A NEW CHEBYSHEV FAMILY WITH APPLICATIONS TO ABEL EQUATIONS

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Abstract. This paper proves that a family of functions, defined through some definite integrals, forms an extended complete Chebyshev system. The key point of our proof consists in reducing the study of certain Wronskians to the Gram determinants of a suitable set of new functions. Our result is applied to give upper bounds for the number of isolated periodic solutions that some perturbed generalized Abel equations can exhibit.

1. Introduction and main results

In this paper we introduce the family of analytic functions

$$I_{k,\alpha}(y) := \int_a^b \frac{g^k(t)}{(1-yg(t))^\alpha} dt, \tag{1}$$

$k = 0, 1, \dots, n$, and prove that they form an *extended complete Chebyshev system* (for short, an ECT-system). As we will see, in contrast to what happens in other papers, we do not need to perform the explicit integration in $I_{k,\alpha}$ to prove our result. In fact our proof uses the standard characterization of ECT-systems trough the computation of several Wronskians, see Theorem 2.1. The key point of our approach is to show that these Wronskians coincide with some Gram determinants associated to the usual inner product in $\mathcal{L}^2([a,b])$,

$$\langle u, v\rangle = \int_a^b u(t)v(t)\, dt,$$

for a suitable set of functions $u_0, u_1, \dots, u_n$. As far as we know this is the first time that this method is used to prove that a set of functions is an ECT-system.

We use the results obtained for the ECT-system to determine upper bounds for the number of isolated 2π-periodic solutions that appear when we perform a first order analysis in ε of the generalized Abel equations

$$\frac{dx}{dt} = \frac{\cos(t)}{q-1}\, x^q + \varepsilon P_n(\cos(t), \sin(t))\, x^p, \tag{2}$$

where $q, p \in \mathbb{N} \setminus \{0, 1\}, q \neq p$, and P_n is a polynomial of degree n. Recall that for the usual Abel equation $\{q, p\} = \{2, 3\}$. This type of results are useful to understand which is the number of isolated 2π-periodic solutions of Abel type

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differential equations in terms of the degrees of the trigonometrical polynomials appearing in them. In turn, these results are strongly related with the Hilbert sixteenth problem for planar polynomial differential equations, see [3, 4, 5, 7]. Our result improves previous results on (2) given in [1, 3, 7].

To state our main results we need to fix some notations. Given $k \in \mathbb{N}, \alpha, a, b \in \mathbb{R}$ and any continuous, non identically zero, function $g(t)$ on $[a,b]$, we consider the new analytic function $I_{k,\alpha}(y)$ defined in (1), on the open interval J given by the connected component of the set $\{y \in \mathbb{R} : 1 - yg(t) > 0 \text{ for all } t \in [a,b]\}$ that contains the origin. For instance, note that if $m := \min_{t\in[a,b]} g(t) < 0$ and $M := \max_{t\in[a,b]} g(t) > 0$ then $J = (1/m, 1/M)$. Our first result shows that, varying k, and for almost all α, the above set of functions is an ECT-system. See next section for the precise definition.

Theorem A. *For any $n \in \mathbb{N}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$, the set of ordered functions $(I_{0,\alpha}, I_{1,\alpha}, \ldots, I_{n,\alpha})$, defined in (1), is an ECT-system on J. When $\alpha \in \mathbb{Z}^-$ it is an ECT-system on J if and only if $n \le -\alpha$. In particular, for the cases for which the set of functions is an ECT-system, any non-trivial function of the form*

$$\Phi_\alpha(y) := \sum_{k=0}^{n} a_k I_{k,\alpha}(y),$$

with $a_k \in \mathbb{R}$, has at most n zeros on J counted with multiplicities.

