HEISENBERG UNIQUENESS PAIRS IN THE PLANE. THREE PARALLEL LINES.

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Abstract. A Heisenberg uniqueness pair is a pair (Γ, Λ), where Γ is a curve in the plane and Λ is a set in the plane, with the following property: any bounded Borel measure μ in the plane supported on Γ, which is absolutely continuous with respect to arc length, and whose Fourier transform $\widehat{\mu}$ vanishes on Λ, must automatically be the zero measure. We characterize the Heisenberg uniqueness pairs for Γ being three parallel lines $\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\}$ with $\alpha < \beta < \gamma$, $(\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}$.

1. Introduction

The Heisenberg uncertainty principle states that both a function and its Fourier transform cannot be too localized at the same time (see (2) and (3)). M. Benedicks in (1) proved that given a non trivial function $f \in L^1(\mathbb{R}^n)$, the Lebesgue measure of the set of points where $f \neq 0$ and the set of points where the Fourier transform $\widehat{f} \neq 0$ can not be simultaneously finite. In this paper we consider a similar problem for measures supported on a subset of $\mathbb{R}^2$.

Let Γ be a smooth curve in the plane $\mathbb{R}^2$ and Γ a subset in $\mathbb{R}^2$. In (4), Hedenmalm and Montes-Rodríguez posed the problem of deciding when is it true that

$$\widehat{\mu}_{|\Lambda} = 0 \text{ implies } \mu = 0$$

for any Borel measure μ supported on Γ and absolutely continuous with respect to the arc length measure on Γ, where

$$\widehat{\mu}(\xi, \eta) = \int_{\mathbb{R}^2} e^{\pi i <(x,y),(\xi,\eta)>} d\mu(x, y).$$

If this is the case then (Γ, Λ) is called a *Heisenberg Uniqueness Pair* (HUP).

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When Γ is the circle Lev (6) and Sjölin (7) independently, characterized the HUP for some "small" sets Λ.

In (4) Hedenmalm and Montes-Rodríguez characterized the HUP in the cases

- Γ is the hyperbola $xy = 1$ and $\Lambda = (\alpha\mathbb{Z} \times \{0\}) \cup (\{0\} \times \beta\mathbb{Z})$, for $\alpha, \beta > 0$.
- Γ two parallel lines in $\mathbb{R}^2$

In this note we present a result generalizing this last case. We characterize the HUP for Γ being three parallel lines

$$\Gamma = \mathbb{R} \times \{\alpha, \beta, \gamma\} \text{ with } \alpha < \beta < \gamma,\ (\gamma - \alpha)/(\beta - \alpha) \in \mathbb{N}.$$

2. Three parallel lines

Given a set $E \subset \mathbb{R}$ and a point $\xi \in E$, let define

- $\mathcal{A}^E_{loc}(\xi) = \{$ Functions χ defined on E such that there exist a small interval I_ξ around ξ and a function $\varphi \in L^1(\mathbb{R})$ such that $\chi(\zeta) = \widehat{\varphi}(\zeta),$ for $\zeta \in I_\xi \cap E\}$.

- $P^{1,p}[\mathcal{A}^E_{loc}](\xi) = \{$ Functions χ defined on E such that there exist an interval I_ξ around ξ and functions $\varphi_0, \varphi_1 \in L^1(\mathbb{R})$ with $\chi^p(\zeta) + \widehat{\varphi}_1(\zeta)\chi(\zeta) + \widehat{\varphi}_0(\zeta) = 0$, for $\zeta \in I_\xi \cap E\}$.

Wiener's lemma (5, p. 57) states that if $\chi \in \mathcal{A}^E_{loc}(\xi)$ and $\chi(\xi) \neq 0$ then $1/\chi \in \mathcal{A}^E_{loc}(\xi)$. Observe also that if $\chi \in \mathcal{A}^E_{loc}(\xi)$ then $\chi \in P^{1,p}[\mathcal{A}^E_{loc}](\xi)$.

Due to invariance under translation and rescaling (see (4)) it will be sufficient to study the case when $\Gamma = \mathbb{R} \times \{0, 1, p\}$ for $p \in \mathbb{N}$, $p > 1$.

Given a set $\Lambda \subset \mathbb{R}^2$, we say that μ is an *admissible measure* if μ is a Borel measure in the plane absolutely continuous with respect to arc length with $supp\mu \subset \Gamma$ and $\widehat{\mu}_{|\Lambda} = 0$.

