C*-ALGEBRAS OF STABLE RANK ONE AND THEIR CUNTZ SEMIGROUPS

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Abstract. The uncovering of new structure on the Cuntz semigroup of a C*-algebra of stable rank one leads to several applications: We answer affirmatively, for the class of stable rank one C*-algebras, a conjecture by Blackadar and Handelman on dimension functions, the Global Glimm Halving problem, and the problem of realizing functions on the cone of 2-quasitraces as ranks of Cuntz semigroup elements. We also gain new insights into the comparability properties of positive elements in C*-algebras of stable rank one.

1. Introduction

A great deal of a C*-algebra's structure is encoded in its Cuntz semigroup. This is perhaps most apparent in the construction of the Cuntz semigroup from Hilbert C*-modules by Coward, Elliott and Ivanescu. In this picture, elements of the Cuntz semigroup are equivalence classes of countably generated Hilbert C*-modules under a certain relation (see [CEI08]). This equivalence relation, although in general weaker than isomorphism, agrees with the isomorphism relation for C*-algebras of stable rank one (that is, C*-algebras whose invertible elements form a dense set). Thus, the Cuntz semigroup of a C*-algebra of stable rank one is simply the set of isomorphism classes of countably generated Hilbert C*-modules (over the C*-algebra) endowed with the order induced by the embedding of Hilbert C*-modules, and with the addition operation induced by the direct sum of Hilbert C*-modules.

In this paper we investigate Cuntz semigroups of C*-algebras of stable rank one. By unraveling fine structural properties of these objects, we are able to resolve relevant questions on dimension functions and on divisibility and comparability properties of C*-algebras of stable rank one. These results push further the work by the fourth author in [Thi17].

Stable rank one is a strong form of finiteness. Nevertheless, C*-algebras of stable rank one are ubiquitous in manifold C*-algebra constructions (see [AP15, DHR97, Rør04]). Further, stable rank one does not constitute a regularity property of the kind encountered in the Elliott classification program of simple nuclear C*-algebras, such as $\mathcal{Z}$-stability or finite nuclear dimension. For example, Toms's examples of non-regular C*-algebras in [Tom06, Tom08] have stable rank one. Our results bring about new insights into the structure of these elusive objects.

Given a C*-algebra A, let us denote its Cuntz semigroup by $\mathrm{Cu}(A)$. One of our key results is as follows:

Theorem (3.8)**.** *Let A be a separable C*-algebra of stable rank one. Then every pair of elements in $\mathrm{Cu}(A)$ has an infimum. Further, addition in $\mathrm{Cu}(A)$ is distributive over the infimum operation.*

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Directly stated in terms of Hilbert C*-modules, the existence of infima in $\mathrm{Cu}(A)$ reads as follows: given countably generated Hilbert C*-modules E and F over A, there exists a countably generated Hilbert C*-module $E \wedge F$, unique up to isomorphism, that embeds in both E and F, and such that if a countably generated Hilbert C*-module H embeds in both E and F then it embeds in $E \wedge F$. This remarkable property proves especially useful when combined with the properties of the Cuntz semigroup encapsulated in the abstract axioms of Cu-semigroups. Equipped with these tools, we tackle a number of questions which we describe next.

1.1. *A conjecture by Blackadar and Handelman.* Let A be a unital C*-algebra. Let $W(A)$ denote the subsemigroup of $\mathrm{Cu}(A)$ of classes $[H]$ such that $H \hookrightarrow A^n$ for some $n \in \mathbb{N}$. A map $d\colon W(A) \to [0,\infty)$ is called a dimension function if it is additive, order-preserving and $d([A]) = 1$. More concretely, a dimension function assigns to each topologically finitely generated Hilbert C*-module a non-negative real number in such a way that this number behaves as expected with respect to direct sums and inclusions and such that A (as a module over itself) has dimension 1. Denote by $DF(A)$ the set of all dimension functions endowed with the topology of pointwise convergence. Blackadar and Handelman conjectured in [BH82] that $DF(A)$ is a Choquet simplex for all C*-algebras A. This conjecture has been confirmed in a number of instances, but it remains open in general; see [Per97, BPT08, ABPP14, dS16]. The existence of infima (or rather, the Riesz Interpolation Property) in the Cuntz semigroup readily implies that $DF(A)$ is a Choquet simplex. We thus confirm the Blackadar-Handelman conjecture for all unital C*-algebras of stable rank one (Theorem 4.1).

1.2. *The Global Glimm Halving Problem.* A result of Glimm says that if a C*-algebra A has an irreducible representation of dimension at least $k \in \mathbb{N}$, then there exists a non-zero *-homomorphism from $M_k(C_0((0,1]))$ into A. The Global Glimm Halving problem (for unital C*-algebras) asks whether there exists a *-homomorphism of $M_k(C_0((0,1]))$ into A with full range (that is, the range generates A as a closed two-sided ideal) provided that A has no nonzero, finite dimensional representations. This problem remains open in general. It is answered affirmatively in [ER06] for C*-algebras of real rank zero. It is also considered in [BK04] and [RR13]. In [RR13], the Global Glimm Halving problem is translated into an equivalence of divisibility properties on the Cuntz semigroup of the C*-algebra. We rely on this alternative formulation in order to solve the problem affirmatively for C*-algebras of stable rank one. In the unital case, this reads as follows:

Theorem (5.7, 9.1)**.** *Let A be a unital C*-algebra of stable rank one, and let $k \in \mathbb{N}$. Then A has no nonzero representations of dimension less than k if and only if there exists a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to A$ with full range.*

1.3. *Realizing functions on $QT(A)$ as ranks of Cuntz semigroup elements.* Let $QT(A)$ denote the set of lower semicontinuous $[0,\infty]$-valued 2-quasitraces on a C*-algebra A. It is well known, stemming from the work of Blackadar and Handelman in [BH82] that each $\tau \in QT(A)$ gives rise to a function $d_\tau\colon \mathrm{Cu}(A) \to [0,\infty]$, that preserves addition, order and suprema of increasing sequences. More precisely, given a countably generated Hilbert C*-module H,

$$d_\tau([H]) = \sup\big\{\tau(\langle x, x\rangle) : x \in H,\ \|x\| \leqslant 1\big\}.$$

Let us now fix an element $[H] \in \mathrm{Cu}(A)$ and consider the map $QT(A) \to [0,\infty]$ given by $\tau \mapsto d_\tau([H])$. This is called the *rank* induced by $[H]$.

The realization problem asks to describe the functions on $\mathrm{QT}(A)$ that arise as ranks of elements of $\mathrm{Cu}(A)$. A variation on this problem, more frequently encountered in the literature, considers the functions induced by Cuntz semigroup elements on the Choquet simplices of tracial states, or 2-quasitracial states, and again asks exactly which functions are obtained in this way. Ranks of Cuntz semigroup elements are linear, lower semicontinuous, and satisfy a technical approximation property whose definition we defer to § 6.4. The collection of all functions with these properties is denoted by $L(QT(A))$. One can then ask, more concretely, whether all functions in $L(QT(A))$ can be realized as ranks of Cuntz semigroup elements. An affirmative answer for simple, unital, separable C*-algebras of stable rank one is given by the fourth author in [Thi17]. We extend the techniques of [Thi17] to remove the assumption of simplicity and the existence of a unit and obtain:

Theorem (7.10)**.** *Let A be a separable C*-algebra of stable rank one that has no nonzero, elementary ideal-quotients (that is, there are no closed, two-sided ideals $J \subseteq I$ of A such that I/J is a nonzero elementary C*-algebra). Then every function in $L(QT(A))$ can be realized as the rank of a Cuntz semigroup element.*

We prove a similar theorem in the traditional set-up where one seeks to realize functions on the 2-quasitracial states as ranks; see Theorems 7.11 and 9.3.

1.4**.** *Comparability properties.* Comparability properties in the Cuntz semigroup, such as strict comparison, m-comparison, or finite radius of comparison, measure degrees of regularity of the C*-algebra. For simple nuclear C*-algebras, the Toms-Winter conjecture asserts the equivalence of the strict comparison property with 'harder' forms of regularity such as $\mathcal{Z}$-stability and finite nuclear dimension. Regularity in the Cuntz semigroup, however, may be encountered in C*-algebras that are both non-nuclear and tensorially prime (for example, the reduced C*-algebra of the free group in infinitely many generators has strict comparison).

The additional structure in the Cuntz semigroup brought about by the stable rank one property entails that seemingly different comparability properties are in fact equivalent. Although our results do not require the assumption of simplicity, we highlight here the simple unital case (see Section 8 for the relevant definitions):

Theorem (cf. Theorem 8.12, Theorem 8.13)**.** *Let A be a simple, unital, separable C*-algebra of stable rank one.*

(i) *A has finite radius of comparison in the sense of Toms (*[Tom06]*) if and only if the subsemigroup $W(A)$ consists precisely of the elements in* $\mathrm{Cu}(A)$ *whose rank is a bounded function on the set of 2-quasitracial states.*
(ii) *If A has either m-comparison for some $m \in \mathbb{N}$ (in the sense defined by Winter in* [Win12]*) or local weak comparison (in the sense defined by Kirchberg and Rørdam in* [KR14]*) then A has strict comparison.*

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2. Preliminaries

2.1. *The Cuntz semigroup.* Let A be a C*-algebra. Denote by A_+ the positive elements in A. Let us recall the definition of the Cuntz semigroup of A in terms of positive elements: Given $a, b \in A_+$, one says that a is *Cuntz smaller* than b, or that a is *Cuntz subequivalent* to b, denoted $a \precsim b$, if there exists a sequence $(d_n)_n$ in A such that $d_n^* b d_n \to a$. The elements a and b are *Cuntz equivalent*, denoted $a \sim b$, if $a \precsim b$ and $b \precsim a$. This is an equivalence relation. Let $[a]$ denote the equivalence class of a. The Cuntz semigroup of A is defined as

$$\mathrm{Cu}(A) := \big\{ [a] : a \in (A \otimes \mathcal{K})_+ \big\}.$$

That is, $\mathrm{Cu}(A)$ is the set of Cuntz equivalence classes of positive elements in the C*-algebra $A \otimes \mathcal{K}$. (Here, and in the sequel, $\mathcal{K}$ denotes the C*-algebra of compact operators on the Hilbert space $\ell^2(\mathbb{N})$.) The Cuntz semigroup $\mathrm{Cu}(A)$ is endowed with the order $[a] \leqslant [b]$ if $a \precsim b$ and the addition operation $[a] + [b] := [a' + b']$, where $a', b' \in (A \otimes \mathcal{K})_+$ are chosen in such a way that $a \sim a'$, $b \sim b'$ and $a'b' = 0$ (such elements always exist). In this way, $\mathrm{Cu}(A)$ is an abelian, partially ordered semigroup.

We will focus largely on Cuntz semigroups of C*-algebras of stable rank one. As pointed out in the introduction, in this case the Cuntz semigroup is isomorphic to the set of isomorphism classes of countably generated Hilbert C*-modules over the C*-algebra. In this picture, and if A has stable rank one, given Hilbert C*-modules H_1 and H_2 over A, we have $[H_1] \leqslant [H_2]$ if H_1 embeds in H_2 as a Hilbert C*-submodule and $[H_1] + [H_2] := [H_1 \oplus H_2]$; see [CEI08, Theorem 3].

2.2. *The category* Cu. Some of the properties of the Cuntz semigroup of a C*-algebra can be abstracted into a category termed Cu, whose objects are called *abstract Cuntz semigroups*, or simply Cu-*semigroups*. We recall the main definitions.

Let S be an abelian, partially ordered semigroup. Given $x, y \in S$, let us write $x \ll y$ if whenever $(y_n)_n$ is an increasing sequence in S such that the supremum $\sup_n y_n$ exists and satisfies $y \leqslant \sup_n y_n$, then there exists n_0 such that $x \leqslant y_{n_0}$. This is a transitive relation on S, sometimes called the *way-below relation* or also the *compact containment relation*; see [GHK+03, Definition I-1.1, p.49] and [APT18b, Paragraph 2.1.1, p.11] for details.

The semigroup S is called a Cu-semigroup if it satisfies the following axioms:

- (O1) Every increasing sequence in S has a supremum.
- (O2) For each $x \in S$ there exists a sequence $(x_n)_n$ such that $x_n \ll x_{n+1}$ for every n, and $x = \sup_n x_n$.
- (O3) If $x' \ll x$ and $y' \ll y$, then $x' + y' \ll x + y$.
- (O4) If $(x_n)_n$ and $(y_n)_n$ are increasing sequences in S, then $\sup_n(x_n + y_n) = \sup_n x_n + \sup_n y_n$.

We call a sequence $(x_n)_n$ satisfying $x_n \ll x_{n+1}$ for all n a $\ll$-*increasing sequence*. It is sometimes called a *rapidly increasing sequence*.

Given Cu-semigroups S and T, a Cu-*morphism* from S to T is a map $S \to T$ that preserves addition, order, the relation $\ll$ and suprema of increasing sequences. The category Cu has as objects the Cu-semigroups, and as morphisms the Cu-morphisms.

2.3. It was proved in [CEI08] that the Cuntz semigroup $\mathrm{Cu}(A)$ of a C*-algebra A is a Cu-semigroup. Further, every *-homomorphism $\varphi\colon A \to B$ between C*-algebras induces a Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ by sending the class of $a \in (A \otimes \mathcal{K})_+$ to the class of $\varphi(a) \in (B \otimes \mathcal{K})_+$. This defines a functor from the category of C*-algebras to the category Cu. By [APT18b, Corollary 3.2.9.], this functor preserves arbitrary inductive limits (sequential inductive limits are covered by [CEI08, Theorem 2]). Further, the functor preserves (co)products and ultraproducts; see [APT18a].

2.4. *Almost algebraic order.* The Cuntz semigroup of a C*-algebra is known to satisfy an additional axiom which we now describe.

Let S be a Cu-semigroup. We say that S has *almost algebraic order*, or that S satisfies axiom (O5), if given $x', x, y', y, z \in S$ such that $x + y \leqslant z$, $x' \ll x$, and $y' \ll y$, there exists $w \in S$ such that $x' + w \leqslant z \leqslant x + w$ and $y' \ll w$.

If A is a C*-algebra, then $\mathrm{Cu}(A)$ satisfies (O5); see [RW10] and [APT18b, Section 4, p.31ff].

2.5. A Cu-semigroup S is said to have *weak cancellation* if for every $x, y, z \in S$, the condition $x + z \ll y + z$ implies $x \ll y$. If A is a C*-algebra with stable rank one, then $\mathrm{Cu}(A)$ has weak cancellation; see [RW10].

2.6. Let S be a Cu-semigroup. Recall that S is said to be *countably based* if there exists a countable subset $B \subseteq S$ such that every element in S is the supremum of a $\ll$-increasing sequence with elements in B. If A is a separable C*-algebra, then $\mathrm{Cu}(A)$ is countably based; see for example [APS11], or [Rob13, Proposition 5.1.1]. One important consequence of having a countably based semigroup is recorded in the following basic result:

Lemma. *Every upward directed set in a countably based* Cu*-semigroup has a supremum.*

2.7. Let S be a Cu-semigroup. Recall that an ideal of S is an order-hereditary submonoid I of S that is closed under suprema of increasing sequences. We define $x \leqslant_I y$ to mean that $x \leqslant y + z$ for some $z \in I$, and write $x \sim_I y$ if both $x \leqslant_I y$ and $y \leqslant_I x$ happen. The quotient S/I is defined as $S/{\sim_I}$. We refer to [APT18b, Section 5.1] for details.

If A is a C*-algebra and I is a closed, two-sided ideal of A, then the inclusion map $I \to A$ induces a Cu-morphism $\mathrm{Cu}(I) \to \mathrm{Cu}(A)$ that identifies $\mathrm{Cu}(I)$ with an ideal in $\mathrm{Cu}(A)$. Further, it was proved in [CRS10] that the quotient map $A \to A/I$ induces an isomorphism $\mathrm{Cu}(A)/\mathrm{Cu}(I) \cong \mathrm{Cu}(A/I)$.

The following proposition is a crucial ingredient in the proofs of Theorems 3.5, 3.8, and 7.2. By embedding the Cuntz semigroup of a C*-algebra as an ideal of a larger Cuntz semigroup, it introduces suitable compact elements associated to elements of the original Cuntz semigroup.

2.8. **Proposition.** *Let A be a stable C*-algebra, and let $a \in A_+$. Then there exists a C*-algebra B and a projection $p_a \in B$ such that:*

(i) *A is a closed, two-sided ideal of B.*
(ii) *For $x \in \mathrm{Cu}(A)$, we have $x \leqslant [a]$ in $\mathrm{Cu}(A)$ if and only if $x \leqslant [p_a]$ in $\mathrm{Cu}(B)$.*
(iii) *If A has stable rank one, then so has B.*

Proof. Since the Hilbert C*-module $H = \overline{aA}$ is singly generated, it follows from Kasparov's stabilization theorem that H is a direct summand of $\ell^2(A)$, that is, there is a Hilbert C*-module H' such that $\overline{aA} \oplus H' \cong \ell^2(A)$. On the other hand, since A is stable, $\ell^2(A) \cong A$ as Hilbert C*-modules. Thus, $\overline{aA}$ is isomorphic to a complemented Hilbert C*-submodule of A. Denote by $M(A)$ the multiplier algebra of A and let $p_a \in M(A)$ be the projection onto this submodule. Then $\overline{aA} \cong p_a A$.

Now let $B = C^*(p_a, A) \subseteq M(A)$. By construction A is a closed two-sided ideal of B and thus (i) is verified.

(ii): Let $x \in \mathrm{Cu}(A)$. Since A is stable, there exists $b \in A_+$ such that $x = [b]$. Suppose that $x \leqslant [p_a]$ in $\mathrm{Cu}(B)$. Then $b \precsim p_a$ in B, and thus for every $\varepsilon > 0$ there exists $v \in p_a B$ such that $(b - \varepsilon)_+ = v^* v$. As $v^* v \in A$, we also have $v \in A$, and therefore $v \in p_a B \cap A = p_a A \cong \overline{aA}$. Hence, $(b-\varepsilon)_+ \precsim a$ in A for all $\varepsilon > 0$, from which we conclude that $x \leqslant [a]$ in $\mathrm{Cu}(A)$. The converse is clear once we observe that $[a] \leqslant [p_a]$ in $\mathrm{Cu}(B)$.

(iii): Assume that A has stable rank one. By construction, $B/A \cong \mathbb{C}$. Thus, B is an extension of A and $\mathbb{C}$, which both have stable rank one. Using [Rie83, Theorem 4.11], it follows that B has stable rank one. □

3. Riesz Interpolation and infima

In this section, we prove that the Cuntz semigroup $\mathrm{Cu}(A)$ of any C*-algebra A of stable rank one has the Riesz Interpolation Property. If A is also separable, then it follows that every pair of elements in $\mathrm{Cu}(A)$ has an infimum. Further, this semilattice structure is compatible with addition; see Theorem 3.8.

In the sequel, we write multiple inequalities in the compact form $x_1, x_2 \leqslant y_1, y_2$.

