BIFURCATION VALUES FOR A FAMILY OF PLANAR VECTOR FIELDS OF DEGREE FIVE

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Abstract. We study the number of limit cycles and the bifurcation diagram in the Poincaré sphere of a one-parameter family of planar differential equations of degree five $\dot{\mathbf{x}} = X_b(\mathbf{x})$ which has been already considered in previous papers. We prove that there is a value $b^* > 0$ such that the limit cycle exists only when $b \in (0, b^*)$ and that it is unique and hyperbolic by using a rational Dulac function. Moreover we provide an interval of length $27/1000$ where b^* lies. As far as we know the tools used to determine this interval are new and are based on the construction of algebraic curves without contact for the flow of the differential equation. These curves are obtained using analytic information about the separatrices of the infinite critical points of the vector field. To prove that the Bendixson-Dulac Theorem works we develop a method for studying whether one-parameter families of polynomials in two variables do not vanish based on the computation of the so called double discriminant.

Keywords: Polynomial planar system, Uniqueness of limit cycles, Bifurcation, Phase portrait on the Poincaré sphere, Dulac function, Double discriminant.

1. Introduction and main results

Consider the one-parameter family of quintic differential systems

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& -x + (a - x^2)(y + y^3), \quad a \in \mathbb{R}. \end{cases} \tag{1}$$

Notice that without the term y^3, (1) coincides with the famous van der Pol system.

2010 AMS Subject classifications: 34C07 (Primary), 34C23, 34C25, 37C27, 13P15 (Secondary).

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Prepublicació Núm 8, febrer 2012.
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http://www.uab.cat/matematiques
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This family was studied in [24] and the authors concluded that it has only two bifurcation values, 0 and a^*, and exactly four different global phase portraits on the Poincaré disc. Moreover, they concluded that there exists $a^* \in (0, \sqrt[3]{9\pi^2/16}) \approx (0, 1.77)$, such that the system has limit cycles only when $0 < a < a^*$ and then if the limit cycle exists, is unique and hyperbolic. Later, it was pointed out in [11] that the proof of the uniqueness of the limit cycle had a gap and a new proof was presented.

System (1) has no periodic orbits when $a \leq 0$ because in this case the function x^2+y^2 is a global Lyapunov function. Thus, from now on, we restrict our attention to the case $a > 0$ and for convenience we write $a = b^2$, with $b > 0$. That is, we consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (b^2 - x^2)(y + y^3), \quad b \in \mathbb{R}^+ \cup \{0\}. \end{cases} \tag{2}$$

Therefore the above family has limit cycles if and only if $b \in (0, b^*)$ with $b^* = \sqrt{a^*}$ and $b^* \in (0, \sqrt[6]{9\pi^2/16}) \approx (0, 1.33)$. Following [24] we also know that the value $b = 0$ corresponds to a Hopf bifurcation and the value b^* to the disappearance of the limit cycle in an unbounded polycycle. By using numerical methods it is not difficult to approach the value b^*. Nevertheless, as far as we know there are no analytical tools to obtain the value b^*. This is the main goal of this paper.

We have succeed in finding an interval of length 0.027 containing b^* and during our study we have also realized that there was a bifurcation value missed in the previous studies. Our main result is:

Theorem 1.1. *Consider system* (2). *Then there exist two positive numbers* $\hat{b}$ *and* b^* *such that:*

(a) *It has a limit cycle if and only if* $0 < b < b^*$. *Moreover, when it exists, it is unique, hyperbolic and stable.*
(b) *The only bifurcation values of the system are 0,*$\hat{b}$ *and* b^*. *In consequence there are exactly six different global phase portraits on the Poincaré disc, which are the ones showed in Figure 1.*
(c) *It holds that* $0.79 < \hat{b} < b^* < 0.817$.

The phase portraits missed in [24] are (ii) and (iii) of Figure 1.

The key steps in our proof of Theorem 1.1 are the following:

- Give analytic asymptotic expansions of the separatrices of the critical points at infinity, see Section 2.
- Use these expansions to construct explicit piecewise rational curves, and prove that they are without contact for the flow given by (2). These curves allow to control the global relative positions of the separatrices of the infinite critical points, see Section 5.
- Provide an alternative proof of the uniqueness and hyperbolicity of the limit cycle, which is based in the construction of an explicit rational Dulac function, see Section 4.

By solving numerically the differential equations we can approach the bifurcation values given in the theorem, see Remark 2.6. We have obtained that $\hat{b} \approx 0.8058459066$, $b^* \approx 0.8062901027$ and then $b^* - \hat{b} \approx 0.000444$. As we have said

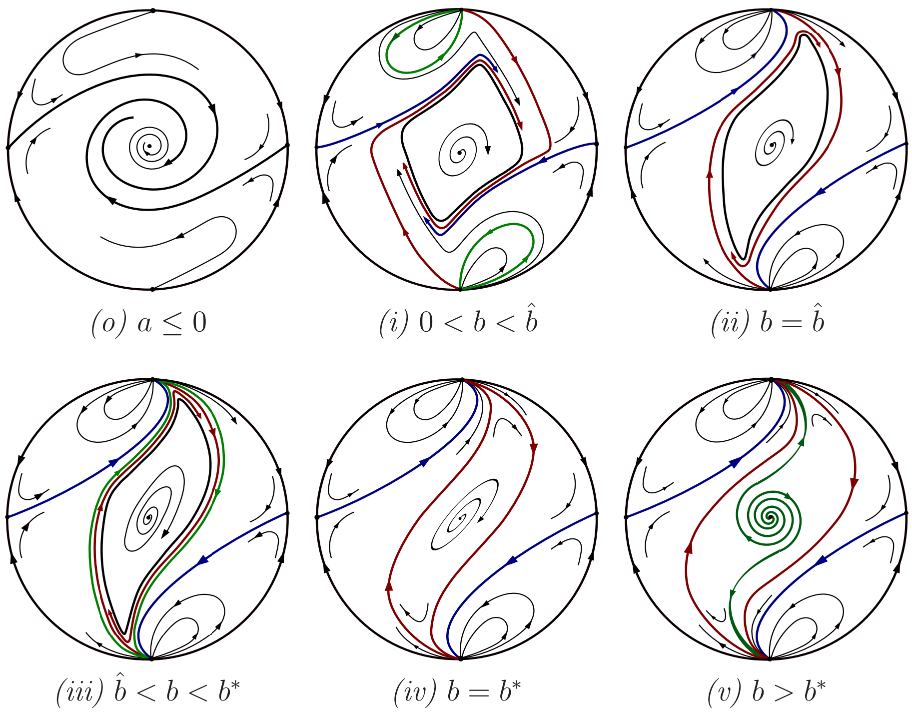

FIGURE 1. Phase portraits of systems (1) and (2). When $a \geq 0$, then $b = \sqrt{a}$.

the main goal of this paper is to get an analytic approach to the more relevant value b^*, because it corresponds to the disappearance of the limit cycle.

Although all our efforts have been focused on system (2), the tools that we introduce in this work can be applied to other families of polynomial vector fields and they can provide an analytic control of the bifurcation values for these families.

As we will see, our approach is not totally algorithmic and following it we do not know how to improve the interval presented in Theorem 1.1 for the values $\hat{b}$ and b^* .

One of the main computational difficulties that we have found has been to prove that certain polynomials in x, y and b, with high degree, do not vanish on some given regions. To treat this question, in Appendix II we propose a general method that uses the so called double discriminant and that we believe that can be useful in other settings, see for instance [1, 22]. In our context this discriminant turns out to be a huge polynomial in b^2 with rational coefficients. In particular we need to control, on a given interval with rational extremes, how many reals roots has a polynomial of degree 965, with enormous rational coefficients. Although Sturm algorithm theoretically works, in practical our computers can not deal with this problem using it. Fortunately we can utilize a kind of bisection procedure based on the Descartes rule ([12]) to overcome this difficulty, see Appendix I.

2. Structure at infinity

As usual, for studying the behavior of the solutions at infinity of system (2) we use the Poincaré compactification. That is, we will use the transformations

$(x, y) = (1/z, u/z)$ and $(x, y) = (v/z, 1/z)$, with a suitable change of time to transform system (2) into two new polynomial systems, one in the (u, z)-plane and another one in the (v, z)-plane respectively (see [2] for details). Then, for understanding the behavior of the solutions of (2) near infinity we will study the structure of the critical points of the transformed systems which are localized on the line $z = 0$. Recall that these points are the *critical points at infinity* of system (2) and their separatrices play a key role for knowing the bifurcation diagram of the system. In fact, it follows from the works of Markus [16] and Newmann [17] that it suffices to know the behavior of these separatrices, the type of finite critical points and the number and type of periodic orbits to know the phase portraits of the system. We obtain the following result:

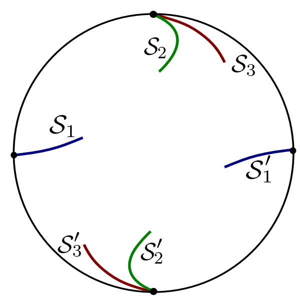

FIGURE 2. Separatrices at infinity for system (2).

Theorem 2.1. *System* (2) *has six separatrices at infinity, which we denote by* $\mathcal{S}_1, \mathcal{S}_2$, $\mathcal{S}_3, \mathcal{S}_1'$, $\mathcal{S}_2'$ *and* $\mathcal{S}_3'$, *see Figure 2. Moreover:*

(*i*) *Each* $\mathcal{S}_k'$ *is the image of* $\mathcal{S}_k$ *under the transformation* $(x, y) \to (-x, -y)$.

(*ii*) *The separatrices* $\mathcal{S}_2$ *and* $\mathcal{S}_3$ *near infinity are contained in the curve* $\{y - \phi(x) = 0\}$ *where* $\phi(x) = \tilde{\phi}(x - b)/(x - b)^2$, $\tilde{\phi}(u)$ *is an analytic function at the origin that satisfies*

$$\tilde{\phi}(u) = \frac{1}{b} - \frac{1}{3b^2}u + \frac{1}{9b^3}u^2 - \frac{359}{27b^4}u^3 + O(u^4). \quad (3)$$

In particular, $\mathcal{S}_2$ *corresponds to* $x \lesssim b$ *and* $\mathcal{S}_3$ *to* $x \gtrsim b$.

(*iii*) *The separatrix* $\mathcal{S}_1$ *near infinity is contained in the curve* $\{y - \varphi(x) = 0\}$ *where* $\varphi(x) = \tilde{\varphi}(1/x)$ *and* $\tilde{\varphi}$ *is an analytic function at the origin that satisfies*

$$\tilde{\varphi}(u) = -u - (b^2 - 1)u^3 - (b^4 - 3b^2 + 2)u^5 + O\left(u^7\right). \quad (4)$$

Remark 2.2. *In the statements* (*ii*) *and* (*iii*) *of Theorem 2.1 the Taylor expansions of the functions* $\tilde{\phi}$ *and* $\tilde{\varphi}$ *can be obtained up to any given order. In fact, in Section 5 we will use the approximation of* $\tilde{\phi}$ *until order 16.*

As a consequence of the above theorem we have the following result:

Corollary 2.3. *All the possible relative positions of the separatrices of system* (2) *in the Poincaré disc are given in Figure 3.*

To prove the above theorem we need some preliminary lemmas.

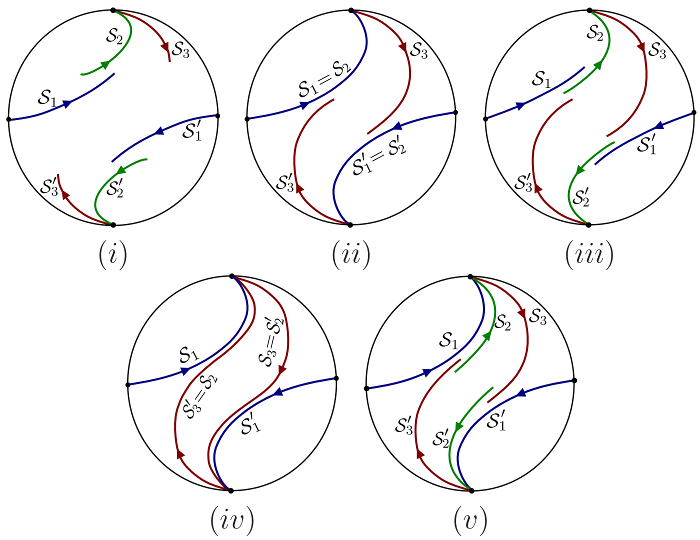

FIGURE 3. Relative position of the separatrices of system (2).

Lemma 2.4. *By using the transformation* $(x, y) = (1/z, u/z)$ *and the change of time* $dt/d\tau = 1/z^4$ *system* (2) *is transformed into the system*

$$\begin{cases} u' = -(1+u^2)z^4 - u(1-b^2z^2)(u^2+z^2), \\ z' = -uz^5, \end{cases} \tag{5}$$

where the prime denotes the derivative respect to τ. *The origin is the unique critical point of* (5) *and it is a saddle. Moreover the stable manifold is the u-axis, the unstable manifold,* $\mathcal{S}_1$, *is locally contained in the curve* $\{u - \psi(z) = 0\}$, *where* $\psi(z)$ *is an analytic function at the origin that satisfies*

$$\psi(z) = -z^2 - (b^2-1)z^4 - (b^4 - 3b^2 + 2)z^6 + O(z^8), \tag{6}$$

see Figure 4.

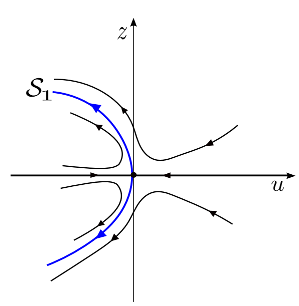

FIGURE 4. Phase portrait of system (5).

Proof. From the expression of (5) it is clear that the origin is its unique critical point. For determining its structure we will use the directional blow-up since the linear part of the system at this point vanishes identically.

The u-directional blow-up is given by the transformation $u = u$, $q = z/u$; and by using the change of time $dt/d\tau = u^2$, system (5) becomes

$$\begin{cases} \dot{u} = -u - (1 - b^2u^2)uq^2 - (1 - b^2u)u^2q^4 - u^4q^4, \\ \dot{q} = q + (1 - b^2u^2)q^3 + (1 - b^2u)uq^5. \end{cases} \tag{7}$$

This system has a unique critical point at origin and it is a saddle with eigenvalues ± 1.

