

# The maximal Beurling transform associated with squares

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## Abstract

It is known that the improved Cotlar's inequality  $B^*f(z) \leq CM(Bf)(z)$ ,  $z \in \mathbb{C}$ , holds for the Beurling transform  $B$ , the maximal Beurling transform  $B^*f(z) = \sup_{\varepsilon > 0} \left| \int_{|w| > \varepsilon} f(z-w) \frac{1}{w^2} dw \right|$ ,  $z \in \mathbb{C}$ , and the Hardy–Littlewood maximal operator  $M$ . In this note we consider the maximal Beurling transform associated with squares, namely,  $B_S^*f(z) = \sup_{\varepsilon > 0} \left| \int_{w \notin Q(0, \varepsilon)} f(z-w) \frac{1}{w^2} dw \right|$ ,  $z \in \mathbb{C}$ ,  $Q(0, \varepsilon)$  being the square with sides parallel to the coordinate axis of side length  $\varepsilon$ . We prove that  $B_S^*f(z) \leq CM^2(Bf)(z)$ ,  $z \in \mathbb{C}$ , where  $M^2 = M \circ M$  is the iteration of the Hardy–Littlewood maximal operator, and  $M^2$  cannot be replaced by  $M$ .

## 1 Introduction

Although we only work with the Beurling transform and its iterations, we begin by considering the question for more general Calderón-Zygmund singular operators. Let  $T$  be a smooth homogeneous Calderón-Zygmund singular integral operator on  $\mathbb{R}^n$  with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\}, \quad (1)$$

where  $\Omega$  is a homogeneous function of degree 0 whose restriction to the unit sphere  $S^{n-1}$  is  $C^\infty$  and satisfies the cancellation property

$$\int_{|x|=1} \Omega(x) d\sigma(x) = 0,$$

$\sigma$  being the normalized surface measure in  $S^{n-1}$ . Let  $Tf$  be the principal value convolution operator

$$Tf(x) = \text{p.v.} \int f(x-y)K(y) dy \equiv \lim_{\varepsilon \rightarrow 0} T^\varepsilon f(x), \quad (2)$$

where  $T^\varepsilon f$  is the truncated operator at level  $\varepsilon$  defined by

$$T^\varepsilon f(x) = \int_{|y|>\varepsilon} f(x-y)K(y) dy = \int_{|x-y|>\varepsilon} f(y)K(x-y) dy. \quad (3)$$

For  $f \in L^p$ ,  $1 \leq p < \infty$ , the limit in (2) exists for almost all  $x$ . One says that the operator  $T$  is even (or odd) if the kernel (1) is even (or odd), that is, if  $\Omega(-x) = \Omega(x)$ ,  $x \in \mathbb{R}^n \setminus \{0\}$  (or  $\Omega(-x) = -\Omega(x)$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ ). Let  $T^*$  be the maximal singular integral

$$T^*f(x) = \sup_{\varepsilon>0} |T^\varepsilon f(x)|, \quad x \in \mathbb{R}^n.$$

The classical Cotlar's inequality (e.g. [Gr1, p. 295]) asserts that

$$T^*f(x) \leq C(M(Tf)(x) + M(f)(x)), \quad x \in \mathbb{R}^n,$$

where  $M$  denotes the Hardy–Littlewood maximal operator. In [MV] one proved the pointwise inequality

$$B^*f(z) \leq M(Bf)(z), \quad z \in \mathbb{C}, \quad (4)$$

where  $Bf$  is the Beurling transform of  $f$  defined by

$$Bf(z) = \text{p.v.} \frac{-1}{\pi} \int \frac{f(z-w)}{w^2} dw, \quad z \in \mathbb{C}, \quad (5)$$

$dw$  denoting the 2-dimensional Lebesgue measure. The work [MOV] addresses the issue of characterizing even smooth homogeneous Calderón–Zygmund operators for which the inequality (4) is true. In particular, it shows that  $T^*f(x) \leq CM(Tf)(x)$ ,  $x \in \mathbb{R}^n$ , if and only if  $\|T^*f\|_2 \leq C\|Tf\|_2$ ,  $f \in L^2(\mathbb{R}^n)$ . The analogous question for odd operators is treated in [MOPV], showing that  $T^*f(x) \leq CM^2(Tf)(x)$ ,  $x \in \mathbb{R}^n$ , if and only if  $\|T^*f\|_2 \leq C\|Tf\|_2$ ,  $f \in L^2(\mathbb{R}^n)$ . (See [BMO] for an  $L^p$  version of these results.)