3.1. The following axiom was introduced in [Thi17]. A Cu-semigroup S is said to satisfy axiom (O6+) if for every $a, b, c, x', x, y', y \in S$ satisfying

$$a \leqslant b + c\,, \quad x' \ll x \leqslant a, b\,, \quad \text{and} \quad y' \ll y \leqslant a, c,$$

there exist $e, f \in S$ such that

$$a \leqslant e + f\,, \quad x' \ll e \leqslant a, b\,, \quad \text{and} \quad y' \ll f \leqslant a, c.$$

Axiom (O6+) is a strengthening of the axiom (O6) of almost Riesz decomposition introduced in [Rob13]. Unlike (O6), which is known to hold for the Cuntz semigroup of any C*-algebra, there are C*-algebras whose Cuntz semigroup does not satisfy (O6+). However, it was shown in [Thi17, Theorem 6.4] that the Cuntz semigroup of any C*-algebra of stable rank one satisfies (O6+).

3.2. **Lemma.** *Let S be a* Cu*-semigroup, and let $B \subseteq S$ be an order-hereditary subset of S that is closed under suprema of increasing sequences. Define*

$$B_\ll = \left\{ x \in S : \textit{there is } y \in B \textit{ such that } x \ll y \right\}.$$

If $B_\ll$ is an upward directed set, then this is also the case for B.

Proof. Let $x, y \in B$. Choose $\ll$-increasing sequences $(x_n)_n$ and $(y_n)_n$ in S such that $x = \sup_n x_n$ and $y = \sup_n y_n$. Then $x_n, y_n \in B_\ll$ for each n. Since $B_\ll$ is upward directed, there exists $z_1 \in B_\ll$ such that $x_1, y_1 \leqslant z_1$. Suppose that, for $n \geqslant 2$, there are $z_1 \leqslant z_2 \leqslant \ldots \leqslant z_n$ in $B_\ll$ such that $x_n, y_n \leqslant z_n$. Using again that $B_\ll$ is upward directed, we may choose $z_{n+1} \in B_\ll$ such that $x_{n+1}, y_{n+1}, z_n \leqslant z_{n+1}$. Now let $z = \sup_n z_n$. By construction $x, y \leqslant z$. Further z belongs to B since by assumption this set is closed under suprema of increasing sequences. □

The lemma below is contained in [Thi17], though not explicitly stated. We reproduce the proof here for convenience.

3.3. **Lemma.** *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5) and (O6+), and let $e, x \in S$. Assume that e is compact. Then the set*

$$\left\{ z \in S : z \leqslant e, x \right\}$$

is upward directed.

Proof. Since the set $\{z \in S : z \leqslant e, x\}$ is order-hereditary and closed under suprema of increasing sequences, it suffices to show by Lemma 3.2 that the set

$$\left\{ z' \in S : \text{there is } z \in S \text{ such that } z' \ll z \leqslant e, x \right\}$$

is upward directed.

Let $z_1', z_2' \in S$ be such that there are $z_1, z_2 \in S$ with

$$z_1' \ll z_1 \leqslant e, y, \quad \text{and} \quad z_2' \ll z_2 \leqslant e, y.$$

First, by (O5), there exists $w \in S$ such that $z_1' + w \leqslant e \leqslant z_1 + w$. Since $z_1 \leqslant y$, we obtain $e \leqslant y + w$. We now apply (O6+) to this inequality and $z_2' \ll z_2 \leqslant e, y$. Thus there exists $x \in S$ such that $e \leqslant x + w$ and $z_2' \ll x \leqslant e, y$. Hence

$$z_1' + w \leqslant e \ll e \leqslant x + w.$$

Since e is compact, we may use weak cancellation in S to obtain $z_1' \ll x$. Hence, $z_1', z_2' \ll x \leqslant e, y$. Choose $x' \in S$ with $z_1', z_2' \ll x' \ll x$. Then x' has the desired properties. □

3.4**.** Recall that an ordered semigroup S has the *Riesz Interpolation Property* if given $x, y, u, v \in S$ such that $x, y \leqslant u, v$, then there exists $z \in S$ with $x, y \leqslant z \leqslant u, v$.

3.5. **Theorem.** *Let A be a C*-algebra of stable rank one. Then $\mathrm{Cu}(A)$ has the Riesz Interpolation Property.*

Proof. Let $x, y \in \mathrm{Cu}(A)$. We must show that the set $\{z \in \mathrm{Cu}(A) : z \leqslant x, y\}$ is upward directed. If x is compact, this follows from Lemma 3.3. We next reduce the general case to this case relying on Proposition 2.8.

We may assume that A is stable. Choose $a \in A_+$ such that $x = [a]$. Applying Proposition 2.8 for A and a, we obtain a C*-algebra B with stable rank one that contains A as a closed, two-sided ideal, and a projection $p_a \in B$ such that $z \in \mathrm{Cu}(A)$ satisfies $z \leqslant x$ if and only if $z \leqslant [p_a]$. Since $[p_a]$ is compact in $\mathrm{Cu}(B)$, and since B has stable rank one, it follows from Lemma 3.3 that the set $\{z \in \mathrm{Cu}(B) : z \leqslant [p_a], y\}$ is upward directed. The inclusion $A \subseteq B$ identifies $\mathrm{Cu}(A)$ with an ideal in $\mathrm{Cu}(B)$. We claim that

$$\big\{z \in \mathrm{Cu}(A) : z \leqslant x, y\big\} = \big\{z \in \mathrm{Cu}(B) : z \leqslant [p_a], y\big\},$$

from which the result will follow.

Indeed, the inclusion '$\subseteq$' follows using that $x \leqslant [p_a]$. To prove the converse inclusion, take $z \in \mathrm{Cu}(B)$ such that $z \leqslant [p_a], y$. Since $\mathrm{Cu}(A)$ is an ideal of $\mathrm{Cu}(B)$ and $y \in \mathrm{Cu}(A)$, we have $z \in \mathrm{Cu}(A)$. Now, since also $z \leqslant [p_a]$, we may use Proposition 2.8 (ii) to conclude that $z \leqslant x$. □

3.6**.** *Inf-semilattice ordered semigroups.* Recall that a partially ordered set S is called an *inf-semilattice*, or also a *meet-semilattice*, if for every pair of elements x and y of S, the greatest lower bound of the set $\{x, y\}$ exists in S. We shall follow the usual notation and denote such infimum by $x \wedge y$.

We further say that a partially ordered semigroup S is *inf-semilattice ordered* if S is an inf-semilattice and addition is distributive over the meet operation, that is,

$$(x + z) \wedge (y + z) = x \wedge y + z, \tag{3.1}$$

for all $x, y, z \in S$.

3.7. **Lemma.** *Let A be a stable C*-algebra and let $a \in A_+$. Let the C*-algebra B and the projection $p_a \in B$ be as in Proposition 2.8. Let $x \in \mathrm{Cu}(A)$ such that $[p_a] \wedge x$ exists in $\mathrm{Cu}(B)$. Then $[a] \wedge x$ exists in $\mathrm{Cu}(A)$ and*

$$[a] \wedge x = [p_a] \wedge x.$$

Proof. Let $w = [p_a] \wedge x$. Since $w \leqslant x$ and since $\mathrm{Cu}(A)$ is an ideal of $\mathrm{Cu}(B)$, we obtain $w \in \mathrm{Cu}(A)$. Now, we also have that $w \leqslant [p_a]$. Hence, $w \leqslant [a]$ by Proposition 2.8(ii). Thus, w is a lower bound for $[a]$ and x.

To show that w is the largest lower bound, let $y \in \mathrm{Cu}(A)$ satisfy $y \leqslant [a]$ and $y \leqslant x$. Then $y \leqslant [p_a]$ in $\mathrm{Cu}(B)$, again by Proposition 2.8(ii). Therefore $y \leqslant [p_a] \wedge x = w$. Hence, $[a] \wedge x = [p_a] \wedge x$, as desired. □

3.8. **Theorem.** *Let A be a separable C*-algebra of stable rank one. Then $\mathrm{Cu}(A)$ is an inf-semilattice ordered semigroup.*

Proof. Without loss of generality, we may assume that A is stable. By Theorem 3.5, $\mathrm{Cu}(A)$ has the Riesz Interpolation Property. Thus, given $x, y \in \mathrm{Cu}(A)$ the set $\{z \in \mathrm{Cu}(A) : z \leqslant x, y\}$ is upward directed. Since A is separable, $\mathrm{Cu}(A)$ is countably based. Applying § 2.6, we conclude that $\{z \in \mathrm{Cu}(A) : z \leqslant x, y\}$ has a supremum, which is precisely $x \wedge y$. Thus, $\mathrm{Cu}(A)$ is an inf-semilattice.

In order to prove that (3.1) holds, we only need to show that

$$(x+z) \wedge (y+z) \leqslant x \wedge y + z, \tag{3.2}$$

for all $x, y, z \in \mathrm{Cu}(A)$, as the opposite inequality is straightforward.

We will first prove (3.2) in the case that both x and z are compact elements and then, through successive generalizations, extend this to the general case.

Step 1: We show that the inequality (3.2) is valid when x and z are compact.

Let $w = (x+z) \wedge (y+z)$. Choose $w' \in \mathrm{Cu}(A)$ such that $w' \ll w$. Applying (O5) for the inequality $w' \ll w \leqslant x+z$, we find $v \in \mathrm{Cu}(A)$ such that $w'+v \leqslant x+z \leqslant w+v$. We get $x + z \leqslant y + z + v$. As A has stable rank one, $\mathrm{Cu}(A)$ has cancellation of compact elements, and since z is compact by assumption, we obtain $x \leqslant y + v$. By (O6+), $x \leqslant x \wedge y + v$. Adding z on both sides we get $x + z \leqslant x \wedge y + v + z$. Hence, using that $x + z$ is compact,

$$w' + v \leqslant x + z \ll x + z \leqslant x \wedge y + z + v\,.$$

It now follows from weak cancellation that $w' \leqslant x \wedge y + z$. Since w' is arbitrary satisfying $w' \ll w$, the inequality (3.2) holds.

Step 2: We show that the inequality (3.2) is valid when x is compact.

Write $x + z = [b]$, with $b \in A_+$. Let B be a C*-algebra of stable rank one and let $p_b \in B$ be a projection, as in Proposition 2.8. Let $f = [p_b] \in \mathrm{Cu}(B)$, which is compact. We have $x + z \leqslant f$ and by Lemma 3.7, $f \wedge w = (x + z) \wedge w$ for all $w \in \mathrm{Cu}(A)$. Since $x \leqslant f$ and x is compact, $x + z' = f$ for some compact element $z' \in \mathrm{Cu}(B)$. Then $x + z \leqslant f = x + z'$, and so by cancellation of compact elements in $\mathrm{Cu}(B)$ we have $z \leqslant z'$. Since x and z' are compact in $\mathrm{Cu}(B)$, we may apply Step 1 to conclude

$$(x + z') \wedge (y + z') \leqslant x \wedge y + z'.$$

Since $z \leqslant z'$, we get

$$(x + z) \wedge (y + z) \leqslant x \wedge y + z'.$$

Hence, by (O6+), it follows that

$$(x + z) \wedge (y + z) \leqslant x \wedge y + z' \wedge (x + z).$$

The proof of Step 2 will be complete once we show $z' \wedge (x + z) = z$. By cancellation of compact elements, and since x is compact by assumption, this is equivalent to showing that $z' \wedge (x + z) + x = z + x$. Since x and z' are compact elements in $\mathrm{Cu}(B)$, we may use Step 1 again to obtain

$$z' \wedge (x + z) + x = (z' + x) \wedge (x + z + x).$$

Now, we apply Lemma 3.7 at the second step and conclude

$$(z' + x) \wedge (x + z + x) = f \wedge (x + z + x) = (z + x) \wedge (x + z + x) = z + x.$$

Therefore $z' \wedge (x + z) + x = z + x$, as desired.

Step 3: We show that the inequality (3.2) holds in general.

Choose $a \in A_+$ such that $x = [a]$. Applying Proposition 2.8 for A and a, we obtain a C*-algebra B with stable rank one that contains A as a closed, two-sided ideal, and a projection $p_a \in B$ such that $z \in \mathrm{Cu}(A)$ satisfies $z \leqslant x$ if and only if $z \leqslant [p_a]$. Let $e = [p_a] \in \mathrm{Cu}(B)$. By Step 2, (3.2) holds in $\mathrm{Cu}(B)$ with e in place of x. This means that

$$(e + z) \wedge (y + z) \leqslant e \wedge y + z$$

for any $y \in \mathrm{Cu}(A)$.

Now, by Lemma 3.7 we have $e \wedge y = x \wedge y$. Therefore, the right hand side of the above inequality is precisely $x \wedge y + z$. On the other hand, the left hand side dominates $(x+z) \wedge (y+z)$. This proves the inequality in general. □

3.9. Let S be an inf-semilattice ordered semigroup, and let $x_i^{(k)} \in S$ for $k = 1, \ldots, n$ and $i = 1, \ldots, N_k$. It follows from (3.1) and induction that

$$\sum_{k=1}^{n} \Big(\bigwedge_{i=1}^{N_k} x_i^{(k)} \Big) = \bigwedge_{(i_1,\ldots,i_n)} \Big(\sum_{k=1}^{n} x_{i_k}^{(k)} \Big),$$

where $(i_1, \ldots, i_n)$ on the right hand side runs through $\{1, \ldots, N_1\} \times \cdots \times \{1, \ldots, N_n\}$.

3.10. If S is an inf-semilattice ordered Cu-semigroup, then S satisfies (O6+). Indeed, if we are given elements $a, b, c, x', x, y', y \in S$ such that $a \leqslant b + c$, $x' \ll x \leqslant a, b$ and $y' \ll y \leqslant a, c$, then let $e = a \wedge b$ and $f = a \wedge c$. We clearly have $x' \ll x \leqslant e$ and $y' \ll y \leqslant f$. On the other hand, applying the formula obtained in § 3.9 at the second step, we obtain

$$a \leqslant (2a) \wedge (a+c) \wedge (a+b) \wedge (b+c) = a \wedge b + a \wedge c = e + f\,.$$

3.11. Let S be an inf-semilattice ordered Cu-semigroup. Given $x \in S$ and an increasing sequence $(y_n)_n$ in S, we have

$$\sup_n (x \wedge y_n) = x \wedge \sup_n y_n.$$

Indeed, the inequality '$\leqslant$' follows since for each $k \in \mathbb{N}$ we have $x \wedge y_k \leqslant x \wedge \sup_n y_n$. To show the converse inequality, let $z' \in S$ be such that $z' \ll x \wedge \sup_n y_n$. Since $z' \ll \sup_n y_n$, there exists $k \in \mathbb{N}$ such that $z' \leqslant y_k$. Since also $z' \leqslant x$, we obtain $z' \leqslant x \wedge y_k \leqslant \sup_n (x \wedge y_n)$. Finally, passing to the supremum over all $z' \ll x \wedge \sup_n y_n$, the desired inequality follows.

As an immediate application of Theorem 3.8, we obtain that quotient maps preserve infima. If I is a closed, two-sided ideal of a C*-algebra A, we denote by $\pi_I \colon A \to A/I$ the quotient map.

3.12. **Corollary.** *Let A be a separable C*-algebra of stable rank one, and let I be a closed, two-sided ideal of A. Then $\mathrm{Cu}(\pi_I) \colon \mathrm{Cu}(A) \to \mathrm{Cu}(A/I)$ preserves infima.*

Proof. We view $\mathrm{Cu}(I)$ as an ideal of $\mathrm{Cu}(A)$. Since I is separable, $\mathrm{Cu}(I)$ has a largest element that we denote by ω_I. Notice that $2\omega_I = \omega_I$, and thus $\omega_I + \mathrm{Cu}(A)$ is an ordered subsemigroup of $\mathrm{Cu}(A)$. It was proved in [CRS10] that $\mathrm{Cu}(\pi_I)$ is an ordered semigroup isomorphism from $\omega_I + \mathrm{Cu}(A)$ to $\mathrm{Cu}(A/I)$. It therefore suffices to show that the map $x \mapsto x + \omega_I$ from $\mathrm{Cu}(A)$ to the subsemigroup $\omega_I + \mathrm{Cu}(A)$ preserves infima. Indeed, for $x, y \in \mathrm{Cu}(A)$ if follows from Theorem 3.8 that

$$(x + \omega_I) \wedge (y + \omega_I) = x \wedge y + \omega_I.$$ □

Another application of Theorem 3.8 allows us to compute the Cuntz semigroup of a particular case of pullbacks (see also [APS11]).

3.13. **Corollary.** *Let A be a separable C*-algebra of stable rank one, and let $I, J \subseteq A$ be closed, two-sided ideals of A. Then*

$$\mathrm{Cu}(A/(I \cap J)) \cong \mathrm{Cu}(A/I) \oplus_{\mathrm{Cu}(A/(I+J))} \mathrm{Cu}(A/J),$$

where the right side denotes the pullback semigroup of pairs $(\bar{s}, \bar{t}) \in \mathrm{Cu}(A)/\mathrm{Cu}(I) \oplus \mathrm{Cu}(A)/\mathrm{Cu}(J)$ such that $\bar{s}$ and $\bar{t}$ agree when mapped to $\mathrm{Cu}(A/(I+J))$.

Proof. As in the proof of Corollary 3.12, given an ideal K of a separable C*-algebra B we denote by ω_K the largest element in $\mathrm{Cu}(K)$ and we identify $\mathrm{Cu}(B/K)$ with $\mathrm{Cu}(B)+\omega_K$. Thus $\mathrm{Cu}(\pi_K)$ is identified with the map $\mathrm{Cu}(B)\to\mathrm{Cu}(B)+\omega_K$ given by $z\mapsto z+\omega_K$.

Observe that $\omega_{I+J}=\omega_I+\omega_J$. Therefore, the map $\mathrm{Cu}(A/I)\to\mathrm{Cu}(A/(I+J))$ is identified with the map

$$\mathrm{Cu}(A)+\omega_I\to\mathrm{Cu}(A)+\omega_I+\omega_J \text{ given by } z\mapsto z+\omega_J.$$

Likewise, the map $\mathrm{Cu}(A/J)\to\mathrm{Cu}(A/(I+J))$ is identified with the map $\mathrm{Cu}(A)+\omega_J\to\mathrm{Cu}(A)+\omega_I+\omega_J$ given by $z\mapsto z+\omega_I$.

Now, denote by S the algebraic pullback of the diagram

$$\begin{tikzcd} & \mathrm{Cu}(A)+\omega_I \arrow[d] \\ \mathrm{Cu}(A)+\omega_J \arrow[r] & \mathrm{Cu}(A)+\omega_I+\omega_J \end{tikzcd}$$

We clearly have a map $\mathrm{Cu}(A)+\omega_{I\cap J}\to S$, given by $z\mapsto(z+\omega_I,z+\omega_J)$. It suffices to show that given $(z_1,z_2)\in S$, there exists a unique element $z\in\mathrm{Cu}(A)+\omega_{I\cap J}$ such that $z+\omega_I=z_1$ and $z+\omega_J=z_2$.

