Differentiability properties of Riesz potentials of finite measures and non-doubling Calderón–Zygmund theory

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Abstract

We study differentiability properties of the Riesz potential, with kernel of homogeneity $2 - d$ in $\mathbb{R}^d$, $d \geq 3$, of a finite Borel measure. In the plane we consider the logarithmic potential of a finite Borel measure. We introduce a notion of differentiability in the capacity sense, where capacity is Newtonian capacity in dimension $d \geq 3$ and Wiener capacity in the plane. We require that the first order remainder at a point is small when measured by means of a normalized weak capacity "norm" in balls of small radii centered at the point. This implies L^p differentiability in the Calderón–Zygmund sense for $1 \leq p < d/d - 2$. If $d \geq 3$, we show that the Riesz potential of a finite Borel measure is differentiable in the capacity sense except for a set of zero C^1-harmonic capacity. The result is sharp and depends on deep results in non-doubling Calderón–Zygmund theory. In the plane the situation is different. Surprisingly there are two distinct notions of differentiability in the capacity sense. For each of them we obtain the best possible result on the size of the exceptional set in terms of Hausdorff measures. Finally we obtain, for $d \geq 3$, results on Peano second order differentiability in the sense of capacity with exceptional sets of zero Lebesgue measure.

1 Introduction

Calderón and Zygmund applied their celebrated results on singular integrals to understand differentiability properties of functions defined on subsets of $\mathbb{R}^d$. Besides the foundational paper [CZ1], where logarithmic potentials in the plane and Riesz potentials in higher dimensions were considered, one may consult [CZ2] and the book [S], in which the central results known up to the seventies were presented. A recent interesting paper on the subject is [ABC]. The setting for our results is as follows.

Let μ be a Borel finite measure in $\mathbb{R}^d$ and, in dimension $d \geq 3$, consider the Riesz potential

$$(1) \qquad u(x) = (P\mu)(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} \, d\mu(y), \quad x \in \mathbb{R}^d.$$

In the plane we take the logarithmic potential

$$(2) \qquad u(z) = (P\mu)(z) = \int_{\mathbb{C}} \log \frac{1}{|z-w|} \, d\mu(w), \quad z \in \mathbb{C}.$$

The kernel chosen in all dimensions $d \geq 2$ is a constant multiple of the fundamental solution of the Laplacian in $\mathbb{R}^d$. The distributional gradient of the potential u is

$$(3) \qquad \nabla u = -(d-2) \frac{x}{|x|^d} * \mu, \quad d \geq 3,$$

and

$$\nabla u = -\frac{z}{|z|^2} * \mu = -\frac{1}{\bar{z}} * \mu, \quad d = 2.$$

Since the kernel in the preceding identities is locally integrable, ∇u is a locally integrable function, hence well defined a.e. The second derivatives of u in the sense of distributions are given in dimension $d > 2$ by

$$(4) \qquad \partial^2_{jj} u = -(d-2) \text{ p. v.} \frac{|x|^2 - d\,x_j^2}{|x|^{d+2}} * \mu - \frac{1}{d}(d-2)\omega_{d-1}\,\mu, \quad 1 \leq j \leq d,$$

where ω_{d-1} is the $d-1$-dimensional surface measure of the unit sphere in $\mathbb{R}^d$, and

$$(5) \qquad \partial_{jk} u = d(d-2) \text{ p. v.} \frac{x_j x_k}{|x|^{d+2}} * \mu, \quad 1 \leq j \neq k \leq d.$$

In dimension $d = 2$, setting $z = x + iy$, one gets

$$(6) \qquad \frac{\partial^2}{\partial x^2} u = \text{p. v.} \frac{x^2 - y^2}{|z|^4} * \mu - \pi\mu,$$

$$(7) \qquad \frac{\partial^2}{\partial y^2} u = \text{p. v.} \frac{y^2 - x^2}{|z|^4} * \mu - \pi\mu,$$

and

$$(8) \qquad \frac{\partial^2}{\partial x \, \partial y} u = \text{p. v.} \frac{2xy}{|z|^4} * \mu.$$

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The key fact is that one can give at almost all points a sense to all right hand sides in (4)–(8). Indeed, the principal value singular integrals exist a.e. after the results of [CZ1] and one can assign to the measure μ in (4), (6) and (7) at the point x the density

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{|B(x,r)|},$$

which exists a.e. in $\mathbb{R}^d$. We denote by $|E|$ the d-dimensional Lebesgue measure of the measurable set E. Calderón and Zygmund proved that if $d = 2$ and μ is absolutely continuous with density locally in LlogL, then u has a second differential in the Peano sense a.e. One obtains the same conclusion for $d \geq 3$ if μ is assumed to be absolutely continuous with density in $L^q(\mathbb{R}^d)$, $q > d/2$. Recall that a function u defined in a neighborhood of a point a has a second differential in the Peano sense if there exists constants A_i, $1 \leq i \leq d$, and B_{jk}, $1 \leq j, k \leq d$, such that

$$u(x) = u(a) + \sum_{i=1}^{d} A_i(x_i - a_i) + \sum_{j,k=1}^{d} B_{jk}(x_j - a_j)(x_k - a_k) + \varepsilon(|x - a|)\,|x - a|^2$$

for a certain function $\varepsilon(t)$ which tends to 0 with t.

Brilliant work by many people during the last decade has shown that most of Calderón–Zygmund theory holds in very general contexts in which the classical homogeneity assumption is dropped. It is enough that the underlying measure m be a positive locally finite Borel measure in $\mathbb{R}^d$ satisfying a growth condition

$$m(B(x,r)) \leq C\,r^n, \quad 0 < r < R,$$

R being the diameter of the support of m and $0 < n \leq d$. Hence m is not necessarily doubling. See, for instance, [T2, Chapter 2] and the many references given there. It appears then appropriate to explore what new differentiability results might the general non-doubling Calderón–Zygmund theory make available. We consider a variant of the notion of differentiability in the L^p sense in which we require the remainder to tend to zero in the weak capacitary "norm".

Definition 1. *Let u be a real function defined in a neighborhood of a point $a \in \mathbb{R}^d$. Given real numbers $A_1, \ldots, A_d$ set*

$$Q(x) = \frac{|u(x) - u(a) - \sum_{i=1}^{d} A_i(x_i - a_i)|}{|x - a|}.$$

We say that u is differentiable in the capacity sense at the point a provided there exist real numbers $A_1, \ldots, A_d$ such that

(9) $$\lim_{r \to 0} \frac{\sup_{t>0} t\,\mathrm{Cap}(\{x \in B(a,r) : Q(x) > t\})}{\mathrm{Cap}(B(a,r))} = 0.$$

Here Cap *stands for Wiener capacity in the plane and Newtonian capacity associated with the kernel $1/|x|^{d-2}$ in higher dimensions. See section 2 for precise definitions.*

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In other words, we require that the normalized weak capacity norm in the ball $B(a, r)$ of the quotient $Q(x)$ tends to 0 with r. This makes sense for potentials $u = P(\mu)$ of finite Borel measures, because they satisfy the inequality

$$(10) \qquad \operatorname{Cap}(\{x \in \mathbb{R}^d : P(\mu)(x) > t\}) \leq \frac{\|\mu\|}{t}, \quad 0 < t,$$

and so, in particular they are defined except for a set of capacity zero.

The notion of differentiability in the capacity sense can be weakened by replacing the denominator $|x - a|$ in $Q(x)$ by r and then rescaling t. We get the following.

Definition 2. *Let u be a real function defined in a neighborhood of a point $a \in \mathbb{R}^d$. We say that u is differentiable in the weak capacity sense at the point a provided there exist real numbers $A_1, \ldots, A_d$ such that*

$$\lim_{r \to 0} \frac{\sup_{t > 0} t \operatorname{Cap}(\{x \in B(a, r) : |u(x) - u(a) - \sum_{i=1}^d A_i(x_i - a_i)| > t\})}{r \operatorname{Cap}(B(a, r))} = 0.$$

A simple argument, which consists in expressing a ball as a union of dyadic annuli, gives readily that the above two notions of differentiability coincide if $d \geq 3$. Instead they are different in the plane as we will discuss later. This is due to the fact that $\operatorname{Cap}(B(a, r)) = \frac{1}{\log \frac{1}{r}}$, in the plane, while in dimensions $d \geq 3$ the dependence of the capacity of a ball on the radius is via a power: $\operatorname{Cap}(B(a, r)) = c_d r^{d-2}$.

Our first result concerns differentiability in the capacity sense of Riesz potentials of finite measures in $\mathbb{R}^d$, $d \geq 3$.

Theorem 1. *For a positive finite Borel measure μ in $\mathbb{R}^d$, $d \geq 3$, the Riesz potential*

$$u(x) = \int_{\mathbb{R}^d} \frac{1}{|x - y|^{d-2}} \, d\mu(y), \quad x \in \mathbb{R}^d,$$

is differentiable in the capacity sense at the point $a \in \mathbb{R}^d$ if and only if

$$(11) \qquad \lim_{r \to 0} \frac{\mu(B(a, r))}{r^{d-1}} = 0$$

and the principal value

$$(12) \qquad \operatorname{p. v.} \int \frac{a - y}{|a - y|^d} \, d\mu(y) = \lim_{\varepsilon \to 0} \int_{|y - a| > \varepsilon} \frac{a - y}{|a - y|^d} \, d\mu(y)$$

exists.

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Hence differentiability in the capacity sense is exactly equivalent to vanishing of the $(d-1)$-dimensional density of the measure and existence of the principal value (12). Existence of the principal values brings into the picture singular integrals with respect to a non-doubling underlying measure. This will happen when one is dealing with measures μ which are as spread as the vanishing of the $(d-1)$-dimensional density allows. In this case the kernel $a - y/|a-y|^d$ will behave as a singular Calderón–Zygmund kernel with respect to μ and there is no reason to expect μ to be doubling.

Our second result asserts that the Riesz potential of each finite Borel measure μ in $\mathbb{R}^d$, $d \geq 3$ is differentiable in the capacity sense except in a set whose size is controlled by an appropriate set function. This set function is called C^1 harmonic capacity and is defined as follows.

The C^1 harmonic capacity of a compact set $E \subset \mathbb{R}^d$ is

(13) $$\kappa^c(E) = \sup |\langle T, 1 \rangle|$$

where the supremum is taken over those distributions T supported on E such that $T * x/|x|^d$ is a continuous vector valued function on $\mathbb{R}^d$ satisfying $\|(T * x/|x|^d)(x)\| \leq 1$, $x \in \mathbb{R}^d$. The terminology refers to the fact that convolving such a distribution with the fundamental solution of the Laplacian one gets a harmonic function on $\mathbb{R}^d \setminus E$ of class $C^1(\mathbb{R}^d)$.

It is readily seen that $\kappa^c(E) = 0$ if and only each function of class $C^1(\mathbb{R}^d)$ harmonic on $\mathbb{R}^d \setminus E$ is linear. The homogeneity of the set function κ^c is $d-1$, that is, $\kappa^c(\lambda E) = \lambda^{d-1}\kappa^c(E)$. Deep results of [RT] show that C^1 harmonic capacity can be described in terms of positive measures supported on the set, having null $(d-1)$-dimensional density and enjoying the property that the singular integral operator determined by the vectorial kernel $x/|x|^d$ is bounded on the L^2 Lebesgue space of the measure. The description is rather explicit and in particular shows that κ^c is semiadditive, i. e.,

$$\kappa^c(E \cup F) \leq C \left(\kappa^c(E) + \kappa^c(F) \right),$$

for a dimensional constant C independent of the compact sets E and F. If F is an arbitrary subset of $\mathbb{R}^d$, then $\kappa^c(F)$ is defined as the supremum of $\kappa^c(E)$ over all compact subsets E of F. See section 2 for more details.

Theorem 2. *For each finite Borel measure μ in $\mathbb{R}^d$, $d \geq 3$, the Riesz potential*

$$u(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} \, d\mu(y), \quad x \in \mathbb{R}^d,$$

is differentiable in the capacity sense at κ^c almost all points.

The result is sharp. In fact the unit sphere in $\mathbb{R}^d$ can be shown to be the set of points at which the potential of a positive finite Borel measure is not differentiable in the

capacity sense. The maximal dimension of a set of vanishing κ^c capacity is $d-1$ and the sphere is, in some sense, the biggest such set that one can imagine.

Somehow surprisingly, in dimension $d = 2$ the result one finds is different. This is due to the usual difficulties associated with the logarithmic kernel. Indeed, there are two separate results, each dealing with one of the two notions of differentiability in the capacity sense we have in the plane.

