A general form of Green Formula and Cauchy Integral Theorem

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Abstract

We prove a general form of Green Formula and Cauchy Integral Theorem for arbitrary closed rectifiable curves in the plane. We use Vitushkin's localization of singularities method and a decomposition of a rectifiable curve in terms of a sequence of Jordan rectifiable sub-curves due to Carmona and Cufí.

1 Introduction

In this paper we prove a general form of Green Formula and Cauchy Integral Theorem for arbitrary closed rectifiable curves in the plane. A closed rectifiable curve is a complex valued mapping γ of bounded variation defined on the unit circle $\mathbb{T}$. We adopt the standard abuse of notation consisting in denoting by γ also the image of the unit circle under the mapping. Recall that the winding number or index of a closed rectifiable curve γ with respect to a point $z \notin \gamma$ is

$$\operatorname{Ind}(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}.$$

Set

$$D = \{z \in \mathbb{C} : \operatorname{Ind}(\gamma, z) \neq 0\}$$

and

$$D_0 = \{z \in \mathbb{C} : \operatorname{Ind}(\gamma, z) = 0\}.$$

The sets D and D_0 are open and a countable union of connected components of $\mathbb{C} \setminus \gamma$. We let dA stand for planar Lebesgue measure and $\overline{\partial} = \partial/\partial\overline{z}$ for the usual Cauchy-Riemann operator. We then have the following.

Theorem. *Let γ be a closed rectifiable curve and let f be a continuous function on $D \cup \gamma$ such that the $\overline{\partial}$ derivative of f in D, in the sense of distributions, belongs to $L^2(D)$. Then*

$$\int_\gamma f(z)\, dz = 2i \int_D \overline{\partial} f(z) \operatorname{Ind}(\gamma, z)\, dA(z).$$

Prepublicació Núm 09, juliol 2013.
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http://www.uab.cat/matematiques
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If D is empty then the right hand side in (1) is 0 and the identity is straightforward. Notice that the integral in the right hand side is absolutely convergent because the function $\mathrm{Ind}(\gamma, z)$ is in $L^2(\mathbb{C})$. This was proven with the best constant in [CC1]. It is also a consequence of the Sobolev imbedding Theorem for $p = 1$ and the fact that $\mathrm{Ind}(\gamma, z)$ is a function of bounded variation. Indeed

$$\overline{\partial}(\mathrm{Ind}(\gamma, z)) = \frac{dz}{2i} \qquad \text{and} \qquad \partial(\mathrm{Ind}(\gamma, z)) = -\frac{\overline{dz}}{2i}.$$

It is not true in general that $\mathrm{Ind}(\gamma, z) \in L^p(\mathbb{C})$ for some $p > 2$. Our proof works also under the assumption that $\overline{\partial} f(z) \, \mathrm{Ind}(\gamma, z) \in L^1(D)$.

Corollary. *Let γ be a closed rectifiable curve and let f be a holomorphic function on D which is continuous on $D \cup \gamma$. Then*

$$\int_\gamma f(z)\, dz = 0.$$

With the extra hypothesis that $\mathrm{Ind}(\gamma, z)$ is bounded on $\mathbb{C} \setminus \gamma$ the Corollary was proven by Nöbeling in 1949 [N]. In fact, the corollary is proven in [M] as a consequence of an approximation theorem of γ by chains formed by boundaries of squares contained in D. Michael's approximation theorem coupled with a regularization argument can be used to give a proof of the Theorem above. Our proof, which we found before becoming aware of [M], keeps the curve fixed and, instead, the function f is suitably approximated.

Combining the Theorem with a well known result of Fesq [F] (see also [Co]) one obtains an appealing statement in which no distributions theory is involved. Fesq proved the following. Assume that a function f is defined and continuous on an open set Ω and has partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ at each point of $\Omega \setminus E$, where E is a countable union of closed sets of finite length (one dimensional Hausdorff measure). Assume further that the measurable function $\overline{\partial} f(z) = \frac{1}{2}(\frac{\partial f}{\partial x}(z) + i\frac{\partial f}{\partial y}(z))$ defined for $z \in \Omega \setminus E$, belongs to $L^1_{loc}(\Omega)$. We emphasize that now $\overline{\partial} f(z)$ is not defined in the sense of distributions but only pointwise. Then

$$\int_{\partial Q} f(z)\, dz = 2i \int_Q \overline{\partial} f(z)\, dA(z), \tag{1}$$

for each square Q with closure contained in Ω. It is not difficult to realize that (1) implies that the pointwise $\overline{\partial}$-derivative of f on Ω is indeed the distributional $\overline{\partial}$-derivative of f on Ω. For the sake of completeness a proof of this simple fact is presented in section 4. Therefore we obtain the following variation of the Theorem.