In the definition (3) of the truncated operator  $T^\varepsilon f$  we are taking the kernel  $K$  outside a ball of radius  $\varepsilon$ . We also could define a truncation outside a cube of sidelength  $\varepsilon$ . Let  $Q(x, \varepsilon)$  be the cube, with sides parallel to axis, centered at  $x$  of side length  $\varepsilon$  and define

$$T_Q^\varepsilon f(x) = \int_{y \notin Q(0, \varepsilon)} f(x-y)K(y) dy = \int_{y \notin Q(x, \varepsilon)} f(y)K(x-y) dy,$$

and consider the associated maximal singular integral

$$T_S^*f(x) = \sup_{\varepsilon>0} |T_Q^\varepsilon f(x)|, \quad x \in \mathbb{R}^n.$$

By simple geometry one checks that

$$\begin{aligned} T_Q^\varepsilon f(x) &= T^{\sqrt{n}\varepsilon/2} f(x) + \int_{B(0, \sqrt{n}\varepsilon/2) \setminus Q(0, \varepsilon)} f(x-y)K(y) dy, \\ T^\varepsilon f(x) &= T_Q^{2\varepsilon} f(x) + \int_{Q(0, 2\varepsilon) \setminus B(0, \varepsilon)} f(x-y)K(y) dy, \end{aligned}$$

and therefore

$$\begin{aligned} T_S^* f(x) &\leq T^* f(x) + CM(f)(x), \\ T^* f(x) &\leq T_S^* f(x) + CM(f)(x). \end{aligned}$$

Consequently, from the  $L^p$ -theory point of view, the maximal operators  $T^*$  and  $T_S^*$  are equivalent. On the other hand, we wonder if there is any pointwise inequality that relates  $T_S^* f(x)$  with  $Tf(x)$  like in (4). In this note we provide an answer for the Beurling transform defined in (5) and for its  $k$ th iteration  $B^k = B \circ \dots \circ B$ . The Fourier multiplier of  $B$  is  $\frac{\bar{\xi}}{\xi}$ , or in other words,  $\widehat{Bf}(\xi) = \frac{\bar{\xi}}{\xi} \hat{f}(\xi)$  and then  $B$  is an isometry on  $L^2(\mathbb{C})$ . The kernel  $b_k$  of  $B^k$  may be computed explicitly, for instance via a Fourier transform argument [St, p. 73], and one obtains

$$b_k(z) = \frac{(-1)^k k}{\pi} \frac{\bar{z}^{k-1}}{z^{k+1}}, \quad z \neq 0.$$

Similarly we get that the kernel of the inverse operator,  $(B^k)^{-1}$ , is precisely the conjugate kernel:  $\frac{(-1)^k k}{\pi} \frac{z^{k-1}}{\bar{z}^{k+1}}$ .

**Theorem.** (a) *If  $k$  is odd, then*

$$(B^k)_S^* f(z) \leq CM^2(B^k f)(z), \quad z \in \mathbb{C},$$

*where  $C$  is a constant depending on  $k$  and  $M^2 = M \circ M$  the iterated maximal operator.*

(b) *If  $k$  is even, for any positive constant  $C$ , for any  $z \in \mathbb{C}$  and for any positive integer  $j$  there exists a function  $f$  in  $L^2(\mathbb{C})$  such that*

$$(B^k)_S^* f(z) \geq CM^j(B^k f)(z),$$

*where  $M^j = M \circ \dots \circ M$  is the  $j$ th iteration of  $M$ .*

Notice that in the right hand side of (4) there is the maximal operator, but in (a) of the Theorem the iteration of  $M$  appears. The reason is that  $B^{-1}(b_1 \chi_{\mathbb{C} \setminus D(0,1)})$  is a bounded function with compact support, but  $B^{-1}(b_1 \chi_{\mathbb{C} \setminus Q(0,1)})$  is an unbounded BMO function without compact support. We also provide an example to show that in part (a) of the above theorem for  $k = 1$  we cannot replace the iterated Hardy–Littlewood maximal operator  $M^2 = M \circ M$  by  $M$ . We believe that similar examples should also give optimality for the case  $k = 3, 5, \dots$ . Property (b) of the Theorem is also satisfied for the Cauchy transform on a Lipschitz graph when the graph is not a line (see [Gi]).

We adhere to the usual convention of denoting by  $C$  a positive constant, independent of the relevant parameters involved, and which may vary from an occurrence to another. Our notation and terminology are standard. For example,  $A \approx B$  means that the two quantities  $A$  and  $B$  satisfy the relation  $C^{-1}A \leq B \leq CA$ , for some constant  $C \geq 1$ .