Let φ be the measure function

$$(14) \qquad \varphi(t) = \begin{cases} t \frac{1}{\log \frac{1}{t}}, & 0 < t \le e^{-1}, \\ t, & e^{-1} \le t, \end{cases}$$

and H^φ the associated Hausdorff measure. We then have

Theorem 3. *For each finite Borel measure μ in $\mathbb{C}$ the logarithmic potential*

$$u(z) = \int_\mathbb{C} \log \frac{1}{|z - w|} \, d\mu(w), \quad z \in \mathbb{C},$$

is differentiable in the weak capacity sense at H^φ almost all points.

The result is sharp in the scale of Hausdorff measures. Given a measure function Φ with the property that $\Phi(r)/\varphi(r) \to \infty$ as $r \to 0$ and satisfying another minor assumption, then there exists a finite Borel measure whose logarithmic potential is not differentiable in the weak capacity sense on a set of positive H^Φ-measure. See Theorem 6 in section 5 for a precise statement.

The next result deals with differentiability in the capacity sense in the plane. Let ψ stand for the measure function

$$(15) \qquad \psi(t) = \begin{cases} t \frac{1}{\log^2(\frac{1}{t})}, & 0 < t \le e^{-1}, \\ t, & e^{-1} \le t. \end{cases}$$

Theorem 4. *For each finite Borel measure μ in $\mathbb{C}$ the logarithmic potential*

$$u(z) = \int_\mathbb{C} \log \frac{1}{|z - w|} \, d\mu(w), \quad z \in \mathbb{C},$$

is differentiable in the capacity sense at H^ψ almost all points.

As before, the result is sharp in the scale of Hausdorff measures. Given a measure function Ψ with the property that $\Psi(r)/\psi(r) \to \infty$ as $r \to 0$ and satisfying another minor assumption, then there exists a finite Borel measure whose logarithmic potential is not differentiable in the capacity sense on a set of positive H^Ψ-measure. See Theorem 7 in section 6 for a precise statement.

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We turn now to Peano second order differentiability. Given a function u, a point $a \in \mathbb{R}^d$ and real numbers A_i, $1 \le i \le d$, and B_{jk}, $1 \le j, k \le d$, set

$$(16) \quad D(x) = \frac{1}{|x-a|^2} \left| u(x) - u(a) - \sum_{i=1}^d A_i(x_i - a_i) - \sum_{j,k=1}^d B_{jk}(x_j - a_j)(x_k - a_k) \right|.$$

Definition 3. *Let u be a real function defined in a neighbourhood of a point $a \in \mathbb{R}^d$. We say that u is differentiable of the second order in the capacity sense at the point a provided there exist real numbers A_i $1 \le i \le d$ and B_{jk}, $1 \le j, k \le d$, such that*

$$(17) \quad \lim_{r \to 0} \frac{\sup_{t>0} t \, \mathrm{Cap}(\{x \in B(a,r) : D(x) > t\})}{\mathrm{Cap}(B(a,r))} = 0.$$

We also mention for the record that there is the corresponding notion of second order differentiability in the weak capacity sense, which consists in requiring that

$$\lim_{r \to 0} \frac{\sup_{t>0} t \, \mathrm{Cap}(\{x \in B(a,r) : \tilde{D}(x) > t\})}{r^2 \, \mathrm{Cap}(B(a,r))} = 0,$$

where $\tilde{D}(x) = \left| u(x) - u(a) - \sum_{i=1}^d A_i(x_i - a_i) - \sum_{j,k=1}^d B_{jk}(x_j - a_j)(x_k - a_k) \right|$.

Theorem 5. *(i) For each finite Borel measure μ in $\mathbb{R}^d$, $d \ge 3$, the Riesz potential*

$$u(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} \, d\mu(y), \quad x \in \mathbb{R}^d,$$

is differentiable of the second order in the capacity sense at almost all points (with respect to Lebesgue measure in $\mathbb{R}^d$).

(ii) There exists a finite Borel measure in $\mathbb{C}$ such that the logarithmic potential of μ is not differentiable of the second order in the weak capacity sense at almost all points of $\mathbb{R}^d$.

The preceding result could have been proved in the sixties and its proof follows standard arguments from [S] for part (i) and an idea of Calderón from [C] for part (ii).

In [ABC] one proves that the Riesz potential $1/|x|^{d-1} * \mu$ of a finite Borel measure μ is differentiable in the L^p sense, $1 \le p < d/(d-1)$, at almost all points. One can also adopt in this context our notion of differentiability in the capacity sense, where this time the capacity involved is the one related to the kernel $1/|x|^{d-1}$, say C_{d-1}. Since one has $C_{d-1}(E) \ge c|E|^{(d-1)/d}$, it turns out that differentiability in the capacity sense implies differentiability in the L^p sense for the range $1 \le p < d/(d-1)$. The argument for the proof of Theorem 5 can be adapted easily to obtain differentiability in the C_{d-1}-capacity sense almost everywhere. Thus one has a slightly better result.

The paper is organized as follows. In section 2 we collect a series of background facts on capacities, singular integral operators on subsets of $\mathbb{R}^d$, and Cantor sets. In sections 3 and 4 we prove Theorems 1 and 2, respectively. In section 5 we prove Theorem 3. Sharpness of Theorem 3 is established in Theorem 6 by means of a construction, inspired by work of Calderón in [C]. Section 6 is devoted to Theorem 4 and its sharpness, established in Theorem 7. The proof of Theorem 5 is in section 7.

Our terminology and notation are standard. For instance, we use the letter C to denote a positive constant, which may vary at each occurrence, and which is independent of the relevant parameters. Usually C depends only on dimension. We use the symbol $A \simeq B$ to indicate that for some constant $C > 1$ one has $C^{-1} B \leq A \leq C B$.

2 Background facts

2.1 Wiener and Newtonian capacities.

If E is a compact subset of R^d, $d \geq 3$, the Newtonian capacity of E is

$$(18) \qquad \qquad \mathrm{Cap}(E) = \sup \mu(E)$$

where the supremum is taken over all positive finite Borel measures supported on E such that the Riesz potential $P(\mu)$ of μ satisfies $P(\mu)(x) \leq 1$, $x \in \mathbb{R}^d$. There is an equivalent definition involving the notion of energy. The energy of a measure ν is

$$(19) \qquad \qquad V(\nu) = \iint \frac{1}{|x - y|^{d-2}} \, d\nu(x) \, d\nu(y)$$

and one has

$$(20) \qquad \mathrm{Cap}(E) = (\inf\{V(\nu) : \text{support of } \nu \subset E \text{ and } \|\nu\| = 1\})^{-1}.$$

It can easily be seen that $\mathrm{Cap}(B(a, r)) = c_d \, r^{d-2}$ (see [AG]).

In the plane one would like to make the same definitions with the Riesz kernel replaced by the logarithm. The difficulty is that the kernel changes sign and this causes inconveniences. One way to proceed is to consider only subsets of the disc centered at the origin of radius 1/2, so that $|z - w| \leq 1$ and $\log \frac{1}{|z-w|} \geq 0$. Then the Wiener capacity is (18) with the kernel $1/|x - y|^{d-2}$ replaced by $\log \frac{1}{|z-w|}$. The energy of a measure is (19) with the same change in the kernel. The relation (20) holds true. We have

$$\mathrm{Cap}(B(a, r)) = \frac{1}{\log \frac{1}{r}}, \quad |a| < 1/4, \quad 0 < r < 1/4.$$

Note that the definition of C^1 harmonic capacity is similar in structure to that of Wiener or Newtonian capacities. In (13) the supremum is taken on all distributions with

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support in the set E whose potential satisfies a certain inequality and in (18) only positive measures are considered. This is a minor difference: one can show that $\mathrm{Cap}(E)$ is the supremum of $|\langle T, 1 \rangle|$ over all distributions T supported on E such that the potential $1/|x - y|^{d-2} * T$ is a function in $L^\infty(\mathbb{R}^d)$ with norm bounded by 1 (see, for instance, [V]). The essential difference lies in the fact that the kernel involved in the definition of C^1 harmonic capacity is vectorial and each of its components is a kernel of variable sign. Then subtle cancellation phenomena have to be taken into account, which explains the enormous difficulties arising in the study of C^1 harmonic capacity. See subsection 2.3 below.

2.2 Singular integrals on subsets of $\mathbb{R}^d$.

Let m be a positive finite Borel measure. Set, for $f \in L^2(m)$,

$$R_\varepsilon(fm)(x) = \int_{|y-x|>\varepsilon} \frac{x-y}{|x-y|^d} f(y)\, dm(y), \quad x \in \mathbb{R}^d, \quad \varepsilon > 0.$$

We say that the operator R with kernel $x/|x|^d$ is bounded on $L^2(m)$ if there exists a constant C such that

$$(21) \qquad \int |R_\varepsilon(fm)(x)|^2\, dm(x) \le C \int |f(x)|^2\, dm(x), \quad \varepsilon > 0.$$

In other words, the truncated operators R_ε are uniformly bounded in $L^2(m)$. If m has no atoms, then a necessary condition for boundedness is the growth condition

$$m(B(x,r)) \le C\, r^{d-1}, \quad x \in \mathbb{R}^d, \quad 0 < r.$$

Our differentiability theorems depend on the existence of the principal values

$$(22) \qquad \mathrm{p.\,v.} \int \frac{x-y}{|x-y|^d}\, dm(y) = \lim_{\varepsilon \to 0} R_\varepsilon(m)(x).$$

In classical Calderón–Zygmund theory existence of principal values is a consequence of the L^2 estimate (21), but in the non-doubling context we are considering existence of principal values is a much subtler issue. A general result which applies to our situation appeals to the vanishing of $(d-1)$-dimensional density, that is,

$$(23) \qquad \lim_{r \to 0} \frac{m(B(x,r))}{r^{d-1}} = 0, \quad x \in \mathbb{R}^d.$$

It was proven in [MV] that (21) and (23) imply existence of the principal values (22) m-a.e. This in turn yields, by classical Calderón–Zygmund theory arguments, the m a.e. existence of the principal values

$$\mathrm{p.\,v.} \int \frac{x-y}{|x-y|^d}\, d\nu(y) = \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \frac{x-y}{|x-y|^d}\, d\nu(y),$$

for each finite Borel measure ν.

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2.3 C^1 harmonic capacity.

Consider the quantity

$$\kappa^o(E) = \sup m(E)$$

where the supremum is taken over all positive finite Borel measures m supported on E such that $m(B(x,r)) \leq r^{d-1}$, $x \in \mathbb{R}^d$, $0 < r$, $\lim_{r \to 0} \frac{m(B(x,r))}{r^{d-1}} = 0$, $x \in \mathbb{R}^d$, and the operator R is bounded on $L^2(m)$ with constant 1 (that is, (21) holds with $C = 1$). In [RT] one shows that there exists a constant depending only on dimension such that

$$(24) \qquad\qquad C^{-1} \kappa^o(E) \leq \kappa^c(E) \leq C \kappa^o(E)$$

for all compact sets $E \subset \mathbb{R}^d$. This is a deep result, depending on previous work of Tolsa on semi-additivity of analytic capacity in the plane. In fact, C^1 harmonic capacity in the plane turns out to be comparable to continuous analytic capacity. We will use (24) in combination with the previous subsection to conclude that if $\kappa^c(E) > 0$ then there exists a non-zero finite Borel measure m supported on E with zero $(d-1)$-dimensional density for which the principal values (22) exist m-a.e.

2.4 L^2 boundedness of R.

There is a non-trivial sufficient condition for boundedness of the operator R on $d = 2$ found by Mattila in [M]. If a positive finite Borel measure m in $\mathbb{C}$ satisfies the growth condition

$$m(B(x,r)) \leq C \, \varphi(r), \quad 0 < r < e^{-1},$$

where φ is the function (14), then the operator R is bounded on $L^2(m)$. The proof of a more general result is a calculation based on Menger curvature. This will be used in combination with the result of subsection 2.2 to conclude that the principal values in (22) exist m-a.e.

2.5 Cantor sets.

Along the paper we will make a couple of constructions to show sharpness of our theorems, which involve planar Cantor sets. Now we recall the definition. Take a sequence $(\lambda_n)_{n=1}^\infty$ such that $0 < \lambda_n < 1/2$. Start with the unit square $Q_0 = [0,1] \times [0,1]$. Take 4 squares contained in Q_0, with sides of length λ_1 parallel to the coordinate axis, each with a vertex in common with Q_0. Repeat the operation in each of these 4 squares with the dilation factor λ_2 in place of λ_1. We obtain 16 squares of side length $\lambda_1 \lambda_2$. Proceeding inductively we get at the n-th generation 4^n squares Q_j^n, $1 \leq j \leq 4^n$, of side length $\sigma_n = \prod_{k=1}^n \lambda_k$. Define the Cantor set associated with the sequence $(\lambda_n)_{n=1}^\infty$ to be $K = \cap_{n=1}^\infty \left(\cup_{j=1}^{4^n} Q_j^n \right)$. There is a unique Borel measure μ supported on K such that $\mu(Q_j^n) = 1/4^n$ for all j and n. This measure plays the role of canonical measure on the Cantor set.