In [7] it is proved that when $g(t) = \sin(t)$ and $[a,b] = [0, 2\pi]$, the function Φ_1 has n zeros in a neighborhood of $y = 0$ and in [3] this result is extended to any Φ_α, for $\alpha \in \mathbb{Q}^+$. Afterwards some of these local results are improved in [1]. In this last paper the functions Φ_1 and $\Phi_{-1/2}$ are explicitly computed and their global number of zeroes in $J = (-1, 1)$ is studied. The authors got the following results:

$$\begin{aligned}\Phi_1(y) = \sum_{k=0}^{n} a_k\, I_{k,1}(y) &= \sum_{k=0}^{n} a_k \int_0^{2\pi} \frac{\sin^k(t)}{1 - y\sin(t)}\, dt \\ &= \frac{1}{y^{2n}\sqrt{1-y^2}} \left(P_{2n}(y) + Q_{2n}(y)\sqrt{1-y^2}\right),\end{aligned} \tag{3}$$

and

$$\begin{aligned}\Phi_{-1/2}(y^2) = \sum_{k=0}^{n} a_k\, I_{k,-1/2}(y^2) &= \sum_{k=0}^{n} a_k \int_0^{2\pi} \sin^k(t)\sqrt{1 - y^2\sin(t)}\, dt \\ &= \frac{\sqrt{1+y^2}}{r^{2n}} \left(P_n(r^2)\mathcal{K}(r) + Q_n(r^2)\mathcal{E}(r)\right),\end{aligned} \tag{4}$$

where P_j and Q_j are some polynomials of degree j, $r = \sqrt{2}|y|/\sqrt{1+y^2}$, and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the two usual elliptic functions

$$\mathcal{K}(r) = \int_0^1 \frac{1}{\sqrt{1-\mu^2}\sqrt{1-r^2\mu^2}}\, d\mu, \quad \mathcal{E}(r) = \int_0^1 \frac{\sqrt{1-r^2\mu^2}}{\sqrt{1-\mu^2}}\, d\mu,$$

see [2]. Then by using the expressions (3) and (4), they prove that the function $\Phi_1(y)$ has at most n zeros in $(-1,1)$, taking into account their multiplicities, and that this upper bound is sharp, and that the function $\Phi_{-1/2}(y^2)$ can have at most $4n+2$ zeros in J and that there are examples having at least $2n$ zeros (also in J).

Theorem A proves that, for any g and α as in the statement, n is the sharp upper bound for the number of zeros of $\Phi_\alpha(y)$ in the whole interval J. Notice that for $\Phi_\alpha(y^2)$ the upper bound goes to $2n$ in the suitable interval.

Our result on the generalized Abel equations (2) is given in the next theorem. As we can see in item (i) the relation between the Abel equations and the functions Φ_α is that precisely they appear when we study the first order terms in ε of the solution of these differential equations. In fact this was our original motivation to prove Theorem A.

Theorem B. *Let $x=\varphi(t,\rho,\varepsilon)$ be the solution of the generalized Abel equation* (2),

$$\frac{dx}{dt}=\frac{\cos(t)}{q-1}\,x^q+\varepsilon P_n(\cos(t),\sin(t))\,x^p,$$

where $q,p\in\mathbb{N}\setminus\{0,1\}, q\neq p$, and P_n is a polynomial of degree n, with initial condition $x(0)=\rho,\ |\rho|<1$. Then:

(i) It holds that

$$\varphi(2\pi,\rho,\varepsilon)=\rho+\varepsilon\rho^p\Phi_\alpha(\rho^{q-1})+O(\varepsilon^2), \tag{5}$$

where Φ_α is the function introduced in Theorem A, for some real constants $a_0,a_1,\ldots,a_n$, $g(t)=\sin(t)$ and $\alpha=(p-q)/(q-1)$.

(ii) For $\varepsilon=0$ and $|\rho|<1$ all the solutions of (2) *are 2π-periodic.*

(iii) The simple zeros in $(-1,1)\setminus\{0\}$ of the function $\Phi_\alpha(\rho^{q-1})$, give rise to initial conditions of isolated 2π-periodic solutions of (2) *that tend to these zeros when ε goes to 0.*

(iv) For $\varepsilon\neq 0$, small enough, and $\Phi_\alpha(\rho^{q-1})\not\equiv 0$, the maximum number of non-zero isolated 2π-periodic solutions obtained in item (iii) is n, when q is even, and $2n$, when q is odd. Moreover in both cases these upper bounds are sharp.

This paper is organized as follows: some preliminary results on ECT-systems and the proof of Theorem A are given in next section. Section 3 deals with the generalized Abel equation (2).