Let $z_1\in\mathrm{Cu}(A)+\omega_i$ and $z_2\in\mathrm{Cu}(A)+\omega_J$. Set $z:=z_1\wedge z_2$. Using Theorem 3.8 at the first step, we obtain

$$z_1\wedge z_2+\omega_I=(z_1+\omega_I)\wedge(z_2+\omega_I)=z_1\wedge(z_1+\omega_J)=z_1.$$

Symmetrically, $z_1\wedge z_2+\omega_J=z_2$. Observe also that $z\in\omega_{I\cap J}+\mathrm{Cu}(A)$. Indeed, since $z_1=z_1+\omega_I$ and $\omega_{I\cap J}+\omega_I=\omega_I$, we get

$$z_1+\omega_{I\cap J}=z_1+\omega_I+\omega_{I\cap J}=z_1+\omega_I=z_1,$$

and similarly $z_2+\omega_{I\cap J}=z_2$. Applying Theorem 3.8 again, we get

$$z+\omega_{I\cap J}=(z_1\wedge z_2)+\omega_{I\cap J}=(z_1+\omega_{I\cap J})\wedge(z_2+\omega_{I\cap J})=z_1\wedge z_2=z.$$

Finally, suppose that $z'\in\omega_{I\cap J}+\mathrm{Cu}(A)$ is such that $z'+\omega_I=z_1$ and $z'+\omega_J=z_2$. Notice that, since A has stable rank one, $\omega_{I\cap J}=\omega_I\wedge\omega_J$. Then, using Theorem 3.8 at third step, we obtain

$$z'=z'+\omega_{I\cap J}=z'+\omega_I\wedge\omega_J=(z'+\omega_I)\wedge(z'+\omega_J)=z_1\wedge z_2=z. \qquad \square$$

3.14. Note that Corollary 3.13 fails to hold if we drop the stable rank one hypothesis. For example, set $A:=M_2(C(S^2))$ and take $I=M_2(C_0(U))$ and $J=M_2(C_0(V))$, where U and V are disjoint open caps of the sphere. Let $p,q\in M_2(C(S^2))$ be rank one projections with different class in $K_0(C(S^2))$. (For instance, p is $e_{11}\otimes 1$ and q is the Bott projection.) Then the images of p and q are Cuntz equivalent in A/I and A/J, but $[p]\neq[q]$.

4. A conjecture of Blackadar and Handelman

Let A be a C*-algebra. Using upper-left corner embeddings $M_n(A)\to M_{n+1}(A)$, set $M_\infty(A):=\bigcup_n M_n(A)$, which has the structure of a local C*-algebra. Recall that the classical (non-complete) Cuntz semigroup $W(A)$ of A is defined as

$$W(A)=M_\infty(A)_+/\!\sim;$$

see [Cun78]. It can also be described as the subsemigroup of $\mathrm{Cu}(A)$ of those classes $[a]$ with a representative $a\in M_\infty(A)_+$. In the case that A has stable rank one, it was proved in [ABP11, Lemma 3.4] that $W(A)$ is a hereditary subset of $\mathrm{Cu}(A)$, and thus may alternatively be described as

$$W(A)=\big\{x\in\mathrm{Cu}(A): x\leqslant n[a] \text{ for some } a\in A_+, n\in\mathbb{N}\big\}.$$

The Grothendieck group of $W(A)$ is denoted by $K_0^*(A)$. It is a partially ordered group with positive cone $K_0^*(A)^+ = \{[x]-[y] : y \leqslant x \text{ in } W(A)\}$.

Assume that A is unital. A *state* on $(W(A),[1_A])$ is an additive, order-preserving map $\lambda\colon W(A)\to[0,\infty)$ with $\lambda([1_A])=1$. We use $\mathrm{St}(W(A),[1_A])$ to denote the set of states on $(W(A),[1_A])$. There is a natural bijection between $\mathrm{St}(W(A),[1_A])$ and states on $(K_0^*(A),K_0^*(A)^+,[1_A])$.

In [Cun78], Cuntz introduced a *dimension function* on A as a certain map $M_\infty(A)\to[0,\infty)$. The set of all dimension functions on A is denoted by $DF(A)$. It is easy to see that there are bijections

$$DF(A)\cong \mathrm{St}\big(W(A),[1_A]\big)\cong \mathrm{St}\big(K_0^*(A),K_0^*(A)^+,[1_A]\big).$$

In [BH82], Blackadar and Handelman conjectured that $DF(A)$ is always a Choquet simplex. This has been confirmed for various classes of C*-algebras: in [Per97, Corollary 4.4] for unital C*-algebras with real rank zero and stable rank one; in [ABPP14, Theorem 4.1] for certain C*-algebras with stable rank two; in [dS16, Theorem 3.4] for unital C*-algebras with finite radius of comparison and finitely many extreme quasitraces.

In view of the results obtained, it was asked in [ABPP14, Problem 3.13] for which C*-algebras A the group $K_0^*(A)$ is an interpolation group. We solve this problem affirmatively for C*-algebras of stable rank one.

4.1. **Theorem.** *Let A be a unital C*-algebra of stable rank one. Then $K_0^*(A)$ is an interpolation group and $DF(A)$ is a Choquet simplex.*

Proof. By Theorem 3.8, we know that $\mathrm{Cu}(A)$ has the Riesz Interpolation Property. This property passes to $W(A)$ since $W(A)$ is hereditary in $\mathrm{Cu}(A)$ (see also the results in [ABPP14, Section 3]). Now apply [Per97, Lemma 4.2] to conclude that $K_0^*(A)$ is an interpolation group. Finally, using for example [Goo86, Theorem 10.17], we obtain that $DF(A)$ is a Choquet simplex. □

5. The Global Glimm halving Problem

In this section we solve the *Global Glimm Halving Problem* for C*-algebras of stable rank one.

5.1. *Global Glimm Halving Problem.* The global Glimm Halving Problem has been posed in various forms (see, for example, [BK04, ER06]). One such formulation is as follows: If A is a unital C*-algebra without finite dimensional representations, is there a *-homomorphism $\varphi\colon M_2(C_0((0,1]))\to A$ with full range? (Recall that a subset of a C*-algebras is called full if it generates the C*-algebra as a closed two-sided ideal.) In Theorem 5.7 we answer this question affirmatively for separable C*-algebras of stable rank one. We even obtain a sharper result that characterizes when A has irreducible representations of a given finite dimension. Further, in Theorem 9.1 we remove the separability assumption. We first establish results on divisibility of elements of Cu-semigroups. These results then translate into a solution of the Global Glimm Halving Problem.

5.2. *Divisibility in the Cuntz semigroup.* Let S be a Cu-semigroup and let $x\in S$. Let us recall two divisibility properties introduced in [RR13]. Given $k,n\in\mathbb{N}$, we say that x is (k,n)-*divisible* if for each $x'\in S$ satisfying $x'\ll x$ there exists $y\in S$ such that $ky\leqslant x$ and $x'\ll ny$. We say that x is *weakly* (k,n)-*divisible* if for each $x'\in S$ with $x'\ll x$ there exist $y_1,\ldots,y_n\in S$ such that $ky_j\leqslant x$ for all j, and $x'\leqslant\sum_{j=1}^n y_j$. Clearly, any (k,n)-divisible element is weakly (k,n)-divisible. In Theorem 5.5 below we obtain a result in the converse direction.

Given x and y in a partially ordered semigroup S, we say that y *dominates* x, and write $x\propto y$, if there exists $n\in\mathbb{N}$ such that $x\leqslant ny$.

5.3. **Lemma.** *Let S be an inf-semilattice ordered* Cu*-semigroup, and let $x, y_1, \ldots, y_n$ be elements in S such that $x \propto y_k$ for $k = 1, \ldots, n$. Then $x \propto \bigwedge_k y_k$. In fact, if $x \leqslant N y_k$ for all k, then $x \leqslant M(\bigwedge_{k=1}^n y_k)$ where $M = n(N-1)+1$.*

Proof. It is enough to prove the last assertion. Assume $N \in \mathbb{N}$ is such that $x \leqslant N y_k$ for $k = 1, \ldots, n$. Set $M = n(N-1)+1$. By (3.9), we have

$$M \bigwedge_{k=1}^{n} y_k = \sum_{j=1}^{M} \bigwedge_{k=1}^{n} y_k = \bigwedge \left(\sum_{k=1}^{M} y_{i_k} \right),$$

where the infimum on the right hand side runs through all sums with M terms taken from the set $\{y_1, \ldots, y_n\}$. Since $M = n(N-1)+1$, each of these sums contains at least one of the y_k repeated N times, whence it is greater than or equal to x. Thus, $M(\bigwedge_{i=1}^n y_k)$ is greater than or equal to x, as desired. □

5.4. **Lemma.** *Let S be an inf-semilattice ordered* Cu*-semigroup satisfying (O5) and weak cancellation. Let $k \in \mathbb{N}$ and let $x', x, y_1, \ldots, y_n \in S$ such that $x' \ll x$, $x' \leqslant \sum_{j=1}^n y_j$, and $k y_j \leqslant x$ for each j. Then there exist $z_1, \ldots, z_k$ such that $\sum_{j=1}^k z_j \leqslant x$ and $x' \propto z_j$ for each j. More precisely, we have $x' \leqslant M z_j$ where*

$$M = \max\{n^r(k-r) + n^{r-1} : r = 1, \ldots, k\}.$$

Proof. We will prove the result by induction over k. The case $k=1$ is trivial taking $z_1 = x$. Let us assume $k > 1$ and that the result holds for $k-1$.

Let $x', x, y_1, \ldots, y_n$ be as in the statement of the lemma. Choose $y_1', \ldots, y_n' \in S$ such that $y_j' \ll y_j$ for each j, and such that $x' \ll \sum_{j=1}^n y_j'$. For each j, choose $y_j'' \in S$ such that $y_j' \ll y_j'' \ll y_j$. Apply (O5) in $(k-1)y_j' \ll (k-1)y_j'' \leqslant x$ to obtain $w_j \in S$ such that

$$(k-1)y_j' + w_j \leqslant x \leqslant (k-1)y_j'' + w_j\,.$$

Multiplying by k in $x \leqslant (k-1)y_j'' + w_j$ we get

$$kx \leqslant (k-1)k y_j'' + k w_j\,.$$

Since $(k-1)k y_j'' \ll (k-1)x$, we get by weak cancellation that $x \leqslant k w_j$.

Set $w = \bigwedge_{j=1}^n w_j$. By Lemma 5.3 we have $x \leqslant (n(k-1)+1)w$. Choose $w', w'' \in S$ such that $w' \ll w'' \ll w$ and $x' \leqslant (n(k-1)+1)w'$. Using (O5) again, we obtain $\tilde{x} \in S$ such that $w' + \tilde{x} \leqslant x \leqslant w'' + \tilde{x}$. For each j, we have

$$(k-1)y_j' + w_j \leqslant x \leqslant \tilde{x} + w''\,.$$

Since $w'' \ll w_j$, we get by weak cancellation that $(k-1)y_j' \leqslant \tilde{x}$. Hence, $\sum_{j=1}^n y_j' \leqslant n\tilde{x}$. Observe also that, by § 3.9,

$$n\left(\big(\sum_{j=1}^{n} y_j'\big) \wedge \tilde{x} \right) = \bigwedge_{k=0}^{n} \left((n-k)\big(\sum_{j=1}^{n} y_j'\big) + k\tilde{x} \right).$$

Further, any of the terms of the infimum on the right hand side is greater than $\sum_{j=1}^n y_j'$. Since $x' \ll \sum_{j=1}^n y_j'$, we have $x' \ll n((\sum_{j=1}^n y_j') \wedge \tilde{x})$. Choose $\tilde{x}'$ such that $\tilde{x}' \ll (\sum_j y_j') \wedge \tilde{x}$ and $x' \leqslant n\tilde{x}'$. By construction, we can apply induction on $\tilde{x}', \tilde{x}, y_1', \ldots, y_n'$ to find $z_1, \ldots, z_{k-1}$ such that $\sum_{i=1}^{k-1} z_i \leqslant \tilde{x}$ and $\tilde{x}' \leqslant M_0 z_i$ for $i = 1, \ldots, n$, where

$$M_0 = \max\{n^s(k-1-s) + n^{s-1} : s = 1, \ldots, k-1\}.$$

Set $z_k = w'$. We have

$$\sum_{j=1}^{k} z_j \leqslant \tilde{x} + w' \leqslant x.$$

Moreover, $x' \leqslant n\tilde{x}' \leqslant nM_0 z_j$ for $j = 1, \dots, k-1$ and $x' \leqslant (n(k-1)+1)z_k$. Since $M = \max\{M_0 n, n(k-1)+1\}$, this completes the proof of the induction step. □

5.5. **Theorem.** *Let S be an inf-semilattice ordered* Cu*-semigroup satisfying (O5) and weak cancellation. Let $k \in \mathbb{N}$ and let $x \in S$. Then x is weakly (k,n)-divisible for some $n \in \mathbb{N}$ if and only if x is (k,N)-divisible for some $N \in \mathbb{N}$.*

Proof. The backward implication is clear. To show the converse, let $x' \in S$ satisfy $x' \ll x$. By assumption, there exist $y_1, \dots, y_n \in S$ such that $ky_j \leqslant x$ for all j, and $x' \leqslant \sum_{j=1}^n y_j$. Apply Lemma 5.4 to obtain $M \in \mathbb{N}$ and $z_1, \dots, z_k \in S$ such that $kz_j \leqslant x$ and $x' \leqslant Mz_j$ for each j. Set $N = k(M-1)+1$ and $z = \bigwedge z_i$. Then $kz \leqslant x$ and $x' \leqslant Nz$ by Lemma 5.3. □

The following result is an improved version of [RR13, Lemma 2.5] that is available for C*-algebras with stable rank one.

5.6. **Lemma.** *Let A be a C*-algebra with stable rank one, let $x \in \mathrm{Cu}(A)$, let $b \in A_+$, and let $k \in \mathbb{N}$ such that $kx \leqslant [b]$. Then there exists a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to \overline{bAb}$ with $[\varphi(e_{11} \otimes \iota)] = x$.*

Proof. We may assume that A is stable and $x \neq 0$. Given $c, d \in A_+$, we write $c \approx d$ if there exists $r \in A$ with $c = r^*r$ and $rr^* = d$. Since A has stable rank one, we have $c \precsim d$ (Cuntz subequivalence) if and only if $c \approx d' \in \overline{dAd}$, for some d'; see for example [ORT11, 6.2].

Choose pairwise orthogonal elements $a_1, \dots, a_k \in A_+$ with $[a_j] = x$ for each j. Then $\sum_j [a_j] = [\sum_j a_j] = kx \leqslant [b]$. Choose $r \in A$ with $\sum_j a_j = r^*r$ and $rr^* \in \overline{bAb}$. Let $r = v|r|$ be the polar decomposition of r in A^{**}. Set $b_j := v^* a_j v$ for each j. Then $b_1, \dots, d_k$ are pairwise orthogonal elements in $\overline{bAb}$ satisfying $[b_j] = [a_j] = x$ for each j. Set $c_1 := b_1/\|b_1\|$. For $j = 2, \dots, k$, we use that $c_1 \precsim b_j$ to choose $c_j \in \overline{b_j A b_j}$ with $c_1 \approx c_j$. Then $c_1, c_2, \dots, c_k$ are pairwise orthogonal, pairwise equivalent (in the sense of $\approx$) elements in $\overline{bAb}$. As noted in [RR13, Remark 2.3], we obtain a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to \overline{bAb}$ satisfying $[\varphi(e_{jj} \otimes \iota)] = c_j$. In particular, $[\varphi(e_{11} \otimes \iota)] = [c_1] = [a_1] = x$. □

5.7. **Theorem.** *Let A be a unital separable C*-algebra of stable rank one, and let $k \in \mathbb{N}$. Then A has no nonzero representations of dimension less than k if and only if there exists a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to A$ with full range.*

Proof. If $\pi\colon A \to M_j(\mathbb{C})$ is a representation with $j < k$ and $\varphi\colon M_k(C_0((0,1])) \to A$ is any *-homomorphism, then $\pi \circ \varphi = 0$. Thus, $\ker(\pi)$ contains the ideal generated by the range of any such φ. If there exists $\varphi\colon M_k(C_0((0,1])) \to A$ with full range then π must be the zero representation. This proves the easy direction.

Suppose now that A has no nonzero representations of dimension less than k. Let $1 \in A$ be the unit of A. We have by [RR13, Theorem 5.3] that $[1]$ is weakly (k,n)-divisible in $\mathrm{Cu}(A)$ for some $n \in \mathbb{N}$. But $\mathrm{Cu}(A)$ is an inf-semilattice ordered Cu-semigroup, since A is separable and of stable rank one. We thus obtain from Theorem 5.5 that $[1]$ is (k,N)-divisible for some $N \in \mathbb{N}$. Since $[1]$ is (k,N)-divisible and A has stable rank one, we can choose $c \in A_+$ such that $k[c] \leqslant [1]$ and $[1] \leqslant N[c]$. By Lemma 5.6, there exists a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to A$ such that $[\varphi(e_{11} \otimes \iota)] = [c]$. This *-homomorphism is as desired. □

5.8. It is possible to adapt the previous proof to nonunital C*-algebras. In this case, however, rather than a *-homomorphism with full range, we obtain for each $a \in A$ in the Pedersen ideal of A a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to A$ such that the ideal generated by the range of φ contains a (assuming that A has no nonzero representations of dimension less than k).

This can be improved if we start with the assumption that A has no *elementary quotients*. In this case we can get $\varphi\colon M_k(C_0((0,1])) \to A$ with full range for each $k \in \mathbb{N}$, even in the nonunital case. We prove this in Theorem 5.11 below. We first establish an improved form of divisibility of full elements (Theorem 5.10) which will also be used in Section 8.

5.9. Let S be a Cu-semigroup. Recall that $x \in S$ is said to be *soft* if for all $x' \in S$ with $x' \ll x$ we have $(k+1)x' \leqslant kx$ for some $k \in \mathbb{N}$. The following result is essentially [ERS11, Proposition 6.4], but we include a proof for completeness.

Lemma. *Let S be a* Cu*-semigroup, and let $(x_j)_j$ be a sequence in S such that $x_j \propto x_{j+1}$ for each j. Then $x := \sum_{j=1}^\infty x_j$ is soft.*

Proof. Let $x' \in S$ satisfy $x' \ll \sum_{j=1}^\infty x_j$. Then there exists n such that $x' \leqslant \sum_{j=1}^n x_j$. We can now find $k \in \mathbb{N}$ such that $\sum_{j=1}^n x_j \leqslant kx_{n+1}$ and hence

$$(k+1)x' \leqslant kx' + \sum_{j=1}^n x_j \leqslant kx' + kx_{n+1} \leqslant k\sum_{j=1}^{n+1} x_j \leqslant k\sum_{j=1}^{\infty} x_j = kx. \qquad \square$$

Recall that a C*-algebra is termed *elementary* if it is isomorphic to the C*-algebra of compact operators on some Hilbert space.

