REMOVABLE SINGULARITIES FOR ANALYTIC FUNCTIONS IN THE LITTLE ZYGMUND SPACE

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Abstract. Using estimates of the law of the iterated logarithm type, a sharp sufficient condition for a compact set to be removable for analytic functions in the little Zygmund space is given.

1. Introduction and statement of results.

One classical problem in complex analysis is the description, in metric or geometric terms, of those plane compact sets K with the property that all functions, analytic outside K, which belong to some space of functions X can be extended analytically to the entire plane. In this case the set K is said to be removable for the analytic functions in X or more briefly X-removable. Of course, the description of such sets depends on the space X considered.

The problem we want to study in this paper is the case when X is the little Zygmund class, space that we define here for the reader's convenience.

The Zygmund space $\Lambda_*(\mathbb{C})$ is defined as the set of complex valued functions f in $\mathbb{C}$ which are bounded and such that

$$\|f\|_* = \sup_{z,h\in\mathbb{C}} \frac{|f(z+h)+f(z-h)-2f(z)|}{2|h|} < \infty \,. \tag{1.1}$$

It is known that the boundedness of f and (1.1) together imply that f is continuous with modulus of continuity verifying $\omega_f(\delta) = O(\delta|\log\delta|)$ as $\delta \to 0^+$. On the other hand it is easy to prove that $\Lambda_*(\mathbb{C})$, equipped with the norm $\|f\|_* + \|f\|_\infty$, is a Banach space.

The closure of bounded $\mathcal{C}^\infty$ functions in $\Lambda_*(\mathbb{C})$ is an important subspace which we denote by $\lambda_*(\mathbb{C})$ and it is called the little Zygmund space. It can be proved that such space consists in all Zygmund functions f for which the function

$$\eta(\delta) = \sup_{z\in\mathbb{C}} \sup_{|h|<\delta} \frac{|f(z+h)+f(z-h)-2f(z)|}{2|h|} \tag{1.2}$$

tends to zero as $\delta \to 0$. It is clear that η is an increasing function and it follows from its definition that it satisfies the doubling condition $\eta(2\delta) \le 2\eta(\delta)$ for $\delta > 0$.

The Zygmund space is closely related to the Lipschitz spaces (see [16]). Nevertheless the behaviour with respect to removability problems is essentially different. In order to describe sets that are Lipschitz-removable, let us remember the definition of Hausdorff measure.

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Given an increasing positive continuous function φ such that $\varphi(0) = 0$ the φ-Hausdorff measure of a set $E \subset \mathbb{C}$ is defined to be

$$H_\varphi(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum \varphi(\operatorname{diam}(B_j)) \,:\, B_j \text{ are open disks, } E \subset \cup B_j \,,\ \operatorname{diam}(B_j) < \varepsilon \right\}.$$

If in the last limit we let $\varepsilon \to \infty$ we obtain the definition of Hausdorff content, which we denote by $M_\varphi(E)$. Obviously $M_\varphi(E) \le H_\varphi(E)$ but it is not difficult to prove that $M_\varphi(E) = 0$ if and only if $H_\varphi(E) = 0$. In the case that $\varphi(t) = t^\beta$ for some $\beta > 0$, it is usual to denote $H_\varphi(E) = H_\beta(E)$ and the same for the Hausdorff content. We refer the reader to the book [13] for more information about Hausdorff measures.

Dolzenko [3] proved that a compact set K is Lip_α-removable ($0 < \alpha < 1$) if and only if $H_{1+\alpha}(K) = 0$. The case $\alpha = 1$ was solved by Uy in [17] and he proved that sets that are Lip_1-removable are those with zero area. On the other hand, when α goes to zero, the space that must be considered is BMO and for this case, Kaufman [8] proved that a compact set K is BMO-removable if and only if $H_1(K) = 0$.

The situation for the Zygmund space is completely different. Using the fact that the Cauchy transform of a bounded function belongs to $\Lambda_*(\mathbb{C})$ (see [6, p. 80]) it is not difficult to prove that compact sets of positive area are nonremovable for the analytic functions in $\Lambda_*(\mathbb{C})$. On the other hand, Uy [19] proved the existence of compact sets of zero area but nonremovable for $\Lambda_*(\mathbb{C})$ (see [1]). Finally, Kaufmann [9] proved that does not exist any measure function φ with the property $M_\varphi(K) = 0$ if and only if K is Λ_*-removable. Nevertheless in [1] it was shown that if $M_\Phi(K) = 0$ with

$$\Phi(t) = t^2 \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}} \tag{1.3}$$

then K is Λ_*-removable and that the function Φ is sharp in some sense.

In [4], a sufficient condition for Λ_*-removability was given in terms of a notion of porosity.

In many aspects, the role played by λ_* in Λ_* is the same as the played by the space lip_α in Lip_α. Consequently, in order to contextualize our problem is natural to ask what is the lip_α-removability condition. For this we require to remember the definition of the lower Hausdorff content.

$$M^*_\varphi(E) = \sup\{M_\psi(E) \,:\, 0 \le \psi \le \varphi,\ \psi \text{ is increasing, } \lim_{t \to 0^+} \frac{\psi(t)}{\varphi(t)} = 0\}.$$

It follows from the Comparison Theorem for Hausdorff measures that if E has measure H_φ σ-finite, then $M^*_\varphi(E) = 0$.

Using the notion of lower Hausdorff content, O'Farrell [15] proved that a compact set $K \subset \mathbb{C}$ is lip_α-removable if and only if $M^*_{1+\alpha}(K) = 0$. The corresponding result for VMO-removability was proved by Verdera in [20]. He showed that a compact set K is VMO-removable if and only if $M^*_1(K) = 0$.

The purpose of this paper is to prove a sharp sufficient condition for the λ_*-removability in terms of a lower Hausdorff content.

First of all we have to mention that in [4] it is noted that it is possible to see, using the main Lemma in [14], that a compact set of positive area is nonremovable for the analytic functions in $\lambda_*(\mathbb{C})$.

Our results are the following.

Theorem 1. *Let $K \subset \mathbb{C}$ be a compact set such that $M^*_\Phi(K) = 0$, being Φ the function defined in (1.3). Then K is λ_*-removable.*

Theorem 2. *For any continuous measure function Ψ such that $\Psi(t) = o(\Phi(t))$ as $t \to 0^+$ there exists a compact set $K \subset \mathbb{C}$ such that $M^*_\Psi(K) = 0$ and K is nonremovable for λ_*.*

The paper is organized as follows. In section 2 we prove Theorem 1. The proof of Theorem 2 is provided in section 3. The key point for this is Proposition 2. Since the proof of this one is rather technical we have decided to pospone it until the last section.

