$ we obtain the desired result. $\square$

Now we have the necessary ingredients to show the convergence of the finite-dimensional distributions of the family of processes $\{X^H\}_{H\in V_0}$.

Proposition 3.12. *Let $\{X^H\}_{H\in V_0}$ be the family of Russo-Vallois stochastic integrals defined in (7). Suppose that the family of stochastic processes $\{u^H\}_{H\in V_0}$ satisfies Block A and Block B of hypotheses if $H_0 > \frac{1}{2}$ and, in the case $H_0 = \frac{1}{2}$, blocks A and B of hypotheses (with (B2') replacing (B2)) and Condition C. Moreover, assume that*

$$(u^H, B^H) \xrightarrow{\mathscr{L}} (u^{H_0}, B^{H_0}),$$

in $(\mathscr{C}([0,T]))^2$ when $H \to H_0$. Then the finite-dimensional distributions of X^H converge to those of X^{H_0} when $H \to H_0$.

Proof. • Case $H_0 \in (\frac{1}{2}, 1)$:

For all $t_1, \ldots, t_m \in [0,T]$ and $g \in \mathscr{C}_b^1(\mathbb{R}^m)$ we write

$$|E[g(X^H_{t_1}, \ldots, X^H_{t_m})] - E[g(X^{H_0}_{t_1}, \ldots, X^{H_0}_{t_m})]| \leqslant T_1(\varepsilon, H) + T_2(\varepsilon, H) + T_3(\varepsilon),$$

where

$$T_1(\varepsilon, H) = |E[g(X^H_{t_1}, \ldots, X^H_{t_m})] - E[g(X^{H,\varepsilon}_{t_1}, \ldots, X^{H,\varepsilon}_{t_m})]|,$$

$$T_2(\varepsilon, H) = |E[g(X^{H,\varepsilon}_{t_1}, \ldots, X^{H,\varepsilon}_{t_m})] - E[g(X^{H_0,\varepsilon}_{t_1}, \ldots, X^{H_0,\varepsilon}_{t_m})]|$$

and

$$T_3(\varepsilon) = |E[g(X^{H_0,\varepsilon}_{t_1}, \ldots, X^{H_0,\varepsilon}_{t_m})] - E[g(X^{H_0}_{t_1}, \ldots, X^{H_0}_{t_m})]|.$$

On one hand we have that $T_1(\varepsilon, H)$ is bounded by

$$C_g \max_{j=1,\ldots,m} \sup_{H\in V_0} E|X^H_{t_j} - X^{H,\varepsilon}_{t_j}|.$$

By using Proposition 3.5, we have that this term tends to zero when $\varepsilon \to 0$.

In an analogous way we can show that $T_3(\varepsilon)$ is bounded by

$$C_g \max_{j=1,\ldots,m} \sup_{H\in V_0} E|X^{H_0}_{t_j} - X^{H_0,\varepsilon}_{t_j}|$$

and using again Proposition 3.5, we obtain that the term $T_3(\varepsilon)$ tends to zero when $\varepsilon \to 0$.

Thus, given $\eta > 0$ we can take a small enough $\varepsilon > 0$ such that $T_1(\varepsilon, H) < \frac{\eta}{3}$ and $T_3(\varepsilon) < \frac{\eta}{3}$.

Finally, the remaining term $T_2(\varepsilon, H)$ tends to zero when $H \to H_0$. In fact, as a consequence of Lemma 3.11 for the $\varepsilon > 0$ taken above we have the following convergence

$$\mathscr{L}(X^{H,\varepsilon}_{t_1}, \ldots, X^{H,\varepsilon}_{t_m}) \longrightarrow \mathscr{L}(X^{H_0,\varepsilon}_{t_1}, \ldots, X^{H_0,\varepsilon}_{t_m}),$$

when $H \to H_0$. This completes the proof in the case $H_0 > \frac{1}{2}$.