The z-directional blow-up is given by the transformation $r = u/z$, $z = z$. Doing the change of time $dt/d\tau = -z^2$, system (5) becomes

$$\begin{cases} \dot{r} = z + (1 - b^2z^2)(r + r^3), \\ \dot{z} = rz^4. \end{cases} \tag{8}$$

This system has a unique critical point at the origin which is semi-hyperbolic. We will use the results of [2, Theorem 65] to determine its type. By applying the linear change of variables $r = -\xi + \eta$, $z = \xi$ system (8) is transformed into

$$\begin{cases} \dot{\xi} = (\eta - \xi)\xi^4, \\ \dot{\eta} = \eta - N(\xi, \eta), \end{cases}$$

where $N(\xi, \eta) = (\eta - \xi)(b^2\xi^2 - \xi^4) + (\eta - \xi)^3(b^2\xi^2 - 1)$. It is easy to see that if $\eta = n(\xi)$ is the solution of $\eta - N(\xi, \eta) = 0$ passing for the origin, then $n(\xi) = -(b^2 - 1)\xi^3 - (b^4 - 3b^2 + 2)\xi^5 + O(\xi^7)$. Thus $(n(\xi) - \xi)\xi^4 = -\xi^5 + O(\xi^7)$. Therefore from [2, Theorem 65] we know that the origin is a semi-hyperbolic saddle. Moreover, its stable manifold is the η-axis and its unstable manifold is given by

$$\eta = -(b^2 - 1)\xi^3 - (b^4 - 3b^2 + 2)\xi^5 + O(\xi^7).$$

In the plane (r, z) the local expression of this manifold is

$$r = -z - (b^2 - 1)z^3 - (b^4 - 3b^2 + 2)z^5 + O(z^7).$$

Finally, in the (u, z)-plane the unstable manifold is contained in the curve (6) and from the analysis of phase portraits of systems (7) and (8) we obtain that the local phase portrait of system (5) is the one given in Figure 4. □

Lemma 2.5. *By using the transformation* $(x, y) = (v/z, 1/z)$ *and the change of time* $dt/d\tau = 1/z^4$ *system* (2) *is transformed into the system*

$$\begin{cases} v' = v(1 + z^2)(v^2 - b^2z^2) + (1 + v^2)z^4, \\ z' = z(1 + z^2)(v^2 - b^2z^2) + vz^5, \end{cases} \tag{9}$$

where the prime denotes the derivative respect to τ. *System* (9) *has a unique critical point at the origin and its local phase portrait is the one showed in Figure 5. Moreover, the separatrices* $\mathcal{S}_2$ *and* $\mathcal{S}_3$ *are locally contained in the curve* $\{v - g(U) = 0\}$ *where* $U = z/v - 1/b$ *and* $g(U)$ *is an analytic function at the origin that satisfies*

$$g(U) = b^6U^2 - \frac{10}{3}b^7U^3 + \frac{22}{3}b^8U^4 + O\left(U^5\right). \tag{10}$$

Proof. From the expression of system (9) it is clear that the origin is its unique critical point. As in Lemma 2.4 we will use the directional blow-up technique to determine its structure since the linear part of the system at this point is identically zero.

It is well-known, see [2], that since at the origin $z'v - v'z = -z^5 + O(z^6)$, all the solution, arriving or leaving the origin have to be tangent to $z = 0$. So it

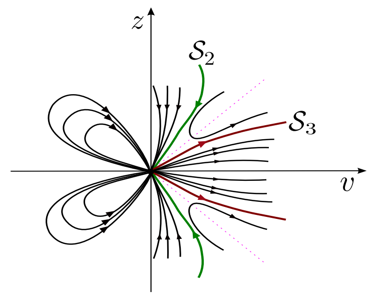

FIGURE 5. Topological local phase portrait of system (9). All the solutions are tangent to the v-axis but for aesthetical reasons this fact is not showed in the figure.

suffices to consider the v-directional blow-up given by the transformation $v = v$, $s = z/v$. Performing it, together with the change of time $dt/d\tau = -v^3$, system (9) is transformed into

$$\begin{cases} \dot{v} = -(1+v^2s^2)(1-b^2s^2) - vs^4(1+v^2), \\ \dot{s} = s^5. \end{cases} \tag{11}$$

This system has not critical points. However, by studying the vector field on the s-axis we will obtain relevant information for knowing the phase portrait of system (9). If $s = 0$ then $\dot{v} = -1$ and $\dot{s} = 0$, that is, the v axis is invariant. If $v = 0$ then $\dot{v} = -1+b^2s^2$ and $\dot{s} = s^5$, this implies that $\dot{v} = 0$ if $s = \pm 1/b$. In addition, a simple computation shows that $\ddot{v} > 0$ at the points $(0, \pm 1/b)$. Therefore the solutions through these points are as it is showed in Figure 6.(a), and by the continuity of solutions with respect to initial conditions, we have that the phase portrait of system (9), close to these points, is as it is showed in Figure 6.(b).

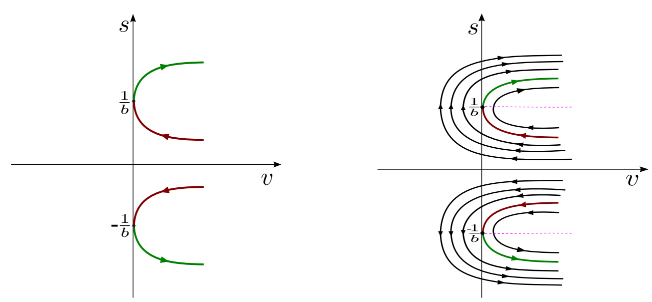

FIGURE 6. Local phase portrait of system (11).

Then by using the transformation $(v, z) = (v, sv)$ and the phase portrait showed in Figure 6.(b) we can obtain the phase portrait of system (9). Recall that the mapping swaps the second and the third quadrants in the v-directional blow-up. In addition, taking into account the change of time $dt/d\tau = -v^3$ it follows that the vector field in the first and fourth quadrant of the plane (v, z) has the opposite direction to the showed in the (v, s)-plane. Therefore the local phase portrait of (9) is the showed in Figure 5.

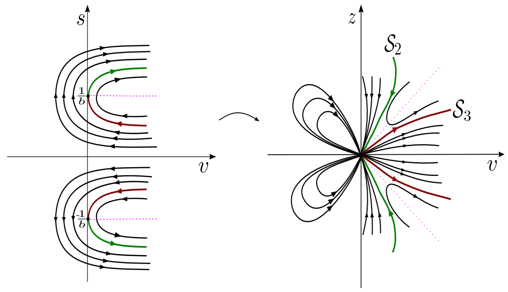

FIGURE 7. Transformation between system (11) and system (9).

To show that the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3$ are contained in the curve (10) we proceed as follows. First, we will obtain the curve that contains the solution through the point $(0, 1/b)$ in the plane (v, s). Second, by using the transformation $(v, z) = (v, sv)$ we will obtain the corresponding curve in the (v, z)-plane and we will show that such curve is exactly the curve given by (10).

Since $\dot{s}$ is positive in $(0, \infty)$, the solution through the point $(0, 1/b)$ (respectively $(0, -1/b)$) is contained in the curve $\{v - g(s) = 0\}$ (respectively $\{v - \tilde{g}(s) = 0\}$), where $g(s)$ (respectively $\tilde{g}(s)$) is an analytical function defined in an open neighborhood of the point, moreover it is clear that $g(1/b) = 0$ and $g'(1/b) = 0$. Consider the Taylor series of $g(s)$ around $(1/b)$:

$$g(s) = \sum_{i=2}^{\infty} \frac{g^{(i)}\left(\frac{1}{b}\right)}{i!}\left(s - \frac{1}{b}\right)^i. \tag{12}$$

Since the curve $\{v - g(s) = 0\}$ is invariant then $\langle \nabla(v - g(s)), \widetilde{X}\rangle = 0$ at all the points of $\{v - g(s) = 0\}$, where $\widetilde{X}$ is the vector field associated to system (11). Thus, we have a function, $\langle \nabla(v - g(s)), \widetilde{X}\rangle$, for which all its coefficients have to be zero. From this observation we obtain linear recurrent equations in the coefficients, $g^{(i)}(1/b)$ of $g(s)$. Simple computations show that the first 3 terms of the Taylor series of $g(s)$ are:

$$b^6\left(s - \tfrac{1}{b}\right)^2 - \tfrac{10}{3}b^7\left(s - \tfrac{1}{b}\right)^3 + \tfrac{22}{3}b^8\left(s - \tfrac{1}{b}\right)^4.$$

Thus, in the plane (v, z), the curve corresponding to $\{v - g(s) = 0\}$ is

$$\left\{v - b^6\left(\tfrac{z}{v} - \tfrac{1}{b}\right)^2 + \tfrac{10}{3}b^7\left(\tfrac{z}{v} - \tfrac{1}{b}\right)^3 - \tfrac{22}{3}b^8\left(\tfrac{z}{v} - \tfrac{1}{b}\right)^4 + O\left(\left(\tfrac{z}{v} - \tfrac{1}{b}\right)^5\right) = 0\right\}.$$

Finally, if $U = z/v - 1/b$, we obtain (10). □

Remark 2.6. *The proof of the above lemma gives a natural way for finding a numerical approximation of the value* b^*. *Notice that in the coordinates* (v, s) *the point* $(0, 1/b)$ *corresponds to both separatrices* $\mathcal{S}_2$ *and* $\mathcal{S}_3$. *Since it is a regular point we can start our numerical method (we use a Taylor method) without initial errors and then follow the flow of the system, both forward and backward for given fixed times, say* $t^+ > 0$ *and* $t^- < 0$. *We arrive to the points* $(v^\pm, s^\pm)$ *with* $s^\pm \neq 0$ *for* $t = t^\pm$, *respectively. These two points have associated two different points* $(x^\pm, y^\pm)$ *in the plane* (x, y), *because of the transformation* $(v, s) = (x/y, 1/x)$. *Now, we integrate numerically the system (2) with initial conditions* $(x^\pm, y^\pm)$ *to continue* *obtaining approximations of the separatrices* $\mathcal{S}_2$ *and* $\mathcal{S}_3$*, respectively. The next step is to compare the points of intersection* $\tilde{x}^+ = \tilde{x}^+(b) < 0$ *and* $\tilde{x}^- = \tilde{x}^-(b) > 0$ *of these approximations with the* x*-axis.*

We consider the function $b \to \Pi(b) := x^+(b) + \tilde{x}^-(b)$ *and we use the bisection method to find one approximate zero of* Π*. Note that if* $\Pi(\bar{b}) = 0$ *then* $\mathcal{S}_2' = \mathcal{S}_3$ *and by the symmetry of the system* $\mathcal{S}_3' = \mathcal{S}_2$*, and therefore* $b^* = \bar{b}$*. Taking* $b_0 = 0.8062901027$*,* $t^+ = 0.05$ *and* $t^- = -0.5$ *we obtain that* $\tilde{x}^+(b_0) + \tilde{x}^-(b_0) \approx -4.58036036 \times 10^{-11}$ *and so* $b^* \approx b_0$*.*

Following a similar procedure, but now using Lemma 2.4 to have an initial condition almost on $\mathcal{S}_1$*, we get that* $\hat{b} \approx 0.8058459066$*.*

Proof of Theorem 2.1. (i) The result follows because system (2) is invariant by the transformation $(x, y) \to (-x, -y)$.

(ii). From (10) and by using the change of variables $(v, z) = (x/y, 1/y)$ we obtain that the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3$ are contained in the curve

$$\left\{ \frac{x}{y} - b^6 \left(\frac{1}{x} - \frac{1}{b} \right)^2 + \frac{10}{3} b^7 \left(\frac{1}{x} - \frac{1}{b} \right)^3 - \frac{22}{3} b^8 \left(\frac{1}{x} - \frac{1}{b} \right)^4 + O\left(\left(\frac{1}{x} - \frac{1}{b} \right)^5 \right) = 0 \right\},$$

or equivalently

$$\left\{ y - \overline{\phi}(x) = 0 \right\}, \tag{13}$$

where

$$\overline{\phi}(x) = \frac{x}{b^6 \left(\frac{1}{x} - \frac{1}{b} \right)^2 - \frac{10}{3} b^7 \left(\frac{1}{x} - \frac{1}{b} \right)^3 + \frac{22}{3} b^8 \left(\frac{1}{x} - \frac{1}{b} \right)^4 + O\left(\left(\frac{1}{x} - \frac{1}{b} \right)^5 \right)}.$$

We can write the function $\overline{\phi}(x)$ as

$$\overline{\phi}(x) = \left(\frac{1}{(x-b)^2} \right) \overline{\phi}_1(x), \tag{14}$$

where

$$\overline{\phi}_1(x) = \frac{b^2 x^3}{b^6 + \frac{10}{3} b^7 \left(\frac{x-b}{bx} \right) + \frac{22}{3} b^8 \left(\frac{x-b}{bx} \right)^2 + O\left(\left(\frac{x-b}{bx} \right)^3 \right)}.$$

The function $\overline{\phi}_1(x)$ is analytical at $x = b$ and it is not difficult to see that it has the following Taylor expansion

$$\overline{\phi}_1(x) = \frac{1}{b} - \frac{(x-b)}{3b^2} + \frac{(x-b)^2}{9b^3} - \frac{359(x-b)^3}{27b^4} + O((x-b)^4).$$

Then (14) can be written as

$$\overline{\phi}(x) = \frac{1}{b(x-b)^2} - \frac{1}{3b^2(x-b)} + \frac{1}{9b^3} - \frac{359}{27b^4}(x-b) + O((x-b)^2).$$

Hence from (13) and taking $\overline{\phi}(x) = \tilde{\phi}(x-b)/(x-b)^2$ we complete the proof.

The proof of (iii) follows by applying the previous ideas, considering the expression given by (6) and the change of variables $(u, z) = (y/x, 1/x)$. □

3. Proof of Theorem 1.1

We start proving a preliminary result that is a consequence of some general properties of semi-complete family of rotated vector fields with respect one parameter, SCFRVF for short, see [7, 18].

Proposition 3.1. *Consider system* (2) *and assume that for* $b = \bar{b} > 0$ *it has no limit cycles. Then there exists* $0 < b^* \leq \bar{b}$ *such that the system has limit cycles if and only if* $b \in (0, b^*)$. *Moreover, for* $b = b^*$ *its phase portrait is like (iv) in Theorem 1.1 and when* $b > b^*$ *it is like (v) in Theorem 1.1.*

Proof. It is easy to see that the system has a limit cycle for $b \gtrsim 0$, which appears from the origin through an Andronov-Hopf bifurcation.

If we denote by $X_b(x,y) = (P_b(x,y), Q_b(x,y))$ the vector field associated to (2) then

$$\frac{\partial}{\partial b^2} \arctan\left(\frac{Q_b(x,y)}{P_b(x,y)}\right) = \frac{P_b(x,y)\frac{\partial Q_b(x,y)}{\partial b^2} - Q_b(x,y)\frac{\partial P_b(x,y)}{\partial b^2}}{P_b^2(x,y) + Q_b^2(x,y)}$$

$$= \frac{y^2(1+y^2)}{P_b^2(x,y) + Q_b^2(x,y)} \geq 0.$$

This means that system (2) is a SCFRVF with respect to the parameter b^2.