Theorem (pointwise version). *Let γ be a closed rectifiable curve. Let f be a continuous function on $D \cup \gamma$ whose partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at each point of $D \setminus E$, where E is a countable union of closed sets of finite length (one* *dimensional Hausdorff measure), and such that $\overline{\partial} f \in L^2(D)$, where $\overline{\partial} f$ is defined pointwise almost everywhere on D. Then*

$$\int_\gamma f(z)\, dz = 2i \int_D \overline{\partial} f(z) \operatorname{Ind}(\gamma, z)\, dA(z).$$

In [M] a weaker version of the preceding result is proven under the extra assumption that $\partial f(z)/\partial x \operatorname{Ind}(\gamma, z)$ and $\partial f(z)/\partial y \operatorname{Ind}(\gamma, z)$ are in $L^1(D)$.

In section 2 we present the proof of the Theorem and we leave for section 3 the discussion of the Main Lemma. The main tool in the proof is the method of separation of singularities due to Vitushkin (see [G], [V] or [Vi]). This is quite natural because in many instances Cauchy Integral Theorem is reduced to the case in which more regular functions are involved via uniform approximation of the given data. For instance, if D is a Jordan domain and γ its boundary, then one can approximate f, uniformly on D, by polynomials in z, for which the result is obvious.

Vitushkin's method produces a large finite sum. The terms in this sum are divided into three classes and in estimating the class which involves more directly the curve we decompose γ in a sum, in most cases infinite, of Jordan curves. This decomposition is a consequence of [CC2, Theorem 4] and reads as follows.

Theorem (Carmona and Cufí). *For each closed rectifiable curve γ such that $D \neq \emptyset$ there exists a sequence (maybe finite) of Jordan curves $(\gamma_n)_{n=1}^\infty$ with the property that $\gamma_n \subset \gamma$, $dz_\gamma = \sum_{n=1}^\infty dz_{\gamma_n}$ and $\sum_{n=1}^\infty l(\gamma_n) \leq l(\gamma)$.*

Here $l(\gamma)$ stands for the length of the curve γ and $dz_\gamma = \sum_{n=1}^\infty dz_{\gamma_n}$ means that $\int_\gamma f(z)\, dz = \sum_{n=1}^\infty \int_{\gamma_n} f(z)\, dz$, for any continuous function f on γ.

The estimate we are looking for is then reduced to the case of a Jordan curve, which is dealt with in the Main Lemma.

A word on the existing literature on Green Formula and Cauchy Integral Theorem is in order. Burckel, in his well-known comprehensive book on Classical Complex Analysis [B, p. 341], states that the more general Cauchy Theorem he knows is that due to Nöbeling ([N]), in which the index of the curve is assumed to be bounded. He seems to be unaware of Michael's article [M], which is apparently widely unknown. We believe that the general form of Green Formula and Cauchy Integral Theorem involving arbitrary rectifiable curves and functions defined in the minimal domain $D \cup \gamma$ deserves to be better known.

2 Proof of the Theorem

We first describe Vitushkin's scheme to separate singularities of functions. The first step is the construction of partitions of unity subordinated to special coverings of the plane by discs of equal radii.

Lemma 1. *Given any $\delta > 0$ there exists a countable family of discs (Δ_j) of radius δ and a family of functions $\varphi_j \in C_0^\infty(\Delta_j)$ such that* *(i)* $\mathbb{C} = \cup_j \Delta_j$.

(ii) The family (Δ_j) *is almost disjoint, that is, for some constant* C *each* $z \in \mathbb{C}$ *belongs to at most* C *discs* Δ_j *(in fact we can take* $C = 21$*).*

(iii) $\sum_j \varphi_j = 1$, $0 \le \varphi_j$ *and* $|\nabla \varphi_j(z)| \le C\delta^{-1}$, $z \in \mathbb{C}$, *where* C *is an absolute constant.*

For the proof, take a grid of squares of side length $\delta/2$ and regularize the characteristic function of each square with an appropriate approximation of the identity (see [V, p.440-441], [G] or [Vi].