- Case $H_0 = \frac{1}{2}$:

  For all $t_1, \dots, t_m \in [0,T]$ and $g \in \mathscr{C}_b^1(\mathbb{R}^m)$ we write also

$$|E[g(X_{t_1}^H, \dots, X_{t_m}^H)] - E[g(X_{t_1}^{1|2}, \dots, X_{t_m}^{1|2})]| \leqslant T_1(\varepsilon, H) + T_2(\varepsilon, H) + T_3(\varepsilon),$$

  where

$$T_1(\varepsilon, H) = |E[g(X_{t_1}^H, \dots, X_{t_m}^H)] - E[g(X_{t_1}^{H,\varepsilon}, \dots, X_{t_m}^{H,\varepsilon})]|,$$

$$T_2(\varepsilon, H) = |E[g(X_{t_1}^{H,\varepsilon}, \dots, X_{t_m}^{H,\varepsilon})] - E[g(X_{t_1}^{1|2,\varepsilon}, \dots, X_{t_m}^{1|2,\varepsilon})]|$$

  and

$$T_3(\varepsilon) = |E[g(X_{t_1}^{1|2,\varepsilon}, \dots, X_{t_m}^{1|2,\varepsilon})] - E[g(X_{t_1}^{1|2}, \dots, X_{t_m}^{1|2})]|.$$

  On one hand we have that $T_1(\varepsilon, H)$ is bounded by

$$C_g \max_{j=1,\dots,m} E|X_{t_j}^H - X_{t_j}^{H,\varepsilon}|$$

  and the term $T_3(\varepsilon)$ is majorized by

$$C_g \max_{j=1,\dots,m} E|X_{t_j}^{1|2} - X_{t_j}^{1|2,\varepsilon}|.$$

  Using Proposition 3.9 we have that given $\rho > 0$ we can take $\varepsilon > 0$ and $\eta > 0$ such that if $H \in \left(\frac{1}{2}, \frac{1}{2} + \eta\right)$ then $T_1(\varepsilon, H) < \rho$ and also $T_3(\varepsilon) < \rho$.

  Taking the ε chosen above and applying also Lemma 3.11, we deduce that the term $T_2(\varepsilon, H)$ converges to zero when $H \to H_0$. This fact finishes the proof.

$\square$

As a consequence of Proposition 3.3 and Proposition 3.12 we obtain the main result of our work.

Theorem 3.13. *Let $\{u^H\}_{H \in V_0}$ be a family of stochastic processes with continuous trajectories such that satisfy Block A and Block B of hypotheses if $H_0 > \frac{1}{2}$ and, in the case $H_0 = \frac{1}{2}$, blocks A and B of hypotheses (with (B2') replacing (B2)) and Condition C. Moreover, assume that*

$$(u^H, B^H) \xrightarrow{\mathscr{L}} (u^{H_0}, B^{H_0}), \tag{38}$$

in $(\mathscr{C}([0,T]))^2$ when $H \to H_0$. Then the family of the laws of the Russo-Vallois stochastic integrals $\{X^H\}_{H \in V_0}$ defined in (7) converges weakly to the law of X^{H_0} in the space of continuous functions $\mathscr{C}([0,T])$ when $H \to H_0$.

4 Example

Now we give an example of a simple family of stochastic processes $\{u^H\}_{H\in V_0}$ with continuous trajectories to which Theorem 3.13 can be applied.

Example 4.1. *Let* $\{u^H\}_{H\in V_0}$ *be the family of stochastic processes defined by* $u^H = \{u_t^H := B_{f(t)}^H,\ t \in [0,T]\}$ *where* $f : [0,T] \to \mathbb{R}_+$ *is a continuous function on* $[0,T]$ *and* $V_0 = \left[\frac{1}{2}, 1\right)$ *. Let* $\{X^H, H \in V_0\}$ *be the family of Russo-Vallois stochastic integrals defined in (7) for these* u^H*. Then,*

$$X^H \xrightarrow{\mathscr{L}} X^{H_0},$$

when $H \to H_0$ *in* $\mathscr{C}([0,T])$ *for each* $H_0 \in [\frac{1}{2}, 1)$.

Proof. We will check that the family $\{u^H\}$ satisfies the assumptions of Theorem 3.13.

- Condition (A1): There exists $p > 2$ such that

$$\int_0^T \sup_{H\in V_0} E|u_s^H|^p ds < +\infty.$$

  In fact, for all $p > 0$,

$$\int_0^T \sup_{H\in V_0} E|B_{f(s)}^H|^p ds \leqslant C_p \int_0^T \sup_{H\in V_0} |f(s)|^{pH} ds < +\infty,$$

  because f is a continuous function on $[0,T]$.