2. Preliminary results and proof of Theorem A

Let $f_0,f_1,\ldots,f_n$ be functions on an open interval J of $\mathbb{R}$. It is said that $(f_0,f_1,\ldots,f_n)$ is an extended complete Chebyshev system (ECT-system) on J if, for all $k=0,1,\ldots,n$, any nontrivial linear combination $a_0f_0(y)+a_1f_1(y)+\cdots+a_kf_k(y)$ has at most k isolated zeros on J counted with multiplicities. Here "T" stands for Tchebycheff, which is one of the transcriptions of the Russian name Chebyshev.

A very useful characterization of ECT-systems is given in the following theorem, see [6, 8]:

Theorem 2.1. *Let $f_0, f_1, \dots, f_n$ be analytic functions defined on an open interval J of $\mathbb{R}$. Then $(f_0, f_1, \dots, f_n)$ is an ECT-system on J if and only if for each $k = 0, 1, \dots, n$, and all $y \in J$, the Wronskian*

$$W(f_0(y), f_1(y), \dots, f_k(y)) := \begin{vmatrix} f_0(y) & f_1(y) & \cdots & f_k(y) \\ f_0'(y) & f_1'(y) & \cdots & f_k'(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(y) & f_1^{(k)}(y) & \cdots & f_k^{(k)}(y) \end{vmatrix}$$

is different from zero.

Next well-known result of linear algebra will also be a key point in our proof.

Theorem 2.2. *Let $v_0, v_1, \dots, v_n$ be elements of a vectorial space E endowed with an inner product $\langle\, ,\rangle$. Then*

$$G(v_0, v_1, \dots, v_n) := \begin{vmatrix} \langle v_0, v_0\rangle & \langle v_0, v_1\rangle & \cdots & \langle v_0, v_n\rangle \\ \langle v_1, v_0\rangle & \langle v_1, v_1\rangle & \cdots & \langle v_1, v_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_0\rangle & \langle v_n, v_1\rangle & \cdots & \langle v_n, v_n\rangle \end{vmatrix} \geq 0$$

and it is zero if and only if the vectors $v_0, v_1, \dots, v_n$ are linearly dependent. The determinant $G(v_0, v_1, \dots, v_n)$ is usually called the Gram determinant.

In fact we will use the above result when E is the space of continuous functions on a closed interval $[a, b]$ and the inner product is $\langle u, v\rangle = \int_a^b u(t)v(t)\,dt$. In this context G is also called the *integral Gram determinant*, see [9, pp. 45-48].

Before proving Theorem A we need a preliminary result about the successive derivatives of $I_{k,\alpha}(y)$ and a recurrence that allows to express $I_{k,\alpha}(y)$ in terms of several $I_{\tilde{k},\tilde{\alpha}}(y)$ with $\tilde{k} < k$ and $\tilde{\alpha} < \alpha$.

Lemma 2.3. *(i) For any $k \geq 0$ and $\ell \geq 1$,*

$$I_{k,\beta}' = \beta I_{k+1,\beta+1}, \quad I_{k,\beta}^{(\ell)} = \prod_{j=0}^{\ell-1}(\beta + j) I_{k+\ell,\beta+\ell}.$$

(ii) For any $k \geq 1$ and $m \leq k$,

$$I_{k,\beta} = \frac{1}{y}\left(I_{k-1,\beta} - I_{k-1,\beta-1}\right) = \frac{1}{y^m}\sum_{j=0}^{m}(-1)^j\binom{m}{j} I_{k-m,\beta-j}.$$

Proof. It is easy to see that the functions $I_{k,\beta}$ are analytic for $y \in J$ and

$$I_{k,\beta}^{(\ell)}(y) = \int_a^b \frac{\partial^\ell}{\partial y^\ell}\frac{g^k(t)}{(1 - yg(t))^\beta}\,dt.$$

Then the statement (i) follows differentiating (1) with respect to y, one or ℓ times, respectively.

