Quadratic differential systems possessing two parallel invariant affine lines

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Abstract

In this paper we classify the family $\boldsymbol{QSL^{2p}}$ of quadratic differential system possessing two parallel invariant affine lines according to their configurations of invariant lines. Actually our classification also includes systems that are limit points in the parameter space of $\boldsymbol{QSL^{2p}}$, i.e. it covers the closure $\overline{\boldsymbol{QSL^{2p}}}$ of $\boldsymbol{QSL^{2p}}$ in the parameter space. We obtained a total of 93 such configurations. Our classification is done in terms of polynomial invariants and it provides us with an algorithm to decide whether or not a quadratic system belongs to the family and if it does then it gives us what its configuration of invariant lines.

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1 Introduction

We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \quad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p, q \in \mathbb{R}[x, y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$, and their associated vector fields

$$\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}. \tag{2}$$

We call *degree* of a system (S) the integer $\deg(S) = \max(\deg(P), \deg(Q))$. We call quadratic (respectively cubic) differential system such a polynomial system of degree two (respectively three). We shall sometimes use quadratic system instead of quadratic differential system. Each such system generates a complex differential vector field when the variables range over $\mathbb{C}$. We recall the following definitions:

Definition 1.1. *Let $F : U \longrightarrow \mathbb{R}$, $U \subseteq \mathbb{R}^2$ be a C^1 function on an open set U. If F is constant on all solutions curves $(x(t), y(t))$ in U of a system (S), we say that F is a first integral on U of (S). If there exists such an F which is nonconstant on any open subset of U we say that this system is integrable on U.*

Remark 1.1. *We note that such a C^1 function $F : U \longrightarrow \mathbb{R}$ is a first integral on U of* (1) *if and only if for all solutions $(x(t), y(t))$ with values in U of* (1) *defined when t is in an open interval of $\mathbb{R}$, we have $\dfrac{dF(x(t), y(t))}{dt} = 0$ for all t in this interval, or equivalently*

$$\tilde{D}F \equiv p(x, y)\frac{\partial F}{\partial x} + q(x, y)\frac{\partial F}{\partial y} = 0. \tag{3}$$

Darboux used algebraic invariant curves $f_i(x, y) = 0$, $i = 1, 2, \dots, s$, over $\mathbb{C}$ to construct first integrals of the form $F = \prod_{i=1}^{s} f_i(x, y)^{\lambda_i}$ for some $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathbb{C}^s \setminus \{0\}$ for complex systems (S). *By an analytic first integral F on an open set U of $\mathbb{C}^2$ of a complex vector field* (2) we mean an analytic function F on U such that $\tilde{D}F = 0$.

Definition 1.2. *An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ is invariant for a polynomial differential system (S) if there exists a polynomial $K(x, y) \in \mathbb{C}[x, y]$, called the cofactor of $f(x, y)$ such that the following identity is satisfied in $\mathbb{C}[x, y]$*

$$p(x, y)\frac{\partial f}{\partial x} + q(x, y)\frac{\partial F}{\partial y} = f(x, y)K(x, y).$$

Even though we may be interested in real systems, the extension of real systems to complex systems proves to be very valuable and could provide real first integrals obtained by using complex algebraic invariant curves according to the theory of Darboux which without the use of complex invariant curves could not have been obtained. Indeed, consider the following:

Example (This example is obtained by taking $g = 1$ in item 32 of Table 1 from [25].)

$$dx/dt = x^2 + 1, \quad dy/dt = x + y$$

This system clearly has two invariant lines which are complex $x \pm i = 0$ with respective co-factors $x \mp i$. This system was proved to be integrable in [25] having the inverse Darboux integrating factor $(x + i)^{1+i/2}(x - i)^{1-i/2}$ and a real first integral $\mathcal{F}_{32}$ in [25]. But this first integral cannot be calculated by using only the real algebraic invariant conic $x^2 + 1 = 0$.

To real systems we can associate their corresponding complex systems and to these we can associate a complex differential equation on the complex projective plane (see [21], Section 2). For each affine invariant curve $f(x, y) = 0$, its projective completion is also invariant for the complex differential equation in the complex projective plane. Furthermore the real (complex) line at infinity is an invariant line of the affine plane of the real (complex) projective plane.

Darboux [12] constructed his theory of *geometric integrability* of planar polynomial differential systems, based on the concept of invariant algebraic curves. In [12] Darboux and gave a sufficient condition of integrability for complex systems (S) in terms of invariant algebraic curves of the system.

Theorem 1.1. *If a polynomial differential system (real or complex)* (S) *has* s *invariant algebraic curves* $f_i(x,y)=0$, $i=1,2,\ldots,s$ *such that* $s \geq m(m+1)/2$ *where* $m=\deg(S)$, *then either we have a first integral of the form* $F=\prod_{i=1}^{s} f_i(x,y)^{\lambda_i}$ *for some* $\lambda=(\lambda_1,\ldots,\lambda_s)\in \mathbb{C}^s\setminus\{0\}$.

Poincaré admired the work of Darboux and wrote two articles on this subject in the 1890's ([19, 20]). His work was followed by work of Painlevé, Autonne at the end of the 19th century and of Dulac ([13]) at the beginning of 20th century. After their work, except for some isolated examples, the integrability theory of Darboux did not advance until the last quarter of the 20th and the beginning of the 21st century when Darboux's theory was further expanded to include also the multiplicity of the curves and interest in invariant algebraic curves of planar polynomial systems has been steadily growing. This is natural as on one side this is an interesting theory involving at the same time algebraic geometric concepts and differential equations with reputedly very difficult problems to solve. On the other side integrable systems are worthy objects of study as although the cases of integrable systems are rare, as Arnold said in [1], (page 405) "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...".

In the first part of his article [12] Darboux developed his *geometric theory of integrability* over the complex projective space. Apart from the theorem mentioned above. Darboux also proves the following theorem incolving singular points of the system which helps to diminish the number of invariant curves involved in the expression of the first integral:

Theorem 1.2. *Suppose that a system* (1) *of degree* m *has* p *algebraic solutions* $f_1(x,y)=0,\ldots f_p(x,y)=0$ *which do not pass through* q *singular points of the system and assume that we have p=m(m+1)/2+1-q. Then there exist complex numbers* $\lambda_1,\ldots,\lambda_p$ *not all zero, such that* $f_1^{\lambda_1}\ldots f_p^{\lambda_p}$ *is a first integral of the system.*

In the Second part of the article, Darboux applies his theory to the particular case of quadratic systems. For example he calculates the integrals of quadratic systems having two, three or four invariant lines not all passing through the same point. His calculation are obtained by choosing the lines in convenient position and assuming that the systems admit an integrating factor product of powers of lines.

The family of quadratic systems which we denote by **QS** is the first nonlinear class of polynomial differential systems. Of the three classical problems on these systems, Hilbert's 16th problem, the problem of Poincaré and the problem of the center, only this last one was solved for **QS** and more than a century after their formulation the other two problems are still open today. Although it is the simplest non-linear class of polynomial suystems we are still far from understanding this class. To gain insight into this family, in recent years subfamilies of **QS** began to be studied from a global viewpoint using a variety of methods among them algebraic and geometric, but also numerical or involving substantial symbolic calculations. In particular families of quadratic systems possessing invariant algebraic curves began to be studied, the simplest ones being those possessing invariant lines.

Every system in **QS** possesses an invariant line, the line at infinity. This line could be simple, or multiple in which case producing several distinct lines in perturbations.

The notion of multiplicity of an invariant line of a system (1) has been introduced in [21]. In the fundamental article [9] several notions of multiplicity of an invariant algebraic curve of a polynomial systems were introduced and they were proven to be equivalent in the case of *algebraic solutions* which are algebraic invariant curves defined by irreducible polynomials. If a system has a finite number of invariant lines $f_i(x,y)=0$, $i=1,...,k$, of respective multiplicities $m_1,...,m_k$, we call *total multiplicity of the invariant lines* of (S), the number $M=\sum_i m_i+m_\infty$ where m_∞ is the multiplicity of the line at infinity. Since in any system (1) the line at infinity is invariant we always have $m_\infty \geq 1$ and in particular we have this for any system in $\boldsymbol{QS}$.

At the beginning of this century a systematic study of non-degenerated quadratic systems possessing invariant algebraic curves was initiated by Schlomiuk and Vulpe. In the series of articles [2,21,23,25–27] the authors studied the class $\boldsymbol{QSL}_{\geq 4}$ of quadratic systems having invariant lines, including the line at infinity, of total multiplicity at least four. We see in [21] that the maximum number of invariant lines, including the line at infinity of non-degenerate quadratic systems is six.

The next step is the study of the subfamily $\boldsymbol{QSL}_{\geq 3}$ of $\boldsymbol{QS}$ which is the family of all non-degenerate quadratic differential systems with invariant lines of total multiplicity at least three. The study of this class began with work on the Lotka-Volterra systems, a family important for applications. This is the class of all quadratic differential systems that have two real invariant lines intersecting at a finite point. In [28,29] the authors completed the study of this class by giving its bifurcation diagram in the 12-dimensional space of the coefficients of quadratic systems (1).

It now remains to complete the study of the family $\boldsymbol{QSL}_{\geq 3}$ which is split in several cases according to the total multiplicity of the invariant affine lines. Another subfamily of $\boldsymbol{QSL}_{\geq 3}$ is the family of non-degenerate real quadratic systems possessing two complex invariant lines intersecting at a real finite point. This study was completed in [5,31]. To complete the study of $\boldsymbol{QSL}_{\geq 3}$ we need to study the family of quadratic systems having either two parallel lines or a unique double affine line, or an affine line and the line at infinity double or the line at infinity triple. Except for the last case, we always have at least one affine line. We recall that for all systems in $\boldsymbol{QS}$ possessing a real affine invariant line the Hilbert's 16th problem is solved as we have:

Theorem 1.3 ([10,11])**.** *Every real quadratic differential system possessing a real affine invariant line has at most one limit cycle.*

As we prove in the Section 8 for the family $\boldsymbol{QSL}_{\geq 2}$ the maximum number of limit cycles is also one and so the Hilbert number of this class H($\boldsymbol{QSL}_{\geq 2}$)=1. In view of this theorem the road is open for the problem of topologically classifying the family of $\boldsymbol{QSL}_{\geq 2}$ which is part of the motivation of the paper.

Apart from this motivation we mention that this family contains the families of both Bernoulli and Ricatti differential systems and it is worthwhile to view these systems from this angle.

In this article we study one of the subfamilies of $\boldsymbol{QSL}_{\geq 3}$ listed above, namely the family $\mathbf{QSL}^{2p}$ of quadratic differential systems which we define below.

The next assertion is almost evident

Lemma 1.1. *If a quadratic system* (1) *possesses two distinct parallel invariant affine lines then this systems could be brought via an affine transformation to the form*

$$\begin{aligned} &\dot{x} = a + cx + gx^2 \equiv p(x),\\ &\dot{y} = b + ex + fy + lx^2 + 2mxy + ny^2. \end{aligned} \tag{4}$$

Proof: Assume that an arbitrary quadratic system possesses two distinct parallel invariant lines

$$\alpha x + \beta y + \gamma_{1,2} = 0, \quad \alpha, \beta, \gamma_{1,2} \in \mathbb{C}.$$

Since these two parallel invariant lines (real or complex) intersect at infinity at a real singularity we deduce that there exists a non-zero $\lambda \in \mathbb{C}$ such that

$$\lambda\alpha x + \lambda\beta y + \lambda\gamma_{1,2} = 0, \;\; \Rightarrow \;\; \tilde{\alpha}x + \tilde{\beta}y + \tilde{\gamma}_{1,2} = 0, \;\; \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}, \; \tilde{\gamma}_{1,2} \in \mathbb{C}.$$

Therefore applying a real linear transformation

$$q \in GL(2,\mathbb{R}): \quad x_1 = \tilde{\alpha}x + \tilde{\beta}y, \quad y_1 = -\tilde{\beta}x + \tilde{\alpha}y, \quad \det(q) = \tilde{\alpha}^2 + \tilde{\beta}^2 \neq 0$$

we arrive at a system possessing in the direction $x_1 = 0$ two parallel invariant lines (real or complex). This means that the first equation of new system in the variables x_1, y_1 does not depend on the variable y_1, i.e. it must be of the form (4). This completes the proof of Lemma 1.1. ∎

We denote by $\boldsymbol{QSL^{2p}}$ the class of non-degenerate quadratic systems which via an affine transformation could be brought fo the canonical form (4) even if for these systems we could have $g = 0$ or $g = c = 0$. This notation is motivated by the fact that although in the case $g = 0$ and $c \neq 0$ (respectively $g = c = 0$) systems (4) possess only one (respectively do not possess any) invariant line in the direction $x = 0$ nevertheless applying a perturbation $p(x,\varepsilon) = a+cx+\varepsilon x^2$ (respectively $p(x,\varepsilon,\nu) = a+\nu x+\varepsilon x^2$) with small parameters $|\varepsilon|, |\nu| \ll 1$ we obtain systems possessing two distinct parallel invariant lines in the direction $x = 0$.

We need the notion of configuration of invariant algebraic curves of a polynomial system (1) which was defined in [26]. We recall here this definition.

Definition 1.3. *Consider a real polynomial differential system* (S) *endowed with a finite number of invariant algebraic curves* $f_i(x,y) = 0$, $i = 1,\dots,k$ *over* $\mathbb{C}$. *We call configuration of invariant curves of* (S) *the set of curves* $f_1 = 0,\dots,f_k = 0$ *and the line at infinity, each endowed with its own multiplicity, together with all the real singular points of* (S) *situated on these curves, each one of them endowed with its own multiplicity.*

The configurations we consider in this work are configurations of invariant lines.

The goal of this paper is to classify the family $\boldsymbol{QSL^{2p}}$ according to the configurations of invariant lines the systems possess.

To make this statement precise we first need to say when two configurations $\mathcal{C}_1$, $\mathcal{C}_2$ of invariant lines of two quadratic systems (S_1) and (S_2) are to be considered as distinct, respectively when two such configurations are to be considered equivalent. The next Section contains the necessary definition.

Our goal is not only to list all possible distinct configurations which occur for systems belonging to the family $\boldsymbol{QSL^{2p}}$ but also and for each such configuration to give necessary and sufficient conditions for a system in this family to have the given configuration of invariant lines. We want these conditions to be independent of the normal form in which the system may be presented. In other words we want these conditions to be affinely invariant, expressed in terms of polynomial invariants. Furthermore we also want to give the bifurcation diagram of the configurations in terms of polynomial invariants in the 12-dimensional parameter space of coefficients of the systems in $\boldsymbol{QSL}$, where by $\boldsymbol{QSL}$ we denote the family of all quadratic differential systems that possess an invariant line.

A second and longer term goal is to give the topological classification of the family $\boldsymbol{QSL^{2p}}$.

A quadratic differential system can be identified with a point in $\mathbb{R}^{12}$, the ordered sequence of its 12 coefficients. The group of affine transformations and time rescaling acts on $\mathbf{QS}$. The moduli space $\mathbf{QS}/_{\simeq}$ of $\mathbf{QS}$ modulo this action is 5 dimensional. The family $\mathbf{QSL}^{2p}$ is located on an algebraic hypersurface of $\mathbb{R}^{12}$. Consider now the family $\mathbf{QSL}_{\geq 2}$ of all quadratic systems with invariant lines of total multiplicity at least two. Our final goal is to give the topological classification of this family. The moduli space $\mathbf{QSL}_{\geq 2}/\simeq$ under this group action is four dimensional. So far only one 4-dimensional sub-family of $\mathbf{QS}/_{\simeq}$ was topologically classified (see the paper [6] where this was done) but this turned out to be a very simple case namely we only needed to study three 3-dimensional slices of this space. As Hilbert's 16th problem is solved for the family $\mathbf{QSL}^{2p}$ we have a good chance to obtain this topological classification and even more, the bifurcation diagram of this class up to limit cycles and graphics.

The paper is organized as follows: In Section 2 we give the necessary preliminary notions we need. In Section 3 we define the main invariant polynomials associated to the class $\boldsymbol{QSL^{2p}}$. In Section 4 we present some preliminary results involving the use of invariant polynomials. Section 5 is dedicated to the construction of the invariant criteria for a quadratic systems to belong to the class $\boldsymbol{QSL^{2p}}$. In Section 6 we give a complete classification of quadratic systems in $\boldsymbol{QSL^{2p}}$ according to their configurations of invariant lines. In Section 7 we prove that all 122 detected configurations for the family $\boldsymbol{QSL^{2p}}$ are non-equivalent in the sens of Definition 2.1. And finally in Section 8 that the maximum number of limit cycles for the family $\mathbf{QSL}_{\geq 2}$ is one and so the Hilbert number of this class H($\mathbf{QSL}_{\geq 2}$)=1.

2 On configurations of invariant lines of a polynomial differential system

In the notion of configuration of invariant algebraic curves of a system (1), the real singular points of the system also intervene and we are led to ask questions such as: how many finite distinct such points could we have on an invariant algebraic curves, what is the maximum multiplicity that a finite singular point on an algebraic curve could have, what is the maximum multiplicity that an infinite singular point of an algebraic curve could have? These numbers are affine invariants of dynamical nature. The bounds for such numbers depend of course on the degree of the system and on the degree of the curve.

The notion of multiplicity of an invariant line of a polynomial system (1) was defined in [21]. This concept was extended to the notion of multiplicity of an invariant algebraic curve of a differential system (1) in [9] where several notions of multiplicity were given and shown to be equivalent in case the algebraic invariant curve is irreducible.

To study the relationship between the dynamical concepts and the geometric or topological characteristics of the curve such as its class, its order, its genus provides us with more motivation to further pursue this work.

Consider two polynomial differential systems (S_1) and (S_2) such that each has a finite set of singular points and a finite set of invariant lines, including the line at infinity. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be the two configurations of invariant lines of (S_1) and (S_2).

Definition 2.1. *We say that two configurations $\mathcal{C}_1, \mathcal{C}_2$, of (S_1) and (S_2) formed by invariant lines (including the line at infinity) are equivalent if and only if there is a bijection ϕ between the two sets of invariant lines sending the line at infinity of $\mathcal{C}_1$ to the line at infinity of $\mathcal{C}_2$, sending a line with coefficients in $\mathbb{R}$ of (S_1) to a line with coefficients in $\mathbb{R}$ of (S_2). In addition the map preserves the multiplicities of the invariant lines, and for each invariant line L of $\mathcal{C}_1$ there is a one to one correspondence ϕ_L between the set of real singular points of (S_1) situated on the line L and the set of real singular points of the system (S_2) situated on the line $\phi(L)$ which preserves the multiplicities of the singular points and their order and sends a real singular point at infinity to a real singular point at infinity. In addition we have the following:*

(i) When we list in a counterclockwise sense the real singular points at infinity on (S_1) starting from a point p on the Poincaré disk, $p_1 = p, ..., p_l$, this correspondence preserves the multiplicities of the singular points and preserves or reverses the orientation.

(ii) We consider the total curves

$$\mathcal{F} : \prod F_j(X;Y;Z)^{m_i} Z^m = 0; \mathcal{F}' : \prod F'_j(X;Y;Z)^{m'_i} Z^m = 0$$

where $F_i(X;Y;Z) = 0$ (respectively $F'_i(X;Y;Z) = 0$) are the projective completions of the lines $\mathcal{L}_i$ (respectively $\mathcal{L}'_i$) and $m_i; m'_i$ are the multiplicities of the curves $F_i = 0$; $F'_i = 0$ and m, m' are respectively the multiplicities of $Z = 0$ in the first and in the second system. Then, there is a one-to-one correspondence between the real singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singular points of the total curves.

Definition 2.2. *Consider a real planar polynomial differential system (5). We call* ***configuration of invariant straight lines*** *of this system, the set of (complex) invariant straight lines (which may have real coefficients), including the line at infinity of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.*

3 The main invariant polynomials associated to the class $\boldsymbol{QSL^{2p}}$

We consider the class of real quadratic polynomial differential systems

$$\begin{aligned}\dot{x} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(\tilde{a},x,y),\\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(\tilde{a},x,y)\end{aligned} \tag{5}$$

where

$$\begin{aligned}&p_0 = a, \quad p_1(x,y) = cx + dy, \quad p_2(x,y) = gx^2 + 2hxy + ky^2,\\ &q_0 = b, \quad q_1(x,y) = ex + fy, \quad q_2(x,y) = lx^2 + 2mxy + ny^2\end{aligned}$$

and with $\max(\deg(p),\deg(q)) = 2$. It is known that on the set $\boldsymbol{QS}$ acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [22]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on $\boldsymbol{QS}$. We can identify the set $\boldsymbol{QS}$ of systems (5) with a subset of $\mathbb{R}^{12}$ via the map $\boldsymbol{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (5) the 12–tuple $\tilde{a} = (a,c,d,g,h,k,b,e,f,l,m,n)$ of its coefficients. We associate to this group action polynomials in x,y and parameters which behave well with respect to this action, the GL–comitants (GL–invariants), the T–comitants (affine invariants) and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [22] (see also [3]).

Next we define the following 38 invariant polynomials associated to the class $\boldsymbol{QSL^{2p}}$:

$$\begin{aligned}&\Big\{\mu_0,\ldots,\mu_4,\ \mathbf{D},\ \mathbf{R},\ \mathbf{U},\ \eta,\ B_1,\ B_2,\ B_3,\ \widetilde{M},\ C_2,\ \theta,\ \theta_3,\ \theta_5,\ \widetilde{K},\\ &\widetilde{N},\ \widetilde{D},\ H_1,\ H_3,\ldots,H_{12},\ H_{15},\ H_{16},\ D_1,\ N_1,\ N_2,\ N_5,\ N_6\Big\}.\end{aligned} \tag{6}$$

According to [3] (see also [8]) we apply the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[\tilde{a},x,y]$ with

$$\begin{aligned}\mathbf{L}_1 &= 2a\frac{\partial}{\partial c} + c\frac{\partial}{\partial g} + \frac{1}{2}d\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial e} + e\frac{\partial}{\partial l} + \frac{1}{2}f\frac{\partial}{\partial m},\\ \mathbf{L}_2 &= 2a\frac{\partial}{\partial d} + d\frac{\partial}{\partial k} + \frac{1}{2}c\frac{\partial}{\partial h} + 2b\frac{\partial}{\partial f} + f\frac{\partial}{\partial n} + \frac{1}{2}e\frac{\partial}{\partial m},\end{aligned}$$

to construct several invariant polynomials from the set. More precisely using this operator and the affine invariant $\mu_0 = \operatorname{Res}_x\big(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,4, \quad \text{where} \quad \mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).$$

Using these invariant polynomials we define some new ones, which according to [3] are responsible for the number and multiplicities of the finite singular points of (5):

$$\begin{aligned}
\mathbf{D} &= \Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\
\mathbf{P} &= 12\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\\
\mathbf{R} &= 3\mu_1^2 - 8\mu_0\mu_2,\\
\mathbf{S} &= \mathbf{R}^2 - 16\mu_0^2\mathbf{P},\\
\mathbf{T} &= 18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 2\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR},\\
\mathbf{U} &= \mu_3^2 - 4\mu_2\mu_4.
\end{aligned}$$

In what follows we also need the so-called *transvectant of order k* (see [15], [18]) of two polynomials f, $g \in \mathbb{R}[\tilde{a},x,y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}}.$$

In order to construct the remaining invariant polynomials contained in the set (6) we first need to define some elementary bricks which help us to construct these elements of the set.

We remark that the following polynomials in $\mathbb{R}[\tilde{a},x,y]$ are the simplest invariant polynomials of degree one with respect to the coefficients of the differential systems (5) which are GL-comitants:

$$\begin{aligned}
C_i(x,y) &= yp_i(x,y) - xq_i(x,y),\ \ i=0,1,2;\\
D_i(x,y) &= \frac{\partial}{\partial x}p_i(x,y) + \frac{\partial}{\partial y}q_i(x,y),\ \ i=1,2.
\end{aligned}$$

Apart from these simple invariant polynomials we shall also make use of the following nine GL-invariant polynomials:

$$\begin{aligned}
&T_1 = (C_0,C_1)^{(1)}, \quad T_2 = (C_0,C_2)^{(1)}, \quad T_3 = (C_0,D_2)^{(1)},\\
&T_4 = (C_1,C_1)^{(2)}, \quad T_5 = (C_1,C_2)^{(1)}, \quad T_6 = (C_1,C_2)^{(2)},\\
&T_7 = (C_1,D_2)^{(1)}, \quad T_8 = (C_2,C_2)^{(2)}, \quad T_9 = (C_2,D_2)^{(1)}.
\end{aligned}$$

These are of degree two with respect to the coefficients of systems (5).