- Condition (A2): There exists $p > 2$ such that

$$\sup_{H\in V_0} \int_0^T \sup_{x\in[0,T]} E|D_r^H u_x^H|^p dr = K_D < +\infty.$$

  We know that $D_r^H B_{f(s)}^H = 1_{[0,f(s)]}(r)$. Thus,

$$\sup_{H\in V_0} \int_0^T \sup_{s\in[0,T]} E|D_r^H B_{f(s)}^H|^p dr = \sup_{H\in V_0} \int_0^T \sup_{s\in[0,T]} 1_{[0,f(s)]}(r) dr \leqslant T < +\infty,$$

- Condition (B1)

$$\lim_{\delta\to 0} \sup_{H\in V_0} \sup_{|y|<\delta} \int_0^T E|u_s^H - u_{s-y}^H|^2 ds = 0.$$

  In this case, we must take care of the prolongation of the process u^H by zero out of the interval $[0,T]$ (this can be done defining $f(s) = 0$ for any $s \in [0,T]^c$). Suppose $\delta < \frac{T}{3}$. Then, we have that

$$
\begin{aligned}
\sup_{H\in V_0}\sup_{|y|<\delta}\int_0^T E|u_s^H - u_{s-y}^H|^2 ds &\leqslant \sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_0^\delta E|u_s^H|^2 ds + \sup_{-\delta<y<0}\int_{T-\delta}^T E|u_s^H|^2 ds\Big\}\\
&+ \sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_\delta^T E|u_s^H - u_{s-y}^H|^2 ds + \sup_{-\delta<y<0}\int_0^{T-\delta} E|u_s^H - u_{s-y}^H|^2 ds\Big\}\\
&\leqslant \sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_0^\delta |f(s)|^{2H} ds + \sup_{-\delta<y<0}\int_{T-\delta}^T |f(s)|^{2H} ds\Big\}\\
&+ \sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_\delta^T |f(s)-f(s-y)|^{2H} ds + \sup_{-\delta<y<0}\int_0^{T-\delta} |f(s)-f(s-y)|^{2H} ds\Big\}.
\end{aligned}
\tag{39}
$$

Letting $\delta \to 0$, the first term in (39):

$$
\sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_0^\delta |f(s)|^{2H} ds + \sup_{-\delta<y<0}\int_{T-\delta}^T |f(s)|^{2H} ds\Big\},
$$

converges to zero because $|f(t)|^{2H}$ can be bounded uniformly in H and in t and the length of the intervals $[0,\delta]$ and $[T-\delta, T]$ tends to zero.

On the other hand the remaining term in (39)

$$
\sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_\delta^T |f(s)-f(s-y)|^{2H} ds + \sup_{-\delta<y<0}\int_0^{T-\delta} |f(s)-f(s-y)|^{2H} ds\Big\}
$$

can be bounded by

$$
C_{f,T}\left(\sup_{0<y<\delta}\int_\delta^T |f(s)-f(s-y)| ds + \sup_{-\delta<y<0}\int_0^{T-\delta} |f(s)-f(s-y)| ds\right).
$$

Since f is uniformly continuous, this expression tends to 0 when $\delta \to 0$.

- <u>Condition (B2) (or Condition (B2'))</u>: For any $p \geqslant 2$

$$
\lim_{\delta\to 0}\sup_{H\in V_0}\sup_{|y|<\delta}\int_0^T\int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H|^p dr ds = 0.
$$

We have that

$$\sup_{H\in V_0}\sup_{|y|<\delta}\int_0^T\int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H|^p drds$$
$$\leqslant \sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_0^\delta\int_0^T E|D_r^H u_s^H|^p drds + \sup_{-\delta<y<0}\int_{T-\delta}^T\int_0^T E|D_r^H u_s^H|^p drds\Big\}$$
$$+\sup_{H\in V_0}\Big\{\sup_{0<y<\delta}\int_\delta^T\int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H|^p drds + \sup_{-\delta<y<0}\int_0^{T-\delta}\int_0^T E|D_r^H u_s^H - D_r^H u_{s-y}^H|^p drds\Big\}$$
$$\leqslant \sup_{0<y<\delta}\int_0^\delta\int_0^T |\mathbf{1}_{[0,f(s)]}(r)|drds + \sup_{-\delta<y<0}\int_{T-\delta}^T\int_0^T |\mathbf{1}_{[0,f(s)]}(r)|drds$$
$$+\sup_{0<y<\delta}\int_\delta^T\int_0^T |\mathbf{1}_{[0,f(s)]}(r) - \mathbf{1}_{[0,f(s-y)]}(r)|drds$$
$$+\sup_{-\delta<y<0}\int_0^{T-\delta}\int_0^T |\mathbf{1}_{[0,f(s)]}(r) - \mathbf{1}_{[0,f(s-y)]}(r)|drds.$$

The first two summands of the above expression tends to zero when $\delta \to 0$.