We will recall two properties of SCFRVF. The first one is the so called *non-intersection property.* It asserts that if γ_1 and γ_2 are limit cycles corresponding to different values of b, then $\gamma_1 \cap \gamma_2 = \emptyset$.

The second one is called *planar termination principle*: [19, 20] if varying the parameter we follow with continuity a limit cycle generated from a critical point $\mathbf{p}$, we get that the union of all the limit cycles covers a 1-connected open set $\mathcal{U}$, whose boundaries are $\mathbf{p}$ and a cycle of separatrices of X_b. The corners of this cycle of separatrices are finite or infinite critical points of X_b. Since in our case X_b only has the origin as a finite critical point we get that $\mathcal{U}$ has to be unbounded. Notice that in this definition, when a limit cycle goes to a semistable limit cycle then we continue the other limit cycle that has collided with it. This limit cycle has to exist, again by the properties of SCFRVF.

If for some value of $b = \bar{b} > 0$ the system has no limit cycle it means that the limit cycle starting at the origin for $b = 0$, has disappeared for some b^*, $0 < b^* \leq \bar{b}$ covering the whole set $\mathcal{U}$. Since $\mathcal{U}$ fills from the origin until infinity, from the non intersection property, the limit cycle cannot either exist for $b \geq b^*$, as we wanted to prove.

Since for $b > 0$ the origin is a repellor, by Corollary 2.3 we know by the Poincaré-Bendixson Theorem that the phase portraits (i),(ii) and (iii) in Figure 1 have at least one limit cycle. Then, the phase portraits for $b \geq b^*$ have to be like (iv) or (v) in the same figure. Since the phase portrait (iv) is the only one having a cycle of separatrices it corresponds to $b = b^*$. Again by the properties of SCFRVF, the phase portrait (iv) does not appear again for $b > b^*$. Hence, for $b > b^*$ the phase portrait has to be like (v) and the proposition follows. □

Remark 3.2. *In Lemma 4.3 we will give a simple proof that when* $b = 1$ *system* (2) *has no limit cycles, based on the fact that for this value of the parameter it has the hyperbola* $xy + 1 = 0$ *invariant by the flow. From the above proposition it follows* *that $b^* < 1$. This result already improves the upper bound of b^*, given in* [24], $\sqrt[6]{9\pi^2/16} \approx 1.33$. *Theorem 1.1 improves again this upper bound, but as we will see, the proof is much more involved.*

Proof of Theorem 1.1. Recall that for $a \leq 0$ the function $V(x,y) = x^2 + y^2$ is a global Lyapunov function for system (1) and therefore the origin is global asymptotically stable. Then it is easy to see that its phase portrait is like (o) in Figure 1.

To prove the theorem we list some of the key points that we will use and that will be proved in the forthcoming sections:

($\mathbf{R}_1$) System (2) has at most one limit cycle for $b \in (0, 0.817]$ and when it exists it is hyperbolic and attractor, see Section 4.

($\mathbf{R}_2$) System (2) has an odd number of limit cycles, with multiplicities taken into account, when $b \leq 0.79$ and the configuration of its separatrices is like (i) in Figure 3, see Proposition 5.1.

($\mathbf{R}_3$) System (2) has an even number of limit cycles, with multiplicities taken into account, when $b = 0.817$ and the configuration of its separatrices is like (v) in Figure 3, see again Proposition 5.1.

The theorem for $b \geq b^*$ is a consequence of Proposition 3.1. Notice that again by this proposition and ($\mathbf{R}_3$), $b^* < 0.817$. Hence, the limit cycles can exist only when $b \in (0, b^*) \subset (0, 0.817]$ and by ($\mathbf{R}_1$) when they exist then there is only one and it is hyperbolic and attractor.

As a consequence of ($\mathbf{R}_2$) and the uniqueness and hyperbolicity of the limit cycle we have that the phase portrait for $b \leq 0.79$ is like (i) in Figure 1.

To study the phase portraits for the remaining values of b, that is $b \in (0.79, b^*)$, first notice that all of them have exactly one limit cycle, which is hyperbolic and stable. So it only remains to know the behavior of the infinite separatrices. We denote by $x_2(b)$ and $x_3'(b)$ the points of intersection of the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3'$ of system (2) with the x-axis (when they exist), see also the forthcoming Figure 13. Notice that for $b > b^*$, $x_3'(b) < x_2(b) < 0$ and $x_3'(b^*) = x_2(b^*) < 0$. The properties of the SCFRVF imply that $x_2(b)$ is monotonous increasing and that $x_3'(b)$ is monotonous decreasing. Hence for $b \lesssim b^*$ the phase portrait of the system is like (iii) in Figure 1. Since we already know that for $b = 0.79$ the phase portrait is like (i), it should exists at least one value, say $b = \hat{b}$, with phase portrait (ii). Since for SCFRVF the solution for a given value of b, say $b = \bar{b}$, becomes a curve without contact for the system when $b \neq \bar{b}$, we have that the phase portraits corresponding to heteroclinic orbits, that is (ii) and (iv) of Figure 1, only appear for a single value of b (in this case $\hat{b}$ and b^*, respectively). Therefore, the theorem follows. □

4. Uniqueness of the limit cycle for $b \leq 817/1000$

In this section we will prove the uniqueness of the limit cycle of system (2) when $b \leq 0.817$. The idea of the proof is to find a suitable rational Dulac function for applying the following generalization of Bendixson–Dulac criterion.

Proposition 4.1. *Consider the C^1-differential system*

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \tag{15}$$

and let $\mathcal{U} \subset \mathbb{R}^2$ *be an open region with boundary formed by finitely many algebraic curves. Assume that:*

(I) There exists a rational function $V(x,y)$ *such that*

$$M := \frac{\partial V}{\partial x}P + \frac{\partial V}{\partial y}Q - V\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \tag{16}$$

does not change sign on $\mathcal{U}$*. Moreover* M *only vanishes on points, or curves that are not invariant by the flow of* (15)*.*

(II) All the connected components of $\mathcal{U} \setminus \{V = 0\}$*, except perhaps one, say* $\widetilde{\mathcal{U}}$*, are simple connected. The component* $\widetilde{\mathcal{U}}$*, if exists, is 1-connected.*

Then the system has at most one limit cycle in $\mathcal{U}$ *and when it exists is hyperbolic and it is contained in* $\widetilde{\mathcal{U}}$*. Moreover its stability is given by the sign of* $-VM$ *on* $\widetilde{\mathcal{U}}$*.*

The above statement is a simplified version of the one given in [9] adapted to our interests. Similar results can be seen in [4, 10, 14, 25].

Remark 4.2. *Looking at the proof of Proposition 4.1 we also know that:*

(i) The Dulac function used in the proof is $1/V$*.*

(ii) In the region $\mathcal{U}$*, the curve* $\{V(x,y) = 0\}$ *is without contact for the flow of* (15)*. In particular, by the Bendixson-Poincaré Theorem, the ovals of the set* $\{V(x,y) = 0\}$ *must surround some of the critical points of the vector field.*

To give an idea of how we have found the function V that we will use in our proof we will first study the van der Pol system and then the uniqueness in our system when $b \leq 0.615$. Although we will not use these two results, we believe that to start studying them helps to a better understanding of our approach.

4.1. **The van der Pol system.** Consider the Van der Pol system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (b^2 - x^2)y. \end{cases} \tag{17}$$

Due to the expression of the above family of differential equations, in order to apply Proposition 4.1, it is natural to start considering functions of the form

$$V(x,y) = f_2 y^2 + f_1(x)y + f_0(x).$$

For this type of functions, the corresponding M is a polynomial of degree 2 in y, with coefficients being functions of x. In particular the coefficient of y^2 is

$$f_1'(x) + f_2(b^2 - x^2).$$

Taking $f_1(x) = (x^2 - 3b^2)f_2 x/3$ we get that it vanishes. Next, fixing $f_2 = 6$, and imposing to the coefficient of y to be zero we obtain that $f_0(x) = 6x^2 + c$, for any constant c. Finally, taking $c = b^2(3b^2 - 4)$, we arrive to

$$V_b(x,y) = 6y^2 + 2(x^2 - 3b^2)xy + 6x^2 + b^2(3b^2 - 4). \tag{18}$$

From (16) of Proposition 4.1, the corresponding M, which only depends on x, is

$$M_b(x,y) = 4x^4 + b^2(3b^2 - 4)(x^2 - b^2).$$

It is easy to see that for $b \in (0, 2/\sqrt{3}) \approx (0, 1.15)$, $M_b(x,y) > 0$. Notice that $V_b(x,y) = 0$ is quadratic in y and so is not difficult to see that it has at most one oval, see Figure 8 for $b = 1$. Then we can apply Proposition 4.1 to prove the uniqueness and hyperbolicity of the limit cycle for these values of b.

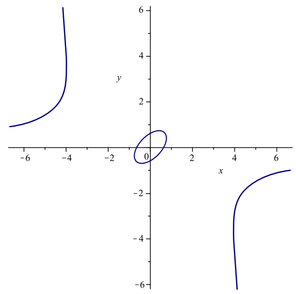

FIGURE 8. The algebraic curve $V_b(x, y) = 0$ with $b = 1$.

We remark that taking a more suitable polynomial Dulac function, it is possible to prove the uniqueness of the limit cycle for all values of b, see [5, p. 105]. We have only included this explanation as a first step towards the construction of a suitable rational Dulac function for our system (2).

4.2. **System** (2) **with** $b \le 651/1000$**.** By making some modifications to the function V_b given by (18), we get an appropriate function for system (2). Consider

$$V_b(x, y) = [2x^3 + 6b^2(1 - b^2)x]y^3 + 6(1 - b^2)y^2 + 2(x^2 - 3b^2)xy + 6(1 - b^2)x^2 + b^2(3b^2 - 4).$$

Computing the double discriminant $\triangle^2(V_b)$ of the function V_b, introduced in Appendix II, we get that

$$\triangle^2(V_b) = b^2(3b^2 - 4)(b^2 - 1)^{15}(P_{19}(b^2))^2,$$

where P_{19} is a polynomial of degree 19. By using for instance the Sturm method, we prove that the smallest positive root of $\triangle^2(V_b)$ is greater than 0.85. Therefore by Proposition 5.7 we know that for $b \in (0, 0.85]$ the algebraic curve $V_b(x, y) = 0$ has no singular points and therefore the set $\{V_b(x, y) = 0\} \subset \mathbb{R}^2$ is a finite disjoint union of ovals and smooth curves diffeomorphic to open intervals.

By applying Proposition 4.1 to system (2) with $V = V_b$, we get that

$$\begin{aligned} M_b(x, y) = {} & 6[(2 - 3b^2)x^4y^2 - 2b^2(2 - b^2)x^3y^3 + (2 - b^2)x^2y^4] + 2(2 - 3b^2)x^4 \\ & -3b^2(14 - 15b^2)x^2y^2 + 12b^4(2 - b^2)xy^3 - b^2(4 - 9b^2)x^2 \\ & +3b^4(2 - 3b^2)y^2 + b^4(4 - 3b^2). \end{aligned} \quad (19)$$

In Subsection 5.5 of Appendix II we prove that M_b does not vanish on $\mathbb{R}^2$ for $b \in (0, 0.651]$. Then by Remark 4.2 all the ovals of $\{V_b(x, y) = 0\}$ must surround the origin, which is the unique critical point of the system. Since the straight line $x = 0$ has at most two points on the algebraic curve $V_b(x, y) = 0$, it can have at most one closed oval surrounding the origin. Then by Proposition 4.1 it follows the uniqueness, stability and hyperbolicity of the limit cycle of system (2) for these values of the parameter b.

4.3. **System (2) with $b \le 817/1000$.** The hyperbola $xy+1=0$ will play an important role in the study of this case. We first prove a preliminary result.

Lemma 4.3. *Consider system (2).*

(I) For $b \neq 1$ the hyperbola $xy+1=0$ is without contact for its flow. In particular its periodic orbits never cut it.

(II) For $b=1$ the hyperbola $xy+1=0$ is invariant for its flow and the system has not periodic orbits.

Proof. Define $F(x,y)=xy+1$ and set $X=(P,Q):=(y,-x+(b^2-x^2)(y+y^3))$. Simple computations give that for $x\neq 0$,

$$(F_xP+F_yQ)|_{y=-1/x}=\frac{1+x^2}{x^2}\left(1-b^2\right).$$

Therefore (I) follows and we have also proved that when $b=1$, the hyperbola is invariant by the flow.

(II) When $b=1$,

$$F_xP+F_yQ=KF, \tag{20}$$

where $K=K(x,y)=y^2-x^2-xy(xy-1)$ is the so called *cofactor* of the invariant curve $F=0$.

Let us prove that the system has no limit cycle. Recall that the origin is repeller. Therefore if we prove that any periodic orbit Γ of the system is also repeller we will have proved that there is no limit cycle.

This will follow if we show that

$$\int_0^T \operatorname{div}(X)(\gamma(t))\,dt>0, \tag{21}$$

where $\gamma(t):=(x(t),y(t))$ is the time parametrization of Γ and $T=T(\Gamma)$ its period.

To prove (21) notice that the divergence of X can be written as $\operatorname{div}(X)=3K+2x^2+1-3xy$. Then,

$$\int_0^T \operatorname{div}(X)(\gamma(t))\,dt=3\int_0^T K(x(t),y(t))dt+\int_0^T(2x(t)^2+1)dt-3\int_0^T x(t)y(t)dt.$$

Observe that from (20) we have that

$$\begin{aligned}\int_0^T K(x(t),y(t))\,dt &= \int_0^T \frac{F_x(x(t),y(t))\dot{x}+F_y(x(t),y(t))\dot{y}}{F(x(t),y(t))}dt\\ &= \int_0^T \frac{d}{dt}\ln|F(x(t),y(t))|dt=\ln|F(x(t),y(t))|\Big|_0^T=0\end{aligned}$$

and that

$$\int_0^T x(t)y(t)dt=\int_0^T x(t)\dot{x}(t)dt=\frac{x^2(t)}{2}\Big|_0^T=0.$$

Therefore

$$\int_0^T \operatorname{div}(X)(\gamma(t))\,dt=\int_0^T(2x(t)^2+1)dt>0,$$

as we wanted to see. □

Theorem 4.4. *System (2) for $b\in(0,0.817]$ has at most one limit cycle. Moreover when it exists it is hyperbolic and attractor.*

Proof. Based on the function V_b used in the Subsection 4.2 we consider the function $V_b(x,y) = \widehat{V}_b(x,y)/(5+6b^{18}x^2)$, where

$$\begin{aligned} \widehat{V}_b(x,y) \;=\; & \tfrac{1}{2}\,b^{18}x^6 + \tfrac{1}{2}\,b^{18}x^4y^2 + \left(1+\tfrac{1}{2}\,b^{12}\right)x^3y^3 + \left(1+\tfrac{3}{2}\,b^2\right)x^3y \\ & -\left(\tfrac{3}{5}\,b^{10} + \tfrac{5}{3}\,b^{14} + 2\,b^{16}\right)x^2y^2 + \left(3\,b^2 - 3\,b^4 + \tfrac{21}{10}\,b^6\right)xy^3 \\ & + \left(3 - 3\,b^2 + 2\,b^4\right)x^2 - b^2\left(3 - \tfrac{1}{10}\,b^4\right)xy + \left(3 - 3\,b^2 + 2\,b^4\right)y^2 \\ & + \tfrac{3}{2}\,b^4 - 2\,b^2. \end{aligned} \tag{22}$$

We have added the non-vanishing denominator to increase a little bit the range of values for which Proposition 4.1 works. Indeed, it can be seen that the above function, but without the denominator, is good for showing that the system has at most one limit cycle for $b \leq 0.811$.

