FOLIATIONS AND WEBS INDUCING GALOIS COVERINGS

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Abstract. Motivated by a previous work of Cerveau and Déserti, we introduce the notion of Galois holomorphic foliation on the complex projective space as those whose Gauss map is a Galois covering when restricted to an appropriate Zariski open subset. First, we establish general criteria assuring that a rational map between projective manifolds of the same dimension defines a Galois covering. Then, these criteria are used to characterize Galois foliations on $\mathbb{P}^2$ belonging to certain classes, which include homogeneous foliations. We also give a geometric characterization of Galois foliations in terms of their inflection divisor and their singularities.

1. Introduction

In this article we introduce the notion of Galois holomorphic foliation on the complex projective space. Our main objective is to establish general criteria characterizing those foliations that are Galois.

Focusing on the two dimensional case considered in [9], let $\mathcal{F}$ be a holomorphic foliation in the complex projective plane $\mathbb{P}^2$. The degree $\deg \mathcal{F}$ of the foliation is the number of tangencies of $\mathcal{F}$ with a generic line of $\mathbb{P}^2$ and the Gauss map $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$ of the foliation, sending $x \in \mathbb{P}^2$ into the tangent line of $\mathcal{F}$ at x, is a well defined rational map whose indeterminacy points are just the singularities of the foliation. If the foliation is non degenerated then the restriction of $\mathcal{G}_{\mathcal{F}}$ to a suitable Zariski open subset W of $\mathbb{P}^2$ is a covering map of order $\deg \mathcal{F} > 0$.

The determination of finite subgroups of the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ of birational transformations of $\mathbb{P}^2$ is a classical topic, nevertheless it continues to be an active field of current research (cf. [2, 14]). In [9], Cerveau and Deserti addressed the problem of finding non-trivial birational deck transformations of the covering associated to a foliation $\mathcal{F}$, that is, birational maps $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ fulfilling $\mathcal{G}_{\mathcal{F}} \circ \tau = \mathcal{G}_{\mathcal{F}}$. Their aim was to construct periodic elements of $\mathrm{Bir}(\mathbb{P}^2)$ in an effective way. In particular they associated a birational involution to each quadratic foliation and trivolutions to certain classes of cubic foliations. In all these cases the restriction of $\mathcal{G}_{\mathcal{F}}$ to the Zariski open set W is necessarily a Galois covering. It is therefore a

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natural question to determine the Galois foliations of $\mathbb{P}^2$, that is those foliations in $\mathbb{P}^2$ whose Gauss map defines a Galois covering. This is the original purpose of this article. We will see that for every Galois foliation the deck transformations of its Gauss map are birational. This fact provides non-trivial parametric realizations of the symmetry groups of regular polyhedra into the Cremona group (cf. Remark 5.22).

We also prove that the set $\mathbb{G}_d$ of Galois foliations of degree d on $\mathbb{P}^2$ is a quasi-projective variety. This raises the question of describing its irreducible components in geometric terms. This problem is of similar nature to the study of the irreducible components of the space of codimension one foliations on $\mathbb{P}^n$ for $n \geq 3$ (cf. [10]) and the study of the irreducible components of the space of flat webs (cf. [26]).

We are specially concerned with the problem of characterizing Galois foliations on $\mathbb{P}^2$ in terms of geometric elements of $\mathcal{F}$. More precisely, in terms of the singular set, the set of inflection points and the generic polar curve, i.e. the preimage by the Gauss map $\mathcal{G}_{\mathcal{F}}$ of a generic line in $\check{\mathbb{P}}^2$. For instance, one of the results that we obtain on this question is the following (cf. Corollary 5.13):

Theorem A. *Let $\mathcal{F}$ be a degree d foliation on $\mathbb{P}^2$. Assume that the following assertions are satisfied:*

(1) at an inflection point p, the tangency order of the leaf through p with its tangent line attains its maximum d;
(2) for each singular point s of $\mathcal{F}$ the following trichotomy holds:
- *s is a non-degenerate radial singularity of order d, or*
- *the generic polar of $\mathcal{F}$ has a single branch at s with multiplicity d, or*
- *s is a non-radial singularity and each branch at s of the generic polar curve of $\mathcal{F}$ is smooth.*

Then $\mathcal{G}_{\mathcal{F}}$ is Galois with cyclic deck transformation group. Moreover, when d is prime the converse is also true.

With the main purpose of studying Galois foliations, we first consider the more general setting of arbitrary rational maps $\mathcal{G}\colon X \dashrightarrow Y$ between complex projective manifolds of the same dimension. Our first result is the following (cf. Theorem 3.11).

Theorem B. *Let $\mathcal{G}\colon X \dashrightarrow Y$ be a dominant rational map between complex connected projective manifolds of the same dimension and let d be its degree. There is a proper Zariski closed subset $\Lambda_{\mathcal{G}}$ of Y such that, if we denote $Y_\Lambda = Y \setminus \Lambda_{\mathcal{G}}$ and $X_\Lambda = \mathcal{G}^{-1}(Y_\Lambda)$, then $\mathcal{G}$ is defined on X_Λ, the morphism $\mathcal{G}|_{X_\Lambda}\colon X_\Lambda \to Y_\Lambda$ is a covering map of degree d and the following property is fulfilled: for every connected and locally path connected subset $V \subset Y_\Lambda$ inducing an epimorphism $\pi_1(V) \twoheadrightarrow \pi_1(Y_\Lambda)$ the map $\mathcal{G}|_U\colon U \to V$, where $U = \mathcal{G}^{-1}(V)$, is a d-sheeted covering, its monodromy group is independent of $V \subset Y$ and every deck transformation of $\mathcal{G}|_U$ extends to a birational map of X.*

As a consequence, the deck transformation group of any of these coverings coincides with the group of birational transformations of X defined as $\mathrm{Deck}(\mathcal{G}) = \{\tau \in \mathrm{Bir}(X) \mid \mathcal{G} \circ \tau = \mathcal{G}\}$. We say that the rational map $\mathcal{G}$ is Galois if $\mathrm{Deck}(\mathcal{G})$ acts transitively on the generic fibre.

The problem of deciding if the rational map $\mathcal{G}\colon X \dashrightarrow Y$ is Galois is simpler for curves since a dominant rational map between algebraic curves is just a branched covering. Hence we are led to reduce the question to the one-dimensional case. We carry out this reduction in different ways along Subsection 3.2. In particular we obtain the following reduction. Using Bertini's theorem and a version of Lefschetz's hyperplane theorem due to Hamm and Lê, we know that there are smooth hyperplane curves $\ell^X \subset X$ and $\ell^Y \subset Y$ such that the natural maps $\pi_1(\ell^X_\Lambda) \to \pi_1(X_\Lambda)$ and $\pi_1(\ell^Y_\Lambda) \to \pi_1(Y_\Lambda)$ are epimorphisms, where $\ell^X_\Lambda = \ell^X \setminus \mathcal{G}^{-1}(\Lambda_\mathcal{G})$ and $\ell^Y_\Lambda = \ell^Y \setminus \Lambda_\mathcal{G}$. We denote by $p\colon \mathcal{G}^*(X_\Lambda) = X_\Lambda \times_{Y_\Lambda} X_\Lambda \to X_\Lambda$ the pull-back covering. In this situation we prove (cf. Theorem 3.16):

Theorem C. *The following assertions are equivalent:*

1. *the rational map $\mathcal{G}\colon X \dashrightarrow Y$ is Galois,*
2. *the restricted covering $\mathcal{G}^{-1}(\ell^Y_\Lambda) \to \ell^Y_\Lambda$ is Galois,*
3. *the restricted pull-back covering $p^{-1}(\ell^X_\Lambda) \to \ell^X_\Lambda$ is trivial.*

A branched covering $\mathcal{G}\colon C \to C'$ between Riemann surfaces is said to be of regular type if for any given $x \in C'$ all the points in $\mathcal{G}^{-1}(x)$ have the same ramification index. If the branched covering is Galois then it is necessarily of regular type but the converse is not true except for particular cases which include $C = \mathbb{P}^1$. A dominant rational map $\mathcal{G}\colon X \dashrightarrow Y$ between projective manifolds of the same dimension is also a branched covering, in the sense of [31], outside a suitable Zariski closed subset of X. In this general case one can also define in a natural way the notion of being of regular type, which takes into account the behavior of the map at the indeterminacy locus. The precise definition is given in Subsection 3.4. Using Theorem C we obtain the following result (cf. Theorem 3.33) which states that, for a concrete rational selfmap of $\mathbb{P}^n$, being Galois could be decided just by studying its ramification and indeterminacy loci.

Theorem D. *A dominant rational map $\mathcal{G}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is Galois if and only if it is of regular type.*

Whereas the pull-back by a dominant rational map of a (singular) foliation, or a web, is always well defined, their push-forward is not. Nevertheless we show that if $\mathcal{G}\colon X \dashrightarrow Y$ is a dominant rational map of degree d between projective manifolds of the same dimension and if $\mathcal{F}$ is a codimension one holomorphic foliation on X, which is in general position with respect to $\mathcal{G}$, then there is a uniquely defined d-web $\mathcal{G}_*\mathcal{F}$ on Y such that, on Y_Λ, $\mathcal{G}_*\mathcal{F}$ is given by the superposition of the d local foliations defined by the covering map $\mathcal{G}|_{X_\Lambda}$ (cf. Proposition 4.6). The precise definition of foliation in general position with respect to a rational map is given in subsection 4.1.

For a given web $\mathcal{W}$ on an projective manifold Y there is a naturally associated projective manifold $Z_\mathcal{W}$ endowed with a foliation $\mathcal{C}_\mathcal{W}$ in general position with respect to a surjective morphism $\pi_\mathcal{W} : Z_\mathcal{W} \to Y$ such that the web $\mathcal{W}$ is the direct image of the foliation $\mathcal{C}_\mathcal{W}$. It is shown that the birational type of the triple $(Z_\mathcal{W}, \mathcal{C}_\mathcal{W}, \pi_\mathcal{W})$ is unique (cf. Theorem 4.8). This property allows us to define Galois webs as those whose associated morphism $\pi_\mathcal{W}$ is a Galois rational map and we prove that every finite group can be realized as the monodromy group of a Galois web (cf. Theorem 4.14).

Recall that a d-web is called totally decomposable if it is the superposition of d global foliations. A characterization of the property for $\mathcal{G}$ of being Galois in terms of foliations in the source space X is given by the following statement (cf. Theorem 4.12).

Theorem E. *Let $\mathcal{G}\colon X \dashrightarrow Y$ be a given dominant rational map between projective manifolds of the same dimension and let $\mathcal{F}$ be a codimension one foliation on X which is in general position with respect to $\mathcal{G}$. Then $\mathcal{G}$ is Galois if and only if the web $\mathcal{G}^*\mathcal{G}_*\mathcal{F}$ on X is totally decomposable.*

We apply this result to the case of the Gauss map of a foliation on the projective space obtaining an explicit family of Galois foliations in each dimension and degree (cf. Proposition 4.20). The construction of these examples are in the spirit of generalizing the goals and methods of [9] to foliations in arbitrary dimension.

We turn back now to the original motivation of this article, that is the characterization of Galois foliations $\mathcal{F}$ in the complex projective plane. In an affine chart, $\mathcal{F}$ is defined by a vector field $X = A(x,y)\partial_x + B(x,y)\partial_y$, where A and B are polynomials. In this situation, Theorem D is equivalent to say that $\mathcal{F}$ is Galois if and only if the polynomial in t

$$P(x,y,t) = \det\begin{pmatrix} A(x,y) & A\big(x+tA(x,y), y+tB(x,y)\big) \\ B(x,y) & B\big(x+tA(x,y), y+tB(x,y)\big) \end{pmatrix}$$

decomposes totally over the field $\mathbb{C}(x,y)$. This criterium, already stated in [9], implies that Galois property defines a Zariski closed set inside the space of degree d foliations. It can be used to determine if a given class of foliations is Galois or not, as the following example shows (cf. Example 5.7).

Example F. *The vector fields*

$$(\alpha u^d + \beta v^d)\partial_x + (\gamma u^d + \delta v^d)\partial_y + (\lambda u^d + \mu v^d)(x\partial_x + y\partial_y)$$

determine a family of degree d Galois foliations on $\mathbb{P}^2$ with cyclic monodromy group, where $(\alpha,\gamma,\lambda), (\beta,\delta,\mu) \in \mathbb{C}^3$ are linearly independent vectors and $u, v \in \mathbb{C}[x,y]$ are $\mathbb{C}$-linearly independent polynomials with $\deg u, \deg v \le 1$.

Then we use Theorem D to give a geometric characterization of Galois foliations on $\mathbb{P}^2$ in terms of its inflection divisor $\mathcal{I}_{\mathcal{F}}$, its singular locus $\Sigma_{\mathcal{F}}$ and the generic polar curve of $\mathcal{F}$ (cf. Theorems 5.11 and 5.12 for the explicit statement) implying in particular Theorem A.

In subsection 5.3 we treat the case of homogeneous foliations in $\mathbb{P}^2$. That is, those foliations whose coefficients A and B are homogeneous polynomials of a same degree d. Such a foliation $\mathcal{F}$ is invariant by the flow associated to the radial vector field and it turns out that in order to decide if $\mathcal{F}$ is Galois we can perform a dimensional reduction in the following way. Let $\widetilde{\mathbb{P}}^2$ be the manifold obtained by blowing up the origin of $\mathbb{C}^2 \subset \mathbb{P}^2$, which is a singularity of the foliation. Then the lift $\widetilde{\mathcal{G}}_{\mathcal{F}}$ to $\widetilde{\mathbb{P}}^2$ of the Gauss map of the foliation is defined on the the exceptional divisor $E \cong \mathbb{P}^1$. Moreover $\widetilde{\mathcal{G}}_{\mathcal{F}}$ maps E onto a copy of $\mathbb{P}^1$ and we prove that the homogeneous foliation $\mathcal{F}$ is Galois if and only if the restricted morphism $\widetilde{\mathcal{G}}_{\mathcal{F}}|_E\colon \mathbb{P}^1 \to \mathbb{P}^1$ is Galois. Then we can use the classification of the Galois ramified coverings of $\mathbb{P}^1$ by itself, that goes back to Klein (cf. Theorem 3.36). The left-right equivalence between rational functions on $\mathbb{P}^1$ preserve Galois property and translates into a natural action of $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ on the space of homogeneous foliations. We obtain the following result (cf. Theorem 5.17).

Theorem G. *The homogeneous Galois foliations of degree d consists of the orbits by the left-right action of $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ of the following ones:*
(1) $x^d\partial_x + y^d\partial_y$ *for every d,*
(2) $(x^n + y^n)^2\partial_x + (x^n - y^n)^2\partial_y$ *if $d = 2n$ is even,*
(3) $(x^4 + 2i\sqrt{3}x^2y^2 + y^4)^3\partial_x + (x^4 - 2i\sqrt{3}x^2y^2 + y^4)^3\partial_y$ *if $d = 12$,*
(4) $(x^8 + 14x^4y^4 + y^8)^3\partial_x + (xy(x^4 - y^4))^4\partial_y$ *if $d = 24$,*
(5) $(x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20})^3\partial_x + (xy(x^{10} + 11x^5y^5 - y^{10}))^5\partial_y$ *if $d = 60$.*

In fact, foliations in Example F are deformations of those in the orbit of $x^d\partial_x + y^d\partial_y$ in the above theorem. Analogous deformations (cf. Definition 5.20) can be considered for the remaining orbits.

More generally, we consider foliations in $\mathbb{P}^2$ admitting a continuous group of automorphisms. In this more general setting we prove that there is a non-constant morphism $\widehat{\mathcal{G}} : \mathbb{P}^1 \to \mathbb{P}^1$, of the same degree than $\mathcal{G}$, so that $\mathrm{Deck}(\mathcal{G}) \simeq \mathrm{Deck}(\widehat{\mathcal{G}})$. In particular, $\mathcal{F}$ is Galois if and only if $\widehat{\mathcal{G}}$ is Galois (cf. Corollary 5.25).

Concerning the problem of describing the irreducible components of the space $\mathbb{G}_d$ of degree d Galois foliations, we consider a discrete numeric invariant, the genus of the polar curve, that is shown to be generically constant along each component (cf. Proposition 5.5). We also introduce a combinatorial invariant, the branching type of the foliation (cf. Definitions 3.22 and 3.29), that we also expect to be generically constant along the irreducible components of $\mathbb{G}_d$. Finally, with the help of an algebraic manipulator we show that $\mathbb{G}_3$ is reducible (Proposition 5.33), answering partially a question raised in [9].

This article is organized as follows. In Section 2 we collect the basic properties of Galois coverings that will be used in the sequel. Section 3 is devoted to the general properties of rational maps between projective manifolds of the same dimension focusing on those that are Galois. In particular we prove Theorems B, C and D. In Section 4 we describe the behavior of a foliation or a web under a rational map and we prove Theorem E. In the last section we apply the above results to the Gauss map of a planar foliation, giving concrete examples of Galois foliations on $\mathbb{P}^2$ and proving Theorems A and G.

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2. Galois coverings

Let $\varpi : E \to B$ be a d-sheet covering over a connected and locally path connected topological space B. Fix a base point $p_0 \in B$ and its fibre $F = \varpi^{-1}(p_0) = \{p_1, \ldots, p_d\}$. We consider its deck transformation group $D = \{\tau : E \xrightarrow{\sim} E \,:\, \varpi \circ \tau = \varpi\}$ acting on the left on F by restriction. In fact, the restriction map is a monomorphism $D \hookrightarrow \mathfrak{S}(F)$, where $\mathfrak{S}(F)$

is the permutation group of F. We also consider the monodromy anti-representation $\bar{\mu} : \pi_1(B, p_0) \to \mathfrak{S}(F)$ of ϖ defined by $\bar{\mu}([\gamma])(p) = \widetilde{\gamma}_p(1)$ for each $p \in F$, where $\widetilde{\gamma}_p(t)$ is a path in E starting at $\widetilde{\gamma}_p(0) = p$ and projecting onto $\gamma = \varpi(\widetilde{\gamma}_p)$. The anti-morphism $\bar{\mu}$ defines a *right* action of the fundamental group of B on F. We define the monodromy representation $\mu : \pi_1(B, p_0) \to \mathfrak{S}(F)$ as the morphism $\gamma \mapsto \mu(\gamma) = \bar{\mu}(\gamma^{-1})$. Its image subgroup, denoted by M, is called the monodromy group of ϖ. It is clear that if E is connected then the action of D is free and the action of M is transitive. Consequently, if E is connected then $|D| \leq d$ and $|M| \geq d$. Identifying $F \simeq \{1, \ldots, d\}$ we can consider both D and M as subgroups of the symmetric group $\mathfrak{S}_d$.

It is well known (see for instance [15, Théorème 4.6.8]) that the covariant functor from the category of coverings over B into the category of left $\pi_1(B, p_0)$-sets, which sends a covering over B to its fibre F over p_0 endowed with the left action given by the monodromy representation, is an equivalence of categories. Indeed, for every left $\pi_1(B, p_0)$-set F, coming from a morphism $\mu : \pi_1(B, p_0) \to \mathfrak{S}(F)$, we can consider the suspension covering $E = \widetilde{B} \times_\mu F \to B$ whose monodromy representation is μ. On the other hand, if a map σ between the fibres over p_0 of two coverings E_1 and E_2 over B is equivariant with respect their monodromy representations (i.e. $\sigma(\mu_1(\gamma)(p)) = \mu_2(\gamma)(\sigma(p))$) then we can extend σ to a morphism $E_1 \to E_2$ of covering maps over B by means of the classical lifting path method. This allows to characterize the image of D inside $\mathfrak{S}(F)$ as the permutations commuting with all the elements of the monodromy group as F. Cukierman states in [12]:

Proposition 2.1. *Let $\varpi : E \to B$ be a covering with fibre F, deck transformation group $D \hookrightarrow \mathfrak{S}(F)$ and monodromy group $M \subset \mathfrak{S}(F)$. Then D is the centralizer $Z(M)$ of M in $\mathfrak{S}(F)$.*

Remark 2.2. In general, there are no inclusion between the subgroups D and M. In fact, it follows from Proposition 2.1 that their intersection $D \cap M = Z(M) \cap M = C(M)$ is the centre of M. Thus, $M \subset D$ if and only if M is abelian. In addition, if the covering ϖ is finite and E is connected then M is abelian if and only if $M = D$ because $|D| \leq \deg \varpi \leq |M|$.

Given a connected and locally path connected space B' and a continuous map $f : B' \to B$ the pull-back covering of $\varpi : E \to B$ by f is defined as $\varpi' : E' = f^*E = E \times_B B' = (\varpi \times f)^{-1}(\Delta_B) \to B'$, where Δ_B is the diagonal of $B \times B$ and ϖ' is the restriction of the second projection onto B'. The map $f' : E \times_B B' \to E$ induced by the first projection is a bijection when it is restricted to a fibre of ϖ'. Thus, given a point $p_0' \in f^{-1}(p_0)$, we can canonically identify the fibre $\varpi'^{-1}(p_0')$ with $\varpi^{-1}(p_0) = F$ via f'. Then the monodromy representation of $\varpi' : E' \to B'$ is the composition $\mu \circ f_* : \pi_1(B', p_0') \to \pi_1(B, p_0) \to \mathfrak{S}(F)$. In particular, the monodromy group M' of ϖ' is a subgroup of the monodromy group M of ϖ. Let D (resp. D') be the deck transformation group of ϖ (resp. ϖ'). There is a natural monomorphism $D \hookrightarrow D'$ given by $\tau \mapsto \tau'$, where $\tau'(e, b') = (\tau(e), b')$. On the other hand, if E and E' are connected and $f'(p_1') = p_1$ then $\varpi'_*(\pi_1(E', p_1')) = f_*^{-1}(\varpi_*(\pi_1(E, p_1)))$.

Proposition 2.3. *With the notations introduced above, if $f_* : \pi_1(B') \to \pi_1(B)$ is a epimorphism then*

(a) $M' = M$,
(b) D and D' are canonically isomorphic,
(c) if E and E' are connected then $f'_* : \pi_1(E') \to \pi_1(E)$ is an epimorphism.

Proof. The first assertion is clear, the second one is a consequence of Proposition 2.1 and the last one follows by noting that $\varpi_* \circ f'_* = f_* \circ \varpi'_*$ and by applying f_* to the equality $\varpi'_*(\pi_1(E', p'_1)) = f_*^{-1}(\varpi_*(\pi_1(E, p_1)))$. □

Remark 2.4. The total space of the pull-back $\varpi' : E' \to B'$ of a connected covering $\varpi : E \to B$ by a continuous map $f : B' \to B$ is not necessarily connected. In particular, if we take $B' = E$ and $f = \varpi$ then there are natural bijections

$$D \simeq \{E'_0 \in \pi_0(E') \,|\, \varpi'_{|E'_0} \text{ is bijective}\} \simeq \mathrm{Fix}(M'),$$

where D is the deck transformation group of ϖ, M' is the monodromy group of ϖ' and $\mathrm{Fix}(M') = \{p \in F \,|\, \forall m' \in M',\ m'(p) = p\}$. Indeed, for each connected component E'_0 of $E' := \varpi^* E$ such that $\varpi_{|E'_0}$ is bijective there is a section $\sigma : E \to E'_0 \subset E' \subset E \times E$ that we can write as $\sigma(p) = (p, \tau(p))$. Then $\tau : E \to E$ is a deck transformation of ϖ. On the other hand, the map $E \times D \to \varpi^* E$ defined by $(p, \tau) \mapsto (p, \tau(p))$ is always injective.

Remark 2.5. With the precedent notations, if E and $E' = f^* E$ are connected, ϖ is finite and $f'_* : \pi_1(E') \to \pi_1(E)$ is an epimorphism then D and D' are canonically isomorphic. Indeed, the monodromy groups of $\varpi^* E$ and $\varpi'^* E'$ coincide. We conclude by noting that the natural morphism $D \hookrightarrow D'$ is injective and $|D| = |D'| < \infty$ by Remark 2.4.

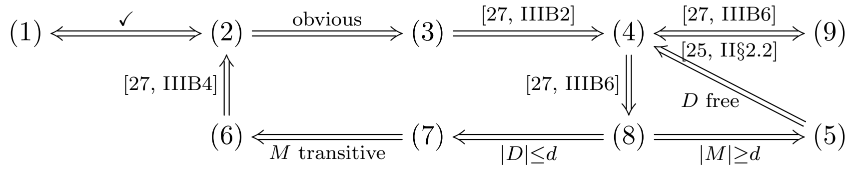
The following result is a compilation of well-known facts (see for instance [27, §III.B] and [25, §II.2]) and some additional remarks.

Theorem 2.6. *For a connected d-sheet covering $\varpi : E \to B$ the following assertions are equivalent:*

1. *the pull-back covering $\varpi^* E = E \times_B E \to E$ is trivial (i.e. a product),*
2. *$\varpi_*(\pi_1(E, p_1)) = \ker \mu$ (or equivalently, $\varpi_*(\pi_1(E, p_1)) \subset \ker \mu$),*
3. *$\varpi_*(\pi_1(E, p_1))$ is a normal subgroup of $\pi_1(B, p_0)$,*
4. *D acts transitively on the fibre,*
5. *$|D| \geq d$ (or equivalently, $|D| = d$),*
6. *M acts freely on the fibre,*
7. *$|M| \leq d$ (or equivalently, $|M| = d$),*
8. *the groups D and M are isomorphic,*
9. *ϖ induces an homeomorphism $E/D \xrightarrow{\sim} B$.*

When properties (1)-(8) are satisfied we say that the covering ϖ is Galois. In this case $D \simeq M \simeq \pi_1(B, p_0)/\varpi_*(\pi_1(E, p_1))$.

Proof. First, recall that a covering is trivial if and only if its monodromy is trivial. Thus, (1) and (2) are equivalent because the monodromy $\mu \circ \varpi_*$ of $\varpi^* E$ is trivial if and only if $\varpi_*(\pi_1(E, p_1)) \subset \ker \mu$. The general scheme of the proof is the following:



[(5) ⇒ (4)]: If $D = \{\tau_1, \ldots, \tau_d\}$ and $p_1 \in \varpi^{-1}(p_0)$ then $\{\tau_j(p_1)\}_{j=1}^{d}$ has cardinal d because D acts freely.
[(7) ⇒ (6)]: Since $M = \{\sigma_1, \ldots, \sigma_d\}$ acts transitively on $\{1, \ldots, d\}$, for each $i \in \{1, \ldots, d\}$ there is a unique $\sigma_j \in M$ such that $\sigma_j(1) = i$. Without loss of generality we can assume that $j = i$. If $\sigma_k(i) = i$ then $\sigma_k \circ \sigma_i = \sigma_i$. Hence $\sigma_k = \mathrm{id}$ and M acts freely on the fibre. □

From Remark 2.2 and Theorem 2.6 we immediately obtain the following:

Corollary 2.7. *If $\varpi : E \to B$ is a connected d-sheet covering with abelian monodromy group $M \subset \mathfrak{S}_d$ then ϖ is Galois. In particular, if M is cyclic then $M \simeq \mathbb{Z}_d$.*

Obviously the converse of Corollary 2.7 is not true in general. Hence part (b) of Theorem 3 in [38] asserting that the subgroups D and M coincide if and only if ϖ is Galois, is wrong. However, it becomes true when d is prime thanks to Cauchy's theorem:

Corollary 2.8. *When $\deg \varpi$ is prime, ϖ is Galois if and only if M is cyclic.*

Remark 2.9. Although it is customary to consider the Galois property only for connected coverings there is a natural definition of non-connected Galois covering $\varpi : E \to B$ by requiring

(0) the existence of a connected Galois covering $\varpi_0 : E_0 \to B$ and a discrete set $C \simeq \pi_0(E)$ such that $E \simeq E_0 \times C$ and ϖ is the composition of the first projection $E \to E_0$ with ϖ_0.

It can be easily checked that condition (0) is equivalent to conditions (1)-(4) in Theorem 2.6. Moreover, it implies (but if $|C| > 1$ it is not equivalent to) the part of assertions (5)-(7) and (9) which are not in parenthesis. In fact, under condition (0) we have $D_E \simeq \mathfrak{S}(C) \oplus D_{E_0}^{\oplus|C|}$ so that $|D_E| = |D_{E_0}|^{|C|}|C|! > |D_{E_0}||C| = \deg(\varpi)$ and $|M_E| = |M_{E_0}| = \deg \varpi_0 = \frac{\deg \varpi}{|C|} < \deg \varpi$, assuming that $|C| > 1$. Hence, in that case condition (8) does not hold.

Using the above notion of non-connected Galois coverings we can state the following result which will be used later.

Proposition 2.10. *Let $\varpi : E \to B$ be a covering, let $B' \subset B$ a connected and locally path connected subset, set $E' = \varpi^{-1}(B') \subset E$ and consider a connected component E'_0 of E'. Then the restrictions maps ϖ' and ϖ'_0 of ϖ to E' and E'_0 are coverings. If ϖ is Galois then so are ϖ' and ϖ'_0 and there are natural inclusions $D'_0 \hookrightarrow D \hookrightarrow D'$ between the deck transformations groups D, D' and D'_0 of ϖ, ϖ' and ϖ'_0 respectively.*

Proof. The first assertion is straightforward. Clearly, each $\tau \in D$ fulfills $\tau(E') = E'$, which implies that $\tau_{|E'} \in D'$, and if $\tau(E'_0) \cap E'_0 \neq \emptyset$ then $\tau(E'_0) = E'_0$ and $\tau_{|E'_0} \in D'_0$. These considerations imply that if D acts transitively on its fibre F then D' and D'_0 acts also transitively on its corresponding fibres $F' = F$ and $F'_0 \subset F$.

Given $\tau'_0 \in D'_0$, by the transitivity of D there is a unique $\tau \in D$ such that $\tau(p) = \tau'_0(p)$. Then $\tau_{|E'_0} = \tau'_0$ and the correspondence $\tau'_0 \mapsto \tau$ defines the first inclusion. The second monomorphism is given by the restriction of maps. Last isomorphism follows from Remark 2.9. □

3. Galois rational maps

In this section we consider dominant rational maps between projective manifolds of the same dimension. The Gauss map associated to (non degenerated) codimension one holomorphic foliations on $\mathbb{P}^n$ provide examples of that kind of maps, which are shown to be finite coverings when restricted to suitable Zariski open subsets. The map is said to be of Galois type if the finite covering is Galois and we address the problem of finding sufficient conditions assuring that such a rational map has that property.