## 2 Proof of the Theorem

We proceed as in [MOPV]. By translating and dilating one reduces the proof to

$$|(B^k)_Q^2 f(0)| \leq CM^2(B^k f)(0), \quad (6)$$

where  $(B^k)_Q^2 f(0)$  is the truncated integral at level 2. Denote the square  $Q(0, 2) = [-1, 1] \times [-1, 1]$  by  $Q_0$ . As usual,  $\chi_E$  denotes the characteristic function of the set  $E$ . The idea is to obtain an identity of the form

$$b_k(z)\chi_{\mathbb{C} \setminus Q_0}(z) = B^k(a_k)(z),$$

for some function  $a_k$ . Since  $B^k$  is an invertible operator, we have

$$a_k = (B^k)^{-1}(b_k \chi_{\mathbb{C} \setminus Q_0})$$

and so  $a_k \in \text{BMO}(\mathbb{C})$ . We claim that if  $k$  is odd we have the decay estimate

$$|a_k(z)| \leq \frac{C_k}{|z|^3}, \quad \text{if } |z| > 3. \quad (7)$$

Before proving the claim we show how it yields (a) of the Theorem. We argue as in [MOPV, p. 3675]. For any  $f \in L^2(\mathbb{C})$  we have

$$\begin{aligned} (B^k)_Q^2 f(0) &= \int_{z \notin Q_0} f(z) b_k(-z) dz = \int f(z) b_k(z) \chi_{\mathbb{C} \setminus Q_0}(z) dz \\ &= \int f(z) B^k(a_k)(z) dz = \int B^k f(z) a_k(z) dz \\ &= \int_{|z| < 3} B^k f(z) (a_k(z) - (a_k)_{D(0,3)}) dz \\ &\quad + (a_k)_{D(0,3)} \int_{|z| < 3} B^k f(z) dz + \int_{|z| > 3} B^k f(z) a_k(z) dz \\ &:= I + II + III, \end{aligned} \quad (8)$$

where, as usual,  $(a_k)_{D(0,3)} = |D(0, 3)|^{-1} \int_{D(0,3)} a_k(z) dz$ . To estimate the term  $I$  we use the generalized Hölder's inequality and the pointwise equivalence  $M_{L(\log L)} f(x) \simeq M^2 f(x)$  ([P]) to get

$$|I| \leq C \|a_k\|_{\text{BMO}} \|B^k f\|_{L(\log L), D(0,3)} \leq CM^2(B^k f)(0).$$

Clearly,

$$|II| \leq CM(B^k f)(0).$$

Finally, from the decay of  $a_k$  we obtain

$$|III| \leq C \int_{|z| > 3} \frac{|B^k f(z)|}{|z|^3} dy \leq CM(B^k f)(0),$$

by using a standard argument which consists in estimating the integral on the annuli  $\{3^j \leq |z| < 3^{j+1}\}$ . Therefore we get (6) and part (a) of the Theorem is proved. Now we turn to the proof of the claim. We express  $a_k$  as

$$\begin{aligned} a_k &= (B^k)^{-1}(b_k \chi_{\mathbb{C} \setminus Q_0}) = (B^k)^{-1}(b_k - b_k \chi_{Q_0}) \\ &= \delta_0 - (B^k)^{-1}(b_k \chi_{Q_0}), \end{aligned}$$

where  $\delta_0$  is the Dirac delta at the origin. We have to determine the decay of  $(B^k)^{-1}(b_k \chi_{Q_0})(z)$  when  $|z| > 3$ :

$$\begin{aligned} (B^k)^{-1}(b_k \chi_{Q_0})(z) &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |w|} \overline{b_k(z-w)} (b_k \chi_{Q_0})(w) dw \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |w|} (\overline{b_k(z-w)} - \overline{b_k(z)}) (b_k \chi_{Q_0})(w) dw \\ &\quad + \overline{b_k(z)} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |w|} (b_k \chi_{Q_0})(w) dw \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon < |w|} (\overline{b_k(z-w)} - \overline{b_k(z)}) (b_k \chi_{Q_0})(w) dw + \overline{b_k(z)} B^k(\chi_{Q_0})(0) \\ &:= I_k + II_k. \end{aligned} \tag{9}$$

On the one hand, since

$$|\overline{b_k(z-w)} - \overline{b_k(z)}| \leq \frac{C|w|}{|z|^3}, \quad |z| > 3, \quad |w| \leq \sqrt{2},$$

we obtain

$$|I_k| \leq C \int_{w \in Q_0} \frac{|w|}{|z|^3} \frac{1}{|w|^2} dw = \frac{C}{|z|^3}.$$

On the other hand, we will check in the next Lemma that when  $k$  is odd  $B^k(\chi_{Q_0})(0) = 0$ . Then  $II_k = 0$  and so we get (7).

**Lemma.** *Let  $Q$  be any square, with sides parallel to axis, centered at 0. Then*

- (a)  $\overline{(B^k \chi_Q)(z)} = (B^k \chi_Q)(\bar{z})$ .
- (b)  $(B^k \chi_Q)(iz) = (-1)^k (B^k \chi_Q)(z)$ .
- (c)  $(B^k \chi_Q)(0) = 0$  if  $k$  is odd, and  $(B^k \chi_Q)(0) \neq 0$  if  $k$  is even.