If μ is an admissible measure then there exist functions $f, g, h \in Ł^1(\mathbb{R})$ such that

$$\widehat{\mu}(\xi, \eta) = \widehat{f}(\xi) + e^{\pi i \eta}\widehat{g}(\xi) + e^{p\pi i \eta}\widehat{h}(\xi), \quad \text{for any} \quad (\xi, \eta) \in \mathbb{R}^2.$$

Observe also that $\widehat{\mu}$ is 2-periodic with respect to the second variable. So, for any set $\Lambda \subset \mathbb{R}^2$, we may consider the periodized set

$$\mathcal{P}(\Lambda) = \{(\xi, \eta) \text{ such that } (\xi, \eta + 2k) \in \Lambda \text{ for some } k \in \mathbb{Z}\}$$

and it follows that (Γ, Λ) is a HUP if and only if $(\Gamma, \overline{\mathcal{P}(\Lambda)})$ is a HUP, where $\overline{\mathcal{P}(\Lambda)}$ stands fore the closure of $\mathcal{P}(\Lambda)$ in $\mathbb{R}^2$.

We may think without loss of generality that Λ is a closed set in $\mathbb{R}^2$, 2-periodic with respect to the second coordinate.

We then have the following result.

Theorem 1. *Let $\Gamma = \mathbb{R} \times \{0, 1, p\}$, for some $p \in \mathbb{N}$, $p > 1$ and $\Lambda \subset \mathbb{R}^2$, closed and 2-periodic with respect to the second variable, then (Γ, Λ) is a Heisenberg uniqueness pair if and only if*

$$\mathfrak{F} := \Pi^3(\Lambda) \cup (\Pi^2(\Lambda) \setminus \Pi^{2^*}(\Lambda)) \cup (\Pi^1(\Lambda) \setminus \Pi^{1^*}(\Lambda)) \tag{2.1}$$

is dense in $\mathbb{R}$.

$\Pi(\Lambda)$ means the projection of Λ on the axe $\mathbb{R} \times \{0\}$ and given a point $\xi \in \Pi(\Lambda)$, $Img(\xi)$ corresponds to the set of points $\eta \in [0, 2)$ with $(\xi, \eta) \in \Lambda$. The sets in $\mathfrak{F}$ are defined as follows:

- $\Pi^1(\Lambda) = \{\, \xi \in \Pi(\Lambda)$ such that there is a unique $\eta_0 \in Img(\xi)\}$
- $\Pi^{1^*}(\Lambda) = \{\, \xi \in \Pi^1(\Lambda)$ such that $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda)}](\xi)\,\}$, where $\chi(\xi) = e^{\pi i \eta_0}$
- $\Pi^2(\Lambda) = \{\, \xi \in \Pi(\Lambda)$ such that there are two different points $\eta_0, \eta_1 \in Img(\xi)$ and if there is another point $\eta_2 \in Img(\xi)$ then $\frac{e^{p\pi i\eta_1} - e^{p\pi i\eta_0}}{e^{\pi i\eta_1} - e^{\pi i\eta_0}} = \frac{e^{p\pi i\eta_2} - e^{p\pi i\eta_0}}{e^{\pi i\eta_2} - e^{\pi i\eta_0}}\,\}$
- $\Pi^{2^*}(\Lambda) = \{\, \xi \in \Pi^2(\Lambda)$ such that $\tau, \Phi \in \mathcal{A}_{loc}^{\Pi^2(\Lambda)}(\xi)$, where $\tau(\xi) = \frac{e^{p\pi i\eta_1} - e^{p\pi i\eta_0}}{e^{\pi i\eta_1} - e^{\pi i\eta_0}}$ and $\Phi(\xi) = e^{\pi i\eta_0}\frac{e^{p\pi i\eta_1} - e^{p\pi i\eta_0}}{e^{\pi i\eta_1} - e^{\pi i\eta_0}} - e^{p\pi i\eta_0}\,\}$
- $\Pi^3(\Lambda) = \{\, \xi \in \Pi(\Lambda)$ such that there are at least three different points $\eta_0, \eta_1, \eta_2 \in Img(\xi)$ with $\frac{e^{p\pi i\eta_1} - e^{p\pi i\eta_0}}{e^{\pi i\eta_1} - e^{\pi i\eta_0}} \neq \frac{e^{p\pi i\eta_2} - e^{p\pi i\eta_0}}{e^{\pi i\eta_2} - e^{\pi i\eta_0}}\,\}$

The following technical lemma is easy to prove and shows that the functions τ and Φ are well defined for $\xi \in \Pi^2(\Lambda)$.