2. Proof of Theorem 1.

The ideas we are going to use in the proof of Theorem 1 are the same as the used in [1], that is, we will make use of the techniques of Stochastic Processes introduced by Makarov in [11] and [12]. Nevertheless the case λ_* is a little bit more delicate and requires a deeper treatment. Here we summarize the most important facts that we shall use concerning this theory.
As in [1] let us denote by Q_0 the unit square in $\mathbb{C}$ and by $\mathcal{D}_n$ the set of dyadic squares of the form

$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \times \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), \ 0 \le k, j < 2^n .$$

A dyadic martingale (S_n), $n \ge 0$, is a sequence of complex integrable functions defined on Q_0 such that for any $n \ge 0$, the function S_n is constant on every square of $\mathcal{D}_n$ and for any $R \in \mathcal{D}_n$ one has $\int_R S_n\, dm = \int_R S_{n+1}\, dm$.
If (S_n) is a dyadic martingale, we shall denote by (S^*_n) its maximal function, that is

$$S^*_n(z) = \max_{1 \le j \le n} |S_j(z)| .$$

An important fact is that if (S_n) is a dyadic martingale and $p \in (1, \infty)$ then

$$\int_{Q_0} (S^*_n)^p\, dm(z) \le C \int_{Q_0} |S_n|^p\, dm(z) ,$$

where C only depends on p. This is known as the Maximal Theorem for martingales.
If φ is a complex integrable function defined on Q_0, the conditional expectation $E(\varphi|\mathcal{D}_n)$ is the function which is constant on every square of $\mathcal{D}_n$ and takes the values

$$E(\varphi|\mathcal{D}_n)_{|Q} \equiv \frac{1}{|Q|} \int_Q \varphi\, dm ,$$

for any $Q \in \mathcal{D}_n$.
The exponential transform of a dyadic martingale (S_n) is the sequence defined by

$$Z_n = \frac{\exp(S_n)}{\Pi_{k=1}^n E\left(\exp(S_k - S_{k-1})|\mathcal{D}_{k-1}\right)} . \tag{2.1}$$

It is not hard to prove that the exponential transform of a dyadic martingale is again a dyadic martingale.
The following Proposition is the key in the proof of Theorem 1 (compare to Theorem A in [1] and Theorem 3.1 in [12]).

Proposition 1. *Let η be a positive increasing function such that it satisfies a doubling condition $\eta(2t) \le C\eta(t)$, $\lim_{t\to 0}\eta(t) = 0$ and $\int_0^1 \frac{\eta(s)^2}{s}\,ds = \infty$. Let (M_n) be a real dyadic martingale so that $M_0 = c > 0$ and $\|M_{n+1} - M_n\|_\infty = O(\eta(2^{-n}))$. Then*

$$H_\Psi(\{z \in Q_0 \,:\, M_n(z) > 0 \text{ for all } n \ge 0\}) > 0\,,$$

where

$$\Psi(t) = t^2\sqrt{\int_t^1 \frac{\eta(s)^2}{s}\,ds\,\log\log\int_t^1 \frac{\eta(s)^2}{s}\,ds}\,. \tag{2.2}$$

Remark. If $\displaystyle\int_0^1 \frac{\eta(s)^2}{s}\,ds$ is finite, H_Ψ is, up to a constant, the Lebesgue measure. On the other hand the convergence of the integral and the assumption on the increments of (M_n) together imply that the martingale converges a.e. to a function in the L^2-norm. A standard argument using the stopping time $\tau = \inf\{n \ge 0 \,:\, M_n < 0\}$ allows us to obtain the corresponding result.

For this reason, previous result has only a significant relevance if $\displaystyle\int_0^1 \frac{\eta(s)^2}{s}\,ds = \infty$. Assuming that this proposition has been proved, let us derive the proof of Theorem 1.

Proof of Theorem 1. Suppose K is a compact set with

$$M^*_\Phi(K) = 0\,. \tag{2.3}$$

We can assume that $K \subset (0,1)\times(0,1)$. If K is not λ_*-removable there exists a function $f \in \lambda_*(\mathbb{C})$ which is analytic on $\mathbb{C}\setminus K$ and so that f cannot be extended holomorphically to $\mathbb{C}$. Consequently there exists a square Q for which

$$\int_{\partial Q} f(z)\,dz \ne 0\,.$$

Multiplying by a suitable constant we can assume that

$$\mathrm{Re}\int_{\partial Q_0} f(z)\,dz = c > 0\,.$$

For any $n \ge 0$ let us define the following sequence of functions (S_n) on Q_0,

$$S_n(z) = \frac{1}{|Q|}\mathrm{Re}\int_{\partial Q} f(\zeta)\,d\zeta\,,$$

where Q is the unique square in $\mathcal{D}_n$ which contains z. It is not difficult to check that (S_n) is a dyadic martingale that verifies $\|S_{n+1} - S_n\|_\infty \le C\eta(2^{-n})$, where $C > 0$ is an absolute constant and η is the function associated to f defined in (1.2). On the other hand, if $z \in Q_0 \setminus K$ there exists a natural number N so that for any $n \ge N$, the square of $\mathcal{D}_n$ which contains the point z is disjoint from K. Since f is analytic on the complement of K we obtain $S_n(z) = 0$ for any $n \ge N$. Consequently, the compact set K contains the set of points $z \in Q_0$ for which $S_n(z) > 0$ for all $n \ge 0$.

By assumption, $f \in \lambda_*(\mathbb{C})$ and thus, the function η defined in (1.2) verifies the hypotheses of Proposition 1. Moreover the martingale (S_n) is also in the hypotheses of this Proposition. Consequently, the compact set K verifies that $H_\Psi(K) > 0$, being Ψ the function (2.2). Since $\lim_{t\to 0}\eta(t) = 0$,

$$\lim_{t\to 0}\frac{1}{\log\frac{1}{t}}\int_t^1 \frac{\eta(s)^2}{s}\,ds = 0\,,$$

which implies that $M^*_\Phi(K) > 0$ and this contradicts (2.3). □

Thus, in order to prove Theorem 1 it is enough to prove Proposition 1. For getting this purpose, we will need several technical lemmas. From now on the function η that appears in the statement of Proposition 1 will remain fixed. We can suppose, without lose of generality that $\eta(1) \le 1$.

We shall use the following notation.

$$V(t) = \int_t^1 \frac{\eta(s)^2}{s}\,ds\,. \tag{2.4}$$

In view of the doubling property of function η it is clear that $V(2^{-n})$ is comparable to $\sum_{j=0}^n \eta(2^{-j})^2$.

Lemma 1. *Let (M_n) be the martingale of Proposition 1. There exists a constant β so that if $(Z_n^{(t)})$ is the exponential transform of (tM_n) $(t>0)$, then*

$$Z_n^{(t)} \ge \exp(tM_n - \beta t^2 V(2^{-n}))\,.$$

Proof. As in [1] let us denote by α the maximum of the function

$$f(x_1,\dots,x_4) = \frac{4\log\frac{1}{4}\sum_{j=1}^4 e^{x_j}}{\sum_{j=1}^4 x_j^2}\,,$$

restricted to the set $\{(x_1,\dots,x_4)\,:\,\sum_{j=1}^4 x_j = 0, \sum_{j=1}^4 x_j^2 \ne 0\}$.