We next define a list of T-comitants:

$$\begin{aligned}
\hat{A}(\tilde{a}) =\ & (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144,\\
\widehat{B}(\tilde{a},x,y) =\ & \Big\{16D_1(D_2,T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2,T_9)^{(1)}(3D_1D_2\\
& - 5T_6 + 9T_7) + 2(D_2,T_9)^{(1)}\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0,T_5)^{(1)}\big)\\
& + 6(D_2,T_7)^{(1)}\big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5)\\
& + C_2(9T_4 + 96T_3)\big] + 6(D_2,T_6)^{(1)}\big[32C_0T_9 - C_1(12T_7 + 52D_1D_2)\\
& - 32C_2D_1^2\big] + 48D_2(D_2,T_1)^{(1)}(2D_2^2 - T_8) + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9)\\
& - 32D_1T_8(D_2,T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7) - 16D_1(C_2,T_8)^{(1)}(D_1^2 + 4T_3)\\
& + 12D_1(C_1,T_8)^{(2)}(C_1D_2 - 2C_2D_1) + 12D_1(C_1,T_8)^{(1)}(T_7 + 2D_1D_2)\\
& + 96D_2^2\big[D_1(C_1,T_6t)^{(1)} + D_2(C_0,T_6)^{(1)}\big] - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9)\\
& - 16D_1D_2T_3(2D_2^2 + 3T_8) + 6D_1^2D_2^2(7T_6 + 2T_7) - 252D_1D_2T_4T_9\Big\}/(2^8 3^3),
\end{aligned}$$

$$
\begin{aligned}
\widehat{D}(\tilde{a},x,y) =\ & \big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6)-(C_1,T_5)^{(1)}-9D_1^2C_2\\
& +6D_1(C_1D_2-T_5)\big]/36,\\
\widehat{E}(\tilde{a},x,y) =\ & \big[D_1(2T_9-T_8)-3(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\big]/72,\\
\widehat{F}(\tilde{a},x,y) =\ & \big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0(D_2,T_9)^{(1)}-9D_2^2T_4\\
& +288D_1\widehat{E}-24(C_2,\widehat{D})^{(2)}+120(D_2,\widehat{D})^{(1)}-36C_1(D_2,T_7)^{(1)}\\
& +8D_1(D_2,T_5)^{(1)}\big]/144,\\
\widehat{K}(\tilde{a},x,y) =\ & (T_8+4T_9+4D_2^2)/72,\\
\widehat{H}(\tilde{a},x,y) =\ & (-T_8+8T_9+2D_2^2)/72,
\end{aligned}
$$

as well as the following affine invariants (which serve as bricks for constructing the needed invariant polynomials):

$$
\begin{aligned}
& A_2(\tilde{a}) = (C_2,\widehat{D})^{(3)}/12, && A_8(\tilde{a}) = \big((\widehat{D},\widehat{H})^{(2)},D_2\big)^{(1)}/8,\\
& A_{11}(\tilde{a}) = (\widehat{F},\widehat{K})^{(2)}/4, && A_{20}(\tilde{a}) = \big((C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16,\\
& A_{21}(\tilde{a}) = \big((\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(2)}/16, && A_{39}(\tilde{a}) = \big(((\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
& A_{42}(\tilde{a}) = \big(((\widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{aligned}
$$

Now we can define the remaining invariant polynomials of the set (6):

$$
\begin{aligned}
\widetilde{K}(\tilde{a},x,y) =& 4\widehat{K} \equiv \text{Jacob}\left(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\right),\\
\widetilde{M}(\tilde{a},x,y) =& (C_2,C_2)^{(2)} \equiv 2\text{Hess}\left(C_2(\tilde{a},x,y)\right),\\
\widetilde{N}(\tilde{a},x,y) =& \widetilde{K} - 4\widehat{H},\\
\widetilde{D}(\tilde{a},x,y) =& \widehat{D},\\
\eta(\tilde{a}) =& (\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \text{Discrim}\left(C_2(\tilde{a},x,y)\right),\\
\theta(\tilde{a}) =& -(\widetilde{N},\widetilde{N})^{(2)}/2 \equiv \text{Discrim}\left(\widetilde{N}(\tilde{a},x,y)\right);\\
\theta_3(\tilde{a}) =& A_8 + A_{11},\\
\theta_5(\tilde{a},x,y) =& 2C_2\left(T_6,T_7\right)^{(1)} - \left(2C_1D_2 + T_5\right)\left(C_1, D_2^2\right)^{(2)};\\
B_1(\tilde{a}) =& \text{Res}_x\left(C_2,\widetilde{D}\right)/y^9 = -2^{-9}3^{-8}\left(B_2,B_3\right)^{(4)},\\
B_2(\tilde{a},x,y) =& (B_3,B_3)^{(2)} - 6B_3(C_2,\widetilde{D})^{(3)},\\
B_3(\tilde{a},x,y) =& (C_2,\widetilde{D})^{(1)} \equiv \text{Jacob}\left(C_2,\widetilde{D}\right),\\
H_1(\tilde{a}) =& -\left((C_2,C_2)^{(2)},C_2)^{(1)},\widetilde{D}\right)^{(3)},\\
H_3(\tilde{a},x,y) =& (C_2,\widetilde{D})^{(2)},\\
H_4(\tilde{a}) =& \left((C_2,\widetilde{D})^{(2)},(C_2,D_2)^{(1)}\right)^{(2)},\\
H_5(\tilde{a}) =& \left((C_2,C_2)^{(2)},(\widetilde{D},\widetilde{D})^{(2)}\right)^{(2)} + 8\left((C_2,\widetilde{D})^{(2)},(\widetilde{D},D_2)^{(1)}\right)^{(2)},\\
H_6(\tilde{a},x,y) =& 16N^2(C_2,\widetilde{D})^{(2)} + H_2^2(C_2,C_2)^{(2)},\\
H_7(\tilde{a}) =& (\widetilde{N},C_1)^{(2)},\\
H_8(\tilde{a}) =& 9\left((C_2,\widetilde{D})^{(2)},(\widetilde{D},D_2)^{(1)}\right)^{(2)} + 2\left[(C_2,\widetilde{D})^{(3)}\right]^2,\\
H_9(\tilde{a}) =& -\left[\!\left[\widetilde{D},\widetilde{D}\right)^{(2)},\widetilde{D},\right)^{(1)},\widetilde{D}\right)^{(3)},\\
H_{10}(\tilde{a}) =& \left((\widetilde{N},\widetilde{D})^{(2)},\ D_2\right)^{(1)},\\
H_{11}(\tilde{a},x,y) =& 8\widehat{H}\left[(C_2,\widetilde{D})^{(2)} + 8(\widetilde{D},D_2)^{(1)}\right] + 3\left[(C_1,\,2\widehat{H}-\widetilde{N})^{(1)} - 2D_1\widetilde{N}\right]^2,\\
H_{12}(\tilde{a},x,y) =& (\widetilde{D},\widetilde{D})^{(2)} \equiv \text{Hessian}(\widetilde{D}),\\
H_{15}(\tilde{a}) =& \left((\widetilde{D},\widetilde{D})^{(2)},\ \widetilde{H}\right)^{(1)},\\
H_{16}(\tilde{a}) =& 14A_2^4 - A_2^2(10A_{20} + 33A_{21}) - 2A_2(15A_{39} + A_{42}),\\
N_1(a,x,y) =& C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},\\
N_2(a,x,y) =& D_1(C_1,C_2)^{(2)} - \left((C_2,C_2)^{(2)},C_0\right)^{(1)},\\
N_5(a,x,y) =& \left[(D_2,C_1)^{(1)} + D_1D_2\right]^2 - 4\left(C_2,C_2\right)^{(2)}\left(C_0,D_2\right)^{(1)},\\
N_6(a,x,y) =& 8D + C_2\left[8(C_0,D_2)^{(1)} - 3(C_1,C_1)^{(2)} + 2D_1^2\right].
\end{aligned}
$$

We remark that the above invariant polynomials (except H_{15} and H_{16}) were constructed and used in [27], [25] and [7] and only the invariant polynomials H_{15} and H_{16} are defined here.

Remark 3.1. *Since we have used the above invariant polynomials in several articles (see for example, [4, 5, 7, 21–23, 25–29]) as well as in the book [3] it happens that the same invariant polynomials are applied under different names. So in order to avoid some confusion concerning the invariant polynomials used in the classification we have done here, we present the correspondence of the names from the articles [21–23, 25, 26]) (at the left) with notations from this paper(at the right):*

$$\mu \to \mu_0; \quad N \to \widetilde{N}; \quad M \to \widetilde{M}; \quad K \to \widetilde{K}; \quad D \to \widetilde{D}.$$

4 Preliminary results involving the use of polynomial invariants

The following two lemmas reveal the geometrical meaning of the invariant polynomials B_1, B_2, B_3, θ and $\widetilde{N}$.

Lemma 4.1 ([21]). *For the existence of an invariant straight line in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).*

Lemma 4.2 ([21]). *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a system (5) corresponding to $\boldsymbol{a} \in \mathbb{R}^{12}$ is the condition $\theta(\boldsymbol{a}) = 0$ (respectively, $\widetilde{N}(\boldsymbol{a}, x, y) = 0$).*

We remark that the invariant polynomials $\mu_i(\tilde{a}, x, y)$ $(i = 0, 1, \ldots, 4)$ defined earlier are responsible for the total multiplicity of the finite singularities of quadratic systems (5). Moreover they detect whether a quadratic system is degenerate or not. More exactly we have the following lemma.

Lemma 4.3. ([8]) *Consider a quadratic system (S) with coefficients $\boldsymbol{a} \in \mathbb{R}^{12}$. Then:*

(i) The total multiplicity of the finite singularities of this system is $4 - k$ if and only if for every i such that $0 \le i \le k-1$ we have $\mu_i(\boldsymbol{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ and $\mu_k(\boldsymbol{a}, x, y) \neq 0$.

(ii) The system (S) is degenerate (i.e. $\gcd(p, q) \neq$ constant) if and only if $\mu_i(\boldsymbol{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ for every $i = 0, 1, 2, 3, 4$.

On the other hand the invariant polynomials η, $\widetilde{M}$ and C_2 govern the number of real and complex infinite singularities. More precisely, according to [30] (see also [22]) we have the next result.

Lemma 4.4. *The number of infinite singularities (real and complex) of a quadratic system in* **QS** *is determined by the following conditions:*

(i) 3 real if $\eta > 0$;

(ii) 1 real and 2 imaginary if $\eta < 0$;

(iii) 2 real if $\eta = 0$ and $\widetilde{M} \neq 0$;

(iv) 1 real if $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$;

(v) ∞ if $\eta = \widetilde{M} = C_2 = 0$.

Moreover, the quadratic systems (5)*, for each one of these cases, can be brought via a linear transformation to the corresponding case of the following canonical systems* $(\mathbf{S}_I)-(\mathbf{S}_V)$*:*

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+(h-1)xy, \\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+(h+1)xy, \\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+hxy, \\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a+cx+dy+gx^2+hxy, \\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2; \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a+cx+dy+x^2, \\ \dot{y} &= b+ex+fy+xy. \end{cases} \qquad (\mathbf{S}_V)$$

Now we define the affine comitants which are responsible for the existence of invariant lines for a non-degenerate quadratic system (5).

Let us apply a translation $x=x'+x_0$, $y=y'+y_0$ to the polynomials $p(\tilde{a},x,y)$ and $q(\tilde{a},x,y)$. We obtain $\hat{p}(\hat{a}(\tilde{a},x_0,y_0),x',y')=p(\tilde{a},x'+x_0,y'+y_0)$, $\hat{q}(\hat{a}(\tilde{a},x_0,y_0),x',y')=q(\tilde{a},x'+x_0,y'+y_0)$. Let us construct the following polynomials

$$\Gamma_i(\tilde{a},x_0,y_0)\equiv \operatorname{Res}_{x'}\Big(C_i\big(\hat{a}(\tilde{a},x_0,y_0),x',y'\big),C_0\big(\hat{a}(\tilde{a},x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\Gamma_i(\tilde{a},x_0,y_0)\in\mathbb{R}[\tilde{a},x_0,y_0],\ (i=1,2).$$

Notation 4.1.

$$\tilde{\mathcal{E}}_i(a,x,y)=\Gamma_i(\tilde{a},x_0,y_0)|_{\{x_0=x,\ y_0=y\}}\in\mathbb{R}[\tilde{a},x,y]\quad(i=1,2). \tag{7}$$

Observation 4.1. *We note that the polynomials* $\tilde{\mathcal{E}}_1(a,x,y)$ *and* $\tilde{\mathcal{E}}_2(a,x,y)$ *are affine comitants of systems* (5) *and are homogeneous polynomials in the coefficients* $a,\dots,n$ *and non-homogeneous in* x,y *and*

$$\deg_{\tilde{a}}\hat{\mathcal{E}}_1=3,\ \deg_{(x,y)}\hat{\mathcal{E}}_1=5,\quad \deg_a\hat{\mathcal{E}}_2=4,\ \deg_{(x,y)}\hat{\mathcal{E}}_2=6.$$

Notation 4.2. *Let* $\mathcal{E}_i(\tilde{a},X,Y,Z)$ $(i=1,2)$ *be the homogenization of* $\hat{\mathcal{E}}_i(\tilde{a},x,y)$*, i.e.*

$$\mathcal{E}_1(\tilde{a},X,Y,Z)=Z^5\hat{\mathcal{E}}_1(\tilde{a},X/Z,Y/Z),\qquad \mathcal{E}_2(\tilde{a},X,Y,Z)=Z^6\hat{\mathcal{E}}_1(\tilde{a},X/Z,Y/Z)$$

and $\mathcal{H}(\tilde{a},X,Y,Z)=\gcd\Big(\mathcal{E}_1(\tilde{a},X,Y,Z),\ \mathcal{E}_2(\tilde{a},X,Y,Z)\Big)$ *in* $\mathbb{R}[\tilde{a},X,Y,Z]$*.*

The geometrical meaning of these affine comitants is given by the two following lemmas (see [21]):

Lemma 4.5 ([21])**.** *The straight line* $\mathcal{L}(x,y)\equiv ux+vy+w=0$*,* $u,v,w\in\mathbb{C}$*,* $(u,v)\neq(0,0)$ *is an invariant line for a quadratic system* (5) *if and only if the polynomial* $\mathcal{L}(x,y)$ *is a common factor of the polynomials* $\tilde{\mathcal{E}}_1(\boldsymbol{a},x,y)$ *and* $\tilde{\mathcal{E}}_2(\boldsymbol{a},x,y)$ *over* $\mathbb{C}$*, i.e.*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a},x,y)=(ux+vy+w)\widetilde{W}_i(x,y)\quad(i=1,2),$$

where $\widetilde{W}_i(x,y)\in\mathbb{C}[x,y]$*.*

Lemma 4.6. *1) If $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant straight line of multiplicity k for a quadratic system* (5) *then $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ in $\mathbb{C}[x,y]$, i.e. there exist $W_i(\boldsymbol{a},x,y) \in \mathbb{C}[x,y]$ $(i=1,2)$ such that*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a},x,y) = (ux+vy+w)^k W_i(\boldsymbol{a},x,y), \quad i=1,2. \tag{8}$$

2) If the line $l_\infty : Z=0$ is of multiplicity $k>1$ then $Z^{k-1} \mid \gcd(\mathcal{E}_1,\mathcal{E}_2)$, in other words we have $Z^{k-1} \mid \mathcal{H}(\boldsymbol{a},X,Y,Z)$.

In what follows the following Lemma it will be very useful.

Lemma 4.7. *Assume that a quadratic system has the form*

$$\dot{x} = a + cx + dy, \quad \dot{y} = q(x,y),$$

i.e. the quadratic homogeneous part of the first equation vanishes (due to the change $x \leftrightarrow y$ we obtain the systems with vanishing the quadratic homogeneous part of the second equation). Then the infinite invariant line $Z=0$ of this system is of multiplicity 2 if $c^2+d^2 \neq 0$ and it is of multiplicity 3 if $c=d=0$.

Proof: Considering Notation 4.2 For the above systems we calculate $\mathcal{H}(X,Y,Z) = Z$ if $c^2+d^2 \neq 0$ and $\mathcal{H}(X,Y,Z) = aZ^2$ if $c=d=0$. Since $a \neq 0$ (otherwise the system become degenerate), according to Lemma 4.6 (see statement *2)*) we deduce that the statement of Lemma 4.7 is valid. ∎

5 Invariant criteria for a system to belong to the class $\boldsymbol{QSL^{2p}}$

In this section we find necessary and sufficient affine invariant conditions for a quadratic system (5) to belong to the class $\boldsymbol{QSL^{2p}}$.

Theorem 5.1. *An arbitrary quadratic system* (5) *belongs to the class $\boldsymbol{QSL^{2p}}$ if and only if $\theta = B_1 = H_7 = 0$ and one of the following conditions is satisfied:*

- ***(i)*** *If $\eta > 0$ then either $\widetilde{N} \neq 0$, or $\widetilde{N} = 0$, $\theta_3 = 0$.*
- ***(ii)*** *If $\eta < 0$ then $\widetilde{N} \neq 0$.*
- ***(iii)*** *If $\eta = 0$, $\widetilde{M} \neq 0$ then either $\widetilde{N} \neq 0$, or $\widetilde{N} = 0$, $\widetilde{K} \neq 0$, $\theta_3 = 0$, or $\widetilde{N} = \widetilde{K} = 0$, $B_2 \neq 0$, $\theta_5 = 0$, or $\widetilde{N} = \widetilde{K} = B_2 = 0$.*
- ***(iv)*** *If $\eta = \widetilde{M} = 0$, $C_2 \neq 0$ then either $\widetilde{N} \neq 0$, or $\widetilde{N} = B_2 = 0$.*
- ***(v)*** *If $\eta = \widetilde{M} = C_2 = 0$.*

Proof: Consider the generic family of systems (5) as well as the family of systems (4) belonging to the class $\boldsymbol{QSL^{2p}}$. For systems (4) we have

$$\theta = B_1 = H_7 = 0, \tag{9}$$

i.e. these conditions are necessary for a system to belong to the class $\boldsymbol{QSL^{2p}}$.

We examine three cases: $\eta > 0$, $\eta < 0$ and $\eta = 0$.

5.1 The case $\eta > 0$

According to [30] (see also [22]) for $\eta > 0$ any system (5) could be brought via an affine transformation and time rescaling to the following canonical form:

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} =& b + ex + fy + (g-1)xy + hy^2. \end{aligned} \tag{10}$$

Calculations yield

$$\eta = 1,\ \theta = -8(g-1)(h-1)(g+h),\ C_2 = xy(x-y)$$

and therefore considering (9) the condition $\theta = 0$ yields $(h-1)(g-1)(g+h) = 0$. Without loss of generality we can consider $h = 1$. Indeed, if $g = 1$ (respectively, $g + h = 0$) we can apply the linear transformation which will replace the straight line $x = 0$ with $y = 0$ (respectively, $x = 0$ with $y = x$) reducing this case to $h = 1$.

So, $h = 1$ and then we calculate

$$H_7 = 4d(g-1)(g+1),\ \widetilde{N} = (g-1)(g+1)x^2.$$

5.1.1 The subcase $\widetilde{N} \neq 0$

Then $(g-1)(g+1) \neq 0$ and hence the condition $H_7 = 0$ yields $d = 0$ (then $B_1 = 0$). This leads to the following family of systems belonging to the class $\boldsymbol{QSL^{2p}}$:

$$\begin{aligned} \dot{x} =& a + cx + gx^2, \\ \dot{y} =& b + ex + fy + (g-1)xy + y^2. \end{aligned} \tag{11}$$

5.1.2 The subcase $\widetilde{N} = 0$

This gives $(g-1)(g+1) = 0$ and without lost of generality we may assume $g = 1$ as the case $g = -1$ can be brought by a linear transformation to the case $g = 1$.

So for systems (10) we have $g = h = 1$ and after an additional translation (to make $c = f = 0$) we get the systems:

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + y^2. \tag{12}$$

For these systems we calculate

$$B_1 = -\,d^2e^2(4a - 4b + d^2 - e^2), \quad \mu_0 = 1, \quad \theta_3 = -2de$$

and imposing $B_1 = 0$ we consider two possibilities: $\theta_3 \neq 0$ and $\theta_3 = 0$.

5.1.2.1 The possibility $\theta_3 \neq 0$. Then $de \neq 0$ and the condition $B_1 = 0$ implies $4a - 4b + d^2 - e^2 = 0$, i.e. $b = a + (d^2 - e^2)/4$. This leads to the systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = a + (d^2 - e^2)/4 + ex + y^2 \tag{13}$$

possessing the invariant line $L_1 = 2x - 2y + d - e = 0$. We observe that due to $de \neq 0$ the above systems could not have invariant lines in the directions $x = 0$ or $y = 0$. Therefore the unique direction for invariant lines is $y = x$. We claim that systems (13) could not have another invariant line in the direction $y = x$ different from the line $L_1 = 0$. Indeed applying the affine transformation $x_1 = 2x - 2y + d - e, \quad y_1 = y$ which brings the line $L_1 = 0$ to the line $x_1 = 0$ we arrive at the systems

$$\dot{x} = x_1(-2d + x + 4y)/2, \quad \dot{y} = (4a + d^2 - 2de + e^2 + 2ex + 4ey + 4y^2)/4.$$

Since the first equation depends on the variable y we conclude that in the considered case a quadratic system could not belong to the family $\boldsymbol{QSL^{2p}}$.

5.1.2.2 The possibility $\theta_3 = 0$.

In this case $de = 0$ and we may assume $d = 0$ due to the change $(x, y, a, b, d, e) \to (y, x, b, a, e, d)$ in systems (12). Evidently these systems for $d = 0$ belong to the class $\boldsymbol{QSL^{2p}}$ and have the form:

$$\dot{x} = a + x^2, \quad \dot{y} = b + ex + y^2. \tag{14}$$

5.2 The case $\eta < 0$

According to [30] (see also [22]) we consider the following canonical form:

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2. \end{aligned} \tag{15}$$

Calculations yield:

$$\begin{gathered} \eta = -4, \ \theta = 8(1+h)\big[g^2 + (h-1)^2\big], \ C_2 = x(x^2 + y^2), \\ \widetilde{N} = (g^2 - 2h + 2)x^2 + 2g(h+1)xy + (h^2 - 1)y^2 \end{gathered} \tag{16}$$

and hence the condition $\theta = 0$ implies $(h+1)[(h-1)^2 + g^2] = 0$. We consider again two subcases: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

5.2.1 Subcase $\widetilde{N} \neq 0$

Then by (16) the condition $\theta = 0$ yields $h = -1$ and in addition we may assume $f = 0$ due to the translation $(x, y) \to (x, y + f/2)$. Hence, we obtain the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2, \tag{17}$$

for which we calculate $H_7 = 4d(4 + g^2)$. Then the condition $H_7 = 0$ gives $d = 0$ and clearly the above systems belong to the family $\boldsymbol{QSL^{2p}}$.

5.2.2 Subcase $\widetilde{N} = 0$

Then from (16) we have $g = h - 1 = 0$ and without loss of generality we may assume $c = d = 0$ due to the translation $(x, y) \to (x - d/2, y - c/2)$. Hence we obtain the systems

$$\dot{x} = a + 2xy, \quad \dot{y} = b + ex + fy - x^2 + y^2, \tag{18}$$

for which calculations yield:

$$B_1 = \; -4a(e^2+f^2)^2, \quad \theta = H_7 = 0.$$

We observe that for systems (18) we have $C_2 = x(x^2+y^2)$. This means that these systems could have real invariant lines only in the direction $x=0$. So the condition $a=0$ must be fulfilled (then $B_1=0$), however the first equation of (18) contains the variable y, i.e. these systems could not belong to $\boldsymbol{QSL^{2p}}$.

5.3 The case $\eta = 0$

According to [30] (see also [22]) we have to examine two subcases: $\widetilde{M} \neq 0$ and $\widetilde{M} = 0$.

5.3.1 Subcase $\widetilde{M} \neq 0$

Following [30] we consider the family systems

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + hxy, \\ \dot{y} =& b + ex + fy + (g-1)xy + hy^2. \end{aligned} \tag{19}$$

for which calculations yield:

$$\theta = 8h^2(1-g), \quad \mu_0 = gh^2, \quad C_2 = x^2y, \quad \widetilde{N} = (g^2-1)x^2 + 2h(g-1)xy + h^2y^2. \tag{20}$$

We observe that the condition $\widetilde{N}=0$ implies $\mu_0 = \theta = 0$ and so we examine two possibilities: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

5.3.1.1 The possibility $\widetilde{N} \neq 0$. The condition $\theta = 0$ implies $h(g-1) = 0$ and since $\mu_0 = gh^2$ we consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

5.3.1.1.1 The case $\mu_0 \neq 0$. Considering (20) we obtain $h \neq 0$, $g = 1$ and then we may assume $h = 1$ due to the rescaling $y \to y/h$. Moreover, we may assume $c = d = 0$ via the translation $(x, y) \to (x-d, y+2d-c)$. So, we obtain the canonical systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + ex + fy + y^2,$$

for which calculation yields

$$B_1 = -a^2e^2, \quad H_7 = -4e.$$

The condition $H_7 = 0$ gives $e = 0$ (this implies $B_1 = 0$) and applying the change $(x, y) \to (y, x)$ we get the systems (after renotating the parameters)

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b + xy + y^2, \tag{21}$$

which evidently belong to the family $\boldsymbol{QSL^{2p}}$.

5.3.1.1.2 The case $\mu_0 = 0$. Since $\theta = 0$ this implies $h = 0$ and for the systems (19) we have $\widetilde{N} = (g^2-1)x^2 \neq 0$. So $g-1 \neq 0$ and we may assume $e = f = 0$ via the translation $(x, y) \to \big(x + f/(1-g), y + e/(1-g)\big)$. This leads to the systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy,$$

for which we calculate

$$B_1 = -bd^4(g-1)^2g^2, \quad \widetilde{N} = (g^2-1)x^2, \quad H_7 = 4d(g^2-1).$$

So due to $\widetilde{N} \neq 0$ the condition $H_7 = 0$ gives $d = 0$ and this implies $B_1 = 0$. Therefore we arrive at the family of systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g-1)xy, \tag{22}$$

which evidently belong to the class $\boldsymbol{QSL^{2p}}$.

5.3.1.2 The possibility $\widetilde{N} = 0$. In this case we get $h = 0$ and $g^2 - 1 = 0$, i.e. $g = 1$ or $g = -1$.

On the other hand for systems $(\mathbf{S}_{III})$ with $h = 0$ we have $\widetilde{K} = g(g-1)x^2$ and we consider two cases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

1) *The case* $\widetilde{K} \neq 0$. Then $g - 1 \neq 0$ and this implies $g = -1$. In this case we may assume $e = f = 0$ due to the translation $(x, y) \to \big(x + f/2, y + e/2\big)$ and we arrive at the family of systems

$$\dot{x} = a + cx + dy - x^2, \quad \dot{y} = b - 2xy, \tag{23}$$

for which calculations yield:

$$B_1 = -4bd^4, \quad \theta_3 = 2d^2.$$

Since $C_2 = x^2y$ we conclude that the only directions for the existence of invariant lines are $x = 0$ and $y = 0$. It is clear that a system (23) belongs to the class $\boldsymbol{QSL^{2p}}$ if and only if $d = 0$ and this condition is equivalent to $\theta_3 = 0$ (this implies $B_1 = 0$). So we arrive at the following family of systems

$$\dot{x} = a + cx - x^2, \quad \dot{y} = b - 2xy. \tag{24}$$

2) *The case* $\widetilde{K} = 0$. Then $g = 1$ and we may assume $c = 0$ due to the translation $(x, y) \to (x - c/2, y)$. Then we obtain the systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + fy.$$

It is clear that in order to have invariant lines in the direction $x = 0$ (respectively $y = 0$) the condition $d = 0$ (respectively $e = 0$) has to be satisfied. Moreover in each one of these cases the above systems belong to the class $\boldsymbol{QSL^{2p}}$ (for $e = 0$ we apply the additional change $(x, y) \to (y, x)$).