Lemma 2. *Let $x, y, z \in \mathbb{C}$ different with*

$$\tau = \frac{y^p - x^p}{y - x} = \frac{z^p - x^p}{z - x},$$

then

$$\frac{z^p - y^p}{z - y} = \tau \qquad \text{and} \qquad \Phi = x\tau - x^p = y\tau - y^p = z\tau - z^p.$$

Next lemma will be needed for the proof of the necessity of condition (2.1) in Theorem 1.

Lemma 3. *Let I be an interval in $\mathbb{R}$ with $\Pi^{2^*}(\Lambda)$ dense in I, then there exists a subinterval $I' \subset I$ with $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$.*

Proof. Pick an arbitrary point $\widetilde{\xi} \in I \cap \Pi^{2^*}(\Lambda)$, since $\tau, \Phi \in \mathcal{A}_{loc}^{\Pi^2(\Lambda)}(\widetilde{\xi})$ and $\Pi^{2^*}(\Lambda)$ is dense in I, we can extend the functions τ, Φ continuously on a neighborhood of $\widetilde{\xi}$. Let $\widetilde{\eta} \neq \widetilde{\varrho} \in Img(\widetilde{\xi})$, then

$$|\tau(\widetilde{\xi})| = \left| \frac{e^{p\pi i\widetilde{\eta}} - e^{p\pi i\widetilde{\varrho}}}{e^{\pi i\widetilde{\eta}} - e^{\pi i\widetilde{\varrho}}} \right| < p,$$

and since τ is continuous around $\widetilde{\xi}$, there exists a small interval I' around $\widetilde{\xi}$ with $|\tau(\xi)| < p$ for $\xi \in I'$. We will see that $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$.

Given $\xi \in I'$, consider a sequence $\{\xi_k\} \subset \Pi^{2^*}(\Lambda) \cap I'$ with $\xi_k \to \xi$, and for each ξ_k let $\eta_k \neq \varrho_k \in Img(\xi_k)$. There exist subsequences $\{\eta_k^*\}$ and $\{\varrho_k^*\}$ such that $\eta_k^* \to \eta^*$ and $\varrho_k^* \to \varrho^*$ for some $\eta^*, \varrho^* \in [0,2]$. Since the set Λ is closed and $2-$periodic with respect to the second coordinate, we may assume WLOG that $\xi \in \Pi(\Lambda)$ with $\eta^* \neq \varrho^* \in Img(\xi)$. Otherwise,

$$\begin{aligned} |\tau(\xi)| \quad &\longleftarrow |\tau(\xi_k^*)| = \left| e^{(p-1)\pi i\eta_k^*} + e^{(p-2)\pi i\eta_k^*} e^{\pi i\varrho_k^*} + \cdots + e^{(p-1)\pi i\varrho_k^*} \right| \\ &\longrightarrow \left| e^{(p-1)\pi i\eta^*} + e^{(p-1)\pi i\eta^*} + \cdots + e^{(p-1)\pi i\eta^*} \right| = p, \end{aligned}$$

which is a contradiction with the fact that $\xi \in I'$.

So $I' \subset \Pi^2(\Lambda) \cup \Pi^3(\Lambda)$ and since the extended functions τ, Φ are continuous on I' we also have that $\xi \in \Pi^{2^*}(\Lambda)$ for any $\xi \in \Pi^2(\Lambda) \cap I'$, and we can conclude that $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$. □

3. Proof of the main result

This section is devoted to the proof of theorem 1. The proof of the sufficiency of condition (2.1) is rather easy. Let μ be an admissible weight, then there exist functions $f, g, h \in Ł^1(\mathbb{R})$ such that

$$\widehat{\mu}(\xi,\eta) = \widehat{f}(\xi) + e^{\pi i\eta}\widehat{g}(\xi) + e^{p\pi i\eta}\widehat{h}(\xi), \quad \text{for any} \quad (\xi,\eta) \in \mathbb{R}^2.$$

Since $\mathfrak{F}$ is dense in $\mathbb{R}$ we will be done if we show that $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$ for any $\xi \in \mathfrak{F} = \Pi^3(\Lambda) \cup (\Pi^2(\Lambda) \setminus \Pi^{2^*}(\Lambda)) \cup (\Pi^1(\Lambda) \setminus \Pi^{1^*}(\Lambda))$.

If $\xi \in \Pi^3(\Lambda)$, let $\eta_0, \eta_1, \eta_2 \in Img(\xi)$ different. Since $\widehat{\mu}_{|\Lambda} = 0$ and $\frac{e^{p\pi i\eta_1} - e^{p\pi i\eta_0}}{e^{\pi i\eta_1} - e^{\pi i\eta_0}} \neq \frac{e^{p\pi i\eta_2} - e^{p\pi i\eta_0}}{e^{\pi i\eta_2} - e^{\pi i\eta_0}}$ it follows that $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$.