The constant α has been chosen in such a way that the inequality

$$E(\exp(M_k - M_{k-1})|\mathcal{D}_{k-1}) \le \exp(\alpha E((M_k - M_{k-1})^2|\mathcal{D}_{k-1})) \tag{2.5}$$

is optimal.

From last inequality we get

$$\begin{aligned} Z_n^{(t)} &= \frac{\exp(tM_n)}{\Pi_{k=1}^n E\left(\exp(tM_k - tM_{k-1})|\mathcal{D}_{k-1}\right)} \\ &\ge \frac{\exp(tM_n)}{\Pi_{k=1}^n \exp(\alpha t^2 E((M_k - M_{k-1})^2|\mathcal{D}_{k-1}))} \\ &= \exp\left(tM_n - \alpha t^2\sum_{k=1}^n E((M_k - M_{k-1})^2|\mathcal{D}_{k-1})\right) \end{aligned} \tag{2.6}$$

Taking into account that $|M_k - M_{k-1}| \le C\eta(2^{-k})$ we have that

$$\sum_{k=1}^n E((M_k - M_{k-1})^2|\mathcal{D}_{k-1})) \le C\sum_{k=1}^n \eta(2^{-k})^2 \le C_1 V(2^{-n})\,.$$

Using this inequality in (2.6) and writing $\beta = C_1\alpha$, we obtain the desired result. □

Lemma 2. *Under the hypothesis of Proposition 1, if $a > c$ then*

1. $m(\{z\in Q_0\,:\,M_n^*(z) > a\}) \le 2\exp\left(-\frac{(a-c)^2}{4\beta V(2^{-n})}\right)$.
2. *For any positive integer p, $\int_{Q_0}(M_n^*)^{2p}\,dm \le C_1^p V(2^{-n})^p p!$, where C_1 is an absolute constant.*

The proof of this Lemma follows from Lemma 1 using similar arguments as in the proof of Lemma 2 in [1].
The final step in the proof of Proposition 1 requires the construction of a special measure that will be supported on the set $\{z : M_n(z) > 0 \text{ for all } n\}$.
Let us consider the stopping time $\tau(z) = \inf\{n : M_n \leq 0\}$ if last set is nonempty and $\tau(z) = \infty$ if $M_n(z) > 0$ for any n. Using this stopping time, define the following sequence of measures

$$d\mu_n = M_{n\wedge\tau}^+ \, dm \, .$$

It is clear that $\mu_n(Q_0) \geq c > 0$, because $(M_{n\wedge\tau})$ is a dyadic martingale and consequently, has constant expectation. On the other hand, since $|M_n - M_{n-1}| \leq C_2\eta(2^{-n})$, for some constant C_2, we have that $M_{n\wedge\tau}^- \in [0, C_2\eta(1)]$. Consequently

$$\mu_n(Q_0) = \int_{Q_0} M_{n\wedge\tau}^+ \, dm = \int_{Q_0} M_{n\wedge\tau} \, dm + \int_{Q_0} M_{n\wedge\tau}^- \, dm \leq c + C_2\eta(1) \, .$$

Thus we have that

$$c \leq \mu_n(Q_0) \leq c + C_2\eta(1) \, ,$$

for any $n \geq 0$.
Let μ be some limit measure of the sequence (μ_n) in the weak-star topology which will remain fixed trough the rest of this section.
The following Lemma is the analogous of Lemma 3.4 in [12].

Lemma 3. *There exists an absolute constant $C > 0$ such that*

$$\int_{Q_0} (M_{n\wedge\tau})^{2p+1} \, dm \leq C^p p! V(2^{-n})^p \, , \tag{2.7}$$

for any nonnegative integer p.

Proof. For simplicity, let us denote $P_n = M_{n\wedge\tau}$. If n is so that, $V(2^{-n}) \leq 2^{2p}$, the proof is a simple consequence of the second part Lemma 2 and the fact that $(2p)! \leq 4^p (p!)^2$.

$$\begin{aligned}
\int_{Q_0} P_n^{2p+1} \, dm &\leq \left(\int_{Q_0} (M_n^*)^{4p} \, dm\right)^{1/2} \left(\int_{Q_0} (M_n^*)^2 \, dm\right)^{1/2} \\
&\leq \Big(C_1^{2p} V(2^{-n})^{2p} (2p)!\Big)^{1/2} \big(C_1 V(2^{-n})\big)^{1/2} \leq (4C_1)^p C_1^{1/2} V(2^{-n})^p
\end{aligned}$$

and the result follows if C is chosen large enough.
If $V(2^{-n}) > 2^{2p}$ the proof of (2.7) is not so easy and it will be made by induction on p. If $p = 0$, then (2.7) is obvious.
Let us denote by $\triangle P_k = P_k - P_{k-1}$, that is, $(\triangle P_k)$ are the increments of the martingale (P_n). It is clear that $|\triangle P_k| \leq |\triangle M_k| \leq C_2\eta(2^{-k})$. Then

$$\int_{Q_0} P_k^{2p+1} \, dm = \int_{Q_0} (P_{k-1} + \triangle P_k)^{2p+1} \, dm = \sum_{j=0}^{2p+1} \binom{2p+1}{j} \int_{Q_0} P_{k-1}^{2p+1-j} (\triangle P_k)^j \, dm \, .$$

Since (P_n) is a martingale, the term corresponding to $j = 1$ is zero, consequently,

$$\int_{Q_0} P_k^{2p+1} \, dm - \int_{Q_0} P_{k-1}^{2p+1} \, dm = \sum_{j=2}^{2p+1} \binom{2p+1}{j} \int_{Q_0} P_{k-1}^{2p+1-j} (\triangle P_k)^j \, dm \, .$$

Using last identity we have

$$\begin{aligned}
\int_{Q_0} P_n^{2p+1}\,dm &= \sum_{k=1}^{n}\left(\int_{Q_0} P_k^{2p+1}\,dm - \int_{Q_0} P_{k-1}^{2p+1}\,dm\right) + \int_{Q_0} P_0^{2p+1}\,dm \\
&= \sum_{k=1}^{n}\sum_{j=2}^{2p+1}\binom{2p+1}{j}\int_{Q_0} P_{k-1}^{2p+1-j}(\triangle P_k)^j\,dm + \int_{Q_0} P_0^{2p+1}\,dm \\
&= \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j+1}\int_{Q_0} P_{k-1}^{2(p-j)}(\triangle P_k)^{2j+1}\,dm \\
&+ \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j}\int_{Q_0} P_{k-1}^{2(p-j)+1}(\triangle P_k)^{2j}\,dm + \int_{Q_0} P_0^{2p+1}\,dm\,.
\end{aligned}$$