*Proof.* It is easy to check (a) and (b). Just consider the symmetry of the domain with respect to conjugation and with respect to rotation of angle  $\pi/2$ :

$$\begin{aligned} \overline{(B^k \chi_Q)(z)} &= \text{p.v.} \int_Q \overline{b_k(z-w)} dw = \text{p.v.} \int_Q b_k(\bar{z} - \bar{w}) dw \\ &= \text{p.v.} \int_Q b_k(\bar{z} - w) dw = (B^k \chi_Q)(\bar{z}) \end{aligned}$$

and

$$\begin{aligned} (B^k \chi_Q)(iz) &= \text{p.v.} \int_Q b_k(iz - w) dw = \text{p.v.} \int_Q b_k(iz - iw) dw \\ &= (-i)^{2k} \text{p.v.} \int_Q b_k(z - w) dw = (-1)^k (B^k \chi_Q)(z). \end{aligned}$$

From (b), when  $k$  is odd, we have  $(B^k \chi_Q)(0) = -(B^k \chi_Q)(0)$  and so  $(B^k \chi_Q)(0) = 0$ . To get  $(B^k \chi_Q)(0) \neq 0$  when  $k$  is even we have to work a little more. Set  $k = 2j$  and by simplicity let  $Q$  be the square  $Q_0$ . Then

$$\begin{aligned} (B^{2j} \chi_{Q_0})(0) &= \text{p.v.} \int_{Q_0} b_{2j}(w) dw = \int_{Q_0 \setminus D(0,1)} b_{2j}(w) dw \\ &= 4 \int_E b_{2j}(w) dw = 4 \frac{2j}{\pi} \int_E \frac{\bar{w}^{2j-1}}{w^{2j+1}} dw, \end{aligned}$$

where  $E = \{w \in Q_0 \setminus D(0,1) : \text{Re } w > 0 \text{ and } \text{Im } w > 0\}$ . The second equality follows from the cancellation of the kernels and the third one from  $b_{2j}(w) = b_{2j}(iw)$ . Therefore, making the change to polar coordinates  $w = re^{i\theta}$ , with  $1 \leq r \leq \sqrt{2}$  and  $\theta(r) \leq \theta \leq \frac{\pi}{2} - \theta(r)$ ,  $\theta(r) = \arccos(\frac{1}{r})$ , we have

$$\begin{aligned} (B^{2j} \chi_{Q_0})(0) &= 4 \frac{2j}{\pi} \int_1^{\sqrt{2}} \int_{\theta(r)}^{\frac{\pi}{2} - \theta(r)} \frac{(re^{-i\theta})^{2j-1}}{(re^{i\theta})^{2j+1}} d\theta r dr \\ &= 4 \frac{2j}{\pi} \int_1^{\sqrt{2}} \int_{\theta(r)}^{\frac{\pi}{2} - \theta(r)} e^{-i\theta 4j} d\theta \frac{dr}{r} \\ &= \frac{-4}{\pi} \int_1^{\sqrt{2}} \sin(\theta(r) 4j) \frac{dr}{r}. \end{aligned}$$

Expressing  $\sin(\theta(r) 4j)$  in terms of  $\sin(\theta(r))$  and  $\cos(\theta(r))$  and replacing  $\sin(\theta(r)) = \frac{\sqrt{r^2-1}}{r}$  and  $\cos(\theta(r)) = 1/r$  we write

$$\begin{aligned} (B^{2j} \chi_{Q_0})(0) &= \frac{-4}{\pi} \sum_{m=0}^{2j-1} (-1)^m \binom{4j}{2m+1} \int_1^{\sqrt{2}} \frac{1}{r^{4j}} (r^2 - 1)^{\frac{2m+1}{2}} \frac{dr}{r} \\ &= \frac{-4}{\pi} \sum_{m=0}^{2j-1} (-1)^m \binom{4j}{2m+1} \int_0^1 \frac{x^{2m+2}}{(x^2 + 1)^{2j+1}} dx \\ &:= \frac{-4}{\pi} \int_0^1 F_j(x) dx. \end{aligned} \tag{10}$$

Integrating by parts as many times as necessary, one easily obtains, for some  $a, b \in \mathbb{Q}$ ,

$$\int_0^1 \frac{x^{2m+2}}{(x^2 + 1)^{2j+1}} dx = a + b \frac{\pi}{4} \neq 0.$$

Although each factor in (10) is non-zero, there still might be a cancellation in the sum on  $m$ . We will check that  $(B^{2j}\chi_{Q_0})(0) \neq 0$  proving that a primitive of  $F_j(x)$  is some rational function  $R_j(x)$  with integer coefficients minus  $\arctan(x)$ . Thus we will have

$$\int_0^1 F_j(x) dx = R_j(1) - R_j(0) - \frac{\pi}{4} \neq 0,$$

because  $R_j(1) - R_j(0)$  will be a rational number.