If $\xi \in \Pi^2(\Lambda)$, let $\eta_0 \neq \eta_1 \in Img(\xi)$. Since $\widehat{\mu}_{|\Lambda} = 0$ then $\widehat{g}(\xi) = -\tau(\xi)\widehat{h}(\xi)$ and $\widehat{f}(\xi) = \Phi(\xi)\widehat{h}(\xi)$. Suppose $\widehat{h}(\xi) \neq 0$, then by Wiener's lemma $\tau, \Phi \in \mathcal{A}_{loc}^{\Pi^2(\Lambda)}(\xi)$ which implies that $\xi \in \Pi^{2^*}(\Lambda)$. So if $\xi \in \Pi^2(\Lambda) \setminus \Pi^{2^*}(\Lambda)$ then $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$.

Finally, if $\xi \in \Pi^1(\Lambda)$ and $\eta_0 \in Img(\xi)$, since $\widehat{\mu}_{|\Lambda} = 0$ then $\widehat{f}(\xi) + \chi(\xi)\widehat{g}(\xi) + \chi^p(\xi)\widehat{h}(\xi) = 0$, where $\chi(\xi) = e^{\pi i\eta_0}$. Suppose $\widehat{h}(\xi) \neq 0$, then $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda)}](\xi)$ and $\xi \in \Pi^{1^*}(\Lambda)$. Otherwise, if $\widehat{h}(\xi) = 0$ and $\widehat{g}(\xi) \neq 0$

then, by Wiener's lemma $\chi \in \mathcal{A}_{loc}^{\Pi^1(\Lambda)}(\xi)$ and also $\chi^p \in \mathcal{A}_{loc}^{\Pi^1(\Lambda)}(\xi)$, so $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda)}](\xi)$ and $\xi \in \Pi^{1^*}(\Lambda)$. This means that if $\xi \in \Pi^1(\Lambda) \setminus \Pi^{1^*}(\Lambda)$ then $\widehat{f}(\xi) = \widehat{g}(\xi) = \widehat{h}(\xi) = 0$.

For the proof of the necessity of condition (2.1), suppose that the set $\mathfrak{F}$ is not dense in $\mathbb{R}$ and let us pick an open interval I that has empty intersection with $\mathfrak{F}$, ie,

$$\Pi(\Lambda) \cap I = (\Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)) \cap I.$$

We consider three cases:

- There exists a small interval $I_\xi \subset I$ around $\xi \in \Pi^{1^*}(\Lambda)$ such that all the points in $I_\xi \cap \Pi(\Lambda)$ belong to $\Pi^{1^*}(\Lambda)$. Since $\chi \in P^{1,p}[\mathcal{A}_{loc}^{\Pi^1(\Lambda)}](\xi)$, there exist an interval $I' \subset I_\xi$ around ξ and functions $\varphi_0, \varphi_1 \in L^1(\mathbb{R})$ such that

$$\chi^p(\xi^*) + \widehat{\varphi}_1(\xi^*)\chi(\xi^*) + \widehat{\varphi}_0(\xi^*) = 0$$

for any $\xi^* \in I' \cap \Pi(\Lambda)$. Let $h \in L^1(\mathbb{R})$ with $\widehat{h}(\xi) \neq 0$ and $supp\widehat{h} \Subset I'$, and define $f, g \in L^1(\mathbb{R})$ via $\widehat{f} = \widehat{h}\widehat{\varphi_0}$, and $\widehat{g} = \widehat{h}\widehat{\varphi_1}$. Now,

$$\widehat{\mu}(\xi^*, \eta^*) = \quad \widehat{f}(\xi^*) + \widehat{g}(\xi^*)\chi(\xi^*) + \widehat{h}(\xi^*)\chi^p(\xi^*) = 0$$

for $\xi^* \in I' \cap \Pi^{1^*}(\Lambda)$, $\eta^* \in Img(\xi^*)$. Finally, since $supp\widehat{h} \Subset I'$ and $I' \cap \Pi(\Lambda) = I' \cap \Pi^{1^*}(\Lambda)$, we can conclude that $\widehat{\mu}_{|\Lambda} \equiv 0$ and we have that μ is a non trivial admissible measure. So (Γ, Λ) is not a Heisenberg uniqueness pair.