There is a special family of Cantor sets K_β, which is worthwhile keeping in mind as a working example, depending on a parameter $\beta \geq 0$, associated with the sequence

$$\lambda_n = \frac{1}{4}(1 + \frac{\beta}{k}), \quad k = 1, 2, \ldots$$

With this choice of λ_n we have

$$\sigma_n \simeq \frac{n^\beta}{4^n}.$$

The canonical measure μ on K_β satisfies the growth condition

$$(25) \qquad \mu(B(z, r)) \simeq r \frac{1}{\log^\beta(\frac{1}{r})}, \quad z \in K_\beta, \quad 0 < r < 1.$$

Note that the function on the right hand side of (25) is $\varphi(r)$ for $\beta = 1$ and $\psi(r)$ for $\beta = 2$.

For $\beta = 0$ we get the famous "corner quarters" Cantor set, which has positive finite length but zero analytic capacity. For $\beta > 0$ the corresponding Cantor set K_β is a compact set of infinite length and Hausdorff dimension 1.

Combining the results of [MTV] and [T1], we see that for $0 \leq \beta \leq 1/2$ the principal values

$$(26) \qquad \mathrm{p.\,v.} \int \frac{z - w}{|z - w|^2} \, d\mu(w) = \lim_{\varepsilon \to 0} \int_{|w - z| > \varepsilon} \frac{z - w}{|z - w|^2} \, d\mu(w)$$

do not exist μ-a.e. As we will show later, this implies that the logarithmic potential of μ is not differentiable in the weak capacity sense at μ almost all points of K_β.

For $1/2 < \beta$ the operator R with kernel $z/|z|^2$ is bounded on $L^2(\mu)$ (see [MTV]) and so the principal values (26) exist μ a.e. (by subsection 2.2). In this case the logarithmic potential of μ is differentiable in the ordinary sense μ-a.e., as it will be shown later.

Consider a measure function $\Phi \colon [0, \infty) \to [0, \infty)$, that is, a continuous (strictly) increasing function with $\Phi(0) = 0$. Associated with Φ there is a Cantor set K whose canonical measure satisfies $\mu(B(x, r)) \leq C \, \Phi(r)$ for $x \in K$ and $0 < r < 1$, provided one has

$$(27) \qquad \limsup_{r \to 0} \frac{\Phi(2r)}{\Phi(r)} < 4.$$

The construction of the Cantor set proceeds as follows. Define σ_n by $4^{-n} = \Phi(\sigma_n)$ and then set $\lambda_n = \sigma_n/\sigma_{n-1}$. To implement the definition of the Cantor set one needs to check that $\lambda_n < 1/2$. This follows readily for n large enough from (27). Indeed, by (27) there exists a positive constant C, $C < 4$, such that $\Phi(2r) \leq C \, \Phi(r)$ for r sufficiently small. Thus, for n large enough,

$$\Phi(\sigma_n) = \frac{\Phi(\sigma_{n-1})}{4} \leq \frac{C}{4} \Phi(\frac{\sigma_{n-1}}{2}) < \Phi(\frac{\sigma_{n-1}}{2})$$

and so $\sigma_n < \sigma_{n-1}/2$.

If $\sigma_n \leq r < \sigma_{n-1}$ and $x \in K$, then $B(x,r)$ is contained in at most 4^3 squares Q_j^n. Then $\mu(B(x,r)) \leq 4^3 \Phi(r)$.

If the lim sup in (27) is exactly 4 then the preceding construction fails for the function $\Phi(r) = r^2 \log(1/r)^{-1}$, because a measure satisfying $\mu(B(x,r)) \leq C r^2 \log(1/r)^{-1}$, $x \in K$, $0 < r < 1$, is identically zero.

The measure function giving the Cantor set K_β is $\Phi(r) = r/\log^\beta(1/r)$.

In dimension d condition (27) should be modified replacing the upper bound 4 by 2^d.

3 Proof of Theorem 1

3.1 The sufficient condition in Theorem 1.

The reader will recognize in the decomposition we are going to use the basic classical argument in [S, p. 242]. Assume that $a = 0$ to simplify notation. In view of (3) we set $A_i = (d-2) \, \text{p. v.} \int \frac{y_i}{|y|^d} \, d\mu(y)$. We have to show that

$$\lim_{r \to 0} \frac{\sup_{t>0} t \, \text{Cap}(\{x \in B(0,r) : Q(x) > t\})}{\text{Cap}(B(0,r))} = 0,$$

where the quotient $Q(x)$ is

$$Q(x) = \frac{|u(x) - u(0) - \sum_{i=1}^{d} A_i x_i|}{|x|}.$$

Given $r > 0$ and $x \in B(0,r)$ set $\varepsilon = 2|x|$. Then, denoting by $\langle v, w \rangle$ the scalar product of the vectors v and w,

$$Q(x) \leq \frac{|u(x) - u(0) + (d-2)\langle R_\varepsilon(\mu)(0), x \rangle|}{|x|} + (d-2) \sup_{0 < \varepsilon \leq 2r} |R_\varepsilon(\mu)(0) - R(\mu)(0)|$$

$$\equiv A_\varepsilon(x) + T_r.$$

Hence

$$\sup_{t>0} \frac{t \, \text{Cap} \left\{x \in B(0,r) : Q(x) > t\right\}}{\text{Cap}(B(0,r))} \leq \sup_{t>0} \frac{t \, \text{Cap} \left\{x \in B(0,r) : A_\varepsilon(x) > \frac{t}{2}\right\}}{\text{Cap}(B(0,r))}$$

$$+ \sup_{t>0} \frac{t \, \text{Cap} \left\{x \in B(0,r) : T_r > \frac{t}{2}\right\}}{\text{Cap}(B(0,r))}$$

$$\leq 2 \sup_{t>0} \frac{t \, \text{Cap} \left\{x \in B(0,r) : A_\varepsilon(x) > t\right\}}{\text{Cap}(B(0,r))} + 2T_r.$$

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Since $T_r \to 0$ as $r \to 0$ we only need to estimate the first term in the right hand side above. Clearly

$$A_\varepsilon(x) \leq \left| \int_{|y|>\varepsilon} \frac{1}{|x|} \left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|y|^{d-2}} - (d-2) \left\langle \frac{y}{|y|^d}, x \right\rangle \right) d\mu(y) \right|$$

$$+ \left| \int_{|y|<\varepsilon} \frac{1}{|x|} \left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|y|^{d-2}} \right) d\mu(y) \right|$$

$$\equiv B_\varepsilon(x) + C_\varepsilon(x).$$

By the mean value theorem and integration by parts, for each positive integer N one has

$$B_\varepsilon(x) \leq C \int_{|y|>\varepsilon} \frac{|x|}{|y|^d} d\mu(y)$$

$$\leq C\varepsilon \int_\varepsilon^\infty \frac{d\mu B(0,\rho)}{\rho^d}$$

(28)
$$= C\varepsilon \left\{ \left[\frac{\mu B(0,\rho)}{\rho^d} \right]_\varepsilon^\infty + d \int_\varepsilon^\infty \frac{\mu B(0,\rho)}{\rho^{d+1}} d\rho \right\}$$

$$\leq C\varepsilon \left\{ \int_\varepsilon^{\varepsilon N} \frac{\mu B(0,\rho)}{\rho^{d+1}} d\rho + \int_{\varepsilon N}^\infty \frac{\mu B(0,\rho)}{\rho^{d+1}} d\rho \right\}$$

$$\leq C \sup_{0<\rho<2rN} \frac{\mu B(0,\rho)}{\rho^{d-1}} + \frac{C}{N} \sup_{0<\rho} \frac{\mu B(0,\rho)}{\rho^{d-1}}.$$

Since N is arbitrary, in view of (11) we conclude that

$$\lim_{r \to 0} \left(\sup_{|x|<r} B_\varepsilon(x) \right) = 0.$$

We now turn our attention to $C_\varepsilon(x)$. Introducing the absolute value inside the integral

$$C_\varepsilon(x) \leq \frac{1}{|x|} \int_{|y|<\varepsilon} \frac{d\mu(y)}{|y-x|^{d-2}} + \frac{1}{|x|} \int_{|y|<\varepsilon} \frac{d\mu(y)}{|y|^{d-2}} \equiv D_\varepsilon(x) + F_\varepsilon(x).$$

The term $F_\varepsilon(x)$ is estimated readily by

$$F_\varepsilon(x) = \frac{1}{|x|} \int_0^\varepsilon \frac{d\mu B(0,\rho)}{\rho^{d-2}}$$

$$= \frac{1}{|x|} \left\{ \left[\frac{\mu B(0,\rho)}{\rho^{d-2}} \right]_0^\varepsilon + (d-2) \int_0^\varepsilon \frac{\mu B(0,\rho)}{\rho^{d-1}} d\rho \right\}$$

$$\leq C \sup_{0<\rho\leq 2r} \frac{\mu B(0,\rho)}{\rho^{d-1}},$$

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and thus
$$\lim_{r \to 0} \left(\sup_{|x|<r} F_\varepsilon(x) \right) = 0.$$

It remains to bound $D_\varepsilon(x)$ and here is the only place where a capacitary estimate, based on (10), is used. We have
$$D_\varepsilon(x) \le \int_{|y|<2r} \frac{1}{|x-y|^{d-2}} \frac{d\mu(y)}{|y|},$$
and the mass of the measure $d\mu(y)/|y|$ is estimated by
$$\int_{|y|<2r} \frac{d\mu(y)}{|y|} = \int_0^{2r} \frac{d\mu B(0,\rho)}{\rho}$$
$$= \left[\frac{\mu B(0,\rho)}{\rho} \right]_0^{2r} + \int_0^{2r} \frac{\mu B(0,\rho)}{\rho^2} d\rho$$
$$\le C r^{d-2} \sup_{0<\rho<2r} \frac{\mu B(0,\rho)}{\rho^{d-1}}.$$

By (10)
$$\sup_{t>0} \frac{t \operatorname{Cap}\{x \in B(0,r) : D_\varepsilon(x) > t\}}{\operatorname{Cap}(B(0,r))} \le C \sup_{0<\rho<2r} \frac{\mu B(0,r)}{\rho^{d-1}},$$
which tends to 0 with r.

3.2   The necessary condition in Theorem 1.

The Green function for the ball $B(a,r)$ is
$$\frac{1}{|x-a|^{d-2}} - \frac{1}{r^{d-2}}, \quad |x-a| < r.$$

By Poisson–Green formula for u and the ball $B(a,r)$
$$u(a) = \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} (u(x) - \langle A, x-a \rangle)\, d\sigma(x)$$
$$+ c_d \int_{B(a,r)} \left(\frac{1}{|x-a|^{d-2}} - \frac{1}{r^{d-2}} \right) d\mu(x),$$

where $A = (A_1, \ldots, A_d)$ is the gradient in the definition of differentiability in the capacity sense (9) and c_d is a positive constant. Since
$$\frac{1}{|x-a|^{d-2}} - \frac{1}{r^{d-2}} \ge c_d \frac{1}{r^{d-2}}, \quad |x-a| < \frac{r}{2},$$

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we obtain

$$(29) \qquad c_d \frac{1}{r^{d-2}} \mu B\left(a, \frac{r}{2}\right) \leq \frac{1}{\sigma(\partial B(a,r))} \int_{\partial B(a,r)} |u(x) - u(a) - \langle A, x-a \rangle| \, d\sigma(x)$$

$$= \frac{1}{\omega_{d-1} r^{d-1}} \int_0^\infty \sigma \left\{ x \in \partial B(a,r) : |Du(x)| > t \right\} dt,$$

where $\omega_{d-1} = \sigma(S^{d-1})$ and

$$(30) \qquad Du(x) = u(x) - u(a) - \langle A, x-a \rangle, \quad x \in \mathbb{R}^d.$$

Estimating from above the potential of the measure $\chi_E(x) \, d\sigma(x)$ one readily obtains the well known estimate

$$(31) \qquad c_d \, \sigma(E) \leq \mathrm{Cap}(E)^{\frac{d-1}{d-2}}, \quad E \subset \partial B(a,r).$$

Hence, the right hand side of (29) is not greater than

$$(32) \qquad c_d \frac{1}{r^{d-1}} \int_0^\infty \mathrm{Cap}^{\frac{d-1}{d-2}} \left\{ x \in \partial B(a,r) : |Du(x)| > t \right\} dt.$$

We split the integral between 0 and ∞ into two pieces: first we integrate between 0 and T and then between T and ∞. The positive number T will be chosen later. For the integral between 0 and T we estimate the capacity of the set $\{x \in \partial B(a,r) : |Du(x)| > t\}$ by $\mathrm{Cap}(\partial B(a,r)) = c_d r^{d-2}$. Thus

$$\frac{c_d}{r^{d-1}} \int_0^T \mathrm{Cap}^{\frac{d-1}{d-2}} \left\{ x \in \partial B(a,r) : |Du(x)| > t \right\} dt \leq c_d T.$$

Define $\varepsilon(r)$ as

$$\varepsilon(r) = \frac{\displaystyle \sup_{t>0} t \, \mathrm{Cap} \left\{ x \in B(a,r) : |Du(x)| > t \right\}}{r \, \mathrm{Cap}(B(a,r))},$$

so that $\varepsilon(r) \to 0$ as $r \to 0$ if u is differentiable in the capacity sense at a. We get

$$\frac{1}{r^{d-1}} \int_T^\infty \mathrm{Cap}^{\frac{d-1}{d-2}} \left\{ x \in \partial B(a,r) : |Du(x)| > t \right\} dt \leq c_d \left(r\varepsilon(r) \right)^{\frac{d-1}{d-2}} \int_T^\infty \frac{dt}{t^{\frac{d-1}{d-2}}}$$

$$= c_d \left(r\varepsilon(r) \right)^{\frac{d-1}{d-2}} \frac{1}{T^{\frac{1}{d-2}}}.$$

The upper bound we obtain for (32) is

$$c_d \left(T + \left(r\varepsilon(r) \right) \right)^{\frac{d-1}{d-2}} \frac{1}{T^{\frac{1}{d-2}}},$$

which is minimized by $T = r\varepsilon(r)$. Therefore

$$c_d \frac{1}{r^{d-2}} \mu B\left(a, \frac{r}{2}\right) \leq r\varepsilon(r),$$

which yields (11).