3.1. **Dominant rational maps between complex projective manifolds.** Let $\mathcal{G} : X \dashrightarrow Y$ be a rational map between complex projective manifolds with indeterminacy locus $\Sigma_{\mathcal{G}} \subset X$. Since Y is projective, the Zariski closed set $\Sigma_{\mathcal{G}}$ has codimension ≥ 2. Consider the closed graph $\Gamma := \overline{\{(x, \mathcal{G}(x)) \,|\, x \in X \setminus \Sigma_{\mathcal{G}}\}} \subset X \times Y$ and the restrictions p_X and p_Y to Γ of the natural projections from $X \times Y$ onto X and Y respectively. A *desingularization of* $\mathcal{G}$ is a commutative diagram

$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{\beta}\swarrow & & \searrow^{\widetilde{\mathcal{G}}} \\ X & \overset{\mathcal{G}}{\dashrightarrow} & Y \end{array}$$

where $\widetilde{X}$ is a projective manifold, $\widetilde{\mathcal{G}}$ and β are morphisms and β is birational. Without loss of generality we can always assume that β restricts to an isomorphism from $\widetilde{X} \setminus \beta^{-1}(\Sigma_{\mathcal{G}})$ onto $X \setminus \Sigma_{\mathcal{G}}$. To give a desingularization of $\mathcal{G}$ is equivalent to give a desingularization of Γ, i.e. a birational morphism $\delta : \widetilde{X} \to \Gamma$. Indeed, from $(\beta, \widetilde{\mathcal{G}})$ we construct the morphism $\delta := \beta \times \widetilde{\mathcal{G}} : \widetilde{X} \to X \times Y$ having image Γ and from δ we recover $\beta := p_X \circ \delta$ and $\widetilde{\mathcal{G}} := p_Y \circ \delta$. We consider also the *exceptional divisor* $\mathcal{E} := \beta^{-1}(\Sigma_{\mathcal{G}})$ of β which *contains* $\delta^{-1}(\mathrm{Sing}(\Gamma))$. For every subsets $A \subset X$ and $B \subset Y$ we define the image $\mathcal{G}(A) := p_Y(p_X^{-1}(A)) = \widetilde{\mathcal{G}}(\beta^{-1}(A)) \subset Y$ and the preimage $\mathcal{G}^{-1}(B) := p_X(p_Y^{-1}(B)) = \beta(\widetilde{\mathcal{G}}^{-1}(B)) \subset X$. For every rational map $f : X \dashrightarrow Y$ with indeterminacy locus $\Sigma_f \subset X$ we define the following Zariski closed sets:

$$\begin{aligned} \Delta_f &:= \overline{\{x \in X \setminus \Sigma_f \,|\, df_x \text{ is not surjective}\}} \subset X \\ \Lambda_f &:= f(\Delta_f \cup \Sigma_f) \subset Y. \end{aligned} \tag{1}$$

Going back to the original setup we note that

$$\begin{aligned}\Lambda_{\mathcal{G}} &= \mathcal{G}(\Delta_{\mathcal{G}} \cup \Sigma_{\mathcal{G}}) = \mathcal{G}(\Delta_{\mathcal{G}} \setminus \Sigma_{\mathcal{G}}) \cup \mathcal{G}(\Sigma_{\mathcal{G}}) \\ &= \widetilde{\mathcal{G}}(\beta^{-1}(\Delta_{\mathcal{G}} \setminus \Sigma_{\mathcal{G}})) \cup \widetilde{\mathcal{G}}(\beta^{-1}(\Sigma_{\mathcal{G}})) \\ &= \widetilde{\mathcal{G}}(\Delta_{\widetilde{\mathcal{G}}} \setminus \mathcal{E}) \cup \widetilde{\mathcal{G}}(\mathcal{E}) = \widetilde{\mathcal{G}}(\Delta_{\widetilde{\mathcal{G}}} \cup \mathcal{E}) = \Lambda_{\widetilde{\mathcal{G}}} \cup \widetilde{\mathcal{G}}(\mathcal{E}).\end{aligned}$$

Assume from now on that $\mathcal{G} : X \dashrightarrow Y$ is *dominant*, i.e. it has dense image. Since every projective morphism is closed we obtain that $\widetilde{\mathcal{G}}$ is surjective. Moreover, by applying Sard's theorem to $\widetilde{\mathcal{G}} : \widetilde{X} \to Y$ we deduce that the Zariski closed set $\Lambda_{\widetilde{\mathcal{G}}}$ is proper.

Remark 3.1. If $\Sigma_{\mathcal{G}} = \emptyset$ or $\dim X = \dim Y$ then the Zariski closed subset $\Lambda_{\mathcal{G}} \subset Y$ is proper. Indeed, $\dim \widetilde{\mathcal{G}}(\mathcal{E}) \leq \dim \mathcal{E} < \dim X$.

Remark 3.2. The hypothesis that X and Y are non-singular projective varieties is not restrictive. Indeed, if $\mathcal{G}' : X' \dashrightarrow Y'$ is a rational map between singular complex projective varieties then we can desingularize it by taking birational morphisms $\beta_X : X \to X'$ and $\beta_Y : Y \to Y'$ with X and Y complex projective manifolds and we can consider the rational map $\mathcal{G} := \beta_Y^{-1} \circ \mathcal{G}' \circ \beta_X : X \dashrightarrow Y$. On the other hand, the assumption that $\mathcal{G}$ is dominant is not restrictive either because we can always replace Y by a desingularization of the Zariski closure of $\mathcal{G}(X \setminus \Sigma_{\mathcal{G}})$ in Y.

In the sequel we will use the following extension of Lefschetz's hyperplane theorem (cf. [22, Theorem 1.1.3(ii)]):

Theorem 3.3 (Hamm-Lê)**.** *Let X be a complex projective algebraic variety in $\mathbb{P}^N$. Let Z be an algebraic subspace of X such that $X \setminus Z$ is non singular. Then there is an open dense set $\Omega \subset \check{\mathbb{P}}^N$ of complex hyperplanes in $\mathbb{P}^N$ such that for any $H \in \Omega$ the space $X \setminus Z$ has the homotopy type of a space obtained from $H \cap (X \setminus Z)$ by attaching cells of (real) dimension at least equal to $\dim_{\mathbb{C}} X$.*

Definition 3.4. *Let X be a complex projective n-manifold in $\mathbb{P}^N$. A* generic hyperplane curve *ℓ^X of X is a curve on X obtained by intersection with $H_1 \cap \cdots \cap H_{n-1}$ where $(H_1, \ldots, H_{n-1})$ is a generic $(n-1)$-tuple of hyperplanes in $\mathbb{P}^N$.*

Notice that if $X = \mathbb{P}^n$ is linearly embedded in $\mathbb{P}^N$ then ℓ^X is just a generic line. We recall that, for $n \geq 2$, adjoining n-cells to a given topological space preserves any system of generators of its fundamental group and that, by Bertini's theorem, a generic hyperplane section of a projective manifold is smooth. By applying $n-1$ times Theorem 3.3 we deduce the following:

Corollary 3.5. *Let X be a complex n-dimensional projective manifold in $\mathbb{P}^N$. Let Z be an algebraic subspace of X. Then there is a open dense subset $\Omega \subset (\check{\mathbb{P}}^N)^{n-1}$ such that for each $(H_1, \ldots, H_{n-1}) \in \Omega$ the hyperplane curve $\ell^X := X \cap H_1 \cap \cdots \cap H_{n-1}$ is smooth and the inclusion $\ell^X \setminus Z \subset X \setminus Z$ induces an epimorphism $\pi_1(\ell^X \setminus Z) \twoheadrightarrow \pi_1(X \setminus Z)$.*

Corollary 3.6. *Let X be a projective manifold, $U \subset X$ an open Zariski subset and $\Lambda \subset X$ a closed Zariski subset. Then the inclusion $U \setminus \Lambda \subset U$ induces an epimorphism $\pi_1(U \setminus \Lambda) \twoheadrightarrow \pi_1(U)$.*

Proof. Applying Corollary 3.5 to the algebraic subspaces $Z = X \setminus U$ and $Z = (X \setminus U) \cup \Lambda$ of X we deduce that that the horizontal arrows of the following commutative diagram are surjective

$$\begin{array}{ccc} \pi_1(\ell^X \cap U \setminus \Lambda) & \twoheadrightarrow & \pi_1(U \setminus \Lambda) \\ \downarrow & & \downarrow \\ \pi_1(\ell^X \cap U) & \twoheadrightarrow & \pi_1(U). \end{array}$$

The left vertical arrow is also surjective because ℓ^X is a complex curve and $\ell^X \cap (X \setminus U) \cap \Lambda \subset \ell^X \cap \Lambda$ are finite sets. Hence the right vertical arrow induced by the inclusion $U \setminus \Lambda \subset U$ is also an epimorphism. □

Let $f\colon A \to B$ be a smooth proper submersion where A and B are smooth manifolds and B is connected. Then f defines a locally trivial fibration with fibre a compact smooth manifold F and, associated to it, there is a monodromy representation that can be defined as follows. Let $F = f^{-1}(b_0)$ be a fibre and let us fix a connection on A. Each loop in B with base point b_0 determines a diffeomorphism of F which is constructed by the lifting path method and whose isotopy class only depends on the homotopy class of the loop. Thus, there is a well defined morphism $\pi_1(B) \to \mathcal{M}(F)$, where $\mathcal{M}(F)$ denotes the mapping class group of the fibre F. This morphism is independent of the chosen connection and its image $\mathrm{Mon}(f)$ is called the geometric monodromy group of the fibration. We will be mainly interested in the case $\dim A = \dim B$. In that situation f is a covering map, F is a finite set and $\mathcal{M}(F)$ is $\mathfrak{S}(F)$, the group of permutations of F. For more details on the definition and properties of the geometric monodromy we refer to [16, §4.4].

In the sequel, given a rational map $f : X \dashrightarrow Y$ and a subset $\Lambda \subset Y$, we will use the following notations:

$$Y_\Lambda := Y \setminus \Lambda, \quad X_\Lambda := f^{-1}(Y_\Lambda) \quad \text{and} \quad f_\Lambda := f_{|X_\Lambda}. \tag{2}$$

We recall a known result stated in a suitable form for our purposes.

Proposition 3.7. *Let $\mathcal{G} : X \dashrightarrow Y$ be a dominant rational map between projective manifolds and assume Y is connected and $\mathcal{G}(\Sigma_\mathcal{G}) \neq Y$. For every proper Zariski closed set $\Lambda \subsetneq Y$ containing $\Lambda_\mathcal{G}$, the restriction $\mathcal{G}_\Lambda : X_\Lambda \to Y_\Lambda$ is a fiber bundle with a fibre F of dimension $\dim X - \dim Y$. Moreover, the monodromy group of $\mathcal{G}_\Lambda$ does not depend on $\Lambda \supseteq \Lambda_\mathcal{G}$. It will denoted by $\mathrm{Mon}(\mathcal{G})$.*

Proof. Since the restriction $\beta_\Lambda : \beta^{-1}(X_\Lambda) \to X_\Lambda$ of β is an isomorphism and $\widetilde{\mathcal{G}}$ is proper, the map $\mathcal{G}_\Lambda = \widetilde{\mathcal{G}} \circ \beta_\Lambda^{-1}$ is a proper surjective submersion. Ehresmann's theorem asserts that $\mathcal{G}_\Lambda$ is a locally trivial fiber bundle. Connectedness of Y implies that all the fibres are smoothly equivalent, say to F. Last assertion follows by applying Corollary 3.6 and Proposition 2.3. □

Notice that the fibre F of the fiber bundle $\mathcal{G}_\Lambda : X_\Lambda \to Y_\Lambda$ is not necessarily connected. By applying Stein factorization theorem to the surjective morphism $\widetilde{\mathcal{G}}$ we obtain a commutative diagram of complex projective varieties

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \gamma\ } & N \\ {\scriptstyle\beta}\downarrow & \searrow^{\widetilde{\mathcal{G}}} & \downarrow{\scriptstyle\rho} \\ X & \overset{\mathcal{G}}{\dashrightarrow} & Y \end{array} \tag{3}$$

where γ is a morphism with connected fibres, N is an algebraic normal variety and ρ is a finite and surjective morphism, see [20, p. 213]. Consequently ρ is a *branched covering* in the sense of [31]. Any surjective proper finite holomorphic map $f : Z \to W$ from a normal complex space Z onto a connected complex manifold W is a finite branched covering of W (cf. Definition 1.1.1 and Example 1.1.2 in [31]). In fact, all the branched coverings appearing in this article will be of that type. Notice also that $N_\Lambda = \rho^{-1}(Y_\Lambda)$ is naturally identified to $\bigcup_{y\in Y_\Lambda} \pi_0(\mathcal{G}_\Lambda^{-1}(y))$. The following proposition says that the branched covering $\rho : N \to Y$ is canonically associated to $\mathcal{G} : X \dashrightarrow Y$.

Proposition 3.8. *The branched covering $\rho : N \to Y$ associated to the rational map $\mathcal{G} : X \dashrightarrow Y$ by the Stein factorization (3) of a desingularization $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ does not depend on the choice of $\widetilde{\mathcal{G}}$.*

Proof. Let $\widetilde{\mathcal{G}} : \widetilde{X} \to Y$ and $\widetilde{\mathcal{G}}' : \widetilde{X}' \to Y$ two desingularizations of $\mathcal{G}$ and let $\widetilde{\mathcal{G}} = \rho \circ \gamma$ and $\widetilde{\mathcal{G}}' = \rho' \circ \gamma'$ their Stein factorizations given by (3). Since $N_\Lambda \simeq \bigcup_{y\in Y_\Lambda} \pi_0(\mathcal{G}_\Lambda^{-1}(y)) \simeq N'_\Lambda$, the coverings ρ_Λ and ρ'_Λ are equivalent. We conclude that the branched coverings $\rho : N \to Y$ and $\rho' : N' \to Y$ are also equivalent by applying [31, Proposition 1.1.5]. □

From now on we will restrict our attention to the case $\dim X = \dim Y$ and Y *connected*. In this situation, the fibre F is finite of cardinal the *topological degree* d of $\mathcal{G}$ (see [29, §5 Theorem A]), and the morphism $\gamma : \widetilde{X} \to N$ is generically injective (because its generic fibre is 0-dimensional and connected), hence birational.

Definition 3.9. *Let $\mathcal{G} : X \dashrightarrow Y$ be a dominant rational map between projective manifolds of the same dimension. We define the group of birational deck transformations of $\mathcal{G}$ as*

$$\mathrm{Deck}(\mathcal{G}) := \{\tau \in \mathrm{Bir}(X) \,|\, \mathcal{G} \circ \tau = \mathcal{G}\}.$$

Remark 3.10. Given birational maps $a : X' \dashrightarrow X$ and $b : Y \dashrightarrow Y'$, the monodromy and deck transformation groups of the rational maps $\mathcal{G} : X \dashrightarrow Y$ and $\mathcal{G}' = b\circ\mathcal{G}\circ a : X' \dashrightarrow Y'$ are canonically isomorphic. Indeed, conjugation by a induces an isomorphism between the deck transformation groups and the isomorphism between the monodromies follows from Proposition 3.7.

The main result of this section is the following.

Theorem 3.11. *Let $\mathcal{G} : X \dashrightarrow Y$ be a rational dominant map between complex projective manifolds of the same dimension. Assume that Y is connected and denote by d the topological degree of $\mathcal{G}$. Let Λ be a Zariski closed set of Y containing $\Lambda_{\mathcal{G}}$, let V be a connected and locally path connected subset of Y_Λ and set $U = \mathcal{G}^{-1}(V)$. Then*

(1) *the maps* $\mathcal{G}_\Lambda : X_\Lambda \to Y_\Lambda$ *and* $\rho_\Lambda : N_\Lambda \to Y_\Lambda$ *are isomorphic d-sheeted coverings;*
(2) *the monodromy group* $\mathrm{Mon}(\mathcal{G})$ *of* $\mathcal{G}_\Lambda$ *does not depend on* $\Lambda \supset \Lambda_\mathcal{G}$ *and coincides with the monodromy group of* $\mathcal{G}_{|U}$ *if the inclusion* $V \subset Y_\Lambda$ *induces an epimorphism* $\pi_1(V) \twoheadrightarrow \pi_1(Y_\Lambda)$*;*
(3) *if one of the following two conditions is fulfilled*
   (a) *the natural morphism* $\pi_1(V) \to \pi_1(Y_\Lambda)$ *is surjective,*
   (b) X *and* U *are connected and the natural morphism* $\pi_1(U) \to \pi_1(X_\Lambda)$ *is surjective,*

   then the restriction maps induce canonical isomorphisms

$$\mathrm{Deck}(\mathcal{G}) \xrightarrow{\sim} \mathrm{Deck}(\mathcal{G}_\Lambda) \xrightarrow{\sim} \mathrm{Deck}(\mathcal{G}_{|U}).$$

Proof. (1) By Proposition 3.7, the maps $\mathcal{G}_\Lambda$ and ρ_Λ are d-sheeted coverings. Since β_Λ and γ_Λ are isomorphisms, we deduce that $\gamma_\Lambda \circ \beta_\Lambda^{-1} : X_\Lambda \to N_\Lambda$ is an equivalence of coverings by the commutativity of diagram (3).
(2) The first part of the assertion also follows from Proposition 3.7. The second part is a consequence of Proposition 2.3.
(3) The second isomorphism follows also from Proposition 2.3 and Remark 2.5. To see the first isomorphism we use Remark 3.10 and part (1) of the Theorem. It suffices to prove that the restriction map induces an isomorphism $\mathrm{Deck}(\rho) \xrightarrow{\sim} \mathrm{Deck}(\rho_\Lambda)$. To see that we apply Riemann's extension theorem, as in the proof of Proposition 1.1.5 of [31], which shows that every deck transformation $\tau_\Lambda : N_\Lambda \to N_\Lambda$ of ρ_Λ extends holomorphically to $N \setminus \mathrm{Sing}(N)$. Then we use the normality of N to have a biholomorphic extension to the whole N. □

Remark 3.12. The above proof shows that $\mathrm{Deck}(\mathcal{G})$ is naturally identified with $\mathrm{Deck}(\rho)$ and that all the elements of $\mathrm{Deck}(\rho)$ are biholomorphisms of N, where $\rho : N \to Y$ is the branched covering uniquely associated to $\mathcal{G} : X \dashrightarrow Y$. In fact, the pair (N, ρ) can be characterized as the minimal birational model of $(X, \mathcal{G})$ fulfilling the following properties:
(i) ρ and all the elements of $\mathrm{Deck}(\rho)$ are morphisms,
(ii) ρ is a branched covering.

Recall that a branched covering $\rho : N \to Y$ is called Galois if the group $\mathrm{Deck}(\rho)$ acts transitively on every fibre (cf. [31, §1]). In view of the above theorem we can give the following definition.

Definition 3.13. *A dominant rational map* $\mathcal{G} : X \dashrightarrow Y$ *between connected complex projective manifolds of the same dimension is said to be Galois if its associated branched covering* $\rho : N \to Y$ *is Galois, or equivalently, if any of its associated coverings* $\mathcal{G}_\Lambda$ *are Galois.*

We recall that every finite group appears as the Galois group of a Galois branched covering as it was proved by M. Namba in [32]. Notice however that deciding the birational type of the source from those of target and the Galois group is a difficult problem.

Theorem 3.14 (Namba)**.** *For every finite group* G *and every connected complex projective manifold* Y *there exists a Galois branched covering* $\rho : N \to Y$ *whose deck transformation group* $\mathrm{Deck}(\rho)$ *is isomorphic to* G*.*

3.2. **Dimensional reduction.** The aim of this subsection is to find criteria to decide whether a rational map is Galois just by looking at its restriction to an appropriate subvariety, the most interesting case being when the subvariety has dimension one, as a rational dominant map between connected complex curves is just a non constant morphism or, equivalently, a branched covering. We obtain several criteria of that type. They will be useful both, from the theoretical point of view as well as in the applications, for deciding whether the Gauss map of certain classes of foliations are Galois. Thus, we want to relate the behavior of a rational map $\mathcal{G} : X \dashrightarrow Y$, or more generally a desingularization $\widetilde{\mathcal{G}} : \widetilde{X} \to Y$ of it, with its restriction to some irreducible subvariety.

Let V be an irreducible projective subvariety of $\widetilde{X}$ and set $W = \widetilde{\mathcal{G}}(V) \subset Y$. Since V or W may be singular, according to Remark 3.2, we fix desingularizations $\beta_V : \widetilde{V} \to V$ and $\beta_W : \widetilde{W} \to W$ of V and W and we consider the rational dominant map $\widetilde{\mathcal{G}}^V := \beta_W^{-1} \circ \widetilde{\mathcal{G}}_{|V} \circ \beta_V : \widetilde{V} \dashrightarrow \widetilde{W}$. Assume that $\dim V = \dim W$. Notice that this condition does not exclude the case $V \subset \mathcal{E}$. By applying Theorem 3.11 to the rational maps $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}^V$ with

$$\Lambda := \Lambda_{\widetilde{\mathcal{G}}} \cup \beta_W(\Lambda_{\widetilde{\mathcal{G}}^V}) \cup \widetilde{\mathcal{G}}(\mathrm{Sing}(V)) \cup \mathrm{Sing}(W)$$

we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{V}_\Lambda & \xrightarrow{\widetilde{\mathcal{G}}^V_\Lambda} & \widetilde{W}_\Lambda \\ \Big\downarrow{\scriptstyle \beta_V} & & \Big\downarrow{\scriptstyle \beta_W} \\ \widetilde{X}_\Lambda & \xrightarrow{\widetilde{\mathcal{G}}_\Lambda} & Y_\Lambda \end{array}$$

where the horizontal arrows are coverings. Since $\widetilde{\mathcal{G}}(\mathrm{Sing}(V)) \cup \mathrm{Sing}(W) \subset \Lambda$ we can identify $\widetilde{V}_\Lambda \simeq V_\Lambda \subset \widetilde{X}_\Lambda$, $\widetilde{W}_\Lambda \simeq W_\Lambda \subset Y_\Lambda$ and $\widetilde{\mathcal{G}}^V_\Lambda$ with the restriction of $\widetilde{\mathcal{G}}_\Lambda$ to V_Λ. In this situation we have the following result.

Proposition 3.15. *Let $V \subset \widetilde{X}$ be an irreducible projective subvariety.*

(a) Assume that $V \not\subset \Delta_{\widetilde{\mathcal{G}}}$.
- *(i) If $\mathcal{G}$ is Galois then $\widetilde{\mathcal{G}}^V$ is Galois and* $\mathrm{Deck}(\widetilde{\mathcal{G}}^V) \hookrightarrow \mathrm{Deck}(\mathcal{G})$.
- *(ii) If $\widetilde{\mathcal{G}}^V$ is Galois,* $\deg \widetilde{\mathcal{G}}^V = \deg \mathcal{G}$ *and the inclusion $W_\Lambda \subset Y_\Lambda$ induces an epimorphism in the fundamental groups then $\mathcal{G}$ is Galois.*

(b) Assume that $V \subset \Delta_{\widetilde{\mathcal{G}}}$ and $\dim \widetilde{\mathcal{G}}(V) = \dim V$. *If $\mathcal{G}$ is Galois then $\widetilde{\mathcal{G}}^V$ is Galois.*

Proof. Part (a) follows from Propositions 2.10 and 2.3 and Theorem 3.11. Assume now that the hypothesis in (b) are fulfilled. In particular, the restriction $\gamma_{|V} : V \to \gamma(V) \subset N$ is birational. It follows, using Remark 3.12, that V is not contained in the indeterminacy locus Σ_τ of any deck transformation $\tau : \widetilde{X} \dashrightarrow \widetilde{X}$ of $\widetilde{\mathcal{G}}$. Then each element $\tau \in \mathrm{Deck}(\widetilde{\mathcal{G}})$ such that $\tau(V \setminus \Sigma_\tau) \cap V \neq \emptyset$ determines a birational map $V \dashrightarrow V$ inducing an element in $\mathrm{Deck}(\widetilde{\mathcal{G}}^V)$. Since $\mathrm{Deck}(\widetilde{\mathcal{G}})$ acts transitively on the generic fibre of $\widetilde{\mathcal{G}}$ then, by continuity, it also acts transitively on a fibre over $\widetilde{\mathcal{G}}(V)$. Hence $\mathrm{Deck}(\widetilde{\mathcal{G}}^V)$ acts also transitively on its fibre. $\square$

A particular situation in which the precedent result applies to a one-dimensional covering $\widetilde{\mathcal{G}}^V$ is to take $V = \widetilde{\mathcal{G}}^{-1}(C)$ for some curve $C \subset Y$. In fact, thanks to Corollary 3.5 there are generic hyperplane curves $\ell^X \subset X$ and $\ell^Y \subset Y$ such that

$$\pi_1(\ell^X \setminus \mathcal{G}^{-1}(\Lambda)) \twoheadrightarrow \pi_1(X \setminus \mathcal{G}^{-1}(\Lambda)) \quad \text{and} \quad \pi_1(\ell^Y \setminus \Lambda) \twoheadrightarrow \pi_1(Y \setminus \Lambda).$$

We use the conventions introduced in (2), namely $Y_\Lambda = Y \setminus \Lambda$ and $X_\Lambda = \mathcal{G}^{-1}(Y_\Lambda)$, and we denote $\ell^X_\Lambda := \ell^X \cap X_\Lambda$ and $\ell^Y_\Lambda := \ell^Y \cap Y_\Lambda$. Let $p_\Lambda : \mathcal{G}^*_\Lambda(X_\Lambda) = X_\Lambda \times_{Y_\Lambda} X_\Lambda \to X_\Lambda$ be the pull-back covering of $\mathcal{G}_\Lambda$ by $\mathcal{G}_\Lambda$ and consider their corresponding restrictions to ℓ^X_Λ and ℓ^Y_Λ:

$$\begin{array}{ccccccc}
p_\Lambda^{-1}(\ell^X_\Lambda) & \hookrightarrow & \mathcal{G}^*_\Lambda(X_\Lambda) & \longrightarrow & X_\Lambda & \hookleftarrow & \mathcal{G}_\Lambda^{-1}(\ell^Y_\Lambda) \\
{\scriptstyle p^\ell_\Lambda}\downarrow & & {\scriptstyle p_\Lambda}\downarrow & & \downarrow{\scriptstyle \mathcal{G}_\Lambda} & & \downarrow{\scriptstyle \mathcal{G}^\ell_\Lambda} \\
\ell^X_\Lambda & \hookrightarrow & X_\Lambda & \xrightarrow{\mathcal{G}_\Lambda} & Y_\Lambda & \hookleftarrow & \ell^Y_\Lambda.
\end{array}$$

By applying Proposition 2.3 and Theorem 2.6 we immediately obtain the following characterizations:

Theorem 3.16. *Let $\ell^X \subset X$ and $\ell^Y \subset Y$ be generic hyperplane curves. Then the restriction map $\mathrm{Deck}(\mathcal{G}_\Lambda) \to \mathrm{Deck}(\mathcal{G}^\ell_\Lambda)$ is an isomorphism and the following assertions are equivalent*

(1) the rational map $\mathcal{G} \colon X \dashrightarrow Y$ is Galois;
(2) the restricted covering $\mathcal{G}^\ell_\Lambda : \mathcal{G}_\Lambda^{-1}(\ell^Y_\Lambda) \to \ell^Y_\Lambda$ is Galois;
(3) the restricted pull-back covering $p^\ell_\Lambda : p_\Lambda^{-1}(\ell^X_\Lambda) \to \ell^X_\Lambda$ is trivial.

Proposition 3.17. *The restricted covering $\mathcal{G}^\ell_\Lambda$ extends to a uniquely determined branched covering $\mathcal{G}^\ell : S^\ell_{\mathcal{G}} \to \ell^Y$ between compact Riemann surfaces. Moreover, the topological type of the covering $\mathcal{G}^\ell$ does not depend on the choice of the generic hyperplane curve $\ell^Y \subset Y$, once we fix an embedding $Y \subset \mathbb{P}^N$.*

Proof. The fact that the covering $\mathcal{G}^\ell_\Lambda$ extends to a uniquely determined branched covering $\mathcal{G}^\ell$, is well known. In fact, $S^\ell_{\mathcal{G}}$ is the normalization of the Zariski closure of $\mathcal{G}_\Lambda^{-1}(\ell^Y_\Lambda)$. To see the second part, we consider the Grassmannian $Z = G_{N-n+1}(\mathbb{P}^N)$ and its subset

$$Z_Y = \{z \in Z \,|\, z \cap Y \text{ is a smooth curve}\},$$

which contains a Zariski open subset of Z by Bertini's theorem. We also consider the incidence variety $\mathcal{V} = \{(y, z) \in Y \times Z \,|\, y \in z\}$ and we denote by $q^Y : \mathcal{V} \to Y$ and $q^Z : \mathcal{V} \to Z$ the natural projections. Set $\mathcal{W} = X \times_Y \mathcal{V} \simeq (\mathcal{G} \times \mathrm{id}_Z)^{-1}(\mathcal{V}) \subset X \times Z$ and denote by $q^X : \mathcal{W} \to X$ the natural projection. Although $\mathcal{V}$ and $\mathcal{W}$ can have singularities, using Remarks 3.1 and 3.2 we can apply Proposition 3.7 to the rational map $\mathcal{G}^{\mathcal{V}} : X \times_Y \mathcal{V} \dashrightarrow \mathcal{V}$ and the morphisms q^X, q^Y and q^Z defined above. In fact, we can choose compatible closed sets Λ in the different spaces so that the arrows in the following commutative diagram are well defined

$$\begin{array}{ccccc} \mathcal{W}_\Lambda \times_{\mathcal{V}_\Lambda} \ell^Y_\Lambda & \longrightarrow & \ell^Y_\Lambda = (q^Z_\Lambda)^{-1}(z_0) & \longrightarrow & \{z_0\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{W}_\Lambda & \xrightarrow{\mathcal{G}^{\mathcal{V}}_\Lambda} & \mathcal{V}_\Lambda & \xrightarrow{q^Z_\Lambda} & Z_\Lambda \\ {\scriptstyle q^X_\Lambda}\downarrow & & \downarrow{\scriptstyle q^Y_\Lambda} & & \\ X_\Lambda & \xrightarrow{\mathcal{G}_\Lambda} & Y_\Lambda, & & \end{array}$$

and the following conditions are fulfilled: $Z_\Lambda \subset Z_Y$, the restrictions q^X_Λ, q^Y_Λ and q^Z_Λ are locally trivial fiber bundles and $\mathcal{G}^{\mathcal{V}}_\Lambda$ and $\mathcal{G}_\Lambda$ are finite coverings. Notice that

$$\mathcal{W}_\Lambda \times_{\mathcal{V}_\Lambda} \ell^Y_\Lambda = \{(x, y, z_0) \,|\, y = \mathcal{G}_\Lambda(x) \in z_0\} \simeq \mathcal{G}_\Lambda^{-1}(\ell^Y_\Lambda).$$

It is easy to check that the composition $\mathcal{W}_\Lambda \xrightarrow{\mathcal{G}^{\mathcal{V}}_\Lambda} \mathcal{V}_\Lambda \xrightarrow{q^Z_\Lambda} Z_\Lambda$ is a fiber bundle trivializing on the same open subsets $U_i \subset Z_\Lambda$ that $q^Z_\Lambda : \mathcal{V}_\Lambda \to Z_\Lambda$. Fix $z_0, z_1 \in Z_\Lambda$ and choose a path $z_t \subset Z_\Lambda$ joining them. Trivializing simultaneously both fiber bundles over the interval $[0, 1]$ we obtain homeomorphisms $\xi : z_0 \cap Y_\Lambda \xrightarrow{\sim} z_1 \cap Y_\Lambda$ and $\chi : \mathcal{G}_\Lambda^{-1}(z_0 \cap Y_\Lambda) \xrightarrow{\sim} \mathcal{G}_\Lambda^{-1}(z_1 \cap Y_\Lambda)$ making commutative the diagram

$$\begin{array}{ccc} \mathcal{G}_\Lambda^{-1}(z_0 \cap Y_\Lambda) & \xrightarrow{\chi} & \mathcal{G}_\Lambda^{-1}(z_1 \cap Y_\Lambda) \\ {\scriptstyle \mathcal{G}^0_\Lambda}\downarrow & & \downarrow{\scriptstyle \mathcal{G}^1_\Lambda} \\ z_0 \cap Y_\Lambda & \xrightarrow{\xi} & z_1 \cap Y_\Lambda. \end{array}$$

Hence the coverings $\mathcal{G}^0_\Lambda$ and $\mathcal{G}^1_\Lambda$ have the same topological type. □

Remark 3.18. The genus of the compact Riemann surface $S^\ell_{\mathcal{G}}$ depends on the embedding $Y \hookrightarrow \mathbb{P}^N$ but it is a well defined invariant in the case $Y = \mathbb{P}^n$, which is the case for instance for the Gauss map of a foliation on $\mathbb{P}^n$.