To study the algebraic curve $\widehat{V}_b(x,y) = 0$ we proceed like in the previous subsection. The double discriminant introduced in Appendix II is

$$\triangle^2(\widehat{V}_b) = b^{182}(3b^2-4)(4b^{36}+27b^{24}+108b^{12}+108)(P_{152}(b^2))^2,$$

where P_{152} is a polynomial of degree 152. It can be seen that the smallest positive root of $\triangle^2(\widehat{V}_b)$ is greater than 0.88. Therefore by Proposition 5.7 we know that for $b \in (0, 0.88]$ this algebraic curve has no singular points. Hence the set $\{V_b(x,y) = 0\} \subset \mathbb{R}^2$ is a finite disjoint union of ovals and smooth curves diffeomorphic to open intervals.

The function that we have to study in order to apply Proposition 4.1 is

$$M_b(x,y) = \frac{N_b(x,y)}{30(6b^{18}x^2+5)^2} \tag{23}$$

where $N_b(x,y)$ is given in (33) of Subsection 5.6. The denominator of M_b is positive for all $(x,y) \in \mathbb{R}^2$. By Lemma 4.3 we know that the limit cycles of the system must lay in the open region $\Omega = \mathbb{R}^2 \cap \{xy+1>0\}$. In Subsection 5.6 of Appendix II we will prove that N_b does not change sign on the region Ω and if it vanishes it is only at some isolated points.

Notice also that the set $\{\widehat{V}_b(x,y) = 0\}$ cuts the y-axis at most in two points, therefore by the previous results and arguing as in Subsection 4.2, we know that it has at most one oval and that when it exists it must surround the origin.

Therefore we are under the hypotheses of Proposition 4.1, taking $\mathcal{U} = \Omega$, and the uniqueness and hyperbolicity of the limit cycle follows. □

5. Phase portraits for $b \leq 79/100$ and $b = 817/1000$

This section is devoted to find the relative position of the separatrices of the infinite critical points when $b \leq 0.79$ and when $b = 0.817$. The main tool will be the construction of algebraic curves that are without contact for the flow of system (2). These curves are essentially obtained by using the functions $\phi_i(x) := \tilde{\phi}_i(x-b)/(x-b)^2$ and $\varphi_i(x) := \tilde{\varphi}_i(1/x)$ where $\tilde{\phi}_i$ and $\tilde{\varphi}_i$ are the approximations of order i of the separatrices of the infinite critical points, given in the expressions (3) and (4) of Theorem 2.1, respectively. That is, we use algebraic approximations of $\mathcal{S}_i$ and $\mathcal{S}'_i$, for $i = 1, 2, 3$.

As usual for knowing when a vector field X is without contact with a curve of the form $y = \psi(x)$ we have to control the sign of

$$N_\psi(x) := \langle \nabla(y - \psi(x)), X\rangle\big|_{y=\psi(x)}.$$

In this section we will repeatedly compute this function when $\psi(x)$ is either $\varphi_i(x)$, $\phi_i(x)$ or modifications of these functions.

We prove the following result.

Proposition 5.1. *Consider system* (2)*. Then:*

(I) For $b \le 79/100$ the configuration of its separatrices is like (i) in Figure 3. Moreover it has an odd number of limit cycles, taking into account their multiplicities.

(II) For $b = 817/1000$ the configuration of its separatrices is like (v) in Figure 3. Moreover it has an even number of limit cycles, taking into account their multiplicities.

Proof. (I) Consider the two functions

$$\varphi_1(x) = -\frac{1}{x}, \qquad \text{and} \qquad \varphi_2(x) = -\frac{1}{x} - \frac{(b^2-1)}{x^3},$$

which are the corresponding expressions in the plane (x, y) of the first and second approximation of the separatrix $\mathcal{S}_1$.

If $b < 1$ then $(\varphi_1 - \varphi_2)(x) = (b^2 - 1)/x^3 > 0$ for $x < 0$. This implies that the separatrix $\mathcal{S}_1$ in the (x, y)-plane and close to $-\infty$ is below the graphic of $\varphi_1(x)$. Moreover

$$N_{\varphi_1}(x) = -\frac{(x^2+1)(b^2-1)}{x^3} < 0 \qquad \text{for } x < 0.$$

This inequality implies that the separatrix $\mathcal{S}_1$ in the plane (x, y) cannot intersect the graphic of $\varphi_1(x)$ for $x < 0$, see Figure 9.

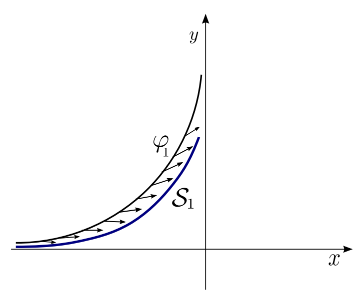

FIGURE 9. Behavior of $\mathcal{S}_1$ for $b < 1$.

Now, we consider the third approximation to the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3$, that is we consider the first three terms in (3). It is given by the graph of the function

$$\phi_3(x) = \frac{(x^2 - 5bx + 13b^2)}{9b^3(x-b)^2}.$$

Let us prove that when $b \in (0, \sqrt{2/3})$, the graphs of $\varphi_1(x)$ and $\phi_3(x)$ intersect at a unique point, (x_0, y_0) with $x_0 < 0$ and $y_0 > 0$. For this is sufficient to show that the function $(\varphi_1 - \phi_3)(x)$ has a unique zero at some $x_0 < 0$.

It is clear that $\lim_{x\to 0^-}(\varphi_1 - \phi_3)(x) = +\infty$ and we have that $(\varphi_1 - \phi_3)(-2b) = (3b^2 - 2)/6b^3$, then for $b < \sqrt{2/3}$, $(\varphi_1 - \phi_3)(-2b) < 0$ hence $(\varphi_1 - \phi_3)(x)$ has a zero at a point x_0 with $-2b < x_0 < 0$. Moreover this zero is unique because the numerator of $(\varphi_1 - \phi_3)(x)$ is a monotonous function.

It also holds that $\nabla(y-\phi_3(x)) = (-\phi_3'(x), 1)$ where $\phi_3'(x) = (-x+7b)/3b^2(-x+b)^3$ is a positive function for $x < 0$, and a simply computation shows that

$$\begin{aligned} N_{\phi_3}(x) &= \tfrac{1}{-729b^9(b-x)^2}\left[(81b^6+1)x^4 + (729b^8 - 405b^6 - 11)bx^3\right. \\ &\quad -9(162b^8 - 108b^6 - 7)b^2x^2 + (729b^8 + 405b^6 - 178)b^3x \\ &\quad \left.-13(81b^6 - 20)b^4\right]. \end{aligned}$$

To control the sign of N_{ϕ_3} we compute the discriminant of its numerator with respect to x. It gives $\mathrm{dis}(N_{\phi_3}(x), x) = b^{12}P_{22}(b^2)$, where P_{22} is a polynomial of degree 22 with integer coefficients.

By using the Sturm method we obtain that $P_{22}(b^2)$ has exactly four real zeros. By Bolzano theorem the positive ones belong to the intervals $(0.7904, 0.7905)$ and $(2.6, 2.7)$.

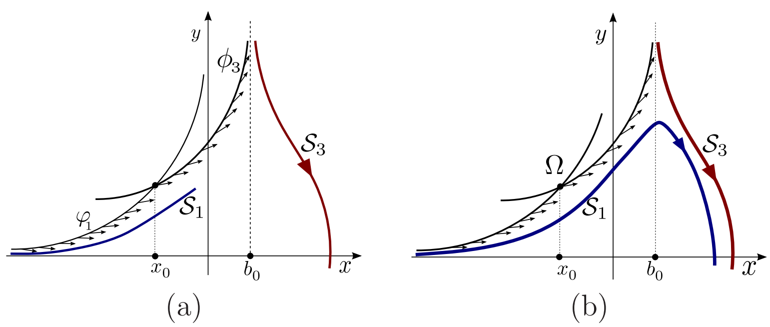

FIGURE 10. Behavior of $\mathcal{S}_1$ and $\mathcal{S}_3$ for $b \le 0.79$

If we fix $b_0 \le 79/100$ then $b_0 < \sqrt{2/3}$ and moreover according to previous paragraph the graphics of $\varphi_1(x)$ and $\phi_3(x)$ intersect at a unique point (x_0, y_0) with $x_0 < 0$ and $y_0 > 0$. Furthermore, $\frac{\partial N_{\phi_3}}{\partial b}(b_0) > 0$ in (x_0, b_0) and $N_{\phi_3} < 0$ in (x_0, b) for all $b \in (0, b_0]$. Therefore the vector field associated to (2) on these curves is the one showed in Figure 10.(a).

From Figure 10.(a) it is clear that the separatrix $\mathcal{S}_1$ cannot intersect the set $\Omega = \{(x, \varphi_1(x)) | -\infty < x \le x_0\} \cup \{(x, \phi_3(x)) | x_0 \le x < b_0\}$. Moreover, since the separatrix $\mathcal{S}_2$ forms an hyperbolic sector together with $\mathcal{S}_3$ we obtain that $\mathcal{S}_1$ cannot be asymptotic to the line $x = b_0$. Hence we must have the situation showed in Figure 10.(b). We know that the origin is a source and from the symmetry of system (2) we conclude that for $b \le 0.79$ the system has an odd number of limit cycles (taking into account multiplicities) and the phase portrait is the one showed in Figure 11.

(II) We start proving the result when $b = b_0 := 89/100$ because the method that we use is the same that for studying the case $b = 817/1000$, but the computations are easier. Recall that we want to prove that the configuration of separatrices is like (v) in Figure 3. That the number of limit cycles must be even (taking into account multiplicities) is then a simply consequence of the Poincaré–Bendixson Theorem, because the origin is a source.

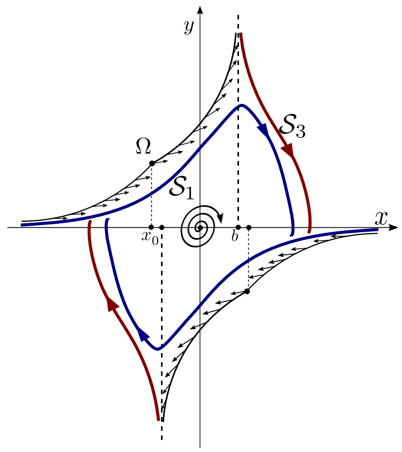

FIGURE 11. For $0 < b \leq 0.79$, system (2) has at least one limit cycle and phase portrait (i) of Figure 1 .

We consider the approximation of eight order to $\mathcal{S}_2$ and $\mathcal{S}_3$ given by the graph of the function $\phi_8(x)$.

By using again the Sturm method it is easy to see that $N_{\phi_8}(x) < 0$ for $x \in (b_0, x_0)$, where $x_0 = 1.924$ is a left approximation to the root of the function $\phi_8(x)$, and $N_{\phi_8}(x) > 0$ for $x \in (x_1, b_0)$, where $x_1 = -2.022$ is a right approximation to the root of the function $N_{\phi_8}(x)$. That is, we have the situation shown in Figure 12.(a). Now, we consider the function $\hat{\phi}_8(x) = \phi_8(x) - 1/(9b^3)$, is clear that $(\phi_8 - \hat{\phi}_8)(x) > 0$. We have $N_{\hat{\phi}_8}(x) > 0$ for $x \in (b_0, x_2)$ where $x_2 = 1.6467$ is a left approximation to the root of the function $\hat{\phi}_8(x)$, moreover the line $x = x_2$ is transversal to the vector field for $y > 0$, thus the separatrix $\mathcal{S}_3$ intersects the x-axis at a point $\bar{x}$ of the interval (x_2, x_0), see again Figure 12.(a).

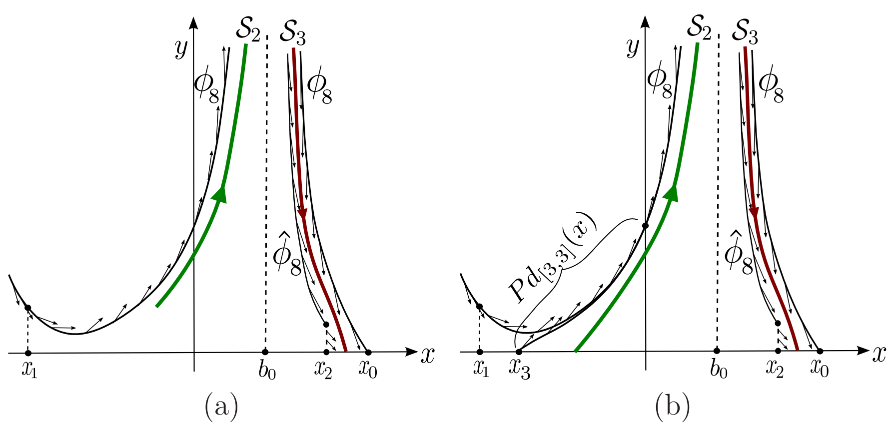

FIGURE 12. Behaviour of $\mathcal{S}_2$ and $\mathcal{S}_3$ for $b \in \{0.817, 0.89\}$.

At this point, the idea is to show that $\mathcal{S}_2$ intersects the x-axis at a point $\hat{x}$, with $-x_2 < \hat{x} < 0$. For proving this, we utilize the Padé approximants method, see [3].