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It remains to prove the existence of the principal value (12). Assume that $a = 0$ to simplify the writing. We know that there exist A_i, $1 \leq i \leq d$, such that

$$\varepsilon(r) = \frac{\sup\limits_{t>0} t \operatorname{Cap} \{x \in B(0,r) : |Du(x)| > t\}}{r \operatorname{Cap}(B(0,r))}$$

tends to 0 with r. Here Du is as in (30) with $a = 0$. Set

$$R_r = (d-2) \int_{|y|>r} \frac{y}{|y|^d} \mu(y), \quad r > 0.$$

Given $r > 0$ and $x \in B(0,r)$, $x \neq 0$, we have

$$|\langle R_{2|x|} - A, x \rangle| \leq Du(x) + |u(x) - u(0) - \langle R_{2|x|}, x \rangle|$$
$$\equiv Du(x) + Eu(x),$$

and

$$\eta(r) := \sup_{t>0} \frac{t \operatorname{Cap} \{x \in B(0,r) : |\langle R_{2|x|} - A, x \rangle| > t\}}{r \operatorname{Cap}(B(0,r))}$$

$$\leq 2\varepsilon(r) + 2 \sup_{t>0} \frac{t \operatorname{Cap} \{x \in B(0,r) : Eu(x) > t\}}{r \operatorname{Cap}(B(0,r))}.$$

In the proof the sufficiency in subsection 3.1 we showed that the second term in the right hand side of the preceding inequality tends to 0 with r. Therefore $\eta(r)$ tends to 0 as r tends to 0.

If $R_r \neq A$ define

$$(33) \qquad K_r = \left\{ x \in \mathbb{R}^d : |x| = \frac{r}{2} \quad \text{and} \quad \left\langle \frac{x}{|x|}, \frac{R_r - A}{|R_r - A|} \right\rangle \geq \frac{1}{\sqrt{2}} \right\}.$$

Observe that K_r is the intersection of the sphere of center 0 and radius $r/2$ with a cone with vertex at 0, axis determined by the unit vector in the direction of $R_r - A$, and aperture $\pi/4$. A dilation argument shows that $\operatorname{Cap}(K_r) = c_d\, r^{d-2}$. Hence, if $R_r \neq A$,

$$\operatorname{Cap} \left\{ x \in B(0,r) : |\langle R_{2|x|} - A, x \rangle| > |R_r - A| \frac{r}{2^{5/2}} \right\} \geq \operatorname{Cap}(K_r) = c_d r^{d-2}.$$

Taking $t = |R_r - A| r / 2^{5/2}$ in the definition of $\eta(r)$ we get

$$\eta(r) \geq c |R_r - A|,$$

and therefore

$$\lim_{r \to 0} R_r = A.$$

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4 Proof of Theorem 2

Let μ be a finite Borel measure in $\mathbb{R}^d$, $d \geq 3$, and let u be its Riesz potential, as in (1). In proving Theorem 2 we can assume, without loss of generality, that μ is positive. Let E be the set of points where u is not differentiable in the capacity sense. Take a positive finite Borel measure m supported on a compact subset of E satisfying $m(B(x,r)) \leq r^{d-1}$, $x \in \mathbb{R}^d$, $0 < r$, $\lim_{r \to 0} m(B(x,r))/r^{d-1} = 0$, $x \in \mathbb{R}^d$, and such that the operator R with kernel $x - y/|x - y|^d$ is bounded on $L^2(m)$. We will show that u is differentiable in the capacity sense m a.e. Hence m must be identically zero and thus $k^c(E) = 0$, as desired.

The Radon–Nikodym decomposition of μ with respect to m is $\mu = f\,m + \mu_s$ where $f \in L^1(m)$ and μ_s is singular with respect to m. On the one hand one has

$$\lim_{r \to 0} \frac{\mu_s(B(a,r))}{m(B(a,r))} = 0$$

at m almost all points a and, on the other hand, m almost all points are Lebesgue points of f. Hence

$$\mu(B(a,r)) \leq C(a)\,m(B(a,r)), \quad 0 < r,$$

at m almost all points a, $C(a)$ being a constant which depends only on the point a. Since the operator R with kernel $x - y/|x - y|^d$ is bounded on $L^2(m)$, by subsection 2.2 the principal value

$$\mathrm{p.\,v.} \int \frac{a - y}{|a - y|^d}\,d\mu(y)$$

exists at m a.e. Hence we can apply the sufficient condition in Theorem 1 to conclude that the potential u of μ is differentiable in the capacity sense at m almost all points, which completes the proof.

Example. Let σ be the surface measure on the unit sphere $S = \{x \in \mathbb{R}^d : |x| = 1\}$. Since σ has non-zero $(d-1)$-dimensional density, one can apply Theorem 1 to conclude that the Riesz potential $1/|x|^{d-2} * \sigma$, $d \geq 3$ is not differentiable in the capacity sense at any point of S. One can avoid appealing to Theorem 1 and make a direct calculation, which works also in dimension $d = 2$ for the logarithmic potential $\log(1/|z|) * \sigma$. Since S has positive and finite $(d-1)$-dimensional Hausdorff measure we get a satisfactory example showing that Theorem 2 is sharp.

5 Proof and sharpness of Theorem 3

Let μ be a finite Borel measure and u its logarithmic potential, as in (2). For the purpose of proving Theorem 3 one can assume, without loss of generality that μ is positive. Let E stand for the set of points at which u is not differentiable in the weak capacity sense. Take a positive finite Borel measure m with compact support contained in E satisfying

the growth condition $m(B(z,r) \leq \varphi(r)$, $0 < r$, where φ is the function in (14). If we see that u is differentiable in the weak capacity sense at m almost all points, then m has to be identically 0 and hence $H^\varphi(E) = 0$.

The Radon–Nikodym decomposition of μ with respect to m has the form $\mu = fm + \mu_s$, with $f \in L^1(m)$ and μ_s singular with respect to m. Given a point a set $\nu = (f - f(a))m + \mu_s$ so that $\mu = \nu + f(a)m$. At m almost all points a one has

(34)
$$|\nu|(B(a,r)) \leq \eta(r)\,\varphi(r)$$

where η is a function depending on a with $\eta(r) \to 0$ as $r \to 0$. We plan to show that the logarithmic potential of ν is differentiable in the weak capacity sense at the point a if (34) is satisfied and the principal value p. v. $(a - w)/|a - w|^2 \, d\nu(w)$ exists. As we mentioned in subsection 2.4 the growth condition fulfilled by m implies that the operator R with kernel $(z - w)/|z - w|^2$ is bounded on $L^2(m)$, which yields m a.e. existence of the principal values p. v. $(a - w)/|a - w|^2 \, d\nu(w)$ for each finite Borel measure ν (by subsection 2.2). Finally we will show that the logarithmic potential of m is differentiable in the ordinary sense m a.e. This will complete the proof of Theorem 3.

We first deal with the logarithmic potential of m.

Lemma 1. *Let m be a positive finite Borel measure such that*

$$m(B(z,r)) \leq \eta(r)\,r, \quad z \in \mathbb{C}, \quad 0 < r,$$

with $\eta(r) \to 0$ as $r \to 0$, and the principal value

$$\mathrm{p.\,v.} \int \frac{a - w}{|a - w|^2} \, d\mu(w)$$

exists at the point a. Then the logarithmic potential of m is differentiable in the ordinary sense at the point a.

Proof. Assume that $a = 0$ and set

$$A = \mathrm{p.\,v.} \int \frac{w}{|w|^2} \, dm(w),$$

$$R_\varepsilon = \int_{|w| > \varepsilon} \frac{w}{|w|^2} \, dm(w), \quad \varepsilon > 0,$$

$$Q(z) = \frac{|u(z) - u(0) - \langle A, z \rangle|}{|z|}, \quad z \in \mathbb{C} \setminus \{0\}.$$

Then

$$Q(z) \leq \frac{|u(z) - u(0) - \langle R_{2|z|}, z \rangle|}{|z|} + |R_{2|z|} - A|, \quad z \in \mathbb{C} \setminus \{0\}.$$

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The second term in the right hand side above tends to 0 with z and the first can be estimated by

$$\frac{1}{|z|} \left| \int_{|w|>2|z|} \left(\log \frac{1}{|w-z|} - \log \frac{1}{|w|} - \left\langle \frac{w}{|w|^2}, z \right\rangle \right) dm(w) \right|$$

$$+ \frac{1}{|z|} \int_{|w|<2|z|} \left(\log \frac{3|z|}{|w-z|} - \log \frac{3|z|}{|w|} \right) dm(w)$$

$$\equiv A(z) + B(z).$$

The term $A(z)$ is treated by the mean value theorem and integration by parts similarly to what was done in the proof of the sufficiency for Theorem 1. One gets

$$A(z) \leq C|z| + C\eta(N|z|) + C\frac{1}{N} \sup_{0<\rho<\frac{1}{4}} \eta(\rho),$$

where N is an arbitrary positive integer. Thus $\lim_{z \to 0} A(z) = 0$. We estimate $B(z)$ by

$$B(z) \leq \frac{1}{|z|} \int_{|w-z|<3|z|} \log \frac{3|z|}{|w-z|} \, dm(w) + \frac{1}{|z|} \int_{|w|<3|z|} \log \frac{3|z|}{|w|} \, dm(w)$$

$$\equiv C(z) + D(z).$$

For $C(z)$ one has

$$C(z) = \frac{1}{|z|} \int_0^{3|z|} \log \frac{3|z|}{\rho} \, dmB(z, \rho)$$

$$= \frac{1}{|z|} \int_0^{3|z|} \frac{mB(z, \rho)}{\rho} \, d\rho$$

$$\leq 3 \sup_{0<\rho<3|z|} \eta(\rho),$$

which yields $\lim_{z \to 0} C(z) = 0$. A similar estimate for $D(z)$ gives that $\lim_{z \to 0} D(z) = 0$, which completes the proof. $\qquad \square$

It remains to deal with the differentiability in the weak capacity sense of the logarithmic potential of ν. We can assume without loss of generality that ν is a positive measure. The following lemma settles the question.

Lemma 2. *Let ν be a positive finite Borel measure such that*

$$\lim_{r \to 0} \frac{\nu(B(a, r))}{\varphi(r)} = 0$$

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and the principal value

$$(35) \qquad \text{p. v.} \int \frac{a - w}{|a - w|^2} \, d\nu(w)$$

exists. Then the logarithmic potential of ν is differentiable in the weak capacity sense at the point a.

Proof. Assume that $a = 0$ and set

$$A = \text{p. v.} \int \frac{w}{|w|^2} \, d\nu(w),$$

$$Du(z) = |u(z) - u(0) - \langle A, z \rangle|,$$

$$R_\varepsilon = \int_{|w| > \varepsilon} \frac{w}{|w|^2} \, d\nu(w), \quad \varepsilon > 0,$$

$$Eu(z) = |u(z) - u(0) - \langle R_{2|z|}, z \rangle|.$$

Then

$$\sup_{t > 0} \frac{t \operatorname{Cap} \{z \in B(0, r) : Du(z) > t\}}{r \operatorname{Cap}(B(0, r))} \leq 2 \sup_{t > 0} \frac{t \operatorname{Cap} \{z \in B(0, r) : Eu(z) > t\}}{r \operatorname{Cap}(B(0, r))}$$

$$+ 2 \sup_{|z| < r} |R_{2|z|} - A|.$$

The second term in the right hand side above tends to 0 as $r \to 0$. To estimate the first note that for $z \in B(0, r)$

$$(36) \qquad Eu(z) \leq Cr\alpha(r) + \int_{|w| < 2r} \log \frac{1}{|w - z|} \, d\nu(w) + \int_{|w| < 2r} \log \frac{1}{|w|} \, d\nu(w),$$

with $\alpha(r) \to 0$ as $r \to 0$. This is proved as in the sufficiency part of Theorem 1.