It remains to detect an invariant polynomial which is responsible for the condition $de = 0$. We calculate

$$B_1 = H_7 = 0, \quad \theta_5 = 96dex^3, \quad B_2 = -648d^4y^4$$

and clearly the condition $de = 0$ is equivalent to $\theta_5 = 0$. However we have two situations geometricaly different: when $d = 0$ the direction of the invariant lines is associated with the double infinite singularity $N_1[0 : 1 : 0]$, whereas when $e = 0$ the corresponding direction is associated with the simple infinite singularity $N_2[1 : 0 : 0]$ (because we have $\mu_0 \neq 0$, i.e. none of the finite singular point have gone to infinity).

Thus we arrive at the systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + ex + fy. \tag{25}$$

if $B_2 = 0$ (i.e. $d = 0$). In the case $B_2 \neq 0$ and $\theta_5 = 0$ (i.e. $e = 0$) after the change $(x, y) \to (y, x)$ and redenoting the parameters we arrive at the family of systems:

$$\dot{x} = a + cx, \quad \dot{y} = b + ex + y^2. \tag{26}$$

Evidently both families above belong to the clas $\boldsymbol{QSL^{2p}}$.

Since all the possibilities in the case $\eta = 0$ and $\widetilde{M} \neq 0$ are examined we deduce that the statement ***(iii)*** of Theorem 5.1 is proved.

5.3.2 Subcase $\widetilde{M} = 0$

In this case we have to discuss two possibilities: $C_2 \neq 0$ and $C_2 = 0$.

5.3.2.1 The possibility $C_2 \neq 0$. According to [22] we consider the following canonical form:

$$\begin{aligned} \dot{x} =& a + cx + dy + gx^2 + hxy, \\ \dot{y} =& b + ex + fy - x^2 + gxy + hy^2, \end{aligned} \tag{27}$$

for which we have $\theta = 8h^3$ and therefore the condition $\theta = 0$ gives $h = 0$. In this case calculations yield:

$$\theta = 0, \quad B_1 = -d^4 g^2(-f^2 - efg + bg^2), \quad H_7 = 4dg^2, \quad C_2 = x^3, \quad \widetilde{N} = g^2 x^2.$$

Due to $C_2 = x^3$ we deduce that the unique direction for the possible invariant lines is $x = 0$ and hence a system (27) with $h = 0$ could belong to the class $\boldsymbol{QSL^{2p}}$ if and only if $d = 0$.

On the other hand the condition $H_7 = 0$ yields $dg = 0$ (this implies $B_1 = 0$) and clearly in the case $\widetilde{N} \neq 0$ (i.e. $g \neq 0$) the condition $H_7 = 0$ is equivalent to $d = 0$. Therefore for $\widetilde{N} \neq 0$ and $H_7 = 0$ we arrive at the family of systems

$$\dot{x} =a + cx + gx^2, \quad \dot{y} = b + ex + fy - x^2 + gxy. \tag{28}$$

Assume now $\widetilde{N} = 0$, i.e. $g = 0$. Then for systems (27) with $h = g = 0$ we calculate $B_2 = -648d^4x^4$ and evidently the condition $d = 0$ is equivalent to $B_2 = 0$. In this case we get the family of systems

$$\dot{x} =a + cx, \quad \dot{y} = b + ex + fy - x^2. \tag{29}$$

5.3.2.2 The possibility $C_2 = 0$. According to [22] we consider the following family of systems:

$$\dot{x} = a + cx + dy + x^2, \quad \dot{y} = b + ex + fy + xy. \tag{30}$$

for which we have $\theta = 0 = B_1$ and $H_7 = 4d$. Obviously it can be observed that for $d = 0$ the above systems belong to the class $\boldsymbol{QSL^{2p}}$.

We claim that if $d \neq 0$ none of systems (30) could be brought via affine transformation to the form (4). Indeed a straightforward calculation shows us that applying any arbitrary affine transformation

$$x_1 = \alpha x + \beta y + \mu, \quad y_1 = \gamma x + \delta y + \nu, \quad \alpha, \beta, \mu, \gamma, \delta, \nu \in \mathbb{R}, \quad \alpha\delta - \beta\gamma \neq 0$$

we arrive at the systems

$$\dot{x}_1 = \tilde{p}_0 + \tilde{p}_1(x_1, y_1) + \frac{\delta x_1^2}{\alpha\delta - \beta\gamma} + \frac{\beta x_1 y_1}{\beta\gamma - \alpha\delta}, \quad \dot{y} = \tilde{q}(x_1, y_1), \tag{31}$$

where $\tilde{p}_i$ $\tilde{p}_i$ and $\tilde{q}$ depend rationally on the parameters $\alpha, \beta, \mu, \gamma, \delta$ and ν and $\tilde{p}_1$ is a linear form in x_1 and y_1. Clearly the above systems will have the form (4) if and only if the first equation does not depend of the variable y and for this it is necessary $\beta = 0$. However in this case we have

$$\tilde{p}_1(x_1, y_1) = \frac{x(\alpha c\delta - \alpha\gamma d - 2\delta\mu)}{\alpha\delta} + \frac{\alpha d y}{\delta}.$$

Since $\alpha \neq 0$ (due to $\alpha\delta - \beta\gamma \neq 0$) we deduce that systems (31) with $\beta = 0$ belong to the class $\boldsymbol{QSL^{2p}}$ if and only if $d = 0$. However this contradicts the condition $d \neq 0$ and this contradiction proves our claim.

Thus all the possibilities are examined and Theorem 5.1 is proved. ∎

6 Classification of systems in $\boldsymbol{QSL^{2p}}$ according to their configurations of invariant lines

According to [21], [23] and [28] we denote by $\mathbf{QSL_i}$ the family of all non-degenerate quadratic differential systems possessing invariant straight lines (including the line at infinity) of total multiplicity i with $i \in \{2, 3, 4, 5, 6\}$. The families $\mathbf{QSL_i}$ with $i \in \{4, 5, 6\}$ were classified topologically using their algebro-geometric structures in [21], [26] for $i = 5, 6$ and in [23] and [25] for $i = 4$.

Regarding the systems in $\mathbf{QSL_3}$ a similar classification is done only for two subfamilies: when there exist two finite invariant lines intersecting in the finite part of the phase plane and either these lines are real (i.e. they are of Lotka-Voltera type [28] and [29]), or they are complex [4].

In the above mentioned papers we have denoted the configurations following the rule: for the class $\mathbf{QSL_i}$ we set *Config.* $i.j_i$ where $i \in \{3, 4, 5, 6\}$ denote the class and j_i depend on the number of configurations in the class $\mathbf{QSL_i}$.

Remark 6.1. *We point out that for the cases $i = 4, 5, 6$ all the configurations are determined and fixed. More precisely for $i = 4$ (respectively $i = 5$; $i = 6$) we have $j_4 = 1, 2, \ldots, 45$*

(respectively $j_5 = 1, 2, \ldots, 30$; $j_6 = 1, 2, \ldots, 11$). On the other hand the quadratic systems in the class $\mathbf{QSL_3}$ *are not investgated completely. Until now we have only the notations Config. 3.1–Config. 3.13. So in the present paper we continue the enumeration of the configurations beginning with Config. 3.14.*

Theorem 6.1. *Assume that a quadratic non-degenerate system (S) belongs to the class of systems* $\boldsymbol{QSL^{2p}}$, *i.e. one of the statements of Theorem 5.1 holds. Then system (S) possesses one of the configurations of invariant lines indicated below if and only if the corresponding conditions are satisfied respectively:*

- ***(i)*** *For $\eta > 0$ the system (S) could possess either one of the configurations Config. 3.14–Config. 3.27 or Config. 4.j for $j \in \{9, 9a, 10, 13, 16, 22, 34\}$ or Config. 5.j for $j \in \{1, 3, 4, 5, 7, 8, 12, 16\}$ or Config. 6.j for $j \in \{1, 2, 5\}$ given in Figure1 if and only if the one of the sets of conditions given in the Diagram 1 is satisfied, correspondingly.*
- ***(ii)*** *For $\eta < 0$ the system (S) could possess either one of the configurations Config. 3.28–Config. 3.41 or Config. 5.j for $j \in \{2, 9, 10\}$ given in Figure 2 if and only if the one of the sets of conditions given in the Diagram 2 is satisfied, correspondingly.*
- ***(iii)*** *For $\eta = 0$ and $\widetilde{M} \neq 0$ the system (S) could possess either one of the configurations Config. 3.42–Config. 3.63 or Config. 4.j for $j \in \{11, 11a, 12, 14, 15, 19, 23, 24, 28, 32, 36, 39\}$ or Config. 5.j for $j \in \{11, 13, 14, 15, 17, 18, 19\}$ or Config. 6.j for $j \in \{8, 9\}$ given in Figure 3 if and only if the one of the sets of conditions given in the Diagram 3 is satisfied, correspondingly.*
- ***(iv)*** *For $\eta = \widetilde{M} = 0$ the system (S) could possess either one of the configurations Config. 3.64–Config. 3.66 or Config. 4.j for $j \in \{31, 37, 38, 44, 46\}$ or Config. 5.j for $j \in \{23, 26, 27, 30\}$ or Config. C_2.j for $j \in \{5, 6, 7, 8, 9\}$ given in Figure 4 if and only if the one of the sets of conditions given in the Diagram 4 is satisfied, correspondingly.*

The proof that all these 122 configurations are non-equivalent, according to our definition of equivalence is done in Section 7.

Proof of Theorem 6.1: Following Theorem 5.1 we consider the cases given by the following three invariant polynomials: η, $\widetilde{M}$ and C_2.

6.1 The case $\eta > 0$

According to the statement (i) of Theorem 5.1 we examine two subcases: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

6.1.1 The subcase $\widetilde{N} \neq 0$

Taking into account the proof of Theorem 5.1 we consider the family of systems (32) for which we have $\widetilde{N} = (g^2 - 1)x^2 \neq 0$. Then due to a translation we may assume $e = f = 0$ and we arrive at the family of systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g-1)xy + y^2, \tag{32}$$

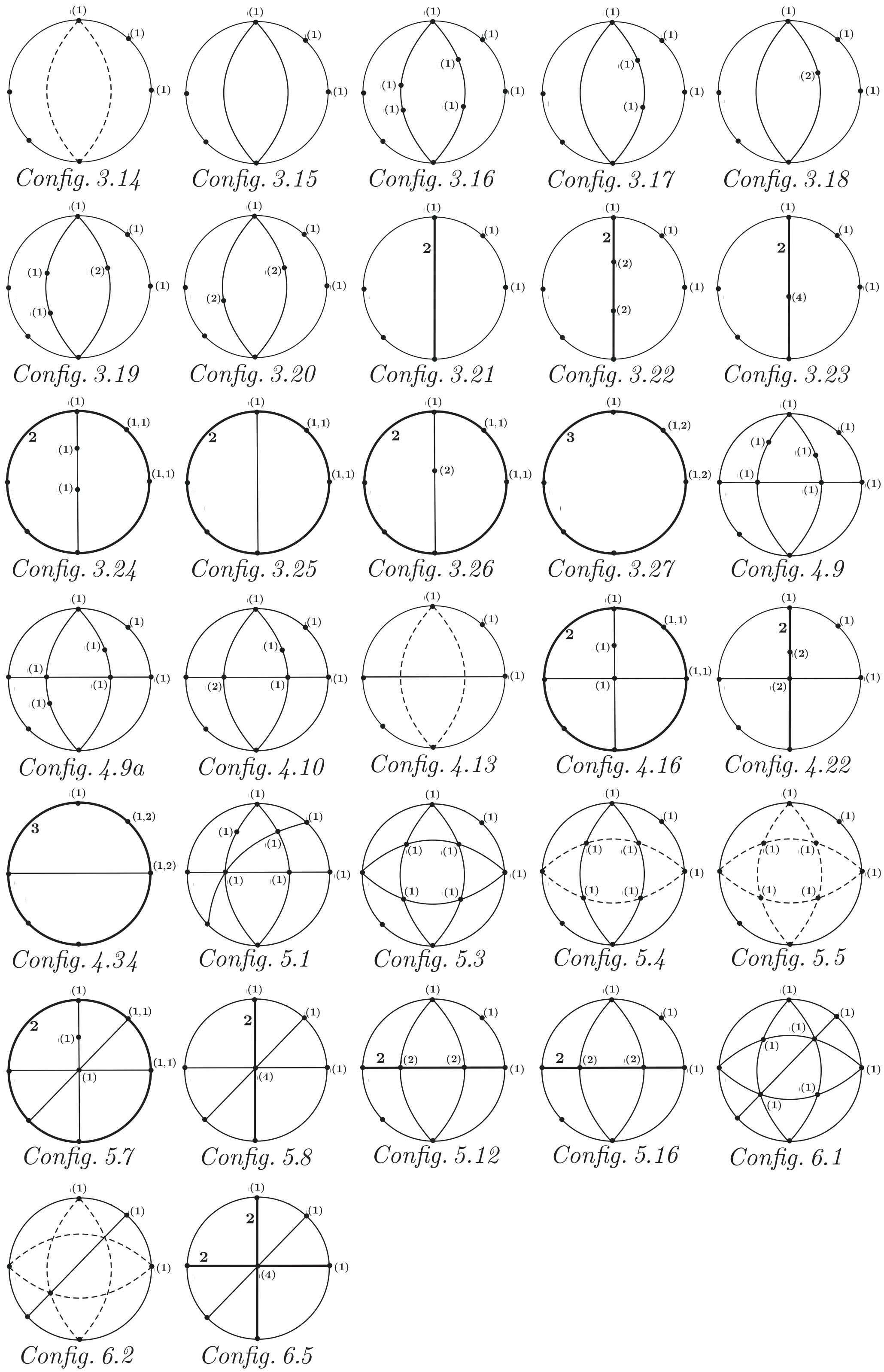

Figure 1: The configurations of quadratic systems in $\mathbf{QSL^{2p}}$ (case $\eta > 0$)

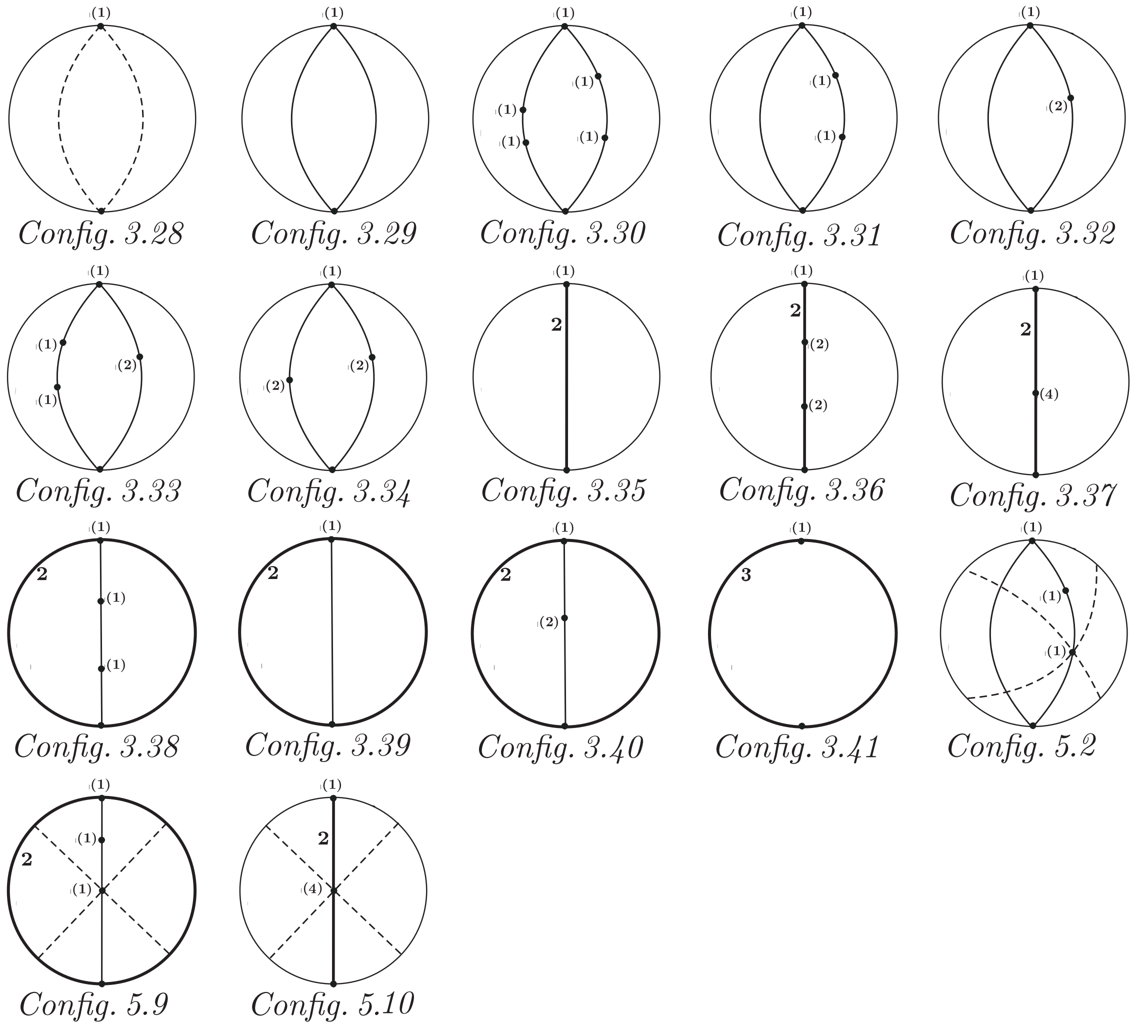

Figure 2: The configurations of quadratic systems in $\mathbf{QSL^{2p}}$ (case $\eta < 0$)

possessing the invariant affine lines $a+cx+gx^2=0$ going in the direction $x=0$. According to [21] in order to have invariant lines in other directions different from $x=0$ the condition $B_2=0$ is necessary. For the above systems we have

$$B_2 = -648b(g-1)^2\big[(b-a)(1+g)^2+c^2\big]x^4 \equiv -648b(g-1)^2\Phi(a,b,c,g)x^4 \qquad (33)$$

and hence these systems remains in the class $\mathbf{QSL_3}$ if $B_2 \neq 0$. So we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

6.1.1.1 The possibility $B_2 \neq 0$.

Then we have only the invariant lines $a+cx+gx^2=0$ and since $\mu_0 = g^2$ we examine two cqses: $\mu_0 \neq 0$ and $\mu_0 = 0$.

6.1.1.1.1 The case $\mu_0 \neq 0$. Then $g \neq 0$ and in order to get a more convenient normal form for the systems for investigation of the invariant lines and their multiplicities we apply the following translation to systems (32): $(x,y) \to \big(x-c/(2g), y+c(g-1)/(4g)\big)$.

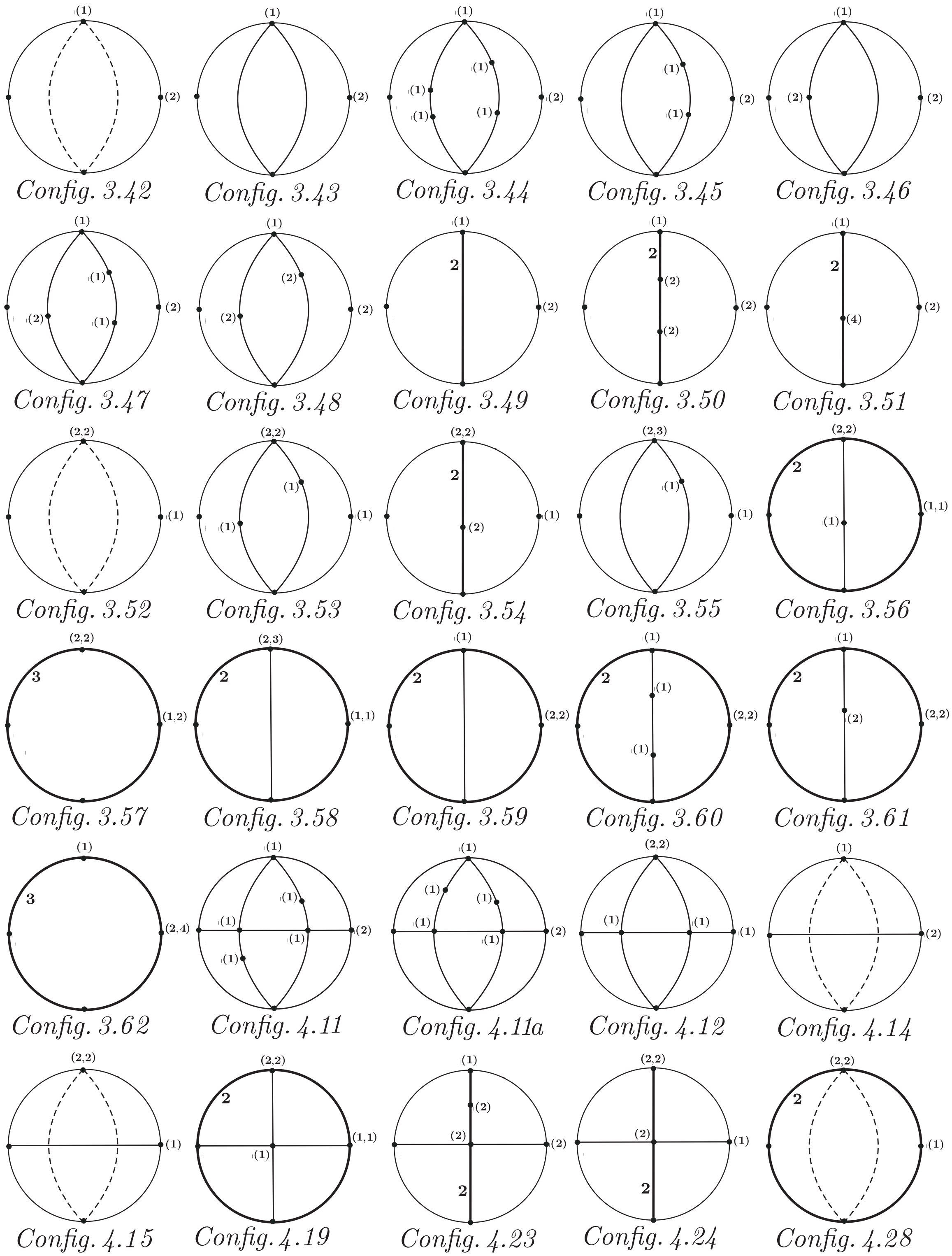

Figure 3: The configurations of quadratic systems in $\mathbf{QSL^{2p}}$ (case $\eta = 0 \neq \widetilde{M}$)

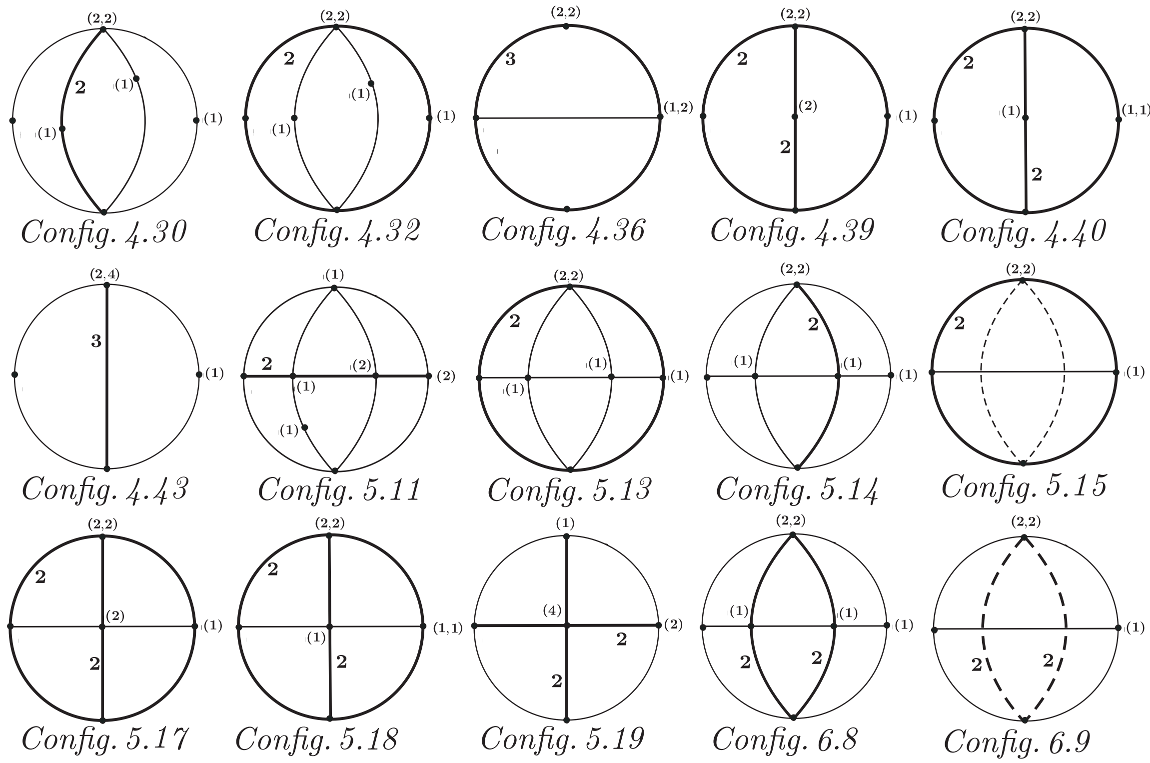

Figure 3 *(continuation)*: The configurations of quadratic systems in $\mathbf{QSL^{2p}}$ (case $\eta = 0 \neq \widetilde{M}$)

Then we arrive at the family of systems

$$\dot{x} = a + gx^2, \quad \dot{y} = b + ex + (g-1)xy + y^2, \tag{34}$$

which possess two parallel invariant lines $a + gx^2 = 0$ which are real (respectively complex; coinciding) if and only if $ag < 0$ (respectively $ag > 0$; $ag = 0$). For the above systems we calculate

$$\eta = 1, \ \mu_0 = g^2, \ \theta = 0, \ \widetilde{N} = (g^2-1)x^2, \ H_{10} = -32ag(g^2-1) \tag{35}$$

and we observe that $\widetilde{N}H_{10} = -32ag(g^2-1)^2x^2$. Hence $\operatorname{sign}(\widetilde{N}H_{10}) = -\operatorname{sign}(ag)$ and we consider three subcases: $\widetilde{N}H_{10} < 0$, $\widetilde{N}H_{10} > 0$ and $H_{10} = 0$ (as $\widetilde{N} \neq 0$).

1: *The subcase* $\widetilde{N}H_{10} < 0$. Then $ag > 0$ and we may assume $a = gu^2 \neq 0$. Moreover we may assume $u = 1$ due to the rescaling $(x, y, t) \to (ux, uy, t/u)$ in systems (34). Thus we arrive at the 3-parameter family of systems

$$\dot{x} = g(x^2+1), \quad \dot{y} = b + ex + (g-1)xy + y^2, \tag{36}$$

which clearly possess two complex invariant lines $x = \pm i$ and all the finite singularities also complex. Thus we get the configuration of invariant lines given by *Config. 3.14* (see Figure 1).