Given a compactly supported continuous function f on the plane, set

$$f_j = \frac{1}{\pi z} * \varphi_j \, \overline{\partial} f, \tag{2}$$

which makes sense, because it is the convolution of the compactly supported distribution $\varphi_j \, \overline{\partial} f$ with the locally integrable function $\frac{1}{\pi z}$. Since $\frac{1}{\pi z}$ is the fundamental solution of the differential operator $\overline{\partial}$, we have $\overline{\partial} f_j = \varphi_j \, \overline{\partial} f$ and thus f_j is holomorphic where f is and off a compact subset of Δ_j. It is easy to see that

$$f_j(z) = \frac{1}{\pi} \int \frac{f(w) - f(z)}{w - z} \, \overline{\partial} \varphi_j(w) \, dA(w)$$

and hence

$$|f_j(z)| \le C \, \omega(f, \delta), \quad z \in \mathbb{C},$$

where C is an absolute constant and $\omega(f, \delta)$ is the modulus of continuity of f. Since the family of functions (φ_j) is a partition of the unity,

$$f = \sum_j f_j. \tag{3}$$

If one defines a singularity of f as a point in the support of $\overline{\partial} f$, then clearly the effect of (3) is to distribute the singularities of f among the discs Δ_j. Notice that the sum in (3) contains only finitely many non-zero terms, because the support of $\overline{\partial} f$ is compact.

Having set up these preliminaries, let us start the proof of the Theorem. We will consider only the case in which D is not empty; otherwise the conclusion is straightforward. Extend the function f to a compactly supported continuous function on the plane, fix a $\delta > 0$ and apply (3). We divide the indexes j into the following three classes :

$$I = \{j : \Delta_j \subset D\},$$
$$II = \{j : \Delta_j \cap \gamma \ne \emptyset\},$$

and

$$III = \{j : \Delta_j \subset D_0\}.$$

For $j \in III$ we have $\int_\gamma f_j(z)\,dz = 0$, because f_j is holomorphic on an open set in which γ is homologous to zero. Hence

$$\sum_{j\in III} \int_\gamma f_j(z)\,dz = 0.$$

For $j \in I$ use the definition of f_j in (2) and Fubini's theorem to get

$$\int_\gamma f_j(z)\,dz = 2i \int_{\mathbb{C}} \varphi_j(w)\,\overline{\partial} f(w)\,\mathrm{Ind}(\gamma, w)\,dA(w).$$

Adding up in $j \in I$ one obtains

$$\sum_{j\in I} \int_\gamma f_j(z)\,dz = 2i \int_{\mathbb{C}} \left(\sum_{j\in I} \varphi_j(w)\right) \overline{\partial} f(w)\,\mathrm{Ind}(\gamma, w)\,dA(w).$$

Since

$$\lim_{\delta\to 0} \sum_{j\in I} \varphi_j(w) = 1, \quad w \in D$$

and $\overline{\partial} f(w)\,\mathrm{Ind}(\gamma, w) \in L^1(D)$, it follows, by dominated convergence, that

$$\lim_{\delta\to 0} \sum_{j\in I} \int_\gamma f_j(z)\,dz = 2i \int_{\mathbb{C}} \overline{\partial} f(w)\,\mathrm{Ind}(\gamma, w)\,dA(w).$$

Therefore

$$\int_\gamma f(z)\,dz = 2i \int_{\mathbb{C}} \overline{\partial} f(z)\,\mathrm{Ind}(\gamma, z)\,dA(z) + \lim_{\delta\to 0} \sum_{j\in II} \int_\gamma f_j(z)\,dz$$

and so to complete the proof it is enough to check that the limit in the above right hand side vanishes. This follows from the inequality

$$\sum_{j\in II} \left| \int_\gamma f_j(z)\,dz \right| \le C\,\omega(f,\delta)\,l(\gamma) + \eta(\delta), \tag{4}$$

where C is an absolute constant and $\eta(\delta)$ a function which tends to zero with δ. To show (4) fix $j \in II$. One has

$$\begin{aligned} \left| \int_\gamma f_j(z)\,dz \right| &= \left| \int_{\gamma\cap\overline{\Delta}_j} f_j(z)\,dz \right| + \left| \int_{\gamma\cap(\overline{\Delta}_j)^c} f_j(z)\,dz \right| \\ &\le C\,\omega(f,\delta)\,l(\gamma\cap\overline{\Delta}_j) + \left| \int_{\gamma\cap(\overline{\Delta}_j)^c} f_j(z)\,dz \right|. \end{aligned} \tag{5}$$