We will prove statement (ii) by induction on m. The case $m = 1$ follows multiplying by $(1 - yg(t))$ the numerator and the denominator of the integrand of (1):

$$
\begin{aligned}
I_{k-1,\beta-1}(y) &= \int_a^b \frac{g^{k-1}(t)}{(1-yg(t))^{\beta-1}} \frac{(1-yg(t))}{(1-yg(t))} dt \\
&= \int_a^b \frac{g^{k-1}(t)}{(1-yg(t))^{\beta}} dt - y \int_a^b \frac{g^{k}(t)}{(1-yg(t))^{\beta}} dt = I_{k-1,\beta} - yI_{k,\beta},
\end{aligned}
$$

then solving $I_{k,\beta}$ of the previous equality we obtain the relation given in the statement.

Suppose that the relation of $I_{k,\beta}$ is satisfied until m, then the next equalities hold:

$$
\begin{aligned}
I_{k,\beta}(y) =& \frac{1}{y^m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} I_{k-m,\beta-j} \\
=& \frac{1}{y^m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{y} \left(I_{k-m-1,\beta-j} - I_{k-m-1,\beta-j-1} \right) \\
=& \frac{1}{y^{m+1}} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} I_{k-m-1,\beta-j} - \sum_{j=0}^{m} (-1)^j \binom{m}{j} I_{k-m-1,\beta-j-1} \right) \\
=& \frac{1}{y^{m+1}} \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} I_{k-m-1,\beta-j} + \sum_{\ell=1}^{m+1} (-1)^\ell \binom{m}{\ell-1} I_{k-m-1,\beta-\ell} \right) \\
=& \frac{1}{y^{m+1}} \Bigg(I_{k-m-1,\beta} + \sum_{j=0}^{m} (-1)^j \left(\binom{m}{j} + \binom{m}{j-1} \right) I_{k-m-1,\beta-j} \\
& \qquad + I_{k-m-1,\beta-(m+1)} \Bigg) \\
=& \frac{1}{y^{m+1}} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} I_{k-(m+1),\beta-j}.
\end{aligned}
$$

□

The following result relates a Wronskian with the determinant of a symmetric matrix that finally will be a Gram determinant.

Lemma 2.4. *Let $I_{0,\alpha},\dots,I_{n,\alpha}$ be the functions defined by* (1). *Then for $y\neq 0$,*

$$W_n := W(I_{0,\alpha},I_{1,\alpha},\dots,I_{n,\alpha}) = \begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ I'_{0,\alpha} & I'_{1,\alpha} & \cdots & I'_{n,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ I^{(n)}_{0,\alpha} & I^{(n)}_{1,\alpha} & \cdots & I^{(n)}_{n,\alpha} \end{vmatrix}$$

$$= \frac{\prod_{j=0}^{n-1}(\alpha+j)^{n-j}}{y^{\frac{(1+n)n}{2}}} \begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ I_{0,\alpha+1} & I_{1,\alpha+1} & \cdots & I_{n,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{0,\alpha+n} & I_{1,\alpha+n} & \cdots & I_{n,\alpha+n} \end{vmatrix}$$

$$= \frac{\prod_{j=0}^{n-1}(\alpha+j)^{n-j}}{(-1)^{\frac{(1+n)n}{2}} y^{(1+n)n}} \begin{vmatrix} I_{0,\alpha} & I_{0,\alpha-1} & \cdots & I_{0,\alpha-n} \\ I_{0,\alpha+1} & I_{0,\alpha} & \cdots & I_{0,\alpha-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{0,\alpha+n} & I_{0,\alpha+n-1} & \cdots & I_{0,\alpha} \end{vmatrix}$$

$$= \frac{\prod_{j=0}^{n-1}(\alpha+j)^{n-j}}{y^{(1+n)n}} \begin{vmatrix} I_{0,\alpha-n} & I_{0,\alpha-n+1} & \cdots & I_{0,\alpha-1} & I_{0,\alpha} \\ I_{0,\alpha-n+1} & I_{0,\alpha-n+2} & \cdots & I_{0,\alpha} & I_{0,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{0,\alpha-1} & I_{0,\alpha} & \cdots & I_{0,\alpha+n-2} & I_{0,\alpha+n-1} \\ I_{0,\alpha} & I_{0,\alpha+1} & \cdots & I_{0,\alpha+n-1} & I_{0,\alpha+n} \end{vmatrix}. \tag{6}$$