For certain rational maps, the property of being Galois can be checked by looking at its restriction to a curve that is not generic. The following example exhibits a class of rational maps in which Proposition 3.15 can be applied to a subvariety V contained in the exceptional divisor $\mathcal{E} \subset \widetilde{X}$. This example will be used in Subsection 5.3.

Example 3.19. Let $A, B \in \mathbb{C}_d[x, y]$ be coprime homogeneous polynomials of degree d and let $u, v \in \mathbb{C}_1[x, y]$ be linear polynomials, not both zero. Define $C = uA + vB$ and consider the rational map $\mathcal{G} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $\mathcal{G}([x, y, z]) = [A(x, y)z, B(x, y)z, C(x, y)]$. The topological degree of $\mathcal{G}$ is d because, for generic $[a, b, c] \in \mathbb{P}^2$, the system of equations $\left\{\frac{A}{C} = \frac{a}{c}, \frac{B}{C} = \frac{b}{c}\right\}$ is equivalent to $\left\{\frac{B}{A} = \frac{b}{a}, au + bv = c\right\}$ whose solutions are the intersection points of the set $aB - bA = 0$, consisting in d lines through the point $O = [0, 0, 1]$, with the straight line $au + bv = c$, which generically does not contains O. On the other hand, it is easy to see that the image of the line $z = 0$ by $\mathcal{G}$ is the point $O' = [0, 0, 1]$ in the target. The indeterminacy points of $\mathcal{G}$ are $\{C(x, y) = 0, z = 0\}$ and $\{A(x, y) = B(x, y) = 0\} = \{O\}$. It can be easily checked that we obtain a desingularization $\beta : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of $\mathcal{G}$ by blowing up once each one of these points. In fact, in the affine charts (t,x) of $\widetilde{\mathbb{P}}^2$ corresponding to $z=1$, $y=tx$ and $a=1$ in the target $\mathbb{P}^2$, the morphism $\widetilde{\mathcal{G}} = \mathcal{G}\circ\beta : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ is written as $\widetilde{\mathcal{G}}(t,x) = (f(t), xg(t))$, where $f(t) = \frac{B(1,t)}{A(1,t)}$ and $g(t) = \frac{C(1,t)}{A(1,t)}$. Since $\det(d\widetilde{\mathcal{G}})(t,x) = f'(t)g(t) \not\equiv 0$, the divisor $\Delta_{\widetilde{\mathcal{G}}}$ consists in the strict transform of a finite number of lines through O and the line $z=0$. Since the image by $\mathcal{G}$ of a line through O is a line through O', we deduce that $\Lambda_{\widetilde{\mathcal{G}}} = \widetilde{\mathcal{G}}(\Delta_{\widetilde{\mathcal{G}}})$ consists in a finite number of lines through O' and a finite number of points in the line $c=0$. Let $V \subset \widetilde{\mathbb{P}}^2$ be the exceptional divisor over O which in the chart (t,x) has equation $x=0$. Its image $W = \widetilde{\mathcal{G}}(V)$ by $\widetilde{\mathcal{G}}(t,x) = [A(1,t), B(1,t), xC(1,t)]$ is just the line $c=0$. Thus, in this case $\widetilde{\mathcal{G}}^V = \widetilde{\mathcal{G}}_{|V}$ and the set $\Lambda = \Lambda_{\widetilde{\mathcal{G}}} \cup \Lambda_{\widetilde{\mathcal{G}}^V} \cup \widetilde{\mathcal{G}}(\mathrm{Sing}(V)) \cup \mathrm{Sing}(W) = \Lambda_{\widetilde{\mathcal{G}}}$. In addition, the inclusion $W \setminus \Lambda_{\widetilde{\mathcal{G}}} \subset \mathbb{P}^2 \setminus \Lambda_{\widetilde{\mathcal{G}}}$ induces an epimorphism of fundamental groups. Finally, thanks to the hypothesis $\gcd(A,B)=1$ we have $\deg \widetilde{\mathcal{G}}_{|V} = d$. By applying Proposition 3.15 we deduce that $\mathcal{G}$ is Galois if and only if the morphism $\widetilde{\mathcal{G}}_{|V} : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\widetilde{\mathcal{G}}_{|V}(x,y) = [A(x,y), B(x,y)]$ is Galois. □

Another situation in which we can made a dimensional reduction is the following. Let $\mathcal{G} : X \dashrightarrow Y$, $f : X \dashrightarrow \widehat{X}$, $g : Y \dashrightarrow \widehat{Y}$ and $\widehat{\mathcal{G}} : \widehat{X} \to \widehat{Y}$ be dominant rational maps between connected complex projective manifolds such that the following diagrams commutes

(4)
$$\begin{array}{ccc} X & \overset{\mathcal{G}}{\dashrightarrow} & Y \\ {\scriptstyle f}\,\vdots & & \vdots\,{\scriptstyle g} \\ \widehat{X} & \overset{\widehat{\mathcal{G}}}{\dashrightarrow} & \widehat{Y} \end{array}$$

Assume further that $\dim X = \dim Y > \dim \widehat{X} = \dim \widehat{Y}$, that f, g have generic connected fibres, and that the topological degrees of $\mathcal{G}$ and $\widehat{\mathcal{G}}$ coincide. Then one has

Proposition 3.20. *Under the above hypothesis, there is a natural isomorphism between* $\mathrm{Deck}(\mathcal{G})$ *and* $\mathrm{Deck}(\widehat{\mathcal{G}})$*. In particular,* $\mathcal{G}$ *is Galois if and only if* $\widehat{\mathcal{G}}$ *is Galois.*

Proof. The hypothesis $\deg \mathcal{G} = \deg \widehat{\mathcal{G}}$ is equivalent to say that the restriction of $\mathcal{G}$ to a generic fibre of f is injective. Hence the rational map $f \times \mathcal{G} : X \dashrightarrow \widehat{X} \times Y$ is generically injective. Its image is contained in $\widehat{X} \times_{\widehat{Y}} Y$ thanks to the commutativity of diagram (4). By Proposition 3.7 and Theorem 3.11 we can choose compatible closed sets Λ in each space so that the restrictions $f_\Lambda : X_\Lambda \to \widehat{X}_\Lambda$ and $g_\Lambda : Y_\Lambda \to \widehat{Y}_\Lambda$ are well defined locally trivial fibrations with connected fibres and $\mathcal{G}_\Lambda : X_\Lambda \to Y_\Lambda$ and $\widehat{\mathcal{G}}_\Lambda : \widehat{X}_\Lambda \to \widehat{Y}_\Lambda$ are finite coverings of the same degree. Hence the map $f_\Lambda \times \mathcal{G}_\Lambda : X_\Lambda \to \widehat{X}_\Lambda \times_{\widehat{Y}_\Lambda} Y_\Lambda$ is surjective and we conclude that $f \times \mathcal{G}$ is a birational map. Consider the morphism $\iota : \mathrm{Deck}(\widehat{\mathcal{G}}) \to \mathrm{Deck}(\mathcal{G})$ given by $\widehat{\tau} \mapsto (f \times \mathcal{G})^{-1} \circ (\widehat{\tau} \times \mathrm{id}_Y) \circ (f \times \mathcal{G})$. In order to conclude the proof, it suffices to construct the inverse map $\iota^{-1} : \mathrm{Deck}(\mathcal{G}) \to \mathrm{Deck}(\widehat{\mathcal{G}})$. We will see that if $\tau \in \mathrm{Bir}(X)$ satisfies $\mathcal{G} \circ \tau = \mathcal{G}$ then

τ sends a generic fibre F of f into another generic fibre F' of f, inducing a birational map $\widehat{\tau} : \widehat{X} \dashrightarrow \widehat{X}$. Let $q \in \widehat{X}$ be a generic point and let $p_0, p_1 \in F$ be two points in the fibre of f over q. Since F is connected we can choose a path p_t joining p_0 and p_1 inside F. Then

$$\widehat{\mathcal{G}}(f(\tau(p_t))) = g(\mathcal{G}(\tau(p_t))) \subset g(\mathcal{G}(F)) = g(G) = \{\widehat{\mathcal{G}}(q)\} \subset \widehat{Y}.$$

Since $\widehat{\mathcal{G}}$ is finite and $f(\tau(p_t))$ is connected we deduce that $f(\tau(p_t))$ consists in a single point of $\widehat{X}$ that we define as $\widehat{\tau}(q)$. It is clear that $f \circ \tau = \widehat{\tau} \circ f$. Taking a local holomorphic section $\sigma : \widehat{V} \subset \widehat{X} \to X$ of f, we see that $\widehat{\tau} = f \circ \tau \circ \sigma$ is holomorphic on the Zariski open set $\widehat{U}$ consisting in the image by f of its generic fibres. Since $\widehat{\mathcal{G}} \circ \widehat{\tau} = \widehat{\mathcal{G}}$ on $\widehat{U}$, we have that $\widehat{\tau} \in \mathrm{Deck}(\widehat{\mathcal{G}}_{|\widehat{U}})$. The fact that $\widehat{\tau}$ extends to a birational map $\widehat{\tau} : \widehat{X} \dashrightarrow \widehat{X}$ follows from assertion (3) in Theorem 3.11. Direct computations show that the map $\tau \mapsto \widehat{\tau}$ is the inverse of $\iota : \mathrm{Deck}(\widehat{\mathcal{G}}) \to \mathrm{Deck}(\mathcal{G})$. □

As an application, we discuss a particular case that will be used in Section 5.

Example 3.21. Let X be a holomorphic vector field on $\mathbb{P}^2$ and let us consider its associated one-parameter group $\{\phi_t\} \subset \mathrm{Aut}(\mathbb{P}^2)$ and the corresponding dual one-parameter group $\{\check{\phi}_t\} \subset \mathrm{Aut}(\check{\mathbb{P}}^2)$. Assume that the foliation defined by X has a rational first integral $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. One can see that the foliation defined by $\{\check{\phi}_t\}$ also admits a rational first integral $\check{f} : \check{\mathbb{P}}^2 \dashrightarrow \mathbb{P}^1$. In fact, we can choose f and $\check{f}$ with connected fibres. Assume now that $\mathcal{G} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$ is a rational map satisfying the relation

$$\mathcal{G} \circ \phi_t = \check{\phi}_t \circ \mathcal{G}. \tag{5}$$

Then $\mathcal{G}$ sends orbits of ϕ_t into orbits of $\check{\phi}_t$ and, since f and $\check{f}$ have connected fibres, there exists a uniquely defined holomorphic map $\widehat{\mathcal{G}} : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\widehat{\mathcal{G}} \circ f = \check{f} \circ \mathcal{G}$. Moreover, $\mathcal{G}$ and $\widehat{\mathcal{G}}$ have the same topological degree. Indeed, the restriction of $\mathcal{G}$ to a generic fibre of f is injective because if p is generic then

$$\mathcal{G}(\phi_{t_1}(p)) = \mathcal{G}(\phi_{t_2}(p)) \Rightarrow \check{\phi}_{t_1 - t_2}(\mathcal{G}(p)) = \mathcal{G}(p).$$

This implies that $\check{\phi}_{t_1-t_2} = \mathrm{id}_{\check{\mathbb{P}}^2}$ and consequently $\phi_{t_1-t_2} = \mathrm{id}_{\mathbb{P}^2}$. Hence the hypothesis of Proposition 3.20 are satisfied. Then $\mathrm{Deck}(\mathcal{G}) \simeq \mathrm{Deck}(\widehat{\mathcal{G}})$ and $\mathcal{G} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$ is Galois if and only if $\widehat{\mathcal{G}} : \mathbb{P}^1 \to \mathbb{P}^1$ is Galois. □

3.3. **Branched coverings.** Let N be a normal projective variety and let Y be a projective manifold of the same dimension. Let $\rho : N \to Y$ be a surjective finite morphism. Then ρ is a branched covering in the sense of [31] (cf. loc. cit. Definition 1.1.1 and Example 1.1.2). The ramification locus $\Delta_\rho \subset N$ and the branching locus $\Lambda_\rho \subset Y$ of ρ are

$$\Delta_\rho = \{\det(d\rho) = 0\} \qquad \text{and} \qquad \Lambda_\rho = \rho(\Delta_\rho).$$

Notice that Δ_ρ has a natural divisor structure. The normal form of ρ at a generic point of an irreducible component $D = \{z = 0\}$ of Δ_ρ is given by $\rho(z, w) = (z^r, w)$ (see [31, Theorem 1.1.8]). The positive integer r is constant along $D \setminus \rho^{-1}(\mathrm{Sing}(\Lambda_\rho))$ (see [31, Corollary 1.1.13]), and it is called the *ramification index* ϱ_D of ρ along D. Notice that $\det(d\rho)(z, w) = z^{\varrho_D - 1} u(z, w)$

with $u(0,w) \neq 0$. Consequently, we can write down the divisor structure of the ramification locus as $\Delta_\rho = \sum_D (\varrho_D - 1)D$. We define the ramification index of an irreducible component D of $\rho^{-1}(\Lambda_\rho)$ which is not contained in Δ_ρ as $\varrho_D = 1$.

We fix an embedding $Y \subset \mathbb{P}^N$ in order to consider the generic hyperplane curve $\ell^Y \subset Y$ and we define the *degree* of a hypersurface $C \subset Y$ as the intersection product $\deg C := C \cdot \ell^Y$. Notice that this definition of $\deg C$ depends on the embedding $Y \subset \mathbb{P}^N$.

Definition 3.22. *Let $\{C_i\}_{i\in I}$ be the set of irreducible components of Λ_ρ and let $\{D_{ij}\}_{j=1}^{k_i}$ denote the set of irreducible components of $\rho^{-1}(C_i)$. We define the* branching type $\mathcal{B}_\rho$ *and the* extended branched type $\mathcal{B}_\rho^+$ *of ρ as the unordered sequences*

$$\mathcal{B}_\rho := [(\varrho_{D_{i1}}, \ldots, \varrho_{D_{ik_i}})]_{i\in I} \quad \text{and} \quad \mathcal{B}_\rho^+ := [(\deg C_i; \varrho_{D_{i1}}, \ldots, \varrho_{D_{ik_i}})]_{i\in I}.$$

We say that the branching type of ρ is regular *if it has the form $\mathcal{B}_\rho = [(\varrho_i)_{d/\varrho_i}]_{i\in I}$, where ϱ_i divides $d = \deg \rho$ and $(\varrho_i)_k$ stands for a sequence of k times ϱ_i. We say that the branching type of ρ is* extremal *if it has the form $\mathcal{B}_\rho = [(d)_1, \ldots, (d)_1]$.*

Proposition 3.23. *Let $\rho : N \to Y$ be a branched covering of degree d. For a generic choice of a hyperplane curve $\ell \subset Y$, the curve $S_\rho^\ell = \rho^{-1}(\ell)$ is smooth and its genus $g(S_\rho^\ell)$ is given by the formula*

$$2 - 2g(S_\rho^\ell) = (2 - 2g(\ell))d - \sum_i \deg C_i \sum_j (\varrho_{D_{ij}} - 1), \tag{6}$$

depending only on the extended branching type $\mathcal{B}_\rho^+$ and the genus $g(\ell)$ of ℓ.

Notice that the restriction of ρ to S_ρ^ℓ is the one-dimensional branched covering $\rho^\ell : S_\rho^\ell \to \ell$ associated to ρ considered in Proposition 3.17.

Proof. To show the smoothness of S_ρ^ℓ, let us consider the codimension ≥ 2 subvariety $Z = \mathrm{Sing}(\Lambda_\rho)$ of Y, which contains $\rho(\mathrm{Sing}(\Delta_\rho)) \cup \rho(\mathrm{Sing}(N))$ thanks to [31, Corollary 1.1.10]. By applying $n-2$ times Bertini's theorem, we can choose $H_1, \ldots, H_{n-2}$ hyperplanes in $\mathbb{P}^N$ such that $S^Y := Y \cap H_1 \cap \cdots \cap H_{n-2}$ is a smooth surface. Consider the (non necessarily irreducible) curve $C^Y := \Lambda_\rho \cap S^Y$. Now, again by Bertini's theorem, we can choose a generic hyperplane H_{n-1} in $\mathbb{P}^N$ avoiding the set of isolated points $Z \cap S^Y = Z \cap H_1 \cap \cdots \cap H_{n-2}$ and such that the curve $\ell := S^Y \cap H_{n-1}$ is smooth and H_{n-1} meets C^Y transversely in regular points (cf. [23, Theorem 17.16 and Exercice 14.6]). Using the normal form $\rho(z,w) = (z^\varrho, w)$ at a generic point of Δ_ρ we conclude that $\rho^{-1}(\ell)$ is smooth.

For such a choice of the hyperplane curve ℓ, it cuts each irreducible component C_i of Λ_ρ in $\deg C_i$ different points. The genus $g(S_\rho^\ell)$ of S_ρ^ℓ can be computed from the extended branching type $\mathcal{B}_\rho^+$ of ρ by the Riemann-Hurwitz formula. □

In the context of Riemann surfaces, there is the notion of branched covering of regular type (see for instance [25, §II6]). That notion makes sense in our more general setting as follows:

Definition 3.24. *A branched covering ρ is of* regular type *if its branching type is regular, or equivalently, if its ramification indices are constant along each fibre (over the generic points of Λ_ρ).*

Here, for a generic point of Λ_ρ, it is meant a point belonging to a dense Zariski open subset $\Lambda_\rho^{\mathrm{reg}}$ of Λ_ρ. In fact, we can take $\Lambda_\rho^{\mathrm{reg}} = \Lambda_\rho \setminus \mathrm{Sing}(\Lambda_\rho)$ because $\rho(\mathrm{Sing}(\Delta_\rho)) \subset \mathrm{Sing}(\Lambda_\rho)$ by [31, Corollary 1.1.10].

Remark 3.25. A branched covering ρ of prime degree d is of regular type if and only if $\mathcal{B}_\rho$ is extremal, i.e. $\varrho_D = d$ for each irreducible component $D \subset \Delta_\rho$.

Clearly, if ρ is Galois then ρ is of regular type. The converse is not true in general (even in the case of Riemann surfaces, see [25, II§6 Example 14]). In [21, Lemma 1] L. Greenberg shows that if the source N is a connected and simply connected Riemann surface (for instance if $N = \mathbb{P}^1$) then regular type implies Galois. In [38] it is claimed that if the target Y is $\mathbb{P}^1$ the above implication also holds but unfortunately this is not true as the following counter-example shows.

Example 3.26. Let $Y_0 = \mathbb{P}^1 \setminus \bigcup_{i=0}^{3} D_i$ be the complement of four disjoint open disks in $\mathbb{P}^1$ with boundaries γ_i and let $\mu : \pi_1(Y_0) \simeq \mathbb{Z}[\gamma_1] * \mathbb{Z}[\gamma_2] * \mathbb{Z}[\gamma_3] \to \mathfrak{S}_4$ be the morphism given by $\mu(\gamma_1) = (1234)$, $\mu(\gamma_2) = (12)(34)$ and $\mu(\gamma_3) = (14)(23)$. Define $\rho_0 : N_0 = \widetilde{Y_0} \times_\mu \{1,2,3,4\} \to Y_0$ to be the suspension covering associated to μ which is not Galois because the monodromy group $M = \mathrm{Im}\,\mu$ has order $8 > 4$. Notice that $[\gamma_0]^{-1} = [\gamma_1\gamma_2\gamma_3] \overset{\mu}{\mapsto} (1432)$. It is clear that $\rho_0^{-1}(\gamma_i) = \delta_i$ is a circle and $\rho_{0|\delta_i}$ is a $4:1$ map for $i = 0,1$; on the other hand, if $i = 2,3$ then $\rho_0^{-1}(\gamma_i) = \delta_i^+ \sqcup \delta_i^-$ are two disjoint circles and $\rho_{0|\delta_i^\pm}$ is a $2:1$ map. Consequently, we can glue disks Δ_i, $i = 0,1$, and $\Delta_i^\pm$, $i = 2,3$, to N_0 in order to obtain a compact Riemann surface N and a branched covering $\rho : N \to \mathbb{P}^1$ extending ρ_0 with branched points $q_0, q_1, q_2, q_3 \in \mathbb{P}^1$ and ramification points $p_0, p_1, p_2^\pm, p_3^\pm \in N$ with ramification indices $4,4,2,2$ respectively. Hence $\mathcal{B}_\rho = [(2)_2, (2)_2, (4)_1, (4)_1]$ and consequently ρ is a degree 4 branched covering of regular type. Riemann-Hurwitz formula implies that N has genus 2. □

Remark 3.27. Let $\rho : N \to Y$ be a connected branched covering of degree ≤ 3. Then ρ is Galois if and only if it is of regular type. Indeed, if $\deg \rho \leq 2$ then ρ is always Galois and there is nothing to prove. If $\deg \rho = 3$ and ρ is of regular type then its monodromy group M is contained in the alternate group $A_3 = \{\mathrm{Id}, (123), (132)\}$ so that $|M| \leq 3$ and we apply Condition (7) in Theorem 2.6. Consequently, Example 3.26 has the minimal degree 4 for a regular type non-Galois branched covering. On the other hand, Riemann-Hurwitz formula implies that the minimal genus is also 2. This implies in particular that every regular type branched covering of degree 4 from the torus to the sphere is Galois.

In the following result the difference between Galois coverings and coverings of regular type is enlightened.

Proposition 3.28. *Let $\rho : N \to Y$ be a finite branched covering and let $(N \times_Y N)_\nu \to N \times_Y N$ be the normalization of the fibered product $N \times_Y N$. Then the composition $p : (N \times_Y N)_\nu \to N \times_Y N \to N$ is a branched covering and*

(a) ρ is of regular type if and only if p is unramified;

(b) ρ is Galois if and only if p is trivial, i.e. the restriction of p to each irreducible component of $(N \times_Y N)_\nu$ is an isomorphism.

Proof. Notice that property (a) can be checked locally in Y. Let $N_0 := \bigsqcup_{i=1}^{r} \mathbb{D}_i^n \hookrightarrow N$ be the preimage by ρ of a polydisk $Y_0 := \mathbb{D}^n \hookrightarrow Y$ such that $\{0\} \times \mathbb{D}^{n-1} = \Lambda_\rho \cap Y_0$ and the restriction ρ_i of ρ to the polydisk $\mathbb{D}_i^n$ writes as $\rho_i(x_i, u) = (x_i^{n_i}, u)$. Then

$$N_0 \times_Y N_0 = \bigsqcup_{i,j=1}^{r} \{(x_i, u, y_j, v) \in \mathbb{D}^{2n} \,|\, x_i^{n_i} = y_j^{n_j},\, u = v\} \hookrightarrow N \times_Y N$$

is the preimage in $N \times_Y N$ of $N_0 \hookrightarrow N$ by the projection $N \times_Y N \to N$. The preimage $\mathcal{P}_0$ of N_0 by p is nothing more than the normalization of $N_0 \times_Y N_0$. If $n_i = n_j$ for all $i, j = 1, \ldots, r$ then $\mathcal{P}_0$ is a disjoint union of polydisks $\{x_i = \zeta^k y_j\} \times \mathbb{D}^{n-1}$, over which $p(x_i, y_j, u) = (x_i, u)$ is an isomorphism, where ζ is a primitive n_i-root of the unity. This shows that if ρ is of regular type then p is unramified. To prove the converse, assume that $n_i \neq n_j$. Then $\{(x_i, y_j) \in \mathbb{D}^2 \,|\, x_i^{n_i} = y_j^{n_j}\} \times \mathbb{D}^{n-1}$ decomposes as k branches of type $x_i^{n'_i} = \zeta' y_j^{n'_j}$, and where $n_i = n'_i k$, $n_j = n'_j k$, $\gcd(n'_i, n'_j) = 1$ and ζ' is a primitive k-root of the unity. The normalization morphism of each branch takes the form $\mathbb{D}^n \ni (z, w) \mapsto (z^{n'_j}, z^{n'_i}, w)$. Hence the restriction of p to the normalization of this branch writes as $p(z, w) = (z^{n'_i}, w)$ which ramifies if $n'_i > 1$. Finally, if $n_i \neq n_j$ there is always a connected component of the preimage of $\{0\} \times \mathbb{D}^{n-1}$ with $n'_i > 1$.

To prove assertion (b) denote by $\rho_\Lambda : N_\Lambda \to Y_\Lambda$ the (maximal) restricted unramified covering induced by $\rho : N \to Y$. Since $N \setminus N_\Lambda$ has real codimension two, the number of irreducible components of $(N \times_Y N)_\nu$ coincide with the number of connected components of $N_\Lambda \times_{Y_\Lambda} N_\Lambda$. We conclude by using the characterization (1) in Theorem 2.6. $\square$

3.4. **Rational maps of regular type.** Let $\mathcal{G} : X \dashrightarrow Y$ be a rational dominant map between connected complex projective manifolds of the same dimension. We fix a desingularization $\widetilde{\mathcal{G}} : \widetilde{X} \to Y$ of $\mathcal{G}$ as in Subsection 3.1 and its Stein factorization (3). Recall that N and ρ are uniquely determined by $\mathcal{G}$ (cf. Proposition 3.8). Since $\det(d\widetilde{\mathcal{G}}) = \det(d\rho \circ \gamma) \det(d\gamma)$, we have the following equality of divisors $\Delta_{\widetilde{\mathcal{G}}} = \gamma^* \Delta_\rho + \Delta_\gamma$. It follows that

$$\Lambda_\rho = \rho(\Delta_\rho) \subset \widetilde{\mathcal{G}}(\Delta_{\widetilde{\mathcal{G}}}) = \Lambda_{\widetilde{\mathcal{G}}} \subset \Lambda_{\mathcal{G}} = \Lambda_{\widetilde{\mathcal{G}}} \cup \widetilde{\mathcal{G}}(\mathcal{E}) \subset Y.$$

For a given irreducible component D of $\Delta_{\widetilde{\mathcal{G}}}$ there are two possibilities:

(a) D is not included in Δ_γ. In that case γ is a local biholomorphism at a generic point of D. Consequently, as ρ is a branched covering, there are local coordinates (z, w) such that $D = \{z = 0\}$ and $\widetilde{\mathcal{G}}(z, w) = (z^k, w)$ with $k > 1$. We say that D is a ramification component of $\widetilde{\mathcal{G}}$ with ramification index $\varrho_D := k$. Notice that $\dim \widetilde{\mathcal{G}}(D) = \dim D$.

(b) D is a component of Δ_γ. In that case, $\dim \widetilde{\mathcal{G}}(D) < \dim D$ and we say that D is a contracting component of $\widetilde{\mathcal{G}}$.

Definition 3.29. *We define the* ramification divisor $\mathcal{R}_{\widetilde{\mathcal{G}}} = \sum_D (\varrho_D - 1)D$ *where D ranges the ramification components of $\Delta_{\widetilde{\mathcal{G}}}$. It coincides with the strict transform of Δ_ρ by γ. For each $\varrho > 1$ we define the reduced (or empty) hypersurface $\mathcal{R}^\varrho_{\widetilde{\mathcal{G}}} = \bigcup_{\varrho_D=\varrho} D \subset \widetilde{X}$, so that we can write*

$$\mathcal{R}_{\widetilde{\mathcal{G}}} = \sum_{\varrho>1} (\varrho - 1)\mathcal{R}^\varrho_{\widetilde{\mathcal{G}}}. \tag{7}$$

The contracting divisor *of $\widetilde{\mathcal{G}}$ is the divisor defined as $\mathcal{C}_{\widetilde{\mathcal{G}}} := \Delta_{\widetilde{\mathcal{G}}} - \mathcal{R}_{\widetilde{\mathcal{G}}}$, which is supported on the contracting components of $\widetilde{\mathcal{G}}$. The* branching type *of $\mathcal{G}$ is defined as the branching type of ρ. In particular, we say that $\mathcal{G}$ is of* regular type *if ρ is of regular type.*

Notice that $\overline{\Lambda_{\widetilde{\mathcal{G}}} \setminus \Lambda_\rho}$ is included in $\widetilde{\mathcal{G}}(\mathcal{C}_{\widetilde{\mathcal{G}}})$ which has codimension greater than 1.

Remark 3.30. Since γ is birational, if $f = 0$ is a local reduced equation of an irreducible component $D \subset \mathcal{R}_{\widetilde{\mathcal{G}}}$ then we can compute the ramification index ϱ_D of ρ along $\gamma(D)$ directly from $\widetilde{\mathcal{G}}$ by writing $\det(d\widetilde{\mathcal{G}}) = f^{\varrho_D - 1}u$ with f not dividing u. Notice that these ramification indices may include some ones corresponding to components of $\mathcal{R}_{\widetilde{\mathcal{G}}} \subset \widetilde{X}$ that are contracted by β.

We are interested in deciding if a rational map $\mathcal{G} : X \dashrightarrow Y$ is of regular type just by looking at geometric elements contained in X and not in $\widetilde{X}$. With that purpose we introduce the following algebraic subsets of X:

$$\mathcal{R}^\varrho_{\mathcal{G}} := \overline{\beta(\mathcal{R}^\varrho_{\widetilde{\mathcal{G}}} \setminus \mathcal{E})}$$

and $\Sigma^\rho_{\mathcal{G}} \subset \Sigma_{\mathcal{G}}$ which is the union of the irreducible components S of $\Sigma_{\mathcal{G}}$ such that $\overline{\beta^{-1}(S) \setminus \mathcal{C}_{\widetilde{\mathcal{G}}}} \subset \mathcal{R}^\varrho_{\widetilde{\mathcal{G}}}$. In other words, all the ramification components of $\beta^{-1}(S)$ share the same ramification index ϱ if and only if $S \subset \Sigma^\varrho_{\mathcal{G}}$.

Let us consider the following condition on $\mathcal{G}$:

Hypothesis 3.31. *For every irreducible components S of $\Sigma_{\mathcal{G}}$ and D and D' of $\overline{\beta^{-1}(S) \setminus \mathcal{C}_{\widetilde{\mathcal{G}}}}$ we have $\widetilde{\mathcal{G}}(D) = \widetilde{\mathcal{G}}(D')$.*

We will see in Section 5 that this hypothesis is always fulfilled by the Gauss map associated to foliations on $\mathbb{P}^2$.