5.10. **Theorem.** *Let A be a separable C*-algebra of stable rank one that has no elementary quotients. Then for every full element $x \in \mathrm{Cu}(A)$ and every $n \in \mathbb{N}$ there exists a soft full element $z \in \mathrm{Cu}(A)$ such that $nz \leqslant x$.*

Proof. The assumption that A has no elementary quotients is equivalent to the assumption that no full hereditary subalgebra of A has nonzero finite dimensional representations. This in turn is equivalent to saying that for every full element $x \in \mathrm{Cu}(A)$ and every $k \in \mathbb{N}$, the element x is weakly (k,n)-divisible for some n. By Theorem 5.5, this implies that every full element x is (k,N)-divisible for some N.

To prove our conclusion, it suffices to consider the case $n = 2$. Let $x \in \mathrm{Cu}(A)$ be full. Choose a $\ll$-increasing sequence $(x_j)_j$ with supremum x. Since x is $(3,N)$-divisible for some $N \in \mathbb{N}$, there exists $z_1 \in \mathrm{Cu}(A)$ such that $3z_1 \leqslant x$ and $x_2 \propto z_1$. Choose $z_1', z_1'' \in \mathrm{Cu}(A)$ such that $z_1' \ll z_1'' \ll z_1$ and $x_1 \propto z_1'$. By (O5) applied to $2z_1' \ll 2z_1'' \leqslant x$, we find $w_1 \in \mathrm{Cu}(A)$ such that

$$2z_1' + w_1 \leqslant x \leqslant 2z_1'' + w_1.$$

Hence, $x + 2x = 3x \leqslant 6z_1'' + 3w_1$. Since $6z_1'' \ll 2x$, we get by weak cancellation that $x \leqslant 3w_1$. In particular, w_1 is a full element.

Since $x_3 \ll x \leqslant 3w_1$ and $x_2 \ll x_3$, there are elements $w_1'', w_1' \in \mathrm{Cu}(A)$ such that $w_1'' \ll w_1' \ll w_1$ and $x_2 \ll 3w_1''$. Since also $2z_1' \ll x \leqslant 3w_1$, we may further assume that $z_1' \ll 3w_1''$.

Using that w_1 is full and hence $(3,N)$-divisible for some N, we find $z_2 \in \mathrm{Cu}(A)$ such that $3z_2 \leqslant w_1$ and $w_1' \propto z_2$. Choose $z_2', z_2'' \in \mathrm{Cu}(A)$ such that $z_2' \ll z_2'' \ll z_2$ and also $w_1'' \propto z_2'$. In particular, we obtain $x_2 \propto w_1'' \propto z_2'$ and likewise, $z_1' \propto z_2'$. Arguing as above, we find a full element $w_2 \in \mathrm{Cu}(A)$ such that $2z_2' + w_2 \leqslant w_1$.

Continuing in this way we build a sequence $z_1', z_2', \ldots$ such that $x_j, z_j' \propto z_{j+1}'$ and

$$2z_1' + 2z_2' + \cdots \leqslant x.$$

Let $z = \sum_{j=1}^\infty z_j'$. We deduce from $z_j' \propto z_{j+1}'$ for all j that z is soft. We deduce from $x_j \propto z_j$ for all j that z is full. $\square$

5.11. **Theorem.** *Let A be a separable C*-algebra of stable rank one that has no elementary quotients. Then for each $k \in \mathbb{N}$ there exists a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to A$ with full range.*

Proof. Let $a \in A_+$ be full, and let $k \in \mathbb{N}$. Then $x := [a]$ is full in $\mathrm{Cu}(A)$. Using Theorem 5.10, we obtain a full element $z \in \mathrm{Cu}(A)$ with $kz \leqslant x$. By Lemma 5.6, there exists a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to \overline{aAa} \subseteq A$ such that $[\varphi(e_{11} \otimes \iota)] = z$. This *-homomorphism has full range. □

6. The cone of functionals and its dual

In this section we provide basic results on the cone $F(S)$ of functionals on a Cu-semigroup S, and its dual $L(F(S))$. We formulate the problem of realizing functions in $L(F(S))$ as ranks of elements in S, which will be tackled in Section 7. The main result of this section is Theorem 6.10, which shows that the natural map $S \to L(F(S))$ preserves infima. This is used repeatedly in the following sections.

6.1. *Functionals.* Let S be a Cu-semigroup. A map $\lambda\colon S \to [0,\infty]$ is called a *functional* if λ is additive, order-preserving, $\lambda(0) = 0$, and it also preserves the suprema of increasing sequences. Let us denote as customary the set of all functionals on S by $F(S)$.

A functional λ in $F(S)$ is said to be *densely finite* if every element of S can be written as a supremum of an increasing sequence in $\{x \in S : \lambda(x) < \infty\}$. This is equivalent to saying that $\lambda(x) < \infty$ whenever there exists $\tilde{x} \in S$ with $x \ll \tilde{x}$. We denote by $F_0(S)$ the set of densely finite functionals.

The set $F(S)$ is endowed with operations of addition and scalar multiplication by positive real numbers (both defined pointwise). Further, $F(S)$ is equipped with a topology that, in terms of convergence, is described as follows: Given $\lambda \in F(S)$ and a net $(\lambda_i)_{i\in I}$ in $F(S)$, we have $\lambda_i \to \lambda$ if

$$\limsup \lambda_i(x') \leqslant \lambda(x) \leqslant \liminf \lambda_i(x) \text{ for all } x', x \in S \text{ such that } x' \ll x.$$

With this topology, $F(S)$ is a compact Hausdorff space; see [ERS11, Theorem 4.8].

Given a C*-algebra A, there is a natural bijection between $F(\mathrm{Cu}(A))$ and the set $QT(A)$ of $[0,\infty]$-valued, lower semicontinuous 2-quasitraces on A; see [ERS11, Theorem 4.4]. This bijection sends $\lambda \in F(S)$ to $\tau_\lambda\colon A_+ \to [0,\infty]$, given by

$$\tau_\lambda(a) := \int_0^{\|a\|} \lambda([(a-t)_+])dt,$$

for $a \in A_+$. Conversely, $\tau \in \mathrm{QT}(A)$ corresponds to $\lambda_\tau \in F(S)$ given by $\lambda_\tau([a]) = \sup_n \tau(a^{1/n})$, for $a \in A_+$. Under this bijection, $F_0(\mathrm{Cu}(A))$ corresponds to the set of densely finite 2-quasitraces on A.

6.2. *Extreme functionals and chisels.* Let S be a Cu-semigroup. A densely finite functional $\lambda \in F_0(S)$ is said to be *extreme* if whenever $\mu \in F(S)$ and $C \in [0,\infty)$ satisfy $\mu \leqslant C\lambda$, then there exists $c \in [0,\infty)$ such that $\mu = c\lambda$. Notice that the zero functional is extreme.

Let $\lambda \in F_0(S)$ be an extreme functional. We define the *chisel* σ_λ at λ as the function $\sigma_\lambda\colon F(S) \to [0,\infty]$ given by $\sigma_0 = 0$ and, if λ is not the zero functional, then

$$\sigma_\lambda(\mu) = \begin{cases} c, & \text{if } \mu = c\lambda \text{ and } c \in [0,\infty) \\ \infty, & \text{otherwise} \end{cases},$$

for $\mu \in F(S)$. It is straightforward to check that σ_λ is both linear (with respect to the cone structure in $F(S)$) and lower semicontinuous.

6.3. *Edwards' condition.* Let S be a Cu-semigroup, and let $\lambda \in F(S)$. We say that S satisfies *Edwards' condition* for λ if

$$\inf\big\{\lambda_1(x) + \lambda_2(y) : \lambda = \lambda_1 + \lambda_2\big\} = \sup\big\{\lambda(z) : z \leqslant x, y\big\}, \tag{6.1}$$

for all $x, y \in S$. If $\lambda \in F_0(S)$ is extreme, it is not hard to show that S satisfies Edward's condition for λ if and only if

$$\min\big\{\lambda(x), \lambda(y)\big\} = \sup\big\{\lambda(z) : z \leqslant x, y\big\}$$

for all $x, y \in S$. This form of Edwards' condition appears in [Thi17, Definition 4.1]. Notice that if every pair of elements in S has an infimum, then Edward's condition for λ may be phrased as follows:

$$\min\big\{\lambda(x), \lambda(y)\big\} = \lambda(x \wedge y).$$

In [APRT18], we introduce and study Edwards' condition for arbitrary functionals on a Cu-semigroup. It is shown there that if A is a C*-algebra, then $\mathrm{Cu}(A)$ satisfies Edwards' condition for all functionals on $\mathrm{Cu}(A)$. In the next section, we will only use that $\mathrm{Cu}(A)$ satisfies Edwards' condition for all extreme, densely finite functionals.

6.4. *Dual of* $F(S)$. Let S be a Cu-semigroup. We now describe the appropriate notion of dual for the cone $F(S)$. Denote by $\mathrm{Lsc}(F(S))$ the set of functions $F(S) \to [0, \infty]$ that are additive, order-preserving, homogeneous (with respect to positive scalars) and lower semicontinuous. We endow $\mathrm{Lsc}(F(S))$ with pointwise order, pointwise addition, and pointwise scalar multiplication by nonzero positive scalars. Given $x \in S$, we define the function $\hat{x}\colon F(S) \to [0, \infty]$ by evaluation, namely:

$$\hat{x}(\lambda) := \lambda(x), \text{ for } \lambda \in F(S).$$

Then $\hat{x}$ belongs to $\mathrm{Lsc}(F(S))$. We call $\hat{x}$ the *rank* of x. Further, the map $S \to \mathrm{Lsc}(F(S))$ defined by $x \mapsto \hat{x}$ preserves addition, order, and suprema of increasing sequences.

The *realification* of S, denoted by S_R, was introduced in [Rob13] as the smallest subsemigroup of $\mathrm{Lsc}(F(S))$ that is closed under suprema of increasing sequences and contains all elements of the form $\frac{1}{n}\hat{x}$ for $x \in S$ and $n \in \mathbb{N}$. It was proved in [Rob13, Proposition 3.1.1] that S_R is a Cu-semigroup, and that S_R satisfies (O5) whenever S does; see also [APT18b, Proposition 7.5.6].

Given $f, g \in \mathrm{Lsc}(F(S))$, we write $f \lhd g$ if $f \leqslant (1-\varepsilon)g$ for some $\varepsilon > 0$ and if f is continuous at each $\lambda \in F(S)$ satisfying $g(\lambda) < \infty$. We denote by $L(F(S))$ the subsemigroup of $\mathrm{Lsc}(F(S))$ consisting of those $f \in \mathrm{Lsc}(F(S))$ that can be written as the pointwise supremum of a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Lsc}(F(S))$ such that $f_n \lhd f_{n+1}$ for all $n \in \mathbb{N}$. We always have $S_R \subseteq L(F(S))$ by [Rob13, Proposition 3.1.6]. If S satisfies (O5), then $S_R = L(F(S))$ by [Rob13, Theorem 3.2.1].

Thus, given a C*-algebra A, we have a natural map

$$\mathrm{Cu}(A) \to \mathrm{Cu}(A)_R = L(F(\mathrm{Cu}(A))),$$

given by $[a] \mapsto \widehat{[a]}$, where $\widehat{[a]}(\lambda) = \lambda([a])$ for all $\lambda \in F(\mathrm{Cu}(A))$.

We remark that the realification S_R can be identified with the tensor product of Cu-semigroups $S \otimes [0, \infty]$ as defined and studied in [APT18b].

6.5. *The problem of realizing functions as ranks.* Let S be a Cu-semigroup. Recall that the function $\hat{x} \in L(F(S))$ is called the rank of $x \in S$. The problem of realizing functions on $F(S)$ as ranks of elements in S consists of finding necessary and sufficient conditions for the map $x \mapsto \hat{x}$ to be a surjection from S to $L(F(S))$. In Theorem 7.10 we solve this problem when S is the Cuntz semigroup of a separable C*-algebra of stable rank one.

6.6. *The problem of realizing full functions as ranks.* Let S be a Cu-semigroup. A function $f \in L(F(S))$ is said to be *full* if it is a full element of the Cu-semigroup $L(F(S))$, that is, if $g \leqslant \infty \cdot f$ for all $g \in L(F(S))$. It is easy to see that f is full if and only if $f(\lambda) = 0$ implies $\lambda = 0$, that is, if f is strictly positive on the nonzero functionals.

A variation on the problem of realizing functions on $F(S)$ as ranks is as follows: Under what conditions is the map $x \mapsto \hat{x}$ a surjection from the subsemigroup of full elements of S to the subsemigroup of full elements of $L(F(S))$? In Theorems 7.8 and 7.11 we address this problem when S is the Cuntz semigroup of a C*-algebra of stable rank one.

Assume that S contains a full, compact element u. In this case, the subsemigroup of full elements of $L(F(S))$ admits a somewhat more concrete description, which we now give. Let $F_u(S)$ denote the set of functionals normalized at u, that is, the set of $\lambda \in F(S)$ such that $\lambda(u) = 1$. Then $F_u(S)$ is a compact, convex set. Let $\mathrm{LAff}(F_u(S))^\sigma_{++}$ denote the set of affine functions $f\colon F_u(S) \to (0,\infty]$ such that $f^{-1}((t,\infty])$ is open and σ-compact for all $t \in \mathbb{R}$.

6.7. **Proposition.** *Let S be a* Cu*-semigroup, and let $u \in S$ be a full, compact element. Then the restriction map $f \mapsto f|_{F_u(S)}$ is a bijection from the set of full functions in $L(F(S))$ to $\mathrm{LAff}(F_u(S))^\sigma_{++}$.*

Proof. Let $f \in L(F(S))$ be full. As pointed out in § 6.6, f is non-zero on $F_u(S)$. Since f is lower semicontinuous and $F_u(S)$ is compact, f attains a minimum. Hence, the range of $f|_{F_u(S)}$ is contained in $(0,\infty]$. Let $t \in (0,\infty)$. By the lower semicontinuity of f, we get that $f^{(-1)}((t,\infty]) \cap F_u(S)$ is open in $F_u(S)$. To see that this set is also σ-compact, write $f = \sup f_n$, where $f_n \lhd f_{n+1}$ for all n. The functions f_n are finite and continuous on $F_u(S)$. Hence,

$$f^{-1}((t,\infty]) \cap F_u(S) = \bigcup_{n,m} (f_n^{-1}([t+\frac{1}{m},\infty]) \cap F_u(S)),$$

where the sets on the right side are compact. It follows that the restriction $f|_{F_u(S)}$ belongs to $\mathrm{LAff}(F_u(S))^\sigma_{++}$.

Since $\hat{u} \leqslant Mf$ for some constant $M > 0$, we have $f(\lambda) = \infty$ for all λ such that $\lambda(u) = \infty$. It is then clear that the map $f \mapsto f|_{F_u(S)}$ is injective on the set of full functions. It remains to show that this map is surjective. Let $g \in \mathrm{LAff}(F_u(S))^\sigma_{++}$. By [Alf71, Corollary I.1.4], there exists an increasing net of affine, continuous functions $g_i\colon F_u(S) \to (0,\infty]$ with supremum g. Exploiting the σ-compactness of the sets $g^{(-1)}((t,\infty])$, we can choose from this net an increasing sequence $(g_n)_n$ with supremum g; see [TT15, Lemma 4.2]. Next, multiplying if necessary the functions g_n by scalars, we can arrange for $g_n \leqslant (1-\varepsilon_n)g_{n+1}$ for some $\varepsilon_n > 0$ and all n, while maintaining that $g = \sup_n g_n$. For each n define $\tilde{g}_n\colon F(S) \to [0,\infty]$ by

$$\tilde{g_n}(\lambda) = \begin{cases} \lambda(u)f(\lambda(u)^{-1}\lambda), & \text{if } \lambda(u) < \infty \\ 0, & \text{if } \lambda = 0 \\ \infty, & \text{otherwise.} \end{cases} \tag{6.2}$$

Then $\tilde{g}_n \lhd \tilde{g}_{n+1}$ for all n. Hence, $\tilde{g} := \sup_n \tilde{g}_n$ belongs to $L(F(S))$. We have $\tilde{g}|_{F_u(S)} = g$, proving the desired surjectivity. □

6.8. **Lemma.** *Let S be an inf-semilattice ordered* Cu*-semigroup, let $x, y \in S$, and let $n \in \mathbb{N}$. Then $[2^n(x \wedge y)]^\wedge = [(2^n x) \wedge (2^n y)]^\wedge$ in $L(F(S))$.*

Proof. By induction on n, it is enough to show the case $n = 1$. Using § 3.9 we have

$$3(x \wedge y) = (3x) \wedge (2x+y) \wedge (2y+x) \wedge (3y).$$

Similarly,

$$(2x \wedge 2y) + (x \wedge y) = (3x) \wedge (2x+y) \wedge (2y+x) \wedge (3y).$$

This proves that $3(x \wedge y) = (2x \wedge 2y) + (x \wedge y)$, which implies

$$\widehat{2(x \wedge y)} + \widehat{x \wedge y} = \widehat{(2x) \wedge (2y)} + \widehat{x \wedge y}.$$

Since $x \wedge y \leqslant 2(x \wedge y)$ and $x \wedge y \leqslant (2x) \wedge (2y)$, we can cancel $\widehat{x \wedge y}$ to obtain the desired equality. $\square$

6.9. It is not always the case that $2x \wedge 2y = 2(x \wedge y)$ for all x, y in the Cuntz semigroup of a separable C*-algebra of stable rank one. Take for example a separable C*-algebra A of stable rank one and with torsion in $K_0(A)$. Then there exist compact elements $e, f \in \mathrm{Cu}(A)$ such that $2e = 2f$ but $e \neq f$. We have $2e \wedge 2f \neq 2(e \wedge f)$. Indeed, suppose for a contradiction that $2e \wedge 2f = 2(e \wedge f)$. Then

$$2e = 2(e \wedge f) = e \wedge f + e \wedge f \leqslant e + f.$$

By cancellation of compact elements, we obtain $e \leqslant f$, and a symmetrical argument proves $f \leqslant e$, which is impossible.

6.10. **Theorem.** *Let S be a countably based, inf-semilattice ordered* Cu*-semigroup satisfying (O5). Then the map $S \to L(F(S))$, given by $x \mapsto \hat{x}$, preserves infima.*

Proof. By § 3.10, S satisfies (O6), and thus [Rob13, Theorem 4.2.2] implies that $L(F(S))$ is an inf-semilattice ordered Cu-semigroup.

Let $x, y \in S$. The inequality $\widehat{x \wedge y} \leqslant \hat{x} \wedge \hat{y}$ is straightforward. To prove the converse inequality, let $f \in L(F(S))$ such that $f \leqslant \hat{x}, \hat{y}$. We will prove that $f \leqslant \widehat{x \wedge y}$. Since $L(F(S)) = S_R$ by the observations made in § 6.1, it suffices to assume that $f = \frac{1}{2^k}\hat{z}$ for some $z \in S$ and $k \in \mathbb{N}$. Let $\varepsilon > 0$, and let $z' \in S$ such that $z' \ll z$. By [Rob13, Proposition 2.2.6] there exists $N \in \mathbb{N}$ such that $1/2^N < \varepsilon$ and

$$(2^N - 1)z' \leqslant 2^{N+k}x, \quad \text{and} \quad (2^N - 1)z' \leqslant 2^{N+k}y.$$

Hence, $(2^N - 1)z' \leqslant 2^{N+k}x \wedge 2^{N+k}y$. Passing to $L(F(S))$ and using Lemma 6.8, we obtain

$$(2^N - 1)\widehat{z'} \leqslant 2^{N+k}\widehat{x \wedge y}.$$

Therefore, by our choice of N, we have

$$\frac{1-\varepsilon}{2^k}\widehat{z'} \leqslant \frac{2^N - 1}{2^{N+k}}\widehat{z'} \leqslant \widehat{x \wedge y}.$$

Since this holds for all $\varepsilon > 0$ and $z' \in S$ such that $z' \ll z$, the result follows. $\square$

7. Realizing functions as ranks

In this section we solve the problems of realizing (full) functions on the cone $F(\mathrm{Cu}(A))$ as ranks of Cuntz semigroup elements when A is a C*-algebra of stable rank one.