The third term in the right hand side of (36) is

$$(37) \qquad \int_0^{2r} \log \frac{1}{\rho} \, d\nu B(0, \rho) = (\log \frac{1}{2r})\nu B(0, 2r) + \int_0^{2r} \frac{\nu B(0, \rho)}{\rho} \, d\rho \leq Cr \sup_{0 < \rho < 2r} \eta(\rho),$$

where $\eta(\rho) = \nu(B(0, \rho)/\varphi(\rho)$. The second term in the right hand side of (36) is the logarithmic potential $P(\chi_{B(0,2r)}\nu)$ of the measure $\chi_{B(0,2r)}\nu$. This is estimated via (10) and we obtain

$$\sup_{t > 0} \frac{t \operatorname{Cap} \{z \in B(0, r) : P(\chi_{B(0,2r)}\nu) > t\}}{r \operatorname{Cap}(B(0, r))} \leq C \log(\frac{1}{r}) \frac{\nu(B(0, 2r))}{r}$$

$$\leq C\eta(2r).$$

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Therefore gathering all previous inequalities

$$\sup_{t>0} \frac{t \operatorname{Cap}\{z \in B(0,r) : Eu(z) > t\}}{r \operatorname{Cap}(B(0,r))} \leq C\alpha(r) + \sup_{0<\rho<2r} \eta(\rho),$$

which tends to 0 with r. $\qquad\square$

There are a couple of necessary conditions for differentiability in the weak capacity sense which provide interesting examples of positive measures with non-differentiable logarithmic potentials. The first is the complete analogue of the necessary condition in Theorem 2 concerning the vanishing of $(d-1)$-dimensional density.

Lemma 3. *Let μ be a positive finite Borel measure such that its logarithmic potential is differentiable in the weak capacity sense at the point $a \in \mathbb{C}$. Then*

$$(38) \qquad \lim_{r\to 0} \frac{\mu(B(a,r))}{r} = 0.$$

Then the logarithmic potential of the arc length measure on $S = \{z \in \mathbb{C} : |z| = 1\}$ is not differentiable in the weak capacity sense at any point of S. In the same vein, the logarithmic potential of the length measure on the corner quarters Cantor set K_0 is not differentiable in the weak capacity sense at any point of K_0.

Proof of Lemma 3. The argument presented for the necessary condition in Theorem 1 works perfectly well in dimension $d = 2$. Indeed, one can replace (31) by

$$c\,\sigma(E) \leq \exp\left(-\frac{1}{\operatorname{Cap}(E)}\right), \quad E \subset \partial B(a,r),$$

where c stands for a small positive constant, and argue similarly. There is, however, an alternative argument which goes as follows. Using the notation introduced in the proof of the necessary condition in Theorem 1 and recalling that the Green function of the disc of center a and radius r is $\log(r/|z-a|)$ one gets

$$c\,\mu B\left(a, \frac{r}{2}\right) \leq \frac{1}{2\pi r} \int_{\partial B(a,r)} |Du(x)|\, d\sigma(x).$$

See (29) and (30). Then, for at least one point $p = p(r) \in \partial B(a,r)$, we have, for a smaller constant c,

$$c\,\mu B\left(a, \frac{r}{2}\right) \leq |Du(p)|.$$

We claim that

$$(39) \qquad \operatorname{Cap}\left\{x \in B(a,r) : |Du(x)| > c\,\mu B\left(a, \frac{r}{4}\right)\right\} \geq c\,\operatorname{Cap}(B(a,r)).$$

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Let us finish the argument assuming (39). Taking $t = c\,\mu B(a, \frac{r}{4})$ we get

$$\varepsilon(r) \geq \frac{c\,\mu B\left(a, \frac{r}{4}\right) \operatorname{Cap}\left\{x \in B(a,r) : |Du(x)| > c\,\mu B\left(a, \frac{r}{4}\right)\right\}}{r \operatorname{Cap} B(a,r)}$$

$$\geq c\,\frac{\mu B\left(a, \frac{r}{4}\right)}{r},$$

which gives (38).

To show the claim take ρ, $\frac{r}{2} < \rho < r$. Then there exists $p = p(\rho)$ with $|p - a| = \rho$ and

$$|Du(p)| > c\,\mu B\left(a, \frac{\rho}{2}\right) \geq c\,\mu B\left(a, \frac{r}{4}\right).$$

The mapping

$$p \longrightarrow \phi(p) = |p - a|$$

is Lipschitz with constant 1 and

$$\phi\left\{x \in B(a,r) : |Du(x)| > c\,\mu B\left(a, \frac{r}{4}\right)\right\} \supseteq \left[\frac{r}{2}, r\right].$$

Since Lipschitz mappings with constant 1 do not increase capacity we conclude that

$$\operatorname{Cap}\left\{x \in B(a,r) : |Du(x)| > c\,\mu\left(a, \frac{r}{4}\right)\right\} \geq \operatorname{Cap}\left(\left[\frac{r}{2}, r\right]\right) \simeq c \operatorname{Cap} B(a,r). \qquad \square$$

We do not know if the existence of principal values is a necessary condition for differentiability in the capacity sense in dimension $d = 2$. We can prove, however, the following.

Lemma 4. *Let μ be a positive finite Borel measure such that its logarithmic potential is differentiable in the weak capacity sense at the point $a \in \mathbb{C}$. Assume also that one of the following two conditions is satisfied*

(i)
$$\mu(B(z, r)) \leq C\,\eta(r)\,r, \quad z \in \mathbb{C}, \quad 0 < r,$$
with $\eta(r) \to 0$ as $r \to 0$.

(ii)
$$\lim_{r \to 0} \frac{\mu(B(a, r))}{\varphi(r)} = 0.$$

Then the principal value

(40)
$$\mathrm{p.\,v.} \int \frac{a - w}{|a - w|^2}\, d\mu(w)$$

exists.

Let K be a Cantor set satisfying the condition

$$\sum_{n=1}^{\infty} \frac{1}{(4^n \sigma_n)^2} = \infty.$$

In the scale of the Cantor sets K_β this is equivalent to $0 \leq \beta \leq 1/2$. Then the operator R with kernel $(z - w)/|z - w|^2$ is unbounded on $L^2(\mu)$, where μ is the canonical measure on K (see [MTV]) and the principal value (40) does not exist for μ almost all points $a \in K$ (see [T1]). Hence, by Lemma 4 the logarithmic potential of μ is not differentiable in the weak capacity sense at μ almost all points $a \in K$.

Proof of Lemma 4. The proof parallels that of the necessary condition in Theorem 1. If u is the logarithmic potential of μ and $a = 0$, then one proves there that, setting

$$Eu(z) = |u(z) - u(0) - \langle R_{2|z|}, z \rangle|,$$

one has

$$\lim_{r \to 0} \frac{\sup_{t>0} t \, \mathrm{Cap} \, \{z \in B(0, r) : |Eu(z)| > t\}}{r \, \mathrm{Cap}(B(0, r))} = 0.$$

This is proven in Lemma 1 under the assumption (i) and in Lemma 2 under the assumption (ii). The rest of the proof is the same, except for the fact that now the set K_r of (33) satisfies $\mathrm{Cap}(K_r) \simeq 1/\log(1/r)$. $\qquad\square$

Theorem 3 is sharp in the scale of Hausdorff measures. This is the content of the following result.

Theorem 6. *Let $\Phi \colon [0, \infty) \to [0, \infty)$, $\Phi(0) = 0$, be a continuous (strictly) increasing function such that*

(41)
$$\limsup_{t \to 0} \frac{\Phi(2r)}{\Phi(r)} < 4,$$

and

$$M(r) := \frac{\Phi(r)}{\varphi(r)} \to \infty, \quad as \quad r \to \infty.$$

Then there exists a compact set K with $H^\Phi(K) > 0$ and a finite Borel measure whose logarithmic potential is not differentiable in the weak capacity sense at H^Φ almost all points of K.

This means that you cannot get any condition better than $H^\varphi(E) = 0$ on the set E of points of non differentiability in the weak capacity sense for the logarithmic potential of a finite Borel measure. In particular, there exists a finite Borel measure whose logarithmic potential is not differentiable in the weak capacity sense on a set of positive C^1 harmonic capacity (that is, positive continuous analytic capacity). Hence the size of the exceptional sets may be larger in dimension 2 than in higher dimensions. See subsection 2.5 for a comment on condition (41).

Proof of Theorem 6. Let K be the Cantor set associated with Φ and let μ be its canonical measure (see subsection 2.5). We aim at constructing a finite Borel measure ν whose logarithmic potential is not differentiable in the weak capacity sense at μ almost all points.

Given a positive integer n take another large positive integer N_n to be determined later. Given a square $Q_j^{N_n}$ of generation N_n, $1 \le j \le 4^{N_n}$, choose a square of generation $N_n + n$ (it is not important which one is chosen). Denote the center of the chosen square of generation $N_n + n$ contained in $Q_j^{N_n}$ by p_j^n, $1 \le j \le 4^{N_n}$. Set

$$E_n = \bigcup_{j=1}^{4^{N_n}} B(p_j^n, \sqrt{2}\,\sigma_{N_n+n}),$$

$$D_m = \bigcup_{n=m}^{\infty} E_n,$$

and

$$D = \bigcap_{m=1}^{\infty} D_m.$$

Clearly $\mu(E_n) = 4^{N_n} 4^{-(N_n+n)} = 4^{-n}$, and this is the only reason why we have descended n more generations after N_n. Hence $\mu(D_m) \le \sum_{n=m}^{\infty} 4^{-n}$ and $\mu(D) = 0$.

Take $a \in K \setminus D$. Then $a \ne p_j^n$, for all n and j, because $p_j^n \notin K$. Since $a \notin D$, $a \notin D_m$ for some m, and so $a \notin B(p_j^n, \sqrt{2}\,\sigma_{N_n+n})$ for all $n \ge m$ and all j.

We proceed now to define the finite Borel measure whose logarithmic potential is not differentiable in the weak capacity sense at all points $a \notin D$. First note that if B is the unit disc $B(0,1)$ we have

(42) $$L(z) := \chi_B(z) \log \frac{1}{|z|} = \log \frac{1}{|z|} * \left(\delta_0 - \frac{d\sigma}{2\pi} \right)$$

where δ_0 is the Dirac delta at the origin and $d\sigma$ the arc-length measure on the unit circle $\{z : |z| = 1\}$. The second identity in (42) can be shown by computing the Laplacian of L and recalling that $1/(2\pi) \log |z|$ is the fundamental solution of the Laplacian in the plane. Translating and dilating we get

$$L(\frac{1}{\rho}(z - p)) = \log \frac{1}{|z|} * \left(\delta_p - \frac{d\sigma_{p,\rho}}{2\pi\rho} \right), \quad p \in \mathbb{C}, \quad 0 < \rho,$$

where δ_p is the Dirac delta at the point p and $d\sigma_{p,\rho}$ is arc-length measure on $\partial B(p, \rho)$. Define

$$\nu = \sum_{n=1}^{\infty} \frac{1}{n^2\, 4^{N_n}} \sum_{j=1}^{4^{N_n}} \left(\delta_{p_j^n} - \frac{d\sigma_{p_j^n, \sqrt{2}\,\sigma_{N_n+n}}}{2\pi\sqrt{2}\,\sigma_{N_n+n}} \right)$$

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which is a finite Borel measure because $\|\nu\| \leq \sum_{n=1}^{\infty} 2/n^2$. The logarithmic potential of ν is

$$u(z) = \sum_{n=1}^{\infty} \frac{1}{n^2 \, 4^{N_n}} \sum_{j=1}^{4^{N_n}} L(\frac{1}{\sqrt{2}\,\sigma_{N_n+n}}(z - p_j^n)).$$

To simplify notation write

$$S_n(z) = \frac{1}{4^{N_n}} \sum_{j=1}^{4^{N_n}} L(\frac{1}{\sqrt{2}\,\sigma_{N_n+n}}(z - p_j^n)).$$

Given $a \in K \setminus D$ as before, we have $a \notin B(p_j^n, \sqrt{2}\,\sigma_{N_n+n})$ for all $n \geq m$ and all j. Thus $S_n(a) = 0$, for all $n \geq m$ and consequently

$$u(z) - u(a) = \sum_{n=1}^{m-1} \frac{1}{n^2}\left(S_n(z) - S_n(a)\right) + \sum_{n=m}^{\infty} \frac{1}{n^2} S_n(z).$$