Proposition 3.32. *Let $\mathcal{G} : X \dashrightarrow Y$ be a dominant rational map between projective manifolds of the same dimension and consider the following assertions:*

(1) $\mathcal{G}$ is of regular type,

(2) for each generic point $y \in \Lambda_\rho$ there is $\varrho | d$, $\varrho > 1$, such that

$$\widetilde{\mathcal{G}}^{-1}(y) \subset \mathcal{R}^\varrho_{\widetilde{\mathcal{G}}} \setminus \mathcal{C}_{\widetilde{\mathcal{G}}},$$

(3) for each generic point $y \in \Lambda_\rho$ there is $\varrho | d$, $\varrho > 1$, such that

$$\mathcal{G}^{-1}(y) := \beta(\widetilde{\mathcal{G}}^{-1}(y)) \subset (\mathcal{R}^\varrho_{\mathcal{G}} \setminus \Sigma_{\mathcal{G}}) \cup \Sigma^\varrho_{\mathcal{G}}.$$

Then (1) and (2) are equivalent and are implied by (3). In addition, if $\mathcal{G}$ satisfies Hypothesis 3.31 then the three assertions are equivalent.

Proof. By definition, the branched covering ρ is of regular type if and only if for generic $y \in \Lambda_\rho$ there is $\varrho | d$, $\varrho > 1$, such that $\rho^{-1}(y) \subset \mathcal{R}^\varrho_\rho = \gamma(\mathcal{R}^\varrho_{\widetilde{\mathcal{G}}})$. This is equivalent to assertion (1) because if $y \in \Lambda_\rho$ is generic then $y \in \Lambda_\rho \setminus \widetilde{\mathcal{G}}(\mathcal{C}_{\widetilde{\mathcal{G}}})$ and γ is an isomorphism outside $\mathcal{C}_{\widetilde{\mathcal{G}}}$.

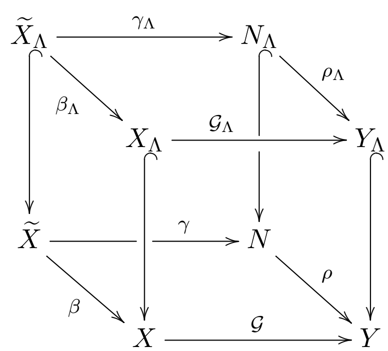
On the other hand, from assertion (3) we deduce that, for $y \in \Lambda_\rho \setminus \widetilde{\mathcal{G}}(\mathcal{C}_{\widetilde{\mathcal{G}}})$ and $\widetilde{x} \in \widetilde{\mathcal{G}}^{-1}(y)$, the point $\beta(\widetilde{x})$ belongs either to $\mathcal{R}^\varrho_{\mathcal{G}} \setminus \Sigma_{\mathcal{G}}$ or to $\Sigma^\varrho_{\mathcal{G}}$. In both cases $\widetilde{x} \in \mathcal{R}^\varrho_{\widetilde{\mathcal{G}}} \setminus \mathcal{C}_{\widetilde{\mathcal{G}}}$.

Finally, from assertion (2) we deduce that, for generic $y \in \Lambda_\rho \setminus \widetilde{\mathcal{G}}(\mathcal{C}_{\widetilde{\mathcal{G}}})$ and $\widetilde{x} \in \widetilde{\mathcal{G}}^{-1}(y)$, the point $\widetilde{x}$ belongs either to $\beta^{-1}(\mathcal{R}^\varrho_{\mathcal{G}} \setminus \Sigma_{\mathcal{G}})$ or to an irreducible component $D \subset \beta^{-1}(S)$ having ramification index ϱ, where S is an irreducible component of $\Sigma_{\mathcal{G}}$. Hypothesis 3.31 implies that for every irreducible component D' of $\overline{\beta^{-1}(S) \setminus \mathcal{C}_{\widetilde{\mathcal{G}}}}$ there is $\widetilde{x}' \in D'$ such that $\widetilde{\mathcal{G}}(\widetilde{x}') = y$ and consequently $\varrho_{D'} = \varrho$. Hence $\beta(\widetilde{x}) \in S \subset \Sigma^\varrho_{\mathcal{G}}$. □

The following result can be considered a generalization of [21, Lemma 1] to arbitrary dimension:

Theorem 3.33. *Let $\mathcal{G} : X \dashrightarrow Y$ be a dominant rational map between connected complex projective manifolds of the same dimension. Assume that X admits a rational generic hyperplane curve $\ell^X \simeq \mathbb{P}^1$ (for instance if $X = \mathbb{P}^n$). Then, $\mathcal{G}$ is of regular type if and only if $\mathcal{G}$ is Galois.*

Proof. Consider the following commutative diagram.



Let $\ell^X \subset X$ be a rational generic hyperplane curve. Since β_Λ and γ_Λ are isomorphisms, the curve $\ell^N_\Lambda := \gamma_\Lambda(\beta_\Lambda^{-1}(\ell^X_\Lambda))$ is isomorphic to $\ell^X_\Lambda := \ell^X \cap X_\Lambda$ and its closure ℓ^N in N is a rational (maybe singular) curve. Moreover, by Corollary 3.5, the natural inclusion induces an epimorphism $\pi_1(\ell^N_\Lambda) \simeq \pi_1(\ell^X_\Lambda) \twoheadrightarrow \pi_1(X_\Lambda) \simeq \pi_1(N_\Lambda)$. If $\mathcal{G}$ is of regular type then $p : (N \times_Y N)_\nu \to N$ is an unramified covering by Proposition 3.28(a). Let $p_\ell : (N \times_Y N)_\nu \times_N \ell^N_\nu \to \ell^N_\nu$ be the pull-back covering of p by the composition $\ell^N_\nu \to \ell^N \hookrightarrow N$ of the normalization of ℓ^N and the inclusion of ℓ^N into N. Since the base space $\ell^N_\nu = \mathbb{P}^1$ of p_ℓ is simply connected we deduce that p_ℓ is the trivial covering (i.e. a product). By Lemma 2.3 be deduce that the covering p_Λ :

$(N \times_Y N)_\Lambda \to N_\Lambda$ is also trivial. From Proposition 3.28(b) we conclude that $\rho_\Lambda : N_\Lambda \to Y_\Lambda$ is Galois. Hence $\mathcal{G}$ is also Galois because $\mathrm{Deck}(\rho_\Lambda) \simeq \mathrm{Deck}(\rho) \simeq \mathrm{Deck}(\mathcal{G})$. □

The above result, Remark 3.25 and Corollary 2.8 imply the following.

Corollary 3.34. *Let $\mathcal{G} : \mathbb{P}^n \dashrightarrow Y$ be a dominant rational map with topological degree d. If all the ramification indices of ρ are equal to d then $\mathcal{G}$ is Galois with cyclic monodromy group. When d is prime the converse is also true. More precisely, if $\mathcal{G}$ is Galois then all the ramification indices of ρ are equal to d and the monodromy group is cyclic.*

However there are examples of cyclic Galois rational maps whose ramification indices are less than its degree:

Example 3.35. If $\mathcal{G}_i : X_i \dashrightarrow Y_i$ are Galois rational maps of degree $d_i > 1$ with cyclic monodromy group, $i = 1, 2$, and $\gcd(d_1, d_2) = 1$ then $\mathcal{G} := \mathcal{G}_1 \times \mathcal{G}_2 : X_1 \times X_2 \dashrightarrow Y_1 \times Y_2$ is Galois and $\mathrm{Deck}(\mathcal{G}) \simeq \mathrm{Deck}(\mathcal{G}_1) \oplus \mathrm{Deck}(\mathcal{G}_2) \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \simeq \mathbb{Z}_d$ with $d = \deg \mathcal{G} = d_1 d_2$ but the ramification indices of $\mathcal{G}$ are $\leq \max(d_1, d_2) < d$. □

3.5. **Galois rational maps from the projective line into itself.** The simplest situation to test Galois property for rational maps is the case where the source and the target are both the Riemann sphere. In this section we recall the classification of the rational maps $\mathcal{G} : \mathbb{P}^1 \to \mathbb{P}^1$ which define Galois coverings. This classification, used in what follows, provides the normal form of the Galois rational maps as well as their corresponding branching types. If we regard $\mathbb{P}^1$ as the unit sphere $\mathbb{S}^2$, then the deck transformation group of $\mathcal{G}$ is conjugate to a finite subgroup of the group $\mathrm{SO}_3 = \mathrm{PSU}_2$, which is the maximal compact subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and whose finite subgroups are well-known: cyclic, dihedral, tetrahedral, octahedral and icosahedral. In fact, for each finite subgroup G of $\mathrm{PSL}_2(\mathbb{C})$ there is a Galois branched covering $\mathcal{G} : \mathbb{P}^1 \to \mathbb{P}^1$ whose deck transformation group (also called Galois group) is just G. More precisely, the following classification goes back to Klein [24, Chapter IV], see also [36, Theorem 3.6.2, pp. 43–44 and 65–66] for a modern exposition:

Theorem 3.36. *Let $\mathcal{G} : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree d Galois rational map with deck transformation group G. Then $\mathcal{G}$ (resp. G) is left-right-equivalent (resp. conjugated) to one of the rational functions (resp. triangular subgroups of $\mathrm{SO}_3 \subset \mathrm{PSL}_2(\mathbb{C})$) appearing in Table 1, where*

$$T(p,q,r) = \langle \sigma, \tau \mid \sigma^p = \tau^q = (\sigma\tau)^r = 1 \rangle, \quad \zeta_n = e^{\frac{2i\pi}{n}}, \quad \phi = \frac{\sqrt{5}-1}{2} \quad \textit{and}$$

$$\mathcal{G}_{C_n}(z) = z^n, \quad \mathcal{G}_{D_n} = \frac{(z^n+1)^2}{4z^n}, \quad \mathcal{G}_T(z) = \left(\frac{z^4 + 2i\sqrt{3}\, z^2 + 1}{z^4 - 2i\sqrt{3}\, z^2 + 1} \right)^3,$$

$$\mathcal{G}_O(z) = \frac{(z^8 + 14z^4 + 1)^3}{108 z^4 (z^4 - 1)^4}, \quad \mathcal{G}_I(z) = \frac{(z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1)^3}{-1728 z^5 (z^{10} + 11z^5 - 1)^5}.$$

	d	$\mathcal{G}$	$\mathcal{B}_{\mathcal{G}}$	G	$\sigma(z)$	$\tau(z)$
Cyclic	n	$\mathcal{G}_{C_n}$	$[(n)_1, (n)_1]$	$C_n = T(1,n,n)$	z	$\zeta_n z$
Dihedral	$2n$	$\mathcal{G}_{D_n}$	$[(2)_n, (2)_n, (n)_2]$	$D_n = T(2,2,n)$	$\frac{1}{z}$	$\frac{\zeta_n}{z}$
Tetrahedral	12	$\mathcal{G}_T$	$[(2)_6, (3)_4, (3)_4]$	$A_4 = T(2,3,3)$	$-z$	$\frac{z+i}{z-i}$
Octahedral	24	$\mathcal{G}_O$	$[(2)_{12}, (3)_8, (4)_6]$	$S_4 = T(2,3,4)$	$\frac{iz-1}{z-i}$	$\frac{z+i}{z-i}$
Icosahedral	60	$\mathcal{G}_I$	$[(2)_{30}, (3)_{20}, (5)_{12}]$	$A_5 = T(2,3,5)$	$\frac{\phi-z}{\phi z+1}$	$\frac{(\phi-z)\zeta_5}{\phi z+1}$

TABLE 1. Klein's classification of Galois rational functions on $\mathbb{P}^1$.

4. FOLIATIONS AND WEBS

Given a codimension one holomorphic foliation $\mathcal{F}$ on the projective space $\mathbb{P}^n$, its associated Gauss map $\mathcal{G}_{\mathcal{F}}$ induces a well defined web $\operatorname{Leg}\mathcal{F}$ on the dual space $\check{\mathbb{P}}^n$ which is called the Legendre transform of $\mathcal{F}$ provided that $\mathcal{G}_{\mathcal{F}}$ is dominant. In this section we study the direct image of foliations and webs by more general rational maps. In particular we deduce that the foliation $\mathcal{F}$ is Galois, which means that the Gauss map $\mathcal{G}_{\mathcal{F}}$ is Galois, if and only the web $\mathcal{G}_{\mathcal{F}}^* \operatorname{Leg}\mathcal{F}$ is totally decomposable. This criterion will be the starting point of the discussion of Galois foliations on the projective plane carried out in Section 5.

In order to give a precise meaning to the direct image of a foliation (or more generally a web) we introduce a definition of a web with multiplicities that generalizes the usual notion of web given for instance in [35, §1.3.1 and §1.3.3].

Definition 4.1. *For a positive integer k, a codimension one k-web (with multiplicities) $\mathcal{W}$ on a complex manifold Y is given by an open cover $\{V_i\}$ of Y and k-symmetric forms $\omega_i \in \operatorname{Sym}^k\Omega^1_Y(V_i)$ subject to the conditions*

(a) for each non-empty intersection $V_i \cap V_j$ there exists a non-vanishing function $g_{ij} \in \mathcal{O}^*_Y(V_i \cap V_j)$ such that $\omega_i = g_{ij}\omega_j$;
(b) the zero set $\operatorname{Sing}(\omega_i)$ of ω_i has codimension at least two;
(c) the germ of ω_i at every generic point of V_i is a product of k germs of integrable 1-forms $\omega_{i\alpha}$, $\alpha = 1, \ldots, k$.

If the 1-forms $\omega_{i\alpha}$ are two by two not collinear at the generic point of V_i, we say that $\mathcal{W}$ is reduced. *In that case, the subset of Y where the non-collinearity condition fails is called the* discriminant *of $\mathcal{W}$ and it is denoted by $\Delta(\mathcal{W})$. The* singular set *$\Sigma_{\mathcal{W}}$ of $\mathcal{W}$ is defined by $\Sigma_{\mathcal{W}} \cap V_i = \operatorname{Sing}(\omega_i)$ and it is contained in $\Delta(\mathcal{W})$.*

The usual definition of web is that of reduced web in our terminology. Notice that for $k = 1$ we recover the usual definition of (singular) codimension one foliation $\mathcal{F}$ (see [4, 6]). In that case $\Delta(\mathcal{F}) = \Sigma_{\mathcal{F}}$ is just the singular set of $\mathcal{F}$. For arbitrary $k \geq 2$, a reduced k-web always looks like locally as the superposition of k foliations, but not necessarily globally. If this is the case we say that the web is *totally decomposable.* In fact, there is a *monodromy representation* $\mu_{\mathcal{W}} : \pi_1(Y \setminus \Delta(\mathcal{W})) \to \mathfrak{S}_k$ of $\mathcal{W}$ which determines the irreducible subwebs of $\mathcal{W}$ and whose triviality is equivalent to the total decomposability of $\mathcal{W}$ (see [35, §1.3.3 and §1.3.4]). In the reduced case, condition (c) allows us to define the *tangent set* $T_y\mathcal{W}$ of $\mathcal{W}$ at a point $y \in U_i \setminus \Delta(\mathcal{W})$ as the union of the k different kernels at y of the linear factors of $\omega_i(y)$.

Remark 4.2. Notice that the functions $\{g_{ij}\}$ form a cocycle with values in $\mathcal{O}_Y^*$ determining a line bundle $L \to Y$. The collection $\{\omega_i\}$ defines an element in $H^0(Y, \mathrm{Sym}^k\Omega_Y^1 \otimes L)$ which can be interpreted as a meromorphic k-symmetric form ω on Y by setting $\omega = \frac{\omega_i}{f_i} = \frac{\omega_j}{f_j}$, once we fix a meromorphic section of L, i.e. a collection of meromorphic functions f_i on V_i fulfilling $f_i = g_{ij} f_j$. The set $\mathcal{P}_k(Y)$ of meromorphic k-symmetric forms on Y can be identified with the space of degree k homogeneous polynomials in $n = \dim Y$ variables over the field $\mathbb{C}(Y)$ of meromorphic functions on Y. Hence we can consider the prime decomposition $\omega = \prod_\alpha \omega_\alpha^{e_\alpha}$ of ω in $\mathcal{P}_k(Y)$. Each prime factor ω_α defines a reduced (and irreducible) web $\mathcal{W}_\alpha$ on Y such that $\mathcal{W}$ is the superposition of the webs $\mathcal{W}_\alpha$ with multiplicities e_α. We define the *reduction* $\mathcal{W}_{\mathrm{red}}$ of $\mathcal{W}$ as the reduced web given by the superposition of the webs $\mathcal{W}_\alpha$ (without multiplicities), which is defined by the meromorphic symmetric form $\omega' = \prod_\alpha \omega_\alpha$. We denote $\mathcal{W} = \boxtimes_\alpha \mathcal{W}_\alpha^{e_\alpha}$ and $\mathcal{W}_{\mathrm{red}} = \boxtimes_\alpha \mathcal{W}_\alpha$. We define the discriminant of $\mathcal{W}$ as $\Delta(\mathcal{W}) = \Delta(\mathcal{W}_{\mathrm{red}})$ and the tangent set of $\mathcal{W}$ at a generic point $y \in Y$ as $T_y\mathcal{W} = T_y\mathcal{W}_{\mathrm{red}}$.

4.1. **Inverse and direct image of foliations and webs.** Let $\mathcal{G} : X \dashrightarrow Y$ be a dominant rational map between complex projective manifolds and let $\mathcal{G}_U : U \to Y$ be its restriction to the dense Zariski open subset $U = X \setminus \Sigma_{\mathcal{G}}$, where $\Sigma_{\mathcal{G}}$ is the indeterminacy locus of $\mathcal{G}$. Let $\mathcal{W}$ be a k-web (with multiplicities) on Y given as in Definition 4.1. As $\mathcal{G}_U$ is a morphism we can consider the open cover $\{U_i = \mathcal{G}_U^{-1}(V_i)\}$ of U and the pull-back $\eta_i = \mathcal{G}_U^*\omega_i \in \mathrm{Sym}^k\Omega_U^1(U_i)$ of the k-symmetric forms ω_i. Since $d\mathcal{G} \not\equiv 0$ the symmetric forms η_i do not vanish identically and we can write them as $\eta_i = f_i\eta_i'$ with $f_i \in \mathcal{O}_U(U_i)$ and $\mathrm{codim}(\mathrm{Sing}(\eta_i')) \geq 2$. Thus, the collection $\{U_i, \eta_i'\}$ defines a k-web $\mathcal{W}'$ on U (with the same multiplicities as $\mathcal{W}$). Since the indeterminacy set $\Sigma_{\mathcal{G}}$ of $\mathcal{G}$ has codimension ≥ 2, an application of Levi's extension theorem (see for instance [6, Remarque 2.17]) allows us to extend $\mathcal{W}'$ to a k-web $\mathcal{G}^*\mathcal{W}$ on X, which is called the *inverse image* (or *pull-back*) of $\mathcal{W}$ by $\mathcal{G}$.

Remark 4.3. One can define a notion of web on a singular projective variety X as a web on its smooth part $X \setminus \mathrm{Sing}(X)$ having the property that it extends to a global web on any of the desingularizations of X (see [35, §1.4.3]). It turns out that if $\mathcal{G} : X \dashrightarrow Y$ is a dominant rational map from a complex projective singular variety X into a projective complex manifold Y endowed with a web $\mathcal{W}$ then the pull-back $\mathcal{G}^*\mathcal{W}$ defines a web on the smooth part of X which extends to any desingularization $\delta : \widetilde{X} \to X$ of X by means of $(\mathcal{G} \circ \delta)^*\mathcal{W}$.

From now on we will suppose that $\dim X = \dim Y$ and that Y is connected and we denote by d the topological degree of $\mathcal{G} : X \dashrightarrow Y$. Let $\mathcal{F}$ be a holomorphic foliation on X of codimension 1. In Proposition 4.6 below we prove that the *direct image* $\mathcal{G}_*\mathcal{F}$ of the foliation $\mathcal{F}$ by such a rational map $\mathcal{G}$ is a well-defined d-web (with multiplicities) on Y. In order to decide whether the web $\mathcal{G}_*\mathcal{F}$ is reduced we introduce the following definition.

Definition 4.4. *We say that a holomorphic foliation $\mathcal{F}$ on X is in* general position *with respect to a rational dominant map $\mathcal{G} : X \dashrightarrow Y$, or that $\mathcal{F}$ is $\mathcal{G}$*-general*, if for generic $y \in Y$ the set of tangent spaces $\{d\mathcal{G}_x(T_x\mathcal{F}) \mid x \in \mathcal{G}^{-1}(y)\}$ has exactly $\deg \mathcal{G}$ elements.*

Clearly, the set of $\mathcal{G}$-general foliations is open. The following result shows that it is non-empty.

Proposition 4.5. *For every dominant rational map $\mathcal{G} : X \dashrightarrow Y$ between projective manifolds of the same dimension $n \geq 2$ there exists a codimension one $\mathcal{G}$-general foliation on X.*

Proof. Fix $y_0 \in Y \setminus \Lambda_{\mathcal{G}}$ and consider the fibre $\mathcal{G}^{-1}(y_0) = \{x_1, \ldots, x_d\} \subset X$. Let us fix an embedding $X \subset \mathbb{P}^N$ and let us consider an affine chart $\mathbb{A}^N \subset \mathbb{P}^N$ containing the points x_i, $i = 1, \ldots, d$. There exists a linear projection $\bar{f} : \mathbb{A}^N \to \mathbb{A}^2$ such that $p_i = \bar{f}(x_i)$ are pairwise different points and $\ker d\bar{f}_{x_i} + T_{x_i}X = T_{x_i}\mathbb{A}^N$. Let $f : X \dashrightarrow \mathbb{P}^2$ be the restriction of $\bar{f}$ to X. Consider the codimension two subspaces $\ell_j := d\mathcal{G}_{x_j}(\ker df_{x_j}) \subset T_{y_0}Y$. For each $j = 1, \ldots, d$, we fix pairwise different codimension one subspaces h_j of $T_{y_0}Y$ containing ℓ_j. Consider the one-dimensional subspaces $r_j = df_{x_j}(d\mathcal{G}_{x_j}^{-1}(h_j)) \subset T_{p_j}\mathbb{P}^2$. We fix affine coordinates (u, v) on $\mathbb{A}^2 \subset \mathbb{P}^2$ such that $p_j = (u_j, v_j)$ with $u_i \neq u_j$ if $i \neq j$ and r_j has equation $v = a_j u + b_j$ with $a_j, b_j \in \mathbb{C}$. Let $p(u)$ be a polynomial such that $p(u_j) = a_j$. The vector field $\partial_u + p(u)\partial_v$ defines a foliation $\mathcal{F}_0$ on $\mathbb{P}^2$ such that $T_{p_j}\mathcal{F}_0 = r_j$. Then $\mathcal{F} = f^*\mathcal{F}_0$ is a $\mathcal{G}$-general foliation on X because $d\mathcal{G}_{x_j}(T_{x_j}\mathcal{F}) = h_j \subset T_{y_0}Y$ are pairwise different subspaces. □

Proposition 4.6. *Let $\mathcal{G} : X \dashrightarrow Y$ be a dominant rational map of degree d between projective manifolds of the same dimension and let $\mathcal{F}$ be a codimension one holomorphic foliation on X. There is a unique d-web (with multiplicities) $\mathcal{G}_*\mathcal{F}$ on Y, called the* direct image *of $\mathcal{F}$ by $\mathcal{G}$, such that $T_y\mathcal{G}_*\mathcal{F} = \bigcup_{x \in \mathcal{G}^{-1}(y)} d\mathcal{G}_x(T_x\mathcal{F}) \subset T_yY$ for generic $y \in Y$. Moreover, if $\mathcal{F}$ is $\mathcal{G}$-general then the web $\mathcal{G}_*\mathcal{F}$ is reduced.*

Proof. We follow the ideas sketched in [35, §1.3.2]. Let $\widetilde{\mathcal{G}} : \widetilde{X} \to Y$ be a desingularization of $\mathcal{G}$. Using the notations introduced in Subsection 3.4, we fix an open set $V \subset Y \setminus \Lambda_{\widetilde{\mathcal{G}}}$ such that $\mathcal{G}^{-1}(V) = \bigsqcup_{m=1}^{d} U_m$, $\mathcal{G}|_{U_m}$ is bijective onto V and there are holomorphic 1-forms ω_m on U_m defining $\mathcal{F}$. Then $\omega_V := \prod_{m=1}^{d} (\mathcal{G}|_{U_m}^{-1})^* \omega_m$ is an element of $\mathrm{Sym}^d \Omega^1_V$. These d-symmetric forms differ by a non-vanishing multiplicative function in each non-empty intersection. Hence they define a d-web (with multiplicities) $\mathcal{W}_0$ on $Y \setminus \Lambda_{\widetilde{\mathcal{G}}}$.

In order to extend $\mathcal{W}_0$ to the generic point of $\Lambda_\rho \subset \Lambda_{\widetilde{\mathcal{G}}}$ we we will use the local normal form of the branched covering $\rho : N \to Y$ given by the Stein factorization (3) of $\widetilde{\mathcal{G}}$. Let $y \in \Lambda_\rho$ be a generic point and let $V \subset Y$ be a neighborhood of y such that $\mathcal{G}^{-1}(V) = \bigsqcup_{j=1}^{k} U_j$, $\mathcal{G}|_{U_j}(z, w) = (z^{\varrho_j}, w)$ and $\mathcal{F}|_{U_j}$ is defined by the holomorphic 1-form $\omega_j = a_j(z, w)dz + b_j(z, w)dw$.

Then

$$\omega'_V := \prod_{j=1}^{k}\prod_{i=1}^{\varrho_j}\left(a_j(z^{\frac{1}{\varrho_j}}\zeta_j^i, w)z^{\frac{1}{\varrho_j}-1}\frac{\zeta_j^i}{\varrho_j}dz + b_j(z^{\frac{1}{\varrho_j}}\zeta_j^i, w)dw\right),$$

is a univalued meromorphic d-symmetric form on V, where ζ_j is a primitive ϱ_j-root of unity. Multiplying ω'_V by a suitable meromophic function on V we obtain a holomorphic d-symmetric form ω_V on V with $\operatorname{codim}(\operatorname{Sing}(\omega_V)) \geq 2$. These symmetric forms define an extension of $\mathcal{W}_0$ to $Y \setminus \widetilde{\mathcal{G}}(\mathcal{C}_{\widetilde{\mathcal{G}}})$.

Finally as $\widetilde{\mathcal{G}}(\mathcal{C}_{\widetilde{\mathcal{G}}})$ has codimension ≥ 2, we can extend $\mathcal{W}_0$ to the whole Y by using the standard argument based on Levi's extension theorem for meromorphic functions (see again [6, Remarque 2.17]).

Last assertion is clear from the above construction. □

Remark 4.7. Notice that if $\mathcal{L}$ is a leaf of $\mathcal{F}$ then $\mathcal{G}(\mathcal{L})$ is a leaf of $\mathcal{G}_*\mathcal{F}$, i.e. $T_y\mathcal{G}(\mathcal{L}) \subset T_y\mathcal{W}$ for generic $y \in \mathcal{G}(\mathcal{L})$.

It turns out that every reduced web $\mathcal{W}$ is the direct image of a *canonical* foliation $\mathcal{C}_{\mathcal{W}}$. More precisely, we have the following result.

Theorem 4.8. *For every reduced web $\mathcal{W}$ on Y there is a complex projective manifold $Z_{\mathcal{W}}$ of the same dimension as Y, a surjective morphism $\pi_{\mathcal{W}} : Z_{\mathcal{W}} \to Y$ and a $\pi_{\mathcal{W}}$-general foliation $\mathcal{C}_{\mathcal{W}}$ on $Z_{\mathcal{W}}$ such that $(\pi_{\mathcal{W}})_*\mathcal{C}_{\mathcal{W}} = \mathcal{W}$ and fulfilling the following universal property:*

For every dominant rational map $\mathcal{G} : X \dashrightarrow Y$ with $\dim X = \dim Y$ and every foliation $\mathcal{F}$ on X such that $(\mathcal{G}_*\mathcal{F})_{\mathrm{red}} = \mathcal{W}$ there exists a unique rational map $\mathcal{G}^Z : X \dashrightarrow Z_{\mathcal{W}}$ such that $\mathcal{G} = \pi_{\mathcal{W}} \circ \mathcal{G}^Z$ and $\mathcal{G}^Z_*\mathcal{F} = \mathcal{C}^e_{\mathcal{W}}$, with $e = \deg \mathcal{G}^Z$:

$$\begin{array}{ccc} & & (Z_{\mathcal{W}}, \mathcal{C}_{\mathcal{W}}) \\ & \overset{\mathcal{G}^Z}{\nearrow} & \downarrow \pi_{\mathcal{W}} \\ (X, \mathcal{F}) & \overset{\mathcal{G}}{\dashrightarrow} & (Y, \mathcal{W}). \end{array}$$

Moreover, if $\mathcal{F}$ is $\mathcal{G}$-general then $\mathcal{G}^Z$ is birational. In particular, the triple $(Z_{\mathcal{W}}, \mathcal{C}_{\mathcal{W}}, \pi_{\mathcal{W}})$ is unique modulo birational transformations.

In addition, the discriminant $\Delta(\mathcal{W})$ of $\mathcal{W}$ is contained in the set $\Lambda_{\pi_{\mathcal{W}}}$ of critical values of $\pi_{\mathcal{W}}$ and the monodromy representation $\mu_{\pi_{\mathcal{W}}}$ of $\pi_{\mathcal{W}}$ is the composition

$$\begin{array}{ccc} \pi_1(Y \setminus \Lambda_{\pi_{\mathcal{W}}}) & \twoheadrightarrow & \pi_1(Y \setminus \Delta(\mathcal{W})) \\ & \underset{\mu_{\pi_{\mathcal{W}}}}{\searrow} & \downarrow \mu_{\mathcal{W}} \\ & & \mathfrak{S}_d \end{array}$$

of the epimorphism induced by the inclusion $Y \setminus \Lambda_{\pi_{\mathcal{W}}} \subset Y \setminus \Delta(\mathcal{W})$ and the monodromy representation $\mu_{\mathcal{W}}$ of $\mathcal{W}$. In particular, the monodromy groups of $\mathcal{W}$ and $\pi_{\mathcal{W}}$ coincide.

This result allows us to define the direct image $\mathcal{G}_*\mathcal{W}$ of a reduced web $\mathcal{W}$ on X by a dominant rational map $\mathcal{G} : X \dashrightarrow Y$ with $\dim X = \dim Y$ by setting $\mathcal{G}_*\mathcal{W} = (\mathcal{G} \circ \pi_{\mathcal{W}})_*\mathcal{C}_{\mathcal{W}}$.

The proof of Theorem 4.8 is based on Proposition 4.9 below. We recall that the manifold T^*Y has a canonical 1-form α whose exterior differential $d\alpha$ defines its usual symplectic structure. The kernel of the 1-form α induces a canonical contact distribution $\mathcal{C}_Y$ on $\mathbb{P}T^*Y$.