Let us use the following notations.

$$\begin{aligned}
\Sigma_1 &= \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j+1}\int_{Q_0} P_{k-1}^{2(p-j)}(\triangle P_k)^{2j+1}\,dm\,, \\
\Sigma_2 &= \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j}\int_{Q_0} P_{k-1}^{2(p-j)+1}(\triangle P_k)^{2j}\,dm\,,
\end{aligned}$$

that is

$$\int_{Q_0} P_n^{2p+1}\,dm = \Sigma_1 + \Sigma_2 + c^{2p+1}\,.$$

In order to estimate Σ_1 we shall use Lemma 2 and the fact that $\eta(2^{-k})^{2j+1} \leq \eta(2^{-k})^2$ for any k and $j \geq 1$.

$$\begin{aligned}
(2.8)\quad \Sigma_1 &\leq \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j+1}\int_{Q_0} |P_{k-1}|^{2(p-j)}|\triangle P_k|^{2j+1}\,dm \\
&\leq \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j+1} C_1^{p-j}(p-j)!V(2^{-k-1})^{p-j}C_2^{2j+1}\eta(2^{-k})^{2j+1} \\
&\leq \sum_{j=1}^{p}\binom{2p+1}{2j+1} C_1^{p-j}C_2^{2j+1}(p-j)!\sum_{k=1}^{n} V(2^{-k+1})^{p-j}\eta(2^{-k})^2\,.
\end{aligned}$$

Now for estimating Σ_2, let us note first that if $P_{k-1}(z) < 0$ then $\triangle P_k(z) = 0$. Consequently,

$$\begin{aligned}
\int_{Q_0} P_{k-1}^{2(p-j)+1}(\triangle P_k)^{2j}\,dm &= \int_{\{P_{k-1}\geq 0\}} P_{k-1}^{2(p-j)+1}(\triangle P_k)^{2j}\,dm \\
&\leq C_2^{2j}\eta(2^{-k})^{2j}\int_{\{P_{k-1}\geq 0\}} P_{k-1}^{2(p-j)+1}\,dm \\
&= C_2^{2j}\eta(2^{-k})^{2j}\int_{Q_0} P_{k-1}^{2(p-j)+1}\,dm \\
&- C_2^{2j}\eta(2^{-k})^{2j}\int_{\{P_{k-1}<0\}} P_{k-1}^{2(p-j)+1}\,dm \\
&\leq C_2^{2j}\eta(2^{-k})^{2j}\int_{Q_0} P_{k-1}^{2(p-j)+1}\,dm + C_2^{2p+1}\eta(2^{-k})^{2p+1}\,.
\end{aligned}$$

Using last inequality and the induction hypothesis we have

$$\begin{aligned}
\Sigma_2 &\le \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j}\left(C_2^{2j}\eta(2^{-k})^{2j}\int_{Q_0}P_{k-1}^{2(p-j)+1}\,dm + C_2^{2p+1}\eta(2^{-k})^{2j}\right)\\
&\le \sum_{k=1}^{n}\sum_{j=1}^{p}\binom{2p+1}{2j}\left(C_2^{2j}\eta(2^{-k})^{2j}C^{p-j}(p-j)!V(2^{-(k-1)})^{p-j}+C_2^{2p+1}\eta(2^{-k})^{2j}\right)\\
&\le \sum_{j=1}^{p}\binom{2p+1}{2j}C_2^{2j}C^{p-j}(p-j)!\left(\sum_{k=1}^{n}V(2^{-(k-1)})^{p-j}\eta(2^{-k})^2\right)\\
&+\ 2^{2p}C_2^{2p+1}\sum_{k=1}^{n}\eta(2^{-k})^2 .
\end{aligned}$$

Now observe that $\sum_{k=1}^{n}V(2^{-(k-1)})^{p-j}\eta(2^{-k})^2$ is less than or equal to

$$\int_{2^{-n}}^{1}V(t)^{p-j}\frac{\eta(t)^2}{t}\,dt=\frac{V(2^{-n})^{p-j+1}}{p-j+1}\,,$$

consequently, using the last fact, inequality (2.8) and the estimate for Σ_2 we have

$$\begin{aligned}
\int_{Q_0}P_n^{2p+1}\,dm \le \sum_{j=1}^{p}\left(\binom{2p+1}{2j+1}C_1^{p-j}C_2^{2j+1}+\binom{2p+1}{2j}C_2^{2j}C^{p-j}\right)(p-j)!\frac{V(2^{-n})^{p-j+1}}{p-j+1}\\
+\ 2^{2p+1}C_2^{2p+1}V(2^{-n})+c^{2p+1} .
\end{aligned}$$

In the last sum, we separate the term corresponding to $j=1$ and estimate the others in the trivial way.

$$\begin{aligned}
\int_{Q_0}P_n^{2p+1}\,dm &\le (p-1)!\left[\binom{2p+1}{3}C_1^{p-1}C_2^3+\binom{2p+1}{2}C_2^2C^{p-1}\right]\frac{V(2^{-n})^p}{p}\\
&+\ (p-2)!V(2^{-n})^{p-1}2^{2p+1}\left(C_1^{p-2}C_2^{2p+1}+C_2^{2p}C^{p-2}\right)\\
&+\ 2^{2p+1}C_2^{2p+1}V(2^{-n})+c^{2p+1}\\
&\le 2(p+1)!V(2^{-n})^pC_1^{p-1}C_2^3+3p!C_2^2C^{p-2}V(2^{-n})^p\\
&+\ (p-2)!V(2^{-n})^{p-1}2^{2p+1}\left(C_1^{p-2}C_2^{2p+1}+C_2^{2p}C^{p-2}\right)\\
&+\ 2^{2p+1}C_2^{2p+1}V(2^{-n})+c^{2p+1}\\
&= p!V(2^{-n})^pC^p\left[2(p+1)\frac{C_1^{p-1}}{C^p}+\frac{3}{C}+\frac{2^{2p+1}}{p(p-1)C^2V(2^{-n})}\right.\\
&+\left.\frac{c^{2p+1}}{p!V(2^{-n})^pC^p}+\frac{2^{2p+2}}{p!V(2^{-n})^{p-1}C^p}\right]\le p!V(2^{-n})^pC^p\,,
\end{aligned}$$

where the last inequality follows if C has been chosen large enough so that the sum inside the parenthesis is smaller than 1. $\square$

From Lemma 3 we deduce that

$$\int_{Q_0}|M_{n\wedge\tau}|^{2p+1}\le C^pp!V(2^{-n})^p\,, \tag{2.9}$$

because on the set $\{M_{n\wedge\tau}<0\}$ we have that $M_{n\wedge\tau}>-\eta(1)$.