Recall that given a function $f(x)$, its Padé approximant $\mathrm{Pd}_{[n,m]}(f)(x,x_0)$ of order (n,m) at a point x_0, or simply $\mathrm{Pd}_{[n,m]}(f)(x)$ when $x_0=0$, is a rational function of the form $F_n(x)/G_m(x)$, where F_n and G_m are polynomials of degrees n and m, respectively, and such that

$$\left| f(x) - \frac{F_n(x)}{G_m(x)} \right| = O\left((x-x_0)^{n+m+1}\right).$$

Consider the Padé approximant $\mathrm{Pd}_{[3,3]}(\phi_8)$. It satisfies that $\mathrm{Pd}_{[3,3]}(\phi_8)(0)=\phi_8(0)$ and by the Sturm method it can be seen that there exists $x_3<0$ such that $\mathrm{Pd}_{[3,3]}(\phi_8)(x_3)=0$, $\mathrm{Pd}_{[3,3]}(\phi_8)$ is positive and increasing on the interval $(x_3,0)$ and a left approximation to x_3 is -1.595. Moreover it is easy to see that $N_{\mathrm{Pd}_{[3,3]}(\phi_8)}(x)>0$ for $x\in(x_3,0)$. Therefore $\mathcal{S}_2$ cannot intersect neither the graph of $y=\mathrm{Pd}_{[3,3]}(\phi_8)(x)$ in $(x_3,0)$ nor the graph of $\phi_8(x)$ in $[0,b_0)$. Hence $\mathcal{S}_2$ intersects the x-axis in a point $\hat{x}$ contained in the interval $(x_3,0)$. This implies that $-x_2<\hat{x}<0$ as we wanted to see, because $-x_2<x_3$. Hence the behavior of the separatrices is like Figure 12.(b). See also Figure 13.

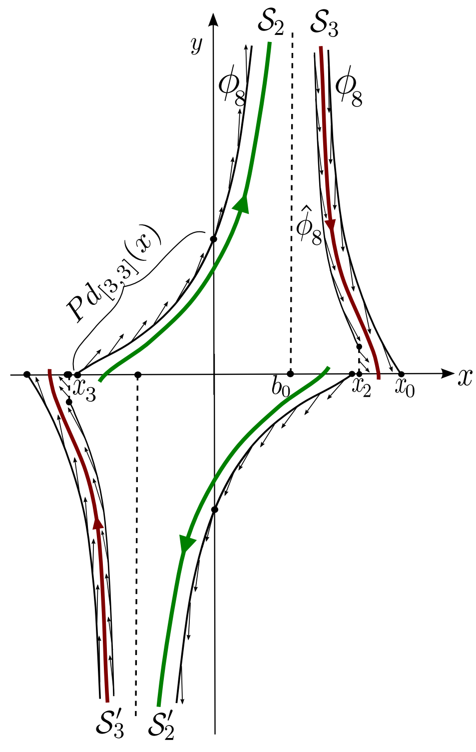

FIGURE 13. Behavior of $\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_2'$ and $\mathcal{S}_3'$ for $b\in\{0.817, 0.89\}$.

When $b_0=817/1000$ we follow the same ideas. For this case we consider the functions $\phi_{16}(x)$ and $\hat{\phi}_{16}(x)=\phi_{16}(x)-1/(9b^3)$. Recall that the graphic of $\phi_{16}(x)$ is the sixteenth order approximation to $\mathcal{S}_2$ and $\mathcal{S}_3$. It is not difficult to prove that $N_{\hat{\phi}_{16}}>0$ on the interval (b_0,x_2), with $x_2=1.6421$ and since the line $x=x_2$ is transversal to X for $y>0$, $\mathcal{S}_3$ intersects the x-axis at a point $\bar{x}>x_2$. Also we have that $N_{\phi_{16}}>0$ on the interval $(-3/100,b_0)$ and using the Padé approximant $\mathrm{Pd}_{[5,1]}(\phi_{16})(x,-3/100)$ we obtain that $\mathcal{S}_2$ intersect to the x-axis in a point $\hat{x} \in (x_3, 0)$ with $x_3 > -1.638$. This implies that $-x_2 < \hat{x} < 0$ as in the case $b = 0.89$. Hence we have the same situation that in Figure 13. $\square$

Remark 5.2. *As it is shown in the proof of Theorem 1.1, the values* 0.79 *and* 0.817*, obtained in the previous proposition, provide a lower and an upper bound for* b^**. We have tried to shrink the interval where* b^* *lies using higher order approximations of the separatrices, but we have not been able to diminish its size.*

Appendix I: The Descartes method

Given a real polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and a real interval $I = (\alpha, \beta)$ such that $P(\alpha)P(\beta) \neq 0$, there are two well-known methods for knowing the number of real roots of P in I: the Descartes rule and the Sturm method.

Theoretically, when all the $a_i \in \mathbb{Q}$ and $\alpha, \beta \in \mathbb{Q}$, the Sturm approach solves completely the problem. If all the roots of P are simple it is possible to associate to it a sequence of $n+1$ polynomials, the so called *Sturm sequence*, and knowing the signs of this sequence evaluated at α and β we obtain the exact number of real roots in the interval. If P has multiple roots it suffices to start with $P/(\gcd(P, P'))$, see [23, Sec. 5.6].

Nevertheless when the rational numbers have big numerators and denominators and n is also big, the computers have not enough capacity to perform the computations to get the Sturm sequence. On the other hand the Descartes rule is not so powerful but a careful use, in the spirit of bisection method, can many times solve the problem.

To recall the Descartes rule we need to introduce some notation. Given an ordered list of real numbers $[b_0, b_1, \ldots, b_{n-1}, b_n]$ we will say that it has C changes of sign if the following holds: denote by $[c_0, c_1, \ldots, c_{m-1}, c_m]$, $m \leq n$ the new list obtained from the previous one after removing the zeros and without changing the order of the remaining terms. Consider the m non-zero numbers $\delta_i := c_i c_{i+1}$, $i = 0, \ldots, m-1$. Then C is the number of negative δ_i.

Theorem 5.3 (Descartes rule)**.** *Let C be the number of changes of sign of the list of ordered numbers*

$$[a_0, a_1, a_2, \ldots, a_{n-1}, a_n].$$

Then the number of positive zeros of the polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$, counted with their multiplicities, is $C - 2k$, for some $k \in \mathbb{N} \cup \{0\}$.

Corollary 5.4. *With the notations of Theorem 5.3 if $C = 0$ then $P(x)$ has not positive roots and if $C = 1$ it has exactly one simple positive root.*

In order to apply Descartes rule to arbitrary open intervals we introduce the following definition:

Definition 5.5. *Given a real polynomial $P(x)$ and a real interval (α, β) we construct a new polynomial*

$$N_\alpha^\beta(P)(x) := (x+1)^{\deg P} P\left(\frac{\beta x + \alpha}{x+1}\right).$$

We will call $N_\alpha^\beta(P)$*, the normalized version of* P *with respect to* (α,β)*. Notice that the number of real roots of* $P(x)$ *in the interval* (α,β) *is equal to the number of real roots of* $N_\alpha^\beta(P)(x)$ *in* $(0,\infty)$*.*

The method suggested in [12] consists in writing $(\alpha,\beta)=\bigcup_{i=1}^k(\alpha_i,\alpha_{i+1})$, with $\alpha=\alpha_1<\alpha_2<\cdots<\alpha_k<\alpha_{k+1}=\beta$ in such a way that on each (α_i,α_{i+1}) it is possible to apply Corollary 5.4 to the normalized version of the polynomial. Although there is no systematic way of searching a suitable decomposition, we will see that a careful use of these type of ideas has been good enough to study the number and localization of the roots for a huge polynomial of degree 965, see Subsection 5.6 in Appendix II.

Appendix II: A method for controlling the sign of polynomials in two variables

The main result of this appendix is a new method for controlling the sign of families of polynomials with two variables. As a starting point we prove a simple result for one-parameter families of polynomials in one variable.

Let $G_b(x)$ be a one-parametric family of polynomials. As usual, we write $\triangle_x(P)$ to denote the discriminant of a polynomial $P(x)=a_nx^n+\cdots+a_1x+a_0$, that is,

$$\triangle_x(P)=(-1)^{\frac{n(n-1)}{2}}\frac{1}{a_n}\operatorname{Res}(P(x),P'(x)),$$

where $\operatorname{Res}(P,P')$ is the resultant of P and P'.

Lemma 5.6. *Let*

$$G_b(x)=g_n(b)x^n+g_{n-1}(b)x^{n-1}+\cdots+g_1(b)x+g_0(b),$$

be a family of real polynomials depending also polynomially on a real parameter b *and set* $\Omega=\mathbb{R}$*. Suppose that there exists an open interval* $I\subset\mathbb{R}$ *such that:*

(i) There is some $b_0\in I$*, such that* $G_{b_0}(x)>0$ *on* Ω*.*
(ii) For all $b\in I$*,* $\triangle_x(G_b)\neq0$*.*
(iii) For all $b\in I$*,* $g_n(b)\neq0$*.*

Then for all $b\in I$*,* $G_b(x)>0$ *on* Ω*.*

Moreover if $\Omega=\Omega_b=(c(b),\infty)$ *for some smooth function* $c(b)$*, the same result holds changing* Ω *by this new* Ω_b *if we add the additional hypothesis*

(iv) For all $b\in I$*,* $G_b(c(b))\neq0$*.*

Proof. The key point of the proof is that the roots (real and complex) of G_b depend continuously of b, because $g_n(b)\neq0$. Notice that hypotheses (iii) and (iv) prevent that moving b some root enters in Ω either from infinity or from the boundary of Ω, respectively. On the other hand if moving b some real roots appear from $\mathbb{C}$, they do appear trough a double real root that is detected by the vanishing of $\triangle_x(G_b)$. Since by item (ii), $\triangle_x(G_b)\neq0$ no real root appears in this way. Hence, for all $b\in I$, the number of real roots of any G_b is the same. Since by item (i) for $b=b_0$, $G_{b_0}>0$ on Ω, the same holds for all $b\in I$. □

To state the corresponding result for families of polynomials with two variables inspired in the above lemma, see Proposition 5.12, we need to prove some results about the iterated discriminants (to replace hypothesis (ii) of the lemma) and to recall how to study the infinity of planar curves (to replace hypothesis (iii)).

5.1. **The double discriminant.** Let $F(x,y)$ be a complex polynomial on $\mathbb{C}^2$. We write F as

$$F(x,y) = a_n y^n + a_{n-1}y^{n-1} + a_{n-2}y^{n-2} + \ldots + a_1 y + a_0, \tag{24}$$

where $a_i = a_i(x) \in \mathbb{C}[x]$. Then

$$\triangle_y(F) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} \operatorname{Res}(F, \partial F/\partial y),$$

and this resultant can be computed as the determinant of the Sylvester matrix of dimension $(2n-1)\times(2n-1)$, see [6],

$$S = \begin{pmatrix} a_n & 0 & 0 & 0 & na_n & 0 & 0 & 0 \\ a_{n-1} & a_n & 0 & 0 & (n-1)a_{n-1} & na_n & 0 & 0 \\ a_{n-2} & a_{n-1} & \ddots & 0 & (n-2)a_{n-2} & (n-1)a_{n-1} & \ddots & 0 \\ \vdots & & \ddots & a_n & \vdots & & \ddots & na_n \\ & \vdots & & a_{n-1} & & \vdots & & (n-1)a_{n-1} \\ a_0 & & & & a_1 & & & \\ 0 & a_0 & & \vdots & 0 & a_1 & & \vdots \\ 0 & 0 & \ddots & & 0 & 0 & \ddots & \\ 0 & 0 & 0 & a_0 & 0 & 0 & 0 & a_1 \end{pmatrix}.$$

We will write $\triangle^2_{y,x}(F) = \triangle_x(\triangle_y(F))$. Analogously we can compute $\triangle^2_{x,y}(F)$. This so called double discriminant plays a special role in the characterization of singular curves of $\{F(x,y)=0\}$ and it is also used in applications, see for instance [1, 13, 22]. In particular we prove the following result.

Proposition 5.7. *Let $F(x,y)$ be a complex polynomial on $\mathbb{C}^2$. If $\{F(x,y)=0\} \subset \mathbb{C}^2$ has a singular point, that is, if there exists a point $(x_0,y_0) \in \mathbb{C}^2$ such that $F(x_0,y_0) = \partial F(x_0,y_0)/\partial x = \partial F(x_0,y_0)/\partial y = 0$, then $\triangle^2_{y,x}(F) = \triangle^2_{x,y}(F) = 0$.*

Proof. We write $F(x,y)$ in the form (24). Without lost of generality we assume that $(x_0,y_0) = (0,0)$. Then from the assumptions it follows that $a_0(0) = a_0'(0) = 0$ and $a_1(0) = 0$, that is, $a_0(x) = x^2\hat{a}_0(x)$ and $a_1(x) = x\hat{a}_1(x)$, with both $\hat{a}_i$ also polynomials.

By using the Sylvester matrix S defined above, we have that

$$\det S = (-1)^n a_0 \det(S(2n-1 \mid n-1)) + a_1 \det(S(2n-1 \mid 2n-1)), \tag{25}$$

where $S(i \mid j)$ means the matrix obtained from S by removing the i-th row and the j-th column.

Notice that the elements of the last row of $S(2n-1 \mid 2n-1)$ are only $0, a_0$ and a_1. Therefore, developing the determinant of this matrix from this row we get that $\det(S(2n-1 \mid 2n-1)) = xQ(x)$, for some polynomial $Q(x)$.

Hence, by using (25), we get that $\det S = x^2 P(x)$ with $P(x)$ another polynomial. This implies that $\triangle_y(F)$ has a double zero at $x=0$ and hence $\triangle^2_{y,x}(F) = 0$.

Analogously we can prove that $\triangle_x(F)$ has a double zero at $y=0$ and hence $\triangle^2_{x,y}(F) = 0$. □

Corollary 5.8. *Consider a one-parameter family of polynomials $F_b(x,y)$, depending also polynomially on b. The values of b such that the algebraic curve*

$F_b(x,y)=0$ has some singular point in $\mathbb{C}^2$ have to be zeros of the polynomial

$$\triangle^2(F_b) := \gcd\left(\triangle^2_{x,y}(F_b), \triangle^2_{y,x}(F_b)\right).$$

By simplicity we will also call the polynomial $\triangle^2(F_b)$, *double discriminant of the family $F_b(x,y)$*. As far as we know the above necessary condition for detecting algebraic curves with singular points is new.

Remark 5.9. *(i) Notice that if in Corollary 5.8, instead of imposing that for $b \in I$, $\triangle^2(F_b) \neq 0$, it suffices to check only that either $\triangle^2_{x,y}(F_b) \neq 0$ or $\triangle^2_{y,x}(F_b) \neq 0$.*

(ii) The converse of the Proposition 5.7 is not true. For instance if we consider the polynomial $F(x,y) = x^3y^3 + x + 1$ then $\triangle^2_{y,x}(F) = \triangle^2_{x,y}(F) = 0$, however $F_x(x,y) = 3x^2y^3 + 1$ and $F_y(x,y) = 3x^3y^2$ hence $\{F(x,y) = 0\}$ does not have singular points.