Given  $n < d$ , we define

$$I(d, 2n) := \int \frac{x^{2n}}{(x^2 + 1)^d} dx.$$

It is not necessary to fix the constant of the primitive because we will value a definite integral. Integrating by parts we have

$$I(d, 2n) = \frac{-x^{2n-1}}{2(d-1)(x^2+1)^{d-1}} + \frac{2n-1}{2(d-1)} I(d-1, 2(n-1)).$$

Iterating the procedure we obtain

$$I(d, 2n) = \frac{(2n-1)!}{2^{n-1}(n-1)!} \frac{(d-n-1)!}{2^n(d-1)!} I(d-n, 0) + R(x), \quad (11)$$

where  $R(x)$  is some rational function defined on  $\mathbb{R}$  and  $R(0) = 0$ . Given  $k \geq 2$  one easily gets

$$I(k, 0) = \frac{(2k-3)!}{2^{2k-3}(k-1)!(k-2)!} I(1, 0) + R(x). \quad (12)$$

Now, from (10) together with (11) and (12),

$$\begin{aligned} \int F_j(x) dx = I(1, 0) & \left\{ \left( \sum_{m=0}^{2j-2} (-1)^m \binom{4j}{2m+1} \frac{(2m+1)!(4j-2m-3)!}{m!(2j)!(2j-m-2)!2^{4j-2}} \right) \right. \\ & \left. - \frac{(4j)!}{(2j-1)!(2j)!2^{4j-1}} \right\} + R_j(x). \end{aligned}$$

Performing some computations (see the Appendix), we get exactly what we wanted,

$$\left( \sum_{m=0}^{2j-2} (-1)^m \binom{4j}{2m+1} \frac{(2m+1)!(4j-2m-3)!}{m!(2j)!(2j-m-2)!2^{4j-2}} \right) - \frac{(4j)!}{(2j-1)!(2j)!2^{4j-1}} = -1, \quad (13)$$

that is,

$$\int F_j(x) dx = -I(1, 0) + R_j(x) = -\arctan(x) + R_j(x).$$

□

Let us prove assertion (b) of the Theorem. Recall that now  $k$  is even. By (8) one has

$$|(B^k)_Q^2 f(0)| \lesssim M^j(B^k f)(0)$$

if and only if

$$|III| \lesssim M^j(B^k f)(0). \quad (14)$$

By (9), when  $|z| > 3$ ,

$$a_k(z) = \overline{b_k(z)}(B^k \chi_{Q_0})(0) + O\left(\frac{1}{|z|^3}\right) := \alpha_k \frac{z^{k-1}}{\bar{z}^{k+1}} + O\left(\frac{1}{|z|^3}\right),$$

where  $\alpha_k$  is a non-zero constant that depends on  $k$ . Consequently, (14) holds if and only if

$$\left| \int_{|z|>3} \frac{z^{k-1}}{\bar{z}^{k+1}} B^k f(z) dz \right| \lesssim M^j(B^k f)(0), \quad f \in L^2.$$

Since that  $B^k$  is invertible in  $L^2$ , this is equivalent to

$$\left| \int_{|z|>3} \frac{z^{k-1}}{\bar{z}^{k+1}} G(z) dz \right| \lesssim M^j(G)(0), \quad f \in G^2. \quad (15)$$

But (15) is false. Indeed, let  $G$  be a function with compact support and  $0 \leq G \leq 1$ . Obviously,  $M^j(G) \leq 1$ . On the other hand, since  $\frac{z^{k-1}}{\bar{z}^{k+1}}$  does not belong to  $L^1(\mathbb{C})$ , we can make the left-hand side of (15) as big as we want.

### 3 Counterexample

This section is devoted to prove that condition (a) in the Theorem is sharp for  $k = 1$ . More precisely, we will prove that there exists a function  $f$  such that for each constant  $C > 0$  there exists a point  $z \in \mathbb{C}$  satisfying

$$B_S^* f(z) > CM(Bf)(z). \quad (16)$$

We choose  $f := B^{-1}(\chi_{Q_0})$ . For  $|z| > 2$  one has  $M(Bf)(z) = M(\chi_{Q_0})(z) \approx \frac{1}{|z|^2}$ . So, in order to get inequality (16) it is sufficient to prove that, for some  $z$ , one has

$$B_S^* f(z) \geq C \frac{\log |z|}{|z|^2}. \quad (17)$$

For  $m > 2$  (for instance,  $m = 5$ ), take  $\alpha \gg m$ , set  $z = \alpha + i\alpha$  and consider the truncated operator

$$(B)_Q^{2(\alpha+m)} f(z) = \frac{-1}{\pi} \int_{w \notin Q(z, 2(\alpha+m))} \frac{f(w)}{(z-w)^2} dw.$$