By an *ideal-quotient* of a C*-algebra A we mean a quotient of the form I/J, where $J \subseteq I$ are closed-two sided ideals of A. Ideal-quotients thus arise as ideals of the quotients of A or as quotients of its ideals.

7.1. **Proposition.** *Let A be a C*-algebra. Then the following statements hold:*

(i) *If A has a nonzero, elementary ideal-quotient then there exists $\lambda \in F(\mathrm{Cu}(A))$ with*
$$\{\hat{x}(\lambda) : x \in \mathrm{Cu}(A)\} = \{0, 1, \ldots, \infty\}.$$
(ii) *If A is separable and has a nonzero, elementary quotient then there exists a densely finite $\lambda \in F(\mathrm{Cu}(A))$ such that*
$$\{\hat{x}(\lambda) : x \in \mathrm{Cu}(A) \text{ and } x \text{ is full}\} = \{1, \ldots, \infty\}.$$

Proof. (i): Assume that I and J are closed, two-sided ideals such that $J \subseteq I$ and I/J is elementary. Then $\mathrm{Cu}(I/J) \cong \overline{\mathbb{N}}$ and thus the quotient map $I \xrightarrow{\pi} I/J$ induces a Cu-morphism $\mathrm{Cu}(\pi) \colon \mathrm{Cu}(I) \to \mathrm{Cu}(I/J) \cong \overline{\mathbb{N}}$. Now let $\lambda \colon \mathrm{Cu}(A) \to [0, \infty]$ be given by $\lambda(x) = \mathrm{Cu}(\pi)(x)$ if $x \in \mathrm{Cu}(I)$ and $\lambda(x) = \infty$ otherwise. It is easy to verify that λ is a functional on $\mathrm{Cu}(A)$ with the desired property.

(ii): Let I be a closed, two-sided ideal such that A/I is elementary. Let $\lambda \in F(\mathrm{Cu}(A))$ be the functional obtained in (i), that is, $\lambda = \mathrm{Cu}(\pi)$, where $\pi\colon A \to A/I$. If $x \in \mathrm{Cu}(A)$ is full then $\hat{x}(\lambda) \neq 0$, so that $\hat{x}(\lambda) \in \{1, 2, \dots, \infty\}$. To complete the proof it suffices to show that there exists a full x such that $\lambda(x) = 1$. Since λ is onto, there exists $x_0 \in \mathrm{Cu}(A)$ such that $\lambda(x_0) = 1$. Let $\omega_I \in \mathrm{Cu}(I)$ be the largest element of $\mathrm{Cu}(I)$, which exists since I is separable. Set $x = x_0 + \omega_I$. Clearly, $\lambda(x) = \lambda(x_0) = 1$. Moreover, x is full, for if $y \in \mathrm{Cu}(A)$ then $\lambda(y) \leqslant \infty \in \overline{\mathbb{N}}$, from which we deduce that $y \leqslant \infty \cdot x_0 + \omega_I = \infty \cdot x$. □

In view of the previous proposition, it is clear that in order to realize every element of $L(F(\mathrm{Cu}(A)))$ in the form $\hat{x}$, with $x \in \mathrm{Cu}(A)$, we must assume that A has no nonzero, elementary ideal-quotients. Similarly, if A is unital, and $F_u(\mathrm{Cu}(A))$ is the set of functionals normalized at $[1_A]$, then in order to realize elements of $\mathrm{LAff}(F_u(\mathrm{Cu}(A)))_{++}$ in the form $\hat{x}|_{F_u(\mathrm{Cu}(A))}$ with $x \in \mathrm{Cu}(A)$ full, we must assume that A has no nonzero, finite dimensional representations. As we show below, if A is has stable rank one, then these are the only obstructions.

In the proof of the following theorem we borrow ideas from the closely related [Thi17, Lemma 8.3].

7.2. **Theorem.** *Let A be a separable C*-algebra of stable rank one, and let $f \in L(F(\mathrm{Cu}(A)))$. Then the set*

$$I_f = \big\{x \in \mathrm{Cu}(A) : \widehat{x'} \ll f \text{ for all } x' \ll x\big\}$$

has a supremum.

Proof. Since A is separable, $\mathrm{Cu}(A)$ is countably based. Thus, as noted in § 2.6, it suffices to show that I_f is upward directed. Clearly I_f is hereditary. It is also closed under the suprema of increasing sequences. For suppose that $x = \sup_n x_n$, where $(x_n)_n$ is an increasing sequence in I_f. Let $x' \ll x$. Then $x' \ll x_n$ for some n, and so $\widehat{x'} \ll f$ by the definition of I_f. This shows that $x \in I_f$. By Lemma 3.2, in order to show that I_f is upward directed it suffices to show that the set $G_f = \{x' \in \mathrm{Cu}(A) \colon x' \ll x \text{ and } x \in I_f\}$ is upward directed. We prove this below. We remark that G_f can be alternatively described as follows:

$$G_f = \big\{x \in S : \text{there exists } y \in S \text{ such that } x \ll y \text{ and } \hat{y} \ll f\big\}.$$

In order to see this, let $x \in G_f$. Then there exist y', y such that $x \ll y' \ll y$ and $y \in I_f$. Then $\widehat{y'} \ll f$, and thus x belongs to the right hand side of the equality above. Conversely, if x is such that $x \ll y$ and $\hat{y} \ll f$ for some y, then clearly $y \in I_f$ and therefore $x \in G_f$.

We now prove that G_f is upward directed. Let $x_1, x_2 \in G_f$. Choose elements y_1, y_1', y_2, y_2' such that

$$x_1 \ll y_1' \ll y_1, \quad x_2 \ll y_2' \ll y_2, \quad \text{and} \quad \widehat{y_1}, \widehat{y_2} \ll f.$$

Since $L(F(\mathrm{Cu}(A)))$ is equal to $\mathrm{Cu}(A)_R$ (the realification of $\mathrm{Cu}(A)$), we have $f = \sup_n \hat{d}_n/k_n$, for suitable $d_n \in \mathrm{Cu}(A)$ and $k_n \in \mathbb{N}$. We may thus find $d \in \mathrm{Cu}(A)$ and $k \in \mathbb{N}$ such that

$$\widehat{y_1}, \widehat{y_2} \ll \frac{\hat{d}}{k} \leqslant f.$$

We will construct $w \in \mathrm{Cu}(A)$ such that $x_1, x_2 \leqslant w$ and $\hat{w} \leqslant \frac{\hat{d}}{k}$. Arranging for $w \in G_f$ after this is easy.

Since $y_1, y_2 \leqslant \infty \cdot d$, there exists $n \in \mathbb{N}$ such that $y_1', y_2' \leqslant nd$. We apply the construction from Proposition 2.8 to A and $d \in \mathrm{Cu}(A)$ to obtain a C*-algebra B of stable rank one and a full projection $p_d \in B$ such that A is an ideal of B, and such that for any $x \in \mathrm{Cu}(A)$ we have $x \leqslant d$ precisely when $x \leqslant [p_d]$ in $\mathrm{Cu}(B)$. Set $e = [p_d]$, which is a full, compact element in $\mathrm{Cu}(B)$.

Then $y_1', y_2' \leqslant nd \leqslant n[p_d] = ne$. By (O5), there exist $z_1, z_2 \in \mathrm{Cu}(B)$ such that

$$x_1 + z_1 \leqslant ne \leqslant y_1' + z_1,$$
$$x_2 + z_2 \leqslant ne \leqslant y_2' + z_2.$$

Set $z = z_1 \wedge z_2$. Note that $z \leqslant ne$. Let $\varepsilon_0 > 0$ be such that $\widehat{y_1}, \widehat{y_2} \leqslant \frac{1-\varepsilon_0}{k}\hat{d}$ and set $g = \frac{1-\varepsilon_0}{k}\hat{d}$. Next, choose $0 < \varepsilon < \varepsilon_0$ such that $\varepsilon\hat{z} \leqslant (\varepsilon_0 - \varepsilon)\frac{\hat{e}}{k}$. (Such an ε exists since $z \leqslant ne$.) Then

$$(1+\varepsilon)g + \varepsilon\hat{z} \leqslant (1+\varepsilon)(1-\varepsilon_0)\frac{\hat{e}}{k} + (\varepsilon_0 - \varepsilon)\frac{\hat{e}}{k} \leqslant \frac{\hat{e}}{k}.$$

We have $n\hat{e} \leqslant \widehat{y_1'} + \widehat{z_1} \leqslant g + \widehat{z_1}$ and similarly $n\hat{e} \leqslant g + \widehat{z_2}$. Using at the first step that $L(F(\mathrm{Cu}(B)))$ is an inf-semilattice ordered Cu-semigroup (by [Rob13, Theorem 4.2.2]) and using Theorem 6.10 at the second step, we obtain

$$n\hat{e} \leqslant g + \hat{z}_1 \wedge \hat{z}_2 = g + \hat{z}.$$

Next, since $e \ll e$, it follows from [Rob13, Lemma 2.2.5] that

$$n\hat{e} \ll (1+\varepsilon)g + (1+\varepsilon)\hat{z}.$$

Choose $z' \in \mathrm{Cu}(B)$ with $z' \ll z$ and $n\hat{e} \ll (1+\varepsilon)g + (1+\varepsilon)\widehat{z'}$. Applying (O5) to $z' \ll z \leqslant ne$, find $w' \in \mathrm{Cu}(B)$ such that $z' + w' \leqslant ne \leqslant z + w'$. Then

$$x_1 + z \leqslant x_1 + z_1 \leqslant ne \ll ne \leqslant z + w'.$$

Recall that B has stable rank one, and thus $\mathrm{Cu}(B)$ has weak cancellation. Therefore, we have $x_1 \leqslant w'$, and similarly, $x_2 \leqslant w'$. On the other hand,

$$\widehat{z'} + \widehat{w'} \leqslant n\hat{e} \ll (1+\varepsilon)g + (1+\varepsilon)\widehat{z'}.$$

Therefore $\widehat{w'} \leqslant (1+\varepsilon)g + \varepsilon\hat{z} \leqslant \frac{1}{k}\hat{e}$.

Let $\omega_A \in \mathrm{Cu}(A)$ be the largest element of $\mathrm{Cu}(A)$. Set $w = w' \wedge \omega_A$, which belongs to $\mathrm{Cu}(A)$ since the inclusion $A \to B$ identifies $\mathrm{Cu}(A)$ with an ideal in $\mathrm{Cu}(B)$. Using $x_1, x_2 \leqslant w'$, we get $x_1, x_2 \leqslant w$. Applying Theorem 6.10 at the first step, and using that $\frac{1}{k}\widehat{\omega_A} = \widehat{\omega_A}$ at the third step, we obtain

$$\hat{w} = \widehat{w'} \wedge \widehat{\omega_A} \leqslant \frac{\hat{e}}{k} \wedge \widehat{\omega_A} = \frac{1}{k}\widehat{e \wedge \omega_A} = \frac{1}{k}\hat{d}.$$

Thus, w is as desired. $\square$

7.3. *The map α.* Let A be a separable C*-algebra of stable rank one. In view of Theorem 7.2, we define $\alpha\colon L(F(\mathrm{Cu}(A))) \to \mathrm{Cu}(A)$ by

$$\alpha(f) = \sup\big\{x \in \mathrm{Cu}(A) : \widehat{x'} \ll f \text{ for all } x' \ll x\big\}.$$

In the course of the proof of Theorem 7.2 we have also shown that

$$\alpha(f) = \sup\big\{x \in \mathrm{Cu}(A) : x \ll y \text{ and } \hat{y} \ll f \text{ for some } y\big\}.$$

Yet another description of α that we will find useful is as follows:

$$\alpha(f) = \sup\big\{x \in \mathrm{Cu}(A) : \hat{x} \leqslant (1-\varepsilon)f \text{ for some } \varepsilon > 0\big\}.$$

To see this, notice first that the set on the right hand side contains the set of all x such that $\hat{x} \ll f$, of which $\alpha(f)$ is the least upper bound. It thus suffices to show that $\alpha(f)$ is an upper bound of the set on the right hand side. Suppose that x satisfies $\hat{x} \leqslant (1-\varepsilon)f$ for some $\varepsilon > 0$. By [Rob13, Lemma 2.2.5], we have $\widehat{x'} \ll f$ for every $x' \ll x$. Hence, $x \in I_f$, from which $x \leqslant \alpha(f)$, as desired.

We use the map α to solve the problems on realizing (full) elements of $L(F(\mathrm{Cu}(A)))$ as ranks of Cuntz semigroup elements when A is separable and of stable rank one.

We show that, under suitable hypotheses, $f = \hat{z}$ for $z = \alpha(f)$. We first prove this is the case when f is the chisel of an extreme densely finite functional (see § 6.2 and Lemma 7.7 below) and then extend this to arbitrary (full) functions.

7.4. **Proposition.** *Let A be a separable C*-algebra, and let $\lambda \in F_0(\mathrm{Cu}(A))$ be an extreme densely finite functional. Then the chisel σ_λ at λ belongs to $L(F(\mathrm{Cu}(A)))$.*

Proof. If $\lambda = 0$ the proposition holds trivially. Assume thus that $\lambda \neq 0$. We first show that the set

$$\big\{f \in L(F(\mathrm{Cu}(A))) : f(\lambda) < 1\big\} \tag{7.1}$$

is upward directed. Let $f_1, f_2 \in L(F(\mathrm{Cu}(A)))$ satisfy $f_1(\lambda), f_2(\lambda) < 1$. Assume that $f_1(\lambda) \leqslant f_2(\lambda)$. Note that $F(L(F(\mathrm{Cu}(A)))) \cong F(\mathrm{Cu}(A))$, using for example Proposition 3.1.1 and Theorem 3.2.1 in [Rob13]. Further, $\mathrm{Cu}(A)$ satisfies Edwards' condition for λ; see § 6.3. Therefore, the map $f \mapsto f(\lambda)$, regarded as a functional on $L(F(\mathrm{Cu}(A)))$, satisfies $(f_1 \wedge f_2)(\lambda) = f_1(\lambda)$. Choose $\varepsilon > 0$ such that $f_2(\lambda) + \varepsilon < 1$. Next, choose $g \lhd f_1 \wedge f_2$ with $g(\lambda) > f_1(\lambda) - \varepsilon$. Finally, we apply [Rob13, Lemma 3.3.2] to obtain h such that $g + h = f_1 + f_2$ and such that the support of h is the same as that of $f_1 + f_2$. We have

$$f_1 + h \geqslant g + h = f_1 + f_2.$$

Hence, since the supports of h and $f_1 + f_2$ agree, we may use cancellation in $L(F(\mathrm{Cu}(A)))$ to conclude that $h \geqslant f_2$. Symmetrically, $h \geqslant f_1$. On the other hand,

$$f_1(\lambda) - \varepsilon + h(\lambda) \leqslant g(\lambda) + h(\lambda) = f_1(\lambda) + f_2(\lambda),$$

from which we deduce that $h(\lambda) \leqslant f_2(\lambda) + \varepsilon < 1$.

Since $L(F(\mathrm{Cu}(A)))$ is a countably based Cu-semigroup, the upward directed set in (7.1) has a supremum, which we now proceed to prove is precisely σ_λ. To this end, it suffices to show that for any $\mu \in F(\mathrm{Cu}(A))$ such that $\mu \notin [0, \infty)\lambda$ and any $C > 0$ there exists $f \in L(F(\mathrm{Cu}(A)))$ such that $f(\lambda) < 1$ and $f(\mu) > C$. To show this, choose $x \in \mathrm{Cu}(A)$ such that $0 < \lambda(x) < \infty$, which is possible as $\lambda \neq 0$ by assumption. Since μ is not a scalar multiple of λ and the latter is extreme by assumption, we have $\mu \not\leqslant 4C\lambda$. Let $y \in \mathrm{Cu}(A)$ be such that $4C\lambda(y) < \mu(y)$. If $\lambda(y) = 0$, then $f = \frac{2C}{\mu(y)}\hat{y}$ is as desired. Suppose that $\lambda(y) > 0$. Set

$$f = \frac{1}{4\lambda(x)} \cdot \hat{x} + \frac{1}{4\lambda(y)} \cdot \hat{y}.$$

Clearly then $f(\lambda) = 1/4 + 1/4 < 1$. Also,

$$f(\mu) \geqslant \frac{1}{4} \cdot \frac{\mu(y)}{\lambda(y)} > C.$$

Hence, f is as desired. □

7.5. **Lemma.** *Let A be a C*-algebra, and let $\lambda, \mu \in F_0(\mathrm{Cu}(A))$ be densely finite functionals, with λ extreme and $\mu \notin [0, \infty)\lambda$. Then for every $\varepsilon > 0$ there exists $w \in \mathrm{Cu}(A)$ such that $\lambda(w) < \varepsilon$ and $\mu(w) > 1/\varepsilon$.*

Proof. If $\lambda = 0$ the lemma follows easily. Let us thus assume that $\lambda \neq 0$. We can reduce the proof to the case that μ is also extreme. To this end, we find $\mu' \leqslant \mu$ that is extreme and not a scalar multiple of λ. Such a μ' must exist since $F_0(\mathrm{Cu}(A))$ is a well-capped cone (see [APRT18]). It is then clear that it suffices to prove the lemma for the pair of functionals λ and μ'. Changing notation, we assume that μ is also an extreme functional.

Choose $x \in \mathrm{Cu}(A)$ such that $0 < \lambda(x) < \infty$ and $0 < \mu(x) < \infty$. This is possible as both λ and μ are densely finite, and thus if $0 < \lambda(z) < \infty$ and $0 < \mu(w) < \infty$, there are $z', w' \in \mathrm{Cu}(A)$ with $z \ll z'$ and $w \ll w'$. Then $x := z + w$ satisfies the required conditions. Normalize λ and μ so that $\lambda(x) = \mu(x) = 1$. (The normalized functionals are multiples of the original functionals by fixed scalars not depending on ε; thus, the proof of the lemma may be reduced to the normalized functionals.) We have $\lambda \nleq \mu$ since μ is extreme and not a scalar multiple of λ. Let $y \in \mathrm{Cu}(A)$ be such that $\mu(y) < \lambda(y) < \infty$. Set $\delta = \lambda(y) - \mu(y)$.