2: *The subcase* $\widetilde{N}H_{10} > 0$. Then $ag < 0$ and we may assume $a = -gu^2 \neq 0$. Moreover we may assume $u = 1$ due to the rescaling $(x, y, t) \to (ux, uy, t/u)$ in systems systems (34)

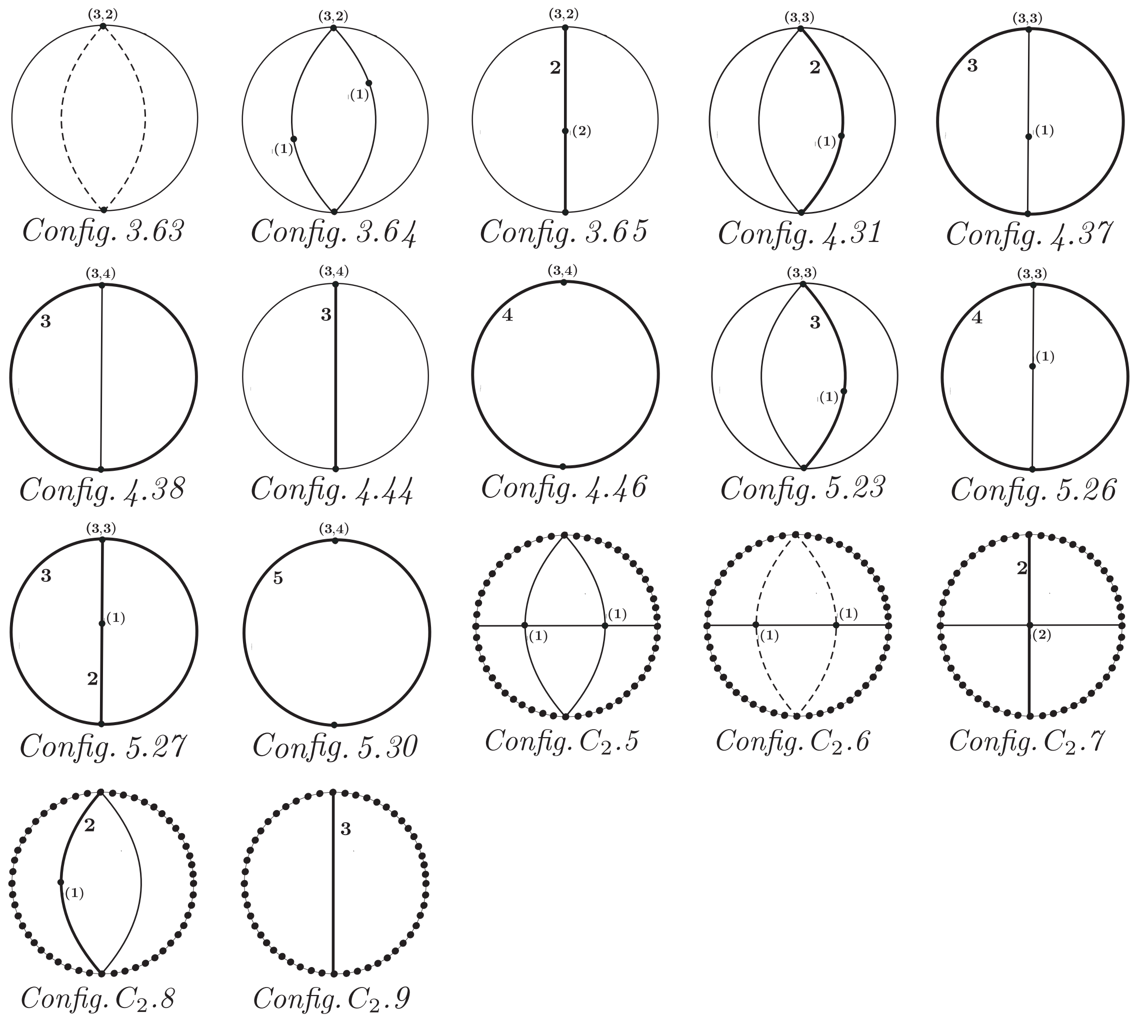

Figure 4: The configurations of quadratic systems in $\mathbf{QSL^{2p}}$ (case $\eta = \widetilde{M} = 0$)

with $a = -gu^2$. Thus we arrive at the 3-parameter family of systems

$$\dot{x} = g(x^2 - 1), \quad \dot{y} = b + ex + (g-1)xy + y^2, \tag{37}$$

possessing two real invariant lines $x = \pm 1$ and the finite singularities $M_{1,2}(x_{1,2}, y_{1,2})$ and $M_{3,4}(x_{3,4}, y_{3,4})$, where

$$\begin{aligned} &x_{1,2} = 1,\ y_{1,2} = \big(1 - g \pm \sqrt{V_1}\,\big)/2; \qquad V_1 = (g-1)^2 - 4(b+e);\\ &x_{3,4} = -1,\ y_{3,4} = \big(g - 1 \pm \sqrt{V_2}\,\big)/2; \quad V_2 = (g-1)^2 - 4(b-e). \end{aligned} \tag{38}$$

On the other hand for the above systems we have $\mathbf{D} = -768g^6V_1V_2$ and $H_{15} = 128g^4(V_1 + V_2)$. So we examine three possibilities: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

2.1: *The possibility* $\mathbf{D} < 0$. Then we obtain $V_1V_2 > 0$ and since $\operatorname{sign}(V_1 + V_2) = \operatorname{sign}(H_{15})$ we discuss two cases $H_{15} < 0$ and $H_{15} > 0$.

2.1.1: *The case* $H_{15} < 0$. Then $V_1 < 0$ and $V_2 < 0$ and clearly all 4 finite singularities (38) are complex. As a result we get the configuration given by *Config. 3.15*.

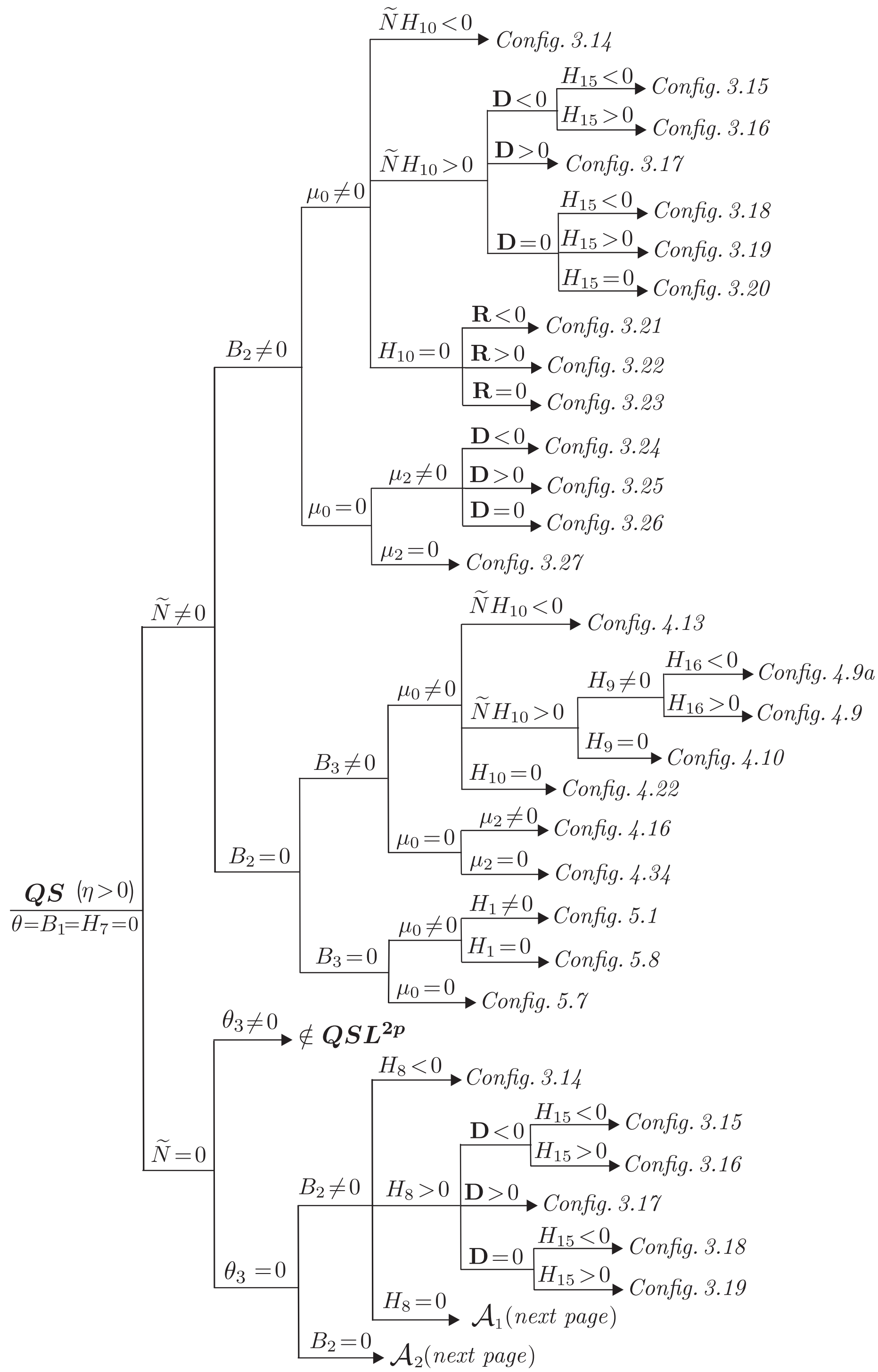

Diagram 1: The invariant criteria for configurations of systems in $\mathbf{QSL^{2p}}$ (case $\eta > 0$)

2.1.2: *The case* $H_{15} > 0$. Then $V_1 > 0$ and $V_2 > 0$ and this implies the existence of 4 real distinct finite singularities. Therefore we obtain the configuration *Config. 3.16*.

2.2: *The possibility* $\mathbf{D} > 0$. In this case we obtain $V_1V_2 < 0$ and without losing the

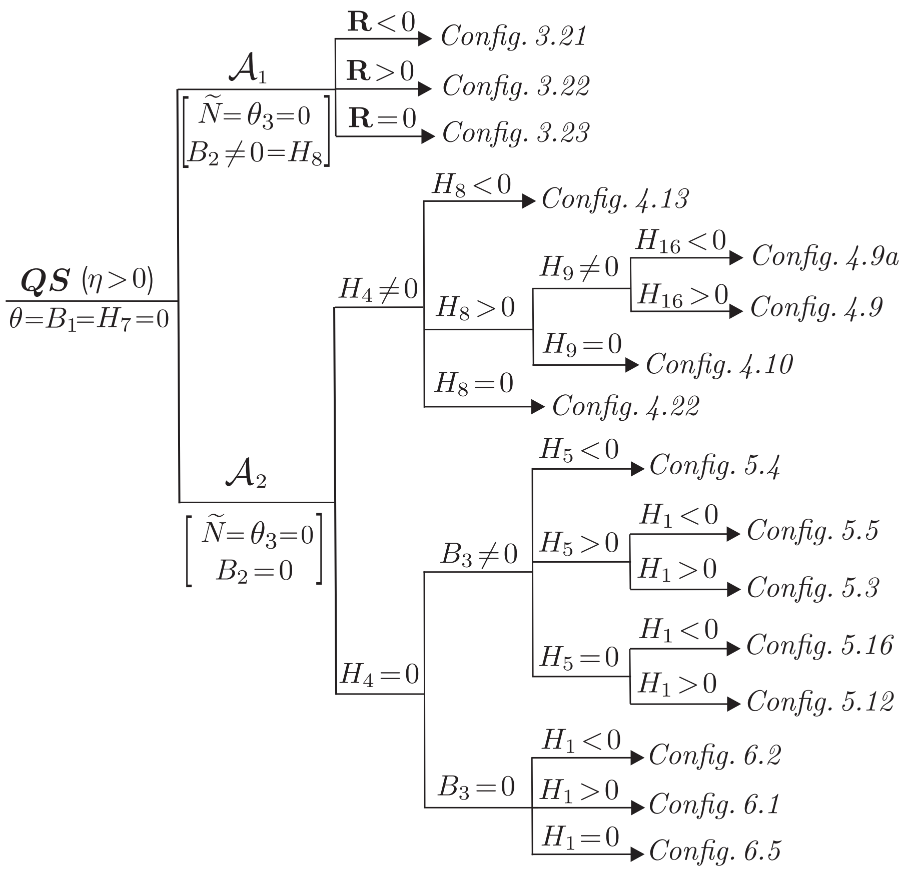

Diagram 1 *(continuation)*: The invariant criteria for configurations of systems in $\mathbf{QSL^{2p}}$ (case $\eta > 0$)

generality we may assume $V_1 > 0$ due to the rescaling $(x, y, t) \to (-x, -y, -t)$ which transfer V_2 to V_1. So on the line $x = 1$ we have two real singularities whereas on the line $x = -1$ are located two complex singular points. In such a way we arrive at the configuration *Config. 3.17*.

2.3: *The possibility* $\mathbf{D} = 0$. Then we obtain $V_1V_2 = 0$ and again without losing the generality we may assume $V_1 = 0$ due to the rescaling $(x, y, t) \to (-x, -y, -t)$ which transfer V_2 to V_1. In this case we have $\text{sign}\,(V_2) = \text{sign}\,(H_{15})$ if $V_2 \neq 0$. Therefore on the line $x = 1$ we have a double real singularity, whereas on the line $x = -1$ there are located either two complex singular points (if $H_{15} < 0$) or two real ones (if $H_{15} > 0$) or one real double (if $H_{15} = 0$) singularity, on this line. We note that the condition $\mathbf{D} = H_{15} = 0$ implies $V_1 = V_2 = 0$ and this leads to the conditions $b = (g-1)^2/4$ and $e = 0$ and then we obtain $B_2 = -81(g^2-1)^4x^4/2$, i.e. the condition $B_2 \neq 0$ is still valid.

Thus in the case $\mathbf{D} = 0$ we get three configurations: *Config. 3.18* if $H_{15} < 0$; *Config. 3.19* if $H_{15} > 0$ and *Config. 3.20* if $H_{15} = 0$.

3: *The subcase* $H_{10} = 0$. Considering (35) we get $a = 0$ and then systems (34) become:

$$\dot{x} = gx^2, \quad \dot{y} = b + ex + (g-1)xy + y^2, \tag{39}$$

possessing the double invariant line $x = 0$ and two double singular points $M_{1,2}(0, \pm\sqrt{-b})$.

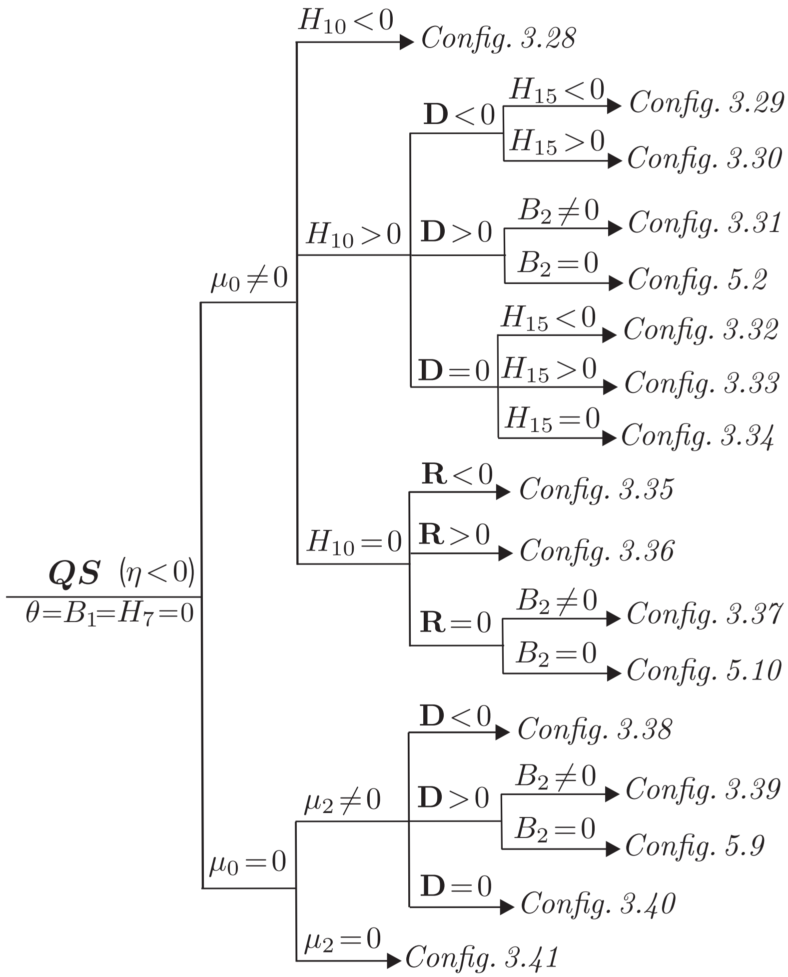

Diagram 2: The invariant criteria for configurations of systems in $\mathbf{QSL^{2p}}$ (case $\eta < 0$)

On the other hand for these systems we have $\mathbf{R} = -16bg^4x^2$ and hence $\operatorname{sign}(\mathbf{R}) = -\sin(b)$ in the case $\mathbf{R} \neq 0$ and due to $\mu_0 \neq 0$ (i.e. $g \neq 0$) the condition $b = 0$ is equivalent to $\mathbf{R} = 0$. In this last case when $b = 0$ for systems (39) we get $B_2 = -648e^4x^4 \neq 0$ and this means that these systems are not homogeneous.

Therefore we deduce that in the case $H_{10} = 0$ we arrive at the following three configurations: *Config. 3.21* if $\mathbf{R} < 0$; *Config. 3.22* if $\mathbf{R} > 0$ and *Config. 3.23* if $\mathbf{R} = 0$.

6.1.1.1.2 The case $\mu_0 = 0$. Then $g = 0$ and systems (32) become:

$$\dot{x} = a + cx, \quad \dot{y} = b - xy + y^2 \tag{40}$$

for which we calculate $\mu_0 = \mu_1 = 0$ and $\mu_2 = -c^2(x - y)y$. So we examine two subcases: $\mu_2 \neq 0$ and $\mu_2 = 0$.

1: *The subcase $\mu_2 \neq 0$.* Then $c \neq 0$ and hence the above systems possess one real invariant affine line $cx + a = 0$. Considering Lemma 4.7 we deduce that the infinite line is a double one.

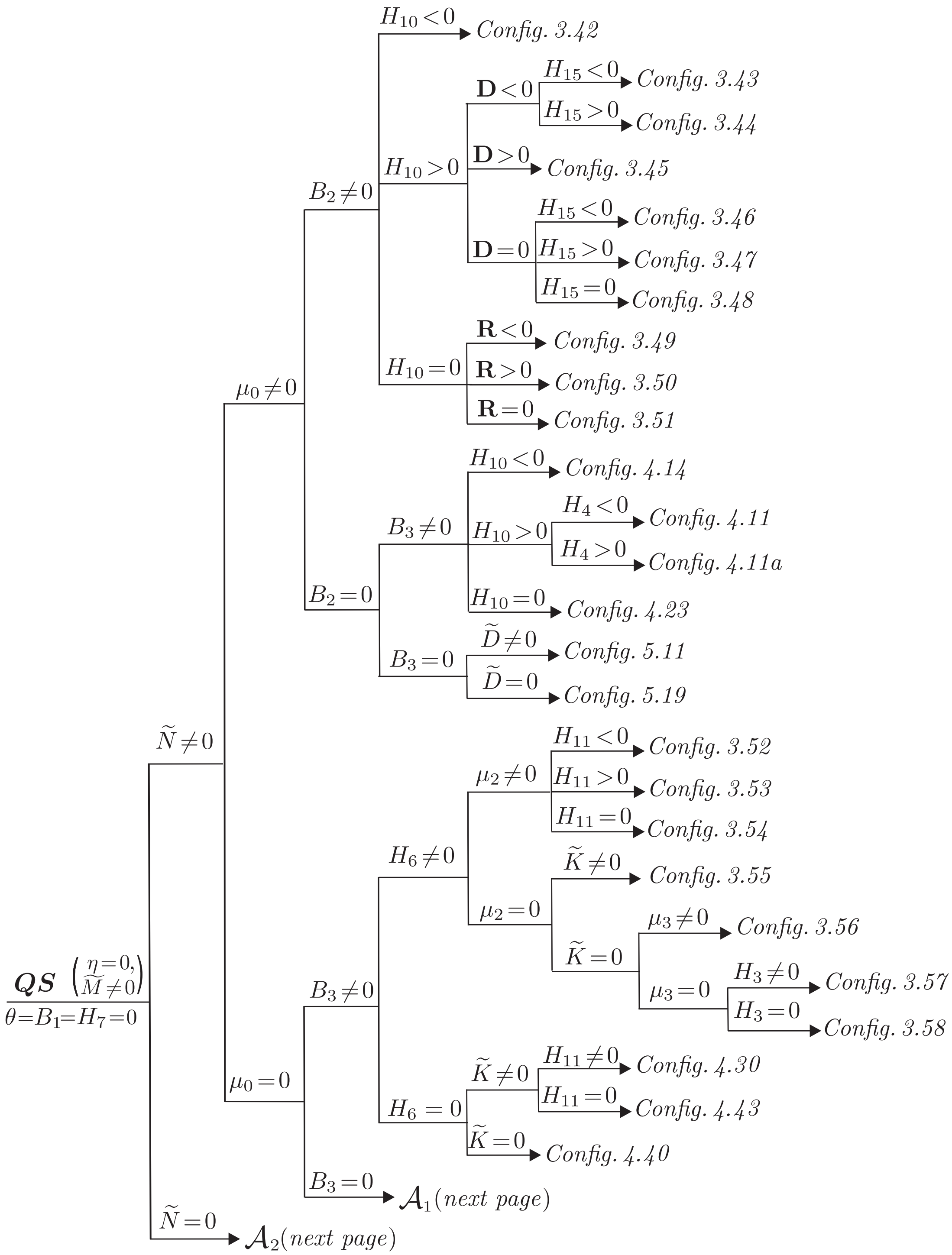

Diagram 3: The invariant criteria for configurations of systems in $\mathbf{QSL^{2p}}$ (case $\eta = 0 \neq \widetilde{M}$)

Since $c \neq 0$ we may assume $c = 1$ due to the rescaling $(x, y, t) \to (cx, cy, t/c)$ and we arrive at the 2-parameter family of systems

$$\dot{x} = a + x, \quad \dot{y} = b - xy + y^2 \tag{41}$$

,

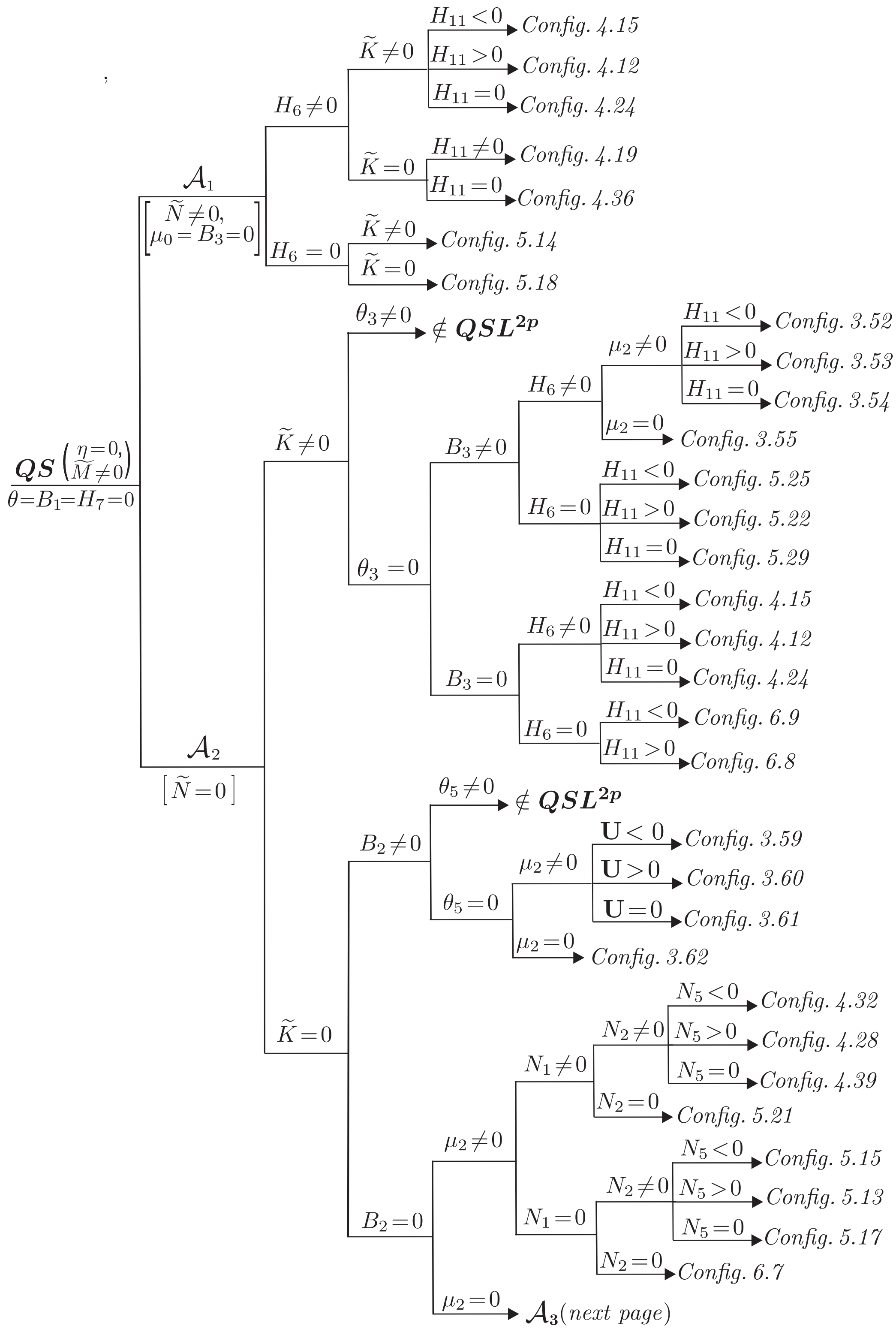

Diagram 3 *(continuation)*: The invariant criteria for configurations of systems in $\mathbf{QSL^{2p}}$ (case $\eta=0\neq\widetilde{M}$)