Proof. Using the expression for the derivatives given in Lemma 2.3(i) we can write

$$W_n = \begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ I'_{0,\alpha} & I'_{1,\alpha} & \cdots & I'_{n,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ I^{(n)}_{0,\alpha} & I^{(n)}_{1,\alpha} & \cdots & I^{(n)}_{n,\alpha} \end{vmatrix}$$

$$= \begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ \alpha I_{1,\alpha+1} & \alpha I_{2,\alpha+1} & \cdots & \alpha I_{n+1,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=0}^{n-1}(\alpha+j) I_{n,\alpha+n} & \prod_{j=0}^{n-1}(\alpha+j) I_{n+1,\alpha+n} & \cdots & \prod_{j=0}^{n-1}(\alpha+j) I_{2n,\alpha+n} \end{vmatrix}$$

$$= \prod_{j=0}^{n-1}(\alpha+j)^{n-j} \begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ I_{1,\alpha+1} & I_{2,\alpha+1} & \cdots & I_{n+1,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n,\alpha+n} & I_{n+1,\alpha+n} & \cdots & I_{2n,\alpha+n} \end{vmatrix}. \tag{7}$$

If we denote the i-row of the previous determinant by $R_i = [I_{i,\alpha+i}, I_{i+1,\alpha+i},\dots, I_{i+n,\alpha+i}]$ for $i=0,\dots,n$, using the expression given in Lemma 2.3(ii) and taking

$k = i + j$, $\beta = \alpha + i$ and $m = i$ for each j-component of R_i, we can write

$$
\begin{aligned}
R_i = \Bigg[& \frac{1}{y^i}\left(I_{i-i,\alpha+i} + \sum_{j=1}^{i}\binom{i}{j}(-1)^j I_{i-i,\alpha+i-j}\right), \\
& \frac{1}{y^i}\left(I_{i+1-i,\alpha+i} + \sum_{j=1}^{i}\binom{i}{j}(-1)^j I_{i+1-i,\alpha+i-j}\right), \dots, \\
& \frac{1}{y^i}\left(I_{i+n-i,\alpha+i} + \sum_{j=1}^{i}\binom{i}{j}(-1)^j I_{i+n-i,\alpha+i-j}\right)\Bigg] \\
= & \frac{1}{y^i}\Bigg([I_{0,\alpha+i}, I_{1,\alpha+i}, \dots, I_{n,\alpha+i}] \\
& \qquad + \sum_{j=1}^{i}\binom{i}{j}(-1)^j\,[I_{0,\alpha+i-j}, I_{1,\alpha+i-j}, \dots, I_{n,\alpha+i-j}]\Bigg) \\
= & \frac{1}{y^i}\left(\widehat{R}_i + \sum_{j=1}^{i}\binom{i}{j}(-1)^j \widehat{R}_{i-j}\right)
\end{aligned}
$$

where $\widehat{R}_\ell = [I_{0,\alpha+\ell}, I_{1,\alpha+\ell}, \dots, I_{n,\alpha+\ell}]$ for all ℓ and $\widehat{R}_0 = R_0$. Then using the elementary properties of the determinants we can write (7) as

$$
\begin{aligned}
W_n &= \prod_{j=0}^{n-1}(\alpha+j)^{n-j}\begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ \frac{1}{y}I_{0,\alpha+1} & \frac{1}{y}I_{1,\alpha+1} & \cdots & \frac{1}{y}I_{n,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{y^n}I_{0,\alpha+n} & \frac{1}{y^n}I_{1,\alpha+n} & \cdots & \frac{1}{y^n}I_{n,\alpha+n}\end{vmatrix} \\
&= \frac{\prod\limits_{j=0}^{n-1}(\alpha+j)^{n-j}}{y^{\frac{(1+n)n}{2}}}\begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ I_{0,\alpha+1} & I_{1,\alpha+1} & \cdots & I_{n,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{0,\alpha+n} & I_{1,\alpha+n} & \cdots & I_{n,\alpha+n}\end{vmatrix},
\end{aligned}
\tag{8}
$$

that coincides with the first equivalent determinant of the statement.