By definition  $B_Q^* f(z) \geq |(B)_Q^{2(\alpha+m)} f(z)|$  and then we will have (17) if we prove

$$|(B)_Q^{2(\alpha+m)} f(z)| \gtrsim \frac{\log |z|}{|z|^2}. \quad (18)$$

The idea is to decompose  $(B)_Q^{2(\alpha+m)} f(z)$  as a sum of certain terms. All terms, except one, can be bounded by  $C|z|^{-2}$  and the exceptional term will be of order  $|z|^{-2} \log |z|$ . We begin by writing the equality

$$(B)_Q^{2(\alpha+m)} f(z) = B^{\sqrt{2}(\alpha+m)} f(z) - \frac{1}{\pi} \int_E \frac{f(w)}{(w-z)^2} dw,$$

where  $E$  is the set  $B(z, \sqrt{2}(\alpha+m)) \setminus Q(z, 2(\alpha+m))$ . From the pointwise inequality (4) the first term is bounded by  $\frac{C}{|z|^2}$  and we just care about the second one. Set  $E = A_1 \cup A_2 \cup B$  where (see Figure 1)

$$\begin{aligned} A_1 &:= \{w \in E : \operatorname{Re}(w-z) < 0, \operatorname{Im}(w-z) < 0 \text{ and } \operatorname{Im}(w+m+im) > 0\}, \\ A_2 &:= \{w \in E : \operatorname{Re}(w-z) < 0, \operatorname{Im}(w-z) < 0 \text{ and } \operatorname{Im}(w+m+im) < 0\}, \\ B &:= E \setminus (A_1 \cup A_2). \end{aligned}$$

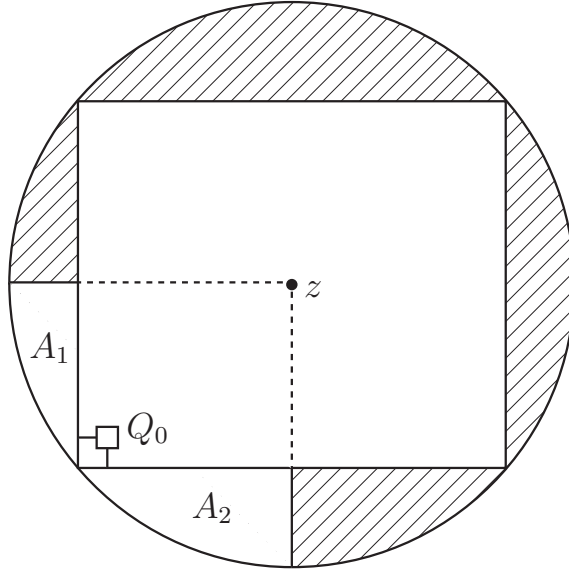


Figure 1

Thus

$$\int_E \frac{f(w)}{(w-z)^2} dw = \int_{A_1 \cup A_2} \frac{f(w)}{(w-z)^2} dw + \int_B \frac{f(w)}{(w-z)^2} dw.$$

On the set  $B$  we have  $|f(w)| = |B^{-1}(\chi_{Q_0})(w)| \leq \frac{C}{|w|^2}$  and then

$$\left| \int_B \frac{f(w)}{(w-z)^2} dw \right| \leq \int_B \frac{C}{|w-z|^2 |w|^2} dw \leq \frac{C|B|}{|z|^4} = \frac{C}{|z|^2}.$$

So, to get (18) it remains to prove that

$$\left| \int_{A_1 \cup A_2} \frac{f(w)}{(w-z)^2} dw \right| \geq C \frac{\log |z|}{|z|^2}. \quad (19)$$

For any  $w \in A_1 \cup A_2$  we write

$$f(w) = \frac{-1}{\pi} \int_{Q_0} \frac{1}{(\overline{w} - \xi)^2} d\xi = \frac{-1}{\pi} \frac{|Q_0|}{\overline{w}^2} - \frac{1}{\pi} \int_{Q_0} \left( \frac{1}{(\overline{w} - \xi)^2} - \frac{1}{\overline{w}^2} \right) d\xi.$$

By the mean value property, the last integral in the above equality is bounded by  $\frac{C}{|\overline{w}|^3}$ . Therefore, an easy computation gives

$$\left| \int_{A_1 \cup A_2} \left( f(w) + \frac{1}{\pi} \frac{|Q_0|}{\overline{w}^2} \right) \frac{1}{(w-z)^2} dw \right| \leq C \int_{A_1 \cup A_2} \frac{1}{|w|^3} \frac{1}{|w-z|^2} dw \leq \frac{C}{|z|^2}.$$

Consequently, inequality (19) holds if and only if

$$\left| \int_{A_1 \cup A_2} \frac{1}{(z-w)^2} \frac{1}{\overline{w}^2} dw \right| \geq C \frac{\log |z|}{|z|^2}. \quad (20)$$

Notice that, by the symmetry of the sets,  $w \in A_1$  if and only if  $i\overline{w} \in A_2$ . Thus the integral in (20) can be written as

$$\int_{A_1} \left( \frac{1}{(z-w)^2} \frac{1}{\overline{w}^2} + \frac{1}{(z-i\overline{w})^2} \frac{1}{(iw)^2} \right) dw.$$

Set  $A_1^- = \{w \in A_1 : \text{Im}(w) < 0\}$ ,  $A_1^+ = \{w \in A_1 : \overline{w} \in A_1^-\}$  and let  $D = A_1 \setminus (A_1^+ \cup A_1^-)$  (see Figure 2).