- There exists a small interval $I_\xi \subset I$ around $\xi \in \Pi^{2^*}(\Lambda)$ such that all the points in $I_\xi \cap \Pi(\Lambda)$ belong to $\Pi^{2^*}(\Lambda)$. Now there exists a small interval $I' \subset I_\xi$ around ξ and functions $\Phi_1, \tau_1 \in L^1(\mathbb{R})$ such that $\widehat{\tau_1} = \tau$ and $\widehat{\Phi_1} = \Phi$ on $I' \cap \Pi(\Lambda)$. Consider a function $h \in L^1(\mathbb{R})$ with $sppt\, \widehat{h} \subset I'$ and $\widehat{h}(\xi) \neq 0$ and define $f, g \in L^1(\mathbb{R})$ as

$$g = -h * \tau_1 \quad \text{and} \quad f = h * \Phi_1.$$

Now, given a point $\xi^* \in I' \cap \Pi^{2^*}(\Lambda)$, let $\eta^* \neq \varrho^* \in Img(\xi^*)$. Since $\tau(\xi^*) = \frac{e^{p\pi i\eta^*} - e^{p\pi i\varrho^*}}{e^{\pi i\eta^*} - e^{\pi i\varrho^*}}$ and $\Phi(\xi) = e^{\pi i\eta^*}\frac{e^{p\pi i\eta^*} - e^{p\pi i\varrho^*}}{e^{\pi i\eta^*} - e^{\pi i\varrho^*}} - e^{p\pi i\varrho^*}$, we have that

$$\widehat{\mu}(\xi^*, \eta^*) = \quad \widehat{f}(\xi^*) + \widehat{g}(\xi^*)e^{\pi i\eta^*} + \widehat{h}(\xi^*)e^{p\pi i\eta^*} = 0$$

and also that $\widehat{\mu}(\xi^*, \varrho^*) = 0$. So, the corresponding measure μ is a nontrivial admissible measure and (Γ, Λ) is not a Heisenberg uniqueness pair.

- All the intervals $I_\xi \subset I$ contain points in $\Pi^{1^*}(\Lambda)$ and points in $\Pi^{2^*}(\Lambda)$. Ie, the sets $\Pi^{1^*}(\Lambda)$ and $\Pi^{2^*}(\Lambda)$ are dense in $I \cap (\Pi^{1^*}(\Lambda) \cup \Pi^{2^*}(\Lambda)) = I \cap \Pi(\Lambda)$. But this is not possible. In fact, if $\Pi^{2^*}(\Lambda)$ is dense in I, by Lemma 3, there exists a subinterval $I' \subset I$ such that $I' \subset \Pi^{2^*}(\Lambda) \cup \Pi^3(\Lambda)$.

This finishes the proof of the theorem.

4. Examples and further results

Given a point $\xi \in \Pi(\Lambda)$ such that $\sharp\{\eta \in Img(\xi)\} \geq 3$, we will state a criteria to decide whether the point ξ belongs to $\Pi^3(\Lambda)$ or $\Pi^2(\Lambda)$. But before it we prove the following lemma.

Lemma 4. *Given $C \in \mathbb{C}$, there exist at most p different points $\rho^{(k)} \in [0, 2)$, such that for any $j \neq k$,*

$$\frac{x^p - y^p}{x - y} = C, \quad \text{where} \quad x = e^{\pi i \rho^{(k)}},\ y = e^{\pi i \rho^{(j)}}. \tag{4.1}$$

Proof. Observe that fixed C, there exists a constant $C^* \in \mathbb{C}$ such that

$$xC - x^p = C^* \tag{4.2}$$

for any $x = e^{\pi i \rho^{(k)}}$ solution of (4.1). Now it is obvious that there are at most p different solutions $\rho^{(k)} \in [0, 2)$ of the equation (4.2).

□

Corollary 5. *Given a point $\xi \in \Pi(\Lambda)$, if $\sharp\{\eta \in Img(\xi)\} > p$ then $\xi \in \Pi^3(\Lambda)$.*

In particular if Γ consists on three parallel equidistant lines in the plane ($p = 2$) we have

$$\Pi^3(\Lambda) = \{\, \xi \in \Lambda \text{ such that } \sharp\{\eta \in Img(\xi)\} \geq 3 \,\}$$
$$\Pi^2(\Lambda) = \{\, \xi \in \Lambda \text{ such that } \sharp\{\eta \in Img(\xi)\} = 2 \,\}$$

Example 6. *The following example shows that corollary 5 is sharp.*

- *Let $\Lambda = \mathbb{R} \times \{2k/p\}_{k=0,\cdots,p-1}$, then for any $\xi \in \mathbb{R}$,*

$$\sharp\{\eta \in Img(\xi)\} = p$$

and $\xi \in \Pi^{2^*}(\Lambda)$. Observe that in this case, (Γ, Λ) is not a HUP.

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