Inequality (2.9) and the maximal theorem for martingales allows us to obtain that

$$\int_{Q_0} (M^*_{n\wedge\tau})^{2p+1} \le C^p p! V(2^{-n})^p \,. \tag{2.10}$$

Lemma 4. *If $p \ge 0$ then*

$$\int_{Q_0} (M^*_n)^{2p}\, d\mu \le C^p p! V(2^{-n})^p \,,$$

where C is an absolute constant.

Proof. Since M^*_n is constant on every square of $\mathcal{D}_n$, we have that

$$\begin{aligned}
\int_{Q_0} (M^*_n)^{2p}\, d\mu &= \int_{Q_0} (M^*_n)^{2p}\, M^+_{n\wedge\tau}\, dm = \int_{Q_0} (M^*_n)^{2p}\, M_{n\wedge\tau}\, dm + \int_{Q_0} (M^*_n)^{2p}\, M^-_{n\wedge\tau}\, dm \\
&= \int_{Q_0} (M^*_n)^{2p}\, M_{n\wedge\tau}\, dm - \int_{\{\tau\le n\}} (M^*_n)^{2p}\, M_\tau\, dm \\
&= \int_{\{\tau>n\}} (M^*_n)^{2p}\, M_n\, dm - 2\int_{\{\tau\le n\}} (M^*_n)^{2p}\, M_\tau\, dm \\
&\le \int_{\{\tau>n\}} (M^*_n)^{2p+1}\, M_n\, dm + 2\eta(1) \int_{\{\tau\le n\}} (M^*_n)^{2p}\, dm \le C^p p! V(2^{-n})^p \,,
\end{aligned}$$

because of (2.10). $\square$

With these previous results, we can prove Proposition 1.

Proof of Proposition 1. For $\ell \ge 0$, let us consider the following sequence of positive integers.

$$n_\ell = \min\{n \,:\, V(2^{-n}) \ge 2^\ell\} \,.$$

It is clear that

$$2^\ell \le V(2^{-n_\ell}) \le 2^{\ell+1} \,. \tag{2.11}$$

Let us consider the following sequence of sets.

$$A_\ell = \{z \in Q_0 \,:\, M_n(z) \le M\varphi(V(2^{-n})) \text{ for any } n \ge n_\ell\} \,,$$

being $\varphi(s) = \sqrt{s \log\log s}$ and M a constant large enough that will be chosen later. Now observe that

$$\begin{aligned}
Q_0 \setminus A_\ell &\subset \cup_{j\ge\ell}\{z \in Q_0 \,:\, M_k(z) > M\varphi(V(2^{-k})) \text{ for some } k \in [n_j, n_{j+1}]\} \\
&\subset \cup_{j\ge\ell}\{z \in Q_0 \,:\, M^*_{n_{j+1}}(z) > M\varphi(V(2^{-n_j}))\} \,.
\end{aligned}$$

On the other hand, for any positive integer p,

$$\begin{aligned}
\mu(\{M^*_{n_{j+1}}(z) > M\varphi(V(2^{-n_j}))\}) &\le \frac{1}{M^p \varphi(V(2^{-n_j}))^p} \int_{Q_0} \left(M^*_{n_{j+1}}\right)^p d\mu \\
&\le \frac{1}{M^p \varphi(V(2^{-n_j}))^p} \left(\int_{Q_0} \left(M^*_{n_{j+1}}\right)^{2p} d\mu\right)^{1/2} \mu(Q_0)^{1/2} \,.
\end{aligned}$$

By Lemma 4, last expression is less than or equal to

$$\frac{1}{M^p \varphi(V(2^{-n_j}))^p} C^{p/2} (p!)^{1/2} V(2^{-n_{j+1}})^{p/2} \mu(Q_0)^{1/2} \,,$$

which is controlled by

$$\frac{1}{M^p \varphi(2^{-j})^p} C^{p/2} (p!)^{1/2} (2^{j+1})^{p/2} \mu(Q_0)^{1/2} \le \left(\frac{Cp}{M^2 \log j}\right)^{p/2} \,, \tag{2.12}$$

because of (2.11) and Stirling's formula. Now choose p the integer part of $M^2 \log j/2C$. If M is large enough, we have that the right hand term of (2.12) is bounded above by $1/2j^2$. Consequently

$$\mu(\cup_l A_l) = \mu(Q_0)\,.$$

Let B be set $\{z \in Q_0 \,:\, M_n(z) > 0 \text{ for any } n \geq 0\}$. Then by the definition of μ we have that

$$c \leq \mu(Q_0) = \mu(B \cup \bigcup_l A_l) \tag{2.13}$$

Let us consider a covering of B by dyadic squares. If $R \in \mathcal{D}_n$ is one of these squares, then

$$\begin{aligned}\mu(R) &= \lim_{k\to\infty} \int_R M^+_{k\wedge\tau}\, dm = \lim_{k\to\infty} \left(\int_R M_{k\wedge\tau}\, dm + \int_R M^-_{k\wedge\tau}\, dm \right) \\ &\leq \int_R M_{k\wedge\tau}\, dm + \eta(1) m(R) \leq M\varphi(V(2^{-n}))m(R) \\ &\leq M\Psi(\operatorname{diam} R)\,,\end{aligned}$$

where Ψ is the function which appears in the statement of Proposition 1.
Now, returning to (2.13) we obtain that

$$0 < c \leq \sum_j \mu(R_j) \leq M \sum_j \Psi(\operatorname{diam} R_j)\,,$$

for any dyadic covering of B. Consequently, $M_\Psi(B) > 0$. $\square$

3. Proof of Theorem 2.

The proof of Theorem 2 is a consequence of the following two facts.

Proposition 2. *For any increasing function η such that $\eta(0) = 0$, $\eta(2t) \leq C\eta(t)$ and $\int_0 \frac{\eta(s)^2}{s}\, ds = \infty$ there exists a compact set $K \subset (0, 1)$ such that*

(1) *$H_\psi(K) < \infty$, being*

$$\psi(t) = t\sqrt{\int_t^1 \frac{\eta(s)^2}{s}\, ds \log\log \int_t^1 \frac{\eta(s)^2}{s}\, ds}\,. \tag{3.1}$$

(2) *K supports a positive singular measure μ such that*

$$|\mu(I) - \mu(I')| \leq C|I|\eta(|I|)\,, \tag{3.2}$$

for any pair of adjacent intervals I and I' of the same length.

Proposition 3. *Let μ be a positive measure on $\mathbb{C}$ such that for any pair of adjacent squares Q and Q', of the same size and with edges parallel to the axes,*

$$|\mu(Q) - \mu(Q')| = o(m(Q)) \ \text{ as } m(Q) \to 0\,. \tag{3.3}$$

Then the function

$$f(z) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{\zeta - z}$$

is continuous in $\mathbb{C}$ and belongs to $\lambda_*(\mathbb{C})$.

Proposition 3 can be proved following the same scheme as the main result in [18] Assume that Propositions 2 and 3 have been proved.