Observe that adding up on $j \in II$ the first terms in right hand side of the inequality above one gets the desired estimate, namely,

$$\sum_{j \in II} \omega(f,\delta)\, l(\gamma \cap \overline{\Delta}_j) \leq C\, \omega(f,\delta)\, l(\gamma),$$

where we used that the family of discs Δ_j is almost disjoint. However, the obvious estimate for the second term

$$\left| \int_{\gamma \cap (\overline{\Delta}_j)^c} f_j(z)\, dz \right| \leq C\, \omega(f,\delta)\, l(\gamma \cap (\overline{\Delta}_j)^c)$$

does not lead anywhere because the length of γ off the disc $\overline{\Delta}_j$ is not under control. To overcome this difficulty we resort to the next lemma.

Main Lemma. *Let Γ be a closed rectifiable Jordan curve, Δ a disc of radius δ and h a bounded continuous function on $\mathbb{C}$, holomorphic off a compact subset of Δ. Then*

$$\left| \int_{\Gamma \cap (\overline{\Delta})^c} h(z)\, dz \right| \leq 2\pi\, \|h\|_\infty\, \delta, \tag{6}$$

where $\|h\|_\infty$ is the supremum norm of h on the whole plane.

We postpone the proof of the Main Lemma to next section and we proceed to finish the proof of the Theorem.

First of all we prove (4) with $\eta \equiv 0$ assuming that $\gamma = \Gamma$ is a Jordan curve.

Consider the case $\delta \leq \frac{1}{2}\operatorname{diam}(\Gamma)$. Since there is a point in $\Gamma \cap \partial\Delta_j$ the length of the curve in the disc $3\Delta_j$ is larger than 2δ. Combining (5) with the Main Lemma applied to f_j and Δ_j we conclude that

$$\left| \int_\Gamma f_j(z)\, dz \right| \leq C\, \omega(f,\delta)\, l(\Gamma \cap (3\Delta_j)),$$

which yields (4) with $\eta \equiv 0$, because the family of discs $(3\Delta_j)$ is almost disjoint.

If $\delta > \frac{1}{2}\operatorname{diam}(\Gamma)$, then the number of discs Δ_j that intersect Γ is less than an absolute constant. Thus (4) with $\eta \equiv 0$ and γ replaced by Γ also holds in this case.

Now we will reduce the proof of (4) to the case of a Jordan curve by appealing to the decomposition theorem of [CC2] mentioned in the introduction. There is a sequence (maybe finite) of rectifiable Jordan curves $(\gamma_n)_{n=1}^\infty$ such that $\gamma_n \subset \gamma$, $dz_\gamma = \sum_{n=1}^\infty dz_{\gamma_n}$ and $\sum_{n=1}^\infty l(\gamma_n) \leq l(\gamma)$. We need to decompose II into two subsets. Let II_0 be the set of indices $j \in II$ such that D_j does not intersect any γ_n and $II_1 = II \setminus II_0$. Since the set of indices II is finite, we have

$$\begin{aligned} \sum_{j \in II_1} \left| \int_\gamma f_j(z)\, dz \right| &\leq \sum_{j \in II_1} \sum_{n=1}^\infty \left| \int_{\gamma_n} f_j(z)\, dz \right| \\ &= \sum_{n=1}^\infty \sum_{j \in II_1} \left| \int_{\gamma_n} f_j(z)\, dz \right|. \end{aligned}$$

Given $n = 1, 2, \ldots$ set $II_n = \{j : \Delta_j \cap \gamma_n \neq \emptyset\}$. Each $j \in II_1$ belongs to at least one II_n, but may belong to several. Taking into account this remark in the first inequality below and applying (4) with $\eta \equiv 0$ to the Jordan curve γ_n and the function f in the second one, we get

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{j \in II_1} \left| \int_{\gamma_n} f_j(z)\, dz \right| &\leq \sum_{n=1}^{\infty} \sum_{j \in II_n} \left| \int_{\gamma_n} f_j(z)\, dz \right| \\ &\leq \sum_{n=1}^{\infty} C\, \omega(f, \delta)\, l(\gamma_n) \\ &\leq C\, \omega(f, \delta)\, l(\gamma), \end{aligned}$$

which is the right estimate.