Proposition 4.9. *For every holomorphic codimension one d-web $\mathcal{W}$ on a projective manifold Y of dimension n, the Zariski closure of*

$$\{(y,[\eta]) \in \mathbb{P}T^*Y \mid y \notin \Delta(\mathcal{W}),\ \ker\eta \subset T_y\mathcal{W}\}$$

is a projective subvariety $Z_{\mathcal{W}} \subset \mathbb{P}T^*Y$ of dimension n such that the restriction of the contact distribution $\mathcal{C}_Y$ of $\mathbb{P}T^*Y$ to $Z_{\mathcal{W}}$ is integrable and induces a foliation $\mathcal{C}_{\mathcal{W}}$ on $Z_{\mathcal{W}}$. Moreover, the restriction of the natural projection $\pi : \mathbb{P}T^*Y \to Y$ to $Z_{\mathcal{W}}$ is a dominant morphism $\pi_{\mathcal{W}} : Z_{\mathcal{W}} \to Y$ satisfying the following properties:

1. *the topological degree of $\pi_{\mathcal{W}}$ is d, $\mathcal{C}_{\mathcal{W}}$ is $\pi_{\mathcal{W}}$-general and the direct image of $\mathcal{C}_{\mathcal{W}}$ by $\pi_{\mathcal{W}}$ is $\mathcal{W}$,*
2. *the set of critical values $\Lambda_{\pi_{\mathcal{W}}}$ of $\pi_{\mathcal{W}}$ contains the discriminant $\Delta(\mathcal{W})$ of $\mathcal{W}$,*
3. *the restriction $\pi^{\Lambda}_{\mathcal{W}} : Z_{\mathcal{W}} \setminus \pi^{-1}_{\mathcal{W}}(\Lambda_{\mathcal{W}}) \to X \setminus \Lambda_{\mathcal{W}}$ of $\pi_{\mathcal{W}}$ is a d-sheeted covering whose monodromy representation is the composition of the morphism $\pi_1(Y \setminus \Lambda_{\pi_{\mathcal{W}}}) \to \pi_1(Y \setminus \Delta(\mathcal{W}))$ induced by the inclusion and the monodromy representation of $\mathcal{W}$.*

This statement follows from [33] and [7] in the 2-dimensional case, and from [8] and [35, §1.3.2] in the general case.

Proof of Theorem 4.8. The existence of the triple $(Z_{\mathcal{W}}, \mathcal{C}_{\mathcal{W}}, \pi_{\mathcal{W}})$ in Theorem 4.8 follows by desingularizing the corresponding objects in Proposition 4.9. In order to prove the existence of the factorization $\mathcal{G}^Z$ we consider the well-defined dominant rational map

$$\begin{array}{rcl} \mathbb{P}T^*\mathcal{G} : \mathbb{P}T^*X & \dashrightarrow & \mathbb{P}T^*Y \\ (x,[\eta]) & \mapsto & (\mathcal{G}(x), [\eta \circ d\mathcal{G}_x^{-1}]), \end{array}$$

preserving the contact structures. Since $(\mathcal{G}_*\mathcal{F})_{\rm red} = \mathcal{W}$, the restriction $\bar{\mathcal{G}}^Z$ of $\mathbb{P}T^*\mathcal{G}$ to $Z_{\mathcal{F}} \subset \mathbb{P}T^*X$ has image $Z_{\mathcal{W}} \subset \mathbb{P}T^*Y$. Hence $\bar{\mathcal{G}}^Z : Z_{\mathcal{F}} \dashrightarrow Z_{\mathcal{W}}$ is a dominant rational map projecting geometrically the foliation $\mathcal{C}_{\mathcal{F}}$ onto $\mathcal{C}_{\mathcal{W}}$. Consequently $\bar{\mathcal{G}}^Z_*\mathcal{C}_{\mathcal{F}} = \mathcal{C}^e_{\mathcal{W}}$, where $e = \deg\bar{\mathcal{G}}^Z$. We define $\mathcal{G}^Z := \bar{\mathcal{G}}^Z \circ \pi_{\mathcal{F}}^{-1} : X \dashrightarrow Z_{\mathcal{W}}$ which satisfies the desired properties. Notice that if $\mathcal{F}$ is $\mathcal{G}$-general then $\bar{\mathcal{G}}^Z$ is generically injective and consequently $\mathcal{G}^Z$ is birational.

The uniqueness of the rational factorization $\mathcal{G}^Z$ follows from the fact that $\mathcal{C}_{\mathcal{W}}$ is $\pi_{\mathcal{W}}$-general. Indeed, if there were two different rational maps $\mathcal{G}_i^Z : (X,\mathcal{F}) \dashrightarrow (Z_{\mathcal{W}}, \mathcal{C}_{\mathcal{W}})$ fulfilling $\mathcal{G} = \pi_{\mathcal{W}} \circ \mathcal{G}_i^Z$ and $\mathcal{G}^Z_{i*}\mathcal{F} = \mathcal{C}^{e_i}_{\mathcal{W}}$ with $e_i = \deg\mathcal{G}_i^Z$, then, for a generic point $x \in X$ we would have $z_1 = \mathcal{G}_1^Z(x) \neq \mathcal{G}_2^Z(x) = z_2$ and $d\pi_{\mathcal{W}}(T_{z_1}\mathcal{C}_{\mathcal{W}}) = d\pi_{\mathcal{W}}(T_{z_2}\mathcal{C}_{\mathcal{W}}) = d\mathcal{G}(T_x\mathcal{F})$ contradicting the fact that $\mathcal{C}_{\mathcal{W}}$ is $\pi_{\mathcal{W}}$-general.

The remaining properties are consequence of Proposition 4.9. □

Remark 4.10. If $\mathcal{F}$ is $\mathcal{G}$-general then the irreducible subwebs of $\mathcal{G}_*\mathcal{F}$ are in one to one correspondence with the irreducible components of X. On the other hand, it can be checked that $\mathcal{F}$ is not $\mathcal{G}$-general if and only if there exists a rational factorization of $\mathcal{G} : X \overset{\sigma}{\dashrightarrow} Z \overset{\pi}{\dashrightarrow} Y$ with $\deg\sigma > 1$ and a foliation $\mathcal{C}$ on Z such that $\mathcal{F} = \sigma^*\mathcal{C}$. Since $\deg \mathcal{G} = (\deg \sigma)(\deg \pi)$, in the case that $\deg \mathcal{G}$ is prime, every not $\mathcal{G}$-general foliation on X is the pull-back by $\mathcal{G}$ of a foliation on Y.

Remark 4.11. For every reduced web $\mathcal{W}$ on Y the pull-back $\pi_{\mathcal{W}}^*\mathcal{W}$ contains the totally decomposable subweb $\boxtimes_{\tau\in\mathrm{Deck}(\pi_{\mathcal{W}})}\tau^*\mathcal{C}_{\mathcal{W}}$ thanks to Remark 4.7.

For a given dominant rational map $\mathcal{G} : X \dashrightarrow Y$ between connected complex projective manifolds of the same dimension let us consider the following commutative diagram

$$\begin{array}{ccc} X \times_Y X & \overset{q}{\dashrightarrow} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathcal{G}} \\ X & \overset{\mathcal{G}}{\dashrightarrow} & Y\,. \end{array}$$

Fix a foliation $\mathcal{F}$ on X. Since $X \times_Y X = \mathcal{G}^*X$, the projections p and $\mathcal{G}$ are locally equivalent and consequently $\mathcal{F}$ is $\mathcal{G}$-general if and only if the foliation $q^*\mathcal{F}$ on the (possibly singular) variety $X \times_Y X$ is p-general (cf. Remark 4.3). From the commutativity of the above diagram we obtain the following equality of reduced webs $p_*q^*\mathcal{F} = \mathcal{G}^*\mathcal{G}_*\mathcal{F}$, provided $\mathcal{F}$ is $\mathcal{G}$-general. From Theorem 4.8, Remark 2.4, Theorem 3.11 and Remark 4.11 we immediately obtain the following result.

Theorem 4.12. *Let $\mathcal{G} : X \dashrightarrow Y$ be a dominant rational map between connected complex projective manifolds of the same dimension and let $\mathcal{F}$ be a $\mathcal{G}$-general foliation on X. Then the triple $(Z_W, \mathcal{C}_{\mathcal{W}}, \pi_{\mathcal{W}})$ associated to the reduced web $\mathcal{W} = \mathcal{G}^*\mathcal{G}_*\mathcal{F}$ is birationally equivalent to $(X \times_Y X, q^*\mathcal{F}, p)$ and $\boxtimes_{\tau\in\mathrm{Deck}(\mathcal{G})}\tau^*\mathcal{F}$ is the maximal totally decomposable subweb of $\mathcal{G}^*\mathcal{G}_*\mathcal{F}$. In particular, $\mathcal{G}$ is Galois if and only if for every (resp. some) $\mathcal{G}$-general codimension one foliation $\mathcal{F}$ on X, the web $\mathcal{G}^*\mathcal{G}_*\mathcal{F}$ is totally decomposable and, in that case, $\mathcal{G}^*\mathcal{G}_*\mathcal{F}$ coincides with the superposition of the foliations $\tau^*\mathcal{F}$ with $\tau \in \mathrm{Deck}(\mathcal{G}) \subset \mathrm{Bir}(X)$.*

Definition 4.13. *A reduced web $\mathcal{W}$ on a connected complex projective manifold Y is called Galois if the universal projection $\pi_{\mathcal{W}} : Z_{\mathcal{W}} \to Y$ is a Galois rational map.*

Theorem 4.14. *For every finite group G and every connected complex projective manifold Y there is a Galois $|G|$-web with monodromy group isomorphic to G.*

Proof. By Theorem 3.14 (M. Namba) there is a Galois branched covering $\rho : N \to Y$ with monodromy group isomorphic to G. Let $\delta : X \to N$ be a desingularization of the normal variety N and set $\mathcal{G} = \rho \circ \delta : X \to Y$. By Proposition 4.5 there exists a $\mathcal{G}$-general foliation $\mathcal{F}$ on X. Then $\mathcal{W} = \mathcal{G}_*\mathcal{F}$ is a web with $(Z_{\mathcal{W}}, \pi_{\mathcal{W}})$ birationally equivalent to $(X, \mathcal{G})$ so that its monodromy group is isomorphic to G. □

4.2. **Foliations and webs on the projective space.** The rest of this section is devoted to treat the case $X = \mathbb{P}^n$. In that case $\mathbb{P}T^*X$ can be canonically identified with the incidence variety

$$\mathcal{V} = \{(p,h) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \,:\, p \in \check{h}\} \subset \mathbb{P}^n \times \check{\mathbb{P}}^n,$$

where $\check{h}$ (resp. $\check{p}$) is the hyperplane in $\mathbb{P}^n$ (resp. $\check{\mathbb{P}}^n$) corresponding to the point $h \in \check{\mathbb{P}}^n$ (resp. $p \in \mathbb{P}^n$). By symmetry $\mathcal{V}$ is also canonically isomorphic to $\mathbb{P}T^*\check{\mathbb{P}}^n$. Moreover, the contact distributions $\mathcal{C}$ of $\mathbb{P}T^*\mathbb{P}^n$ and $\mathbb{P}T^*\check{\mathbb{P}}^n$ coincide under the identification with $\mathcal{V}$ and

$$\mathcal{C}_{(p,h)} = d\pi^{-1}(T_p\,\check{h}) = d\check{\pi}^{-1}(T_h\,\check{p}) \subset T_{(p,h)}\mathcal{V}, \tag{8}$$

where π and $\check{\pi}$ are the restrictions to $\mathcal{V}$ of the natural projections onto $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$.

For each reduced web $\mathcal{W}$ on $\mathbb{P}^n$ the universal projection $\pi_{\mathcal{W}} : Z_{\mathcal{W}} \to \mathbb{P}^n$ is birationally equivalent to the restriction of π to the (possibly singular) subvariety $Z_{\mathcal{W}} \subset \mathcal{V} \simeq \mathbb{P}T^*\mathbb{P}^n$ considered in Proposition 4.9. Let $\check{\pi}_{\mathcal{W}}$ be the restriction of $\check{\pi}$ to $Z_{\mathcal{W}} \subset \mathcal{V}$. Thanks to formula (8) we see that $\mathcal{C}_{\mathcal{W}}$ is in general position with respect to the projections $\pi_{\mathcal{W}}$ and $\check{\pi}_{\mathcal{W}}$, whenever they are dominant maps.

Definition 4.15. *We say that a reduced web on $\mathbb{P}^n$ is* non-degenerate *if the map $\check{\pi}_{\mathcal{W}} : Z_{\mathcal{W}} \to \check{\mathbb{P}}^n$ is dominant. In that case we can consider the reduced web* $\operatorname{Leg}\mathcal{W} := (\check{\pi}_{\mathcal{W}})_*\mathcal{C}_{\mathcal{W}}$ *on $\check{\mathbb{P}}^n$ which is called the* Legendre transform *of $\mathcal{W}$.*

From now on all the webs considered will be reduced. To every web $\mathcal{W}$ on $\mathbb{P}^n$ we can associate its *characteristic numbers* $d_i(\mathcal{W})$, $i = 0, \dots, n-1$, which can be defined (see [35, §1.4.1]) as the number of pairs $(p,h) \in \mathbb{P}^n \times \check{\mathbb{P}}^n$ such that $p \in \ell_i \subset h \subset \mathbb{P}^n$ and $T_p h \subset T_p\mathcal{W}$, for a given generic linear i-plane $\mathbb{P}^i \simeq \ell_i \subset \mathbb{P}^n$. Notice that $d_0(\mathcal{W})$ counts the number of leaves of $\mathcal{W}$ through a generic point of $\mathbb{P}^n$, that is $\mathcal{W}$ is a $d_0(\mathcal{W})$-web. The integer $d_1(\mathcal{W})$ counts the number of points of a generic line ℓ_1 where the web has a leaf with tangent space containing ℓ_1, i.e. the degree of $\mathcal{W}$. The integer $d_{n-1}(\mathcal{W})$ counts the number of points where the leaves of $\mathcal{W}$ are tangent to a generic hyperplane. If $\mathcal{W}$ is non-degenerate we have $d_i(\operatorname{Leg}\mathcal{W}) = d_{n-1-i}(\mathcal{W})$ for each $i = 0, \dots, n-1$, see [35, §1.4.3] for more details.

For a foliation $\mathcal{F}$ on $\mathbb{P}^n$ we have that $d_0(\mathcal{F}) = 1$ and $d_{n-1}(\mathcal{F})$ is just the topological degree of its Gauss map $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^n \dashrightarrow \check{\mathbb{P}}^n$ defined by $\mathcal{G}_{\mathcal{F}}(p) = T_p\mathcal{F}$, where the tangent space $T_p\mathcal{F}$ of $\mathcal{F}$ at a regular point p of $\mathcal{F}$ is thought as a hyperplane of $\mathbb{P}^n$. Indeed, we have $\mathcal{G}_{\mathcal{F}} = \check{\pi}_{\mathcal{F}} \circ \pi_{\mathcal{F}}^{-1}$. In the case $n = 2$ the topological degree of $\mathcal{F}$ coincides with its usual degree, i.e. the number of tangency points of the leaves of $\mathcal{F}$ with a generic line.

Remark 4.16. The classification of *degenerate* foliations, i.e. foliations whose Gauss map is not dominant, is known in dimension $n \leq 4$: for $n = 2$ they are of degree zero, i.e. pencils of lines, for $n = 3$ see [10] and [18] for $n = 4$.

Definition 4.17. *A non-degenerate codimension one foliation $\mathcal{F}$ on $\mathbb{P}^n$ is said Galois if the web* $\operatorname{Leg}\mathcal{F}$ *is Galois or equivalently the Gauss map $\mathcal{G}_{\mathcal{F}}$ is Galois.*

From Theorem 4.12 we obtain:

Corollary 4.18. *A non-degenerate codimension one foliation $\mathcal{F}$ on $\mathbb{P}^n$ is Galois if and only if the web $\mathcal{G}_{\mathcal{F}}^*\mathrm{Leg}\mathcal{F}$ is totally decomposable. In that case, $\mathcal{G}_{\mathcal{F}}^*\mathrm{Leg}\mathcal{F}$ is the superposition of the foliations $\tau^*\mathcal{F}$ with $\tau \in \mathrm{Deck}(\mathcal{G}_{\mathcal{F}})$.*

Example 4.19. Every foliation $\mathcal{F}$ on $\mathbb{P}^n$ with $d_{n-1}(\mathcal{F}) \in \{1, 2\}$ is Galois because its Gauss map $\mathcal{G}_{\mathcal{F}}$ induces a covering of degree $d_{n-1}(\mathcal{F}) \leq 2$. Notice that, if $n \geq 3$, there are examples of such foliations with $d_1(\mathcal{F}) > 2$. For instance, for each $\nu \geq 2$ consider the exceptional foliation $\mathcal{E}_\nu$ on $\mathbb{P}^3$ (cf. [5]) given in the affine chart (x, y, z) by the integrable 1-form $\imath_{S_\nu}\imath_{X_\nu}(dx\wedge dy\wedge dz)$, where

$$S_\nu = x\partial_x + \nu y\partial_y + (1 - \nu + \nu^2)z\partial_z$$
$$X_\nu = \partial_x + \nu x^{\nu-1}\partial_y + (1 - \nu + \nu^2)y^{\nu-1}\partial_z.$$

We have that $d_1(\mathcal{E}_\nu) = \nu$ and $d_2(\mathcal{E}_\nu) = \nu - 1$. Then, foliations $\mathcal{E}_2$ and $\mathcal{E}_3$ are Galois but $\mathcal{E}_4$ is not, as it can be checked by computing explicitly the divisor $\mathcal{R}_{\mathcal{G}_{\mathcal{E}_4}}$ and applying Proposition 3.32. □

We construct now a family of Galois foliations on $\mathbb{P}^n$ whose Galois character is checked by means of Corollary 4.18. Let $(x_1, \ldots, x_n)$ be an affine chart of $\mathbb{P}^n$ and let $\sum_{i=1}^n A_i(x)dx_i$ be an integrable polynomial 1-form defining a foliation $\mathcal{F}$ on $\mathbb{P}^n$. Consider the affine chart $(y_1, \ldots, y_n)$ of $\check{\mathbb{P}}^n$ such that $\sum_{i=1}^n x_i y_i = 1$ is an affine equation of $\mathcal{V} \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$. In these affine charts, the contact distribution of $\mathcal{V}$ is given by the kernel of the 1-form $\sum_{i=1}^n y_i dx_i = -\sum_{i=1}^n x_i dy_i$ and the Gauss map $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{F}$ is written as

$$\mathcal{G}_{\mathcal{F}}(x_1, \ldots, x_n) = \left(\frac{A_1(x)}{C(x)}, \ldots, \frac{A_n(x)}{C(x)}\right),$$

where $C(x) = \sum_{i=1}^n x_i A_i(x)$. The following result provides examples of Galois foliations on $\mathbb{P}^n$ for each degree $k \geq 1$.

Proposition 4.20. *For each $n \geq 2$ and $k \geq 1$ the foliation $\mathcal{F}_{n,k}$ on $\mathbb{P}^n$ given in an affine chart by the polynomial first integral $\sum_{i=1}^n x_i^{k+1}$ is Galois with $d_1(\mathcal{F}_{n,k}) = k$ and $d_{n-1}(\mathcal{F}_{n,k}) = k^{n-1}$.*

Proof. With the precedent notations, $A_i(x) = x_i^k$ and $C(x) = \sum_{i=1}^n x_i^{k+1}$. For generic $y = (y_1, \ldots, y_n) \in \check{\mathbb{P}}^n$, the tangency points between $\mathcal{F}_{n,k}$ and the hyperplane y are the solutions of the following system of equations:

$$\left(\frac{x_i}{x_n}\right)^k = \frac{y_i}{y_n}, \quad i = 1, \ldots, n-1, \quad \text{and} \quad \frac{x_n^k}{y_n} = C(x),$$

that can be explicitly solved:

$$x_i = x_{ij}(y) := \frac{y_i^{\frac{1}{k}}\zeta^{j_i}}{\sum\limits_{\ell=1}^{n} y_\ell^{1+\frac{1}{k}}\zeta^{j_\ell}}, \quad i=1,\dots,n,$$

where ζ is a primitive k-root of the unity, $j=(j_1,\dots,j_{n-1})\in\mathbb{Z}_k^{n-1}$ and $j_n=0$. Then the k^{n-1}-symmetric form

$$\prod_{j\in\mathbb{Z}_k^{n-1}}\Big(\sum_{i=1}^{n} x_{ij}(y)dy_i\Big)$$

is rational and it defines the dual web $\operatorname{Leg}\mathcal{F}_{n,k}$ on $\check{\mathbb{P}}^n$. Finally the pull-back of $\operatorname{Leg}\mathcal{F}_{n,k}$ by $\mathcal{G}:=\mathcal{G}_{\mathcal{F}_{n,k}}$ is formally the symmetric product, varying $j\in\mathbb{Z}_k^{n-1}$, of the 1-forms

$$\sum_{i=1}^{n}\mathcal{G}^*(x_{ij}(y))\mathcal{G}^*(dy_i).$$

It suffices to notice that

$$\mathcal{G}^*(x_{ij}(y)) = \frac{x_i^k\zeta^{j_i}\sum\limits_{\ell=1}^{n}x_\ell^{k+1}}{\sum\limits_{\ell=1}^{n}x_\ell^{k+1}\zeta^{j_\ell}}$$

are rational functions in order to conclude by applying Corollary 4.18. $\square$

Remark 4.21. It can be checked that $\det(d\mathcal{G}_{\mathcal{F}_{n,k}}(x)) = \frac{(x_1\cdots x_n)^{k-1}}{C(x)^n}$ and $\mathcal{G}(\{x_i=0\})=\{y_i=0\}$. Consequently, the local monodromy of $\mathcal{F}_{n,k}$ has order k. We can also compute explicitly all the elements of the deck transformation group of $\mathcal{G}_{\mathcal{F}_{n,k}}$:

$$\tau_j(x_1,\dots,x_n)=\left(\frac{x_1\zeta^{j_1}\sum\limits_{\ell=1}^{n}x_\ell^{k+1}}{\sum\limits_{\ell=1}^{n}x_\ell^{k+1}\zeta^{j_\ell}},\dots,\frac{x_n\zeta^{j_n}\sum\limits_{\ell=1}^{n}x_\ell^{k+1}}{\sum\limits_{\ell=1}^{n}x_\ell^{k+1}\zeta^{j_\ell}}\right),\quad j\in\mathbb{Z}_k^n,\ j_n=0.$$

We expect this group to be isomorphic to the abelian group $\mathbb{Z}_k^{n-1}$. A particular case in which this assertion is true is for $n=3$ and $k=2$ because $\mathbb{Z}_2\oplus\mathbb{Z}_2$ is the only group of order 4 all whose elements have order 2.

Remark 4.22. Notice that birational geometry on $\mathbb{P}^n$ is not well-behaved with respect to the projective duality. More precisely, for any birational map $\tau\in\operatorname{Deck}(\mathcal{G}_\mathcal{F})$ there exists a lift $\mathbb{P}T^*\tau$ preserving the subvariety $Z_\mathcal{F}\subset\mathbb{P}T^*\mathbb{P}^n\simeq\mathcal{V}$ defined in Proposition 4.9. However, in general there is no rational map $\check{\tau}:\check{\mathbb{P}}^n\dashrightarrow\check{\mathbb{P}}^n$ such that $\mathcal{G}_\mathcal{F}\circ\tau=\mathcal{G}_\mathcal{F}=\check{\tau}\circ\mathcal{G}_\mathcal{F}$. In general there is no relation between $\operatorname{Leg}\mathcal{F}$ and $\operatorname{Leg}\tau^*\mathcal{F}$ as the following example shows.

Example 4.23. Let $\mathcal{F}$ be the degree 2 foliation on $\mathbb{P}^2$ given by the vector field $(x^2-x)\partial_x + (y^2-y)\partial_y$ then its Gauss map $\mathcal{G}_\mathcal{F}(x,y) = \left(\frac{1-y}{x(x-y)}, \frac{x-1}{y(x-y)}\right)$ admits a birational involution $\tau(x,y) = \left(\frac{-x(1-x+y)}{1-x-y}, \frac{-y(1+x-y)}{1-x-y}\right)$ such that $\mathcal{G}_\mathcal{F} \circ \tau = \mathcal{G}_\mathcal{F}$. Then $\mathcal{G}_\mathcal{F}^*\mathrm{Leg}\mathcal{F} = \mathcal{F} \boxtimes \mathcal{F}^\perp$, $\mathcal{F}$ is Galois and the foliation $\mathcal{F}^\perp := \tau^*\mathcal{F}$ is defined by the vector field $(x^2-x)(1+x-3y)\partial_x + (y^2-y)(1+y-3x)\partial_y$. Thus $\mathcal{F}^\perp$ has degree 3 and, using Proposition 5.1, it can be easily checked that $\mathcal{F}^\perp$ is not Galois. □

5. Galois foliations on the projective plane

The aim of this section is to understand the geometry of Galois foliations on the complex projective plane. We begin by giving an algebraic characterization of Galois foliations using the main result of Section 4. In particular we deduce that the space $\mathbb{G}_d$ of degree d Galois foliations is a quasi-projective variety. We also provide a characterization of Galois foliations in terms of geometric elements naturally associated to them by using results of Subsection 3.4. Finally we obtain a full characterization of homogeneous Galois foliations and we also exhibit examples of Galois foliations with other symmetries.

5.1. **Algebraic characterization of Galois foliations.** Recall that a degree d foliation $\mathcal{F}$ on $\mathbb{P}^2$ is given by a 1-form on $\mathbb{C}^3$,

$$\omega = a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz,$$

with a, b, c homogeneous polynomials of degree $d+1$ without common factors and fulfilling $\omega(R) = ax + by + cz = 0$, where $R = x\partial_x + y\partial_y + z\partial_z$ is the radial vector field (see for instance [6, §9.1]). Thus, the space $\mathbb{F}_d$ of degree d foliations on $\mathbb{P}^2$ is a Zariski open subset of the projective space $\overline{\mathbb{F}}_d = \mathbb{P}(U_d)$, where

$$U_d := \{(a,b,c) \in \mathbb{C}_{d+1}[x,y,z]^{\oplus 3} \,|\, ax + by + cz = 0\} \tag{9}$$

and $\mathbb{C}_d[x,y,z]$ is the vector space of degree d homogeneous polynomials in x, y, z. The condition $\omega(R) = 0$ implies that the foliation can also be defined by an homogeneous vector field $Z = \alpha\partial_x + \beta\partial_y + \gamma\partial_z$ of degree d on $\mathbb{C}^3$ fulfilling $\omega = \imath_Z \imath_R(dx \wedge dy \wedge dz)$. In the affine chart $z = 1$, the foliation $\mathcal{F}$ is defined by the polynomial 1-form

$$a(x,y,1)dx + b(x,y,1)dy = -\bar{b}(x,y)dx + \bar{a}(x,y)dy + \bar{c}(x,y)(xdy - ydx),$$

where $\bar{a}, \bar{b}, \bar{c} \in \mathbb{C}[x,y]$, $\deg \bar{a}, \deg \bar{b} \le d$ and $\bar{c}$ is homogeneous of degree d. It can be also defined by the vector field $X = A(x,y)\partial_x + B(x,y)\partial_y$ with $A = \bar{a} + x\bar{c}$ and $B = \bar{b} + y\bar{c}$. The line at infinity $z = 0$ is invariant by $\mathcal{F}$ if and only if $\bar{c} = 0$. We deduce that U_d is isomorphic to the space of vector fields

$$X = A(x,y)\partial_x + B(x,y)\partial_y = \bar{a}(x,y)\partial_x + \bar{b}(x,y)\partial_y + \bar{c}(x,y)(x\partial_x + y\partial_y)$$

with $\bar{a}, \bar{b}, \bar{c} \in \mathbb{C}[x,y]$, $\deg \bar{a}, \deg \bar{b} \le d$ and $\bar{c}$ homogeneous of degree d. The relationship between the vector fields Z and X is given by the identities $\bar{a}(x,y) = \alpha(x,y,1)$, $\bar{b}(x,y) = \beta(x,y,1)$ and $\bar{c}(x,y) = -\gamma(x,y,1)$. The vector field X is said *saturated* if $\gcd(A,B) = 1$. This condition jointly with

$\max(\deg A, \deg B) \geq d$ is equivalent to the condition $\gcd(a,b,c) = 1$ defining $\mathbb{F}_d$.

We consider the Gauss map $\mathcal{G}_{\mathcal{F}} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$ of $\mathcal{F}$ which is written as

$$\mathcal{G}_{\mathcal{F}}([x,y,z]) = [a(x,y,z), b(x,y,z), c(x,y,z)]$$

in homogeneous coordinates. From now on the Gauss map of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ will be denoted simply by $\mathcal{G}$. A criterion to test if $\mathcal{F}$ is Galois is Corollary 4.18 which can be reformulated in the following form, already considered in [9] for the degree 3 case (cf. Proposition 5.2 loc. cit.):

Proposition 5.1. *A foliation $\mathcal{F}$ on $\mathbb{P}^2$ given by the polynomial vector field $X = A(x,y)\partial_x + B(x,y)\partial_y$ is Galois if and only if the polynomial*

$$P(x,y,t) = \det \begin{pmatrix} A(x,y) & A(x+tA(x,y), y+tB(x,y)) \\ B(x,y) & B(x+tA(x,y), y+tB(x,y)) \end{pmatrix} \in \mathbb{C}[x,y,t] \tag{10}$$

decomposes totally over the field $\mathbb{C}(x,y)$. In fact, each one of its rational roots $t = t(x,y) \in \mathbb{C}(x,y)$ determines a birational deck transformation of $\mathcal{G}$:

$$(x,y) \mapsto (x + t(x,y)A(x,y), y + t(x,y)B(x,y)).$$

Besides its practical applications, this criterion allows us to prove the following result.

Proposition 5.2. *The set $\mathbb{G}_d$ of degree d Galois foliations is a Zariski closed subset of $\mathbb{F}_d$.*

Proof. Let V_m be the vector subspace of $\mathbb{C}[x,y]$ of polynomials of degree $\leq m$. Put $\delta := (d+1)^2 - 1 = (d+2)d$ and $\Delta := \delta(d-1) = (d+2)d(d-1)$ and let us consider the map $\mathfrak{p} : U_d \to V_\delta^{\oplus d} \hookrightarrow V_\Delta^{\oplus d}$ given by $\mathfrak{p}(A\partial_x + B\partial_y) = (a_1, \ldots, a_d)$ where $a_i = a_i(x,y)$ are determined by the polynomial $P_X(x,y,t) = \sum_{j=1}^d a_j(x,y)t^j$ given by formula (10). Notice that $P_{\lambda X}(x,y,t) = \lambda^2 P(x,y,\lambda t)$, so that $\mathfrak{p}(\lambda X) = \phi_\lambda(\mathfrak{p}(X))$, with $\phi_\lambda(a_1,\ldots,a_d) = (\lambda^3 a_1, \ldots, \lambda^{d+2}a_d)$. Thus, $\mathfrak{p}$ induces a rational map $\overline{\mathfrak{p}} : \overline{\mathbb{F}}_d \dashrightarrow \mathbb{P}_w(V_\Delta^{\oplus d})$, where $\mathbb{P}_w(V_\Delta^{\oplus d})$ is the weighted projective space associated to the weights $(3,\ldots,d+2)$ in each direct summand V_Δ of $V_\Delta^{\oplus d}$. Notice that $\overline{\mathfrak{p}}$ is defined on the open set $\mathbb{F}_d \subset \overline{\mathbb{F}}_d$.