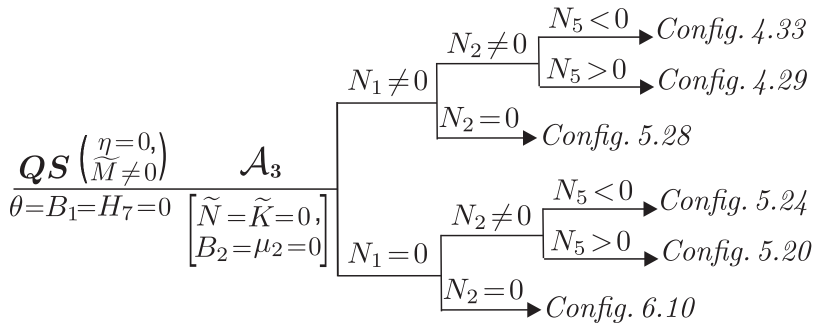

Diagram 3 *(continuation)*: The invariant criteria for configurations of systems in $\mathbf{QSL^{2p}}$ (case $\eta = 0 \neq \widetilde{M}$)

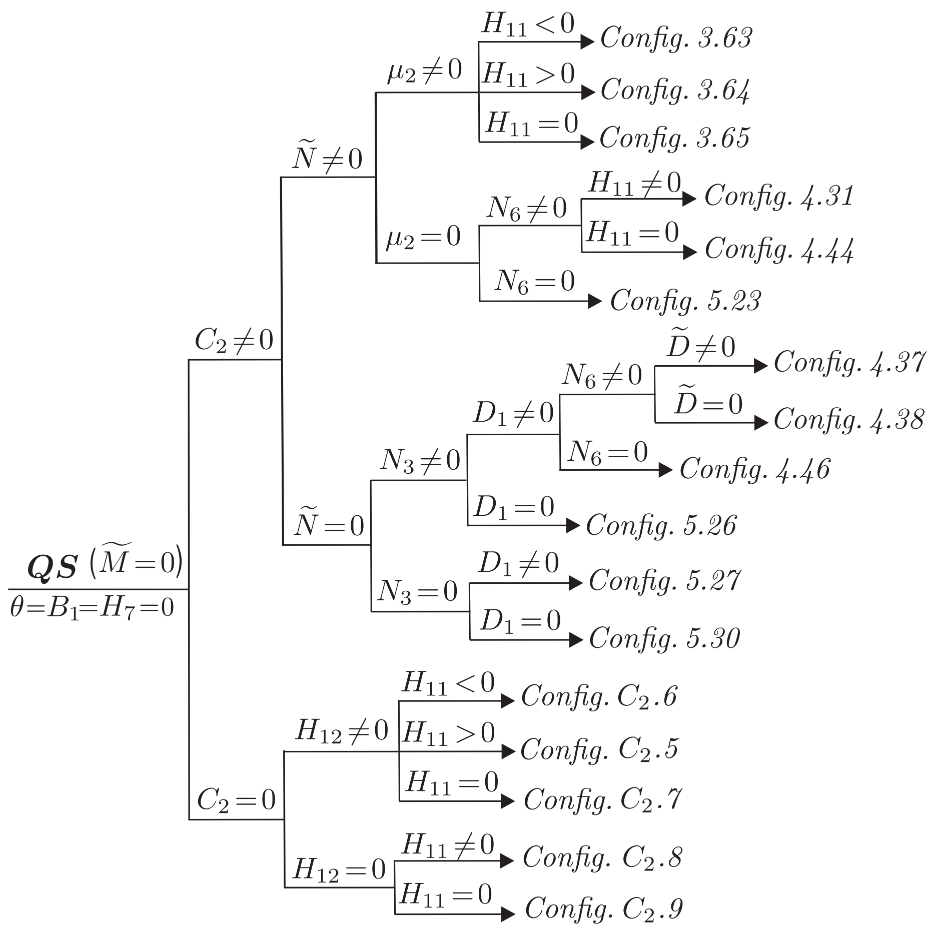

Diagram 4: The invariant criteria for configurations of systems in $\mathbf{QSL^{2p}}$ (case $\eta = 0 = \widetilde{M}$)

which possess two finite singularities $M_{1,2}(x_{1,2}, y_{1,2})$ where

$$x_{1,2} = -a, \quad \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right).$$

We underline that other two finite singularities have gone to infinity and according to [3, Lemma 5.2] (since $\mu_2 = -(x-y)y$) they coalesced with two different infinite singularities of the lines $y = 0$ and $y = x$. As a result we get two infinite double singularities of the type

$(1,1)$ (i.e. one finite and one infinite singularities coalesced).

On the other hand for systems (41) we have $\mathbf{D} = -48(a^2 - 4b)$ and therefore $\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(a^2 - 4b)$, i.e. this invariant polynomial $\mathbf{D}$ is responsible for the kind of the finite singularities: real, complex or coinciding.

Since $B_2 \neq 0$ as we mentioned earlier systems (41) could not possess invariant affine lines in the directions different from $x = 0$. So considering the double infinite invariant line, we arrive at the configuration *Config. 3.24* if $\mathbf{D} < 0$ (then the finite singularities are real distinct); at the configuration *Config. 3.25* if $\mathbf{D} > 0$ (then the finite singularities are complex) and at the configuration *Config. 3.26* if $\mathbf{D} = 0$ (then we have a double real singularity).

2: *The subcase* $\mu_2 = 0$. Then $c = 0$ and systems (40) become

$$\dot{x} = a, \quad \dot{y} = b - xy + y^2 \tag{42}$$

which do not possess finite singularities because $a \neq 0$, otherwise the systems are degenerate. For these systems we have $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 = a^2y^2(x-y)^2$ and according to [3, Lemma 5.2] the four finite singularities have gone to infinity and two of them coalesced with the infinite singularities $N_1[1:0:0]$ (the point of the line $y = 0$), whereas the other two coalesced with the infinite singularities $N_2[1:1:0]$ (the point of the line $y = x$). As a result we obtain at infinity two triple singularities of the type $(1,3)$.

We point out that by Lemma 4.7 the line at infinity $Z = 0$ of the systems (42) is of multiplicity 3. So since $B_2 \neq 0$ we could not have invariant affine lines and hence we get the configuration *Config. 3.27.*

6.1.1.2 The possibility $B_2 = 0$.

Considering (33) and $\widetilde{N} \neq 0$ (i.e. $g - 1 \neq 0$) we get $b\Phi(a,b,c,g) = 0$. We claim that due to an affine transformation in systems (32) we could transfer the condition $\Phi = 0$ to the condition $b = 0$.

Indeed, applying to systems (32) the affine transformation $x_1 = x$, $y_1 = x - y$ we arrive at the systems

$$\dot{x}_1 = a_1 + c_1 + g_1x_1^2, \quad \dot{y}_1 = b_1 + (g_1 - 1)x_1y_1 + y_1^2, \tag{43}$$

where

$$\begin{aligned} &a_1 = -a + \frac{2c^2\left(g^2+1\right)}{(g+1)^4},\ g_1 = -g,\ b_1 = b - a + \frac{c^2}{(g+1)^2},\ c_1 = \frac{c(g-1)^2}{(g+1)^2} \ \Rightarrow \\ &a = -a_1 + \frac{2{c_1}^2\left(g_1^2+1\right)}{(g_1+1)^4},\ g = -g1,\ b = b_1 - a_1 + \frac{c_1^2}{(g_1+1)^2},\ c = \frac{c_1(g_1-1)^2}{(g_1+1)^2}. \end{aligned} \tag{44}$$

Then substituting the expressions for the old parameters we obtain

$$\begin{aligned} \Phi(a,b,e,g) =&(b-a)(1+g)^2 + c^2 = \Big[(b_1 - a_1 + \frac{c_1^2}{(g_1+1)^2}) - (-a_1 + \frac{2{c_1}^2\left(g_1^2+1\right)}{(g_1+1)^4})\Big](1-g_1)^2 \\ &+ \Big[\frac{c_1(g_1-1)^2}{(g_1+1)^2}\Big]^2 = b_1(g_1-1)^2, \quad (g^2-1) = (g_1^2-1) \neq 0. \end{aligned}$$

and this completes the proof of our claim.

So considering the condition $b = 0$ we get the family of systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = y\left[(g-1)x + y\right], \tag{45}$$

possessing an additional invariant line $y = 0$, i.e. these systems belong to the class $\mathbf{QSL_j}$ with $j \geq 4$. So taking into consideration the article [23] (see Table 2) we evaluate the following invariant polynomials:

$$\begin{gathered}
\eta = 1 > 0,\ B_2 = H_7 = \theta = 0,\ \mu_0 = g^2,\ \widetilde{N}H_{10} = 8(g^2-1)^2(c^2-4ag)x^2,\\
H_4 = -48(g-1)\big[c^2 - a(1+g)^2\big],\ B_3 = 3\big[c^2 - a(1+g)^2\big]x^2y^2,\\
H_9 = -576a^2(g-1)^4(c^2-4ag)^2
\end{gathered} \tag{46}$$

and we examine two cases: $B_3 \neq 0$ and $B_3 = 0$.

6.1.1.2.1 The case $B_3 \neq 0$. In this case we observe that this condition implies $H_4 \neq 0$ and since we have $\eta > 0$ and $H_4 \neq 0$, according to [23, Table 2] we deduce that systems (45) possess the following configurations of if and only if the corresponding conditions are satisfied, respectively:

$$\begin{array}{lcl}
\mu_0 \neq 0, \widetilde{N}H_{10} < 0 & \Leftrightarrow & \textit{Config. 4.13};\\
\mu_0 \neq 0, \widetilde{N}H_{10} > 0, H_9 \neq 0 & \Leftrightarrow & \textit{Config. 4.9};\\
\mu_0 \neq 0, \widetilde{N}H_{10} > 0, H_9 = 0 & \Leftrightarrow & \textit{Config. 4.10};\\
\mu_0 \neq 0, \widetilde{N}H_{10} = 0 & \Leftrightarrow & \textit{Config. 4.22};\\
\mu_0 = 0, \mu_2 \neq 0, H_9 \neq 0 & \Leftrightarrow & \textit{Config. 4.16};\\
\mu_0 = 0, \mu_2 \neq 0, H_9 = 0 & \Leftrightarrow & \textit{Config. 4.17};\\
\mu_0 = 0, \mu_2 = 0 & \Leftrightarrow & \textit{Config. 4.34};
\end{array}$$

Remark 6.2. *We point out that in the case $\mu_0 \neq 0$, $\widetilde{N}H_{10} > 0$ and $H_9 \neq 0$ in paper [23] it is presented only one configuration Config. 4.9. However depending on the position of the finite singularities of systems* (45) *with respect to the invariant line $e+(g-1)y = 0$ we could have another configuration different from Config. 4.9 which we denote here by Config. 4.9a (see Figure 1).*

Next we detect the invariant condition for distinguishing these two configurations. We prove the following lemma:

Lemma 6.1. *Assume that for an arbitrary quadratic system the conditions $\eta > 0$, $\theta = H_7 = B_2 = 0$, $\mu_0 B_3 H_4 H_9 \neq 0$ and $\widetilde{N}H_{10} > 0$ are satisfied. Then the configuration of the invariant lines of this system is Config. 4.9a if $H_{16} < 0$ and Config. 4.9 if $H_{16} > 0$.*

Proof: If for a quadratic system the conditions provided by the lemma hold then according to [23, Theorem 4.1, statement (ii)] this system could be brought via an affine transformation and time rescaling to the canonical form

$$\dot{x} = x^2 - 1, \quad \dot{y} = (y+b)\big[y + (1-l)x - b\big], \quad b, l \in \mathbb{R}, \quad (l-1)\big[(l \pm 1)^2 - 4b^2\big] \neq 0.$$

However applying the transformation $(x,y,t,b,l) \to (-x,-y-b,-t,b/2,1-a)$ to the above systems we could simplify them and we arrive at the family of systems

$$\dot{x} = x^2 - 1, \quad \dot{y} = y(y + ax + b), \tag{47}$$

for which we calculate:

$$\eta = (a-1)^2,\ \theta = H_7 = B_2 = 0,\ \mu_0 = 1,\ \widetilde{N} H_{10} = 32a^2(a-2)^2x^2,$$
$$H_4 = -48a\big[(a-2)^2 - b^2\big],\ H_9 = -9216(a^2-b^2)^2,\ B_3 = 3\big[(a-2)^2 - b^2\big]x^2y^2.$$

Thus according to the conditions provided by Lemma 6.1 for the new canonical systems (47) the following condition must be satisfied:

$$a(a-1)(a-2)(a^2-b^2)\big[(a-2)^2-b^2\big] \neq 0.$$

We observe that the above systems posses the invariant lines $x = \pm 1$ and $y = 0$ and the following finite singularities:

$$M_1(-1,0), \quad M_2(-1,a-b), \quad M_3(1,0), \quad M_4(1,-a-b).$$

It is clear that the singularities M_2 and M_4 are located on the invariant lines $x = -1$ and $x = 1$, respectively. And it is important to detect if they are located on the same part with respect to the invariant line $y = 0$ (*Config. 4.9*) or on different parts (*Config. 4.9a*). So in the first case the product of the corresponding ordinates must be positive, i.e. $(a+b)(b-a) > 0$, whereas in the second case it must be negative, i.e. $(a+b)(b-a) < 0$.

On the other hand for systems (47) we have

$$H_{16} = 180(a+b)(b-a)\big[(a-2)^2-b^2\big]^2 \quad \Rightarrow \quad \mathrm{sign}\,(H_{16}) = \mathrm{sign}\,\big((a+b)(b-a)\big)$$

and hence we arrive at the configuration *Config. 4.9a* if $H_{16} < 0$ and *Config. 4.9* if $H_{16} > 0$. This completes the proof of Lemma 6.1. ∎

6.1.1.2.2 The case $B_3 = 0$. Considering (46) this condition gives $a = \dfrac{c^2}{(g+1)^2}$ and hence we get the family of systems

$$\begin{aligned}\dot{x} =& \frac{1}{(g+1)^2}\big[(g+1)x + c\big]\big[g(g+1)x + c\big],\\ \dot{y} =& y\left[(g-1)x + y\right],\end{aligned} \tag{48}$$

which possess four invariant affine lines:

$$(g+1)x + c = 0,\ g(g+1)x + c = 0,\ y = 0,\ (g+1)(x-y) + c = 0.$$

Considering the invariant line at infinity $Z = 0$ we conclude that these systems belong to the class $\mathbf{QSL_5}$. In this case following [21] we calculate.

$$\eta = 1 > 0,\ B_3 = \theta = 0,\ \mu_0 = g^2,\ \widetilde{N} = (g^2-1)x^2,\ H_1 = 576c^2(g-1)^2/(g+1)^2,\ H_6 = 0.$$

Since $\widetilde{N} \neq 0$, according to [21, Table 4] in the case $\mu_0 \neq 0$ we obtain *Config. 5.1* if $H_1 \neq 0$ and *Config. 5.8* if $H_1 = 0$, whereas for $\mu_0 = 0$ we get the unique configuration *Config. 5.7.*

6.1.2 The subcase $\widetilde{N} = 0$

According to Theorem 5.1 in this case the condition $\theta_3 = 0$ must be satisfied and taking into account the proof of this theorem we consider the family of systems (14), i.e. the systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + ex + y^2. \tag{49}$$

for which we have

$$B_2 = 648e^2(4a - 4b - e^2)x^4,\ B_3 = -3x^2\big[e^2x^2 - 2e^2xy + 4(a-b)y^2\big],\ H_4 = 96e^2. \tag{50}$$

We consider two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

6.1.2.1 The possibility $B_2 \neq 0$. In this case we could not have invariant affine lines different from $x^2 + a = 0$. Since for the above systems we have $H_8 = -3456ae^2$ we examine three cases: $H_8 < 0$, $H_8 > 0$ and $H_8 = 0$.

6.1.2.1.1 The case $H_8 < 0$. Then $a > 0$ and hence we have two complex invariant lines and only complex finite singularities. This evidently leads to the configuration *Config. 3.14*

6.1.2.1.2 The case $H_8 > 0$. This condition implies $a < 0$ and we may set $a = -u^2 \neq 0$. So assuming $u = 1$ due to the rescaling $(x, y, t) \to (ux, uy, t/u)$ we arrive at the family of systems

$$\dot{x} = x^2 - 1, \quad \dot{y} = b + ex + y^2. \tag{51}$$

We observe that this family of systems is a subfamily of (37) defined by the condition $g = 1$. So we follow the same steps as in the examination of the family (37) but taking into consideration the condition $g = 1$. Therefore since $B_2 \neq 0$ in this particular case (i.e. $g = 1$) we get the same configurations obtained for systems (37) in the generic case with one exception. More precisely since for systems (51) we have

$$\mathbf{D} = -12288(b - e)(b + e), \quad H_{15} = -1024b$$

then the condition $\mathbf{D} = H_{15} = 0$ is impossible, otherwise we get $b = e = 0$ and this implies $B_2 = 0$ which contradicts our assumption.

Thus we have the next remark.

Remark 6.3. *Assume that for systems* (49) *the condition $B_2 \neq 0$ and $H_8 > 0$ hold. Then these systems possess the indicated configurations of invariant lines if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\textit{Config. 3.15} & \Leftrightarrow & \mathbf{D} < 0 \quad H_{15} < 0; \\
\textit{Config. 3.16} & \Leftrightarrow & \mathbf{D} < 0 \quad H_{15} > 0; \\
\textit{Config. 3.17} & \Leftrightarrow & \mathbf{D} > 0; \\
\textit{Config. 3.18} & \Leftrightarrow & \mathbf{D} = 0 \quad H_{15} < 0; \\
\textit{Config. 3.19} & \Leftrightarrow & \mathbf{D} = 0 \quad H_{15} > 0.
\end{array}$$

6.1.2.1.3 The case $H_8 = 0$. Then $a = 0$ and we arrive at the family of systems

$$\dot{x} = x^2, \quad \dot{y} = b + ex + y^2, \tag{52}$$

which is a subfamily of (39) defined by the condition $g = 1$. So following step by step the three subcases $\mathbf{R} < 0$, $\mathbf{R} > 0$ and $\mathbf{R} = 0$ examined for systems (39) we arrive at the next remark.

Remark 6.4. *Assume that for systems* (49) *the condition $B_2 \neq 0$ and $H_8 = 0$ hold. Then these systems possess the indicated configurations of invariant lines if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{aligned} &\textit{Config. 3.21} \quad \Leftrightarrow \quad \mathbf{R} < 0; \\ &\textit{Config. 3.22} \quad \Leftrightarrow \quad \mathbf{R} > 0; \\ &\textit{Config. 3.23} \quad \Leftrightarrow \quad \mathbf{R} = 0. \end{aligned}$$

6.1.2.2 The possibility $B_2 = 0$. Considering (50) this condition implies $e(4a - 4b - e^2) = 0$ and since $H_4 = 96e^2$ we consider two cases: $H_4 \neq 0$ and $H_4 = 0$.

6.1.2.2.1 The case $H_4 \neq 0$. Then $e \neq 0$ and the condition $B_2 = 0$ gives $4a - 4b - e^2 = 0$, i.e. $b = a - e^2/4$ and we arrive at the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = a - e^2/4 + ex + y^2. \tag{53}$$

These systems possess three invariant affine lines: $x^2 + a = 0$ and $2(x - y) - e = 0$, i.e. these systems possess invariant lines of total multiplicity at least 4. On the other hand calculations yield $B_3 = -3e^2x^2(x - y)^2 \neq 0$ due to $H_4 \neq 0$. This means that systems (53) belong to the class $\boldsymbol{QSL_4}$ and following [23] we calculate

$$\eta = 1 > 0, \ B_2 = \widetilde{N} = 0, \ H_4 = 96e^2, \ H_8 = -3456ae^2.$$

We observe that the condition $B_3 \neq 0$ implies $H_4 \neq 0$ and in the case $H_8 \neq 0$ we have $\operatorname{sign}(H_8) = -\operatorname{sign}(a)$. So according to [23, Table 2] we obtain the configuration *Config. 4.9* if $H_8 > 0$ and $H_9 \neq 0$; *Config. 4.10* if $H_8 > 0$ and $H_9 = 0$; *Config. 4.13* if $H_8 < 0$ and *Config. 4.22* if $H_8 = 0$.

However for $\widetilde{N} = 0$ we also have to include the omitted configuration *Config. 4.9a* as we did in the case $\widetilde{N} \neq 0$ (see Lemma 6.1).

We prove the next lemma.

Lemma 6.2. *Assume that for an arbitrary quadratic system the conditions $\eta > 0$, $\theta = H_7 = B_2 = 0$, $\mu_0 B_3 H_4 H_9 \neq 0$, $\widetilde{N} = 0$ and $H_8 > 0$ are satisfied. Then the configuration of the invariant lines of this system corresponds to Config. 4.9a if $H_{16} < 0$ and to Config. 4.9 if $H_{16} > 0$.*

Proof: If for a quadratic system the conditions provided by the above lemma hold, then as it was shown above this system could be brought via an affine transformation to the form (53), for which the condition $H_8 > 0$ and $B_3 \neq 0$ implies $a < 0$. Then we may consider

$a = -u^2 \neq 0$ and applying the rescaling $(x, y, t) \mapsto (ux, uy, t/u)$ we may assume $a = -1$ and we arrive at the 1-parameter family of systems

$$\dot{x} = x^2 - 1, \quad \dot{y} = -1 - e^2/4 + ex + y^2 \tag{54}$$

possessing three affine invariant lines: $x = \pm 1$ and $L(x, y) = 2x - 2y - e = 0$. The above systems have the following four real finite singularities:

$$M_1(-1, -(e+2)/2), \quad M_2(-1, (e+2)/2), \quad M_3(1, (2-e)/2), \quad M_4(1, (e-2)/2).$$

We observe that the singularity M_1 (respectively M_3) is located at the intersection of the invariant line $x = -1$ (respectively $x = 1$) with the invariant line $L(x, y) = 0$. Now it is important to detect if the remaining singularities M_2 and M_4 are located on the same part with respect to the invariant line $L(x, y) = 0$ (*Config. 4.9*) or on the different parts (*Config. 4.9a*). So in the first case the product $L(x_2, y_2)L(x_4, y_4)$ of the corresponding ordinates must be positive, whereas in the second case it must be negative. We calculate

$$L(x_2, y_2)L(x_4, y_4) = 4(e-2)(e+2).$$

On the other hand for systems (47) we have

$$H_{16} = 180(e-2)e^4(2+e) \quad \Rightarrow \quad \text{sign}\,(H_{16}) = \text{sign}\,\big((e-2)(e+2)\big)$$

and hence we arrive at the configuration *Config. 4.9a* if $H_{16} < 0$ and *Config. 4.9* if $H_{16} > 0$. This completes the proof of Lemma 6.2.

6.1.2.2.2 The case $H_4 = 0$. Then $e = 0$ (this implies $B_2 = 0$) and we get the following 2-parameter family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + y^2 \tag{55}$$

which evidently possess at list five invariant lines, including the line at infinity. Following [26] for these systems we calculate

$$\eta = 1 > 0,\ B_2 = \widetilde{N} = H_4 = 0,\ B_3 = -12(a-b)x^2y^2,\ H_5 = 6144ab,\ H_1 = -1152(a+b)$$

and according to [21] (see Table 4, Table 2, Diagram 2 and Diagram 1) we arrive at the following configurations and phase portraits of systems (55) if and only if the corresponding conditions are satisfied, respectively:

$$\begin{array}{lcl}
B_3 \neq 0,\ H_5 < 0 & \Leftrightarrow & \textit{Config. 5.4};\\
B_3 \neq 0,\ H_5 > 0,\ H_1 < 0 & \Leftrightarrow & \textit{Config. 5.5};\\
B_3 \neq 0,\ H_5 > 0,\ H_1 > 0 & \Leftrightarrow & \textit{Config. 5.3};\\
B_3 \neq 0,\ H_5 = 0,\ H_1 < 0 & \Leftrightarrow & \textit{Config. 5.16};\\
B_3 \neq 0,\ H_5 = 0,\ H_1 > 0 & \Leftrightarrow & \textit{Config. 5.12};\\
B_3 = 0,\ H_1 < 0 & \Leftrightarrow & \textit{Config. 6.2};\\
B_3 = 0,\ H_1 > 0 & \Leftrightarrow & \textit{Config. 6.1};\\
B_3 = 0,\ H_1 = 0 & \Leftrightarrow & \textit{Config. 6.5}.
\end{array}$$

Thus we completed the proof of the statement (i) of Theorem 6.1.

6.2 The case $\eta < 0$

According to the statement (ii) of Theorem 5.1 in this case the condition $\widetilde{N} \neq 0$ must hold. So we consider the family of systems (17) with $d = 0$, i.e. the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2. \tag{56}$$

with $C_2 = x(x^2+y^2)$. So we conclude that we can have real affine invariant lines only in the direction $x = 0$ and in the case $B_2 = 0$ (which is the necessary condition for the existence of invariant lines in the second direction) we could have only complex conjugate invariant lines in the directions $y = \pm ix$.

For these systems calculation yield

$$\eta = -4 < 0,\ \mu_0 = g^2,\ \theta = 0,\ H_7 = 0$$

and we examine two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

6.2.1 The subcase $\mu_0 \neq 0$

Then $g \neq 0$ and we may assume $c = 0$ due to the translation $(x, y) \to (x - c/(2g), y - c/4)$ which conserves the second equation of the above systems (56). So we get the family of systems

$$\dot{x} = a + gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2, \quad g \neq 0, \tag{57}$$

for which calculations yield

$$B_2 = -648\big[e^4 - 2e^2(-4b - 4ag + bg^2) + (a^2 + b^2)(4 + g^2)^2\big]x^4 \equiv -648x^4\widetilde{B}_2.$$

Lemma 6.3. *Assume that for systems* (57) *the condition* $B_2 = 0$ *holds. This implies* $B_3 = 0$ *and then these systems belong to the class* $\boldsymbol{QSL_5}$*. Moreover the condition* $B_2 = 0$ *implies* $H_{10} \geq 0$ *and* $\mathbf{D} \geq 0$*, and if* $H_{10} = 0$ *(which is equivalent to* $\mathbf{D} = 0$*) then we get quadratic homogeneous systems.*

Proof: We observe that the polynomial B_2 is bi-quadratic with respect to the parameter e. Then we calculate

$$\text{Discrim}\,[\widetilde{B}_2, e^2] = -4(-4a + 4bg + ag^2)^2 \leq 0.$$

So in order to impose $B_2 = 0$ the condition $-4a + 4bg + ag^2 = 0$ is necessary and this gives $b = \dfrac{a\left(4 - g^2\right)}{4g}$. Then calculations yield

$$\widetilde{B}_2 = -\frac{81x^4}{2g^2}\left[4e^2g + a(4 + g^2)^2\right]^2, \quad B_3 = \frac{3x^3y}{2g}\left[4e^2g + a(4 + g^2)^2\right]$$

and therefore the condition $\widetilde{B}_2 = 0$ implies $B_3 = 0$ and we get $a = -\dfrac{4e^2g}{\left(g^2 + 4\right)^2}$. In this case we obtain $H_{10} = \dfrac{128e^2g^2}{4 + g^2} \geq 0$ and $\mathbf{D} = \dfrac{3 \cdot 2^{22}e^8g^8}{(4 + g^2)^8} \geq 0$. Due to $g \neq 0$ the condition

$H_{10} = 0$ implies $e = 0$ and then we get $a = b = 0$, i.e. we arrive at the homogeneous quadratic systems.