Recall that $a \neq p_j^n$, for all n and j. If $r > 0$ is small enough, then $p_j^n \notin B(a, r)$, for $n \leq m - 1$ and all j. Therefore

$$\sum_{n=1}^{m-1} \frac{1}{n^2}\left(S_n(z) - S_n(a)\right)$$

is smooth on $B(a, r)$. Consequently the differentiability properties of u at the point a depend only on

$$R(z) := \sum_{n=m}^{\infty} \frac{1}{n^2} S_n(z).$$

Assume that R is differentiable in the weak capacity sense at a. It is a general fact that then R is Lipschitz in the weak capacity sense at the point a. Since $R(a) = 0$ this means that

$$\frac{\sup_{t>0} t \operatorname{Cap}(\{z \in B(a, r) : |R(z)| > t\})}{r \operatorname{Cap}(B(a, r))} \leq C_a, \quad 0 < r < 1/4,$$

for some constant C_a depending only on a. To disprove the preceding inequality we take radii of the form

$$r = r_k = \sqrt{2}\,\sigma_{N_k}, \quad k = 1, 2, \dots$$

For each k the point a belongs to a square $Q_j^{N_k}$ of generation N_k. Hence

$$B(p_j^k, \sqrt{2}\,\sigma_{N_k+k}) \subset B(a, \sqrt{2}\,\sigma_{N_k}).$$

Take $k \geq m$ large enough so that $p_j^n \notin B(a, \sqrt{2}\,\sigma_{N_k})$, for $n \leq m - 1$ and all j.
Then

$$R(z) \geq \frac{1}{k^2} S_k(z) \geq \frac{1}{k^2 4^{N_k}} L(\frac{1}{\sqrt{2}\,\sigma_{N_k+k}}(z - p_j^k)).$$

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The right hand side of the inequality above is larger than t if and only if

$$|z - p_j^k| < \sqrt{2}\,\sigma_{N_k+k}\,e^{-k^2 4^{N_k} t}.$$

Thus

$$\sup_{t>0} t\,\mathrm{Cap}\left\{z \in B(a, \sqrt{2}\,\sigma_{N_k}) : |R(z)| > t\right\} \geq c \sup_{t>0} \frac{t}{k^2 4^{N_k} t + \log \frac{1}{\sqrt{2}\,\sigma_{N_k+k}}}$$

$$= c\,\frac{1}{k^2 4^{N_k}}$$

and

$$\sup_{1/4 > r > 0} \frac{\sup_{t>0} t\,\mathrm{Cap}\left\{z \in B(a, r) : |R(z)| > t\right\}}{r\,\mathrm{Cap}(B(a,r))} \geq c\,\frac{1}{k^2 4^{N_k} \sigma_{N_k}} \log \frac{1}{\sigma_{N_k}}$$

$$= c\,\frac{M(\sigma_{N_k})}{k^2}.$$

Given k take now N_k so that $M(\sigma_{N_k}) \geq k^3$, which is possible because $M(r) \to \infty$ as $r \to 0$. $\qquad\square$

6 Proof and sharpness of Theorem 4

The proof follows the pattern of that of Theorem 3. Following the details of the argument below should provide a clear explanation of the role of the function ψ in (15) as a substitute for the function φ in Theorem 3.

Let μ be a finite Borel measure and u its logarithmic potential. We assume, without loss of generality, that μ is positive. Let E stand for the set of points at which u is not differentiable in the capacity sense. Take a positive finite Borel measure m with compact support contained in E satisfying the growth condition $m(B(z,r) \leq \psi(r)$, $z \in \mathbb{C}$, $0 < r$. If we see that u is differentiable in the capacity sense at m almost all points, then m has to be identically 0 and hence $H^\psi(E) = 0$.

The Radon–Nikodym decomposition of μ with respect to m has the form $\mu = fm + \mu_s$, with $f \in L^1(m)$ and μ_s singular with respect to m. Given a point a set $\nu = (f - f(a))m + \mu_s$ so that $\mu = \nu + f(a)m$. At m almost all points a one has

(43) $$|\nu|(B(a,r)) \leq \eta_a(r)\,\psi(r)$$

where η_a is a function, possibly depending on a, with $\eta_a(r) \to 0$ as $r \to 0$. We plan to show that the logarithmic potential of ν is differentiable in the capacity sense at the point a if (43) holds. This will complete the proof because the logarithmic potential of the measure m is of class $C^1(\mathbb{C})$. This is a consequence of the fact that its gradient $-1/2 \int (\overline{w} - \overline{z})^{-1}\,dm(w)$ is a continuous function, which in turn follows from the uniform growth condition $m(B(z,r) \leq \psi(r)$, $z \in \mathbb{C}$, $0 < r$.

Let us proceed to prove that the logarithmic potential of ν is differentiable in the capacity sense at the point a if (43) holds. If $|\nu|(B(a,r)) \leq C_a \psi(r)$ holds for $0 < r$ with a constant C_a, which may depend on a, then it is easily seen that $\int |w-a|^{-1} \, d|\nu|(w) < \infty$. Hence the principal value p. v. $(a-w)/|a-w|^2 \, d\nu(w)$ exists. Without loss of generality we can assume ν to be a positive measure. Assume that $a = 0$ and set

$$A = \int \frac{w}{|w|^2} \, d\nu(w),$$

$$Qu(z) = \frac{|u(z) - u(0) - \langle A, z \rangle|}{|z|}, \quad z \neq 0,$$

$$R_\varepsilon = \int_{|w|>\varepsilon} \frac{w}{|w|^2} \, d\nu(w), \quad \varepsilon > 0,$$

$$Eu(z) = \frac{|u(z) - u(0) - \langle R_{2|z|}, z \rangle|}{|z|}, \quad z \neq 0.$$

Then

$$\sup_{t>0} \frac{t \operatorname{Cap}\{z \in B(0,r) : Qu(z) > t\}}{\operatorname{Cap}(B(0,r))} \leq \sup_{t>0} \frac{t \operatorname{Cap}\{z \in B(0,r) : Eu(z) > t\}}{\operatorname{Cap}(B(0,r))}$$

$$+ \sup_{|z|<r} |R - R_{2|z|}|.$$

The second term in the right hand side above tends to 0 as $r \to 0$. To estimate the first one notes that

$$Eu(z) \leq C|z| \int_{|w|>2|z|} \frac{d\nu(w)}{|w|^2} + \frac{1}{|z|} \int_{|w|<2|z|} \log \frac{1}{|w-z|} \, d\nu(w)$$

$$+ \frac{1}{|z|} \int_{|w|<2|z|} \log \frac{1}{|w|} \, d\nu(w)$$

$$\equiv A(z) + B(z) + C(z).$$

Set $\eta = \eta_a$ for the sake of notational simplicity. Integrating by parts we get for all positive integers N

$$A(z) \leq C|z| \left[\frac{\nu B(0,\rho)}{\rho} \right]_{2|z|}^{1/4} + C|z| \int_{2|z|}^{1/4} \frac{\nu B(0,\rho)}{\rho^3} \, d\rho$$

$$\leq C|z| \|\nu\| + C|z| \int_{2|z|}^{2|z|N} \frac{\eta(\rho)}{\rho^2 \log^2(\rho)} \, d\rho$$

$$+ C|z| \int_{2|z|N}^{1/4} \frac{\eta(\rho)}{\rho^2 \log^2(\rho)} \, d\rho$$

$$\leq C|z| \|\nu\| + C \frac{\|\eta\|_\infty}{\log^2(2|z|N)} + C \frac{\|\eta\|_\infty}{N}.$$

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Since N is arbitrary we see that

$$\lim_{r \to 0} \sup_{|z|<r} A(z) = 0.$$

The term $C(z)$ is estimated similarly via an integration by parts. We obtain

$$C(z) = \frac{1}{|z|} \left[\log(\frac{1}{\rho}) \nu B(0,\rho) \right]_0^{2|z|} + \frac{1}{|z|} \int_0^{2|z|} \frac{\nu B(0,\rho)}{\rho} \, d\rho$$

$$\leq \frac{\|\nu\|_\infty}{\log \frac{1}{2|z|}} + \frac{1}{|z|} \int_0^{2|z|} \frac{\eta(\rho)}{\log^2(\rho)} \, d\rho$$

$$\leq \frac{\|\eta\|_\infty}{\log \frac{1}{2|z|}} + \frac{\|\eta\|_\infty}{\log^2(2|z|)},$$

and so

$$\lim_{r \to 0} \sup_{|z|<r} C(z) = 0.$$

For the term $B(z)$ we perform a capacity estimate. First, note that

$$B(z) \leq 2 \int_{|w|<2|z|} \log \frac{1}{|w-z|} \frac{d\nu(w)}{|w|},$$

and

$$\int_{|w|<2|z|} \frac{d\nu(w)}{|w|} = \left[\frac{\nu B(0,\rho)}{\rho} \right]_0^{2|z|} + \int_0^{2|z|} \frac{\nu B(0,\rho)}{\rho^2} \, d\rho$$

$$\leq \frac{\|\nu\|_\infty}{\log^2(2|z|)} + \int_0^{2|z|} \frac{\eta(\rho)}{\log^2(\rho)} \frac{d\rho}{\rho}$$

$$\leq \frac{\|\eta\|_\infty}{\log^2(2|z|)} + \frac{1}{\log \frac{1}{2|z|}} \sup_{\rho<2|z|} \eta(\rho).$$

Therefore

$$\sup_{t>0} \frac{t \, \mathrm{Cap}\{z \in B(0,r) : B(z) > t\}}{\mathrm{Cap}(B(0,r))} \leq \frac{1}{\mathrm{Cap}(B(0,r))} \left(\frac{\|\eta\|_\infty}{\log^2(2r)} + \frac{1}{\log \frac{1}{2r}} \sup_{t<2r} \eta(t) \right)$$

$$\leq C \left\{ \frac{\|\eta\|_\infty}{\log \frac{1}{2r}} + \sup_{t<2r} \eta(t) \right\},$$

which tends to 0 with r. It is worth remarking that only in the last inequality we used that $\eta(r)$ tends to 0 with r. $\qquad \square$

Theorem 4 is sharp in the scale of Hausdorff measures, as the next result shows.

Theorem 7. *Let $\Psi \colon [0, \infty) \to [0, \infty)$, $\Psi(0) = 0$, be a continuous (strictly) increasing function such that*

(44)
$$\limsup_{t \to 0} \frac{\Psi(2r)}{\Psi(r)} < 4,$$

and

$$M(r) := \frac{\Psi(r)}{\psi(r)} \to \infty, \quad as \quad r \to \infty.$$

Then there exists a compact set K with $H^{\Psi}(K) > 0$ and a finite Borel measure whose logarithmic potential is not differentiable in the capacity sense at H^{Ψ} almost all points of K.

Therefore there is no condition better than $H^{\psi}(E) = 0$ on the set E of points of non differentiability in the capacity sense for the logarithmic potential of a finite Borel measure. In particular, there exists a finite Borel measure whose logarithmic potential is not differentiable in the capacity sense on a set of positive H^{φ} measure. Thus the two notions of differentiability in the capacity sense are different in dimension 2. Also note that the size of the exceptional sets is definitely larger in dimension 2 than in higher dimensions. See subsection 2.5 for a discussion of condition (44).

Proof of Theorem 7. The proof is similar to that of Theorem 6, although a difficulty appears that requires a new idea. The proof is written to make it accessible to a reader who has not gone through the proof of Theorem 6.

Let K be the Cantor set associated with Ψ and let μ be its canonical measure. We aim at constructing a finite Borel measure ν whose logarithmic potential is not differentiable in the capacity sense at μ almost all points.

Given a positive integer n take another large positive integer N_n to be determined later. Given a square $Q_j^{N_n}$ of generation N_n let $Q_j^{2N_n}$, $1 \le j \le 4^{N_n}$, the squares of generation $2N_n$ contained in $Q_j^{N_n}$. Choose a square of generation $2N_n + n$ inside $Q_j^{2N_n}$ and let p_j^n be its center. It is not important what square is chosen; what matters is that it is a square of generation $2N_n + n$. Descending to generation $2N_n$ instead of N_n is a first difference with respect to the proof of Theorem 6. It will become apparent later why we need to do so. Set

$$E_n = \bigcup_{j=1}^{4^{2N_n}} B(p_j^n, \sqrt{2}\,\sigma_{2N_n+n}),$$

$$D_m = \bigcup_{n=m}^{\infty} E_n,$$

and

$$D = \bigcap_{m=1}^{\infty} D_m.$$

Clearly $\mu(E_n) = 4^{2N_n}4^{-(2N_n+n)} = 4^{-n}$, and this is the only reason why we have descended n more generations after $2N_n$. Hence $\mu(D_m) \le \sum_{n=m}^{\infty} 4^{-n}$ and $\mu(D) = 0$.