If we denote the i-column of the previous determinant by $C_i = [I_{i,\alpha}, I_{i,\alpha+1}, \dots, I_{i,\alpha+n}]^t$ for $i = 0, \dots, n$, using the expression given in Lemma 2.3(ii) and taking

$k=i$, $\beta=\alpha+j$ and $m=i$ for each j-component of C_i, we can write

$$
\begin{aligned}
C_i =& \Bigg[\frac{1}{y^i}\left(\sum_{j=0}^{i-1}\binom{i}{j}(-1)^j I_{i-i,\alpha-j}+(-1)^i I_{i-i,\alpha-i}\right),\\
&\frac{1}{y^i}\left(\sum_{j=0}^{i-1}\binom{i}{j}(-1)^j I_{i-i,\alpha+1-j}+(-1)^i I_{i-i,\alpha+1-i}\right),\\
&\dots,\frac{1}{y^i}\left(\sum_{j=0}^{i-1}\binom{i}{j}(-1)^j I_{i-i,\alpha+n-j}+(-1)^i I_{i-i,\alpha+n-i}\right)\Bigg]^t\\
=&\frac{1}{y^i}\Bigg(\sum_{j=0}^{i-1}\binom{i}{j}(-1)^j\left[I_{0,\alpha-j},I_{0,\alpha+1-j},\dots,I_{0,\alpha+n-j}\right]^t\\
&+(-1)^i\left[I_{0,\alpha-i},I_{0,\alpha+1-i},\dots,I_{0,\alpha+n-i}\right]^t\Bigg)\\
=&\frac{1}{y^i}\sum_{j=0}^{i-1}\binom{i}{j}(-1)^j\widehat{C}_{i-j}+(-1)^i\widehat{C}_i,
\end{aligned}
$$

where $\widehat{C}_\ell=[I_{0,\alpha-\ell},I_{0,\alpha+1-\ell},\dots,I_{0,\alpha+n-\ell}]$ for all ℓ and $\widehat{C}_0=C_0$. Then using again the properties of the determinants we can write (8) as

$$
\begin{aligned}
W_n =& \frac{\prod_{j=0}^{n-1}(\alpha+j)^{n-j}}{y^{\frac{(1+n)n}{2}}}
\begin{vmatrix}
I_{0,\alpha} & \frac{(-1)}{y}I_{0,\alpha-1} & \cdots & \frac{(-1)^n}{y^n}I_{0,\alpha-n}\\
I_{0,\alpha+1} & \frac{(-1)}{y}I_{0,\alpha} & \cdots & \frac{(-1)^n}{y^n}I_{0,\alpha-n+1}\\
\vdots & \vdots & \ddots & \vdots\\
I_{0,\alpha+n} & \frac{(-1)}{y}I_{0,\alpha+n-1} & \cdots & \frac{(-1)^n}{y^n}I_{0,\alpha}
\end{vmatrix}\\
=&\frac{\prod_{j=0}^{n-1}(\alpha+j)^{n-j}}{(-1)^{\frac{(1+n)n}{2}}y^{(1+n)n}}
\begin{vmatrix}
I_{0,\alpha} & I_{0,\alpha-1} & \cdots & I_{0,\alpha-n}\\
I_{0,\alpha+1} & I_{0,\alpha} & \cdots & I_{0,\alpha-n+1}\\
\vdots & \vdots & \ddots & \vdots\\
I_{0,\alpha+n} & I_{0,\alpha+n-1} & \cdots & I_{0,\alpha}
\end{vmatrix}
\end{aligned}
\tag{9}
$$

that is exactly the second equivalent determinant of the statement.

Reordering the columns of determinant (9), changing $(\widehat{C}_0,\widehat{C}_1,\dots,\widehat{C}_{n-1},\widehat{C}_n)$ by $(\widehat{C}_n,\widehat{C}_{n-1},\dots,\widehat{C}_1,\widehat{C}_0)$, we can transform it into a determinant of a symmetric matrix.