On he other hand for these values of the parameters a and b systems (57) possess the following invariant lines:

$$(4+g^2)x = \pm 2e, \quad (g+2i)(y+ix)+e=0, \quad (g-2i)(y-ix)+e=0.$$

So considering the line at infinity these systems belong to the class $\boldsymbol{QSL_5}$ and this completes the proof of our lemma. ∎

It is easy o observe that systems (57) possess two finite invariant lines which are real (respectively complex; coinciding) if $ag < 0$ (respectively $ag > 0$; $a = 0$).

On the other hand for these systems we have $H_{10} = -32ag(4+g^2)$ and hence $\text{sign}\,(H_{10}) = \text{sign}\,(ag)$. So we consider three possibilities: $H_{10} < 0$, $H_{10} > 0$ and $H_{10} = 0$.

6.2.1.1 The possibility $H_{10} < 0$. In this case we get $ag > 0$ and therefore the invariant lines of systems (57) are complex. Then the finite singularities are also complex.

On the other hand considering Lemma 6.3 we deduce that the condition $B_2 \neq 0$ holds, i.e. we could not have invariant lines in other directions. As a result we arrive at the configuration given by *Config. 3.28.*

6.2.1.2 The possibility $H_{10} > 0$. Then $ag < 0$ and we may assume $a = -gu^2 \neq 0$. Moreover we may consider $u = 1$ due to the rescaling $(x, y, t) \to (ux, uy, t/u)$ in systems (57) with $a = -gu^2$. Thus we arrive at the 3-parameter family of systems

$$\dot{x} = g(x^2-1), \quad \dot{y} = b + ex - x^2 + gxy - y^2, \tag{58}$$

possessing two real invariant lines $x = \pm 1$ and the finite singularities $M_{1,2}(x_{1,2}, y_{1,2})$ and $M_{3,4}(x_{3,4}, y_{3,4})$ located on these invariant lines, where

$$\begin{aligned} &x_{1,2} = 1,\ y_{1,2} = \big(g \pm \sqrt{\widetilde{V}_1}\,\big)/2; \qquad \widetilde{V}_1 = g^2 + 4(b+e-1);\\ &x_{3,4} = -1,\ y_{3,4} = \big(-g \pm \sqrt{\widetilde{V}_2}\,\big)/2; \quad \widetilde{V}_2 = g^2 + 4(b-e-1). \end{aligned} \tag{59}$$

On the other hand for the above systems we calculate:

$$\mathbf{D} = -768g^6\widetilde{V}_1\widetilde{V}_2, \quad H_{15} = 128g^4(\widetilde{V}_1 + \widetilde{V}_2) \tag{60}$$

and we discuss three cases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

6.2.1.2.1 The case $\mathbf{D} < 0$. Then $\widetilde{V}_1\widetilde{V}_2 > 0$ and the finite singularities are all either real or complex and this situation is governed by the invariant polynomial H_{15}, because $\text{sign}\,(H_{15}) = \text{sign}\,(\widetilde{V}_1 + \widetilde{V}_2)$.

On the other hand according to Lemma 6.3 we deduce that the condition $B_2 \neq 0$ must be fulfilled in this case, i.e. we do not have other invariant lines apart from $x = \pm 1$. So we examine two subcases: $H_{15} < 0$ and $H_{15} > 0$.

1: *The subcase* $H_{15} < 0$. This implies $\widetilde{V}_1 < 0$ and $\widetilde{V}_2 < 0$, i.e. all four finite singularities are complex. Then we arrive at the configuration of invariant lines of systems (58) given by *Config. 3.29*.

2: *The subcase* $H_{15} > 0$. Then we obtain $\widetilde{V}_1 > 0$ and $\widetilde{V}_2 > 0$, i.e. systems (58) possess four real distinct finite singularities. This leads to the configuration *Config. 3.30*.

6.2.1.2.2 The case D > 0**.** Then $\widetilde{V}_1\widetilde{V}_2 < 0$ and may assume $\widetilde{V}_1 > 0$ and $\widetilde{V}_2 < 0$ due to the rescaling $(x, y, t) \to (-x, -y, -t)$, which change the sign of the parameter e and this transfer the $\widetilde{V}_2$ in $\widetilde{V}_1$. Therefore on the invariant line $x = 1$ we have two real singularities whereas on the line $x = 1$ there are two complex singularities.

On the other hand considering Lemma 6.3 we deduce that in this case we could have the condition $B_2 = 0$ which implies the existence of additional invariant lines. So we discus two subcases: $B_2 \neq 0$ and $B_2 = 0$.

1: *The subcase* $B_2 \neq 0$. So in this case the invariant lines $x = \pm 1$ are the only lines for systems (58) and we get the configuration *Config. 3.31*.

2: *The subcase* $B_2 = 0$. As it is shown in the proof of Lemma 6.3 for systems (57) the condition $b = \dfrac{a\left(4 - g^2\right)}{4g}$ is necessary in order to have the $B_2 = 0$. Therefore for systems (58) with $a = -g$ we get $b = (g^2 - 4)/4)$ and then we calculate:

$$B_2 = \frac{81}{2}(4 - 2e + g^2)^2(4 + 2e + g^2)^2 x^4 = 0.$$

Without lost of generality we may assume $4 - 2e + g^2 = 0$ due to the rescaling $(x, y, t) \to (-x, -y, -t)$, which change the sign of the parameter e. Therefore we obtain $e = (g^2 + 4)/2$ and we arrive at the family of systems

$$\dot{x} = g(x^2 - 1), \quad \dot{y} = (g^2 - 4)/4 + (g^2 + 4)x/2 - x^2 + gxy - y^2, \tag{61}$$

possessing the following 4 invariant affine lines:

$$x = \pm 1, \quad 2(y - ix) + g + 2i = 0, \quad 2(y + ix) + g - 2i = 0.$$

Since the complex lines intersect at the real finite singular point $(1, -g/2)$ located on the invariant line $x = 1$ and on the same line we have the second real singularity $(1, 3g/2)$, we arrive at the unique configuration given by *Config. 5.2* (see [21], Diagram 2 on page 34).

6.2.1.2.3 The case D $= 0$**.** Considering (60) this condition gives $\widetilde{V}_1\widetilde{V}_2 = 0$ due to $\mu_0 = g^2 \neq 0$. Moreover as it is mentioned above, we may assume $\widetilde{V}_1 = 0$ and considering (59) this implies $b = 1 - e - g^2/4$. So we get the systems

$$\dot{x} = g(x^2 - 1), \quad \dot{y} = (4 - 4e - g^2)/4 + ex - x^2 + gxy - y^2 \tag{62}$$

possessing the following fnite singularities:

$$M_1 \equiv M_2(1, g/2), \quad M_{3,4}(-1, -g/2 \pm \sqrt{-2e}).$$

On the other hand we have $H_{15} = -1024eg^4$ and hence this invariant polynomials is responsible for the kind of the finite singularities located on the invariant line $x = -1$. More exactly they are real (respectively complex; coinciding) if $H_{15} > 0$ (respectively $H_{15} < 0$; $H_{15} = 0$).

We claim that in the case under examination the condition $B_2 \neq 0$ must hold. Indeed suppose the contrary, that $B_2 = 0$. Then according to Lemma 6.3 the condition $B_2 = \mathbf{D} = 0$ implies $H_{10} = 0$ that contradicts to $H_{10} > 0$. This completes the proof of our claim.

Considering the double singularity on the invariant line $x = 1$ we obtain *Config. 3.32* if $H_{15} < 0$; *Config. 3.33* if $H_{15} > 0$ and *Config. 3.34* if $H_{15} = 0$.

6.2.1.3 The possibility $H_{10} = 0$.

In this case we get $a = 0$ and then systems (57) become systems

$$\dot{x} = gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2, \tag{63}$$

possessing the double invariant line $x = 0$ and two double singular points $M_{1,2}(0, \pm\sqrt{b})$.

On the other hand for these systems we have $\mathbf{R} = 16bg^4x^2$, i.e. $\operatorname{sign}(\mathbf{R}) = \sin(b)$ and we examine two cases: $\mathbf{R} \neq 0$ and $\mathbf{R} = 0$.

6.2.1.3.1 The case $\mathbf{R} \neq 0$.

Considering Lemma 6.3 we deduce that in the case $\mathbf{R} \neq 0$ we must have $B_2 \neq 0$, otherwise we get homogeneous quadratic systems (i.e. $b = e = 0$) and this implies $\mathbf{R} = 0$. As a result we arrive at the configuration given by *Config. 3.35* if $\mathbf{R} < 0$; by *Config. 3.36* if $\mathbf{R} > 0$.

6.2.1.3.2 The case $\mathbf{R} = 0$.

Then $b = 0$ and we obtain $B_2 = -648e^4x^4$. So in the case $B_2 \neq 0$ we obviously obtain *Config. 3.37*.

Assuming $B_2 = 0$ we have $e = 0$ and this leads to the homogeneous systems (63) with $b = e = 0$ possessing the invariant lines $x = 0$ (double) and $y = \pm ix$. Therefore we have systems with invariant lines of total multiplicity 5 and this leads to the configuration *Config. 5.10* (see [21], Diagram 2 on page 35).

6.2.2 The subcase $\mu_0 = 0$

Then $g = 0$ and considering (56) we may assume $e = 0$ due to the translation $(x, y) \to (x + e/2, y)$. So we arrive at the 3-parameter family of systems

$$\dot{x} = a + cx, \quad \dot{y} = b - x^2 - y^2, \tag{64}$$

for which we have $\mu_0 = \mu_1 = 0$ and $\mu_2 = c^2(x^2 + y^2)$.

6.2.2.1 The possibility $\mu_2 \neq 0$.

According to [3, Lemma 5.2] two finite singularities have gone to infinity and coalesced with two complex infinite singularities. Since $c \neq 0$ we may assume $c = 1$ due to the rescaling $(x, y, t) \to (cx, cy, t/c)$ and we arrive at the 2-parameter family of systems

$$\dot{x} = a + x, \quad \dot{y} = b - x^2 - y^2 \tag{65}$$

which possess the invariant affine line $x = -a$. According to Lemma 4.7 the line at infinity $Z = 0$ of the systems (65) is of multiplicity 2.

We determine that the above systems possess two finite singularities $M_{1,2}\left(-a, \pm\sqrt{b-a^2}\right)$ which could be either real or complex or coinciding depending on the value of $b - a^2$.

On the other hand for systems (65) we calculate

$$\mathbf{D} = -768(a^2 - b), \quad B_2 = -648\big[16a^2 + (1-4b)^2\big]x^4$$

and we have the following remark.

Remark 6.5. *The condition* $B_2 = 0$ *implies* $\mathbf{D} > 0$.

Indeed, assuming $B_2 = 0$ we get $a = 0$ and $b = 1/4$ which gives $\mathbf{D} = 192 > 0$.

We observe that $\operatorname{sign}(\mathbf{D}) = \operatorname{sign}(b - a^2)$ and we discuss three cases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

6.2.2.1.1 The case $\mathbf{D} < 0$. Then $b - a^2 > 0$ and the singularities $M_{1,2}$ are real distinct. By the above remark the condition $B_2 \neq 0$ hold, i.e. we could not have invariant lines in other directions. As a result we arrive at the unique configuration given by *Config. 3.38*.

6.2.2.1.2 The case $\mathbf{D} > 0$. This implies $b - a^2 < 0$ and the singularities $M_{1,2}$ are complex. If $B_2 \neq 0$ we get the unique configuration *Config. 3.39*.

Assume now $B_2 = 0$. Following Remark 6.5 we get $a = 0$ and $b = 1/4$ and we arrive at the system

$$\dot{x} = x, \quad \dot{y} = 1/4 - x^2 - y^2 \tag{66}$$

which possesses 3 invariant affine lines: $x = 0$ and $2(y \pm ix) + 1 = 0$, i.e. we have invariant lines of total multiplicity 5 (the infinite line is double). Taking into account the existence of two finite singularities $(0, \pm 1/2)$ we obtain the configuration *Config. 5.9* (see [21], Diagram 2 on page 35).

6.2.2.1.3 The case $\mathbf{D} = 0$. This condition yields $b = a^2$ and we get the family of systems

$$\dot{x} = a + x, \quad \dot{y} = a^2 - x^2 - y^2$$

which possess the unique finite singularity $(-a, 0)$ and clearly it is double. By Remark 6.5 we deduce that $B_2 \neq 0$ and it is easy to determine that this leads to the unique configuration *Config. 3.40*.

6.2.2.2 The possibility $\mu_2 = 0$. Then $c = 0$ and systems (64) become as systems

$$\dot{x} = a, \quad \dot{y} = b - x^2 - y^2 \tag{67}$$

which do not possess finite singularities because $a \neq 0$, otherwise the systems are degenerate. For these systems we have $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 = a^2(x^2 + y^2)^2$ and according to [3, Lemma 5.2] the four finite singularities have gone to infinity and coalesced with two complex infinite singularities: one couple with $N_1[1:i:0]$ and the second couple with $N_2[1:-i:0]$.

On the other hand according to Lemma 4.7 the line at infinity $Z=0$ of the systems (67) is of multiplicity 3.

Since for systems (67) we have $B_2 = -10368(a^2+b^2)x^4 \neq 0$ (due to $a \neq 0$) we arrive at the unique configuration *Config. 3.41*.

Thus we completes the proof of the statement (ii) of Theorem 6.1.

6.3 The case $\eta = 0$ and $\widetilde{M} \neq 0$

According to the statement (iii) of Theorem 5.1 we examine two subcases: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

6.3.1 The subcase $\widetilde{N} \neq 0$

Taking into account the proof of Theorem 5.1 we have to consider two possibilities: $\mu_0 \neq 0$ and $\mu_0 = 0$.

6.3.1.1 The possibility $\mu_0 \neq 0$. As it was shown in the proof of Theorem 5.1 in this case we arrived at the systems (21), i.e. in what follow we examine the systems

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b + xy + y^2 \tag{68}$$

with $C_2 = -xy^2$. So we have the invariant affine lines $a + cx + x^2 = 0$ in the directions $x = 0$ and in the case $B_2 = 0$ (which is a necessary condition for the existence of invariant lines in the second direction) we could also have such lines in the direction $y = 0$. Fo the above systems we calculate $B_2 = -648b^2x^4$ and we discuss two cases: $B_2 \neq 0$ and $B_2 = 0$.

6.3.1.1.1 The case $B_2 \neq 0$. Then systems (68) possess only two invariant affine lines $a + cx + x^2 = 0$ which are real (respectively complex; coinciding) if $c^2 - 4a > 0$ (respectively, $c^2 - 4a < 0$; $c^2 - 4a = 0$).

On the other hand for these systems we have $H_{10} = 8(c^2 - 4a)$ and hence $\operatorname{sign}(H_{10}) = \operatorname{sign}(c^2 - 4a)$. So we examine three subcases: $H_{10} < 0$, $H_{10} > 0$ and $H_{10} = 0$.

1: *The subcase* $H_{10} < 0$. Then $c^2 - 4a < 0$ and hence the invariant affine lines of systems (68) are complex. Therefore the finite singularities located on these line are also complex and we arrive at the configuration *Config. 3.42*.

2: *The subcase* $H_{10} > 0$. This condition implies $c^2 - 4a > 0$ and for commodity we replace c by $2c$. Then we get the condition $4(c^2 - a) > 0$ and setting $c^2 - a = u^2 \neq 0$, (i.e. $a = c^2 - u^2$), after the transformation $(x, y, t) \to (ux - c, uy, t/u)$ we arrive at the family of systems

$$\dot{x} = x^2 - 1 \quad \dot{y} = b + fy + xy + y^2 \tag{69}$$

for which $B_2 = -648b^2x^4$, i.e. this polynomial keeps its value after the mentioned above transformation. These systems possess two real invariant lines $x = \pm 1$ and the finite singularities $M_{1,2}(x_{1,2}, y_{1,2})$ and $M_{3,4}(x_{3,4}, y_{3,4})$, where

$$\begin{aligned} &x_{1,2} = -1,\ y_{1,2} = \big(1 - f \pm \sqrt{\widehat{V}_1}\,\big)/2; \qquad \widehat{V}_1 = (1-f)^2 - 4b;\\ &x_{3,4} = 1,\ y_{3,4} = \big(-f - 1 \pm \sqrt{\widehat{V}_2}\,\big)/2; \quad \widehat{V}_2 = (1+f)^2 - 4b. \end{aligned} \tag{70}$$

On the other hand for systems (69) we calculate

$$\mathbf{D} = -768\widehat{V}_1\widehat{V}_2, \quad H_{15} = 128(\widehat{V}_1 + \widehat{V}_2) \tag{71}$$

and we discuss three possibilities: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

2.1: *The possibility* $\mathbf{D} < 0$. Then we have $\widehat{V}_1\widehat{V}_2 > 0$, i.e. these expressions have the same sign. More precisely, if $H_{15} < 0$ then $\widehat{V}_1 < 0$ and $\widehat{V}_2 < 0$ and evidently in this case all four finite singularities are complex. If $H_{15} > 0$ then $\widehat{V}_1 > 0$ and $\widehat{V}_2 > 0$ and hence all four finite singularities are real. Considering the condition $B_2 \neq 0$ (i.e. we do not have other invariant affine line apart from $x = \pm 1$) we arrive at *Config. 3.43* if $H_{15} < 0$ and at *Config. 3.44* if $H_{15} > 0$.

2.2: *The possibility* $\mathbf{D} > 0$. In this case we have $\widehat{V}_1\widehat{V}_2 < 0$, i.e. two finite singularities of systems (69) are real and two are complex. Therefore evidently we get the configuration given by *Config. 3.45*.

2.3: *The possibility* $\mathbf{D} = 0$. Then $\widehat{V}_1\widehat{V}_2 = 0$ and without losing the generality we may assume $\widehat{V}_1 = 0$ due to the rescaling $(x, y, t) \to (-x, -y, -t)$ which change the sign of the parameter f and hence transfer $\widehat{V}_2$ to $\widehat{V}_1$.

Thus $\widehat{V}_1 = 0$ and then the singular points $M_{1,2}$ coalesced. On the other hand by (71) we have $\operatorname{sign}(\widehat{V}_2) = \operatorname{sign}(H_{15})$ and therefore we arrive at the *Config. 3.46* if $H_{15} < 0$ and at *Config. 3.47* if $H_{15} > 0$.

Clearly when $H_{15} = 0$ we obtain $\widehat{V}_1 = \widehat{V}_2 = 0$ and hence systems (69) possess two double finite singularities. This leads to *Config. 3.48*.

3: *The subcase* $H_{10} = 0$. Then $c^2 - 4a = 0$ and for comodity we repplace again c by $2c$. Then we get $a = c^2$ and applying the transformation $(x, y, t) \to (-x - c, -y, -t)$ to systems (68) we arrive at the family of systems

$$\dot{x} = x^2 \quad \dot{y} = b + fy + xy + y^2. \tag{72}$$

These systems possess the double invariant line $x = 0$ and two double singular points $M_{1,2}\big(0, -(f \pm \sqrt{f^2 - 4b})/2\big)$. It is clear that these singularities are real (respectively, complex; coinciding) if $f^2 - 4b > 0$ (respectively $f^2 - 4b < 0$; $f^2 - 4b = 0$).

On the other hand for the above systems we have $\mathbf{R} = 4(f^2 - 4b)x^2$ and considering the condition $B_2 \neq 0$ (i.e. we do not have other invariant affine line apart from the double line $x = \pm 1$) we arrive at *Config. 3.49* if $\mathbf{R} < 0$; *Config. 3.50* if $\mathbf{R} > 0$ and at *Config. 3.51* if $\mathbf{R} = 0$.

6.3.1.1.2 The case $B_2 = 0$.

This implies $b = 0$ and we get the family of systems

$$\dot{x} = a + cx + x^2, \quad \dot{y} = y(x + y) \tag{73}$$

possessing the additional invariant line $y = 0$. So these systems possess invariant lines of multiplicity at least four and following [23] we calculate:

$$\eta = 0,\ \widetilde{M} = -8y^2 \neq 0,\ \theta = H_7 = B_2 = 0,\ \mu_0 = 1 \neq 0,$$
$$\widetilde{N} = x^2 \neq 0,\ B_3 = -3ax^2y^2,\ H_{10} = 8(c^2 - 4a).$$

Therefore according to [23] (see Table 2 and Diagram 1) if $B_3 \neq 0$ we obtain the configuration *Config. 4.14* if $H_{10} < 0$; *Config. 4.11* if $H_{10} > 0$ and *Config. 4.23* if $H_{10} = 0$. However in the case $H_{10} > 0$ in [23] a configuration was omitted (see Lemma 6.4 below).

Assume now $B_3 = 0$. Then $a = 0$ and since in this case for the above systems we have $\widetilde{D} = c^2xy^2$, according to [23] (see Table 2 and Diagram 1) we arrive at the configuration *Config. 5.11* if $\widetilde{D} \neq 0$ and *Config. 5.19* if $\widetilde{D} = 0$.

Remark 6.6. *We point out that in the case $\mu_0 \neq 0$, $\widetilde{H}_{10} > 0$ and $B_3 \neq 0$ in paper [23] there appears only one configuration Config. 4.11. However depending on the position of the finite singularities of systems* (73) *with respect to the invariant line $y = 0$ we could have another configuration different from Config. 4.11 which we denote here by Config. 4.11a (see Figure 3).*

We have to distinguish these two configurations. The next lemma holds.

Lemma 6.4. *Assume that for an arbitrary quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\theta = H_7 = B_2 = 0$, $\mu_0 B_3 \neq 0$ and $H_{10} > 0$ are satisfied. Then the configurations of the invariant lines of this system are Config. 4.11 if $H_4 < 0$ and Config. 4.11a if $H_4 > 0$.*

Proof: If for a quadratic system the conditions provided by the lemma hold then according to [23, Theorem 4.1, statement (ii)] this system could be brought via an affine transformation and time rescaling to the canonical form

$$\dot{x} = x(x + y), \quad \dot{y} = (y + k)^2 - 1, \quad k \in \mathbb{R}, \quad k \neq \pm 1. \tag{74}$$

These systems posses the invariant lines $x = 0$ and $y = -k \pm 1$ and the finite singularities:

$$M_{1,2}(0, -k \pm 1), \quad M_3(k - 1, 1 - k), \quad M_4(k + 1, -k - 1).$$

It is clear that the singularities M_3 and M_4 are located on the invariant lines $y = 1 - k$ and $y = -1 - k$, respectively. And it is important to detect if they are located on the same part with respect to the invariant line $x = 0$ (*Config. 4.11a*) or on different parts (*Config. 4.11*). So in the first case the product of the corresponding abscissae must be positive, i.e. $(k - 1)(k + 1) > 0$, whereas in the second case it must be negative, i.e. $(k - 1)(k + 1) < 0$.

On the other hand for systems (74) we have $H_4 = 48(k - 1)(1 + k)$ and we conclude that for $H_4 < 0$ we get *Config. 4.11* and for $H_4 > 0$ we obtain *Config. 4.11a*. This completes the proof of Lemma 6.4. ∎

6.3.1.2 The possibility $\mu_0 = 0$.

As it was shown in the proof of Theorem 5.1 in this case we arrived at the systems (21), i.e. the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g-1)xy, \tag{75}$$

for which we have

$$C_2 = x^2 y,\ B_2 = 0,\ B_3 = -3b(g-1)^2 x^4,\ \widetilde{N} = (g^2-1)x^2. \tag{76}$$

We observe that we have two invariant affine lines in the direction $x = 0$ (which is defined by the double factor of C_2), whereas in the direction $y = 0$ we could have only the line $y = 0$ if $b = 0$. Due to $\widetilde{N} \neq 0$ the last condiion is equivalent to $B_3 = 0$ and therefore we examine two cases: $B_3 \neq 0$ and $B_3 = 0$.

6.3.1.2.1 The case $B_3 \neq 0$.

Then the invariant lines $a + cx + gx^2 = 0$ are the only invariant affine lines of systems (75). However for these systems we have $H_6 = 128(-1 + g)^4 \big[\big] a(1+g)^2 - c^2\big] x^6$ and setting $H_6 = 0$ (i.e. $a = c^2/(1+g)^2$ due to $(g+1) \neq 0$) we obtain the systems

$$\dot{x} = \frac{1}{(g+1)^2}(c + gx + x)(c + g^2 x + gx), \quad \dot{y} = b + (g-1)xy, \tag{77}$$

Then considering the Notation 4.2 for these systems we calculate

$$\mathcal{H}(X,Y,Z) = \frac{1}{(g+1)^2}(c + gX + X)^2(c + g^2X + gX)$$

and hence by Lemma 4.6 the invariant line $c + g^2X + gX = 0$ of the above systems is simple whereas the invariant line $c + gX + X = 0$ is double. So we consider two subcases: $H_6 \neq 0$ and $H_6 = 0$.

1: *The subcase* $H_6 \neq 0$. Then both invariant lines of systems (75) are simple ones and these systems belong to the class $\mathbf{QSL_3}$.

On the other hand for these systems we have $\mu_0 = \mu_1 = 0$, $\mu_2 = ag(g-1)^2x^2$ and by [3, Lemma 5.2] if $\mu_2 \neq 0$ then only two finite singularities have gone to infinity and coalesced with the same infinite singular point $N_2[0:1:0]$ which becomes of multiplicity 4 of the type $(2,2)$. On the other hand the singular point $N_1[1:0:0]$ is an elementary singularity.