(iii) Sometimes $\triangle^2_{y,x}(F) \neq \triangle^2_{x,y}(F)$. For instance this is the case when $F = y^2 + x^3 + bx^2 + bx$ because

$$\triangle^2_{x,y}(F) = -110592b^9(b-4)(b-3)^6 \quad \textit{and} \quad \triangle^2_{y,x}(F) = 256b^3(b-4).$$

Notice that $\triangle^2(F) = b^3(b-4)$.

5.2. **Algebraic curves at infinity.** Let

$$F(x,y) = F^0(x,y) + F^1(x,y) + \cdots + F^n(x,y)$$

be a polynomial on $\mathbb{R}^2$ of degree n. We denote by

$$\widetilde{F}(x,y,z) = z^n F^0(x,y) + z^{n-1}F^1(x,y) + \cdots + F^n(x,y)$$

its homogenization in $\mathbb{RP}^2$.

For studying $\widetilde{F}(x,y,z)$ in $\mathbb{RP}^2$ we can use its expressions in the three canonical charts of $\mathbb{RP}^2$, $\{[x:y:1]\}$, $\{[x:1:z]\}$, and $\{[1:y:z]\}$, which can be identified with the real planes $\{(x,y)\}$,$\{(x,z)\}$, and $\{(y,z)\}$ respectively. Of course the expression in the chart $\{[x:y:1]\}$, that is, in the (x,y)-plane is precisely $F(x,y)$.

We denote by $\widetilde{F}_1(x,z)$ and $\widetilde{F}_2(y,z)$ the expressions of the function $\widetilde{F}$ in the planes $\{(x,z)\}$ and $\{(y,z)\}$, respectively. Therefore $\widetilde{F}_1(x,z) = \widetilde{F}(x,1,z)$ and $\widetilde{F}_2(y,z) = \widetilde{F}(1,y,z)$.

Let $[x^*:y^*:z^*] \in \mathbb{RP}^2$ be a point of $\{\widetilde{F}=0\}$. If $z^* \neq 0$, then $[x^*:y^*:z^*]$ corresponds to a point in $\mathbb{R}^2$, otherwise it is said that $[x^*:y^*:0]$ is a point of F at infinity. Notice that the points at infinity of F correspond to the points $[x^*:y^*:0]$ where $(x^*,y^*) \neq (0,0)$ is a solution of the homogeneous part of degree n of F,

$$\mathcal{H}_n(F(x,y)) = F^n(x,y),$$

that is $F^n(x^*,y^*) = 0$. Equivalently, these are the zeros of $\widetilde{F}_1(x,0)$ and $\widetilde{F}_2(y,0)$. In other words, $[x^*:y^*:0]$ is a point at infinity of F if and only if x^*/y^* is a zero of $\widetilde{F}_1(x,0) = F^n(x,1)$ or y^*/x^* is a zero of $\widetilde{F}_2(y,0) = F^n(1,y)$.

Let $\Omega \subset \mathbb{R}^2$ be an unbounded open subset with boundary $\partial\Omega$ formed by finitely many algebraic curves. It is clear that this subset can be extended to $\mathbb{RP}^2$. We will call the adherence of this extension $\bar{\Omega}$. When a point at infinity of F is also in $\bar{\Omega}$, for short we will say that is a point at infinite which is also in Ω.

5.3. **Isolated points of families of algebraic curves.** To state our main result we need explicit conditions to check when a point of a real algebraic curve $G(x,y)=0$ is isolated. Recall that it is said that a point $\mathbf{p}\in\mathbb{R}^2$ on the curve is *isolated* if there exists an open neighborhood $\mathcal{U}$ of $\mathbf{p}$, such that

$$\mathcal{U}\cap\{(x,y)\in\mathbb{R}^2\,:\,G(x,y)=0\}=\mathbf{p}.$$

Clearly isolated points are singular points of the curve. Next result provides an useful criterion to deal with this question.

Lemma 5.10. *Let $G(x,y)$ be a real polynomial. Assume that $(0,0)\in\{G(x,y)=0\}$ and that there are natural numbers p,q and m, with $\gcd(p,q)=1$, and a polynomial G^0 satisfying $G^0(\varepsilon^pX,\varepsilon^qY)=\varepsilon^mG^0(X,Y)$, and such that for all $\varepsilon>0$,*

$$G(\varepsilon^pX,\varepsilon^qY)=\varepsilon^mG^0(X,Y)+\varepsilon^{m+1}G^1(X,Y,\varepsilon),$$

for some polynomial function G^1. If the only real solution of $G^0(X,Y)=0$ is $(X,Y)=(0,0)$, then the origin is an isolated point of $G(x,y)=0$.

Proof. Assume without loss of generality that $G^0\geq0$. We start proving that $K:=\{(x,y)\in\mathbb{R}^2\,:\,G^0(x,y)=1\}$ is a compact set. Clearly it is closed, so it suffices to prove that it is bounded. Since G^0 is a quasi-homogeneous polynomial we know that there exists a natural number m_0 such that $m=m_0pq$ and $G^0(x,y)=P_{m_0}(x^q,y^p)$, where P_{m_0} is a real homogeneous polynomial of degree m_0. The fact that the only real solution of the equation $G^0(x,y)=0$ is $x=y=0$ implies that P_{m_0} has not linear factors when we decompose it as a product of real irreducible factors. Hence m_0 is even and $P_{m_0}(x,y)=\prod_{i=1}^{m_0/2}(A_ix^2+B_ixy+C_iy^2)$, with $B_i^2-4A_iC_i<0$. As a consequence,

$$G^0(x,y)=\prod_{i=1}^{m_0/2}(A_ix^{2q}+B_ix^qy^p+C_iy^{2p}),\quad\text{with}\quad B_i^2-4A_iC_i<0.\tag{26}$$

Assume, to arrive to a contradiction, that K is unbounded. Therefore it should exist a sequence $\{(x_n,y_n)\}$, tending to infinity, and such that $G^0(x_n,y_n)=1$. But this is impossible because the conditions $B_i^2-4A_iC_i<0$, $i=1,\ldots,m_0/2$, imply that all the terms $A_ix_n^{2q}+B_ix_n^qy_n^p+C_iy_n^{2p}$ in (26) go to infinity. So K is compact.

Let us prove that $(0,0)$ is an isolated point of $\{(x,y)\in\mathbb{R}^2\,:\,G(x,y)=0\}$. Assume, to arrive to a contradiction, that it is not. Therefore there exists a sequence of points $\{(x_n,y_n)\}$, tending to 0 and such that $G(x_n,y_n)=0$ for all $n\in\mathbb{N}$. Consider $G^0(x_n,y_n)=:(g_n)^m>0$. It is clear that $\lim_{n\to\infty}(g_n)^m=0$. Write $(x_n,y_n)=((g_n)^pu_n,(g_n)^qv_n)$. Notice that

$$(g_n)^m=G^0(x_n,y_n)=G^0(g_n^pu_n,g_n^qv_n)=(g_n)^mG^0(u_n,v_n).$$

Then $G^0(u_n,v_n)=1$ and $(u_n,v_n)\in K$, for all $n\in\mathbb{N}$. Therefore, taking a subsequence if necessary, we can assume that

$$\lim_{n\to\infty}(u_n,v_n)=(u^*,v^*)\in K.\tag{27}$$

We have that $0=G(x_n,y_n)=(g_n)^m+(g_n)^{m+1}G^1(u_n,v_n,g_n)$. Dividing by $(g_n)^m$ we obtain that $1=0+g_nG^1(u_n,v_n,g_n)$, and passing to the limit we get that $1=0$ which gives the desired contradiction.

Notice that to prove that $\lim_{n\to\infty}g_nG^1(u_n,v_n,g_n)=0$ we need to know that the sequence $\{(u_n,v_n)\}$ remains bounded and this fact is a consequence of (27). $\square$

We remark that the suitable values p, q and m and the function G^0 appearing in the statement of Lemma 5.10 are usually found by using the Newton diagram associated to G.

We also need to introduce a new related concept for families of curves. Consider a one-parameter family of algebraic curves $G_b(x, y) = 0$, $b \in I$, also depending polynomially of b. Let $(x_0, y_0) \in \mathbb{R}^2$ be an isolated point of $G_b(x, y) = 0$ for all $b \in I$, we will say that (x_0, y_0) is *uniformly isolated* for the family $G_b(x, y) = 0$, $b \in I$ if for each $b \in I$ there exist neighborhoods $\mathcal{V} \subset I$ and $\mathcal{W} \subset \mathbb{R}^2$, of b and (x_0, y_0) respectively, such that for all $b \in \mathcal{V}$,

$$\{(x, y) \in \mathbb{R}^2 \, : \, G_b(x, y) = 0\} \cap \mathcal{W} = (x_0, y_0). \tag{28}$$

Next example shows a one-parameter family of curves that has the origin isolated for all $b \in \mathbb{R}$ but it is not uniformly isolated for $b \in I$, with $0 \in I$,

$$G_b(x, y) = (x^2 + y^2)(x^2 + y^2 - b^2)(x - 1). \tag{29}$$

It is clear that the origin is an isolated point of $\{G_b(x, y) = 0\}$ for all $b \in \mathbb{R}$, but there is no open neighborhood $\mathcal{W}$ of $(0, 0)$, such that (28) holds for any b in a neighborhood of $b = 0$.

Next result is a version of Lemma 5.10 for one-parameter families. In its proof we will use some periodic functions introduced by Lyapunov in his study of the stability of degenerate critical points, see [15]. Let us recall them.

Let $u(\varphi) = \operatorname{Cs}(\varphi)$ and $v(\varphi) = \operatorname{Sn}(\varphi)$ be the solutions of the Cauchy problem:

$$u' = -v^{2p-1},\, v' = u^{2q-1}, \quad u(0) = \sqrt[2q]{1/p} \quad \text{and} \quad v(0) = 0,$$

where the prime denotes the derivative with respect to φ.

Then $x = \operatorname{Cs}(\varphi)$ and $y = \operatorname{Sn}(\varphi)$ parameterize the algebraic curve $px^{2q} + qy^{2p} = 1$, that is $p \operatorname{Cs}^{2q}(\varphi) + q \operatorname{Sn}^{2p}(\varphi) = 1$, and both functions are smooth $T_{p,q}$-periodic functions, where

$$T = T_{p,q} = 2p^{-1/2q} q^{-1/2p} \frac{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{1}{2q}\right)}{\Gamma\left(\frac{1}{2p} + \frac{1}{2q}\right)},$$

and Γ denotes the Gamma function.

Proposition 5.11. *Let $G_b(x, y)$ be a family of real polynomials which also depends polynomially on b. Assume that $(0, 0) \in \{G_b(x, y) = 0\}$ and that there are natural numbers p, q and m, with $\gcd(p, q) = 1$, and a polynomial G_b^0 satisfying $G_b^0(\varepsilon^p X, \varepsilon^q Y) = \varepsilon^m G_b^0(X, Y)$, and such that for all $\varepsilon > 0$,*

$$G_b(\varepsilon^p X, \varepsilon^q Y) = \varepsilon^m G_b^0(X, Y) + \varepsilon^{m+1} G_b^1(X, Y, \varepsilon),$$

for some polynomial function G_b^1. If for all $b \in I \subset \mathbb{R}$, the only real solution of $G_b^0(X, Y) = 0$ is $(X, Y) = (0, 0)$, then the origin is an uniformly isolated point of $G_b(x, y) = 0$ for all $b \in I$.

Proof. Assume without loss of generality that $G_b^0 \geq 0$. Let us write the function $G_b(x, y)$ using the so-called generalized polar coordinates,

$$x = \rho^p \operatorname{Cs}(\varphi),\, y = \rho^q \operatorname{Sn}(\varphi), \quad \text{for} \quad \rho \in \mathbb{R}^+.$$

Then

$$G_b(x,y) = G_b(\rho^p \operatorname{Cs}(\varphi), \rho^q \operatorname{Sn}(\varphi)) \\ = \rho^m G_b^0(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi)) + \rho^{m+1} G_b^1(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi), \rho). \quad (30)$$

Using the same notation that in the proof of Lemma 5.10, with the obvious modifications, we know from (26) that

$$G_b^0(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi)) = \prod_{i=1}^{m_0/2} (A_i(b) \operatorname{Cs}^{2q}(\varphi) + B_i(b) \operatorname{Cs}^q(\varphi) \operatorname{Sn}^p(\varphi) + C_i(b) \operatorname{Sn}^{2p}(\varphi)),$$

with all $B_i^2(b) - 4A_i(b)C_i(b) < 0$. Therefore, it is not difficult to prove that there exists two positive continuous functions, $L(b)$ and $U(b)$ such that

$$0 < L(b) \leq G_b^0(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi)) \leq U(b),$$

due to the periodicity of the Lyapunov functions and the discriminant conditions. Dividing the expression (30) by ρ^m we obtain that the points of $\{G_b(x,y) = (0,0)\} \setminus \{(0,0\}$ are given by

$$G_b^0(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi)) + \rho\, G_b^1(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi), \rho) = 0. \quad (31)$$

Fix a compact neighborhood of b, say $\mathcal{V} \subset I$. Set $L = \min_{x \in \mathcal{V}} L(b)$. Then there exists $\delta > 0$ such that for any $||(x,y)|| \leq \delta$ and any $b \in \mathcal{V}$,

$$|\rho\, G_b^1(\operatorname{Cs}(\varphi), \operatorname{Sn}(\varphi), \rho)| < L/2.$$

Therefore (31) never holds in this region and

$$\{(x,y) \in \mathbb{R}^2 \,:\, G_b(x,y) = 0\} \cap \{(x,y) \in \mathbb{R}^2 \,:\, ||(x,y)|| < \delta\} = (0,0),$$

for all $b \in \mathcal{V}$, as we wanted to prove. □

Notice that, the fact that for all $b \in \mathbb{R}$, the origin of (29) is isolated simply follows plotting the zero level set of G_b. Alternatively, we can apply Lemma 5.10 with $p = 1, q = 1$ and $m = 2$ to prove that the origin is isolated when $b \neq 0$ and with $p = q = 1$ and $m = 4$ when $b = 0$. In any case, Proposition 5.11 can not be used.

5.4. **The method for controlling the sign.**

Proposition 5.12. *Let $F_b(x,y)$ be a family of real polynomials depending also polynomially on a real parameter b and let $\Omega \subset \mathbb{R}^2$ be an open connected subset having a boundary $\partial\Omega$ formed by finitely many algebraic curves. Suppose that there exists an open interval $I \subset \mathbb{R}$ such that:*

(i) For some $b_0 \in I$, $F_{b_0}(x,y) > 0$ on $\Omega \subset \mathbb{R}^2$.
(ii) For all $b \in I$, $\triangle^2(F_b) \neq 0$.
(iii) For all $b \in I$, all points of $F_b = 0$ at infinity which are also in Ω do not depend on b and are uniformly isolated.
(iv) For all $b \in I$, $\{F_b = 0\} \cap \partial\Omega = \emptyset$.