Choose numbers $m, n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} + \varepsilon < m\delta$ and

$$|\lambda(nx) - \lambda(my)| = |n - m\lambda(y)| < \varepsilon\,.$$

Set $z = (nx) \wedge (my)$. Since λ is an extreme functional that satisfies the Edwards' condition (see § 6.3), we have $\lambda(z) = \min\{n, m\lambda(y)\}$. Since $n - \varepsilon < m\lambda(y)$ and clearly $n - \varepsilon < n$, we deduce that $|\lambda(z) - n| < \varepsilon$. Choose $z' \in \mathrm{Cu}(A)$ such that $z' \ll z$ and $|\lambda(z') - n| < \varepsilon$.

Now, by (O5) applied to $z' \ll z \leqslant nx$, there is $w \in S$ such that $z' + w \leqslant nx \leqslant z + w$. Then $\lambda(w) \leqslant \lambda(nx) - \lambda(z') < \varepsilon$. Also,

$$\begin{aligned}\mu(z) + \mu(w) \geqslant \mu(nx) = n = n - \lambda(my) + \lambda(my) &\geqslant -\varepsilon + m\delta + \mu(my) \\ &\geqslant -\varepsilon + m\delta + \mu(z).\end{aligned}$$

Therefore $\frac{1}{\varepsilon} < m\delta - \varepsilon \leqslant \mu(w)$, as desired. □

7.6. **Lemma.** *Let A be a separable C*-algebra of stable rank one that has no nonzero type I quotients, and let $\lambda \in F_0(\mathrm{Cu}(A))$ be a nonzero, densely finite functional.*

(i) *For each $\varepsilon > 0$ the set $\{x \in \mathrm{Cu}(A) : \lambda(x) < \varepsilon\}$ is a full subset of* $\mathrm{Cu}(A)$.
(ii) *The range of λ is* $[0, \infty]$.

Proof. (i): Let $W \subseteq \mathrm{Cu}(A)$ be the ideal generated by $\{x \in \mathrm{Cu}(A) : \lambda(x) < \varepsilon\}$. Let $I \subseteq A$ be the closed, two-sided ideal such that $W = \mathrm{Cu}(I)$. Suppose for the sake of contradiction that I is proper. Let $x \in \mathrm{Cu}(A)$ with $\lambda(x) < \infty$. Find $x'' \ll x' \ll x$ such that $\lambda(x) - \lambda(x'') < \varepsilon$. By (O5), there exists w such that $x'' + w \leqslant x \leqslant x' + w$. Evaluating on λ we get $\lambda(w) < \varepsilon$, whence $w \in W$. Thus, the images of x and x' in $\mathrm{Cu}(A/I)$ agree. It follows that the image of x in $\mathrm{Cu}(A/I)$ is compact.

Next, we show that A/I contains a positive element with spectrum $[0, 1]$. Since A/I is not type I, it follows from Glimm's theorem that there exists a sub-C*-algebra $B \subseteq A/I$ that has a UHF-algebra as a quotient. In a UHF-algebra it is easy to find a positive element $\tilde{b}$ with spectrum $[0, 1]$. Lift $\tilde{b}$ to a positive, contractive element b in B. The b has spectrum $[0, 1]$ in B, and consequently also in A/I. By [BC09, Theorem 3.5], if C is a stably finite C*-algebra and $c \in C_+$, then $[c]$ is compact if and only if 0 is an isolated point of the spectrum of c. Since A/I has stable rank one, it is stably finite, and it follows that $[(b - t)_+] \in \mathrm{Cu}(A/I)$ is not compact for every $t \in (0, 1)$. Let $a \in A_+$ be a lift of b. Then $\lambda([(a - 1/2)_+]) < \infty$ and $[(a - 1/2)_+]$ is mapped to $[(b - 1/2)_+]$ in $\mathrm{Cu}(A/I)$, which is not compact. This contradicts what was proved in the previous paragraph.

(ii): It suffices to show that λ attains arbitrarily small nonzero values. Fix $\varepsilon > 0$. By part (i), $\{x \in \mathrm{Cu}(A) : \lambda(x) < \varepsilon\}$ is a full subset of $\mathrm{Cu}(A)$. So if λ does not attain nonzero values less than ε then it is the zero functional, contradicting our assumption. Thus, there exists $x \in \mathrm{Cu}(A)$ such that $0 < \lambda(x) < \varepsilon$. □

7.7. **Lemma.** *Let A be a separable C*-algebra of stable rank one that has no nonzero type I quotients, let $\lambda \in F_0(\mathrm{Cu}(A))$ be an extreme, densely finite functional, and let σ_λ denote its chisel. Then $\sigma_\lambda = \hat{z}$ for $z := \alpha(\sigma_\lambda)$.*

Proof. We clearly have $\hat{z} \leqslant \sigma_\lambda$. If $\lambda = 0$ then σ_λ is the zero function, and the lemma holds trivially. Assume thus that $\lambda \neq 0$. We first show that z is full. Let $x \in \mathrm{Cu}(A)$ satisfy $\lambda(x) < 1$. Then $\hat{x} \leqslant (1 - \varepsilon)\sigma_\lambda$ for a sufficiently small ε. As noted in § 7.3, we get $x \leqslant \alpha(\sigma_\lambda) = z$. Hence, by Lemma 7.6 (i), z is full.

Let $0 < \varepsilon < 1$. By Lemma 7.6 (ii), there exists $x \in \mathrm{Cu}(A)$ such that $\lambda(x) = 1 - \varepsilon$. Then $\hat{x} \leqslant (1-\varepsilon)\sigma_\lambda$, whence $x \leqslant z$. Evaluating at λ we get $1 - \varepsilon \leqslant \lambda(z)$. Since ε can be arbitrarily small, we obtain $\lambda(z) = 1$, that is, $\hat{z}(\lambda) = 1$.

Let μ be a densely finite functional that is not a scalar multiple of λ, and let $\varepsilon > 0$. By Lemma 7.5, there exists $w \in \mathrm{Cu}(A)$ such that $\lambda(w) < 1$ and $\mu(w) > 1/\varepsilon$. Similar as in the previous paragraph, we get $w \leqslant z$, from which we obtain that $\mu(z) > 1/\varepsilon$. Since ε can be arbitrarily small, we deduce $\mu(z) = \infty$, that is, $\hat{z}(\mu) = \infty$.

We have shown that $\hat{z}(\mu) = \sigma_\lambda(\mu)$ for all μ densely finite. Further, since z is full, this equality holds also for all functionals that are not densely finite, as in this case both sides equal ∞. The lemma is thus proved. ☐

7.8. **Theorem.** *Let A be a separable C*-algebra of stable rank one that has no nonzero type I quotients, and let $f \in L(F(\mathrm{Cu}(A)))$ be a full function. Then $f = \hat{z}$ for $z := \alpha(f)$.*

Proof. Choose $g \lhd f$. Choose $\varepsilon > 0$ such that $g \lhd (1-\varepsilon)f$. We claim that there exists $x \in \mathrm{Cu}(A)$ such that $g \leqslant \hat{x} \leqslant (1-\varepsilon)f$. Once the claim is proved, notice that from $\hat{x} \leqslant (1-\varepsilon)f$ we get $x \leqslant \alpha(f) = z$. Hence, $g \leqslant \hat{z}$ for any g such that $g \lhd f$. Since f is the supremum of all such g, we obtain that $\hat{z} = f$, as desired.

We now prove the claim. If $f = \sigma_0$ (that is, the function equal to ∞ on all nonzero functionals) then we can choose $x = \infty$. We may thus assume that f is finite on some nonzero functionals. Set

$$K := \big\{\lambda \in F(\mathrm{Cu}(A)) : f(\lambda) \leqslant 1\big\}.$$

Observe that K is a closed (whence compact) convex subset of $F(\mathrm{Cu}(A))$. Further, since the function f is full, we have $K \subseteq F_0(\mathrm{Cu}(A))$. Let $\partial_e K$ denote the set of extreme points of K. Notice that since K is a cap of $F_0(\mathrm{Cu}(A))$, every extreme point of K is also an extreme functional in $F_0(\mathrm{Cu}(A))$ (see [APRT18]).

For each $\lambda \in \partial_e K$ set $x_\lambda := \alpha((1-\varepsilon/2)f(\lambda)\sigma_\lambda)$. We know by Lemma 7.7 that $\hat{x}_\lambda = (1-\varepsilon/2)f(\lambda)\sigma_\lambda$. Hence

$$g \ll (1-\frac{\varepsilon}{2})f \leqslant (1-\frac{\varepsilon}{2})f(\lambda)\sigma_\lambda = \hat{x}_\lambda.$$

Choose $x'_\lambda \in \mathrm{Cu}(A)$ such that $x'_\lambda \ll x_\lambda$ and $g \leqslant \widehat{x'_\lambda}$. By the definition of the map α, we have $\widehat{x'_\lambda} \ll (1-\varepsilon/2)f(\lambda)\sigma_\lambda = \widehat{x_\lambda}$. Let us choose $h_\lambda \in L(F(\mathrm{Cu}(A)))$ such that $\widehat{x'_\lambda} \leqslant h_\lambda \leqslant \widehat{x_\lambda}$ and such that h_λ is continuous on K. In order to do this, use that $\widehat{x'_\lambda} \ll \widehat{x_\lambda}$ to choose $x''_\lambda \ll x_\lambda$ such that $\widehat{x'_\lambda} \ll \widehat{x''_\lambda}$, and then let choose h_λ such that $\widehat{x'_\lambda} \leqslant h_\lambda \lhd \widehat{x''_\lambda}$. Since $x''_\lambda \ll x_\lambda$, the function $\widehat{x''_\lambda}$ is finite on the densely finite functionals, and thus h_λ is continuous on all the densely finite functionals and in particular on K.

For each finite set of extreme functionals $F \subseteq \partial_e K$ define

$$h_F := \bigwedge_{\lambda \in F} h_\lambda$$

The set $\{(h_F)|_K : F \subseteq \partial_e K \text{ finite}\}$ is a downward directed family of continuous affine functions on K, whence the pointwise infimum is an upper semicontinuous affine function $h \colon K \to [0,\infty)$. Since $g \leqslant h_F$ for all F, we have $g|_K \leqslant h$. We also have by construction that $h(\lambda) \leqslant (1-\varepsilon/2)f(\lambda)$ for all $\lambda \in \partial_e K$. Since both h and $f|_K$ are affine functions of first Baire class on K, we obtain that $h \leqslant (1-\varepsilon/2)f|_K$.

For any finite set $F \subseteq \partial_e K$, define

$$U_F = \big\{\lambda \in K : h_F(\lambda) < f(\lambda)\big\} \quad \text{and} \quad V_F = \big\{\lambda \in K : h_F(\lambda) < 1\big\},$$

which are open subsets of K as h_F is continuous on K and $f - h_F$ is lower semicontinuous on K. Using the inequality $h \leqslant (1-\varepsilon/2)f|_K$ we see that

$$K = \bigcup_{F \Subset \partial_e K} (U_F \cup V_F).$$

By compactness of K, and since the $U_F \cup V_F$ are upwards directed, there exists a finite subset F of $\partial_e K$ such that $K = U_F \cup V_F$.

We now claim that $h_F \leqslant f$. Let $\lambda \in F(\mathrm{Cu}(A))$. If $f(\lambda) = \infty$ the claim holds trivially, so assume that $f(\lambda) < \infty$. Suppose also that $0 < f(\lambda)$ and let $t = \frac{1}{f(\lambda)}$. Clearly, $t\lambda$ is an element of K. Thus, if $t\lambda \in U_F$, then $h_F(t\lambda) < f(t\lambda)$, and hence $h_F(\lambda) \leqslant f(\lambda)$. If, on the other hand, $t\lambda \in V_F$, then $h_F(t\lambda) < 1 = f(t\lambda)$, whence $h_F(\lambda) \leqslant f(\lambda)$. Finally, suppose that $f(\lambda) = 0$. Then, given $x \in \mathrm{Cu}(A)$, we have $\hat{x} \leqslant \infty f$ because f is full. Therefore, for any $x' \in \mathrm{Cu}(A)$ such that $x' \ll x$, there is $n \in \mathbb{N}$ with $\hat{x}' \leqslant ng$. Thus $\lambda(x') = 0$, and this implies that $\lambda = 0$. Thus the claim is proved.

Now, set $x := \bigwedge_{\lambda \in F} x'_\lambda$. Then, using that $g \leqslant \widehat{x'_\lambda}$ for each extreme functional λ at the first step, using Theorem 6.10 at the second step, and using that $\widehat{x'_\lambda} \leqslant h_\lambda$ at the third step, we obtain

$$g \leqslant \bigwedge_{\lambda \in F} \widehat{x'_\lambda} = \hat{x} \leqslant \bigwedge_{\lambda \in F} h_\lambda = h_F \leqslant f.$$

Thus, x is as desired. □

7.9. Let A be a C*-algebra and let $I \subseteq A$ be a closed, two-sided ideal of A. Recall that we may regard $\mathrm{Cu}(I)$ as an ideal of $\mathrm{Cu}(A)$. Given $\lambda \in F(\mathrm{Cu}(I))$, define $\tilde{\lambda} \in F(\mathrm{Cu}(A))$ by

$$\tilde{\lambda}(x) = \begin{cases} \lambda(x), & \text{if } x \in \mathrm{Cu}(I) \\ \infty, & \text{otherwise} \end{cases}.$$

The assignment $\lambda \mapsto \tilde{\lambda}$ defines an order-embedding $F(\mathrm{Cu}(I)) \to F(\mathrm{Cu}(A))$ which is a right inverse to the restriction map $F(\mathrm{Cu}(A)) \to F(\mathrm{Cu}(I))$, given by $\lambda \mapsto \lambda|_{\mathrm{Cu}(I)}$. Thus, the latter map is surjective. By applying the functor $L(\cdot)$ to the restriction map, we obtain a map $L(F(\mathrm{Cu}(I))) \to L(F(\mathrm{Cu}(A)))$, given by $f \mapsto \tilde{f}$, where $\tilde{f}(\lambda) := f(\lambda|_{\mathrm{Cu}(I)})$ for $\lambda \in F(\mathrm{Cu}(A))$.

7.10. **Theorem.** *Let A be a separable C*-algebra of stable rank one that has no nonzero, elementary ideal-quotients, and let $f \in L(F(\mathrm{Cu}(A)))$. Then $f = \hat{z}$ for $z := \alpha(f)$.*

Proof. The set $W = \{x \in \mathrm{Cu}(A) : \hat{x} \leqslant \infty \cdot f\}$ is an ideal of $\mathrm{Cu}(A)$. Let I be the closed, two-sided ideal of A such that $\mathrm{Cu}(I) = W$. Note that I is a separable C*-algebra of stable rank one that has no nonzero, elementary ideal-quotients, and in particular no nonzero type I quotients. Now, writing $f = \sup \frac{\hat{x}_n}{k_n}$ for suitable $x_n \in \mathrm{Cu}(A)$ and $k_n \in \mathbb{N}$, we see that $x_n \in \mathrm{Cu}(I)$ for all n, and thus $f \in L(F(\mathrm{Cu}(I)))$. It is moreover clear that f is a full element of $L(F(\mathrm{Cu}(I)))$. Let $\alpha_I \colon L(F(\mathrm{Cu}(I))) \to \mathrm{Cu}(I)$ be the map from § 7.3 for I. Set $y := \alpha_I(f)$. By Theorem 7.8, we have $f = \hat{y}$ in $L(F(\mathrm{Cu}(I)))$. Using the observation from § 7.9, it is easy to see that the following diagram commutes:

$$\begin{tikzcd} L(F(\mathrm{Cu}(I))) \arrow[r] \arrow[d, "\alpha_I"] & L(F(\mathrm{Cu}(A))) \arrow[d, "\alpha"] \\ \mathrm{Cu}(I) \arrow[r, hook] & \mathrm{Cu}(A). \end{tikzcd}$$

It follows that $\alpha_I(f) = \alpha(f)$, and consequently $\hat{z} = f$ in $L(F(\mathrm{Cu}(A)))$. □

7.11. **Theorem.** *Let A be a separable, unital C*-algebra of stable rank one that has non nonzero, finite dimensional quotients. Set $u = [1_A]$. Let $F_u(\mathrm{Cu}(A))$ denote the set of functionals $\lambda \in F(\mathrm{Cu}(A))$ normalized at u. Then for each $f \in \mathrm{LAff}(F_u(\mathrm{Cu}(A)))^{\sigma}_{++}$ there exists $z \in \mathrm{Cu}(A)$ such that $\hat{z}|_{F_u(\mathrm{Cu}(A))} = f$.*

Proof. Let I be a closed, two-sided ideal of A such that A/I has type I. Choose a maximal ideal J containing I. Then A/J is simple, unital and has type I, whence it is finite dimensional. It follows that A has no nonzero type I quotients.

We can thus apply Theorem 7.8 to realize full functions in $L(F(\mathrm{Cu}(A)))$. Moreover, by Proposition 6.7, given a function $f \in \mathrm{LAff}(\mathrm{Cu}(A))^{\sigma}_{++}$, there exists a full $\tilde{f} \in L(F(\mathrm{Cu}(A)))$ whose restriction to $F_u(\mathrm{Cu}(A))$ is f. Then $\tilde{f} = \hat{z}$ for $z := \alpha(\tilde{f})$, and so $\hat{z}|_{F_u(\mathrm{Cu}(A))} = f$. ☐

8. Supersoft elements and comparability

In this section we introduce the notion of supersoft elements in Cuntz semigroups of a separable C*-algebras of stable rank one. We use these elements to advance further the study of comparability properties in the Cuntz semigroups of these C*-algebras.

8.1. Let A be a separable C*-algebra of stable rank one. We call $z \in \mathrm{Cu}(A)$ *supersoft* if $\alpha(\hat{z}) = z$. Put differently, $z = \alpha(f)$ and $\widehat{\alpha(f)} = f$ for some $f \in L(F(\mathrm{Cu}(A)))$.

8.2. **Proposition.** *Let A be a separable C*-algebra of stable rank one.*

(i) *If $z \in \mathrm{Cu}(A)$ is supersoft then z is soft.*
(ii) *If $x \in \mathrm{Cu}(A)$ is soft, $z \in \mathrm{Cu}(A)$ is supersoft, and $\hat{x} \leqslant \hat{z}$, then $x \leqslant z$.*
(iii) *If x is soft then $x \leqslant \alpha(\hat{x})$ and $\alpha(\hat{x})$ is supersoft.*

Proof. (i): Let $z = \alpha(\hat{z})$ be supersoft. Let $z' \ll z$. Then $\widehat{z'} \ll \hat{z}$, by the definition of α (see § 7.3). This in turn implies that z is soft (see [APT18b, Proposition 5.3.3]).

(ii): Let $x' \in \mathrm{Cu}(A)$ satisfy $x' \ll x$. Since x is soft, $\widehat{x'} \ll \hat{x} \leqslant \hat{z}$. Thus, $\widehat{x'} \ll \hat{z}$ for every $x' \ll x$. It follows that $x \leqslant \alpha(\hat{z}) = z$.