We have that

$$\sup_{0<y<\delta}\int_\delta^T\int_0^T |\mathbf{1}_{[0,f(s)]}(r) - \mathbf{1}_{[0,f(s-y)]}(r)|drds + \sup_{-\delta<y<0}\int_{T-\delta}^T\int_0^T |\mathbf{1}_{[0,f(s)]}(r) - \mathbf{1}_{[0,f(s-y)]}(r)|drds$$
$$\leqslant \sup_{0<y<\delta}\int_\delta^T\int_0^T \mathbf{1}_{[f(s)\wedge f(s-y),(f(s)\vee f(s-y))]}(r)drds$$
$$+\sup_{-\delta<y<0}\int_0^{T-\delta}\int_0^T \mathbf{1}_{[f(s)\wedge f(s-y),(f(s)\vee f(s-y))]}(r)drds$$
$$\leqslant \sup_{0<y<\delta}\int_\delta^T |f(s)-f(s-y)|ds + \sup_{-\delta<y<0}\int_0^{T-\delta} |f(s)-f(s-y)|ds.$$

Taking into account that f is uniformly continuous we have that the last terms tend to zero, when $\delta \to 0$.

- Finally we will check Condition C.

  We have the following candidates for the derivates $D^{H,+}$ i $D^{H,-}$

$$\lim_{h\to 0^+} D_r^H u_{r+h}^H = \lim_{h\to 0^+}\mathbf{1}_{[0,f(r+h)]}(r) = \mathbf{1}_{[0,f(r)]}(r)$$

and

$$\lim_{h\to 0^-} D_r^H u_{r+h}^H = \lim_{h\to 0^-}\mathbf{1}_{[0,f(r+h)]}(r) = \mathbf{1}_{[0,f(r))}(r),$$

respectively.

So, we will define

$$D_r^{H,+} u_r^H := \mathbf{1}_{[0,f(r)]}(r)$$

and

$$D_r^{H,-} u_r^H := \mathbf{1}_{[0,f(r))}(r).$$

Now we will check that Condition C holds.

For any $r \in [0,T]$ and $\delta \in (0,T/3)$, let $m_{r,\delta} := \min_{y\in[0,\delta]} \{f(r-y), f(r)\}$ and $M_{r,\delta} := \max_{y\in[0,\delta]} \{f(r-y), f(r)\}$. Then,

$$\begin{aligned}
\sup_{H\in V_0} \int_0^T \sup_{0<y<\delta} E|D_r^{H,-} u_r^H - D_r^H u_{r-y}^H| dr &\leqslant \int_0^T \sup_{0<y<\delta} \mathbf{1}_{[f(r-y)\wedge f(r),(f(r-y)\vee f(r))]}(r)dr \\
&\leqslant \int_0^T \mathbf{1}_{[m_{r,\delta},M_{r,\delta}]}(r)dr
\end{aligned}$$

and this term tends to zero when $\delta \to 0$, by the Dominated Convergence Theorem.

In a similar way one can prove that

$$\sup_{H\in V_0} \int_0^T \sup_{-\delta<y<0} E|D_r^{H,+} u_r^H - D_r^H u_{r-y}^H| dr \longrightarrow 0,$$

when $\delta \to 0$.

Now we will check that

$$(u^H, B^H) \xrightarrow{\mathscr{L}} (u^{H_0}, B^{H_0})$$

in the space of continuous functions $(\mathscr{C}([0,T]))^2$ where $H \to H_0$.
Since the function $f : [0,T] \to \mathbb{R}_+$ is continuous, we can assume that it takes values on an interval $[0,T']$ such that $[0,T] \subset [0,T']$.