Proof of Theorem 2. Let us consider a positive increasing function η in the assumptions of Proposition 2 such that

$$\Psi(t) = o(t\psi(t)),$$

being ψ the function defined on (3.1). Let K be the compact set given by this Proposition and define $F = K \times [0,1]$. If we denote by $\eta(t) = t\psi(t)$, it is not difficult to check that $H_\eta(F)$ is finite and thus $M^*_\Psi(F) = 0$. In order to prove that F is not λ_*-removable, first we shall see that it supports a measure ν verifying (3.3).

Let h be a positive $\mathcal{C}^2$ function supported on $(0,1)$ and let ν be the measure $\mu \times h\, dy$. It is clear that ν is supported on F. For $j = 1,2$, let us denote by $Q_j = [a_j, b_j] \times [c_j, d_j]$ two adjacent squares with side length h. We have two cases. First, if $a_1 = a_2$,

$$|\nu(Q_1) - \nu(Q_2)| = \left| \mu([a_1,b_1]) \left(\int_{c_1}^{d_1} h(s)\, ds - \int_{c_2}^{d_2} h(s)\, ds \right) \right| \le Ch^3 \log(1/h) = o(h^2).$$

The second case is $c_1 = c_2$. Then

$$|\nu(Q_1) - \nu(Q_2)| = \int_{c_1}^{d_1} h(s)\, ds\, |\mu([a_1,b_1]) - \mu([a_2,b_2])| \le Ch^2\eta(h) = o(h^2).$$

Now, by Proposition 3, the Cauchy transform of the measure ν belongs to the little Zygmund space. Moreover is analytic on $\mathbb{C} \setminus F$ but not on the entire plane, because ν is not the zero measure. $\square$

4. Estimates on the size of Kahane's compact sets.

The proof of Proposition 2 requires an accurate estimate on the size of a collection of compact sets defined by Kahane in [7], similar to the obtained by Makarov in [10]. Actually, the scheme we are going to follow is the same of Makarov.

Fix the function η appearing in the statement of Proposition 2. For $j \ge 1$ let c_j be the smallest power of $1/2$ which is bigger than $\eta(4^{-j})$. With this definition, (c_j) is a decreasing sequence of positive numbers such that c_j is comparable to $\eta(4^{-j})$ and for any $j \ge 1$, $c_{j+1} = c_j$ or $c_{j+1} = c_j/2$. On the other hand, since $\int_0 \frac{\eta(s)^2}{s}\, ds = \infty$ and η is doubling it turns out that $\sum_{n=1}^\infty c_n^2 = \infty$.

For any $n \ge 0$ let us denote by $\mathcal{Q}_n$ the set of the intervals of the form

$$[\frac{k}{4^n}, \frac{k+1}{4^n}) \;\; 0 \le k < 4^n .$$

We are going to define a sequence of simple functions (S_n) in such a way that for any $n \ge 0$, S_n is constant on every interval of $\mathcal{Q}_n$. Let S_0 be the function which takes the value 1 over the interval $[0,1)$ and assume that $S_1, \dots, S_n$ have been defined. Given $I \in \mathcal{Q}_n$ let $I_1, \dots, I_4$ be the four intervals in $\mathcal{Q}_{n+1}$ which are contained in I and let I_- and I_+ be the intervals of $\mathcal{Q}_n$ which are adjacent to I on the left and on the right respectively.

If $j \in \{1,2,3,4\}$, we define

$$S_{n+1}(I_j) = S_n(I) + \epsilon_j c_{n+1},$$

where $\epsilon_j \in \{-1,1\}$, $\epsilon_1\epsilon_2=-1$, $\epsilon_3\epsilon_4=-1$ and

$$\begin{aligned} \epsilon_1 = 1 &\quad \text{if} \quad S_n(I_-) > S_n(I) \\ \epsilon_1 = -1 &\quad \text{if} \quad S_n(I_-) \le S_n(I) \\ \epsilon_4 = 1 &\quad \text{if} \quad S_n(I_+) > S_n(I) \\ \epsilon_4 = -1 &\quad \text{if} \quad S_n(I_+) \le S_n(I) \end{aligned}$$

With this definition, if $I \in \mathcal{Q}_n$ it is clear that $S_n(I)$ is, choosing the corresponding sequence of signs, equal to $1 \pm c_1 \pm \cdots \pm c_n$.
We define

$$E = \{x \in [0,1) \,:\, S_n(x) > 0 \text{ for any } n \ge 1\}\,. \tag{4.1}$$

For $n \ge 0$, let M_n be the function given by

$$M_n(x) = \begin{cases} S_n(x) & \text{if } S_1(x),\dots,S_n(x) > 0 \\ 0 & \text{otherwise} \end{cases} ,$$

and let $\mu_n = M_n\,dx$. It is clear that (μ_n) is a sequence of probability measures. Let μ be some limit in the weak star topology. From the definition, it follows that μ is supported on E. Moreover the assumptions on η assure that E has Lebesgue measure zero.
In [7] it is proved that the measure μ verifies (3.2). Thus our goal is to prove that $H_\psi(E) < \infty$.

Lemma 5. *There exists a positive constant α such that for any $I \in \mathcal{Q}_m$ and for any $k > 0$,*

$$\mu\big(I \cap \{S_{m+k} - S_m > \big(\sum_{j=m+1}^{m+k} c_j^2\big)^{1/2}\}\big) \ge \alpha\mu(I)\,.$$

Proof. For $n \ge 1$, let us denote $E_n = \operatorname{supp} M_n$ and, for convenience, $\sigma = \big(\sum_{j=m+1}^{m+k} c_j^2\big)^{1/2}$.
If $M_m(I) = 0$ then $\mu(I) = 0$ and there is nothing to prove. Thus assume that $M_m(I) = a > 0$.
It is clear that

$$\mu\big(I \cap \{S_{m+k} - S_m > \sigma\}\big) = \sum_{s>0} \mu\big(I \cap \{S_{m+k} - S_m > \sigma\} \cap \{S_{m+k} = s\}\big)\,,$$

because in the last sum, there is only a finite number of nonvanishing terms. On the other hand the set $I \cap \{S_{m+k} - S_m > \sigma\} \cap \{S_{m+k} = s\}$ is union of intervals of $\mathcal{Q}_{m+k}$. Consequently, for this set, the measures μ and μ_{m+k} take the same value. Thus, if we denote by λ the Lebesgue measure in $[0,1]$,

$$\begin{aligned} \mu\big(I \cap \{S_{m+k} - S_m > \sigma\}\big) &= \sum_{s>0} \mu_{m+k}\big(I \cap \{S_{m+k} - S_m > \sigma\} \cap \{S_{m+k} = s\}\big) \\ &= \sum_{s>0} s\lambda\big(I \cap \{S_{m+k} - S_m > \sigma\} \cap \{S_{m+k} = s\} \cap E_{m+k}\big) \\ &= \sum_{s>S} s\lambda\big(I \cap \{S_{m+k} > a + S\} \cap \{S_{m+k} = s\} \cap E_{m+k}\big) \\ &\ge (a+S)\lambda(I \cap E_{m+k} \cap \{S_{m+k} > a + S\})\,. \end{aligned}$$