$$
W_n=\frac{(-1)^{\left[\frac{n+1}{2}\right]}}{(-1)^{\frac{(1+n)n}{2}}}\frac{\prod_{j=0}^{n-1}(\alpha+j)^{n-j}}{y^{(1+n)n}}
\begin{vmatrix}
I_{0,\alpha-n} & I_{0,\alpha-n+1} & \cdots & I_{0,\alpha-1} & I_{0,\alpha}\\
I_{0,\alpha-n+1} & I_{0,\alpha-n+2} & \cdots & I_{0,\alpha} & I_{0,\alpha+1}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
I_{0,\alpha-1} & I_{0,\alpha} & \cdots & I_{0,\alpha+n-2} & I_{0,\alpha+n-1}\\
I_{0,\alpha} & I_{0,\alpha+1} & \cdots & I_{0,\alpha+n-1} & I_{0,\alpha+n}
\end{vmatrix},
\tag{10}
$$

where [ ] denotes the integer part function. This last expression coincides with the third equivalent determinant of the statement because $\left[\frac{n+1}{2}\right]+\frac{(1+n)n}{2}$ is always an even number. Thus the lemma is proved. $\square$

Next result will be the key point in our proof of Theorem A.

Proposition 2.5. *Let W_n be the Wronskian defined in Lemma 2.4. When α is a negative integer and $n>-\alpha$ then $W_n=0$. Otherwise, W_n does not vanish on J and* $\operatorname{sgn}(W_n)=\operatorname{sgn}\left(\prod_{j=0}^{n-1}(\alpha+j)^{n-j}\right)$.

Proof. When α is a negative integer and $n>-\alpha$ it is clear that $\prod_{j=0}^{n-1}(\alpha+j)^{n-j}=0$. Thus by equality (7) we know that $W_n=0$. Assume that $\prod_{j=0}^{n-1}(\alpha+j)^{n-j}\neq 0$. In this case consider the auxiliary functions $f_i(t)=\frac{1}{(1-yg(t))^{(\alpha-n)/2+i}}$, for $i=0,1,\ldots,n$, which are well defined on J because on this set $1-yg(t)>0$. Notice that

$$\langle f_i,f_j\rangle=\int_a^b\frac{1}{(1-yg(t))^{\alpha-n+i+j}}\,dt=I_{0,\alpha-n+i+j}(y).$$

Hence by using the equivalent expression (6) of the Wronskian, when $y\neq 0$,

$$W_n=\frac{\prod\limits_{j=0}^{n-1}(\alpha+j)^{n-j}}{y^{(1+n)n}}\,G(f_0,f_1,\ldots,f_n),\tag{11}$$

where $G(f_0,f_1,\ldots,f_n)$ is the integral Gram determinant. From Theorem 2.2, it is non-negative and vanishes if and only if the functions f_i are linearly dependent. Since the function $g(t)$ is not identically zero, the functions f_i are linearly independent because they can be written as

$$f_i(t)=\frac{1}{(1-yg(t))^{\frac{\alpha-n}{2}+i}}=\frac{(1-yg(t))^n}{(1-yg(t))^{\frac{\alpha-n}{2}+i+n}}=\frac{(1-yg(t))^{n-i}}{(1-yg(t))^{\frac{\alpha+n}{2}}},\ i=0,\ldots,n$$

and the numerators are polynomials of degree $n,n-1,\ldots,0$ in y, respectively. Then the sign of W_n, on the set $J\setminus\{0\}$, is the sign of $\prod_{j=0}^{n-1}(\alpha+j)^{n-j}$ because the Gram determinant in (11) is always positive and $y^{(1+n)n}>0$.

In the expression of W_n given in (7), it can be seen that the determinant appearing there, evaluated at $y=0$, is also positive because it can also be written as a new integral Gram determinant,

$$\begin{vmatrix} I_{0,\alpha}(0) & I_{1,\alpha}(0) & \cdots & I_{n,\alpha}(0)\\ I_{1,\alpha+1}(0) & I_{2,\alpha+1}(0) & \cdots & I_{n+1,\alpha+1}(0)\\ \vdots & \vdots & \ddots & \vdots\\ I_{n,\alpha+n}(0) & I_{n+1,\alpha+n}(0) & \cdots & I_{2n,\alpha+n}(0)\end{vmatrix}$$

$$=\begin{vmatrix} \int_a^b 1dt & \int_a^b g(t)dt & \cdots & \int_a^b g^n(t)dt\\ \int_a^b g(t)dt & \int_a^b g^2(t)dt & \cdots & \int_a^b g^{n+1}(t)dt\\ \vdots & \vdots & \ddots & \vdots\\ \int_a^b g^n(t)dt & \int_a^b g^{n+1}(t)dt & \cdots & \int_a^b g^{2n}(t)dt\end{vmatrix}=G(1,g,g^2,\ldots,g^n)>0.$$