Take $a \in K \setminus D$. Then $a \ne p_j^n$, for all n and j, because $p_j^n \notin K$. Since $a \notin D$, $a \notin D_m$ for some m, and so $a \notin B(p_j^n, \sqrt{2}\,\sigma_{2N_n+n})$ for all $n \ge m$ and all j.

We proceed now to define the finite Borel measure whose logarithmic potential is not differentiable in the capacity sense at all points $a \in K \setminus D$. Set

$$\nu = \sum_{n=1}^{\infty} \frac{1}{n^2\,4^{2N_n}} \sum_{j=1}^{4^{2N_n}} \left(\delta_{p_j^n} - \frac{d\sigma_{p_j^n, \sqrt{2}\,\sigma_{2N_n+n}}}{2\pi\sqrt{2}\,\sigma_{2N_n+n}} \right),$$

where δ_p is the Dirac delta at the point p and $d\sigma_{p,\rho}$ is the arc length measure on $\partial B(p,\rho)$. Since $\|\nu\| \le 2\sum_{n=1}^{\infty} 1/n^2$, ν is a finite Borel measure. The logarithmic potential of ν is

$$u(z) = \sum_{n=1}^{\infty} \frac{1}{n^2\,4^{2N_n}} \sum_{j=1}^{4^{2N_n}} L(\frac{1}{\sqrt{2}\,\sigma_{2N_n+n}}(z - p_j^n)),$$

where L is the function in (42). To simplify notation write

$$S_n(z) = \frac{1}{4^{2N_n}} \sum_{j=1}^{4^{2N_n}} L\left(\frac{1}{\sqrt{2}\,\sigma_{2N_n+n}}(z - p_j^n) \right).$$

Given $a \in K \setminus D$ as before, we have $a \notin B(p_j^n, \sqrt{2}\,\sigma_{2N_n+n})$ for all $n \ge m$ and all j. Thus $S_n(a) = 0$, for all $n \ge m$ and consequently

$$u(z) - u(a) = \sum_{n=1}^{m-1} \frac{1}{n^2} \left(S_n(z) - S_n(a) \right) + \sum_{n=m}^{\infty} \frac{1}{n^2} S_n(z).$$

Recall that $a \ne p_j^n$, for all n and j. If $r > 0$ is small enough, then $p_j^n \notin B(a,r)$, for $n \le m-1$ and all j. Therefore

$$\sum_{n=1}^{m-1} \frac{1}{n^2} \left(S_n(z) - S_n(a) \right)$$

is smooth on $B(a,r)$. Consequently the differentiability properties of u at the point a depend only on

$$R(z) := \sum_{n=m}^{\infty} \frac{1}{n^2} S_n(z).$$

Assume that R is differentiable in the capacity sense at a. Then R is Lipschitz in the capacity sense at the point a, as a simple argument shows. Since $R(a) = 0$ this means that

$$\frac{\sup_{t>0} t\,\mathrm{Cap}(\{z \in B(a,r) : \frac{|R(z)|}{|z-a|} > t\})}{\mathrm{Cap}(B(a,r))} \le C_a, \quad 0 < r < 1/4,$$

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for some constant C_a depending only on a. To disprove the preceding inequality we take radii of the form

$$r = r_k = \sqrt{2}\,\sigma_{N_k}, \quad k = 1, 2, \ldots$$

with $k \geq m$ large enough so that $p_j^n \notin B(a, \sqrt{2}\,\sigma_{N_k})$, for $n \leq m - 1$ and all j. For each such k the point a belongs to a square Q^{N_k} of generation N_k, which contains 4^{N_k} points p_j^k. Now we classify the $p_j^k \in Q^{N_k}$ according to their distance to a. Denote by Q^{N_k+1} a square of generation $N_k + 1$ contained in Q^{N_k} and not containing a. The square Q^{N_k+1} contains 4^{N_k-1} points p_j^k. If $p_j^k \in Q^{N_k+1}$ and $z \in B(p_j^k, \sqrt{2}\,\sigma_{2N_k+k})$, then $|z - a| < 2\sigma_{N_k}$. We construct inductively pairwise disjoint squares Q^{N_k+l}, $l = 1, 2, \ldots, N_k$, of generation $N_k + l$, contained in Q^{N_k}, containing 4^{N_k-l} points p_j^k, and with the property that if $p_j^k \in Q^{N_k+l}$ and $z \in B(p_j^k, \sqrt{2}\,\sigma_{2N_k+k})$, then $|z - a| < 2\sigma_{N_k+l-1}$. Since

$$B(p_j^k, \sqrt{2}\,\sigma_{2N_k+k}) \subset B(a, \sqrt{2}\,\sigma_{N_k}) = B(a, r), \quad p_j^k \in Q^{N_k},$$

and

$$R(z) \geq \frac{1}{k^2} S_k(z) \geq \frac{1}{k^2 4^{2N_k}} L(\frac{1}{\sqrt{2}\,\sigma_{2N_k+k}}(z - p_j^k)),$$

we get

$$\left\{ z \in B(a, r) : \frac{R(z)}{|z - a|} > t \right\}$$

$$\supset \bigcup_{l=1}^{N_k} \bigcup_{p_j^k \in Q^{N_k+l}} \left\{ z \in B(p_j^k, \sqrt{2}\,\sigma_{2N_k+k}) : \log \frac{\sqrt{2}\,\sigma_{2N_k+k}}{|z - p_j^k|} > t\,k^2\,4^{2N_k}\,2\,\sigma_{N_k+l-1} \right\}$$

$$= \bigcup_{l=1}^{N_k} \bigcup_{p_j^k \in Q^{N_k+l}} B_{lj},$$

where

$$B_{lj} = B\left(p_j^k, \sqrt{2}\,\sigma_{2N_k+k}\,e^{-t\,2\,k^2\,4^{2N_k}\,\sigma_{N_k+l-1}}\right), \quad p_j^k \in Q^{N_k+l}.$$

Lemma 5 below yields that if $t > T_k$ for a large positive number T_k, then the balls B_{lj} are disjoint and

$$(45) \qquad \mathrm{Cap}\left(\bigcup_{l=1}^{N_k} \bigcup_{p_j^k \in Q^{N_k+l}} B_{lj}\right) \geq \frac{1}{2} \sum_{l=1}^{N_k} \sum_{p_j^k \in Q^{N_k+l}} \mathrm{Cap}\,(B_{lj}).$$

The proof of Lemma 5 will be discussed later. It seems worthwhile to make a digression now to explain the need to descend to generation $2N_k$. Should we have proceeded as in the proof of Theorem 6 we would have descended up to generation N_k only, which means

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taking only one term in the union in the left hand side of (45). Thus we would have obtained

$$\frac{\sup_{t>0} t \operatorname{Cap}(\{z \in B(a,r) : \frac{R(z)}{|z-a|} > t\})}{\operatorname{Cap}(B(a,r))} \geq \log\left(\frac{1}{r}\right) \sup_{t>0} \frac{t}{t2k^2 4^{N_k}\sigma_{N_k} + \log 1/\sqrt{2}\,\sigma_{N_k+k}}$$

$$\geq c \log\left(\frac{1}{\sigma_{N_k}}\right) \frac{1}{k^2 4^{N_k}\sigma_{N_k}} \geq \frac{c}{k^2}\frac{M(\sigma_{N_k})}{\log 1/\sigma_{N_k}},$$

which does not conclude.

We proceed to complete the proof using Lemma 5. We have

$$\sup_{t>0} t \operatorname{Cap}(\{z \in B(a,r) : \frac{R(z)}{|z-a|} > t\})$$

$$\geq c \sup_{t>T_k} \sum_{l=1}^{N_k} \sum_{p_j^k \in Q^{N_k+l}} \frac{t}{t\,2\,k^2\,4^{2N_k}\,\sigma_{N_k+l-1} + \log\frac{1}{\sqrt{2}\,\sigma_{2N_k+k}}}$$

$$\geq c \sup_{t>T_k} \sum_{l=1}^{N_k} \frac{t\,4^{N_k-l}}{t\,2\,k^2\,4^{2N_k}\,\sigma_{N_k+l-1} + \log\frac{1}{\sqrt{2}\,\sigma_{2N_k+k}}}$$

$$= \frac{c}{k^2} \sum_{l=1}^{N_k} \frac{4^{N_k-l}}{4^{2N_k}\,\sigma_{N_k+l-1}}$$

$$= \frac{c}{k^2} \sum_{l=1}^{N_k} \frac{1}{4^{N_k+l-1}\,\sigma_{N_k+l-1}}$$

$$= \frac{c}{k^2} \sum_{l=1}^{N_k} \frac{M(\sigma_{N_k+l-1})}{\log^2\left(\frac{1}{\sigma_{N_k+l-1}}\right)}$$

$$\geq \frac{c}{k^2} \inf_{N\geq N_k} M(\sigma_N) \sum_{l=1}^{N_k} \frac{1}{\log^2\left(\frac{1}{\sigma_{N_k+l-1}}\right)}$$

and so, recalling that $r = r_k = \sqrt{2}\sigma_{N_k}$,

(46)
$$\frac{\sup_{t>0} t \operatorname{Cap}(\{z \in B(a,r) : \frac{R(z)}{|z-a|} > t\})}{\operatorname{Cap}(B(a,r))}$$

$$\geq \frac{c}{k^2} \inf_{N\geq N_k} M(\sigma_N) \log\left(\frac{1}{\sigma_{N_k}}\right) \sum_{l=1}^{N_k} \frac{1}{\log^2\left(\frac{1}{\sigma_{N_k+l-1}}\right)}.$$

At this point it is convenient to distinguish two cases. The first is that

(47)
$$\lim_{n\to\infty} \frac{\Psi(\sigma_n)}{\sigma_n} = 0.$$

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Let us check that then, for some positive integer n_0,

$$(48) \qquad n \log 2 \leq \log \frac{1}{\sigma_n} \leq n \log 4, \quad n \geq n_0.$$

The first inequality follows from the definition of Cantor sets which gives $\sigma_n < 2^{-n}$ for all n. The second follows from (47), which yields $4^n \sigma_n \geq 1$, $n \geq n_0$. Introducing (48) in (46) one gets

$$\frac{\sup_{t>0} t \operatorname{Cap}(\{z \in B(a,r) : \frac{R(z)}{|z-a|} > t\})}{\operatorname{Cap}(B(a,r))} \geq \frac{c}{k^2} \inf_{N \geq N_k} M(\sigma_N),$$

and now it only remains to choose N_k large enough so that

$$\inf_{N \geq N_k} M(\sigma_N) \geq k^3.$$

If (47) is not satisfied then for some $\delta > 0$ and for infinitely many indexes n one has $\Psi(\sigma_n)/\sigma_n \geq \delta > 0$. Given x in the Cantor set K let Q^n the square of generation n containing x. Then for the measure μ associated with K we have

$$\frac{\mu(B(x, \sqrt{2}\,\sigma_n))}{\sqrt{2}\,\sigma_n} \geq \frac{\mu(Q^n)}{\sqrt{2}\,\sigma_n} = \frac{1}{\sqrt{2}} \frac{\Psi(\sigma_n)}{\sigma_n} \geq \frac{\delta}{\sqrt{2}},$$

which says that μ has no vanishing linear density at any point of K. Thus the logarithmic potential of μ is not differentiable in the capacity sense at any point of K and we are done in this case without resorting to any complicated measure like ν. $\qquad \square$

We turn now to the discussion of inequality (45).

Lemma 5. *Let $B_j = B(p_j, r_j)$, $1 \leq j \leq N$, a family of disjoint discs of center p_j and radius $r_j < 1$. Let $\delta = \min_{j \neq k} \operatorname{dist}(B_j, B_k)$ and assume that $0 < \delta < 1$. Set $\sigma = \max_j r_j$. If $\sigma \leq \delta^N$, then*

$$(49) \qquad \operatorname{Cap}\left(\bigcup_{j=1}^{N} B_j\right) \geq \frac{1}{2} \sum_{j=1}^{N} \operatorname{Cap}(B_j).$$

To apply Lemma 5 to (45) note that the radius of the disc B_{lj} is

$$\sqrt{2}\,\sigma_{2N_k+k}\, e^{-t\,2\,k^2\,4^{2N_k}\,\sigma_{N_k+l-1}} \leq e^{-t\,4^{2N_k}\,\sigma_{2N_k}}$$

and the distance between two such discs is larger than $\sigma_{2N_k-1} - 2\sigma_{2\,N_k} > 0$. For any fix k the number of discs B_{lj} is less than 4^{N_k}. Hence the hypothesis of Lemma 5 are satisfied if

$$t \geq T_k := \frac{4^{N_k}}{4^{2N_k}\,\sigma_{2N_k}} \log \frac{1}{\sigma_{2N_k-1} - 2\sigma_{2\,N_k}},$$

which is the large number T_k used in the proof of Theorem 7.