Consequently, we have obtained the following estimate.

$$\frac{1}{\mu(I)}\mu\big(I\cap\{S_{m+k}-S_m>\sigma\}\big)\geq\frac{a+\sigma}{a}\frac{\lambda(I\cap E_{m+k}\cap\{S_{m+k}>a+S\})}{\lambda(I)}\,. \tag{4.2}$$

Now observe that last quotient of measures coincides with the probability

$$P(T_1>-a,T_2>-a,\dots,T_k>\sigma)\,,$$

where $T_n=\sum_{j=1}^n c_{m+j}X_j$ with X_j independent Bernouilli random variables. By the Reflexion Principle for random walks ([5]), last probability is equal to

$$P(T_k>\sigma)-P(T_k>2a+\sigma)\,.$$

We get finally that the right hand term of (4.2) is

$$(1+\frac{\sigma}{a})P(1\leq\frac{T_k}{\sigma}\leq 1+2\frac{a}{\sigma})$$

and this is bounded below because of definition of σ and the Berry-Essen uniform version of the Central Limit Theorem (see [2]). $\square$

For convenience, let us define the following sequence of positive integers.

$$V_n=\inf\{k\,:\,\sum_{j=1}^k c_j^2\geq n\}\,. \tag{4.3}$$

Lemma 6. *There exists a positive constant δ so that for any interval $I\in\mathcal{Q}_{V_{2^{n-1}}}$ one has*

$$\mu\big(I\cap\{S_{V_{2^n}}-S_{V_{2^{n-1}}}\geq\delta(2^n\log_2 n)^{1/2}\}\big)\geq\frac{1}{n}\mu(I)\,.$$

Proof. Let k be the biggest integer such that $2^k\leq\frac{\log n}{\log(1/\alpha)}$, and define $\delta=(4\log(1/\alpha))^{-1/2}$ where α is the constant appearing in the statement of Lemma 5. For j between 0 and 2^k let us define $d(j)=2^{n-1}(1+j/2^k)$ and the set

$$C_j=\{S_{V_{d(j)}}-S_{V_{d(j-1)}}>\sqrt{2^{-k}2^{n-1}}\}\,.$$

It is clear that if $x\in\bigcap_{j=1}^{2^k}C_j$ then $S_{V_{2^n}}(x)-S_{V_{2^{n-1}}}(x)>\sqrt{2^k2^{n-1}}$ and taking into account the definition of k we have that x belongs to the set for which we want to estimate the measure. Thus it is enough to prove that

$$\mu\big(I\cap\bigcap_{j=1}^{2^k}C_j\big)\geq\frac{1}{n}\mu(I)\,. \tag{4.4}$$

For this purpose, let j be an integer between 1 and 2^k and suppose that J is an interval of the generation $V_{d(j-1)}$. Then by Lemma 5,

$$\mu(J\cap\{S_{V_{d(j)}}-S_{V_{d(j-1)}}>\sum_{\ell=V_{d(j-1)}+1}^{V_{d(j)}}c_\ell^2\})\geq\alpha\mu(J)\,.$$

Now observe that, by (4.3), the set in (4.4) is exactly C_j, consequently,

$$\mu\big(I\cap\bigcap_{j=1}^{2^k}C_j\big)=\mu\big(C_{2^k}\cap I\cap\bigcap_{j=1}^{2^k-1}C_j\big)\geq\alpha\mu\big(I\cap\bigcap_{j=1}^{2^k-1}C_j\big)\geq\dots\geq\alpha^{2^k}\mu(I)\geq\frac{1}{n}\mu(I)\,,$$

because of the definition of k. $\square$

Now let us consider the following sequence of sets.

$$A_N = \left\{ x \,:\, S_k(x) \le \delta \left(\sum_{j=1}^{k} c_j^2 \log_2 \log_2 \sum_{j=1}^{k} c_j^2 \right)^{1/2} \text{ , for any } k \in [V_{\log_2 N}, V_N] \right\} ,$$

where δ is the constant in Lemma 6.

Lemma 7.

$$\mu(A_{2^N}) = O(\frac{\log N}{N}) \ \textit{as}\ N \to \infty \,.$$

Proof. For $j \ge 1$ let us consider the following set.

$$B_j = \{x \,:\, S_{V_{2^j}}(x) \le \delta\sqrt{2^j \log_2 j}\} \,,$$

being δ the same constant as in lemma 6.

If $x \in A_{2^N}$ then $S_k(x) \le \delta \sqrt{\sum_{j=1}^{k} c_j^2 \log_2 \log_2 \sum_{j=1}^{k} c_j^2}$ for any k between V_N and V_{2^N}. In particular we obtain that x is in B_j for all $\log_2 N < j < N$. Therefore,

$$A_{2^N} \subset \bigcap_{j=\log_2 N}^{N} B_j \,.$$

Now, fixed j, it is clear that B_j is a union of intervals of length $4^{-V_{2^j}}$. If I is an interval of the generation $V_{2^{j-1}}$, then

$$\mu(B_j \cap I) = \mu(I) - \mu(I \setminus B_j) = \mu(I) - \mu(I \cap \{S_{2V_{2^j}} > \delta\sqrt{2^j \log_2 j}\}) \,.$$

Since the martingale (S_n) is positive on the support of μ, we obtain

$$\mu(I \cap \{S_{V_{2^j}} > \delta\sqrt{2^j \log_2 j}\}) \ge \mu(I \cap \{S_{V_{2^j}} - S_{V_{2^{j-1}}} > \delta\sqrt{2^j \log_2 j}\}) \,.$$

By Lemma 6, we have that the previous measure is bigger than $\mu(I)/j$. Consequently,

$$\mu(B_j \cap I) \le (1 - \frac{1}{j})\mu(I) \,.$$

Then, by the additivity of μ we have

$$\mu(\cap_{k=\log_2 N}^{j} B_k) = \mu(B_j \cap \cap_{k=\log_2 N}^{j-1} B_k) \le \frac{j-1}{j} \mu(\cap_{k=\log_2 N}^{j-1} B_k) \,.$$

Iterating this inequality we obtain the desired result, because

$$\mu(A_{2^N}) \le \mu(\cap_{j=\log_2 N}^{N} B_j) \le \prod_{j=log_2 N}^{N} \frac{j-1}{j} = \frac{\log_2 N}{N} \,. \quad \square$$

From Lemma 7 we deduce directly that if $2^n \le N \le 2^{n+1}$, then

$$\mu(A_N) = O(\frac{\log \log N}{\log N}) \,. \tag{4.5}$$

Lemma 8. $\lambda(E_{V_N} \cap A_N) = O(\frac{\log \log N}{\sqrt{N \log N}})$ *as* $N \to \infty$*, being* V_N *be the sequence defined in (4.3).*