If $\mu_2 = 0$ by the same lemma at least three finite singularities have gone to infinity. So we consider two possibilities: $\mu_2 \neq 0$ and $\mu_2 = 0$.

1.1: *The possibility* $\mu_2 \neq 0$. As it is mentioned above, at infinity we have the singular points $N_1[1:0:0]$ (elemental) and $N_2[0:1:0]$ (of multiplicity 4). Clearly systems (75) possess only two invariant affine lines $a + cx + gx^2 = 0$ which are real (respectively complex; coinciding) if $c^2 - 4ag > 0$ (respectively, $c^2 - 4ag < 0$; $c^2 - 4ag = 0$).

On the other hand for these systems we have $H_{11} = 48(g-1)^4(c^2 - 4ag)x^4$ and hence $\operatorname{sign}(H_{11}) = \operatorname{sign}(c^2 - 4a)$. So we examine three cases: $H_{11} < 0$, $H_{11} > 0$ and $H_{11} = 0$.

1.1.1: *The case* $H_{11} < 0$. Then the invariant affine lines are complex as well as the finite singularities. Considering the multiplicity of infinite singularities we arrive at the configuration *Config. 3.52*.

1.1.2: *The case* $H_{11} > 0$. Then $c^2 - 4ag > 0$ and setting $c^2 - 4ag = u^2$ we obtain $a = (c^2 - u^2)/(4g)$. Then applying the transformation $(x, y, t) \to \big((ux - c)/(2g), y, 2gt/u\big)$ we arrive at the family of systems

$$\dot{x} = g(x^2 - 1) \quad \dot{y} = b + fy + (g - 1)xy. \tag{78}$$

These systems possess two real invariant lines $x = \pm 1$ intersecting at infinity at the point $N_2[0 : 1 : 0]$ of multiplicity 4. For the above systems we have

$$\mu_2 = g^2(f + 1 - g)(f - 1 + g)x^2, \quad B_3 = -3b(g - 1)^2x^4, \quad \widetilde{K} = 2g(g - 1)x^2?$$

and due to $\mu_2 \neq 0$ systems (78) possess the following two finite singularities:

$$M_1\big(-1, -\frac{b}{f + 1 - g}\big), \quad M_2\big(1, -\frac{b}{f - 1 + g}\big).$$

So due to $B_3 \neq 0$ we arrive at the unique configuration *Config. 3.53*.

1.1.3: *The case* $H_{11} = 0$. This condition implies $c^2 - 4ag = 0$, i.e. $a = c^2/(4g)$ and we obtain the systems

$$\dot{x} = (c + 2gx)^2/(4g), \quad \dot{y} = b + (g - 1)xy, \tag{79}$$

possessing the double invariant line $c + 2gx = 0$ and the double real singular point $\left(-\frac{c}{2g}, \frac{2bg}{c(g - 1)}\right)$. Taking into consideration the existence of the multiple singularity at infinity (of the type (2,2)) as well as of the simple one we obtain the configuration *Config. 3.54*.

1.2: *The possibility* $\mu_2 = 0$. This condition implies $ag(g - 1) = 0$ and due to $\widetilde{N} \neq 0$ (i.e. $g - 1 \neq 0$) we obtain $ag = 0$. On the other hand for systems (75) we have $\widetilde{K} = 2g(g - 1)x^2$ and we examine two cases: $\widetilde{K} \neq 0$ and $\widetilde{K} = 0$.

1.2.1: *The case* $\widetilde{K} \neq 0$. Then the condition $\mu_2 = 0$ gives $a = 0$ and this leads to the family of systems

$$\dot{x} = x(c + gx) \quad \dot{y} = b + (g - 1)xy \tag{80}$$

for which we calculate

$$\mu_0 = \mu_1 = \mu_2 = 0, \ \mu_3 = bcg(1 - g)x^3, \ \widetilde{K} = 2g(g - 1)x^2,$$
$$B_3 = -3b(g - 1)^2x^4, \ H_6 = -128c^2(g - 1)^4x^6$$

The condition $\widetilde{K}B_3H_6 \neq 0$ implies $\mu_3 \neq 0$ and then by [3, Lemma 5.2] only three finite singularities have gone to infinity and coalesced with the same infinite singular point $N_2[0 : 1 : 0]$ which become of multiplicity 5 of the type $(2, 3)$.

Since $B_3 \neq 0$ (i.e. $b \neq 0$) the above systems possess only two invariant affine lines $x = 0$ and $x = -c/g$. On the other hand these systems possess only one finite singularity

$\left(-\dfrac{c}{g}, \dfrac{bg}{c(g-1)}\right)$ which is located on the invariant line $x = -c/g$. As a result we arrive at the configuration *Config. 3.55*.

1.2.2: *The case* $\widetilde{K} = 0$. Then $g = 0$ and we obtain the family of systems

$$\dot{x} = a + cx, \quad \dot{y} = b - xy, \tag{81}$$

for which we have

$$\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 = -acx^2y,\ B_3 - 3bx^4,\ H_{11} = 48c^2x^4.$$

1.2.2.1: *The subcase* $\mu_3 \neq 0$. Then $ac \neq 0$ and hence the above systems possess one real invariant affine line $cx + a = 0$. We observe that according to Lemma 4.7 the line at infinity $Z = 0$ of the systems (81) is of multiplicity 2.

On the other hand since the invariant polynomial μ_3 contains the factor x^2y, by [3, Lemma 5.2] three finite singularities have gone to infinity, two of them coalescing with the infinite singular point $N_2[0:1:0]$ which becomes of multiplicity 4 of the type $(2, 2)$, whereas the third one coalesced with the singular point $N_1[1:0:0]$ which becomes of multiplicity 2 of the type $(1, 1)$.

So considering the singular point $(-a/c, -bc/a)$ located on the invariant line $cx + a = 0$ and the double infinite line we obtain *Config. 3.56*.

1.2.2.2: *The subcase* $\mu_3 = 0$. This condition gives $ac = 0$ and since $H_{11} = 48c^2x^4$ we examine two possibilities: $H_{11} \neq 0$ and $H_{11} = 0$.

1.2.2.2.1: *The possibility* $H_{11} \neq 0$. Then $c \neq 0$ and we get $a = 0$. This leads to the systems

$$\dot{x} = cx, \quad \dot{y} = b - xy \tag{82}$$

for which we have $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 = -bc^2x^3y \neq 0$ (otherwise we get degenerate systems). So by [3, Lemma 5.2] the above systems possess at infinity the singularities $N_2[0:1:0]$ of multiplicity 5 (of the type $(2, 3)$) and $N_1[1:0:0]$ of multiplicity 2 (of the type $(1, 1)$). Since by Lemma 4.7 the line at infinity is double we obtain the configuration *Config. 3.57*.

1.2.2.2.2: *The possibility* $H_{11} = 0$. Then $c = 0$ and this leads to the systems

$$\dot{x} = a, \quad \dot{y} = b - xy, \tag{83}$$

which according to Lemma 4.7 possess the invariant line at infinity $Z = 0$ of multiplicity 3.

Since for the above systems we have $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 = a^2x^2y^2 \neq 0$, by [3, Lemma 5.2] two finite singularities have gone to infinity coalescing with the infinite singular point $N_2[0:1:0]$ which becomes of multiplicity 4 of the type $(2, 2)$, whereas other two coalesced with the singular point $N_1[1:0:0]$ which becomes of multiplicity 3 of the type $(1, 2)$. As a result we arrive at the configuration *Config. 3.58*.

2: *The subcase* $H_6 = 0$. As mentioned earlier in this case systems (75) belong to the class $\mathbf{QSL_4} \bigcup \mathbf{QSL_5} \bigcup \mathbf{QSL_6}$ and for $H_6 = 0$ we obtain systems (77). So following [23] for systems(77) we calculate:

$$\eta=0,\ \widetilde{M}=-8x^2\neq 0,\ \theta=H_7=B_2=\mu_0=0,\ B_3=-3b(-1+g)^2x^4\neq 0,$$
$$\widetilde{N}=(g^2-1)x^2\neq 0,\ \widetilde{K}=2g(g-1)x^2,\ H_{11}=\frac{48}{(g+1)^2}c^2(g-1)^6x^4.$$

Therefore according to [23] (see Table 2 and Diagram 1) in the case $\widetilde{K}\neq 0$ we arrive at the configuration *Config. 4.30* if $H_{11}\neq 0$ and at the configuration *Config. 4.43* if $H_{11}=0$.

Assuming $\widetilde{K}=0$ we obtain the configuration *Config. 4.40*.

6.3.1.2.2 The case $B_3=0$. Then considering (76) and the condition $\widetilde{N}\neq 0$ we obtain $b=0$ and systems (75) become

$$\dot{x}=a+cx+gx^2,\quad \dot{y}=(g-1)xy. \tag{84}$$

We observe that these systems possess invariant affine lines $a+cx+gx^2$ and $y=0$. So taking into account the line at infinity we conclude that the above systems possess invariant lines of total multiplicity at least four. So following [23] and [21] we calculate:

$$\eta=0,\ \widetilde{M}=-8x^2\neq 0,\ \theta=H_7=B_2=0,\ \mu_0=0,\ B_3=0,\ \widetilde{N}=(g^2-1)x^2\neq 0,$$
$$\widetilde{K}=2g(g-1)x^2\neq 0,\ H_6=128(g-1)^4\big[a(g+1)^2-c^2\big]x^6,\ H_{11}=48(g-1)^4(c^2-4ag)x^4.$$

According to [23] (see Table 2 and Diagram 1) and [21] (see Table 4 and Diagram 2) we have the following lemma.

Lemma 6.5. *Assume that for an arbitrary non-degenerate quadratic system the conditions $\eta=\mu_0=\theta=H_7=B_3=0$ and $\widetilde{M}\widetilde{N}\neq 0$ hold. Then this system could possess one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
H_6\neq 0,\ \widetilde{K}\neq 0,\ H_{11}<0 & \Leftrightarrow & \textit{Config. 4.15};\\
H_6\neq 0,\ \widetilde{K}\neq 0,\ H_{11}>0 & \Leftrightarrow & \textit{Config. 4.12};\\
H_6\neq 0,\ \widetilde{K}\neq 0,\ H_{11}=0 & \Leftrightarrow & \textit{Config. 4.24};\\
H_6\neq 0,\ \widetilde{K}=0,\ H_{11}\neq 0 & \Leftrightarrow & \textit{Config. 4.19};\\
H_6\neq 0,\ \widetilde{K}=0,\ H_{11}=0 & \Leftrightarrow & \textit{Config. 4.36};\\
H_6=0,\ \widetilde{K}\neq 0 & \Leftrightarrow & \textit{Config. 5.14};\\
H_6=0,\ \widetilde{K}=0 & \Leftrightarrow & \textit{Config. 5.18}.
\end{array}$$

6.3.2 The subcase $\widetilde{N}=0$

According to the statement (*iii*) of Theorem 5.1 in this case we have to consider two possibilities: $\widetilde{K}\neq 0$ and $\widetilde{K}=0$.

6.3.2.1 The possibility $\widetilde{K}\neq 0$. According to Theorem 5.1 in this case the condition $\theta_3=0$ must be satisfied and taking into account the proof of this theorem we consider the family of systems (24), i.e. the systems

$$\dot{x}=a+cx-x^2,\quad \dot{y}=b-2xy. \tag{85}$$

For these systems we have

$$\eta = 0,\ \widetilde{M} = -8x^2 \neq 0,\ \mu_0 = \mu_1 = 0,\ \mu_2 = -4ax^2,\ \widetilde{N} = B_2 = 0,\ \mu_3 = -2cx^2(bx + 2ay),$$
$$\widetilde{K} = 4x^2 \neq 0,\ B_3 = -12bx^4,\ H_6 = -2048c^2x^6,\ H_{11} = 768(4a + c^2)x^4 \tag{86}$$

and we examine two cases: $B_3 \neq 0$ and $B_3 = 0$.

6.3.2.1.1 The case $B_3 \neq 0$**.** We observe that in the case $H_6 = 0$ (i.e. $c = 0$) for systems (89) we have $\mathcal{H}(X, Y, Z) = (X^2 - aZ^2)^2$. This means that these systems belong to the class $\mathbf{QSL_5}$ and therefore we discuss two subcases: $H_6 \neq 0$ and $H_6 = 0$.

1: *The subcase* $H_6 \neq 0$. We detect that the family of systems (89) is a subfamily of (75) defined by the condition $g = -1$ (which implies $\widetilde{N} = 0$). So we could follow the steps and the results obtained for the family (75) with $g = -1$ and namely the branch defined by the condition $\widetilde{K}H_6B_3 \neq 0$.

Thus in the case $B_3H_6 \neq 0$ we arrive at the configurations below if and only if the following conditions are satisfied, correspondingly:

$$\begin{array}{lcl} \mu_2 \neq 0,\ H_{11} < 0 & \Leftrightarrow & \textit{Config. 3.52};\\ \mu_2 \neq 0,\ H_{11} > 0 & \Leftrightarrow & \textit{Config. 3.53};\\ \mu_2 \neq 0,\ H_{11} = 0 & \Leftrightarrow & \textit{Config. 3.54};\\ \mu_2 = 0 & \Leftrightarrow & \textit{Config. 3.55}. \end{array}$$

2: *The subcase* $H_6 = 0$. Then $c = 0$ and we get the systems

$$\dot{x} = a - x^2, \quad \dot{y} = b - 2xy. \tag{87}$$

for which following [21] (see Table 4) we evaluate the following invariant polynomials:

$$\eta = 0,\ \widetilde{M} = -8x^2 \neq 0,\ \widetilde{N} = B_2 = 0,\ B_3 = -12bx^4,\ H_2 = 0,\ H_3 = 32ax^2.$$

We observe that for the above systems we have $H_{11} = 3072ax^4$ and then $\operatorname{sign}(H_3) = \operatorname{sign}(H_{11})$. So according to [21] (see Table 4 and Diagram 2) we arrive at the configuration *Config. 5.25* if $H_{11} < 0$; *Config. 5.22* if $H_{11} > 0$ and *Config. 5.29* if $H_{11} = 0$.

6.3.2.1.2 The case $B_3 = 0$**.** Then $b = 0$ and this leads to the family of systems

$$\dot{x} = a + cx - x^2, \quad \dot{y} = -2xy, \tag{88}$$

for which we have

$$\eta = 0,\ \widetilde{M} = -8x^2 \neq 0,\ \widetilde{K} = 4x^2,\ B_3 = \widetilde{N} = 0,\ H_6 = -2048c^2x^6,$$
$$\mu_0 = H_7 = 0,\ H_{11} = 768(4a + c^2)x^4.$$

So according to [23] (see Table 2 and Diagram 1) in the case $H_6 \neq 0$ we obtain *Config. 4.15* if $H_{11} < 0$; *Config. 4.12* if $H_{11} > 0$ and *Config. 4.24* if $H_{11} = 0$.

Assume now $H_6 = 0$, i.e. we have $c = 0$. Therefore we get the family of systems

$$\dot{x} = a - x^2, \quad \dot{y} = -2xy \tag{89}$$

for which following [21] (see Table 2) we calculate

$$\eta = 0,\ \widetilde{M} = -8x^2 \neq 0,\ \widetilde{H} = -4x^2,\ B_3 = \widetilde{N} = 0,\ H_2 = 0,\ H_3 = 32ax^2,\ H_{11} = 3072ax^4.$$

We observe that $\operatorname{sign}(H_{11}) = \operatorname{sign}(H_3)$ and hence comparing with the conditions from Table 2 in [21] we arrive at the configuration *Config. 6.9* if $H_{11} < 0$ and at *Config. 6.8* if $H_{11} > 0$. We notice that $H_{11} \neq 0$, otherwise we get degenerate system.

6.3.2.2 The possibility $\widetilde{K} = 0$. According to the statement (*iii*) of Theorem 5.1 for $\widetilde{N} = \widetilde{K} = 0$ we have to consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

6.3.2.2.1 The case $B_2 \neq 0$. According to Theorem 5.1 in this case the condition $\theta_5 = 0$ must be satisfied and taking into account the proof of this theorem we arrive at the systems (26), i.e. at the systems

$$\dot{x} = a + cx, \quad \dot{y} = b + ex + y^2. \tag{90}$$

for which we have

$$C_2 = -xy^2,\ B_2 = -648e^4x^4 \neq 0,\ \mu_0 = \mu_1 = 0,\ \mu_2 = c^2y^2.$$

So we consider two subcases: $\mu_2 \neq 0$ and $\mu_2 = 0$

***1:** The subcase* $\mu_2 \neq 0$. Then $c \neq 0$ and we may assume $c = 1$ due to the rescaling $(x, y, t) \to (cx, cy, t/c)$. Since $\mu_2 = y^2$, by [3, Lemma 5.2] only two finite singularities have gone to infinity and coalesced with the same infinite singular point $N_1[1:0:0]$ which becomes of multiplicity 4 of the type $(2, 2)$. At the same time the singular point $N_2[0:1:0]$ remains an elementary singularity.

On the other hand according to Lemma 4.7 systems (90) possess the invariant line at infinity $Z = 0$ of multiplicity 2. Obviously we obtain that the above systems (with $c = 1$) possess two finite singularities $(-a, \pm\sqrt{ae - b})$ the kind of which (real or complex or coinciding) is governed by the invariant polynomial $\mathbf{U} = 4(ae - b)x^2y^4$.

Thus summarizing the information discussed above we obtain *Config. 3.59* if $\mathbf{U} < 0$; *Config. 3.60* if $\mathbf{U} > 0$ and *Config. 3.61* if $\mathbf{U} = 0$.

***2:** The subcase* $\mu_2 = 0$. This condition implies $c = 0$ and we arrive at the family of systems

$$\dot{x} = a, \quad \dot{y} = b + ex + y^2, \tag{91}$$

for which we have $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 = a^2y^4 \neq 0$ (otherwise we get degenerate systems). Therefore by [3, Lemma 5.2] all the finite singular points have gone to infinity and coalesced with the same infinite singular point $N_1[1:0:0]$ which becomes of multiplicity 6 of the type $(2, 4)$.

On the other hand according to Lemma 4.7 the infinite invariant line of systems (**??**) is of multiplicity three. Therefore due to $B_2 \neq 0$ (i.e. $e \neq 0$) these systems do not possess any invariant affine lines. As a result we get the configuration *Config. 3.62*.

6.3.2.2.2 The case $B_2 = 0$. According to the proof of Theorem 5.1 in this case we have to examine systems (25), i.e. the systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + ex + fy, \tag{92}$$

for which we have

$$C_2 = x^2y,\ B_2 = B_3 = 0,\ \mu_0 = \mu_1 = 0,\ \mu_2 = f^2x^2.$$

So we consider two subcases: $\mu_2 \neq 0$ and $\mu_2 = 0$.

1: *The subcase* $\mu_2 \neq 0$. Then $f \neq 0$ and we may assume $f = 1$ and $b = 0$ due to the change $(x, y, t) \to (fx, y - b/f, t/f)$. This leads to the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = ex + y, \tag{93}$$

for which $\mu_2 = x^2$. Then by [3, Lemma 5.2] only two finite singularities have gone to infinity and coalesced with the same infinite singular point $N_2[0:1:0]$ which becomes of multiplicity 4 of the type $(2, 2)$. At the same time the singular point $N_1[1:0:0]$ remains an elementary singularity.

On the other hand according to Lemma 4.7 the infinite invariant line of systems (93) is double. As a result we obtain that these systems possess invariant lines of total multiplicity 4.

We observe that for systems (93) the following conditions are satisfied:

$$\eta = 0,\ \widetilde{M} \neq 0,\ \theta = H_7 = B_3 = \mu_0 = \widetilde{N} = \widetilde{K} = 0.$$

So following [23] (see Table 2) for systems (93) we calculate additionally the following invariant polynomials:

$$N_1 = 8ex^4, \quad N_2 = 4(1 + 4a)x, \quad N_5 = -64ax^2,\ \widetilde{D} = -x^2y,\ \widetilde{H} = 0.$$

We examine the two possibilities: $N_1 \neq 0$ and $N_1 = 0$.

1.1: *The possibility* $N_1 \neq 0$. Then in the case $N_2 \neq 0$, since $\widetilde{D} \neq 0$, according to [23] (see Table 2 and Diagram 1) we arrive at the the configurations *Config. 4.32* if $N_5 < 0$; *Config. 4.28* if $N_5 > 0$ and *Config. 4.39* if $N_5 = 0$.

If $N_2 = 0$ we obtain $a = -1/4$ and then considering the Notation 4.2 for systems (93) we calculate

$$\mathcal{H}(X, Y, Z) = \frac{1}{16}Z(2X - Z)^2(2X + Z)$$

and hence by Lemma 4.6 we conclude that these systems possess invariant lines of total multiplicity 5. Considering the above conditions according to [21] (see Table 4 and Diagram 2) we arrive at the configuration *Config. 5.21.*

1.2: *The possibility* $N_1 = 0$. This condition gives $e = 0$ and this implies the existence of an additional invariant affine line $y = 0$, i.e. systems (93) possess invariant lines of total multiplicity 5. In this case if $N_2 \neq 0$ then due to $\widetilde{D} \neq 0$ and $\widetilde{H} = 0$, according to [21] (see

Table 4 and Diagram 2) systems (93) with $e = 0$ possess the configuration *Config. 5.15* if $N_5 < 0$; *Config. 5.13* if $N_5 > 0$ and *Config. 5.17* if $N_5 = 0$.

In the case $N_2 = 0$ (i.e. $a = -1/4$) we get the system

$$\dot{x} = 1/4(-1 + 2x)(1 + 2x), \quad \dot{y} = y$$

for which we calculate

$$\mathcal{H}(X, Y, Z) = \frac{1}{16} ZY(2X - Z)^2(2X + Z).$$

Therefore by Lemma 4.6 we conclude that this system possesses invariant lines of total multiplicity 6. As a result considering [21] (see Table 4 and Diagram 2) we arrive at the configuration *Config. 6.7*.

2: *The subcase* $\mu_2 = 0$. This implies $f = 0$ and we get the family of systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + ex, \tag{94}$$

for which we have

$$\eta = 0, \ \widetilde{M} \neq 0, \ \theta = B_3 = \mu_0 = \widetilde{N} = \widetilde{K} = 0.$$

So following [23] (see Table 2) for systems (94) we calculate additionally the following invariant polynomials:

$$N_1 = 8ex^4, \quad N_2 = 16ax, \quad N_5 = -64ax^2, \ \widetilde{D} = 0.$$

We examine the two possibilities: $N_1 \neq 0$ and $N_1 = 0$.

2.1: *The possibility* $N_1 \neq 0$. Then since $\widetilde{D} = 0$, according to [23] (see Table 2 and Diagram 1) in the case $N_2 \neq 0$ (this implies $N_5 \neq 0$) we arrive at the configurations *Config. 4.33* if $N_5 < 0$ and *Config. 4.29* if $N_5 > 0$.

If $N_2 = 0$ we get $a = 0$ and considering the Notation 4.2 for systems (94) we calculate

$$\mathcal{H}(X, Y, Z) = X^3 Z$$

and hence by Lemma 4.6 the invariant line $x = 0$ of systems (97) is triple whereas the infinite invariant line $Z = 0$ is double. So we have these systems possess invariant line of total multiplicity 5. In this case since $\widetilde{N}_1 \neq 0$ and $N_2 = \widetilde{H} = 0$, according to [21] (see Table 4 and Diagram 2) systems (94) with $a = 0$ possess the unique configuration *Config. 5.28*.

2.2: *The possibility* $N_1 = 0$. This condition gives $e = 0$ and then $b \neq 0$ otherwise we get degenerate systems. In this case for systems (94) we have

$$\mathcal{H}(X, Y, Z) = bZ^2(X^2 + aZ^2)$$

if $a \neq 0$ (i.e. $N_2 \neq 0$) and

$$\mathcal{H}(X, Y, Z) = bX^3 Z^2$$

if $a = 0$ (i.e. $N_2 = 0$).

Therefore considering Lemma 4.6 we conclude that systems (94) possess invariant lines of total multiplicity five (respectively six) if $N_2 \neq 0$ (respectively $N_2 = 0$).

So according to [21] (see Table 4 and Diagram 2) in the case $N_2 \neq 0$ (then $N_5 \neq 0$) we obtain the configurations *Config. 5.24* if $N_5 < 0$ and *Config. 5.20* if $N_5 > 0$.

In the case $N_2 = 0$ following [21] (see Table 2 and Diagram 1) we arrive at the unique configuration *Config. 6.10*.

Since all the possibilities are examined this completes the proof of the statement (iii) of Theorem 6.1.

6.4 The case $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$

According to the statement (iv) of Theorem 5.1 we have to examine two subcases: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

6.4.1 The subcase $\widetilde{N} \neq 0$

Taking into account the proof of Theorem 5.1 we consider the family of systems (28) for which we have $\widetilde{N} = g^2x^2 \neq 0$. Then due to a translation we may assume $e = f = 0$ and we arrive at the family of systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b - x^2 + gxy, \tag{95}$$

for which we calculate

$$C_2 = x^3,\ B_2 = 0,\ B_3 = -3ag^2x^4,\ \widetilde{N} = g^2x^2,\ \mu_0 = \mu_1 = 0,\ \mu_2 = ag^3x^2. \tag{96}$$

We examine two possibilities: $\mu_2 \neq 0$ and $\mu_2 = 0$.

6.4.1.1 The possibility $\mu_2 \neq 0$. Then by [3, Lemma 5.2] only two finite singularities have gone to infinity and coalesced with the unique infinite singular point $N_1[0 : 1 : 0]$ which becomes of multiplicity 5 of the type $(3, 2)$. Clearly systems (95) possess only two invariant affine lines $a + cx + gx^2 = 0$ which are real (respectively complex; coinciding) if $c^2 - 4ag > 0$ (respectively, $c^2 - 4ag < 0$; $c^2 - 4ag = 0$).

On the other hand for these systems we have $H_{11} = 48g^4(c^2 - 4ag)x^4$ and hence $\operatorname{sign}(H_{11}) = \operatorname{sign}(c^2 - 4a)$.

We observe that systems (95) possess two finite singularities $M_{1,2}(x_{1,2}, y_{1,2})$ with the coordinates

$$x_{1,2} = \frac{-c \pm \sqrt{c^2 - 4ag}}{2g}, \quad y_{1,2} = \frac{c(bg - a) \pm (a + bg)\sqrt{c^2 - 4ag}}{2ag^2}.$$

So we conclude that these singularitiffes are real (respectively complex; coinciding) if $c^2 - 4ag > 0$ (respectively, $c^2 - 4ag < 0$; $c^2 - 4ag = 0$). Then considering the multiplicity of the infinite singularity it is clear that we arrive at the configuration *Config. 3.63* if $H_{11} < 0$; textitConfig. 3.64 if $H_{11} > 0$ and textitConfig. 3.65 if $H_{11} = 0$.