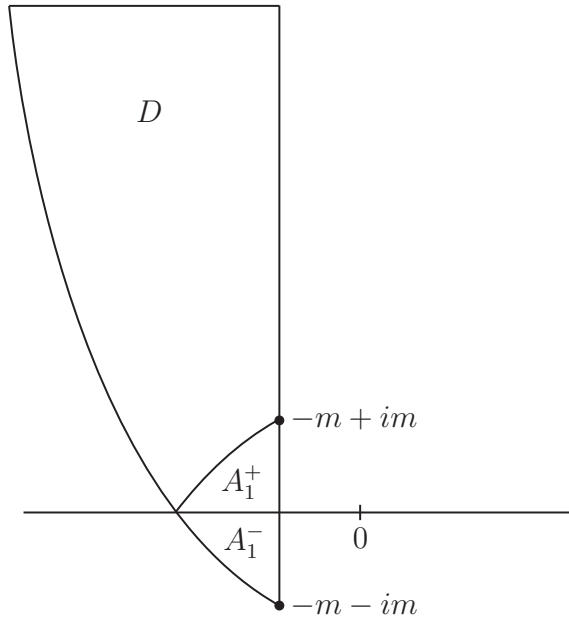


Figure 2

First of all we prove that the integral over  $A_1^+ \cup A_1^-$  is bounded by  $\frac{C}{|z|^2}$ . Using a change of variable in the integral over  $A_1^-$ , one can see that the integral over  $A_1^+ \cup A_1^-$  is

$$\int_{A_1^+} \left( \frac{1}{(z - \bar{w})^2} - \frac{1}{(z - i\bar{w})^2} \right) \frac{1}{w^2} dw + \int_{A_1^+} \left( \frac{1}{(z - w)^2} - \frac{1}{(z - iw)^2} \right) \frac{1}{\bar{w}^2} dw := I + II.$$

We are going to estimate  $|I|$  and  $|II|$  by  $\frac{C}{|z|^2}$ . Indeed,

$$|I| = \left| \int_{A_1^+} \frac{-2\bar{w}^2 + \bar{w}(2z - 2iz)}{w^2(z - \bar{w})^2(z - i\bar{w})^2} dw \right| \leq \frac{C}{|z|^4} \int_{A_1^+} \left( 1 + \frac{|z|}{|w|} \right) dw \leq \frac{Cm}{|z|^3} \leq \frac{C}{|z|^2},$$

where in the first inequality we have used that  $|z - \bar{w}| = |z - i\bar{w}| \approx |z|$ , and in the second one that  $\int_{A_1^+} |w|^{-1} \leq Cm$ . Using similar computations for the second term we get

$$|II| = \left| \int_{A_1^+} \frac{-2w^2 + w(2z - 2iz)}{\bar{w}^2(z - w)^2(z - iw)^2} dw \right| \leq \frac{C}{|z|^2}.$$

So, only remains to compute the integral over  $D$ , which we split again into two terms,

$$\begin{aligned} \int_D \left( \frac{1}{(z - w)^2} \frac{1}{\bar{w}^2} - \frac{1}{(z - i\bar{w})^2} \frac{1}{w^2} \right) dw &= \int_D \frac{1}{(z - w)^2} \left( \frac{1}{\bar{w}^2} - \frac{1}{w^2} \right) dw \\ &+ \int_D \frac{1}{w^2} \left( \frac{1}{(z - w)^2} - \frac{1}{(z - i\bar{w})^2} \right) dw := III + IV. \end{aligned}$$

We note that if  $w \in D$  then  $|z - \bar{w}| = |z - i\bar{w}| \approx |z|$  and  $m \leq |w| \leq 3|z|$ . We treat the term  $|IV|$  as before. In fact, we have

$$\begin{aligned} |IV| &= \int_D \frac{2z(w - i\bar{w}) - \bar{w}^2 - w^2}{w^2(z - w)^2(z - i\bar{w})^2} dw \leq \int_D \frac{2|w|^2 + 4|z||w|}{|w|^2|z - w|^2|z - i\bar{w}|^2} dw \\ &\leq \frac{C}{|z|^4} \int_D \left( 1 + \frac{|z|}{|w|} \right) dw \leq \frac{C|D|}{|z|^4} + \frac{C}{|z|^3} \int_m^{3|z|} 1 dr \leq \frac{C}{|z|^2}. \end{aligned}$$