(iii): Let $x' \in \mathrm{Cu}(A)$ satisfy $x' \ll x$. Since x is soft, $\widehat{x'} \ll \hat{x}$. Hence, $x' \leqslant \alpha(\hat{x})$ (see § 7.3). Passing to the supremum over all $x' \ll x$ we get that $x \leqslant \alpha(\hat{x})$. Hence, $\hat{x} \leqslant \widehat{\alpha(\hat{x})}$. On the other hand, from the definition of α we have that $\widehat{\alpha(f)} \leqslant f$ for any f. Thus, $\widehat{\alpha(\hat{x})} \leqslant \hat{x}$. It follows that $\hat{x} = \widehat{\alpha(\hat{x})}$. Hence, $\alpha(\hat{x})$ is supersoft. ☐

8.3. Let A be a separable C*-algebra of stable rank one. Our results on realizing elements of $L(F(\mathrm{Cu}(A)))$ as ranks guarantee the existence of supersoft elements in $\mathrm{Cu}(A)$:

(1) By Theorem 7.8, if $f \in L(F(\mathrm{Cu}(A)))$ is a full function then $\alpha(f)$ is supersoft, provided that A has no nonzero type I quotients. In particular, this is true if A is unital and has no nonzero, finite dimensional quotients.
(2) By Theorem 7.10, the set of supersoft elements agrees with the range of α, provided that A has no nonzero, elementary ideal-quotients.

For the result below, recall from § 7.3 that, if A is a separable C*-algebra with stable rank one and $f \in L(F(\mathrm{Cu}(A)))$, we have

$$\alpha(f) = \sup I_f, \text{ where } I_f = \big\{ x \in \mathrm{Cu}(A) : \hat{x}' \ll f \text{ for all } x' \ll x \big\}.$$

8.4. **Proposition.** *Let A be a separable C*-algebra of stable rank one. Then, the map $\alpha\colon L(F(\mathrm{Cu}(A))) \to \mathrm{Cu}(A)$ preserves the order, the suprema of increasing sequences, and the infima of pairs of elements.*

Proof. Let $f,g \in L(F(\mathrm{Cu}(A)))$ satisfy $f \leqslant g$. Then $I_f \subseteq I_g$, and thus $\alpha(f) \leqslant \alpha(g)$.

Next, let $(f_n)_n$ be an increasing sequence in $L(F(\mathrm{Cu}(A)))$, and set $f := \sup_n f_n$. Since α is order-preserving, the sequence $(\alpha(f_n))_n$ is increasing in $\mathrm{Cu}(A)$. Set $x := \sup_n \alpha(f_n)$. Since $\alpha(f_n) \leqslant \alpha(f)$ for all n, we have $x \leqslant \alpha(f)$. To prove the converse inequality, take $z \in I_f$. Then $\hat{z} \ll f$, and thus there is $n \in \mathbb{N}$ with $\hat{z} \ll f_n$. This means that $z \in I_{f_n}$ and thus $z \leqslant \alpha(f_n) \leqslant x$. Passing to the supremum over all $z \in I_f$ we get $\alpha(f) \leqslant x$. Hence $x = \alpha(f)$, as desired.

Finally, let us show that $\alpha(f \wedge g) = \alpha(f) \wedge \alpha(g)$. From the fact that α is order preserving we deduce at once that $\alpha(f \wedge g) \leqslant \alpha(f) \wedge \alpha(g)$. Let $0 < \varepsilon < 1$ and suppose that $z \leqslant \alpha((1-\varepsilon)f) \wedge \alpha((1-\varepsilon)g)$. Then

$$\hat{z} \leqslant (1-\varepsilon)f \wedge (1-\varepsilon)g = (1-\varepsilon)(f \wedge g).$$

Hence, $z \leqslant \alpha(f \wedge g)$. It follows that $\alpha((1-\varepsilon)f) \wedge \alpha((1-\varepsilon)g) \leqslant \alpha(f \wedge g)$. Letting $\varepsilon \to 0$ and using that α preserves suprema of increasing sequences, we get that $\alpha(f) \wedge \alpha(g) \leqslant \alpha(f \wedge g)$. □

8.5. **Theorem.** *Let A be a separable C*-algebra of stable rank one, let $x \in \mathrm{Cu}(A)$, and let $f \in L(F(\mathrm{Cu}(A)))$ satisfy $\hat{x} \leqslant \infty f$. Suppose that we are in one of the following cases:*

(i) *A is unital, has no nonzero, finite dimensional quotients, and f is full;*
(ii) *A has no nonzero, elementary ideal-quotients.*

Then

$$\alpha(f + \hat{x}) = \alpha(f) + x.$$

Proof. In both cases, (i) and (ii), we have that $\alpha(f)$ is supersoft, whence soft. By [APT18b, Theorem 5.3.11 (2)], the soft elements form an absorbing subsemigroup of $\mathrm{Cu}(A)$ in the following sense: if $z \leqslant \infty \cdot w$ and w is soft then $z + w$ is soft. Therefore, since $x \leqslant \infty \cdot \alpha(f)$, we have that $\alpha(f) + x$ is soft. Using Proposition 8.2 (ii) at the first step, and that $\widehat{\alpha(f)} = f$ at the second step, we obtain

$$\alpha(f) + x \leqslant \alpha(\widehat{\alpha(f)} + \hat{x}) = \alpha(f + \hat{x}).$$

Let us prove the opposite inequality. Assume first that $\hat{x} \propto f$ (that is, $\hat{x} \leqslant Cf$ for some constant $C > 0$). Let $h \in L(F(\mathrm{Cu}(A)))$ be any function such that $h \lhd f + \hat{x}$. In the case (i), assume also that h is full, and thus in either case we have $\widehat{\alpha(h)} = h$. Choose $\varepsilon > 0$ such that $h \leqslant (1-\varepsilon)f + \hat{x}$. We claim that $h \lhd h + \frac{\varepsilon}{2} f$. Indeed, notice first that $h \propto f$, since $\hat{x} \propto f$. It is then clear that for small enough $\delta > 0$ we have $h \leqslant (1-\delta)(h + \frac{\varepsilon}{2} f)$. Further, if $f(\lambda) < \infty$ then $f(\lambda) + \hat{x}(\lambda) < \infty$ and therefore h is continuous at λ.

Consider the element

$$y = (\alpha((1-\varepsilon)f) + x) \wedge \alpha(h).$$

Then

$$\hat{y} = ((1-\varepsilon)f) + \hat{x}) \wedge h = h.$$

Hence, $h \ll \hat{y} + \frac{\varepsilon}{2} f$ (since $h \lhd h + \frac{\varepsilon}{2} f$). Choose $y' \in \mathrm{Cu}(A)$ such that $y' \ll y$ and $h \ll \hat{y}' + \frac{\varepsilon}{2} f$. Then $y' \ll y \leqslant \alpha(h)$, and thus there exists by (O5) a $z \in \mathrm{Cu}(A)$ such that

$$y' + z \leqslant \alpha(h) \leqslant y + z.$$

Observe then that

$$\hat{y}' + \hat{z} \leqslant h \leqslant \hat{y}' + \frac{\varepsilon}{2} f.$$

It follows that $\hat{z} \leqslant \frac{\varepsilon}{2} f$, and so $z \leqslant \alpha(\varepsilon f)$. Then, using the definition of α at the last step, we obtain

$$\alpha(h) \leqslant y + z \leqslant \alpha((1-\varepsilon)f) + x + \alpha(\varepsilon f) \leqslant \alpha(f) + x.$$

Passing to the supremum over all $h \lhd f + \hat{x}$ and using that α is supremum preserving (Proposition 8.4) we get that $\alpha(f + \hat{x}) \leqslant \alpha(f) + x$, as desired.

Let us finally deal with the case that $\hat{x} \leqslant \infty f$. If $x' \ll x$ then $\hat{x}' \propto f$. Hence $\alpha(f + \hat{x}') = \alpha(f) + x'$. Passing to the supremum over all $x' \ll x$ the theorem follows. ☐

8.6. **Corollary.** *Let A be a separable C*-algebra of stable rank one.*

(i) *If A is unital and has no nonzero, finite dimensional quotients, then α is additive on the set of full elements of $L(F(\mathrm{Cu}(A)))$ and its range is an absorbing subsemigroup of* $\mathrm{Cu}(A)$.

(ii) *If A has no nonzero, elementary ideal-quotients, then α is additive and its range is an absorbing subsemigroup of* $\mathrm{Cu}(A)$.

Proof. (i): This is a straightforward consequence of the previous theorem.

(ii): To show that α is additive, let $f, g \in L(F(S))$. If $x, y \in \mathrm{Cu}(A)$ satisfy $\hat{x} \leqslant (1-\varepsilon)f$ and $\hat{y} \leqslant (1-\varepsilon)g$ for some $\varepsilon > 0$, then $\hat{x} + \hat{y} \leqslant (1-\varepsilon)(f+g)$, which implies that $x + y \leqslant \alpha(f+g)$. Passing to the supremum of all such x and y we obtain the inequality $\alpha(f) + \alpha(g) \leqslant \alpha(f+g)$.

Set $w_f = \alpha(\infty f)$ and $w_g = \alpha(\infty g)$. Then $w_f + w_g$ is idempotent, that is, $2(w_f + w_g) = w_f + w_g$. It is not difficult to check that idempotent elements are supersoft. Hence,

$$\alpha(\infty f) + \alpha(\infty g) = w_f + w_g = \alpha(\hat{w}_f + \hat{w}_g) = \alpha(\infty f + \infty g).$$

Now, using Theorem 8.5 and that α preserves infima (Proposition 8.4),

$$\alpha(f+g) \wedge w_f = \alpha\big((f+g) \wedge (\infty f)\big) = \big(\alpha(f) + \alpha(g)\big) \wedge w_f \leqslant \alpha(f) + \alpha(g).$$

Similarly,

$$\alpha(f+g) \wedge w_g \leqslant \alpha(f) + \alpha(g).$$

Hence,

$$\alpha(f+g) \wedge w_f + \alpha(f+g) \wedge w_g \leqslant 2(\alpha(f) + \alpha(g)).$$

Using the distributivity of addition over infima on the left side we obtain that $\alpha(f+g) \leqslant 2(\alpha(f) + \alpha(g))$.

Let $0 < \varepsilon < 1/2$. Then, using Theorem 8.5 and the inequality just established, we obtain

$$\begin{aligned}
\alpha\big((1-\varepsilon)(f+g)\big) &= \alpha\big((1-2\varepsilon)f\big) + \alpha\big((1-2\varepsilon)g\big) + \alpha\big(\varepsilon(f+g)\big)\\
&\leqslant \alpha\big((1-2\varepsilon)f\big) + \alpha\big((1-2\varepsilon)g\big) + \alpha(2\varepsilon f) + \alpha(2\varepsilon g)\\
&\leqslant \alpha(f) + \alpha(g).
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain that $\alpha(f+g) \leqslant \alpha(f) + \alpha(g)$. ☐

8.7. *Radius of comparison.* Let A be a unital C*-algebra. Set $u := [1_A]$ and recall that we use $F_u(\mathrm{Cu}(A))$ to denote the set of all $\lambda \in F(\mathrm{Cu}(A))$ such that $\lambda(u) = 1$. Recall that the *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is the infimum of the set of $r \in (0, \infty]$ such that

$$\lambda(x) + r \leqslant \lambda(y) \text{ for all } \lambda \in F_u(\mathrm{Cu}(A))) \implies x \leqslant y$$

for all $x, y \in \mathrm{Cu}(A)$ with y full. We will find the following restatement of the definition of $\mathrm{rc}(A)$ more convenient: $\mathrm{rc}(A)$ is the infimum of the set of $r \in (0, \infty]$ such that

$$\hat{x} + r\hat{u} \leqslant \hat{y} \implies x \leqslant y$$

for all $x, y \in \mathrm{Cu}(A)$. Observe that the fullness of y is now automatic since $r\hat{u} \leqslant \hat{y}$ and $r > 0$.

In the result below, we shall be using the general fact that, if $x \in \mathrm{Cu}(A)$, then x is full if and only if $\hat{x} \in L(F(\mathrm{Cu}(A)))$ is full. Recall that $W(A)$ denotes the set of Cuntz classes of positive elements in $M_\infty(A)$.

8.8. **Theorem.** *Let A be a separable, unital C*-algebra of stable rank one that has no nonzero, finite dimensional quotients. Set $u = [1_A]$. Then the following are equivalent:*

(i) $W(A) = \{x \in \mathrm{Cu}(A) : \hat{x} \leqslant n\hat{u}$ *for some* $n \in \mathbb{N}\}$.
(ii) $W(A)$ *contains at least one full supersoft element.*
(iii) *There exists $N \in \mathbb{N}$ such that $\hat{x} \leqslant \hat{u}$ implies $x \leqslant Nu$ for all $x \in \mathrm{Cu}(A)$.*
(iv) *A has finite radius of comparison.*

Proof. (i) $\implies$ (ii): Set $y := \alpha(\hat{u})$, which is a supersoft element. Since $\hat{y} = \hat{u}$, we have by (i) that y is an element of $W(A)$. It remains to see that y is full, but this follows from the fact that $\hat{y} = \hat{u}$ and u is full in $\mathrm{Cu}(A)$.

(ii) $\implies$ (iii): Let $z \in W(A)$ be a full supersoft element. Thus, there exist $m, n \in \mathbb{N}$ such that $u \leqslant mz \leqslant nu$.

Now let $x \in \mathrm{Cu}(A)$ be such that $\hat{x} \leqslant \hat{u}$. Then $\hat{x} \leqslant m\hat{z}$, and thus

$$x \leqslant x + \alpha(\hat{x}) \leqslant x + \alpha(m\hat{z}) = \alpha(\hat{x} + m\hat{z}) \leqslant \alpha(2m\hat{z}) = 2mz \leqslant 2nu.$$

(iii) $\implies$ (iv): Let N be as in (iii). To show that $\mathrm{rc}(A) \leqslant N$, let $x, y \in \mathrm{Cu}(A)$ satisfy $\hat{x} + N\hat{u} \leqslant \hat{y}$. Set $z := \alpha(\hat{u})$. Applying Theorem 8.5, we obtain

$$x + Nu \leqslant x + Nu + z = \alpha(\hat{x} + N\hat{u} + \hat{u}) \leqslant \alpha(\hat{y} + \hat{z}) = y + z.$$

By (iii), we have $z \leqslant Nu$, and therefore $x + Nu \leqslant y + Nu$. Hence, $x \leqslant y$ by cancellation of compact elements.

(iv) $\implies$ (i): Clearly if $x \in W(A)$ then $\hat{x} \leqslant n\hat{u}$ for some $n \in \mathbb{N}$. Suppose conversely that $x \in \mathrm{Cu}(A)$ and $n \in \mathbb{N}$ satisfy $\hat{x} \leqslant n\hat{u}$. Let $N \in \mathbb{N}$ satisfy $N > \mathrm{rc}(A)$. From $\hat{x} + N\hat{u} \leqslant (N+n)\hat{u}$ we deduce that $x \leqslant (N+n)u$. Hence, $x \in W(A)$. ☐

8.9. *Strict comparison and local weak (m, γ)-comparison.* Recall that a C*-algebra A is said to have *strict comparison* if whenever $x, y \in \mathrm{Cu}(A)$ satisfy $\hat{x} \leqslant (1-\varepsilon)\hat{y}$ for some $\varepsilon > 0$, then $x \leqslant y$.

Let us say that A has *strict comparison on full elements* if whenever $x, y \in \mathrm{Cu}(A)$, with y full, satisfy $\hat{x} \leqslant (1-\varepsilon)\hat{y}$ for some $\varepsilon > 0$, then $x \leqslant y$. Clearly, if A is a simple C*-algebra this property agrees with strict comparison.

Suppose now that A is unital. Set $u = [1_A]$. Recall that we denote by $F_u(\mathrm{Cu}(A))$ the set of all $\lambda \in F(\mathrm{Cu}(A))$ such that $\lambda(u) = 1$. Suppose that there exist $m \in \mathbb{N}$ and $\gamma \geqslant 1$ such that if $a, b \in A_+$, with b full, satisfy

$$\gamma \cdot \sup_{\lambda \in F_u(\mathrm{Cu}(A))} \lambda([a]) \leqslant \inf_{\lambda \in F_u(\mathrm{Cu}(A))} \lambda([b])$$

then $[a] \leqslant m[b]$. We then say that A has *local weak (m, γ)-comparison.* The word local here refers to the fact that we do not choose a and b in $A \otimes \mathcal{K}$, just in A. The case when A is simple and $m = 1$ of this property appears in [KR14, Definition 2.1], where it is called 'local weak comparison'. We show below that if A is a separable, unital C*-algebra of stable rank one that has no nonzero, finite dimensional quotients, then local weak (m, γ)-comparison implies strict comparison on full elements.