Define the following functional

$$\begin{array}{rccccl}
\Psi: & \mathscr{C}([0,T']) & \longrightarrow & (\mathscr{C}([0,T]))^2 & & \\
& x & \mapsto & \Psi(x) := & (x\circ f,\ x_{|_{[0,T]}}), &
\end{array}$$

where $x_{|_{[0,T]}}$ denotes the restriction of x to the interval $[0,T]$.

It is easily checked that Ψ is continuous. Moreover, we have

$$(u^H, B^H) = (B^H \circ f, B^H) = \Psi(B^H).$$

Since $\{B^H\}$ converges in law to B^{H_0} in the space $\mathscr{C}([0,T'])$ and Ψ is continuous, we deduce that

$$(u^H, B^H) \xrightarrow{\mathscr{L}} (u^{H_0}, B^{H_0}),$$

in $(\mathscr{C}([0,T]))^2$
We have checked that the assumptions of Theorem 3.13 are satisfied and then the convergence in law that we wanted to prove follows. □

5 Proof of Proposition 2.3

Proof. The proof of this proposition will be decomposed in two steps.
From property (1) of the Skorohod integral we have that for all $\varepsilon > 0$

$$
\begin{aligned}
\frac{1}{2\varepsilon}\int_0^t u_s(B^{1|2}_{s+\varepsilon} - B^{1|2}_{s-\varepsilon})ds &= \frac{1}{2\varepsilon}\int_0^t u_s\delta^{1|2}(\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(\cdot))ds \\
&= \frac{1}{2\varepsilon}\int_0^t \delta^{1|2}(u_s\mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(\cdot))ds + \frac{1}{2\varepsilon}\int_0^t \langle D^{1|2}_{\boldsymbol{\cdot}} u_s, \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(\cdot)\rangle_{L^2([0,T])}ds \\
&= \delta^{1|2}((u\mathbf{1}_{[0,t]})^{\varepsilon}) + \frac{1}{2\varepsilon}\int_0^t \langle D^{1|2}_{\boldsymbol{\cdot}} u_s, \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(\cdot)\rangle_{L^2([0,T])}ds \\
&= \delta^{1|2}((u\mathbf{1}_{[0,t]})^{\varepsilon}) + \frac{1}{2\varepsilon}\int_0^t \int_{s-\varepsilon}^{s+\varepsilon} D^{1|2}_r u_s dr ds \qquad (40)
\end{aligned}
$$

where we have used the notation

$$
v^{\varepsilon}_s = \frac{1}{2\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon} v_x dx.
$$

- Step 1:

  For any $u \in \mathbb{D}^{1,2}(L^2([0,T]))$, and $t \in [0,T]$, the family $(u\mathbf{1}_{[0,t]})^{\varepsilon}$ approximates $u\mathbf{1}_{[0,t]}$ in the space $\mathbb{D}^{1,2}(L^2([0,T]))$ and as a consequence we obtain the convergence in $L^2(\Omega)$ of the indefinite Skorohod integral:

  $$
  \delta^{1|2}((u\mathbf{1}_{[0,t]})^{\varepsilon}) \xrightarrow{\varepsilon\to 0} \delta^{1|2}(u\mathbf{1}_{[0,t]}).
  $$

  In order to show this, we will prove that for any $u \in \mathbb{D}^{1,2}(L^2([0,T]))$ we have that

  $$
  E\big(\|u^{\varepsilon} - u\|^2_{L^2([0,T])}\big) \longrightarrow 0, \quad \text{as } \varepsilon \to 0 \qquad (41)
  $$

  and

  $$
  E\big(\|D^{1|2}u^{\varepsilon} - D^{1|2}u\|_{L^2([0,T]^2)}\big) \longrightarrow 0, \quad \text{as } \varepsilon \to 0. \qquad (42)
  $$

  We will only see (42), the proof of (41) is similar but simpler.