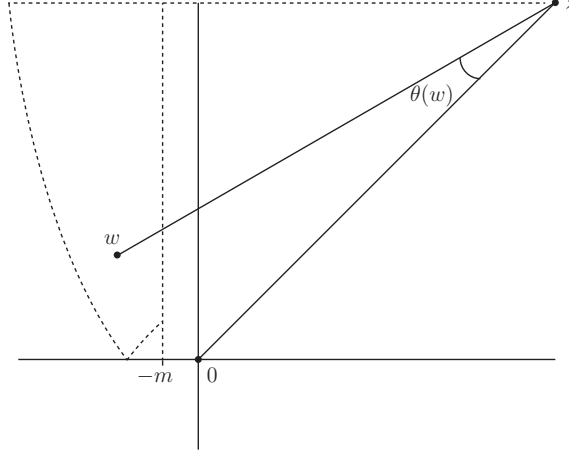


Figure 3

Finally the term  $III$  will give us that our counterexample works. For  $w \in D$  (see Figure 3) we write  $w - z = R(w)e^{i(\frac{5\pi}{4} - \theta(w))}$  where  $R(w) = |w - z| \approx |z|$  and  $0 \leq \theta(w) \leq \frac{\pi}{4}$ . So,  $i(w - z)^{-2} = -e^{i2\theta(w)}(R(w))^{-2}$ . Then we have

$$\begin{aligned} III &= 2i \int_D \frac{1}{(z - w)^2} \frac{\operatorname{Im} w^2}{|w|^4} dw = -2 \int_D \frac{e^{i2\theta(w)}}{R^2(w)} \frac{\operatorname{Im} w^2}{|w|^4} dw \\ &= -2 \int_D \frac{(\cos 2\theta(w) + i \sin 2\theta(w))}{R^2(w)} \frac{\operatorname{Im} w^2}{|w|^4} dw. \end{aligned}$$

Since  $\cos 2\theta(w) \geq 0$  and  $\operatorname{Im} w^2 \leq 0$ ,

$$|\operatorname{Re} III| = 2 \int_D \frac{\cos 2\theta(w)}{R^2(w)} \frac{|\operatorname{Im} w^2|}{|w|^4} dw \geq \frac{C}{|z|^2} \int_D \cos 2\theta(w) \frac{|\operatorname{Im} w^2|}{|w|^4} dw.$$

Fix  $\delta > 0$  and set

$$\tilde{D} := \{w \in D : \cos 2\theta(w) > \delta\} \cap \left\{w \in \mathbb{C} : \frac{\pi}{2} + \frac{\pi}{100} \leq \arg w \leq \pi - \frac{\pi}{100}\right\}.$$

Then, if  $w \in \tilde{D}$  we have  $|\operatorname{Im} w| \geq |w| \frac{\beta}{\sqrt{1+\beta^2}} \geq \frac{\beta|w|}{2}$ , where  $\beta = \tan \frac{\pi}{50}$ . Note that the Lebesgue measure of  $\tilde{D}$  is comparable to the Lebesgue measure of  $D$ . Finally, using polar coordinates we get

$$|\operatorname{Re} III| \geq C \frac{\delta\beta}{|z|^2} \int_{\tilde{D}} \frac{1}{|w|^2} dw \geq C \frac{\delta\beta}{|z|^2} \int_{4m}^{|z|/10} \frac{dr}{r} \geq C \frac{\delta\beta}{|z|^2} \log |z|,$$

which yields (20) and the counterexample.

## 4 Appendix

In this section we will prove, for the sake of the reader's convenience, identity (13). To get it we will write the term in the left hand side of (13) in another way. That is,

$$\begin{aligned}
& \left( \sum_{m=0}^{2j-2} (-1)^m \binom{4j}{2m+1} \frac{(2m+1)!(4j-2m-3)!}{m!(2j)!(2j-m-2)!2^{4j-2}} \right) - \frac{(4j)!}{(2j-1)!(2j)!2^{4j-1}} \\
&= \frac{(4j)!}{(2j)!(2j-1)!2^{4j-1}} \sum_{m=0}^{2j-1} (-1)^m \binom{2j-1}{m} \frac{1}{4j-2m-1} \\
&:= \frac{(4j)!}{(2j)!(2j-1)!2^{4j-1}} S,
\end{aligned}$$

where the last identity defines  $S$ . Thus, only remains to prove that

$$S = -\frac{(4j)!}{(2j)!(2j-1)!2^{4j-1}}. \quad (21)$$

Using twice the trivial fact that

$$\sum_{m=0}^{2j-1} (-1)^m \binom{2j-1}{m} = (1-1)^{2j-1} = 0,$$

we have

$$\begin{aligned}
S &= \sum_{m=0}^{2j-1} (-1)^m \binom{2j-1}{m} \left( \frac{1}{4j-2m-1} - 1 \right) \\
&= (-1)2(2j-1) \sum_{m=0}^{2j-2} (-1)^m \binom{2j-1}{m} \frac{1}{4j-2m-1} \\
&= (-1)2(2j-1) \sum_{m=0}^{2j-2} (-1)^m \binom{2j-1}{m} \left( \frac{1}{4j-2m-1} - \frac{1}{3} \right) \\
&= (-1)^2 2^2 (2j-1)(2j-2) \sum_{m=0}^{2j-3} (-1)^m \binom{2j-1}{m} \frac{1}{4j-2m-1}.
\end{aligned}$$

Iterating this process  $(2j-1)$  times we obtain (21), and so (13) is proved.

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