Then for all $b \in I$, $F_b(x,y) > 0$ on Ω.

Proof. Consider the following set

$$J := \{b \in I \,:\, F_b(x,y) > 0 \quad \text{for all} \quad (x,y) \in \Omega\}.$$

By hypothesis (i), $J \neq \emptyset$ because $b_0 \in J$. Consider now $\bar{b} = \sup J$. We want to prove that $\bar{b} \in \partial I$. If this is true, arguing similarly with $\inf J$ the result will follow.

We will prove the result by contradiction. So assume that $\bar{b} \in I$.

Notice that if $F_{\bar{b}}(x,y)$ takes positive and negative values on Ω, by continuity this would happen for any b near enough to $\bar{b}$. This is in contradiction with the fact that $\bar{b}$ is the supremum of J. Therefore, either $F_{\bar{b}}(x,y) \geq 0$ or $F_{\bar{b}}(x,y) > 0$ in Ω.

In the first case it is clear that a point (x_0,y_0) where $F_{\bar{b}}(x_0,y_0) = 0$ has to be a singular point of the curve $\{F_{\bar{b}}(x,y) = 0\}$. Therefore, by Corollary 5.8, $\triangle^2(F_{\bar{b}}) = 0$ which is in contradiction with (ii).

In the second case it should exist a sequence of real numbers $\{b_n\}$, with $b_n \downarrow \bar{b}$, and a sequence of points $\{(x_n,y_n)\} \in \Omega$ such that $\lim_{n\to\infty} F_{b_n}(x_n,y_n) = 0$.

If the sequence is bounded, renaming it if necessary, we arrive to a convergent sequence. Call $(\bar{x},\bar{y}) \in \overline{\Omega}$ its limit, where $\overline{\Omega}$ denotes the adherence of Ω. Then $F_{\bar{b}}(\bar{x},\bar{y}) = 0$. By hypothesis (iv), the point $(\bar{x},\bar{y}) \notin \partial\Omega$ and we also know that $F_{\bar{b}}(x,y) > 0$ on Ω. Therefore we have a contradiction and the sequence $\{(x_n,y_n)\}$ must be unbounded.

This unbounded sequence can be considered in the projective space $\mathbb{RP}^2$. Then this sequence must converge to a point $\mathbf{p}$ of $F_{\bar{b}}(x,y) = 0$ at infinity, which is also in $\mathcal{U}$. Since by hypothesis (iii) this point is uniformly isolated, there exists a neighborhood $\mathcal{V}$ of $\bar{b}$ and an open neighborhood $\mathcal{W}$ of $\mathbf{p}$ such that this point is the only real point in $\mathbb{RP}^2$ of the homogenization of $F_b(x,y) = 0$. This is in contradiction with the fact $F_{b_n}(x_n,y_n) = 0$ for all n, and the result follows. □

5.5. **Control of the sign of** (19)**.** In this subsection we will prove by using Proposition 5.12, that for $b \in (0, 0.6512)$, the function M_b given in (19) is positive on $\Omega = \mathbb{R}^2$.

To check hypothesis (i), we prove that $M_{1/2} > 0$ for all $\mathbb{R}^2$. For this value,

$$M_{1/2} = \tfrac{15}{2}\,x^4y^2 - \tfrac{21}{4}\,x^3y^3 + \tfrac{21}{2}\,x^2y^4 - \tfrac{123}{16}\,x^2y^2 + \tfrac{21}{16}\,xy^3 + \tfrac{5}{2}\,x^4 - \tfrac{7}{16}\,x^2 + \tfrac{15}{64}\,y^2 + \tfrac{13}{64}.$$

We think $M_{1/2}$ as a polynomial in x and y as a parameter and we apply Lemma 5.6. If $y = 0$ then $M_{1/2}$ reduces to the polynomial $(5/2)x^4 - (7/16)x^2 + 13/64$ which is positive on $\mathbb{R}$. Now, we compute $\triangle_x(M_{1/2})$ and we obtain a polynomial in the variable y of degree 20. By using the Sturm method it is easy to see that it does not have real roots. Moreover, the coefficient of x^4 is $5(3y^2+1)/2 > 0$. Therefore, $M_{1/2} > 0$ on $\mathbb{R}^2$, as we wanted to see.

To check hypothesis (ii) we compute the double discriminant of M_b and we obtain that $\triangle^2_{x,y}(M_b)$ is a polynomial in b of degree 1028, of the following form

$$\begin{aligned}\triangle^2_{x,y}(M_b) =& b^{320}(b^2-2)^{40}(3b^2-2)^5(3b^2-4)(2b^6-4b^4-3b^2+2)\times\\ &\times (b^6-2b^4-3b^2+2)(P_2(b^2))^8(P_6(b^2))^4(P_{32}(b^2))^2(P_{33}(b^2))^6,\end{aligned}$$

where P_i are polynomials of degree i with rational coefficients. By using the Sturm method we localize the real roots of each factor of $\triangle^2_{x,y}(M_b)$ and we obtain that in the interval $(0, 0.6512)$ none of them has real roots. In fact $P_{32}(b^2)$ has a root in $(0.6513, 0.6514)$ and that is the reason for which we can not increase more the value of b. Therefore $\triangle^2_{x,y}(M_b) \neq 0$ for all $b \in (0, 0.6512)$.

Finally we have to check hypothesis (iii). Notice that in this case $\partial\Omega = \emptyset$ and so (iv) follows directly.

The zeros at infinity are given by the directions

$$\mathcal{H}_6(M_b) = 6x^2y^2[(2-3b^2)x^2 - 2b^2(2-b^2)xy + (2-b^2)y^2] = 0.$$

For $|b| < 0.7275$ it has only the non-trivial solutions $x = 0$ and $y = 0$. The homogenization of M_b is

$$\begin{aligned}\widetilde{M}_b \;=\; & 6[(2-3b^2)x^4y^2 - 2b^2(2-b^2)x^3y^3 + (2-b^2)x^2y^4] + 2(2-3b^2)x^4z^2 \\ & -3b^2(14-15b^2)x^2y^2z^2 + 12b^4(2-b^2)xy^3z^2 - b^2(4-9b^2)x^2z^4 \\ & +3b^4(2-3b^2)y^2z^4 + b^4(4-3b^2)z^6,\end{aligned} \tag{32}$$

and hypothesis (iii) is equivalent to prove that $(0,0)$ is an uniformly isolated singularity for $\widetilde{M}^1_b(x,z) = \widetilde{M}_b(x,1,z)$ and that $(0,0)$ is also an uniformly isolated singularity for $\widetilde{M}^2_b(y,z) = \widetilde{M}_b(1,y,z)$.

First we prove this result for $\widetilde{M}^1_b(x,z)$. From (32),

$$\begin{aligned}\widetilde{M}^1_b(x,z) \;=\; & 6[(2-3b^2)x^4 - 2b^2(2-b^2)x^3 + (2-b^2)x^2] + 2(2-3b^2)x^4z^2 \\ & -3b^2(14-15b^2)x^2z^2 + 12b^4(2-b^2)xz^2 - b^2(4-9b^2)x^2z^4 \\ & +3b^4(2-3b^2)z^4 + b^4(4-3b^2)z^6.\end{aligned}$$

Hence,

$$\widetilde{M}^1_b(\varepsilon^2X, \varepsilon Z) = \Big(6(2-b^2)X^2 + 12b^4(2-b^2)XZ^2 + 3b^4(2-3b^2)Z^4\Big)\varepsilon^4 + O(\varepsilon^5).$$

The discriminant with respect to X of the homogeneous polynomial $T(X,W) := 6(2-b^2)X^2 + 12b^4(2-b^2)XW + 3b^4(2-3b^2)W^2$, where $W = Z^2$, is

$$\triangle_X(T) = 72W^2b^4(b^2-2)(2b^6 - 4b^4 - 3b^2 + 2).$$

Since its smallest positive root is greater than 0.673 it holds for $b \in (0, 673)$ that $T(X,W) = 0$ if and only if $(X,W) = (0,0)$. Therefore by Proposition 5.11 the point $(0,0)$ is an uniformly isolated point of the curve $\widetilde{M}^1_b(x,z) = 0$, for these values of b.

For the other point, since

$$\begin{aligned}\widetilde{M}^2_b(y,z) \;=\; & 6[(2-b^2)y^4 - 2b^2(2-b^2)y^3 + (2-3b^2)y^2] + 2(2-3b^2)z^2 \\ & -3b^2(14-15b^2)y^2z^2 + 12b^4(2-b^2)y^3z^2 - b^2(4-9b^2)z^4 \\ & +3b^4(2-3b^2)y^2z^4 + b^4(4-3b^2)z^6,\end{aligned}$$

we have that

$$\widetilde{M}^2_b(\varepsilon Y, \varepsilon Z) = 2(2-3b^2)\Big(3Y^2 + Z^2\Big)\varepsilon^2 + O(\varepsilon^3),$$

and the result follows for $b \in (0, \sqrt{2/3}) \approx (0, 0.816)$, by applying again the same proposition.

So, we have shown that for $b \in (0, 0.6512)$ all the hypotheses of the Proposition 5.12 hold. Therefore we have proved that for $b \in (0, 0.651]$, $M_b(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$.

5.6. **Control of the sign of** (23)**.** The numerator of the function M_b given in (23) is a polynomial of the following form

$$N_b(x,y) = f_0(x,b) + f_1(x,b)y + f_2(x,b)y^2 + f_3(x,b)y^3 + f_4(x,b)y^4, \tag{33}$$

where

$$\begin{aligned} f_0(x,b) &= 90b^{36}x^{10} - 15b^{18}(6b^{20}-5)x^8 + 15b^{18}(24b^4 - 59b^2 + 24)x^6 \\ &\quad -(378b^{24} - 810b^{22} + 360b^{20} - 300b^4 + 675b^2 - 300)x^4 \\ &\quad -15b^2(18b^{22} - 24b^{20} + 21b^4 - 45b^2 + 20)x^2 - 75b^4(-4+3b^2), \end{aligned}$$

$$\begin{aligned} f_1(x,b) &= 180b^{36}x^7 + 12b^{18}(60b^{16} + 50b^{14} + 18b^{10} + 25)x^5 - 20b^{10}(36b^{12} \\ &\quad -54b^{10} + 54b^8 - 30b^6 - 25b^4 - 9)x^3 - 180b^{20}(3b^2-4)x, \end{aligned}$$

$$\begin{aligned} f_2(x,b) &= 270b^{36}x^{10} - 45b^{18}(6b^{20} + 2b^{18} - 5)x^8 + 3b^{18}(30b^{20} + 120b^{16} \\ &\quad +100b^{14} - 90b^{12} + 36b^{10} + 360b^4 - 615b^2 + 335)x^6 - (360b^{36} \\ &\quad +300b^{34} + 108b^{30} + 2214b^{24} - 3690b^{22} + 3435b^{20} + 360b^{18} \\ &\quad -300b^{16} - 250b^{14} + 225b^{12} - 90b^{10} - 900b^4 + 1350b^2 - 900)x^4 \\ &\quad -b^2(468b^{22} - 540b^{20} - 1080b^{18} + 300b^{16} + 250b^{14} + 90b^{10} \\ &\quad +1845b^4 - 3075b^2 + 2475)x^2 - 90b^4(4b^2-5), \end{aligned}$$

$$\begin{aligned} f_3(x,b) &= -180b^{20}(b^{10}-3)x^7 + 30b^2(6b^{34} + 6b^{30} - 24b^{22} + 18b^{20} - 72b^{18} \\ &\quad -5b^{10} + 15)x^5 + 30b^2(24b^{24} - 36b^{22} + 72b^{20} + 10b^{16} + 5b^{12} \\ &\quad -20b^4 + 15b^2 - 60)x^3 - 20b^4(36b^{18} - 54b^{16} + 54b^{14} + 30b^{12} \\ &\quad +25b^{10} + 9b^6 - 30b^4 + 45b^2 - 90)x, \end{aligned}$$

$$\begin{aligned} f_4(x,b) &= 90b^{36}x^8 - 3b^{18}(30b^{20} + 120b^{16} + 100b^{14} + 36b^{10} - 25)x^6 \\ &\quad +b^{10}(360b^{26} + 300b^{24} + 198b^{20} + 360b^{12} - 615b^{10} + 720b^8 \\ &\quad -300b^6 - 250b^4 - 90)x^4 + (-738b^{24} + 1080b^{22} - 1080b^{20} \\ &\quad +300b^{18} + 250b^{16} + 315b^{12} + 300b^4 - 450b^2 + 900)x^2 + 15b^6. \end{aligned}$$

We will prove that $N_b \geq 0$ on $\Omega := \{(x,y) : xy+1>0\}$ for all $b \in (0, 0.817]$ and if it vanishes this only happens at some isolated points. We will use again Proposition 5.12. Notice that $\partial\Omega = \{(x,y) : xy+1=0\}$.

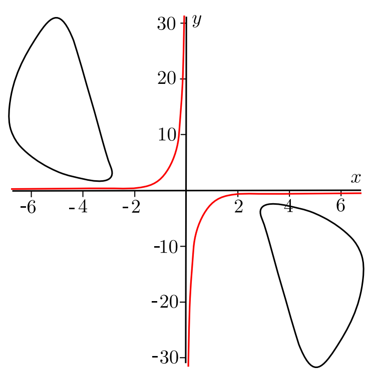

FIGURE 14. Curves $N_b = 0$ and $xy+1=0$ with $b = 0.817$.

It is not difficult to verify that $\{N_b(x,y) = 0\} \cap \{xy+1=0\} = \emptyset$ for $b \in (0, 0.8171)$, see Figure 14. It suffices to see that for these values of b, and $x \neq 0$, the one variable function $N_b(x, 1/x)$, never vanishes. We skip the details. Therefore hypothesis (iv) is satisfied.

For proving that hypothesis (ii) of Proposition 5.12 holds we compute the double discriminant $\triangle^2_{y,x}(N_b)$. It is an even polynomial in b, of degree 21852, of the following form

$$b^{7566}(3b^2-4)(159b^4-380b^2+225)^2(P_{71}(b^2))^2(P_{386}(b^2))^4(P_{587}(b^2))^6(P_{965}(b^2))^2, \tag{34}$$

where P_i are polynomials of degree i with rational coefficients. By using the Sturm method it is easy to see that its first 4 factors do not have real roots in $(0, 0.8171)$. We replace $b^2 = t$ in the next three polynomials to reduce their degrees and we obtain $\mathcal{P}_1(t) := P_{386}(t)$, $\mathcal{P}_2(t) := P_{587}(t)$, and $\mathcal{P}_3(t) := P_{965}(t)$. It suffices to study their number of real roots in $(0, 0.6678]$, because $0.6678 > (0.8171)^2$. Our computers have no enough capacity to get their Sturm sequences. Therefore we will use the Descartes approach as it is explained in Appendix I.