  Observe that for each $r \in [0,T]$

  $$
  D^{1|2}_r u^{\varepsilon} = D^{1|2}_r u * \varphi_{\varepsilon}
  $$

  and then, using Young's inequality for convolutions, we have that

  $$
  \|D^{1|2}_r u^{\varepsilon}\|_{L^2([0,T])} \leqslant \|D^{1|2}_r u\|_{L^2([0,T])}\,\|\varphi_{\varepsilon}\|_{L^1([0,T])} = \|D^{1|2}_r u\|_{L^2([0,T])}\,,
  $$

  which implies that

  $$
  \sup_{0<\varepsilon\leqslant 1}\int_0^T\int_0^T |D^{1|2}_r u^{\varepsilon}_s - D^{1|2}_r u_s|^2 ds\, dr \leqslant 2\,\|D^{1|2}u\|^2_{L^2([0,T]^2)} \in L^1(\Omega).
  $$

So, if we show that

$$\|D^{1|2}u^{\varepsilon} - D^{1|2}u\|^2_{L^2([0,T]^2)} \longrightarrow 0 \quad \text{almost surely,} \tag{43}$$

we can conclude by using the Dominated Convergence Theorem. But, for any $r \in [0,T]$ and any $\omega \in \Omega$

$$\int_0^T |D_r^{1|2} u_s^{\varepsilon} - D_r^{1|2} u_s|^2 ds \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

Moreover

$$\sup_{0<\varepsilon\leqslant 1} \int_0^T |D_{\cdot}^{1|2} u_s^{\varepsilon} - D_{\cdot}^{1|2} u_s|^2 ds \leqslant 2 \int_0^T |D_{\cdot}^{1|2} u_s|^2 ds \in L^1([0,T]),$$

therefore (43) follows by dominated convergence and this finishes the proof of (42).

- Step 2:

  We will study the convergence in $L^1(\Omega)$ of the second summand of (40) to the trace term, that is:

$$\frac{1}{2\varepsilon}\int_0^t \langle D_{\cdot}^{1|2} u, \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\rangle_{L^2([0,T])} ds = \frac{1}{2\varepsilon}\int_0^t \int_0^T D_r^{1|2} u_s \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}(r) dr ds \longrightarrow \int_0^t \frac{1}{2}\nabla_r^{1|2} u_r dr,$$

  when $\varepsilon \to 0$. Recall that

$$\nabla_r^{1|2} u_r = D_r^{1|2,-} u_r + D_r^{1|2,+} u_r.$$

  Taking into account this identity, we will insert the following term

$$D_r^{1|2,-} u_r \mathbf{1}_{\{r<s<r+\varepsilon\}} + D_r^{1|2,+} u_r \mathbf{1}_{\{r-\varepsilon<s<r\}}$$

  into

$$E\Big|\int_0^t \frac{1}{2\varepsilon}\left(\int_{r-\varepsilon}^{r+\varepsilon} D_r^{1|2} u_s ds\right) dr - \int_0^t \frac{1}{2}\nabla_r^{1|2} u_r dr\Big|$$

  and we have that this last expression can be bounded by

$$\begin{aligned} &E\Big|\int_0^t \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} (D_r^{1|2} u_s - D_r^{1|2,-} u_r \mathbf{1}_{\{r<s<r+\varepsilon\}} - D_r^{1|2,+} u_r \mathbf{1}_{\{r-\varepsilon<s<r\}}) ds\right) dr\Big| \\ &+ E\Big|\int_0^t \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} (D_r^{1|2,-} u_r \mathbf{1}_{\{r<s<r+\varepsilon\}} + D_r^{1|2,+} u_r \mathbf{1}_{\{r-\varepsilon<s<r\}} - \frac{1}{2}\nabla_r^{1|2} u_r) ds\right) dr\Big|. \end{aligned} \tag{44}$$

  The first summand of the right-hand side tends to zero when $\varepsilon \to 0$. In fact, we can write

$$\begin{aligned} &E\Big|\int_0^t \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} (D_r^{1|2} u_s^{1|2} - D_r^{1|2,-} u_r^{1|2} \mathbf{1}_{\{r<s<r+\varepsilon\}} - D_r^{1|2,+} u_r^{1|2} \mathbf{1}_{\{r-\varepsilon<s<r\}}) ds\right) dr\Big| \\ &\quad \leqslant E\Big|\int_0^t \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r} (D_r^{1|2} u_s^{1|2} - D_r^{1|2,-} u_r^{1|2}) ds\right) dr\Big| \\ &\quad + E\Big|\int_0^t \left(\frac{1}{2\varepsilon}\int_{r}^{r+\varepsilon} (D_r^{1|2} u_s^{1|2} - D_r^{1|2,+} u_r^{1|2}) ds\right) dr\Big| \\ &\quad \leqslant \int_0^t \sup_{r-\varepsilon<s<r} E|D_r^{1|2} u_s^{1|2} - D_r^{1|2,-} u_r^{1|2}| dr + \int_0^t \sup_{r<s<r+\varepsilon} E|D_r^{1|2} u_s^{1|2} - D_r^{1|2,+} u_r^{1|2}| dr. \end{aligned}$$

Hence, conditions (4) and (5) imply that this term tends to zero when $\varepsilon \to 0$.