Let $\mathfrak{q} : V_\delta^{\oplus 2(d-1)} \to V_\Delta^{\oplus d}$ be the map defined by $\mathfrak{q}(b_1, c_1, \ldots, b_{d-1}, c_{d-1}) = (a_1, \ldots, a_d)$ where $\sum\limits_{j=1}^{d} a_j t^j = t \prod\limits_{i=1}^{d-1} (c_i t - b_i)$. Notice that

$$\mathfrak{q}(\lambda_1 b_1, \lambda_1 c_1, \ldots, \lambda_{d-1}b_{d-1}, \lambda_{d-1}c_{d-1}) = \left(\prod_{i=1}^{d-1} \lambda_i\right) \mathfrak{q}(b_1, c_1, \ldots, b_{d-1}, c_{d-1})$$

and $\mathfrak{q}(b_1, c_1, \ldots, b_{d-1}, c_{d-1}) = (0,\ldots,0)$ if and only if there is $1 \leq i \leq d-1$ such that $b_i = c_i = 0$. Then $\mathfrak{q}$ induces a morphism $\overline{\mathfrak{q}} : \left(\mathbb{P}(V_\delta^{\oplus 2})\right)^{d-1} \to \mathbb{P}(V_\Delta^{\oplus d})$. Let $Z \subset V_\Delta^{\oplus d}$ and $\overline{Z} \subset \mathbb{P}(V_\Delta^{\oplus d})$ be the images of $\mathfrak{q}$ and $\overline{\mathfrak{q}}$ respectively. Since $\overline{\mathfrak{q}}$ is a morphism we deduce that $\overline{Z}$ is a Zariski closed subset. Denote by $f : V_\Delta^{\oplus d} \setminus \{0\} \to \mathbb{P}(V_\Delta^{\oplus d})$, $f_w : V_\Delta^{\oplus d} \setminus \{0\} \to \mathbb{P}_w(V_\Delta^{\oplus d})$ and $g : \left(V_\delta^{\oplus 2} \setminus \{0\}\right)^{d-1} \to \left(\mathbb{P}(V_\delta^{\oplus 2})\right)^{d-1}$ the natural projections. Notice that $Z^* := Z \setminus \{0\}$ is saturated by the maps f and f_w because, if the polynomial $P(t) = \sum_{j=1}^{d} a_j t^j$ decomposes as $t \prod_{i=1}^{d-1} (c_i t - b_i)$ with $b_i, c_i \in V_\delta$, then the same property holds for $\lambda P(t)$ and for $\lambda^2 P(\lambda t)$.

From $\overline{\mathfrak{q}} \circ g = f \circ \mathfrak{q}$ and the surjectivity of g we deduce that $f(Z^*) = \overline{Z}$. Since Z^* is saturated by f and f is surjective we deduce that $Z^* = f^{-1}(\overline{Z})$ is a Zariski closed set of $V_\Delta^{\oplus d} \setminus \{0\}$. Then $\overline{Z}_w := f_w(Z^*)$ is a Zariski closed set of $\mathbb{P}_w(V_\Delta^{\oplus d})$ because Z^* is saturated by f_w. Finally, we deduce from Proposition 5.1 that $\mathbb{G}_d = (\overline{\mathfrak{p}}_{|\mathbb{F}_d})^{-1}(\overline{Z}_w)$ is a Zariski closed set of $\mathbb{F}_d$. □

The first part of the following natural question was asked in [9, §6] for the case $d = 3$:

Question 5.3. *(1) Which is the number, dimension and type of the irreducible components of* $\mathbb{G}_d$*?*

(2) Is the branching type of the Gauss map of a degree d Galois foliation generically constant along each irreducible component of $\mathbb{G}_d$*?*

(3) What are the elements of $\overline{\mathbb{G}}_d \setminus \mathbb{G}_d \subset \overline{\mathbb{F}}_d$*?*

Recall that the *polar curve* of $\mathcal{F}$ with respect to $p \in \mathbb{P}^2$ is defined as $\mathcal{G}^{-1}(\check{p}) = \mathrm{Tang}(\mathcal{F}, R_p)$, where R_p is the radial vector field centered at p. A *generic polar* of $\mathcal{F}$ is a polar curve of $\mathcal{F}$ with respect to a generic point $p \in \mathbb{P}^2$. We know by [30] that the generic polar of a degree d foliation is an irreducible (and reduced) curve of degree $d+1$ and that its genus is generically constant. If the point $p \in \mathbb{P}^2$ is generic and $\ell = \check{p} \subset \check{\mathbb{P}}^2$ denotes its dual line, then the composition of the normalization $S_{\mathcal{G}}^{\ell} \to \mathcal{G}^{-1}(\ell)$ with the restriction of $\mathcal{G}$ to $\mathcal{G}^{-1}(\ell)$ is the branched covering $\mathcal{G}^{\ell} : S_{\mathcal{G}}^{\ell} \to \ell \simeq \mathbb{P}^1$ considered in Proposition 3.17. The branching type of $\mathcal{G}^{\ell}$ is a numerical invariant of $\mathcal{F}$, which contains in particular the genus of $S_{\mathcal{G}}^{\ell}$, cf. Proposition 3.23 and Remark 3.18. In fact, Proposition 3.17 implies that the topological type of the generic polar is constant. In [17] a stronger result is proved: the constancy of the topological embedded type $\mathcal{G}^{-1}(\ell) \subset \mathbb{P}^2$ of the generic polar of a foliation $\mathcal{F}$ on $\mathbb{P}^2$. The following statement gives a precise meaning to the notion of *generic* in this situation that we will use in the sequel.

Lemma 5.4. *If p does not belong to the dual curve $\check{\Lambda}_{\mathcal{G}} \subset \mathbb{P}^2$ of $\Lambda_{\mathcal{G}} \subset \check{\mathbb{P}}^2$ then the polar curve of $\mathcal{F}$ with respect to p is irreducible and its genus does not depend on p.*

Proof. We follow the proof of Proposition 3.17 in the case $X = \mathbb{P}^2$, $Y = \check{\mathbb{P}}^2$, $N = 2$ and $\mathcal{G}$ being the Gauss map of a foliation $\mathcal{F}$ on $\mathbb{P}^2$. In that situation $Z = Z_Y = \check{\check{\mathbb{P}}}^2 = \mathbb{P}^2$, $\mathcal{V} = \{(y, z) \in \check{\mathbb{P}}^2 \times \mathbb{P}^2 \mid z \in \check{y}\}$, $\mathcal{W} \simeq (\mathcal{G} \times \mathrm{id_Z})^{-1}(\mathcal{V})$ and $\mathcal{G}^{\mathcal{V}} \simeq \mathcal{G} \times \mathrm{Id}_Z$ restricted to $(\mathcal{G} \times \mathrm{id}_Z)^{-1}(\mathcal{V})$. Hence

$$\{\det(d\mathcal{G}^{\mathcal{V}}) = 0\} \subset \{(x, z) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \det(d\mathcal{G}_x) = 0,\ z \in \mathcal{G}(x)\},$$

so that $Z \setminus Z_\Lambda \subset \mathcal{G}(\Delta_{\mathcal{G}}) = \check{\Lambda}_{\mathcal{G}}$ and consequently $\mathbb{P}^2 \setminus \check{\Lambda}_{\mathcal{G}} \subset Z_\Lambda$. By Proposition 3.17, the topological type of the desingularization of the polar curve of $\mathcal{F}$ with respect to $p \in Z_\Lambda$ is constant. On the other hand, if $p \notin \check{\Lambda}_{\mathcal{G}}$

then the dual line $\check{p}$ meets transversely $\Lambda_{\mathcal{G}}$, the inclusion induces an epimorphism $\pi_1(\check{p}\setminus\Lambda_{\mathcal{G}}) \twoheadrightarrow \pi_1(\check{\mathbb{P}}^2\setminus\Lambda_{\mathcal{G}})$ and consequently the monodromy group of $\mathcal{G}_{|\mathcal{G}^{-1}(\check{p}\setminus\Lambda_{\mathcal{G}})}$ is transitive. Then $\mathcal{G}^{-1}(\check{p}\setminus\Lambda_{\mathcal{G}})$ is connected and hence $\mathcal{G}^{-1}(\check{p})$ is irreducible. □

Thus, the map $g : \mathbb{F}_d \to \mathbb{Z}^+$ given by the genus of the generic polar of $\mathcal{F}$ is well-defined. Since $g(\mathcal{F})$ can be computed from the extended branching type $\mathcal{B}^+_{\mathcal{F}}$ of $\mathcal{F}$ by formula (6), the following result goes in the direction of an affirmative answer to Question 5.3(2):

Proposition 5.5. *The map $g : \mathbb{F}_d \to \mathbb{Z}^+$ is lower semi-continuous, i.e. the sets $\{\mathcal{F} \in \mathbb{F}_d \,|\, g(\mathcal{F}) \le e\}$ are Zariski closed for all $e \in \mathbb{Z}^+$. In particular, the genus of the generic polar is generically constant along each irreducible component of $\mathbb{G}_d$ and $\{\mathcal{F} \in \mathbb{F}_d \,|\, g(\mathcal{F}) = 0\}$ is a Zariski closed set. Moreover, $\{\mathcal{F} \in \mathbb{F}_d \,|\, g(\mathcal{F}) = \frac{d(d-1)}{2}\}$ is a dense Zariski open set.*

Proof. Consider the rational map $\mathcal{G}_d : \mathbb{P}^2 \times \overline{\mathbb{F}}_d \dashrightarrow \check{\mathbb{P}}^2 \times \overline{\mathbb{F}}_d$ given by $\mathcal{G}_d(p, \mathcal{F}) = (\mathcal{G}_{\mathcal{F}}(p), \mathcal{F})$. Although for any given $\mathcal{F} \in \mathbb{F}_d$, the set $\Delta_{\mathcal{G}_{\mathcal{F}}}$ is closed in $\mathbb{P}^2$, the union $\bigcup_{\mathcal{F}\in\mathbb{F}_d} \Delta_{\mathcal{G}_{\mathcal{F}}} \times \{\mathcal{F}\}$ in not necessarily a closed subset of $\mathbb{P}^2 \times \mathbb{F}_d$. Nevertheless, this union is contained in the Zariski closed set $\mathcal{D}_d := \overline{\{\det(d\mathcal{G}_d) = 0\}}$, and therefore its image is contained in the Zariski closed set $\mathcal{L}_d := \mathcal{G}_d(\mathcal{D}_d) \subset \check{\mathbb{P}}^2 \times \overline{\mathbb{F}}_d$. Since the fibre of $\mathcal{L}_d$ over $\mathcal{F}$ is a curve on $\check{\mathbb{P}}^2$, we deduce that $\mathcal{L}_d$ is an hypersurface. Let $h(x, y, z, \mathcal{F}) = 0$ be a reduced equation of $\mathcal{L}_d$. Let $\check{\mathcal{L}}_d$ be the image of $\mathcal{L}_d$ by the rational map $\check{\mathbb{P}}^2 \times \overline{\mathbb{F}}_d \dashrightarrow \mathbb{P}^2 \times \overline{\mathbb{F}}_d$ given by $(q, \mathcal{F}) \mapsto \Big(\big[\frac{\partial h}{\partial x}(q), \frac{\partial h}{\partial y}(q), \frac{\partial h}{\partial z}(q)\big], \mathcal{F}\Big)$. Clearly, the fibre of $\check{\mathcal{L}}_d$ over $\mathcal{F}$ contains the dual curve $\check{\Lambda}_{\mathcal{F}} \subset \mathbb{P}^2$ of $\Lambda_{\mathcal{F}} := \mathcal{G}_{\mathcal{F}}(\Delta_{\mathcal{G}_{\mathcal{F}}}) \subset \check{\mathbb{P}}^2$. Thus, we can consider the Zariski open set

$$Y := \{([\alpha, \beta, \gamma], [a, b, c]) \in \mathbb{P}^2 \times \mathbb{F}_d \,|\, [\alpha, \beta, \gamma] \notin \check{\Lambda}_{[a,b,c]})\}$$

and the family of projective curves $f : X \to Y$ over Y given by

$$X := \{([x, y, z], [\alpha, \beta, \gamma], [a, b, c]) \in \mathbb{P}^2 \times Y \,|\, (\alpha a + \beta b + \gamma c)(x, y, z) = 0\}.$$

It can be easily checked that X is a non-singular hypersurface of $\mathbb{P}^2 \times Y$. Since each fibre of f is a curve we have $\dim_x X = \dim_{f(x)} Y + \dim_x(f^{-1}(f(x)))$ for every $x \in X$. This condition, jointly with the smoothness of X and Y, implies that the morphism f is flat, see [19, Corollary 3.20]. Then [13, Proposition 2.4] asserts that the map sending $y \in Y$ to the geometric genus of $f^{-1}(y)$ is lower semi-continuous. We conclude by noting that this map is constant along the fibres of the second projection $Y \subset \mathbb{P}^2 \times \mathbb{F}_d \to \mathbb{F}_d$. Last statement follows from [30, Proposition 6.2]. □

Remark 5.6. Let $\mathbb{E}_d$ be the subset of $\mathbb{F}_d$ consisting of foliations $\mathcal{F}$ with extremal extended branching type $\mathcal{B}^+_{\mathcal{F}} = [(c_1; (d)_1), \ldots, (c_k; (d)_1)]$. By Corollary 3.34, $\mathbb{E}_d \subset \mathbb{G}_d$ and $\mathbb{E}_d = \mathbb{G}_d$ when d is prime. Let $\mathbb{E}^c_d$ be the subset of $\mathbb{E}_d$ defined by the equality $c_1 + \cdots + c_k = c$. Then formula (6) shows that $g_{|\mathbb{E}^c_d}$ is constant and equal to $\frac{(c-2)(d-1)}{2}$.

Before going further with Question 5.3 let us present some explicit examples. As remarked in [9], every foliation of degree one or two on $\mathbb{P}^2$ is Galois but it follows from Proposition 5.1 that foliations of degree $d \ge 3$

are generically non-Galois. The following example provides infinite families of Galois foliations in each degree, and all of them have cyclic monodromy group. That family contains as particular cases all the examples considered in [9].

Example 5.7. For all linearly independent vectors (α,γ,λ), $(\beta,\delta,\mu)\in\mathbb{C}^3$ and every $\mathbb{C}$-linearly independent $u,v\in\mathbb{C}[x,y]$ with $\deg u,\deg v\leq 1$, the degree d foliation defined by the saturated vector field

$$(\alpha u^d+\beta v^d)\partial_x+(\gamma u^d+\delta v^d)\partial_y+(\lambda u^d+\mu v^d)(x\partial_x+y\partial_y)$$

is Galois with cyclic monodromy group and generic extremal extended branching type $[(1;(d)_1),(1;(d)_1)]$. In particular the genus of its generic polar is always 0. Indeed, the slope of the vector field takes the form $p(x,y)=\frac{\gamma+\delta w^d+y(\lambda+\mu w^d)}{\alpha+\beta w^d+x(\lambda+\mu w^d)}$, with $w=\frac{v}{u}$. The roots of polynomial (10) for the vector field $X=\partial_x+p(x,y)\partial_y$ are the solutions of the equation $p(x+t,y+tp(x,y))=p(x,y)$, which reduces to $w(x+t,y+tp(x,y))^d=w(x,y)^d$. Using that $\deg u,\deg v\leq 1$, the last equation factorizes as the following d linear equations in the variable t:

$$w(x+t,y+tp(x,y))=\zeta^k w(x,y)\ \text{ with }\ \zeta=e^{\frac{2i\pi}{d}}\ \text{ and }\ k\in\{1,\ldots,d\}. \tag{11}$$

For each $k\in\mathbb{Z}_d$ the rational solution $t=t_k(x,y)\in\mathbb{C}(x,y)$ of (11) associated to ζ^k determines a deck transformation

$$\tau_k(x,y)=(x+t_k(x,y),y+t_k(x,y)p(x,y))$$

of the Gauss map $\mathcal{G}$ which satisfies the relation $w(\tau_k(x,y))=\zeta^k w(x,y)$. If $\tau_k\circ\tau_\ell=\tau_{m(k,\ell)}$ then

$$\zeta^{k+\ell}w=w\circ\tau_k\circ\tau_\ell=w\circ\tau_{m(k,\ell)}=\zeta^{m(k,\ell)}w$$

and consequently $m(k,\ell)=k+\ell$ in the set $\mathbb{Z}_d$ parametrizing the deck transformation group of the foliation. Hence the monodromy group is also cyclic. Finally, a simple but lengthy computation shows that the sets $\mathcal{R}^\varrho_{\mathcal{G}}$ defined in Subsection 3.4 are empty if $\varrho\neq d$ and $\mathcal{R}^d_{\mathcal{G}}=\{uv=0\}$. Its image by $\mathcal{G}$ consists in two straight lines, provided that $\{u=0\}$ and $\{v=0\}$ are not invariant by the foliation, condition which holds generically. Last statement follows from Remark 5.6 and Proposition 5.5. □

Next example is a Galois foliation with non-cyclic monodromy group.

Example 5.8. An explicit computation of the polynomial defined in (10) shows that the degree $d=2n$ foliation $\mathcal{F}$ defined by the vector field

$$(x^n+y^n)^2\partial_x+(x^n-y^n)^2\partial_y$$

is Galois. We will see in Subsection 5.3 that its monodromy group is the dihedral group D_n of order $2n$. Moreover, a straightforward computation shows that $\mathcal{R}^\varrho_{\mathcal{G}}=\emptyset$ if $\varrho\notin\{2,n\}$,

$$\mathcal{R}^2_{\mathcal{G}}=\{(x^n+y^n)(x^n-y^n)=0\}\quad\text{and}\quad\mathcal{R}^n_{\mathcal{G}}=\{xy=0\},$$

so that its extended branching type is given by $[(1;(2)_n),(1;(2)_n),(1;(n)_2)]$ and the genus of its generic polar is also 0. □

5.2. **Geometric characterization of Galois foliations.** We address now the question of characterizing Galois foliations on $\mathbb{P}^2$ in terms of geometric elements naturally associated to the foliation. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ of degree $d > 0$ given by an homogeneous 1-form $\omega = adx + bdy + cdz$ with $a, b, c \in \mathbb{C}_{d+1}[x, y, z]$. Recall that its Gauss map $\mathcal{G} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$ is written

$$\mathcal{G}([x, y, z]) = [a(x, y, z), b(x, y, z), c(x, y, z)]$$

in homogeneous coordinates. We will use Theorem 3.33 which states that a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is Galois if and only if its associated Gauss map $\mathcal{G}$ is of regular type. Therefore, the geometric elements that we are led to consider are the indeterminacy locus $\Sigma_{\mathcal{G}}$ of $\mathcal{G}$, which is just the singular set $\Sigma_{\mathcal{F}}$ of $\mathcal{F}$, and the set $\Delta_{\mathcal{G}}$ that we denote $\Delta_{\mathcal{F}}$ from now on, i.e.

$$\Delta_{\mathcal{F}} := \overline{\{p \in \mathbb{P}^2 \setminus \Sigma_{\mathcal{F}} \mid \det(d\mathcal{G}_p) = 0\}} = \left\{ \begin{vmatrix} \partial_x a & \partial_y a & \partial_z a \\ \partial_x b & \partial_y b & \partial_z b \\ \partial_x c & \partial_y c & \partial_z c \end{vmatrix} = 0 \right\} \subset \mathbb{P}^2.$$

Recall that $\Delta_{\mathcal{F}} = \Delta_{\mathcal{G}}$ has a natural divisor structure.

One can give a geometric interpretation to $\Delta_{\mathcal{F}}$ as follows. Let us consider the inflection locus $\mathcal{I}_{\mathcal{F}}$ of the foliation $\mathcal{F}$, which is the closure of the set of points in $\mathbb{P}^2 \setminus \Sigma_{\mathcal{F}}$ where the leaves of $\mathcal{F}$ have a contact of order greater than one with its tangent line. It is shown in [34] that $\mathcal{I}_{\mathcal{F}}$ has a natural divisor structure defined by the homogeneous equation

$$F(x, y, z) := \begin{vmatrix} x & y & z \\ Z(x) & Z(y) & Z(z) \\ Z^2(x) & Z^2(y) & Z^2(z) \end{vmatrix} = 0,$$

of degree $3d$, where $Z(x, y, z)$ is an homogeneous vector field defining $\mathcal{F}$. Indeed, if $\zeta(t) = (x(t), y(t), z(t))$ is an integral curve of Z and $\gamma(t) = \left(\frac{x(t)}{z(t)}, \frac{y(t)}{z(t)}\right)$ is its projection to the affine chart (x, y) of $\mathbb{P}^2$ then

$$F(\zeta(t)) = \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = z(t)^3 \det(\gamma'(t), \gamma''(t)),$$

whose vanishing characterize the inflection points of the curve $\gamma(t)$. Notice also that, if $\mathcal{F}$ is defined by a vector field $X = A(x, y)\partial_x + B(x, y)\partial_y$ in the affine chart (x, y) and $\gamma(t) = (x(t), y(t))$ is an integral curve of X, then $\det(\gamma'(t), \gamma''(t)) = f(\gamma(t))$, where

$$f(x, y) = \begin{vmatrix} A(x, y) & B(x, y) \\ X(A(x, y)) & X(B(x, y)) \end{vmatrix}. \tag{12}$$

Consequently, $f(x, y) = 0$ is an affine equation for $\mathcal{I}_{\mathcal{F}}$.

Lemma 5.9. *If the foliation $\mathcal{F}$ is not degenerated, there is an equality of divisors $\mathcal{I}_{\mathcal{F}} = \Delta_{\mathcal{F}}$.*

Proof. Let $Z = \alpha\partial_x + \beta\partial_y + \gamma\partial_z$ be a degree d homogeneous vector field defining $\mathcal{F}$ and set $\omega = \imath_Z \imath_R dx \wedge dy \wedge dz = adx + bdy + cdz$. Then $a = y\gamma - z\beta$, $b = z\alpha - x\gamma$, $c = x\beta - y\alpha$ and the equality of divisors follows from Euler identity. □

We can decompose $\mathcal{I}_{\mathcal{F}} = \mathcal{I}_{\mathcal{F}}^{\mathrm{inv}} + \mathcal{I}_{\mathcal{F}}^{\mathrm{tr}}$, where the support of $\mathcal{I}_{\mathcal{F}}^{\mathrm{inv}}$ consists in the union of the invariant lines of $\mathcal{F}$ and the support of $\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}}$ is the closure of the inflection points that are isolated along the leaves of $\mathcal{F}$. For each $\varrho > 1$ we consider the reduced (maybe empty) curves $\mathcal{I}_{\mathcal{F}}^{\varrho} \subset \mathbb{P}^2$ defined by the equality of divisors

$$\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}} = \sum_{\varrho>1} (\varrho - 1)\mathcal{I}_{\mathcal{F}}^{\varrho}.$$

Remark 5.10. For each irreducible component D of $\mathcal{I}_{\mathcal{F}}^{\varrho}$ the ramification index of $\mathcal{G}$ along D is ϱ and it coincides with the tangency order between the leaf of $\mathcal{F}$ through a generic point p of D and its tangent line $T_p\mathcal{F}$. Indeed, the ramification index of $\mathcal{G}$ can be though as the number of local regular preimages $\mathcal{G}^{-1}(q')$ by $\mathcal{G}$ collapsing to $p \in \mathcal{G}^{-1}(q)$ as $q' \to q$, that is, the number of tangency points of order one collapsing to p.

In order to study the indeterminacy locus of $\mathcal{G}$, we consider a composition $\beta : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of blow-ups centered at $\Sigma_{\mathcal{F}}$ which desingularize $\mathcal{G}$, i.e. such that $\widetilde{\mathcal{G}} = \mathcal{G} \circ \beta : \widetilde{\mathbb{P}}^2 \to \check{\mathbb{P}}^2$ is a well-defined morphism. For each $s \in \Sigma_{\mathcal{F}}$ we set $\mathcal{E}_s = \beta^{-1}(s)$. We denote $\mathcal{E}_s^{\mathrm{dom}}$ (resp. $\mathcal{E}_s^{\mathrm{ram}}$) the union of irreducible components D of $\mathcal{E}_s$ such that $\delta_D := \deg(\widetilde{\mathcal{G}}) > 0$ (resp. $\varrho_D > 1$). We also set $\Sigma_{\mathcal{F}}^{\mathrm{ram}} := \{s \in \Sigma_{\mathcal{F}} \,|\, \mathcal{E}_s^{\mathrm{ram}} \neq \emptyset\}$. We recall that ϱ_D is the ramification index of $\widetilde{\mathcal{G}}$ along D and we notice that $\mathcal{E}_s^{\mathrm{dom}} \neq \emptyset$ and that $\mathcal{E}_s^{\mathrm{ram}} \subset \mathcal{E}_s^{\mathrm{dom}}$.

As we pointed out at the beginning of Subsection 3.1, the morphism $\delta := \beta \times \widetilde{\mathcal{G}} : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2 \times \check{\mathbb{P}}^2$ is a desingularization of the graph $\Gamma_{\mathcal{F}} \subset \mathcal{V} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2$ of $\mathcal{G}$ which is canonically identified with the variety $Z_{\mathcal{F}} \subset \mathbb{P}T^*\mathbb{P}^2$ considered in Proposition 4.9. Thus, if $s \in \Sigma_{\mathcal{F}}$ and $\widetilde{x} \in \mathcal{E}_s$ then $\delta(\widetilde{x}) = (\beta(\widetilde{x}), \widetilde{\mathcal{G}}(\widetilde{x})) \in \mathcal{V}$ and $s = \beta(\widetilde{x})$ belongs to the line $\widetilde{\mathcal{G}}(\widetilde{x}) \subset \mathbb{P}^2$, or equivalently $\widetilde{\mathcal{G}}(\widetilde{x}) \in \check{s}$. Since $\widetilde{\mathcal{G}}(\mathcal{E}_s)$ has positive dimension and it is contained in the line $\check{s}$ we deduce that $\widetilde{\mathcal{G}}(\mathcal{E}_s) = \check{s}$. Hence Hypothesis 3.31 is satisfied by the Gauss map of every foliation on $\mathbb{P}^2$. Keeping the notations used in Subsection 3.4, we have that $\mathcal{R}_{\mathcal{G}}^{\varrho} = \mathcal{I}_{\mathcal{F}}^{\varrho}$ and $\Sigma_{\mathcal{G}}^{\rho}$ consists of the singularities $s \in \Sigma_{\mathcal{F}}$ of $\mathcal{F}$ such that each irreducible component of $\mathcal{E}_s^{\mathrm{dom}}$ has ramification index ϱ. To unify the notations we will denote $\Sigma_{\mathcal{G}}^{\rho}$ by $\Sigma_{\mathcal{F}}^{\varrho}$.

Using Theorem 3.33 and Proposition 3.32 we obtain the following characterization of Galois foliations.

Theorem 5.11. *A degree d foliation $\mathcal{F}$ on $\mathbb{P}^2$ is Galois if and only if for each generic $\check{\ell} \in \Lambda_\rho \subset \check{\mathbb{P}}^2$ there is $\varrho | d$, $\varrho > 1$, such that* $\mathrm{Tang}(\mathcal{F}, \ell) \subset (\mathcal{I}_{\mathcal{F}}^{\varrho} \setminus \Sigma_{\mathcal{F}}) \cup \Sigma_{\mathcal{F}}^{\varrho}$.

Remark 5.10 gives a clear geometric meaning to the curve $\mathcal{I}_{\mathcal{F}}^{\varrho}$. It remains to give a geometric characterization of the sets $\Sigma_{\mathcal{F}}^{\varrho}$. To this purpose, we define the *vanishing order* of $\mathcal{F}$ at a point $s \in \Sigma_{\mathcal{F}}$ as

$$\nu_s := \min\{k \geq 1 \,:\, J_s^k X \neq 0\}$$

and the *tangency order* of $\mathcal{F}$ with a generic line passing through $s \in \Sigma_{\mathcal{F}}$ as

$$\tau_s := \min\{k \geq \nu_s \,:\, \det(J_s^k X, R_s) \neq 0\},$$

where X is a saturated vector field defining $\mathcal{F}$, $J_s^k X$ is its k-jet at s and R_s is the radial vector field centered at s.

Fix an affine chart (x, y) on $\mathbb{P}^2$ such that the corresponding line at infinity ℓ_∞ is not contained in the subset $\check{\Lambda}_{\mathcal{G}} \subset \mathbb{P}^2$ and assume that $\mathcal{F}$ is given by the vector field $X = A(x, y)\partial_x + B(x, y)\partial_y$. Let (p, q) the affine chart of $\check{\mathbb{P}}^2$ so that, in the affine chart (x, y, p, q) of $\mathbb{P}^2 \times \check{\mathbb{P}}^2$, the incidence variety $\mathcal{V}$ has equation $y = px+q$. Hence (x, y, p) and (p, q, x) are affine charts of $\mathcal{V}$ where the graph $\Gamma_{\mathcal{F}} \subset \mathcal{V}$ of $\mathcal{F}$ has equations $F(x, y, p) := B(x, y)-pA(x, y) = 0$ and $\check{F}(p, q, x) := B(x, px+q)-pA(x, px+q) = 0$ respectively. The differential 1-form $dy-pdx = dq+xdp$ defines the contact distribution $\mathcal{C}$ on $\mathcal{V}$ inducing the foliation $\mathcal{C}_{\mathcal{F}}$ on $\Gamma_{\mathcal{F}}$. Let $\pi_{\mathcal{F}}$ and $\check{\pi}_{\mathcal{F}}$ be the restrictions to $\Gamma_{\mathcal{F}}$ of the natural projections π and $\check{\pi}$ from $\mathcal{V}$ onto $\mathbb{P}^2$ and $\check{\mathbb{P}}^2$ respectively. Notice that for each $s \in \Sigma_{\mathcal{F}}$ we have $\pi_{\mathcal{F}}^{-1}(s) = \pi^{-1}(s) \subset \Gamma_{\mathcal{F}}$ and $\pi_{\mathcal{F}}^{-1}(s) \subset \mathrm{Sing}(\Gamma_{\mathcal{F}})$ if and only if $\nu_s > 1$. The foliation $\mathcal{F} = (\pi_{\mathcal{F}})_*\mathcal{C}_{\mathcal{F}}$ and the web $\mathrm{Leg}\,\mathcal{F} = (\check{\pi}_{\mathcal{F}})_*\mathcal{C}_{\mathcal{F}}$ are given by the implicit differential equations $F(x, y, \frac{dy}{dx}) = 0$ and $\check{F}(p, q, -\frac{dq}{dp}) = 0$ respectively. (For more details see [26, 3].)

Once the affine chart (x, y) is fixed, we can also interpret $\Gamma_{\mathcal{F}} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2$ as a family of curves on $\mathbb{P}^2$ over $\ell_\infty \simeq \mathbb{P}^1$, whose fibers are the polar curves of $\mathcal{F}$ with respect to the points $[1, p, 0] \in \ell_\infty$, because each of them is given by the equation $F(x, y, p) = 0$, for fixed p.