Proof of Lemma 5. The normalized equilibrium potential of the disc $B_j = B(p_j, r_j)$ is

$$u_j = \frac{1}{\log \frac{1}{r_j}} \log \frac{1}{|z|} * \frac{d\sigma_j}{2\pi r_j},$$

where σ_j stands for the arc-length measure on ∂B_j. Then

$$u_j(z) = \begin{cases} \frac{1}{\log \frac{1}{r_j}} \log \frac{1}{|z-p_j|} & \text{if} \quad |z - p_j| \geq r_j, \\ \\ 1 & \text{if} \quad |z - p_j| \leq r_j. \end{cases}$$

If $z \in B_k$, $k \neq j$ then

$$u_j(z) \leq \frac{\log \frac{1}{\delta}}{\log \frac{1}{\sigma}},$$

and so

$$\sum_{j=1}^{N} u_j(z) \leq 1 + (N-1)\frac{\log \frac{1}{\delta}}{\log \frac{1}{\sigma}} \leq 1 + \frac{N-1}{N} \leq 2, \quad z \in \mathbb{C},$$

which yields (49) by definition of Wiener capacity (18). $\qquad\square$

7 Second order differentiability

Proof of Theorem 5, part (i). Assume that $d \geq 3$. Then the first order derivatives of $1/|x|^{d-2}$ in the distributions sense are the locally integrable functions

$$\partial_i \frac{1}{|x|^{d-2}} = -(d-2)\frac{x_i}{|x|^{d-2}}, \quad 1 \leq i \leq d.$$

The second order derivatives in the distributions sense are given by principal value distributions and the Dirac delta δ_0 at the origin via the identities

$$\partial_{ij} \frac{1}{|x|^{d-2}} = d(d-2) \,\mathrm{p.\,v.}\, \frac{x_i x_j}{|x|^{d+2}}, \quad i \neq j,$$

$$\partial_{ii} \frac{1}{|x|^{d-2}} = -(d-2) \,\mathrm{p.\,v.}\, \frac{|x|^2 - d\,x_i^2}{|x|^{d+2}} + a_d\,\delta_0,$$

where $a_d = -(d-2)\omega_{d-1}/d$ and ω_{d-1} is the $(d-1)$-dimensional surface measure of the unit sphere in $\mathbb{R}^d$.

Assume that φ is a C^∞ function with compact support. Then $u = 1/|x|^{d-2} * \varphi$ is a C^∞ function on $\mathbb{R}^d$ and its second order partial derivatives are

$$\partial_{ij} u(x) = d(d-2) \left(\text{p. v.} \frac{x_i x_j}{|x|^{d+2}} * \varphi \right)(x), \qquad i \neq j, \qquad x \in \mathbb{R}^d,$$

$$\partial_{ii} u(x) = \left(\text{p. v.} \frac{|x|^2 - d\, x_i^2}{|x|^{d+2}} * \varphi \right)(x) + a_d\, \varphi(x), \quad 1 \leq i \leq d, \quad x \in \mathbb{R}^d.$$

In particular, the principal value integrals exist at each point $x \in \mathbb{R}^d$.

Given a finite Borel measure μ in $\mathbb{R}^d$, there is a way of defining first and second derivatives of the potential $u = 1/|x|^{d-2} * \mu$ at a fixed point $a \in \mathbb{R}^d$. For the first order derivatives we only have to require that a is a Lebesgue point of the locally integrable functions

$$\partial_i u = -(d-2)\frac{x_i}{|x|^{d-2}} * \mu, \quad 1 \leq i \leq d.$$

For the second order derivatives

(50)
$$\partial_{ij} u = d(d-2) \left(\text{p. v.} \frac{x_i x_j}{|x|^{d+2}} * \mu \right), \qquad i \neq j, \quad x \in \mathbb{R}^d,$$

$$\partial_{ii} u = -(d-2) \left(\text{p. v.} \frac{|x|^2 - d\, x_i^2}{|x|^{d+2}} * \mu \right) + a_d\, \mu, \qquad 1 \leq i \leq d,$$

it is natural to require existence at the point a of all the above principal value integrals and of the limit

(51)
$$\tilde{\mu}(a) := \lim_{r \to 0} \frac{\mu(B(a,r))}{r^d}.$$

We know, by Lebesgue differentiation theorem and by standard Calderón–Zygmund theory, that the stated conditions are satisfied for almost all points a with respect to d dimensional Lebesgue measure dx. Then the prospective second order Taylor polynomial of u at a

(52)
$$u(a) + \sum_{i=1}^d \partial_i u(a)(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} u(a)(x_i - a_i)(x_j - a_j)$$

is defined at almost all points.

Now we make a convenient reduction. To study differentiability properties of u at a fixed point a it is enough to replace μ by $\chi_B \mu$, with $B = B(a,1)$, because the potentials of μ and $\chi_B \mu$ differ by a smooth function on B. Let $\varphi \in C^\infty$ be a function with compact support in the ball $B(a,2)$ taking the value 1 on B. Then by the Radon–Nikodym decomposition there is a function f in $L^1(B)$ such that

$$\mu = (f - f(a))\varphi\, dx + \mu_s + f(a)\varphi\, dx,$$

where μ_s is the singular part of μ. Since the potential of φdx is smooth on $\mathbb{R}^d$, we can assume that μ is a positive measure which satisfies

$$\tilde{\mu}(a) := \lim_{r \to 0} \frac{\mu(B(a,r))}{r^d} = 0.$$

One of the effects of this assumption is that in the definition of the second order derivatives $\partial_{ii} u$ at the point a one can avoid the second term in (50), which would be the limit (51).

We have to show (17) where $D(x)$ is as in (16) with the second order Taylor polynomial as in (52). The structure of the proof is very similar to that of the sufficiency part in Theorem 1, so we only outline the argument. Take $a = 0$ for simplicity. First we replace the principal value integrals by truncations at level ε, where $\varepsilon = 2|x|$. The difference is a term which tends to 0 with ε. We split the domain of integration of the integral into two pieces, one corresponding to $|y| > \varepsilon$. In that piece one estimates the remainder of the Taylor expansion up to order 2 in terms of third derivatives. The upper bound one gets is

$$C \, \varepsilon \int_{|y|>\varepsilon} \frac{d\mu(y)}{|y|^{d+1}}.$$

This term is estimated by integration by parts introducing a parameter N as is (28). It remains to estimate the integral over $|y| < \varepsilon$ with respect to μ of

$$\frac{1}{|x|^2} \left| \frac{1}{|x-y|^{d-2}} - \frac{1}{|y|^{d-2}} - (d-2) \left\langle \frac{y}{|y|^d}, x \right\rangle \right|,$$

which is not greater than a constant times the sum of the 3 terms

$$\frac{1}{\varepsilon^2} \frac{1}{|x-y|^{d-2}} + \frac{1}{\varepsilon^2} \frac{1}{|y|^{d-2}} + \frac{1}{\varepsilon} \frac{1}{|y|^{d-1}}.$$

The integral over $|y| < \varepsilon$ with respect to $d\mu$ of the second and third terms above is less than or equal to

$$\frac{1}{\varepsilon} \int_{|y|<\varepsilon} \frac{1}{|y|^{d-1}} \, d\mu(y),$$

which is estimated by an integration by parts as in (28). The upper bound one gets is

$$C \sup_{0<\rho<\varepsilon} \frac{\mu(B(0,\rho))}{\rho^d} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

One is left with

$$F(x) = \frac{1}{\varepsilon^2} \int_{|y|<\varepsilon} \frac{1}{|x-y|^{d-2}} \, d\mu(y).$$

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If $x \in B(0, r)$ then
$$F(x) \leq P\left(\chi_{B(0,2r)}(y) \frac{d\mu(y)}{|y|^2}\right)(x),$$

where P is the Newtonian potential of the indicated measure, as in (1). The total mass of the measure $\chi_{B(0,2r)}(y)\, d\mu(y)/|y|^2$ is estimated by an integration by parts and one gets the upper bound
$$\sup_{0<\rho<r} \frac{\mu(B(0,\rho))}{\rho^d} r^{d-2}.$$

Therefore
$$\sup_{t>0} \frac{t \operatorname{Cap}\{x \in B(0,r) : F(x) > t\}}{\operatorname{Cap}(B(0,r))} \leq C \sup_{0<\rho<r} \frac{\mu(B(0,\rho))}{\rho^d},$$

which completes the proof of Theorem 5, part (i). $\qquad \square$

Proof of Theorem 5, part (ii). The construction is practically that of Calderón in [C], so that we will briefly outline the argument. It is enough to construct a finite Borel measure in the unit square $Q = [0,1] \times [0,1]$ whose logarithmic potential is not second order differentiable in the weak capacity sense at almost all points of the square. If this measure has been constructed, then one covers the plane by disjoint dyadic squares Q_n, $n = 0, 1, \ldots$ of side length 1 and one sets $\mu = \sum_{n=0}^{\infty} 1/2^n\, \mu_n$, where μ_n is the translation into Q_n of the measure constructed in the unit square.

Divide the unit square into 4^{n^2} disjoint squares of side length 2^{-n^2}. The vertices of those squares not lying in the boundary of the unit square are of the form $(i2^{-n^2}, j2^{-n^2})$ with $1 \leq i, j \leq (2^{n^2} - 1)$. There are $N_n := (2^{n^2} - 1)^2 \leq 4^n$ such vertices. Denote them by p_{nk}, where the index k varies from 1 to N_n. Let B_{nk} be the ball with center p_{nk} and radius $1/(n2^{n^2})$. Set

$$E_n = \bigcup_{k=1}^{N_n} B_{nk}, \qquad D_m = \bigcup_{n=m}^{\infty} E_n, \qquad D = \bigcap_{m=1}^{\infty} D_m,$$

so that $|E_n| \leq N_n/(n^2 4^{n^2}) \leq 1/n^2$. Hence $|D_m| \to 0$ as $m \to \infty$ and $|D| = 0$. Let L be the function in (42). Define

$$S_n(z) = \frac{1}{N_n} \sum_{k=1}^{N_n} L(n2^{n^2}(z - p_{nk})), \quad z \in \mathbb{C},$$

$$u(z) = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} S_n(z), \qquad z \in \mathbb{C}.$$

Then u is the logarithmic potential of a finite Borel measure supported in the unit square.

We claim that u is not second order differentiable in the weak capacity sense at any point of $Q \setminus D$. Take $a \in Q \setminus D$, so that $a \in Q \setminus D_m$ for some positive integer m. Thus

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$a \in Q$, $a \notin B_{nk}$, $n \geq m$, $1 \leq k \leq N_n$. We consider radii of the form $r = r_q = 1/2^{q^2}$. If q is large enough then r is small enough so that the ball $B(a, r)$ does not contain any p_{nk}, $1 \leq n \leq m-1$, $1 \leq k \leq N_n$. Then

$$u(z) - u(a) - \langle \nabla u(a), z - a \rangle$$

$$= \sum_{n=1}^{m-1} \frac{1}{n^{3/2}} \left(S_n(z) - S_n(a) - \langle \nabla S_n(a), z - a \rangle \right) + \sum_{n=m}^{\infty} \frac{1}{n^{3/2}} S_n(z)$$

and the first term in the right-hand side is smooth on $B(a, r)$. Hence the differentiability properties of u are exactly those of

$$R(z) := \sum_{n=m}^{\infty} \frac{1}{n^{3/2}} S_n(z).$$

Assume that R is second order differentiable in the weak capacity sense at a. Then

(53) $$\frac{\sup_{t>0} t \, \mathrm{Cap}(\{z \in B(a,r) : |R(z)| > t\})}{r^2 \, \mathrm{Cap}(B(a,r))} \leq C_a, \quad 0 < r < 1/4,$$

for some constant C_a depending only on a. To disprove (53) we note that, since $a \in Q$, there is a point $p_{qk} \in B(a, 1/2^{q^2})$. Moreover $|p_{qk} - a| < (1/\sqrt{2})r$, $r = 1/2^{q^2}$.

If $q \geq m$, then

$$R(z) \geq \frac{1}{q^{3/2}} S_q(z) \geq \frac{1}{q^{3/2} N_q} L(q 2^{q^2}(z - p_{qk})).$$

If q is large enough the set $\{z \in B(a,r) : |R(z)| > t\}$ contains the ball of center p_{qk} and radius $1/(e^{tq^{3/2}4^{q^2}} q 2^{q^2})$. Thus the left-hand side of (53) is not less than a constant times

$$q^2 4^{q^2} \sup_{t>0} \frac{t}{q^{3/2} 4^{q^2} t + \log q + q^2 \log 2} = q^2 4^{q^2} \frac{1}{q^{3/2} 4^{q^2}} = q^{1/2},$$

which shows that (53) cannot hold. $\qquad\square$

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