Thus W_n is well defined on the whole J, does not vanish and its sign coincides with the one of $\prod_{j=0}^{n-1}(\alpha+j)^{n-j}$, as we wanted to prove. □

Remark 2.6. *Note that although for $\alpha=-m\in\mathbb{Z}^-$ the functions*

$$I_{k,-m}=\int_a^b g^k(t)(1-y\,g(t))^m\,dt,$$

are well defined for all $y\in\mathbb{R}$ our result only proves that the set $(I_{0,-m},I_{1,-m},\dots,I_{n,-m})$, for $n\le-\alpha=m$, is an ECT-system on J. In fact it is easy to see that, for instance, the functions

$$a_0I_{0,-2}(y)+a_1I_{1,-2}(y),$$

which are polynomials of degree 2 in y, can have two zeros in $\mathbb{R}$, proving that $(I_{0,-2},I_{1,-2})$ is not a ECT-system on the whole $\mathbb{R}$.

Proof of Theorem A. By using Theorem 2.1, we know that to prove the theorem it suffices to show that, under our hypotheses and for any $k=0,1,\dots,n$, the Wronskian of the functions $(I_{0,\alpha},I_{1,\alpha},\dots,I_{k,\alpha})$ does not vanish on J. This is a direct consequence of Proposition 2.5. □

3. Generalized Abel equations

This section is devoted to prove Theorem B.

Proof of Theorem B. (i)-(ii) Following the computations of [1] or [3] it is not difficult to see that

$$\begin{aligned}\varphi(t,\rho,\varepsilon)=&\rho\left(\frac{1}{1-\rho^{q-1}\sin(t)}\right)^{\frac{1}{q-1}}\\&+\varepsilon\left(\frac{\rho}{1-\rho^{q-1}\sin(t)}\right)^{p}\int_0^t\frac{P_n(\cos(s),\sin(s))}{(1-\rho^{q-1}\sin(s))^{\alpha}}\,ds+O(\varepsilon^2).\end{aligned}$$

Note that since $\rho\in(-1,1)$ the flow is well defined for all $t\in\mathbb{R}$. Then

$$\varphi(2\pi,\rho,\varepsilon)=\rho+\varepsilon\rho^p\int_0^{2\pi}\frac{P_n(\cos(t),\sin(t))}{(1-\rho^{q-1}\sin(t))^{\alpha}}\,dt+O(\varepsilon^2).$$

Using that $\cos^{2\ell}(t)=(1-\sin^2(t))^{\ell}$ and $\cos^{2\ell+1}(t)=(1-\sin^2(t))^{\ell}\cos(t)$ and, that for any polynomial $R(z)$,

$$\int_0^{2\pi}\frac{R(\sin(t))\cos(t)}{(1-\rho^{q-1}\sin(t))^{\alpha}}\,dt=0$$

we get that

$$\int_0^{2\pi}\frac{P_n(\cos(t),\sin(t))}{(1-\rho^{q-1}\sin(t))^{\alpha}}\,dt=\int_0^{2\pi}\frac{Q_n(\sin(t))}{(1-\rho^{q-1}\sin(t))^{\alpha}}\,dt$$

where Q_n is a new polynomial of degree n, that we can write as $Q_n(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$. Hence the expression (5) follows.

Item (iii) is a direct consequence of (5) and the Implicit Function Theorem applied to

$$\frac{\varphi(2\pi,\rho,\varepsilon)-\rho}{\varepsilon}=\rho^p\Phi_\alpha(\rho^{q-1})+O(\varepsilon).$$

(iv) Putting $y=\rho^{q-1}$ and taking $J=(-1,1)$ we know from Theorem A that the maximum number of zeros of $\Phi_\alpha(y)$ on J counted with multiplicities is n and that this upper bound is sharp. Since when q is odd $\Phi_\alpha(\rho^{q-1})=\Phi_\alpha((-\rho)^{q-1})$ the result follows. □

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