Consider the normalization $\kappa : \Gamma^\nu_{\mathcal{F}} \to \Gamma_{\mathcal{F}}$ of the graph $\Gamma_{\mathcal{F}}$ of $\mathcal{F}$. By the universal property of the normalization there are birational morphisms $\xi : \widetilde{\mathbb{P}}^2 \to \Gamma^\nu_{\mathcal{F}}$ and $\chi : \Gamma^\nu_{\mathcal{F}} \to N$ such that the composition $\chi \circ \xi$ is the birational morphism $\gamma : \widetilde{\mathbb{P}}^2 \to N$ given by the Stein factorization (3) of the chosen desingularization $\widetilde{\mathcal{G}} : \widetilde{\mathbb{P}}^2 \to \check{\mathbb{P}}^2$ of $\mathcal{G} = \check{\pi}_{\mathcal{F}} \circ \pi_{\mathcal{F}}^{-1} : \mathbb{P}^2 \dashrightarrow \check{\mathbb{P}}^2$. These maps make commutative the following diagram:

$$\begin{array}{ccccc} \widetilde{\mathbb{P}}^2 & \xrightarrow{\ \xi\ } & \Gamma^\nu_{\mathcal{F}} & \xrightarrow{\ \chi\ } & N \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\rho} \\ \mathbb{P}^2 & \xleftarrow{\ \pi_{\mathcal{F}}\ } & \Gamma_{\mathcal{F}} & \xrightarrow{\ \check{\pi}_{\mathcal{F}}\ } & \check{\mathbb{P}}^2. \end{array}$$

In fact, for each $s \in \Sigma_{\mathcal{F}}$ the morphism ξ (resp. χ) collapses exactly each irreducible component of $\mathcal{E}_s \setminus \mathcal{E}_s^{\mathrm{dom}}$ (resp. each invariant line) to a point. This implies that ξ induces a canonical bijection between the sets of irreducible components of the curves $\mathcal{E}_s^{\mathrm{dom}}$ and $\kappa^{-1}(\pi_{\mathcal{F}}^{-1}(s))$. Moreover, the irreducible components of $\kappa^{-1}(\pi_{\mathcal{F}}^{-1}(s))$ are in one-to-one correspondence with the local irreducible components of $\Gamma_{\mathcal{F}}$ along $\pi_{\mathcal{F}}^{-1}(s)$. Notice also that every irreducible component D of $\mathcal{E}_s^{\mathrm{dom}}$ intersects the strict transform of a generic polar curve $\mathcal{P}$ in exactly $\delta_D > 0$ points and on the other hand $\mathcal{P} \cap \mathcal{E}_s = \mathcal{P} \cap \mathcal{E}_s^{\mathrm{dom}}$. Consequently the desingularization of the generic polar curve $\mathcal{P}$ coincides with the desingularization $\beta : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of $\mathcal{G}$, but in general it is different from the reduction of singularities of $\mathcal{F}$, cf. [11].

By [37, Corollaire p. 80], the restriction of $\kappa : \Gamma^\nu_{\mathcal{F}} \to \Gamma_{\mathcal{F}}$ to the preimage by κ of the polar curve $\Gamma_{\mathcal{F}} \cap \{p = p_0\}$ coincide with its normalization for generic $p_0 \in \mathbb{C}$. Fix $s \in \Sigma_{\mathcal{F}}$ and assume we have chosen the affine chart (x, y) so that $s = (0, 0)$ and that x is not a factor of an equation defining the tangent cone of the polar curve $F(x, y, p) = 0$ at s. The local irreducible components of $\Gamma_{\mathcal{F}}$ at $(0, 0, p_0)$ are in canonical bijection with the branches of the polar curve $F(x, y, p_0) = 0$. Moreover, for each branch γ_{p_0}

of $F(x,y,p_0)=0$ at s there are local coordinates $(t,p)\in(\mathbb{C}^2,0)$ on $\Gamma^\nu_{\mathcal{F}}$ such that

$$(\pi_{\mathcal{F}}\circ\kappa)(t,p)=\Big(t^m,\sum_{j\geq m}c_j(p)t^j\Big)$$

is a simultaneous Puiseux parametrization of the corresponding branches γ_p of the polar curves $F(x,y,p)=0$ at s. Here m stands for the multiplicity of γ_p at s for generic p. Notice that the the sum of the multiplicities m of the different branches of $F(x,y,p_0)$ at s is the multiplicity of $F(x,y,p_0)$ at s and it coincides with the vanishing order ν_s of the vector field X at s.

Since for generic $p_0\in\mathbb{C}$, the birational morphisms χ and ξ are local biholomorphisms at the points $(0,p_0)$ and $\xi^{-1}(0,p_0)$ respectively, for each irreducible component D of $\mathcal{E}_s^{\mathrm{dom}}$, the ramification index of $\widetilde{\mathcal{G}}$ along D coincides with the ramification index along $\xi(D)=\{t=0\}$ of the map $\check{\pi}_{\mathcal{F}}\circ\kappa:\Gamma^\nu_{\mathcal{F}}\to\check{\mathbb{P}}^2$, which is written as

$$(\check{\pi}_{\mathcal{F}}\circ\kappa)(t,p)=\Big(p,\sum_{j\geq m}(c_j(p)-p\delta_{jm})t^j\Big).$$

Theorem 5.12. *Let $\mathcal{F}$ be a degree d foliation on $\mathbb{P}^2$ and let us fix $s\in\Sigma_{\mathcal{F}}$. Then $s\in\Sigma^\varrho_{\mathcal{F}}$ if and only if either,*

- *$\tau_s=\nu_s$ and each branch of the generic polar at s has multiplicity ϱ, or*
- *$\tau_s>\nu_s$, $\tau_s-\nu_s+1=\varrho$ and each branch of the generic polar at s, whose strict transform does not meet the exceptional divisor of the first blow up of s, has multiplicity ϱ.*

Moreover,

(1) $s\in\Sigma^{\mathrm{ram}}_{\mathcal{F}}$ if and only if $\tau_s>\nu_s$ or $\tau_s=\nu_s$ and the generic polar of $\mathcal{F}$ has a singular branch at s;

(2) $s\in\Sigma^d_{\mathcal{F}}$ if and only if $(\nu_s,\tau_s)=(1,d)$ or $\nu_s=d$ and the generic polar of $\mathcal{F}$ has a single branch at s.

Proof. Set $\nu=\nu_s$ and $\tau=\tau_s$ and consider an irreducible component D of $\mathcal{E}_s^{\mathrm{dom}}$. With the notations introduced above, we have that ϱ_D is the ramification index of $\check{\pi}_{\mathcal{F}}\circ\kappa$ along $\xi(D)=\{t=0\}$. If $c_m(p)\not\equiv p$ then $\varrho_D=m$. If $c_m(p)\equiv p$ then $y-px$ divides the polynomial $B_\nu(x,y)-pA_\nu(x,y)$. This implies that $A_\nu=xC_{\nu-1}$ and $B_\nu=yC_{\nu-1}$ for some homogeneous polynomial $C_{\nu-1}(x,y)$ of degree $\nu-1$. Consequently $(y-px)^2$ does not divide $B_\nu(x,y)-pA_\nu(x,y)=(y-px)C_{\nu-1}(x,y)$ and there is a unique branch γ_p of $F(x,y,p)=0$ at s, tangent to $y=px$, which is necessarily non-singular at s. Hence the strict transform of γ_p meets the exceptional divisor D_s of the first blow up of s and consequently $D=D_s$. Consider the chart (t,x) around $D_s=\{x=0\}$ given by $y=tx$. Then $\widetilde{\mathcal{G}}(t,x)=(p(t,x),x(t-p(t,x)))$ with $p(t,x)=\frac{B(x,tx)}{A(x,tx)}$. We write

$$X=A\partial_x+B\partial_y=(C_{\nu-1}+\cdots+C_{\tau-2})R+X_\tau+\cdots+X_d+C_dR,$$

where X_τ is not collinear with the radial vector field R. Consequently,

$$\begin{aligned} p(t,x) &= \frac{x^\nu t(C_{\nu-1}(1,t)+xC_\nu(1,t)+\cdots+x^{\tau-\nu+1}C_{\tau-2})+x^\tau B_\tau(1,t)+\cdots+x^{d+1}B_{d+1}(1,t)}{x^\nu(C_{\nu-1}(1,t)+xC_\nu(1,t)+\cdots+x^{\tau-\nu+1}C_{\tau-2})+x^\tau A_\tau(1,t)+\cdots+x^{d+1}A_{d+1}(1,t)}\\ &= t+x^{\tau-\nu}p_{\tau-\nu}(t)+\cdots \end{aligned}$$

with $p_{\tau-\nu}(t) \not\equiv 0$, $q(t,x) = -x^{\tau-\nu+1}p_{\tau-\nu}(t) + \cdots$ and

$$\begin{aligned} \det(d\widetilde{\mathcal{G}}) &= \begin{vmatrix} 1 + x^{\tau-\nu}p'_{\tau-\nu}(t) + \cdots & -p'_{\tau-\nu}(t)x^{\tau-\nu+1} + \cdots \\ (\tau-\nu)p_{\tau-\nu}(t)x^{\tau-\nu-1} + \cdots & -(\tau-\nu+1)p_{\tau-\nu}(t)x^{\tau-\nu} + \cdots \end{vmatrix} \\ &= -(\tau-\nu+1)p_{\tau-\nu}(t)x^{\tau-\nu} + \cdots \end{aligned}$$

Hence, in that case we have $\varrho_{D_s} = \tau - \nu + 1$ by Remark 3.30. We deduce the following dichotomy:

- If $\nu_s = \tau_s$ then ϱ_D is the multiplicity m of the branch of the generic polar whose strict transform meets D.
- If $\nu_s < \tau_s$ then there are two possibilities: $D = D_s$ and $D \neq D_s$. In the first case $\varrho_D = \tau_s - \nu_s + 1$ and in the second one ϱ_D is the multiplicity m of the branch of the generic polar whose strict transform meets D.

First part of the statement follows from these assertions.

By [30, Proposition 5.2] the generic polar curve of a foliation on $\mathbb{P}^2$ is (globally) irreducible. Consider the (necessarily reduced) local prime factorization of $F(x,y,p) = \prod_{i=1}^r f_i(x,y,p)$ in $\mathcal{O}_{\mathbb{P}^2,s}$ for generic $p \in \mathbb{C}$. Now assertions (1) and (2) follow easily from the facts that $m_i = \mathrm{ord}_s f_i$ and $\sum_i m_i = \nu_s$. □

Corollary 5.13. *Let $\mathcal{F}$ be a degree d foliation on $\mathbb{P}^2$. If the following conditions are satisfied*

(1) at an isolated inflection point p along a leaf L of $\mathcal{F}$, the tangency order of L with its tangent line T_pL attains its maximum d;

(2) for each $s \in \Sigma_{\mathcal{F}}$ the following trichotomy holds:
- *$\nu_s = 1$ and $\tau_s = d$, or*
- *$\nu_s = d$ and the generic polar of $\mathcal{F}$ has a single branch at s, or*
- *$\nu_s = \tau_s$ and each branch at s of the generic polar of $\mathcal{F}$ is smooth;*

then $\mathcal{F}$ is Galois with cyclic monodromy group. The converse is also true when d is prime.

Proof. Assertion (1) is equivalent to the equality $\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}} = (d-1)\mathcal{I}_{\mathcal{F}}^{d}$ by Remark 5.10. The two first possibilities in assertion (2) are equivalent to $s \in \Sigma_{\mathcal{F}}^{d}$ and the third one to $s \notin \Sigma_{\mathcal{F}}^{\mathrm{ram}}$, so that assertion (2) is equivalent to the equality $\Sigma_{\mathcal{F}}^{\mathrm{ram}} = \Sigma_{\mathcal{F}}^{d}$. We conclude by applying Corollary 3.34. □

Remark 5.14. It is worth noting that, if $\mathcal{F} \in \mathbb{E}_d \subset \mathbb{G}_d$ with $d \geq 3$ then $\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}} = (d-1)\mathcal{I}_{\mathcal{F}}^{d}$, which implies that $\mathcal{I}_{\mathcal{F}}^{\mathrm{inv}} \neq \emptyset$ when $d \neq 4$. Indeed, if $\mathcal{I}_{\mathcal{F}} = \mathcal{I}_{\mathcal{F}}^{\mathrm{tr}}$ then $3d = (d-1)k$ and $(k,3) = \ell(d,d-1)$ for some $\ell \in \mathbb{Z}$, because $\gcd(d,d-1) = 1$. Hence $\ell = 1$ and $d = 4$. On the other hand, Example 5.8 provides a family of homogeneous Galois foliations of degree $d = 2n$ for which $\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}} \neq \mathcal{I}_{\mathcal{F}}^{d}$. Finally, the hypothesis $\Sigma_{\mathcal{F}}^{\mathrm{ram}} = \Sigma_{\mathcal{F}}^{d}$ cannot be removed from Corollary 5.13 because the foliation $\mathcal{F}$ defined by the vector field

$$x^2(x+3y)\partial_x + y^2(y+3x)\partial_y$$

is convex, i.e. $\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}} = \emptyset$, but it is not Galois. In fact, $\mathcal{F}$ has singular points at infinity $[1,0,0]$, $[0,1,0]$, $[1,-1,0]$ and $[1,1,0]$, all of them with vanishing order $\nu = 1$, and with tangency orders $\tau = 2$, 2, 3 and 1 respectively.

We know from Corollary 4.18 that $\mathcal{F}$ is Galois if and only if $\mathcal{G}^*\mathrm{Leg}\,\mathcal{F}$ is totally decomposable. Let us point out the close relationship between the ramification index of $\widetilde{\mathcal{G}}$ along a component D of $\mathcal{R}_{\widetilde{\mathcal{G}}}$ and the local (resp. semi-local) decomposability of the web $\mathrm{Leg}\,\mathcal{F}$ at the generic point of (resp. along) the component $\widetilde{\mathcal{G}}(D)$ of $\Lambda_\rho = \Delta(\mathrm{Leg}\,\mathcal{F}) \subset \check{\mathbb{P}}^2$.

Let D be an irreducible component of $\mathcal{R}_{\widetilde{\mathcal{G}}}$ and let U_D be a tubular neighborhood of D. By Lemma 4.6, the direct image $\mathcal{W}_D := \widetilde{\mathcal{G}}_*(\widetilde{\mathcal{F}}_{|U_D})$ is an irreducible $\varrho_D\delta_D$-web on the tubular neighborhood $\widetilde{\mathcal{G}}(U_D)$ of $\widetilde{\mathcal{G}}(D) \subset \Lambda_\rho$.

Let C be an irreducible component of Λ_ρ and let V_C be a tubular neighborhood of C. Then $\mathrm{Leg}\,\mathcal{F}|_{V_C} = \boxtimes_D \mathcal{W}_D$, where D ranges the set of irreducible components of $\widetilde{\mathcal{G}}^{-1}(C)$. Let $p \in C$ be a generic point of $C \subset \Lambda_\rho$ and let V_p be a small neighborhood of p. Then, for each irreducible component D of $\widetilde{\mathcal{G}}^{-1}(C)$, we have $\mathcal{W}_D|_{V_p} = \boxtimes_{j=1}^{\delta_D} \mathcal{W}_D^j$, where $\mathcal{W}_D^j$ is an irreducible ϱ_D-web on V_p.

In the case $C = \check{s}$ with $s = (0,0) \in \Sigma_{\mathcal{F}}$ we have that $\check{F}(p,0,x) = x^{\tau_s} u(p,x)$ with $u(p,0) \not\equiv 0$. This implies that, in a tubular neighborhood $V_{\check{s}}$ of the dual line $\check{s} \subset \check{\mathbb{P}}^2$ of s, we can decompose $\mathrm{Leg}\,\mathcal{F}|_{V_{\check{s}}} = \mathcal{W}_{\tau_s}^{\mathrm{inv}} \boxtimes \mathcal{W}_{d-\tau_s}^{\mathrm{tr}}$ where $\Delta(\mathcal{W}_{\tau_s}^{\mathrm{inv}}) = \check{s}$ is totally invariant by $\mathcal{W}_{\tau_s}^{\mathrm{inv}}$ and $\mathcal{W}_{d-\tau_s}^{\mathrm{tr}}$ is transverse to $\check{s}$. In fact, $\mathcal{W}_{\tau_s}^{\mathrm{inv}} = \boxtimes_{D \subset \mathcal{E}_s^{\mathrm{dom}}} \mathcal{W}_D$. Hence

$$\sum_{D \subset \mathcal{E}_s^{\mathrm{dom}}} \delta_D \varrho_D = \tau_s. \tag{13}$$

Remark 5.15. If $\mathcal{F}$ is a Galois foliation of degree d and $s \in \Sigma_{\mathcal{F}}^{\mathrm{ram}}$ with $\tau_s < d$ then there must exist an irreducible component $C \subset \mathcal{I}_{\mathcal{F}}^{\mathrm{tr}}$ such that $\overline{\mathcal{G}(C \setminus \Sigma_{\mathcal{F}})} = \check{s}$. Indeed, we know from the hypothesis $s \in \Sigma_{\mathcal{F}}^{\mathrm{ram}}$ and formula (13) that there is $1 < \varrho \le \tau_s < d$ such that $s \in \Sigma_{\mathcal{F}}^{\varrho}$. Taking a line $\ell \subset \mathbb{P}^2$ such that $\ell \cap \Sigma_{\mathcal{F}} = \{s\}$ we see that $\mathrm{Tang}(\mathcal{F}, \ell)$ must contain points in $\mathcal{I}_{\mathcal{F}}^{\varrho} \setminus \Sigma_{\mathcal{F}}$ thanks to Theorem 5.11.

Example 5.16. Another illustrative example is given by the family of Fermat foliations introduced in [26]. They are defined by the vector fields $(x^d - \varepsilon x)\partial_x + (y^d - \varepsilon y)\partial_y$ with $\varepsilon \neq 0$. All of them are convex, i.e. $\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}} = \emptyset$, but they have radial singularities $s \in \Sigma_{\mathcal{F}}$ with vanishing order $\nu_s = 1$ and tangency order $\tau_s = 2$. From Remark 5.15 we deduce that Fermat foliations of degree $d > 2$ are not Galois. However, their degenerations $x^d\partial_x + y^d\partial_y$, obtained by taking $\varepsilon = 0$, are Galois, as we have seen in Example 5.7. □

5.3. **Homogeneous Galois foliations and their deformations.** In [9] the authors are interested in describing the algebraic set $\mathbb{G}_3$ of degree three Galois foliations. Due to the difficulty of problem in its full generality, they focus on the homogeneous case, for which they dispose of a particularly simple generic normal form depending only on 4 complex parameters:

$$\mathcal{F}_{\alpha;\lambda,\mu,\nu} \,:\, \frac{dx}{x} + \lambda\frac{dy}{y} + \mu\frac{dy - dx}{y - x} + \nu\frac{dy - \alpha dx}{y - \alpha x} = 0, \tag{14}$$

with $\lambda\mu\nu(1 + \lambda + \mu + \nu)\alpha(\alpha - 1) \neq 0$. They prove some partial results about the set $\mathcal{T}$ of $(\alpha; \lambda, \mu, \nu) \in \mathbb{C}^4$ such that the foliation $\mathcal{F}_{\alpha;\lambda,\mu,\nu}$ admits a birational trivolution $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ fullfilling $\mathcal{G} \circ \tau = \mathcal{G}$, where $\mathcal{G}$ is the Gauss map of the foliation. Namely, in [9, Theorem 5.17 and Proposition 5.18] it is stated that the point $(-1;1,1,1) \in \mathcal{T}$ admits a neighborhood U such that $\mathcal{T} \cap U$ is a smooth surface and the intersection of $\mathcal{T}$ with the hyperplane $\alpha = -1$ consists in two irreducible curves meeting at $(-1;1,1,1)$. This subsection is devoted to describe completely the set of homogeneous Galois foliations of arbitrary degree as well as its geometry.

Let $\mathbb{H}_d$ be the set of degree d homogeneous foliations given by saturated vector fields $A(x,y)\partial_x + B(x,y)\partial_y$. It is a Zariski open subset of $\mathbb{P}(\mathbb{C}_d[x,y]^{\oplus 2}) \simeq \mathbb{P}^{2d+1}$. The left-right actions of $\mathrm{PSL}_2(\mathbb{C})$ on the set of rational functions induce a natural action φ of $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{H}_d$ by means of

$$\varphi([\alpha_{ij}],[\beta_{ij}],[A_1,A_2]) = [\beta_{11}A_1^\alpha + \beta_{12}A_2^\alpha, \beta_{21}A_1^\alpha + \beta_{22}A_2^\alpha],$$

where

$$A_i^\alpha(x,y) = A_i(\alpha_{11}x + \alpha_{12}y, \alpha_{21}x + \alpha_{22}y).$$

It is easy to see that the Gauss map of a homogeneous foliation $\mathcal{F}$ defined by $X = A(x,y)\partial_x + B(x,y)\partial_y$ in the affine charts (x,y) on $\mathbb{P}^2$ and (a,b) on $\check{\mathbb{P}}^2$ corresponding to the line $ay - bx = 1$, takes the form

$$\mathcal{G}(x,y) = \left(\frac{A(x,y)}{yA(x,y) - xB(x,y)}, \frac{B(x,y)}{yA(x,y) - xB(x,y)}\right),$$

which is precisely in the class of rational maps considered in Example 3.19, with $u = y$ and $v = -x$. By applying Proposition 3.15 and Theorem 3.36 we obtain the following classification result.

Theorem 5.17. *The subset $\mathbb{H}_d \cap \mathbb{G}_d$ of $\mathbb{H}_d$ is constructible and its irreducible components consist of the orbits by $\varphi : \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C}) \times \mathbb{H}_d \to \mathbb{H}_d$ of the foliations*

(1) $x^d\partial_x + y^d\partial_y$ *for every* d,
(2) $(x^n + y^n)^2\partial_x + (x^n - y^n)^2\partial_y$ *if* $d = 2n$ *is even,*
(3) $(x^4 + 2i\sqrt{3}x^2y^2 + y^4)^3\partial_x + (x^4 - 2i\sqrt{3}x^2y^2 + y^4)^3\partial_y$ *if* $d = 12$,
(4) $(x^8 + 14x^4y^4 + y^8)^3\partial_x + (xy(x^4 - y^4))^4\partial_y$ *if* $d = 24$,
(5) $(x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20})^3\partial_x + (xy(x^{10} + 11x^5y^5 - y^{10}))^5\partial_y$ *if* $d = 60$.

The closure of each orbit is an unirational variety. The first one is 5-dimensional and the rest are 6-dimensional. Each irreducible component corresponds to a different Galois group according to Klein's classification given in Theorem 3.36.

Proof. Since $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ is an irreducible rational quasi-projective variety, we deduce from Chevalley's theorem (cf. [28, §2.6]) that each φ-orbit is an irreducible constructible set and its closure is an unirational variety, i.e. the closure of the image of a dominant rational map from a projective space. The assertion about the dimension in (1) follows from an explicit computation of the differential of the map $\varphi([\alpha_{ij},\beta_{ij}],[x^d,y^d])$ at the identity, which is

$$[(\beta_{11} + \alpha_{11}d)x^d + x^{d-1}y\alpha_{12}d + \beta_{12}y^d, \beta_{21}x^d + y^{d-1}x\alpha_{21}d + (\beta_{22} + \alpha_{22}d)y^d].$$

Its kernel is a 1-dimensional subspace of $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$. Analogous computations can be made in the cases (2)-(5). $\square$

Notice that, for each degree d, the first component of $\mathbb{H}_d \cap \mathbb{G}_d$ considered in the above Theorem consists of the homogeneous foliations appearing in Example 5.7 and the second one contains the foliations considered in Example 5.8. In addition, we can write the deck transformations of $\mathcal{G}$ in terms of $\widehat{\tau}(z) \in \mathrm{Deck}(B(1,z)/A(1,z)) \subset \mathrm{PSL}_2(\mathbb{C})$ in the following way

$$\tau(x,y) = \frac{A(x,y)y - B(x,y)x}{A(x,y)\widehat{\tau}(y/x) - B(x,y)}(1, \widehat{\tau}(y/x)). \tag{15}$$

Specializing the previous result to $d = 3$, the case considered in [9], we deduce that the set $\mathbb{H}_3 \cap \mathbb{G}_3$ of homogeneous degree 3 Galois foliations is the φ-orbit of $x^3\partial_x + y^3\partial_y$, which has dimension 5 inside the 7-dimensional space $\mathbb{H}_3$ and it is saturated by the orbits of the linear group of $\mathrm{GL}_2(\mathbb{C}) \subset \mathrm{PSL}_3(\mathbb{C})$.

Corollary 5.18. *The 4-dimensional slice $\mathcal{S} := \{\mathcal{F}_{\alpha;\lambda,\mu,\nu}\}$ given by the normal form (14) is transverse to the $\mathrm{GL}_2(\mathbb{C})$-orbits and $\mathcal{T} = \mathcal{S} \cap \mathbb{G}_3$ is a rational surface.*

Proof. A computation shows that the subset

$$\Gamma := \{(\alpha,\beta) \in \mathrm{PSL}_2(\mathbb{C})^2 \mid (\alpha,\beta)\cdot[x^3,y^3] \in \mathcal{S}\}$$

is the graph of a morphism $\gamma : U_{\mathcal{S}} \subset \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$, where $U_{\mathcal{S}}$ is a Zariski open set. The previous infinitesimal computation implies that $\mathcal{T} = \Gamma \cdot [x^3, y^3] \subset \mathcal{S}$ has dimension two. Hence $\mathcal{T}$ is an unirational surface. Since in dimension ≤ 2, rational and unirational are equivalent concepts, we conclude that $\mathcal{T}$ is a rational surface. □

The classification of homogeneous Galois foliations given by Theorem 5.17 can be used to obtain a negative test for proving that a given foliation on $\mathbb{P}^2$ is not Galois. It also provides (see Proposition 5.19 below) restrictions to either the type of the singularities of Galois foliations or the finite subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ that can occur as Galois groups of foliations on $\mathbb{P}^2$. For a general account on the finite subgroups of $\mathrm{Bir}(\mathbb{P}^2)$ we refer to [14]. Notice that Theorem 3.14 asserts that every finite group G occurs as the Galois group of a Galois branched covering $\rho : N \to Y$ but it does not give any indication about those that can be realized with rational total space N.

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$, for each singularity $s \in \Sigma_{\mathcal{F}}$ and each $\mathcal{F}$-invariant line $\ell \subset \mathcal{I}_{\mathcal{F}}^{\mathrm{inv}}$ we consider the homogeneous foliations $\mathcal{F}_s$ and $\mathcal{F}_\ell$ defined respectively by:

- $\mathcal{F}_s$ is the saturation of the first non-zero jet of a vector field defining $\mathcal{F}$ at s,
- $\mathcal{F}_\ell$ is the saturation of the top degree homogeneous part of a vector field defining $\mathcal{F}$ in the affine chart $\mathbb{P}^2 \setminus \ell$.

Notice $\mathcal{F}_s$ and $\mathcal{F}_\ell$ are homogeneous foliations on $\mathbb{P}^2$. Therefore, if they are Galois their deck transformation group are of Klein type, that is, appearing in the list given in Theorem 3.36. The relation between the foliations $\mathcal{F}$, $\mathcal{F}_s$ and $\mathcal{F}_\ell$ is given by the following statement.

Proposition 5.19. *Let $\mathcal{F}$ be a Galois foliation on $\mathbb{P}^2$. For each $s \in \Sigma_{\mathcal{F}}$ and $\ell \subset \mathcal{I}_{\mathcal{F}}^{\mathrm{inv}}$ we have that*

(1) if $\deg \mathcal{F}_s > 0$ the homogeneous foliation $\mathcal{F}_s$ is Galois; moreover if the exceptional divisor D_s obtained blowing up once the point s is not a ramification component then the Klein type deck transformation group of $\mathcal{F}_s$ injects into the deck transformation group of $\mathcal{F}$;

(2) if $\deg \mathcal{F}_\ell = \deg \mathcal{F}$ then the homogeneous foliation $\mathcal{F}_\ell$ is also Galois.

Proof. We obtain assertion (1) by applying Proposition 3.15 to $V = D_s \subset \widetilde{\mathbb{P}}^2$. Assertion (2) follows from the fact that $\mathbb{G}_d$ is closed by noting that $\mathcal{F}_\ell = \lim_{\varepsilon\to\infty} h_\varepsilon^* \mathcal{F}$, where $h_\varepsilon \in \mathrm{PSL}_3(\mathbb{C})$ is given by $h_\varepsilon(x,y) = (\varepsilon x, \varepsilon y)$ in the affine chart $\mathbb{P}^2 \setminus \ell$. □

In the case $d_\ell = \deg \mathcal{F}_\ell < \deg \mathcal{F} = d$ we are not able to assure that $\mathcal{F}_\ell$ is Galois. Nevertheless $\mathcal{F}_\ell$ belongs to $\overline{\mathbb{G}}_d \setminus \mathbb{G}_d \subset \overline{\mathbb{F}}_d$ and we expect that $\mathcal{F}_\ell \in \mathbb{G}_{d_\ell}$. An affirmative answer would help to describe an stratification of $\overline{\mathbb{G}}_d$ in terms of families of Galois foliations of lower degree (cf. Question 5.3(3)).

Motivated by Theorem 5.17 and Example 5.7 we consider the following family of deformations of a homogeneous foliation.