On the other hand, one can find a properly expression for the second summand of (44) and check that it is equal to zero

$$
\begin{aligned}
&\Big|\int_0^t \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} (D_r^{1|2,-}u_r \mathbf{1}_{\{r<s<r+\varepsilon\}} + D_r^{1|2,+}u_r \mathbf{1}_{\{r-\varepsilon<s<r\}} - \frac{1}{2}\nabla_r u_r) ds\right) dr\Big| \\
&\quad = \Big|\int_0^t \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} D_r^{1|2,-}u_r \mathbf{1}_{\{r<s<r+\varepsilon\}} ds\right) dr - \frac{1}{2}\int_0^t D_r^{1|2,-}u_r dr \\
&\quad + \int_0^t \left(\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} D_r^{1|2,+}u_r \mathbf{1}_{\{r-\varepsilon<s<r\}} ds\right) dr - \frac{1}{2}\int_0^t D_r^{1|2,+}u_r dr\Big| \\
&\quad = 0.
\end{aligned}
$$

Thus, we obtain the desired convergence in $L^1(\Omega)$ of the trace term

$$
\frac{1}{2\varepsilon}\int_0^t \langle D_{\cdot}^{1|2} u, \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\rangle_{L^2([0,T])} ds \longrightarrow \int_0^t \frac{1}{2}\nabla_r^{1|2} u_r dr,
$$

when $\varepsilon \to 0$.

□

Acknowledgement

The authors are supported by the grant MEC-FEDER Ref. MTM2009-08869 from the Dirección General de Investigación, Ministerio de Educación y Ciencia, Spain.

References

[1] Alòs, E., Mazet, O., Nualart, D. Stochastic calculus with respect to fractional Brownian motion with Hurst parameter lesser than $\frac{1}{2}$ *Stochastic Process. Appl.*, 86 , no. 1, 121–139, 2000.

[2] Alòs, E., Mazet, O., Nualart, D. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, 29, no. 2, 766–801, 2001.

[3] Alòs, E., Nualart, D. Stochastic integration with respect to the fractional Brownian motion. *Stochastics and Stochastics Reports*, 75(3):129–152, 2003.

[4] Billingsley, P. *Convergence of probability measures*. John Wiley & Sons Inc., New York, 1968.

[5] Decreusefond, L., Üstünel, A. S. Stochastic analysis of the fractional Brownian motion. *Potential Anal.*, 10, no. 2, 177–214, 1999.

[6] Jolis, M. On the Wiener integral with respect to the fractional Brownian motion on an interval. *J. Math. Anal. Appl.* 330, no. 2, 1115–1127, 2007.

[7] Jolis, M., Viles, N. Continuity with respect to the Hurst parameter of the laws of the multiple fractional integrals. *Stochastic Process. Appl.* 117, no. 9, 1189–1207, 2007.

[8] Jolis, M., Viles, N. Continuity in law with respect to the Hurst parameter of the local time of the fractional Brownian motion. *J. Theoret. Probab.* 20, no. 2, 133–152, 2007

[9] Jolis, M., Viles, N. Continuity in the Hurst parameter of the law of the Wiener integral with respect to the fractional Brownian motion. *Statist. Probab. Lett.*, doi:10.1016/j.spl.2009.12.011

[10] Nualart, D. The Malliavin calculus and related topics, Probability and its Applications (New York), Second Edition, Springer-Verlag, Berlin, 2006.

[11] Nualart, D.; Pardoux, É. Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields* 78 (1988), no. 4, pp. 535–581

[12] Pipiras, V., Taqqu, M. S. Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli* 7, no. 6, 873–897, 2001.

[13] Russo, P.; Vallois, P. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields* 97 (1993), pp. 403–421