We consider first the polynomial $\mathcal{P}_1(t)$. Its normalized version $N_0^{0.68}(\mathcal{P}_1)$ has all their coefficients positive. Therefore $\mathcal{P}_1(t)$ has no real roots in $(0, 0.68)$ as we wanted to see.

Applying the Descartes rule to the normalized versions of $\mathcal{P}_2(t)$, $N_0^{0.561}(\mathcal{P}_2)$, $N_{0.561}^{0.811}(\mathcal{P}_2)$ and $N_{0.562}^{0.812}(\mathcal{P}_2)$, we obtain that the number of zeros in the intervals $(0, 0.561)$, $(0.561, 0.811)$ and $(0.562, 0.812)$ is 0, 1 and 0 respectively. That is, there is only one root of $\mathcal{P}_2(t)$ in $(0, 0.812)$, it is simple and it belongs to $(0.561, 0.562)$. Refining this interval with Bolzano Theorem we prove that the root is in the interval $(0.5617, 0.5618)$.

Finally to study $\mathcal{P}_3(t)$ we consider $N_0^{11/20}(\mathcal{P}_3)$, $N_{11/20}^{7/12}(\mathcal{P}_3)$ and $N_{7/12}^{52/75}(\mathcal{P}_3)$. By Descartes rule we obtain that the number of zeros of $\mathcal{P}_3$ in the corresponding intervals is 0, 1 and 1 or 3, respectively. By Bolzano Theorem we can localize more precisely these zeros and prove that in the last interval there are exactly 3 zeros. So we have proved that the polynomial $\mathcal{P}_3$ has exactly 4 zeros in the interval $(0, 52/75) \approx (0, 0.693)$, and each one of them is contained in one of the following intervals

$$(0.5614, 0.5615)\,, (0.6678, 0.6679)\,, (0.6690, 0.6700)\,, (0.6870, 0.6880)\,.$$

In brief, for $t \in (0, 0.6678]$ the double discriminant $\triangle_{y,x}(N_b)$ only vanishes at two points $t = t_1$ and $t = t_2$ with $t_1 \in (0.5614, 0.5615)$ and $t_2 \in (0.5617, 0.5618)$. Therefore we are under the hypothesis (ii) of Proposition 5.12 for b belonging to each of the intervals $(0, b_1)$, (b_1, b_2) and $(b_2, 0.8171)$, where

$$b_1 := \sqrt{t_1} \approx 0.749301, \quad b_2 := \sqrt{t_2} \approx 0.749478.$$

To ensure that on each interval we are under the hypotheses (i) of the proposition we prove that N_b does not vanish on Ω for one value of b in each of the above three intervals. We take

$$\frac{1}{2} \in (0, b_1), \quad \frac{7494}{10000} \in (b_1, b_2), \quad \text{and} \quad \frac{3}{4} \in (b_2, 0.8171).$$

We study with detail the case $b = 1/2$. The other two cases can be treated similarly and we skip the details. So we have to study on Ω the sign of the function

$$\begin{aligned} N_{1/2} \quad = \quad & \tfrac{135}{34359738368}x^{10}y^2 + \tfrac{45}{34359738368}x^8y^4 + \tfrac{45}{34359738368}x^{10} + \tfrac{117964485}{137438953472}x^8y^2 \\ & + \tfrac{138195}{268435456}x^7y^3 + \tfrac{39253779}{137438953472}x^6y^4 + \tfrac{39321555}{137438953472}x^8 + \tfrac{45}{17179869184}x^7y \\ & + \tfrac{320504301}{137438953472}x^6y^2 + \tfrac{1932072223485}{17179869184}x^5y^3 - \tfrac{906074381}{8589934592}x^4y^4 + \tfrac{645}{1048576}x^6 \\ & + \tfrac{1229859}{1073741824}x^5y + \tfrac{5315442024413}{8589934592}x^4y^2 - \tfrac{1808748465}{4194304}x^3y^3 + \tfrac{6763995071}{8388608}x^2y^4 \\ & + \tfrac{1258289751}{8388608}x^4 + \tfrac{55625}{262144}x^3y - \tfrac{1910154937}{4194304}x^2y^2 + \tfrac{26361865}{262144}xy^3 + \tfrac{15}{64}y^4 \\ & - \tfrac{316538295}{8388608}x^2 + \tfrac{585}{1048576}xy + \tfrac{45}{2}y^2 + \tfrac{975}{64}. \end{aligned}$$

We consider $N_{1/2}$ as a polynomial in x with coefficients in $\mathbb{R}[y]$ and we apply Lemma 5.6 with $\Omega_y = (-1/y, \infty)$ when $y > 0$ and $\Omega_0 = (-\infty, \infty)$. Notice that for the symmetry of the function there is no need to study the zone $y < 0$ because $N_{1/2}(-x, -y) = N_{1/2}(x, y)$. We introduce the following notation $S_y(x) := N_{1/2}(x, y)$. We prove the following facts:

(i) If we write $S_y(x) = \sum_{i=1}^{1} 0 s_i(y)x^i$, then $s_{10}(y) = k(1+3y^2)$ for some $k \in \mathbb{Q}^+$. Therefore $s_{10}(y) > 0$ for all $y \in \mathbb{R}$.
(ii) If $y = 0$ then $S_0(x)$ is an even polynomial of degree 10 and it is easy to see that $S_0(x) > 0$ over $\mathbb{R}$.
(iii) We already know that $\{S_y(x) = 0\} \cap \partial\Omega = \emptyset$.
(iv) Some computations give that

$$\triangle_x(S_y) = P_{35}(y^2),$$

where P_{35} is a polynomial of degree 35. Moreover, using once more the Sturm method, we get that $P_{35}(y^2)$ has only two positive roots $0 < y_1 < y_2$, with $y_1 \approx 0.588423$ and $y_2 \approx 6065.2946$. From this result it is easy to prove that:
(a) If $y \in [0, y_1) \cup (y_2, \infty)$, then $S_y(x) > 0$.
(b) If $y \in (y_1, y_2)$, then $S_y(x)$ has only two real roots, say $x_1(y) < x_2(y)$, and none of them belongs to the interval $(-1/y, \infty)$. So $S_y(x) > 0$ on $(-1/y, \infty)$.
(c) If $y \in \{y_1, y_2\}$, then $S_y(x)$ has only a real root, $x_1(y)$, which is a double root and $x_1(y) \notin (-1/y, \infty)$. So, again $S_y(x) > 0$ on $(-1/y, \infty)$.

Thus, by Lemma 5.6, the function $N_{1/2}$ is positive on $(x, y) \in \Omega$, as we wanted to see. In fact, its level curves are like the ones showed in Figure 14. The straight lines $y = y_1$ and $y = y_2$ correspond to the lower and upper tangents to the oval contained in the second quadrant.

To be under all the hypotheses of Proposition 5.12 it only remains to study the function $\widetilde{N}_b$ at infinity. We denote by $\widetilde{N}_b(x, y, z)$ its homogenization in $\mathbb{RP}^2$ and by $\widetilde{N}_b^1(x, z)$ and $\widetilde{N}_b^2(y, z)$ the expressions of the function $\widetilde{N}_b$ in the planes $\{(x, z)\}$ and $\{(y, z)\}$, respectively. Since $\mathcal{H}_{12}(N_b) = 90b^{36}x^8y^2[3x^2 + y^2]$, the only non-trivial solutions of $\mathcal{H}_{12}(N_b) = 0$ are $x = 0$ and $y = 0$. Hence these directions give rise to two points of N_b at infinity which are also on the region Ω. They correspond to the points $(0, 0)$ of the algebraic curves $\widetilde{N}_b^1(x, z) = 0$ and $\widetilde{N}_b^2(y, z) = 0$. We have to prove that both points are uniformly isolated.

Similarly that in the previous subsection, we write

$$\widetilde{N}_b^1(\varepsilon X,\varepsilon Z) = \Big(90b^{36}X^8 - 3b^{18}(30b^{20}+120b^{16}+100b^{14}+36b^{10}-25)X^6Z^2 \\ + b^{10}(360b^{26}+300b^{24}+198b^{20}+360b^{12}-615b^{10}+720b^8-300b^6-250b^4-90)X^4Z^4 \\ + (-738b^{24}+1080b^{22}-1080b^{20}+300b^{18}+250b^{16}+315b^{12}+300b^4-450b^2+900)X^2Z^6 \\ + 15b^6Z^8\Big)\varepsilon^8 + O(\varepsilon^9)$$

and

$$\widetilde{N}_b^2(\varepsilon Y,\varepsilon Z) = 90b^{36}(3Y^2+Z^2)\varepsilon^2 + O(\varepsilon^3).$$

By Proposition 5.11, for the second algebraic curve it is clear that for all $b>0$ $(0,0)$ is an isolated point.

For studying the first one we denote by $R(X,Z)$ the homogenous polynomial accompanying ε^8 and we obtain that

$$\triangle_X(R(X,Z)) = Z^{56}b^{150}(P_{71}(b^2))^2,$$

for some polynomial P_{71} of degree 71 and integer coefficients. Since the smallest positive root of this polynomial is greater that 0.92 we can easily prove that for $b<0.92$, $R(X,Z)=0$ if and only if $X=Z=0$. Therefore we can use again Proposition 5.11 and prove that $(0,0)$ is an uniformly isolated point of the curve for these values of b.

So, if we write

$$(0,0.8171) = (0,b_1)\cup\{b_1\}\cup(b_1,b_2)\cup\{b_2\}\cup(b_2,0.8171),$$

we can apply Proposition 5.12 to each one of the open intervals to prove that for $b\in(0,0.817]\setminus\{b_1,b_2\}$ it holds that $N_b(x,y)>0$ for all (x,y) in Ω. By continuity, for the two values $b\in\{b_1,b_2\}$, we obtain that $N_b(x,y)\ge 0$. Since $\triangle_y(N_b)\not\equiv 0$ either it is always positive or it vanishes only at some isolated points, as we wanted to prove.

It can be seen that for $b \gtrsim \hat{b} \approx 0.81722$, $N_b(x,y)$ changes sign on Ω because there appears one oval in the set $\{N_b(x,y)=0\}$. The value $\hat{b}^2 \approx 0.6678492$ corresponds to the root of $\mathcal{P}_3$ in the interval $(0.6678, 0.6679)$ that has appeared in the proof as a root of the double discriminant.

Acknowledgements. The first two authors are partially supported by a MCYT/FEDER grant number MTM2008-03437 and by a CIRIT grant number 2009SGR 410.

References

[1] J. G. Alcazar, J. Schicho, J. R. Sendra, *A delineability–based method for computing critical sets of algebraic surfaces*, Journal of Symbolic Computation, **42**, (2007), 678–691.

[2] A. A. Andronov, E. A. Leontovich, I. I Gordon. and Maier A. G., “Qualitative theory of second-order dynamic systems”, John Wiley & Sons, New York (1973).

[3] G. A. Baker, P. Graves–Morris, “Padé approximants”, Cambridge [etc.]: Cambridge, University Press, (1996) 2nd ed.

[4] L. A. Cherkas, *The Dulac function for polynomial autonomous systems on a plane*, (Russian) Differ. Uravn. **33** (1997), 689–699, 719; Differ. Equations **33** (1997), 692-701.

[5] C. Chicone, “Ordinary differential equations with applications”, Texts in Applied Mathematics, **34**. Springer, New York, (2006) 2nd ed.

[6] D. Cox, J. Little, D. O'Shea, "Using algebraic geometry", New York [etc.]: Springer–Verlag, (1998).
[7] G. F. D. Duff, *Limit-cycles and rotated vector fields*. Ann. of Math. **57**, (1953) 15–31.
[8] F. Dumortier, J. Llibre, J. C. Artés, "Qualitative theory of planar differential systems", Springer-Verlag, Universitext, (2006).
[9] A. Gasull, H. Giacomini, *Upper bounds for the number of limit cycles of some planar polynomial differential systems*, Discrete and continuous dynamical systems, **27** (2010) 217–229.
[10] A. Gasull, H. Giacomini, *A new criterion for controlling the number of limit cycles of some generalized Liénard equations*, Journal of Differential Equations, **185** (2002) 54–73.
[11] M. Han, T. Qian, *Uniqueness of periodic solutions for certain second-order equations*, Acta Mathematica Sinica, English Series, **20** (2004) 247–254.
[12] W. Krandick, K. Mehlhorn, *New bounds for the Descartes method*. J. Symbolic Comput., **41** (2006), 49-66.
[13] D. Lazard, S. McCallum, *Iterated discriminants*, Journal of Symbolic Computation, **44** (2009), 1176–1193.
[14] N. G. Lloyd, *A note on the number of limit cycles in certain two-dimensional systems*, J. London Math. Soc., **20** (1979) 277–286.
[15] A. M. Lyapunov, *Stability of motion*, Mathematics in Science and Engineering, **30**, Academic Press, New York, London (1966).
[16] L. Markus, *Global structure of ordinary differential equations in the plane*, Trans. Amer. Math. Soc., **76** (1954), 127–148.
[17] D. Neumann, *Classification of continuous flows on 2-manifolds*, Proc. Amer. Math. Soc., **48** (1975), 73–81.
[18] L. M. Perko, *Rotated vector fields and the global behavior of limit cycles for a class of quadratic systems in the plane*, J. Differential Equations **18**, (1975) 63–86.
[19] L. M. Perko, *Global families of limit cycles of planar analytic systems*, Trans. Amer. Math. Soc. **322**, (1990) 627–656.
[20] L. M. Perko, *Bifurcation of limit cycles*. Bifurcations of planar vector fields (Luminy, 1989), 315–333, Lecture Notes in Math., **1455**, Springer, Berlin, 1990.
[21] L. M. Perko, "Differential equations and dynamical systems" New York [etc.]: Springer–Verlag, (1996) 2nd ed.
[22] J. Pettigrew, J. A. G. Roberts, *Characterizing singular curves in parametrized families of biquadratics*, Journal of Physics A: Mathematical and Theoretical, **41** (2008) 115203, 28 pp.
[23] J. Stoer, R. Bulirsch, "Introduction to numerical analysis". Translated from the German by R. Bartels, W. Gautschi and C. Witzgall. Springer-Verlag, New York-Heidelberg, 1980.
[24] X. Wang, J. Jiang, P. Yan, *Analysis of global bifurcation for a class of systems of degree five*, Journal of Mathematical Analysis and Applications, **222** (1998) 305–318.
[25] K. Yamato, *An effective method of counting the number of limit cycles*, Nagoya Math. J., **76** (1979) 35–114.