Proof. Let M be the integer part of $\sqrt{\dfrac{N}{\log N}}$ and write,

$$\lambda(E_{V_N}\cap A_N) = \sum_{j=1}^{M}\lambda(E_{V_N}\cap A_N\cap\{j-1<S_{V_N}\le j\}) + \sum_{j>M}\lambda(E_{V_N}\cap A_N\cap\{j-1<S_{V_N}\le j\})\,. \tag{4.6}$$

In order to estimate the second sum of (4.6) let us observe that the set appearing there is union of dyadic intervals of length 4^{-V_N}. Thus

$$\begin{aligned}\lambda(E_{V_N}\cap A_N\cap\{j-1<S_{V_N}\le j\}) &=\le \frac{1}{j-1}\,\mu(E_{V_N}\cap A_N\cap\{S_{V_N}=s\})\\ &\le \frac{1}{j-1}\mu(A_N\cap\{S_{V_N}=s\})\,.\end{aligned}$$

Hence the second sum of (4.6) is less than or equal to

$$\sum_{j>M}\frac{1}{j-1}\mu(A_N\cap\{j-1<S_{V_N}\le j\})\le\frac{1}{M}\mu(A_N)\le C_1\,\frac{\log\log N}{\sqrt{N\log N}}\,,$$

because of (4.5) and the selection of M.
For the first term of (4.6) we use again the Reflexion principle and the central Limit Theorem. If (W_n) is a random walk with increments (c_n), using again the Reflection Principle, the first sum of the right term of (4.6) is less than or equal to

$$\begin{aligned}&\sum_{j=1}^{M}\lambda(E_{V_N}\cap\{j-1<S_{V_N}\le j\})\\ \le\ &\sum_{j=1}^{M}P(W_1>0,\dots,W_{V_N-1}>0,\{j-1<S_{V_N}\le j\}|W_0=1)\\ =\ &\sum_{j=1}^{M}(P(\{j-1<W_{V_N}\le j\}|W_0=1)-P(\{j-1<W_{V_N}\le j\}|W_0=-1))\\ =\ &\sum_{j=1}^{M}(P(\{j-2<W_{V_N}\le j-1\}|W_0=0)-P(\{j<W_{V_N}\le j+1\}|W_0=0))\\ =\ &P(\{-1<W_{V_N}\le 1\}|W_0=0)-P(\{M-1<W_{V_N}\le M+1\}|W_0=0).\end{aligned}$$

By the central Limit theorem and the integral mean value theorem, this last difference becomes

$$C_2\frac{1}{\sqrt{N}}(e^{-x_1^2/2}-e^{-x_2^2/2})\,,$$

where $x_1\in(\frac{-1}{\sqrt{N}},\frac{1}{\sqrt{N}})$ and $x_2\in(\frac{1}{\sqrt{\log N}}-\frac{1}{\sqrt{N}},\frac{1}{\sqrt{\log N}}+\frac{1}{\sqrt{N}})$.
Finally,since $|x_2-x_1|\le\dfrac{2}{\sqrt{\log N}}$, a direct aplication of the mean value theorem gives us the desired estimate that we want to see. $\square$

Now, we are able to prove Proposition 2
Proof of Proposition 2.
Given $\varepsilon > 0$, we are going to see that the set E defined in (4.1) has measure $H_\psi^{(\varepsilon)}(E) < c$, where c does not depend on ε and ψ is the measure function defined on (3.1).
Let N be a positive integer such that $V_{\log_2 N} > \log_4 \varepsilon^{-1}$. Then

$$H_\psi^{(\varepsilon)}(E) \le H_\psi^{(\varepsilon)}(E_{V_N} \setminus A_N) + H_\psi^{(\varepsilon)}(E_{V_N} \cap A_N)\,. \tag{4.7}$$

We are going to estimate the first term of (4.7). Let us observe that $E_{V_N} \setminus A_N$ is $\mathcal{Q}_{V_N}$-measurable. If I is one the dyadic intervals of length 4^{-V_N} contained in $E_{V_N} \setminus A_N$ then for any $x \in I$ we have

$$S_k(x) > \delta \sqrt{\sum_{j=1}^k c_j^2 \log\log \sum_{j=1}^k c_j^2}$$

for some k between $V_{\log_2 N}$ and V_N. Moreover, the first k with this property is the same for all points in I. So we can consider

$$k_I = \min \left\{ k \in [V_{\log_2 N}, V_N] : S_k(x) > \delta \sqrt{\sum_{j=1}^k c_j^2 \log\log \sum_{j=1}^k c_j^2}, x \in I \right\}.$$

Let $J(I)$ be the interval of length 4^{-k_I} containing I. Then for intervals $I_1 \neq I_2$, of length 4^{-V_N} which appears in the decomposition of $E_{V_N} \setminus A_N$, or $J(I_1) = J(I_2)$ either they are disjoint. Let $\mathcal{F}$ the familiy of intervals $J(I_n)$, with $I_n \in \mathcal{Q}_{V_N}$ and $I_n \cap (E_{V_N} \setminus A_N) \neq \emptyset$.
If $J \in \mathcal{F}$ has length 4^{-k}, then

$$\begin{aligned}
\mu(J) &= S_k(J) 4^{-k} \ge \delta 4^{-k} \sqrt{\sum_{j=1}^k c_j^2 \log\log \sum_{j=1}^k c_j^2} \\
&\ge \delta 4^{-k} c_1 \sqrt{\int_{4^{-k}}^1 \frac{\eta(s)^2}{s}\, ds \log\log \int_{4^{-k}}^1 \frac{\eta(s)^2}{s}\, ds} \\
&\ge c_1 \delta \psi(|J|)\,.
\end{aligned}$$

On the other hand, since $k \ge V_{\log_2 N}$ we have that $4^{-k} < \varepsilon$. Hence

$$H_\psi^{(\varepsilon)}(E_{V_N} \setminus A_N) \le \sum_{J \in \mathcal{F}} \psi(|J|) \le \frac{1}{c_1 \delta} \sum_{J \in \mathcal{F}} \mu(J) = \frac{1}{c_1 \delta}\,.$$

For the second term of (4.7) we have

$$\begin{aligned}
H_\psi^{(\varepsilon)}(E_{V_N} \cap A_N) &= \psi(4^{-V_N}) 4^{V_N} P(E_{V_N} \cap A_N) \\
&\le c_1 \sqrt{\sum_{j=1}^{V_N} c_j^2 \log\log \sum_{j=1}^{V_N} c_j^2} P(E_N \cap A_N) \\
&\le c_2 \frac{\log\log N}{\sqrt{\log N}} \le c_3\,,
\end{aligned}$$

by Lemma 8 and the definition of V_N. $\square$

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