8.10. **Lemma.** *Let A be a separable C*-algebra of stable rank one, let $x \in \mathrm{Cu}(A)$ be full, and let $f \in L(F(\mathrm{Cu}(A)))$ satisfy $f \ll \hat{x}$. Then there exist $y, z \in \mathrm{Cu}(A)$ such that $f \leqslant \hat{y}$, $y + z \leqslant x$, and z is full.*

Proof. Choose $w \in \mathrm{Cu}(A)$ satisfying $f \ll \hat{w} \ll \hat{x}$. Set $x_1 := x \wedge w$. By Theorem 6.10, we have $\widehat{x_1} = \hat{x} \wedge \hat{w} = \hat{w}$. Therefore $f \ll \widehat{x_1}$ and $x_1 \leqslant x$. Choose $y \in \mathrm{Cu}(A)$ such that $y \ll x_1$ and $f \leqslant \hat{y}$. Finally, apply (O5) to $y \ll x_1 \leqslant x$ to obtain $z \in \mathrm{Cu}(A)$

such that $y + z \leqslant x \leqslant x_1 + z$. It remains to show that z is full. Denote by W the ideal generated by z, that is, $W = \{z' \in \mathrm{Cu}(A) : z' \leqslant \infty \cdot z\}$. Let $I \subseteq A$ be the closed, two-sided ideal such that $W = \mathrm{Cu}(I)$. Passing to $\mathrm{Cu}(A/I)$ by the quotient map, let us denote the images of x, x_1, and y by $\underline{x}$, $\underline{x}_1$, and $\underline{y}$. We have $\underline{y} = \underline{x}_1 = \underline{x}$, and this element is compact. Since $\widehat{x_1} \ll \hat{x}$, we can choose $\varepsilon > 0$ with $\widehat{x_1} \leqslant (1-\varepsilon)\hat{x}$. Passing to $\mathrm{Cu}(A/I)$ we obtain $\hat{\underline{x}} = (1-\varepsilon)\hat{\underline{x}}$. Thus, $\underline{x}$ is a compact element on which no functional is finite and nonzero. Since A/I is stably finite, we deduce $\underline{x} = 0$. Hence $x \leqslant \infty z$, and since x is full we get that z is full as well. □

8.11. **Lemma.** *Let A be a unital, separable C*-algebra of stable rank one. Set $u = [1_A]$. Let $(z_i)_i$ be a sequence of full, supersoft elements in $\mathrm{Cu}(A)$ such that $\widehat{z_i} \leqslant \hat{u}$ for all i. Then*

$$\sum_{i=1}^{\infty} z_i = \sum_{i=1}^{\infty} (z_i \wedge u).$$

Proof. Applying § 3.9 in $\sum_{i=1}^{n}(z_i \wedge u)$ we obtain the infimum of a number of sums. Since, for each i, the element z_i is full by assumption, we may apply Theorem 8.5 to conclude that those sums whose terms contain at least one u and at least one z_i result in a supersoft element. Further, since $\widehat{z_i} \leqslant \hat{u}$ for all i, this supersoft element is larger than $\sum_{i=1}^{n} z_i$. Hence,

$$\sum_{i=1}^{n}(z_i \wedge u) = (\sum_{i=1}^{n} z_i) \wedge nu.$$

Passing to the supremum over all n we get the desired equality. □

8.12. **Theorem.** *Let A be a unital, separable C*-algebra, of stable rank one that has no nonzero finite dimensional quotients. Then the following conditions are equivalent:*

(i) *A has local weak (m, γ)-comparison for some $m \in \mathbb{N}$ and $\gamma \geqslant 1$.*
(ii) *For each full element $x \in \mathrm{Cu}(A)$ there exists a full, supersoft element $z \in \mathrm{Cu}(A)$ such that $z \leqslant x$.*
(iii) *A has strict comparison on full elements.*
(iv) *The restriction of α to $\{f \in L(F(\mathrm{Cu}(A))) : f$ is full$\}$ is a Cu-morphism into the subsemigroup of full elements of $\mathrm{Cu}(A)$.*

Proof. (i) $\Longrightarrow$ (ii): Let $m \in \mathbb{N}$ and $\gamma \geqslant 1$ such that A has local weak (m, γ)-comparison. Let $x \in \mathrm{Cu}(A)$ be full. As above, set $u = [1_A]$. Replacing x by $x \wedge u$ if necessary, we may assume that $x \leqslant u$. (Note that $x \wedge u$ remains full by Lemma 5.3). By Theorem 5.10, we can choose a sequence $(x_i)_i$ of full elements in $\mathrm{Cu}(A)$ such that $\sum_i mx_i \leqslant x$. Since x_i is full for each i, there exists $n_i \in \mathbb{N}$ such that $u \leqslant n_i x_i$, and we may clearly further assume that $\sum_i \frac{1}{n_i} \leqslant 1$. Set

$$\varepsilon = \sum_{i=1}^{\infty} \frac{1}{\gamma n_i}.$$

Applying Theorem 5.10 again, let $z \in \mathrm{Cu}(A)$ be a full, supersoft element such that $\hat{z} \leqslant \varepsilon\hat{u}$. We claim that $z \leqslant x$. Set $t_i := \frac{1}{\varepsilon\gamma n_i}$ for each $i \in \mathbb{N}$, and observe that $\sum_i t_i = 1$. Set $z_i := \alpha(t_i\hat{z})$ for each i. Since z is full, we see that z_i is full for each i. Using Theorem 8.5 at the second step, Proposition 8.4 at the third step, and that z is supersoft at the last step, we have

$$\sum_{i=1}^{\infty} z_i = \sup_n \sum_{i=1}^{n} \alpha(t_i\hat{z}) = \sup_n \alpha(\sum_{i=1}^{n} t_i\hat{z}) = \alpha(\sup_n \sum_{i=1}^{n} t_i\hat{z}) = \alpha(\hat{z}) = z.$$

Since $\hat{z} \leqslant \hat{u}$, Lemma 8.11 implies

$$z = \sum_{i=1}^{\infty} (z_i \wedge u).$$

By the way we picked the sequence $(t_i)_i$, we have $\gamma n_i \widehat{z_i} \leqslant \hat{u} \leqslant n_i \hat{x}_i$. Since A has local weak (m, γ)-comparison we conclude that $z_i \wedge u \leqslant m x_i$ for all i. Therefore,

$$z = \sum_{i=1}^{\infty} (z_i \wedge u) \leqslant \sum_{i=1}^{\infty} m x_i \leqslant x,$$

as desired.

(ii) $\Longrightarrow$ (iii): Suppose that $x, y \in \mathrm{Cu}(A)$, with y full, satisfy $\hat{x} \leqslant (1-\varepsilon)\hat{y}$ for some $\varepsilon > 0$. Let $x' \in \mathrm{Cu}(A)$ satisfy $x' \ll x$. By Lemma 8.10 there exist $y', w \in \mathrm{Cu}(A)$ such that $\hat{x} \leqslant \widehat{y'}$, $y' + w \leqslant y$, and w is full. By assumption, there exists a full, supersoft element $z \in \mathrm{Cu}(A)$ such that $z \leqslant w$. Then $\hat{x} + \hat{z} \leqslant \widehat{y'} + \hat{z}$, whence $\alpha(\hat{x} + \hat{z}) \leqslant \alpha(\widehat{y'} + \hat{z})$. Now, since z is full we have $\hat{x}, \widehat{y'} \leqslant \infty \hat{z}$. Using Theorem 8.5 in the second and fourth steps, and that z is supersoft in the first and sixth steps we obtain

$$x + z = x + \alpha(\hat{z}) = \alpha(\hat{x} + \hat{z}) \leqslant \alpha(\widehat{y'} + \hat{z}) = y' + \alpha(\hat{z}) = y' + z \leqslant y' + w \leqslant y,$$

and thus $x \leqslant y$, as desired.

(iii) $\Longrightarrow$ (iv) We have already shown that α preserves order and suprema of increasing sequences (Proposition 8.4), and that α is additive on full functions (Theorem 8.5). It remains to show that it preserves the way below relation. Let us show first that if $x, y \in \mathrm{Cu}(A)$, with x full and soft, are such that $\hat{x} \leqslant \hat{y}$, then $x \leqslant y$ (cf. [APT18b, Theorem 5.2.18]). Choose a full element $x' \in \mathrm{Cu}(A)$ such that $x' \ll x$. Since x is soft, $\widehat{x'} \ll \hat{x} \leqslant \hat{y}$. By strict comparison on full elements, $x' \leqslant y$. Passing to the supremum over all full $x' \ll x$, we get $x \leqslant y$. Now let $f, g \in L(F(\mathrm{Cu}(A)))$ be full and such that $f \ll g$. Since $\alpha(g) = \sup_{x \ll \alpha(g)} x$, and $z \mapsto \hat{z}$ is supremum preserving, we can choose $x \ll \alpha(g)$ such that $f \leqslant \hat{x}$. Since $\alpha(f)$ is soft and full, we deduce that $\alpha(f) \leqslant x \ll \alpha(g)$, as desired.

(iii) $\Longrightarrow$ (i): Obvious.

(iv) $\Longrightarrow$ (iii): Suppose that $x, y \in \mathrm{Cu}(A)$, with y full, satisfy $\hat{x} \leqslant (1-\varepsilon)\hat{y}$ for some $\varepsilon > 0$. Let $x' \ll x$. Then $\widehat{x'} \ll \hat{y}$, and thus $\widehat{x'} + \hat{u} \ll \hat{y} + \hat{u}$. We deduce

$$x' + \alpha(\hat{u}) = \alpha(\widehat{x'} + \hat{u}) \ll \alpha(\hat{y} + \hat{u}) = y + \alpha(\hat{u}).$$

By weak cancellation, we get $x' \leqslant y$. Passing to the supremum over all $x' \ll x$, we obtain $x \leqslant y$. ☐

8.13. **Theorem.** *Let A be a separable C^*-algebra of stable rank one that has no nonzero, elementary ideal-quotients. Then, the following conditions are equivalent:*

(i) *There exist $m \in \mathbb{N}$ and $\gamma \geqslant 1$ such that $\gamma \hat{x} \leqslant \hat{y}$ implies $x \leqslant my$ for all $x, y \in \mathrm{Cu}(A)$.*
(ii) *For each $x \in \mathrm{Cu}(A)$ there exists $y \leqslant x$ that is supersoft and such that $\infty y = \infty x$.*
(iii) $\mathrm{Cu}(A)$ *has strict comparison.*

Moreover, these conditions imply that α is a Cu*-morphism.*

Proof. (i) $\Longrightarrow$ (ii): Given $x \in \mathrm{Cu}(A)$, choose x' such that $mx' \leqslant x$ and $\infty x' = \infty x$. Set $y := \alpha(\widehat{x'}/\gamma)$. Then $\gamma \hat{y} = \hat{x}'$. So $y \leqslant mx' \leqslant x$ and $\infty y = \infty x' = \infty x$.

(ii) $\Longrightarrow$ (iii): Let $x, y \in \mathrm{Cu}(A)$ and $\varepsilon > 0$ satisfy $\hat{x} \leqslant (1-\varepsilon)\hat{y}$. Let $x' \ll x$. Then $\widehat{x'} \ll \hat{y}$. By Lemma 8.10, there exist $y', z \in \mathrm{Cu}(A)$ such that $\widehat{x'} \leqslant \widehat{y'}$, $y' + z \leqslant y$, and $y \leqslant \infty z$. Let $w \in \mathrm{Cu}(A)$ be supersoft, such that $w \leqslant z$ and $\infty w = \infty z$. Then

$x'+w$ and $y'+w$ are supersoft, and so $x' \leqslant x'+w \leqslant y'+w \leqslant y$. Passing to the supremum over all $x' \ll x$, we get $x \leqslant y$.

(iii) $\implies$ (i): Obvious.

Lastly, let us show that (iii) implies that α is a Cu-morphism. As in the proof of Theorem 8.12 (iii) $\implies$ (iv), we only need to check preservation of the way below relation. Let $f, g \in L(F(S))$ satisfy $f \ll g$. As in the proof of Theorem 8.12 (iii) $\implies$ (iv), we obtain $x \in S$ such that $f \leqslant \hat{x}$ and $x \ll \alpha(g)$. By [APT18b, Theorem 5.2.18], if elements y, z in a Cu-semigroup with strict comparison satisfy $\hat{y} \leqslant \hat{z}$, and if y is soft, then $y \leqslant z$. Since $\alpha(f)$ is soft, and since $\widehat{\alpha(f)} = f \leqslant \hat{x}$, we get $\alpha(f) \leqslant x$, and then $\alpha(f) \ll \alpha(g)$. □

9. Nonseparable C*-algebras

Here we show that the hypothesis of separability can be dropped in some of the results from the previous sections. To this end, we rely on the model theory of C*-algebras and in particular on the Downward Löwenheim-Skolem Theorem for C*-algebras. For the model theory of C*-algebras we refer the reader to [FHL+16].

Given a C*-algebra A and a C*-subalgebra B, we write $B \prec A$ if B is an *elementary submodel* of A. This means that for every *formula* φ in the language of C*-algebras and every n-tuple $\overline{a}$ in B, we have $\varphi^B(\overline{a}) = \varphi^A(\overline{a})$ (see [FHL+16, Definition 2.3.3]). By the Downward Löwenheim-Skolem Theorem ([FHL+16, Theorem 2.6.2]), every C*-algebra has a separable elementary submodel. Important to us in what follows is that if $B \prec A$ then the induced map $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding ([FHL+16, Lemma 8.1.3]).

The next result removes the separability assumption in Theorem 5.7.

9.1. **Theorem.** *Let A be a unital C*-algebra of stable rank one, and let $k \in \mathbb{N}$. Then A has no nonzero representations of dimension less than k if and only if there exists a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to A$ with full range.*

Proof. The proof of the easy direction in Theorem 5.7 does not make use of the separability hypothesis. Hence, it applies here.

Suppose that A is a unital C*-algebra of stable rank one without nonzero representations of dimension less than k. By [RR13, Corollary 5.4], the element [1] is weakly (k,n)-divisible for some n (see § 5.2). Thus, there exist $a_1, \ldots, a_n \in A_+$ such that $k[a_i] \leqslant [1]$ for all i and $[1] \leqslant \sum_{i=1}^n [a_i]$. Apply the Downward Löwenheim-Skolem Theorem to obtain a separable C*-subalgebra $B \prec A$ that contains $1, a_1, \ldots, a_n$. Since the inclusion of B in A induces an order-embedding of $\mathrm{Cu}(B)$ in $\mathrm{Cu}(A)$ ([FHL+16, Lemma 8.1.3]), the inequalities $k[a_i] \leqslant [1]$ for all i and $[1] \leqslant \sum_{i=1}^n [a_i]$ also hold in $\mathrm{Cu}(B)$. By [RR13, Corollary 5.4], B has no representations of dimension less than k. On the other hand, by [FHL+16, Lemma 3.8.2]), the property of having stable rank one is elementary and so passes to elementary submodels. We can thus apply Theorem 5.7 in B to obtain a *-homomorphism $\varphi\colon M_k(C_0((0,1])) \to B \subseteq A$ whose range is full in B. Since $1 \in B$, the range of φ is also full in A. □

Next we extend Theorem 7.11 to the nonseparable case. We start with a preparatory result.

9.2. **Lemma.** *Let A be a C*-algebra, let $B \prec A$, and let $a, b \in B_+$. Then $\lambda([a]) \leqslant \lambda([b])$ for all $\lambda \in F(\mathrm{Cu}(A))$ if and only if $\lambda([a]) \leqslant \lambda([b])$ for all $\lambda \in F(\mathrm{Cu}(B))$.*

Proof. The backward implication follows directly using that every functional on $\mathrm{Cu}(A)$ restricts to a functional on $\mathrm{Cu}(B)$. To show the converse, suppose that $\lambda([a]) \leqslant \lambda([b])$ for all $\lambda \in F(\mathrm{Cu}(A))$. Let $\varepsilon > 0$ and $\delta > 0$. By [Rob13, Proposition 2.2.6] there exist $M, N \in \mathbb{N}$ such that $M/N \geqslant 1-\delta$ and $M[(a-\varepsilon)_+] \leqslant N[b]$ in $\mathrm{Cu}(A)$. Since the inclusion $A \to B$ induces an order-embedding $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$, this inequality also holds in $\mathrm{Cu}(B)$. Fix $\lambda \in F(\mathrm{Cu}(B))$. Evaluating both sides of $M[(a-\varepsilon)_+] \leqslant N[b]$ on λ we get

$$(1-\delta)\lambda\big([(a-\varepsilon)_+]\big) \leqslant \lambda([b]).$$

Since this holds for all $\delta, \varepsilon > 0$ we conclude that $\lambda([a]) \leqslant \lambda([b])$, as desired. □

9.3. **Theorem.** *Let A be a unital C*-algebra of stable rank one with no finite dimensional quotients. Set $u = [1_A]$ and let $F_u(\mathrm{Cu}(A)) \subseteq F(\mathrm{Cu}(A))$ denote the set of functionals normalized at u. Then for each $f \in \mathrm{LAff}(F_u(\mathrm{Cu}(A)))^\sigma_{++}$ there exists $z \in \mathrm{Cu}(A)$ such that $\hat{z}|_{F_u(\mathrm{Cu}(A))} = f$.*

Proof. Let us regard A embedded in $A\otimes\mathcal{K}$ as the 'upper left corner'. Let $1_A \in A\otimes\mathcal{K}$ denote the unit of A. Given $f \in \mathrm{LAff}(F_u(\mathrm{Cu}(A)))^\sigma_{++}$, apply Proposition 6.7 to obtain $\tilde{f} \in L(F(\mathrm{Cu}(A)))$ that extends f. Find a sequence $(x_i)_i$ in $\mathrm{Cu}(A)$ such that $\sup_i \hat{x}_i/m_i = \tilde{f}$. Choose $a_i \in (A\otimes\mathcal{K})_+$ such that $x_i = [a_i]$ for all i.

Since A has no finite dimensional representations, by [RR13, Corollary 5.4] there exists for each k an $n_k \in \mathbb{N}$ such that $[1_A]$ is weakly (k, n_k)-divisible in $\mathrm{Cu}(A)$. We thus find $b_{k,l} \in A_+$ for $k = 1, 2, \ldots$ and $l = 1, \ldots, n_k$ such that $k[b_{k,l}] \leqslant [1_A]$ for all k, l and $[1_A] \leqslant \sum_{l=1}^{n_k} [b_{k,l}]$ for all k. Apply the Downward Löwenheim-Skolem theorem to obtain a separable elementary submodel $B \prec A\otimes\mathcal{K}$ that contains all a_i, all $b_{k,l}$, and 1_A.

As argued in the proof of Theorem 9.1, B has stable rank one. Further, the inclusion of B in A induces a natural order-embedding $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$.

We claim that B is stable To prove this we use the Hjelmborg-Rørdam criterion for stability established in [HR98, Theorem 2.1]. By the stability of A, for each $b \in B_+$ and $\varepsilon > 0$ there exists $v \in A$ such that $\|a - v^*v\| < \varepsilon$ and $\|a - vv^*\| < \varepsilon$. Since B is an elementary submodel of A, there exists also $\tilde{v} \in B$ fulfilling the same inequalities. Since B is separable, [HR98, Theorem 2.1] implies that B is stable.

Let us show that $1_A \in B$ is full in B. For every $b \in B_+$ we have $[b] \leqslant \infty[1_A]$ in $\mathrm{Cu}(A)$, as 1_A is full in $A\otimes\mathcal{K}$. Using that $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding, we get $[b] \leqslant \infty[1_A]$ in $\mathrm{Cu}(B)$, which implies that 1_A is full in B.

The inequalities $k[b_{k,l}] \leqslant [1_A]$ and $[1_A] \leqslant \sum_{l=1}^{n_k}[b_{k,l}]$ hold in $\mathrm{Cu}(B)$ for all k, l, using again that $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding. Therefore, the element $[1_A]$ is weakly (k, n_k)-divisible in $\mathrm{Cu}(B)$ for all k. By [RR13, Corollary 5.4], the hereditary C*-subalgebra $1_A B 1_A$ has no finite dimensional representations.

By Lemma 9.2, the sequence $(\hat{x}_i/m_i)_i$ is increasing when regarded as a sequence in $L(F(\mathrm{Cu}(B)))$. Let $h \in L(F(\mathrm{Cu}(B)))$ be its supremum. The function h is full, since $\hat{x}_i/m_i$ is full for large enough i. By Theorem 7.11 applied in the C*-algebra B, we have $h = \hat{x}$ for $x := \alpha(h)$. Since B is stable, there exists $c \in B_+$ such that $[c] = x$, and thus $\widehat{[c]} = h$.

We claim that $[c]$, regarded as an element in $\mathrm{Cu}(A)$, satisfies $\widehat{[c]} = \tilde{f}$. By Lemma 9.2, the inequalities $\hat{x}_i/m_i \leqslant \widehat{[c]}$, which hold in $L(F(\mathrm{Cu}(B)))$, also hold in $L(F(\mathrm{Cu}(A)))$ for all i. Passing to the supremum over i, we get that $\tilde{f} \leqslant \widehat{[c]}$. Let $[c'] \in \mathrm{Cu}(B)$ be such that $[c'] \ll [c]$. By the definition of $\alpha(h)$, we have that $\widehat{[c']} \ll h = \widehat{[c]}$ in $L(F(\mathrm{Cu}(B)))$. Hence $\widehat{[c']} \leqslant \hat{x}_i/m_i$ for some i. By Lemma 9.2, this inequality holds also in $L(F(\mathrm{Cu}(A))$. Hence, $\widehat{[c']} \leqslant f$. This holds for $c' = (c-\varepsilon)_+$ and arbitrary $\varepsilon > 0$. Hence, $\widehat{[c]} \leqslant \tilde{f}$ in $L(F(\mathrm{Cu}(A)))$, as desired. □

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