6.4.1.2 The possibility $\mu_2 = 0$.

Due to $\widetilde{N} \neq 0$ (i.e. $g \neq 0$) the condition implies $a = 0$ and we get the systems

$$\dot{x} = x(c + gx), \quad \dot{y} = b - x^2 + gxy, \tag{97}$$

for which we have $B_3 = 0$. Considering the Notation 4.2 for these systems we calculate

$$\mathcal{H}(X, Y, Z) = X(gX + cZ)^2$$

and hence by Lemma 4.6 the invariant line $x = 0$ of systems (97) is simple whereas the invariant line $gx+c = 0$ is double. So these systems possess invariant lines of total multiplicity 4 and considering [23] (see Table 2) we calculate the following invariant polynomials:

$$\eta = \widetilde{M} = 0,\ \theta = B_3 = 0,\ \widetilde{N} = g^2x^2 \neq 0,\ N_6 = 8(c^2 + bg^2)x^3,\ H_{11} = 48c^2g^4x^4.$$

We discuss two cases: $N_6 \neq 0$ and $N_6 = 0$.

6.4.1.2.1 The case $N_6 \neq 0$. According to [23] (see Table 2 and Diagram 1) in this case systems (97) possess the configuration *Config. 4.31* if $H_{11} \neq 0$ and *Config. 4.44* if $H_{11} = 0$.

6.4.1.2.2 The case $N_6 = 0$. This condition implies $c^2 + bg^2 = 0$ and since $g \neq 0$ (due to $\widetilde{N} \neq 0$) we obtain $b = -c^2/g^2$. Then for systems (97) we have

$$\eta = \widetilde{M} = \theta = B_3 = N_6 = 0, \quad \widetilde{N} \neq 0$$

and by [21] (see Table 4 and Diagram 2) we arrive at the configuration *Config. 5.23*.

6.4.2 The subcase $\widetilde{N} = 0$

Considering the proof of Theorem 5.1 we have to examine the family of systems (29) for which we may assume $b = e = 0$ due to the translation $(x, y) \to (x + e/2, y)$. So we arrive at the family of systems

$$\dot{x} = a + cx, \quad \dot{y} = b + fy - x^2. \tag{98}$$

for which considering the Notation 4.2 we calculate

$$\mathcal{H}(X, Y, Z) = Z^2(cX + aZ).$$

Hence by Lemma 4.6 the infinite invariant line $Z = 0$ is triple and considering the invariant affine line $a + cx = 0$ we conclude that the above systems possess invariant lines of multiplicity at least 4. So following [23] (see Table 2) we evaluate the following invariant polynomials:

$$\eta = \widetilde{M} = 0,\ \widetilde{N} = \theta = B_3 = 0,\ N_3 = 3(c - f)x^3,$$
$$N_6 = 8c(c - f)x^3,\ \widetilde{D} = -f^2x^3,\ D_1 = c + f.$$

According to [23] (see Table 2 and Diagram 1) and [21] (see Table 4 and Diagram 2) we have the following lemma.

Lemma 6.6. *Assume that for an arbitrary non-degenerate quadratic system the conditions* $\eta = \widetilde{M} = 0$, $\widetilde{N} = \theta = B_3 = 0$ *hold. Then this system could possess one of the following configurations if and only if the corresponding conditions are satisfied, respectively:*

$$
\begin{array}{lcl}
N_3 \neq 0,\ D_1 \neq 0,\ N_6 \neq 0,\ \widetilde{D} \neq 0 & \Leftrightarrow & \textit{Config. 4.37};\\
N_3 \neq 0,\ D_1 \neq 0,\ N_6 \neq 0,\ \widetilde{D} = 0 & \Leftrightarrow & \textit{Config. 4.38};\\
N_3 \neq 0,\ D_1 \neq 0,\ N_6 = 0 & \Leftrightarrow & \textit{Config. 4.46};\\
N_3 \neq 0,\ D_1 = 0 & \Leftrightarrow & \textit{Config. 5.26};\\
N_3 = 0,\ D_1 \neq 0 & \Leftrightarrow & \textit{Config. 5.27};\\
N_3 = 0,\ D_1 = 0 & \Leftrightarrow & \textit{Config. 5.30}.
\end{array}
$$

Thus we completes the proof of the statement (iv) of Theorem 6.1.

6.5 The case $C_2 = 0$

According to the proof of Theorem 5.1 in this case we arrived at the systems (30) with $d = 0$. Moreover due to a translation we may assume $e = f = 0$ and hence we have to examine the family of systems

$$\dot{x} = a + cx + x^2, \quad \dot{y} = b + xy. \tag{99}$$

For these systems we have $C_2 = 0$, i.e. the infinite line $Z = 0$ is filled ut with singularities. We remark that in [27] the complete classification of the configuration of invariant lines for the whole family of quadratic systems with $C_2 = 0$ was done. So following [27] (see Table 1) we evaluate for systems (99) the following invariant polynomials:

$$H_{10} = 0, \quad H_{11} = -48(4a - c^2)x^4, \quad H_{12} = -8a^2x^2.$$

Then according to [27] (see Table 1 and Diagram 2) we obtain the following lemma.

Lemma 6.7. *Assume that for an arbitrary non-degenerate quadratic system the condition* $C_2 = H_7 = 0$ *holds. Then this system possesses one of the configurations of invariant lines indicated below if and only if the corresponding conditions are satisfied respectively:*

$$
\begin{array}{lcl}
H_{12} \neq 0,\ H_{11} < 0 & \Leftrightarrow & \textit{Config. } C_2.6;\\
H_{12} \neq 0,\ H_{11} > 0 & \Leftrightarrow & \textit{Config. } C_2.5;\\
H_{12} \neq 0,\ H_{11} = 0 & \Leftrightarrow & \textit{Config. } C_2.7;\\
H_{12} = 0,\ H_{11} \neq 0 & \Leftrightarrow & \textit{Config. } C_2.8;\\
H_{12} = 0,\ H_{11} = 0 & \Leftrightarrow & \textit{Config. } C_2.9.
\end{array}
$$

As all the cases were examined we conclude that Theorem 6.1 is proved. ∎

7 Geometric invariants and the proof of the non-equivalence of the 122 configurations

In this section we complete the proof of Theorem 6.1 by showing that all 122 configurations of invariant lines we constructed are non-equivalent according to Definition 2.1.

First of all we point out that in the papers [21] and [23] the non-equivalence of the configurations containing invariant lines of the total multiplicity 4, 5 or 6 was proved. Moreover in [27] the same is done for the configurations of the systems with the infinite line filled up with singularities. So in this section we focus our attention to the configurations of the systems in $\boldsymbol{QSL^{2p}}$ possessing invariant lines of total multiplicity exactly 3. We denote this family by $\boldsymbol{QSL_3^{2p}}$

To prove the non-equivalence of the configurations we obtained for this family we introduce first some notations.

Notation 7.1. *Let* $(S) \in \boldsymbol{QSL}$. *Let us denote*

$$\mathbf{IL}(S) = \left\{ \ l \ \middle| \ \begin{array}{c} l \textit{ is a line in } \mathbf{P}_2(\mathbb{C}) \textit{ such} \\ \textit{that } l \textit{ is invariant for } (S) \end{array} \right\};$$

$$m(l) = \textit{the multiplicity of the invariant line } l \textit{ of } (S).$$

We denote by $Sing_{\mathbb{R}}(S)$ the set of all real singular points of the system (S).

$$N_{\mathbb{R}}^{\mathbf{IL}} = \#\{l \in \mathbf{IL} | l : aX + bY + cZ = 0, a, b, c \in \mathbb{R}\};$$

Remark 7.1. *In defining* $m(l)$ *we assume, of course, that* (S) *has a finite number of invariant lines. We note that the line* $l_\infty : Z = 0$ *is included in* $\mathbf{IL}(S)$ *for any* $(S) \in \boldsymbol{QSL}$. *We also note that both* $m(l)$ *and* $N_{\mathbb{R}}^{\mathbf{IL}}$ *are invariants under the group action.*

Let $\mathcal{F}_i(X, Y, Z) = 0$ be the projective completion of the invariant line $f_i(x, y) = 0$ and let m_i be its multiplicity. Also let m be the multiplicity of the line at infinity $Z = 0$ We introduce the following additional notations.

$$\mathcal{G} : \prod_i \mathcal{F}_i(X, Y, Z)^{m_i} Z^m = 0;$$

$$Sing_{\mathbb{R},\mathcal{G}}(S) = \{w \in Sing_{\mathbb{R}}(S) | w \in \mathcal{G}\}$$

Definition 7.1. *Definition 2.11. Let* V *be an irreducible algebraic variety of dimension* n *over a field* K. *A cycle of dimension* r *or* r*-cycle on* V *is a formal sum* $\sum_W m(W)W$ *where* W *is a subvariety of* V *of dimension* r *which is not contained in the singular locus of* V, $m(W) \in \mathbb{Z}$, *and only a finite number of* $m(W)$*'s are non-zero. We call degree of an* r*-cycle the sum* $\sum_W$. *An* $(n-1)$*-cycle is called a divisor.*

Definition 7.2. *We call type of an* r*-cycle the set of all ordered couples* (n_1, n_2) *where* n_1 *is a coefficient,* $n_1 = m(W)$ *appearing in the* $r - cyle$ *and* n_2 *is the number of* W*'s in the cycle whose coefficient is* $m(W)$.

We denote the type of an r-cycle D by $\mathcal{T}(D)$.

We now introduce three cycles on $\mathbb{P}_2(\mathbb{R})$ which encapsulate the basic features of the configurations, i.e. i) the invariant lines together with their multiplicities and ii) the real singular points located on the invariant lines together with their multiplicities.

The first one of these two cycles is the divisor on $\mathbb{P}_2(\mathbb{R})$ that encapsulates the multiplicities of the invariant lines:

$$D_M = \sum_{l\in IL} m(l)l$$

The second one is the zero-cycle that encapsulates the multiplicities of the real singular points located on the algebraic invariant curves or equivalently on the curve $\mathcal{G}$:

$$D_{\mathbb{R},\mathcal{G}} = \sum_{w\in Sing_{\mathbb{R},\mathcal{G}}} m(w)w$$

The third cycle we need to consider is the multiplicity divisor of real singularities on the line at infinity $l_\infty : Z = 0$

$$D_{\mathbb{R},l_\infty} = \sum_{w\in Sing_{\mathbb{R},l_\infty}} m(w)w$$

The *types* of these three cycles are affinely invariant and we use them to distinguish the configurations.

We first define a few additional geometric invariants.

$$\begin{aligned}
n^\infty &= \#\{w \in Supp\,\mathbf{D}_S(C,Z)\,\big|\, w \in \mathbf{P}_2(\mathbb{C})\};\\
n^\infty_{\mathbb{R}} &= \#\{w \in Supp\,\mathbf{D}_S(C,Z)\,\big|\, w \in \mathbf{P}_2(\mathbb{R})\};\\
N^{\mathbf{IL}}_{\mathbb{R}} &= \#\{l \in Supp\,\mathbf{D}_{\mathbf{IL}}\,\big|\, l : aX+bY+cZ=0, a,b,c\in\mathbb{R}\};\\
m(l_\infty) &= \text{the multiplicity of the invariant line } l_\infty : Z=0 \text{ of } (S);\\
\mathcal{M}_{\mathbf{IL}} &= \max\{m(l)\,|\, l\in \mathbf{IL}(S)\}.
\end{aligned}$$

The note that $n^\infty_{\mathbb{R}} \in \{1,2,3\}$. Furthermore we have $\mathcal{M}_{IL} \in \{1,2,3\}$. Since these are both affinely invariant it is sufficient to distinguish among the configurations that have $(n^\infty_{\mathbb{R}}, \mathcal{M}_{IL}) = (i,j)$ where $i,j \in \{1,2,3\}$.

From the Diagram 5 it is clear that the above defined geometric invariants are sufficient to prove the non-equivalence of all the 53 configurations of invariant lines obtained for the systems in the class $\boldsymbol{QSL_3^{2p}}$.

8 Existence of limit cycles for the class $\mathbf{QSL}_{\geq 2}$

We consider the class of real quadratic polynomial differential systems (5), i.e. the systems

$$\begin{aligned}
\dot{x} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(\tilde{a},x,y),\\
\dot{y} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(\tilde{a},x,y)
\end{aligned} \tag{100}$$

where

$$\begin{aligned}
&p_0 = a,\quad p_1(x,y) = cx+dy,\quad p_2(x,y) = gx^2+2hxy+ky^2,\\
&q_0 = b,\quad q_1(x,y) = ex+fy,\quad q_2(x,y) = lx^2+2mxy+ny^2.
\end{aligned}$$

According to [10, 11] Theorem 1.3 is valid, i.e. every real quadratic differential system (100) possessing a real affine invariant line has at most one limit cycle.

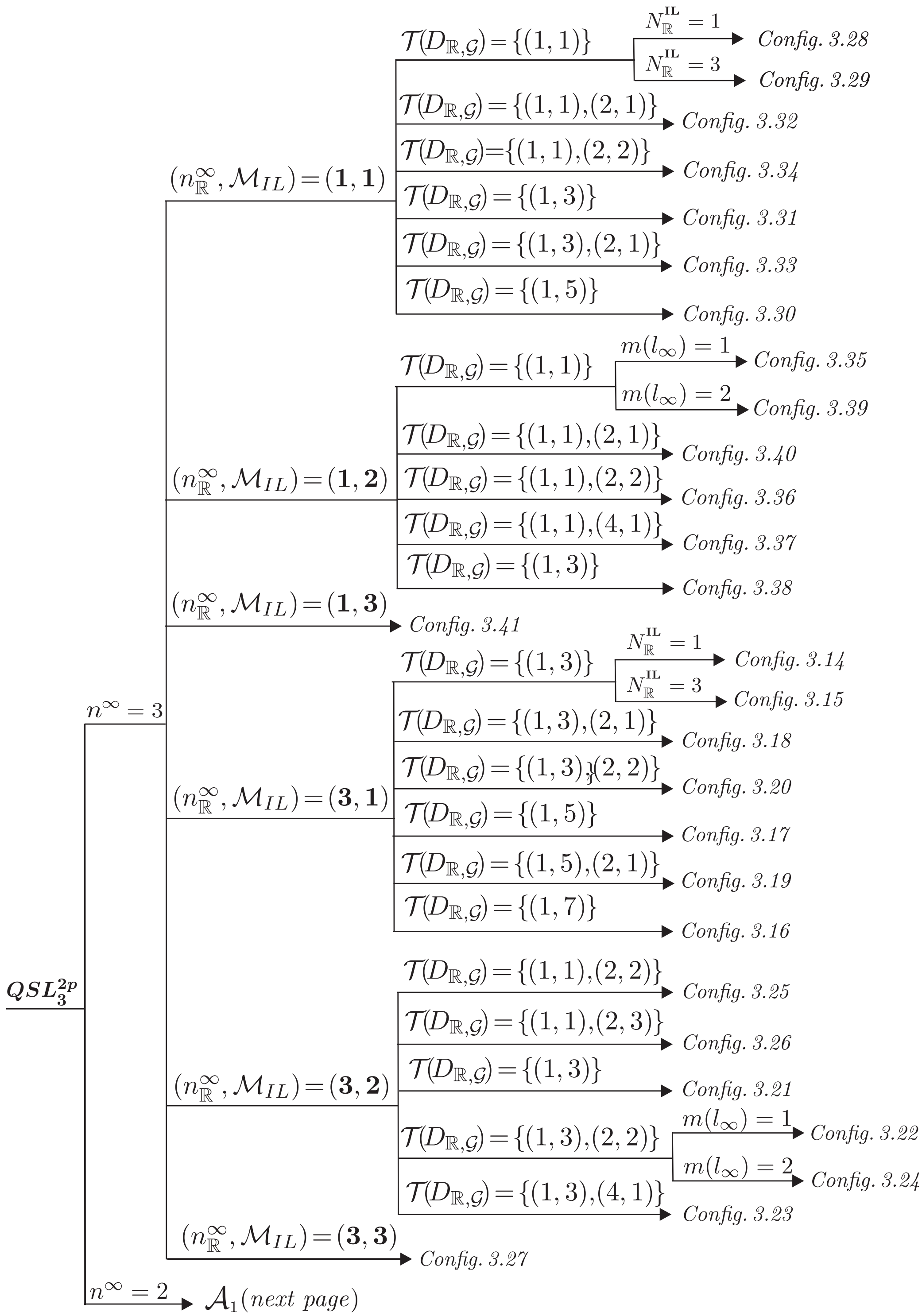

Diagram 5: The non-equivalence of the configurations of systems in $\mathbf{QSL^{2p}}$

We prove here the following theorem.

Theorem 8.1. *Every real quadratic differential system* (5) *belonging to the class* $\boldsymbol{QSL}_{\geq 2}$ *has at most one limit cycle, i.e. the Hilbert number of this class* $H(\boldsymbol{QSL}_{\geq 2})=1$.

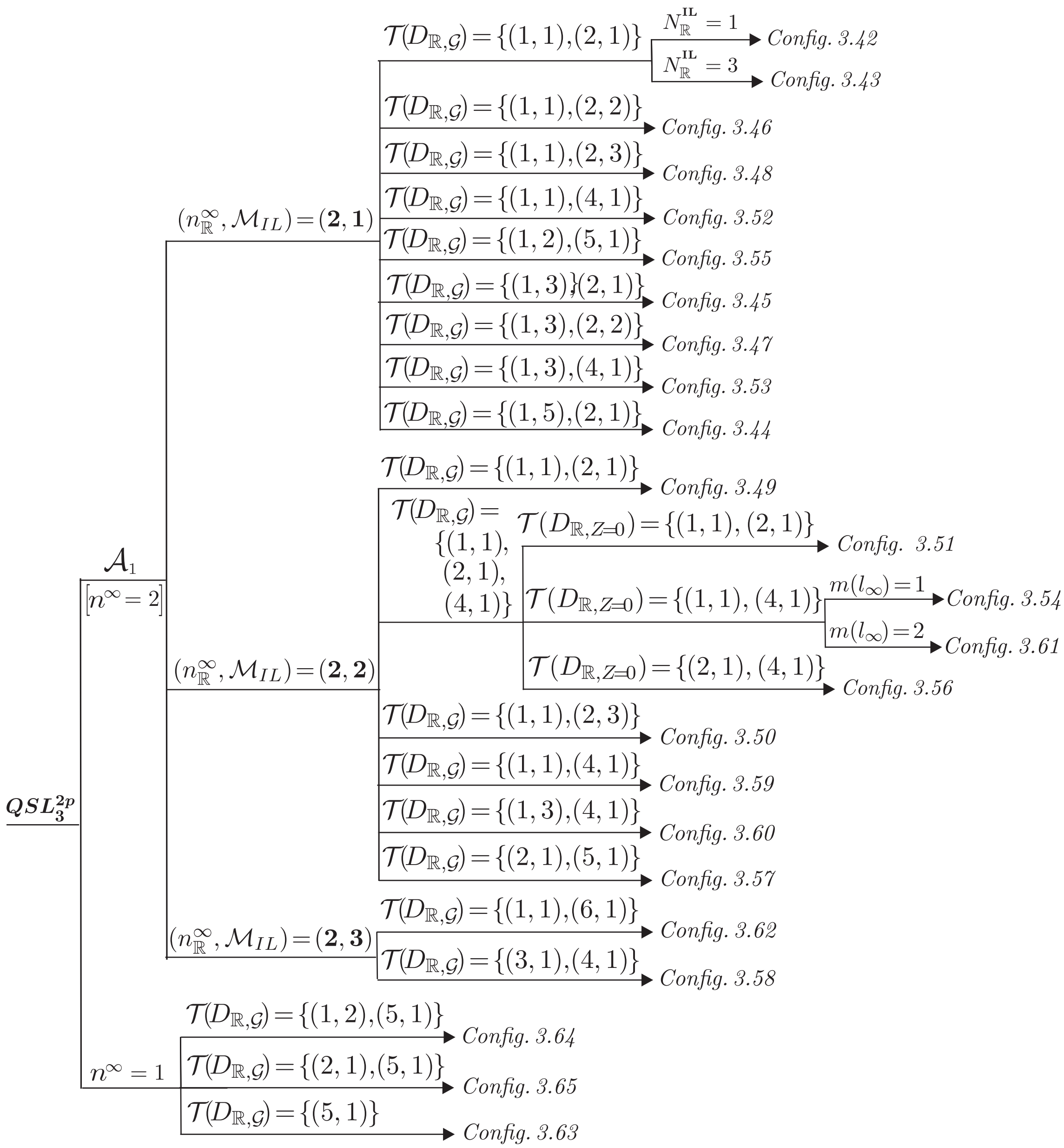

Diagram 5 *(continuation)*: The non-equivalence of the configurations of systems in $\mathbf{QSL^{2p}}$

Proof: In the papers [25] and [26] the topological classification of quadratic systems possessing invariant lines of total multiplicity at least four is given. From this classification it follows that if a quadratic system (S) belongs to the class $\mathbf{QSL}_{\geq 4}$ then it could not possess a limit cycle.

Assume now that a quadratic system (S) possesses invariant line of total multiplicity either 2 or 3. If this system has at least one real invariant affine line then by Theorem 1.3 it can have at most one limit cycle. If (S) does not have any real affine invariant line then we could only have the following possibilities:

(i) system (S) has only complex invariant lines on its phase plane;

(ii) system (S) does not have invariant lines on the finite part of its phase plane.

We examine each one of these possibilities.

(i) Since a complex affine invariant line must have its conjugate we deduce that in this case $(S) \in\mathbf{QSL}_3$, i.e. it possess two complex invariant lines on the phase plane.

If these complex lines have a finite intersecting point, then according to [31] (see also [4]) the system (S) could possess at most one limit cycle.

Assume now that the two complex lines are parallel intersecting at infinity. Then by Lemma 1.1 the system (S) could be brought via an affine transformation to the form (2) and evidently all finite singularities of (S) (if there exist any) are located on the parallel invariant lines and hence have complex coordinates. So no limit cycle could exist.

(ii) Suppose that the system (S) does not possess invariant affine lines. Since this system possesses invariant lines of total multiplicity either 2 or 3 we deduce that the infinite invariant line must be of the multiplicity at least 2.

On the other hand considering Lemma 4.6 (see statement 2) we deduce, that if the line $l_\infty : Z = 0$ is of multiplicity > 1 then $Z \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$. In other words Z is a common factor of the polynomials $\mathcal{E}_1(X, Y, Z)$ and $\mathcal{E}_2(X, Y, Z)$ (see Notation 4.2).

Taking into account the definition of the invariant polynomials $\mathcal{E}_1(X, Y, Z)$ and $\mathcal{E}_2(X, Y, Z)$ (see Notations 4.1 and 4.2) for systems (100) we calculate

$$\begin{aligned}
\mathcal{E}_1(X, Y, Z) =&\frac{1}{2}C_2(X, Y)\widetilde{K}(X, Y) + \phi_1(X, Y)Z + \phi_2(X, Y)Z^2 + \ldots + \phi_5(X, Y)Z^5,\\
\mathcal{E}_2(X, Y, Z) =&C_2(X, Y)\Psi(X, Y) + \psi_1(X, Y)Z + \psi_2(X, Y)Z^2 + \ldots + \psi_6(X, Y)Z^6,
\end{aligned}$$

where

$$\begin{aligned}
C_2(X, Y) =& - lX^3 + (g - 2m)X^2Y + (2h - n)XY^2 + kY^3,\\
\widetilde{K}(X, Y) =& 4\big[(gm - hl)X^2 + (gn - kl)XY + (hn - km)Y^2\big] \equiv 4\big[\alpha X^2 + \beta XY + \gamma Y^2\big],\\
\Psi(X, Y) =& (2g\alpha + l\beta)X^3 + \big[(4h + 2n)\alpha + g\beta + 4l\gamma\big]X^2Y\\
&+ \big[2k\alpha + (2h + n)\beta + 4m\gamma\big]XY^2 + (k\beta + 2n\gamma)Y^3.
\end{aligned}$$

Therefore we conclude that the invariant polynomials $\mathcal{E}_1(X, Y, Z)$ and $\mathcal{E}_2(X, Y, Z)$ have Z as a common factor if and only if the conditions $C_2(X, Y)\widetilde{K}(X, Y) = C_2(X, Y)\Psi(X, Y) = 0$ hold. Since $C_2 = 0$ leads to systems with the line at infinity filled up with singularities (see Lemma 4.4) clearly the condition $C_2 \neq 0$ has to be satisfied.

On the other hand we observe that the condition $\widetilde{K}(X, Y) = 0$ implies $\alpha = \beta = \gamma = 0$ and then $\Psi(X, Y) = 0$. Therefore we conclude that $\widetilde{K}(X, Y) = 0$ is necessary and sufficient for a quadratic system to have the invariant line at infinity of the multiplicity at least 2.

We point out that the geometric meaning of the condition $\widetilde{K}(X, Y) = 0$ was revealed in [21] (see Lemma 42), where it was proved that the degree of gcd $\big(p_2(x, y), q_2(x, y)\big) = 2$ if and only if for systems (5) the condition $\widetilde{K} = 0$ holds. This means that the homogeneous quadratic parts of these systems are proportionally, i.e. we have $up_2(x, y)+vq_2(x, y) = 0$ for $u, v \in \mathbb{R}$, $u^2+v^2 \neq 0$. Then applying the linear transformation $x_1 = ux+vy$, $y_1 = -vx+uy$ with $\det = u^2 + v^2 \neq 0$ we obtain systems having a linear first equation, i.e. we arrive at the systems (keeping the old notations)

$$\dot{x} = a + cx + dy, \quad \dot{y} = q_0 + q_1(x,y) + q_2(x,y). \tag{101}$$

It is clear that the invariant line at infinity for these systems has the multiplicity at least 2, because the simple perturbation of the first equation $\dot{x} = (a + cx + dy)(1 + \varepsilon x)$ ($|\varepsilon| \ll 1$) generates an invariant affine line which tends to infinity when $\varepsilon \to 0$. Evidently that for a sufficiently small parameter ε this perturbation conserves two or more limit cycles if they exist. However this leads to a contradiction with Theorem 1.3.

Thus we conclude that a quadratic system with multiple infinite invariant line could not have more than one limit cycle and this completes the proof of our theorem. ∎

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