Definition 5.20. *Let $\mathcal{F} \in \mathbb{H}_d$ be a homogeneous foliation given by a saturated homogeneous vector field $X = A(x,y)\partial_x + B(x,y)\partial_y$. For every $\mathbb{C}$-linearly independent polynomials $u, v \in \mathbb{C}[x,y]$ of degrees ≤ 1, and every linearly independent vectors $(\alpha,\gamma,\lambda), (\beta,\delta,\mu) \in \mathbb{C}^3$ we consider the* extended left-right deformation *(ELR in short) of $\mathcal{F}$ as the family of foliations given by the vector fields*

$$(\alpha A + \beta B)(u,v)\partial_x + (\gamma A + \delta B)(u,v)\partial_y + (\lambda A + \mu B)(u,v)(x\partial_x + y\partial_y).$$

Proposition 5.21. *If $\mathcal{F}$ is a Galois homogeneous foliation then every element of its ELR-deformation is Galois with the same branching type that $\mathcal{F}$.*

Proof. The polar curve P of the general element of the ELR-deformation of $\mathcal{F}$ with respect to a generic point $(a,b) \in \mathbb{C}^2 \subset \mathbb{P}^2$ is given by the affine equation

$$F(x,y) := \begin{vmatrix} x-a & (\alpha A + \beta B)(u,v) + x(\lambda A + \mu B)(u,v) \\ y-b & (\gamma A + \delta B)(u,v) + y(\lambda A + \mu B)(u,v) \end{vmatrix} = 0.$$

Since u and v are $\mathbb{C}$-linearly independent polynomials of degree ≤ 1, from the equation $\frac{v}{u} = w \in \mathbb{P}^1$ we can express either $y = y_0(w) + y_1(w)x$ or $x = x_0(w) + x_1(w)y$, with $x_i(w), y_i(w) \in \mathbb{C}(w)$. Without loss of generality we can assume that we are in the first situation. From equation $F(x, y_0(w) + y_1(w)x) = 0$ we obtain an explicit rational parametrization $\pi : \mathbb{P}^1 \to P$ given by

$$x(w) = \left.\frac{((\lambda A + \mu B)\, a + \alpha\, A + \beta B)\, y_0 + (\gamma A + \delta B)\, a + (-\alpha A - \beta B)\, b}{-\,((\lambda A + \mu B)\, a + \alpha A + \beta B)\, y_1 + (\lambda A + \mu B)\, b + (\gamma A + \delta B)}\right|_{(1,w)}$$

$$y(w) = y_0(w) + y_1(w)x(w)$$

On the other hand, the pencil $\check{p}$ of lines through p can be parametrized by $t \in \mathbb{P}^1$ by means of $\frac{y-b}{x-a} = t$. By composing $\mathcal{G}_{|P} : P \to \check{p}$ to the left by $\pi : \mathbb{P}^1 \to P$ and to the right with the inverse of $\mathbb{P}^1 \overset{\sim}{\to} \check{p}$ we obtain the rational map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $w \mapsto \frac{(\gamma+b\lambda)A(1,w)+(\delta+b\mu)B(1,w)}{(\alpha+a\lambda)A(1,w)+(\beta+a\mu)B(1,w)}$, which is right equivalent to the Galois rational map $w \mapsto \frac{B(1,w)}{A(1,w)}$ because

$$(\gamma + b\lambda)(\beta + a\mu) - (\delta + b\mu)(\alpha + a\lambda) = \begin{vmatrix} \alpha & \gamma & \lambda \\ \beta & \delta & \mu \\ a & b & -1 \end{vmatrix} \neq 0$$

if $(a, b) \in \mathbb{C}^2$ is generic. □

Remark 5.22. The family of vector fields considered in Example 5.7 consists in the ELR-deformation of the homogeneous Galois foliation defined by $x^d\partial_x + y^d\partial_y$. One can also made explicit the ELR-deformation of each homogeneous Galois foliation given in Theorem 5.17, obtaining, by using formula (15), explicit parametric continuous deformations of faithful representations of the triangular groups C_n, D_n, A_4, S_4 and A_5 into the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$, whose images are not contained in $\mathrm{PSL}_3(\mathbb{C})$. For instance, these considerations applied to the foliation $x^d\partial_x + y^d\partial_y$ provide the following family $h_{\alpha\beta}(x, y) = (X_{\alpha\beta}(x, y), Y_{\alpha\beta}(x, y))$ of order d elements of $\mathrm{Bir}(\mathbb{P}^2)$:

$$X_{\alpha\beta}(x,y) = \frac{\left(y\beta_{11}(\alpha_{11}x+\alpha_{12}y)^d + y\beta_{12}(\alpha_{21}x+\alpha_{22}y)^d - x\beta_{21}(\alpha_{11}x+\alpha_{12}y)^d - x\beta_{22}(\alpha_{21}x+\alpha_{22}y)^d\right)x}{\zeta y\beta_{11}(\alpha_{11}x+\alpha_{12}y)^d + \zeta y\beta_{12}(\alpha_{21}x+\alpha_{22}y)^d - x\beta_{21}(\alpha_{11}x+\alpha_{12}y)^d - x\beta_{22}(\alpha_{21}x+\alpha_{22}y)^d},$$

$$Y_{\alpha\beta}(x,y) = \frac{\left(y\beta_{11}(\alpha_{11}x+\alpha_{12}y)^d + y\beta_{12}(\alpha_{21}x+\alpha_{22}y)^d - x\beta_{21}(\alpha_{11}x+\alpha_{12}y)^d - x\beta_{22}(\alpha_{21}x+\alpha_{22}y)^d\right)\zeta y}{\zeta y\beta_{11}(\alpha_{11}x+\alpha_{12}y)^d + \zeta y\beta_{12}(\alpha_{21}x+\alpha_{22}y)^d - x\beta_{21}(\alpha_{11}x+\alpha_{12}y)^d - x\beta_{22}(\alpha_{21}x+\alpha_{22}y)^d},$$

where ζ is a d-root of the unity and $\alpha = [\alpha_{ij}], \beta = [\beta_{ij}] \in \mathrm{PSL}_2(\mathbb{C})$.

Remark 5.23. Every homogeneous foliations admits the infinitesimal symmetry $R = x\partial_x + y\partial_y$ but the general element of its ELR-deformation does not admit R as infinitesimal symmetry any more. However, it can be checked that the set of all ELR-deformations of every homogeneous foliation contains the special subsets:

(a) $\{P(y)\partial_y + Q(y)(x\partial_x + y\partial_y) \,|\, P, Q \in \mathbb{C}[y]\}$, obtained by taking $u, v \in \mathbb{C}[y]$ and $\alpha = \beta = 0$ and admitting the infinitesimal symmetry $x\partial_x$,
(b) $\{P(y)\partial_x + Q(y)(x\partial_x + y\partial_y) \,|\, P, Q \in \mathbb{C}[y]$, obtained by taking $u, v \in \mathbb{C}[y]$ and $\gamma = \delta = 0$ and admitting the infinitesimal symmetry $y\partial_x$.

5.4. **Foliations with continuous automorphism group.** A natural class of foliations on $\mathbb{P}^2$ including homogeneous foliations is that of foliations $\mathcal{F}$ with a continuous group of automorphisms $\mathrm{Aut}(\mathcal{F}) \subset \mathrm{PSL}_3(\mathbb{C})$. After giving a classification of foliations in that class we establish a general criterion to decide whether they are Galois in terms of a suitable rational map $\mathbb{P}^1 \to \mathbb{P}^1$.

Taking into account that every foliation of degree 1 or 2 is Galois, we can assume that $\mathcal{F}$ has degree ≥ 3. Let $R \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{F})) \subset \mathfrak{X}(\mathbb{P}^2) \simeq \mathfrak{sl}_3(\mathbb{C})$ be a non-trivial infinitesimal automorphism of $\mathcal{F}$. There are four possible Jordan form types for the traceless matrix associated to R:

(a) $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -(\alpha+\beta) \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

In the affine chart (x, y) the corresponding vector field R takes one of the following normal forms:

(a) $R = \alpha x\partial_x + \beta y\partial_y$ with $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$,
(b) $R = y\partial_x$,
(c) $R = y\partial_x + \partial_y$,
(d) $R = (x + y)\partial_x + y\partial_y$.

Let $X = A(x, y)\partial_x + B(x, y)\partial_y$ be a *saturated* polynomial vector field defining $\mathcal{F}$. The fact that $R \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{F}))$ translates into the relation

$$L_R X = \varepsilon\, X, \tag{16}$$

for some rational function $\varepsilon \in \mathbb{C}(x, y)$. Since the poles of ε are contained in the zeroes of the coefficients of X and that vector field is saturated we see that $\varepsilon \in \mathbb{C}[x, y]$. Finally, using that $\deg R = 1$ we deduce that ε must be constant. The following result describes the foliations of degree ≥ 2 having a continuous automorphism group.

Proposition 5.24. *Let $X = A(x, y)\partial_x + B(x, y)\partial_y$ be a saturated polynomial vector field of degree ≥ 2 satisfying $L_R X = \varepsilon X$ for some $R \in \mathfrak{X}(\mathbb{P}^2)$ in the precedent list (a)-(d) of normal forms and for $\varepsilon \in \mathbb{C}$.*

(a) If $R = \alpha x\partial_x + \beta y\partial_y$ then $\beta/\alpha \in \mathbb{Q}$, so that we can assume that $\alpha, \beta \in \mathbb{Z}$ are coprime, $\varepsilon \in \mathbb{Z}\alpha + \mathbb{Z}\beta$ and

$$A(x, y) = \sum_{\alpha i+\beta j=\varepsilon+\alpha} a_{ij} x^i y^j \quad \textit{and} \quad B(x, y) = \sum_{\alpha i+\beta j=\varepsilon+\beta} b_{ij} x^i y^j$$

are quasi-homogenous polynomials of weights (α, β).
(b) If $R = y\partial_x$ then $\varepsilon = 0$ and $X = P(y)\partial_x + Q(y)(x\partial_x + y\partial_y)$ for some coprime polynomials $P, Q \in \mathbb{C}[y]$.
(c) If $R = y\partial_x + \partial_y$ then $\varepsilon = 0$ and $X = P(y^2 - 2x)(y\partial_x + \partial_y) + Q(y^2 - 2x)\partial_x$ for some coprime polynomials $P, Q \in \mathbb{C}[z]$.

In addition,

(d) if $R = (x + y)\partial_x + y\partial_y$, relation $L_R X = \varepsilon X$ does not hold for any saturated polynomial vector field X of degree ≥ 2.

Proof. (a) Writing $A = \sum a_{ij}x^i y^j$ and $B = \sum b_{ij}x^i y^j$, if $L_R X = \varepsilon X$ then

$$\left(\sum a_{ij}(\alpha i + \beta j - \alpha - \varepsilon)x^i y^j\right)\partial_x + \left(\sum b_{ij}(\alpha i + \beta j - \beta - \varepsilon)x^i y^j\right)\partial_y = 0,$$

leading to the claimed form of A and B. It is not difficult to see that if $\beta/\alpha \notin \mathbb{Q}$ and $\deg X \geq 2$ then X cannot be saturated.

(b) If $R = y\partial_x$ then $L_R X - \varepsilon X = (y\partial_x A - B - \varepsilon A)\partial_x + (y\partial_x B - \varepsilon B)\partial_y = 0$ implies that $B = e^{\frac{\varepsilon x}{y}}\bar{Q}(y) \in \mathbb{C}[x,y]$. Hence $\varepsilon = 0$ and $B = \bar{Q} \in \mathbb{C}[y]$. From the ∂_x-component of $L_R X - \varepsilon X = 0$ we obtain that $A = \frac{\bar{Q}(y)x}{y} + P(y) \in \mathbb{C}[x,y]$. Thus, $\bar{Q}(y) = yQ(y)$ for some $Q \in \mathbb{C}[y]$.

(c) If $R = y\partial_x + \partial_y$ then

$$L_R X - \varepsilon X = (y\partial_x A + \partial_y A - B - \varepsilon A)\partial_y + (y\partial_x B + \partial_y B - \varepsilon B)\partial_y = 0$$

implies that $B = e^{\varepsilon y}P(y^2 - 2x)$ and necessarily $\varepsilon = 0$. From the ∂_x-component of $L_R X - \varepsilon X = 0$ we obtain that $A(x,y) = yP(y - x^2) + Q(y^2 - x)$ for some polynomials $P, Q \in \mathbb{C}[z]$.

(d) If $R = (x+y)\partial_x + y\partial_y$ and $X = \sum_{n\geq 0} X_n$ with $X_n = A_n\partial_x + B_n\partial_y$ homogeneous of degree n, then the degree n homogeneous part of $L_R X - \varepsilon X$ is

$$\begin{aligned} 0 &= L_R X_n - \varepsilon X_n = ((x+y)\partial_x A_n + y\partial_y A_n - (\varepsilon+1)A_n - B_n)\partial_x + \\ &\quad ((x+y)\partial_x B_n + y\partial_y B_n - (\varepsilon+1)B_n)\partial_y \\ &= (y\partial_x A_n - (\varepsilon + 1 - n)A_n - B_n)\partial_x + (y\partial_x B_n - (\varepsilon + 1 - n)B_n)\partial_y. \end{aligned}$$

As before, looking at the ∂_y-component we deduce that if $B_n \neq 0$ then $B_n = e^{\frac{(\varepsilon+1-n)x}{y}}Q(y) \in \mathbb{C}[x,y]$. Hence $\varepsilon = n-1$ and $B(y) = Q(y) = qy^n$ for some $q \in \mathbb{C}$. Substituting B in the ∂_y-component of $L_R X_n - \varepsilon X_n$ we easily deduce that $A(x,y) = qxy^{n-1} + py^n$ for some $p \in \mathbb{C}$. Since there is at most one $n \in \mathbb{Z}_+$ such that $\varepsilon = n-1$, we deduce that $X = X_n = y^{n-1}((py + qx)\partial_x + qy\partial_y)$ is not saturated because $\deg X = n \geq 2$.

□

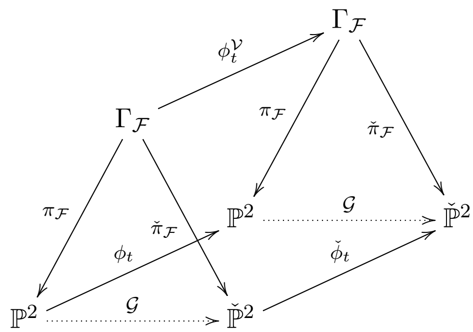
Let ϕ_t be the flow of homographic transformations of $\mathbb{P}^2$ associated to R and let $\check{\phi}_t$ be the dual flow on $\check{\mathbb{P}}^2$ associated to the dual vector field $\check{R}$. We consider on $\check{\mathbb{P}}^2$ the affine chart (a,b) that parametrizes the lines $\{ax + by = 1\}$. We can check that, in the relevant cases (a), (b) and (c), the corresponding flows and dual vector fields are the following:

(a) for $R = \alpha x\partial_x + \beta y\partial_y$ we have $\check{R} = -(\alpha a\partial_a + \beta b\partial_b)$, $\phi_t(x,y) = (xe^{\alpha t}, ye^{\beta t})$ and $\check{\phi}_t(a,b) = (ae^{-\alpha t}, be^{-\beta t})$;

(b) for $R = y\partial_x$ we have $\phi_t(x,y) = (x + ty, y)$, $\check{\phi}_t(a,b) = (a, b - at)$ and $\check{R} = -a\partial_b$;

(c) for $R = y\partial_x + \partial_y$ we have $\phi_t(x,y) = (x + ty + t^2/2, y + t)$, $\check{\phi}_t(a,b) = \left(\frac{a}{1+bt-at^2/2}, \frac{b-ta}{1+bt-at^2/2}\right)$ and $\check{R} = -(ab\partial_a + (a + b^2)\partial_b)$.

Let $\phi_t^{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ denote the flow induced by $\mathbb{P}T^*\phi_t : \mathbb{P}T^*\mathbb{P}^2 \to \mathbb{P}T^*\mathbb{P}^2$ via the identification $\mathcal{V} = \{(p,\ell) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 \mid p \in \ell\} \simeq \mathbb{P}T^*\mathbb{P}^2$. The relation (16) implies that $(d\phi_t)_p(X_p)$ is collinear to $X(\phi_t(p))$ so that $\phi_t^{\mathcal{V}}$ preserves the graph $\Gamma_{\mathcal{F}} \subset \mathcal{V}$ of the foliation $\mathcal{F}$ defined by X. The commutativity of the top faces of the diagram



implies that $\check{\phi}_t \circ \mathcal{G} = \mathcal{G} \circ \phi_t$, which is relation (5) in Example 3.21.

Corollary 5.25. *To every foliation $\mathcal{F}$ on $\mathbb{P}^2$ admitting a continuous group of automorphisms we can associate a non-constant morphism $\widehat{\mathcal{G}} : \mathbb{P}^1 \to \mathbb{P}^1$ so that* $\mathrm{Deck}(\mathcal{G}) \simeq \mathrm{Deck}(\widehat{\mathcal{G}})$. *In particular, $\mathcal{F}$ is Galois $\Longleftrightarrow$ $\widehat{\mathcal{G}}$ is Galois.*

Proof. In cases (a), (b) and (c) the foliations defined by the vector fields R and its dual $\check{R}$ admit explicit primitive rational first integrals

$$\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \qquad \text{and} \qquad \check{\rho} : \check{\mathbb{P}}^2 \dashrightarrow \mathbb{P}^1$$

respectively, and rational sections

$$\sigma : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2 \qquad \text{and} \qquad \check{\sigma} : \mathbb{P}^1 \dashrightarrow \check{\mathbb{P}}^2$$

such that $\rho \circ \sigma = \check{\rho} \circ \check{\sigma} = \mathrm{Id}_{\mathbb{P}^1}$. It can be easily checked that, in the affine charts considered above, these maps are given by

(a) $\rho(x, y) = y^\alpha / x^\beta$, $\sigma(z) = (z^\gamma, z^\delta)$, $\check{\rho}(a, b) = b^\alpha / a^\beta$ and $\check{\sigma}(z) = (z^\gamma, z^\delta)$, where $\gamma, \delta \in \mathbb{Z}$ satisfy Bézout's relation $\alpha\delta - \beta\gamma = 1$,

(b) $\rho(x, y) = y$, $\sigma(z) = (0, z)$, $\check{\rho}(a, b) = a$ and $\check{\sigma}(z) = (z, 0)$,

(c) $\rho(x, y) = y^2 - 2x$, $\sigma(z) = (-z/2, 0)$, $\check{\rho}(a, b) = \frac{b^2+2a}{a^2}$ and $\check{\sigma}(z) = (2/z, 0)$.

Moreover, the Gauss map of the foliation given by the vector field $A(x, y)\partial_x + B(x, y)\partial_y$ is written as

$$\mathcal{G}(x, y) = \left(\frac{-B(x, y)}{C(x, y)}, \frac{A(x, y)}{C(x, y)} \right), \quad \text{with} \quad C(x, y) = yA(x, y) - xB(x, y).$$

Thus we obtain explicit expressions for the map $\widehat{\mathcal{G}} = \check{\rho} \circ \mathcal{G} \circ \sigma : \mathbb{P}^1 \to \mathbb{P}^1$:

$$\text{(17)} \qquad \begin{cases} \text{(a)} & \widehat{\mathcal{G}}(z) = A(z^\gamma, z^\delta)^\alpha (-B(z^\gamma, z^\delta))^{-\beta} C(z^\gamma, z^\delta)^{\beta-\alpha}, \\ \text{(b)} & \widehat{\mathcal{G}}(z) = -\frac{B(0,z)}{C(0,z)} = -\frac{Q(z)}{P(z)}, \\ \text{(c)} & \widehat{\mathcal{G}}(z) = \frac{Q(z)^2 - zP(z)^2}{P(z)^2} = \left(\frac{Q(z)}{P(z)}\right)^2 - z, \end{cases}$$

where A, B take the form given by Proposition 5.24 in each case. Consequently, we are in the hypothesis of Example 3.21 and we can apply Proposition 3.20 in order to conclude. □

Notice that all Galois foliations of this type have Galois group appearing in Klein's classification given by Theorem 3.36. This fact and Proposition 5.19 motivate the following natural question:

Question 5.26. *Are there Galois foliations on $\mathbb{P}^2$ whose Galois group is not of Klein type?*

Remark 5.27. If we set $\alpha = \beta = 1$ in case (a), we obtain the class of homogeneous foliations studied in Subsection 5.3. For every coprime homogeneous polynomials A, B in two variables of the same degree, the homogeneous and type (b) foliations on $\mathbb{P}^2$ given respectively by the vector fields

$$A(x,y)\partial_x + B(x,y)\partial_y \quad \text{and} \quad A(1,y)\partial_x - B(1,y)(x\partial_x + y\partial_y)$$

satisfy that the map $\widehat{\mathcal{G}}$ induced by their Gauss map is $\widehat{\mathcal{G}} = [A,B] : \mathbb{P}^1 \to \mathbb{P}^1$. Moreover, in the homogeneous case we have $\widehat{\mathcal{G}} = [A,B] = \widetilde{\mathcal{G}}_{|D_O}$, where D_O is the exceptional divisor obtained after blowing up once the origin, and we recover Theorem 5.17 in an alternative way. On the other hand, as we have already pointed out in Remark 5.23, cases (a) with $\beta = 0$ and (b) can be thought as degenerations of homogeneous foliations.

Despite the criterion provided by Corollary 5.25 for deciding whether a foliation with an infinitesimal symmetry is Galois and the explicit form of the rational map $\widehat{\mathcal{G}}$ given in (17), it is not easy to find new examples of Galois foliations admitting such a symmetry. This is due to the difficulty of recovering the coefficients A and B based only on the map $\widehat{\mathcal{G}}$. However, we can present some examples and partial results about the quasi-homogeneous case (a) with $0 < \alpha < \beta$.

First of all, notice that the degree d foliation $\mathcal{F}$ given by the vector field

$$x^{d+1}\partial_x + (y^d + x^d y)\partial_y$$

belongs to the Galois family of Example 5.7 and that it is quasi-homogeneous with weights $\alpha = d-1$ and $\beta = d$. Moreover, it can be checked that the foliation $\mathcal{F}$ is convex, i.e. $\mathcal{I}^{\mathrm{tr}}_{\mathcal{F}} = \emptyset$, that $\Sigma^{\mathrm{ram}}_{\mathcal{F}} = \Sigma_{\mathcal{F}} = \{[0,0,1],[0,1,0]\}$ and that $\mathcal{B}^+_{\mathcal{F}} = [(1;(d)_1),(1;(d)_1)]$.

Secondly, thanks to Corollary 3.34 we can present two new explicit Galois quasi-homogeneous foliations of degree $d = 3$.

Example 5.28. The foliation $\mathcal{F}$ given by the vector field

$$X_1 = (y + x^2)\partial_x - \frac{x^3}{3}\partial_y$$

is quasi-homogeneous with weights $\alpha = 1$ and $\beta = 2$. Using formula (12), it can be easily checked that $\mathcal{I}^{\mathrm{tr}}_{\mathcal{F}} = \{x^2(3y+2x^2)^2 = 0\}$. On the other hand, we have that $\Sigma_{\mathcal{F}} = \{s_1 = [0,0,1], s_2 = [0,1,0]\}$ with $\nu_{s_i} = \tau_{s_i} = i$ for $i = 1,2$. It can be checked that the generic polar has i smooth branches at s_i, for $i = 1,2$. By applying Theorem 5.12 we deduce that $\Sigma^{\mathrm{ram}}_{\mathcal{F}} = \emptyset$. Since $\mathcal{G}$ maps $x = 0$ into $p = 0$ and $3y + 2x^2 = 0$ into $3q - p^2 = 0$, its extended branching type is $\mathcal{B}^+_{\mathcal{F}} = [(1;(3)_1),(2;(3)_1)]$, so that $\mathcal{F}$ is Galois and the genus of its generic polar is $g = 1$. □

Example 5.29. The foliation $\mathcal{F}$ given by the vector field

$$X_2 = yx\partial_x + (\zeta y^2 + x^3)\partial_y, \quad \text{with} \quad \zeta = \frac{1 \pm i\sqrt{3}}{2},$$

is quasi-homogeneous with weights $\alpha = 2$ and $\beta = 3$. As before, it can be checked that $\mathcal{I}^{\mathrm{tr}}_{\mathcal{F}} = \{(y^2 - x^3)^2 = 0\}$ and that, for each $s \in \Sigma_{\mathcal{F}} =$

$\{[0,0,1],[0,1,0]\}$, the generic polar has an ordinary double point at s. Hence $\Sigma_{\mathcal{F}}^{\mathrm{ram}}=\emptyset$ again by Theorem 5.12. Since $\mathcal{G}$ maps $\mathcal{I}_{\mathcal{F}}^{\mathrm{tr}}$ into $\{q-\xi p^3=0\}$ with $\xi=\frac{\zeta}{3-6\zeta}$, the extended branching type of $\mathcal{F}$ is $\mathcal{B}_{\mathcal{F}}^{+}=[(3;(3)_1)]$, so that the foliation $\mathcal{F}$ is Galois and the genus of its generic polar is also $g=1$. □

Let us finish this subsection by pointing out a special property fulfilled by every quasi-homogeneous foliation $\mathcal{F}$ with weights $\alpha\neq\beta$. Namely, the envelope of the family of the tangent lines of $\mathcal{F}$ along a fibre $F=\{y^\alpha-zx^\beta=0\}$ of $\rho(x,y)=\frac{y^\alpha}{x^\beta}$ is another fibre $F'=\{y^\alpha-z'x^\beta=0\}$ with $z'=\widehat{\psi}_{\mathcal{F}}(z):=\frac{(-\alpha)^\alpha\beta^{-\beta}(\beta-\alpha)^{\beta-\alpha}}{\widehat{\mathcal{G}}(z)}$. If $p\in F\setminus\{0\}$ then $T_p\mathcal{F}$ is tangent to F' at a uniquely determined point $\psi_{\mathcal{F}}(p)\in F'$. It can be checked that the self-map $\psi_{\mathcal{F}}:\mathbb{P}^2\dashrightarrow\mathbb{P}^2$ is rational. By construction it makes commutative the following diagram

$$\begin{array}{ccc}\mathbb{P}^2 & \overset{\psi_{\mathcal{F}}}{\dashrightarrow} & \mathbb{P}^2\\ {\scriptstyle\rho}\,\vdots & & \vdots\,{\scriptstyle\rho}\\ \mathbb{P}^1 & \xrightarrow{\widehat{\psi}_{\mathcal{F}}} & \mathbb{P}^1\,.\end{array}$$

Example 5.30. If $\mathcal{F}$ is given by the vector field $xy\partial_x+(by^2+cx^3)\partial_y$ then $\psi_{\mathcal{F}}(x,y)=\left(\frac{3x((b-1)y^2+cx^3)}{by^2+cx^3},\frac{2((b-1)y^2+cx^3)}{y}\right)$ and $\widehat{\psi}_{\mathcal{F}}(z)=\frac{4(c+bz)^3}{27z(c+(b-1)z)}$. □

Let $r:\mathbb{P}^2\times\mathbb{P}^2\dashrightarrow\check{\mathbb{P}}^2$ be the rational map defined by considering the line passing through two different points. Then $\mathrm{Id}\times r:\mathbb{P}^2\times\mathbb{P}^2\dashrightarrow\mathbb{P}^2\times\check{\mathbb{P}}^2$ is birational and $(\mathrm{Id}\times r)\circ(\mathrm{Id}\times\psi_{\mathcal{F}})=\mathrm{Id}\times\mathcal{G}$ so that all the relevant topological information of $\mathcal{G}$ is encoded by $\psi_{\mathcal{F}}$, which has the advantage of being a self-map that can be naturally iterated.

Question 5.31. *Which is the relationship between the Galois property of a quasi-homogeneous foliation $\mathcal{F}$ on $\mathbb{P}^2$ and the dynamical behavior of its associated rational self-maps $\psi_{\mathcal{F}}:\mathbb{P}^2\dashrightarrow\mathbb{P}^2$ and $\widehat{\psi}_{\mathcal{F}}:\mathbb{P}^1\to\mathbb{P}^1$?*

5.5. **Reducibility of the space of degree 3 Galois foliations.** Recall that the vector space U_3 defined in (9) is isomorphic to the space of vector fields $X=A(x,y)\partial_x+B(x,y)\partial_y$ with $A,B\in\mathbb{C}[x,y]$, $A=\bar{a}+x\bar{c}$, $B=\bar{b}+y\bar{c}$, $\deg\bar{a},\deg\bar{b}\leq 3$ and $\bar{c}$ homogeneous of degree 3 (see Subsection 5.1). The projectivization $\overline{\mathbb{F}}_3=\mathbb{P}(U_3)$ contains a Zariski open set that can be identified to the space $\mathbb{F}_3$ of degree 3 foliations on $\mathbb{P}^2$. If $X\in U_3$ is such a vector field we will denote $[X]\in\overline{\mathbb{F}}_3$ the foliation defined by X. Let $\mathbb{G}_3\subset\mathbb{F}_3$ be the Zariski closed set of Galois degree 3 foliations.

In order to estimate the dimension of $\mathbb{G}_3$ we can compute an upper bound of the dimension of the tangent space of $\mathbb{G}_3$ at a point $[X]\in\mathbb{G}_3$. To do that, we note that $\mathbb{G}_3$ coincides with the set of foliations $[X]\in\mathbb{F}_3$ such that the t-discriminant $\Delta_X=a_2^2-4a_1a_3\in\mathbb{C}[x,y]$ of the polynomial $P_X(x,y,t)/t\in\mathbb{C}[x,y,t]$ considered in (10), is a square, i.e. $\Delta_X=\delta_X^2$ with $\delta_X\in\mathbb{C}[x,y]$.

Lemma 5.32. *If $[X] \in \mathbb{G}_3$ then $T_{[X]}\mathbb{F}_3 = U_3/\langle X\rangle$ and*

$$T_{[X]}\mathbb{G}_3 \subset \left\{ Y \in U_3 \,|\, \delta_X \text{ divides } \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(\Delta_{X+\varepsilon Y}) \in \mathbb{C}[x,y] \right\} \Big/ \langle X\rangle.$$

Proof. Let V_m denote the space of polynomials in $\mathbb{C}[x,y]$ of degree $\leq m$. Writing $P_X = a_1 t + a_2 t^2 + a_3 t^3$, it is easy to check that $a_1 \in V_9$, $a_2 \in V_{12}$ and $a_3 \in V_{15}$, so that $\Delta_X \in V_{24}$. The map $\mathfrak{s} : V_{12} \to V_{24}$ given by $\delta \mapsto \delta^2$ induces a morphism $\overline{\mathfrak{s}} : \mathbb{P}(V_{12}) \to \mathbb{P}(V_{24})$ whose image $\overline{S}$ is Zariski closed. Then the preimage S of $\overline{S}$ in V_{24} is also Zariski closed. Let $f_1, \ldots, f_k$ be generators of the ideal $I(S)$. Then $f_1 \circ \Delta, \ldots, f_k \circ \Delta$ is a system of equations defining the preimage G_3 of $\mathbb{G}_3$ in U_3. Although we do not know whether $f_i \circ \Delta$ generate the ideal $I(G_3)$ we have

$$\begin{aligned} T_X G_3 \subset \bigcap_{i=1}^k \ker d(f_i \circ \Delta)_X &= \left\{ Y \in U_3 \;\Big|\; \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Delta_{X+\varepsilon Y} \in \bigcap_{i=1}^k \ker(df_i)_{\Delta_X} \right\} \\ &= \left\{ Y \in U_3 \;\Big|\; \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Delta_{X+\varepsilon Y} \in T_{\Delta_X} S \right\}. \end{aligned}$$

Consider $\Delta = \delta^2 \in S \setminus \{0\} \subset V_{24}$ with $\delta \in V_{12} \setminus \{0\}$ and $\Gamma \in T_\Delta V_{24} = V_{24}$. Since, for $\gamma \in T_\delta V_{12} = V_{12}$, $d\mathfrak{s}_\delta(\gamma) = 2\delta\gamma \neq 0$ if $\gamma \neq 0$, it follows that $S \setminus \{0\}$ is smooth and consequently $T_\Delta S = \operatorname{Im} d\mathfrak{s}_\delta$. Hence $\Gamma \in T_\Delta S$ if and only if δ divides Γ. We conclude by taking the quotient by the 1-dimensional subspace $\langle X\rangle$ of $T_X G_3$. □

An explicit computation carried out with `maple` shows that $\dim T_{[X_2]}\mathbb{G}_3 \leq 9$ and consequently $\dim C' \leq 9$ for each irreducible component C' of $\mathbb{G}_3$ containing the point $[X_2] \in \mathbb{G}_3$. On the other hand, the family $E \subset \mathbb{F}_3$ given in Example 5.7 for $d = 3$ is the image of an explicit morphism $\varphi : W \subset \mathbb{P}^{11} \to \mathbb{F}_3$. It can be checked that the rank of $d\varphi$ at the point $[\alpha, \beta, \gamma, \delta, \lambda, \mu, u, v] = [1, 0, 0, 1, 0, 0, x, y]$ is 9 and consequently $\dim E \geq 9$. The following result is a very partial answer to Question 5.3(1) for $d = 3$.

Proposition 5.33. *The Zariski closed set $\mathbb{G}_3$ of degree 3 Galois foliations is reducible. More precisely, the foliation $[X_2] \in \mathbb{G}_3$ given in Example 5.29 and the family $E \subset \mathbb{G}_3$ given in Example 5.7 for $d = 3$, lie in different irreducible components of $\mathbb{G}_3$.*

Proof. Let C be an irreducible component of $\mathbb{G}_3$ containing the irreducible subset $E \subset \mathbb{G}_3$ and let C' be an irreducible component of $\mathbb{G}_3$ containing the point $[X_2] \in \mathbb{G}_3$. Proposition 5.5 implies that $E' := \{[X] \in C' \,|\, g([X]) = 0\}$ is a Zariski closed set of C'. If $C = C'$ then $9 \leq \dim E \leq \dim C = \dim C' \leq 9$ contradicting that E is contained in the *proper* Zariski closed set E' (cf. Example 5.7). □

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