We turn now our attention to the sum over II_0. If $j \in II_0$ then $\operatorname{Ind}(\gamma, z) = \sum_{n=1}^{\infty} \operatorname{Ind}(\gamma_n, z)$ is constant on $\Delta_j \setminus \gamma$. Hence $\Delta_j \setminus \gamma \subset D_0$ or $\Delta_j \setminus \gamma \subset D$.

In the first case we argue as we did for $j \in III$. The infinite cycle $\sum_{n=1}^{\infty} \gamma_n$ is homologous to 0 in an open set on which f_j is holomorphic and therefore $\int_\gamma f_j(z)\, dz = \sum_{n=1}^{\infty} \int_{\gamma_n} f_j(z)\, dz = 0$. This follows by the usual argument to prove Cauchy's Theorem for finite cycles.

To settle the case $\Delta_j \setminus \gamma \subset D$ we need to know that the $\overline{\partial}$-derivative of f on Δ_j does not charge the set $\gamma \cap \Delta_j$.

Lemma 2. *Let Δ be an open disc and assume that $\Delta \setminus \gamma \subset D$. Then the $\overline{\partial}$-derivative of f on Δ in the distributions sense is the function $\overline{\partial} f(z)\, \chi_{\Delta \setminus \gamma}(z)$.*

Proof. It is shown in Appendix 1(section 4) that the conclusion of the Lemma follows from the identity

$$\int_{\partial Q} f(z)\, dz = 2i \int_Q \overline{\partial} f(z)\, \chi_{\Delta \setminus \gamma}(z)\, dA(z), \tag{7}$$

for each closed square Q with sides parallel to the coordinates axis contained in Δ. To prove (7) subdivide Q in dyadic sub-squares. At the n-th generation one has 4^n dyadic sub-squares of Q, denoted by Q_k, $1 \leq k \leq 4^n$. Their side length is $L\, 4^{-n}$, L being the side length of Q. Set $I = \{k : Q_k \cap \gamma = \emptyset\}$ and $J = \{k : Q_k \cap \gamma \neq \emptyset\}$. Then

$$\begin{aligned} \int_{\partial Q} f(z)\, dz &= \sum_{k \in I} \int_{\partial Q_k} f(z)\, dz + \sum_{k \in J} \int_{\partial Q_k} f(z)\, dz \\ &= \sum_{k \in I} 2i \int_{Q_k} \overline{\partial} f(z)\, dA(z) + \sum_{k \in J} \int_{\partial Q_k} \left(f(z) - f(z_k)\right)\, dz \\ &\equiv T_1(n) + T_2(n), \end{aligned}$$

where z_k is any point in Q_k and the last identity is a definition of $T_1(n)$ and $T_2(n)$. On the one hand it is clear that

$$\lim_{n\to\infty} T_1(n) = 2i \int_Q \overline{\partial} f(z)\, \chi_{\Delta\setminus\gamma}(z)\, dA(z)$$

and on the other hand, setting $\epsilon_n = \sqrt{2}L4^{-n}$, we have

$$|T_2(n)| \leq \omega(f, \epsilon_n) \sum_{k\in J} l(\partial Q_k).$$

Take n big enough so that $\operatorname{diam}(2Q_k) = \sqrt{2}\, 2L4^{-n} < \operatorname{diam}(\gamma)$. Then $l(\partial Q_k) \leq 8\, l(2Q_k \cap \gamma)$ and thus

$$|T_2(n)| \leq C\, \omega(f, \epsilon_n) l(\gamma),$$

because the family of squares $2Q_k, 1 \leq k \leq 4^n$, is almost disjoint (with an absolute constant). Letting $n \to \infty$ we get (7). □

Denote by II_2 the set of indices $j \in II_0$ such that $\Delta_j \setminus \gamma \subset D$. For $j \in II_2$, by Fubini's theorem and Lemma 2,

$$\begin{aligned}\int_\gamma f_j(z)\, dz &= 2i \int_{\mathbb{C}} \varphi_j(w)\, \overline{\partial} f(w)\, \chi_{\Delta_j\setminus\gamma}(w)\ \operatorname{Ind}(\gamma, w)\, dA(w)\\ &= 2i \int_D \varphi_j(w)\, \overline{\partial} f(w) \operatorname{Ind}(\gamma, w)\, dA(w),\end{aligned}$$

which is the same relation we found for indices $j \in I$. It is also clear that for some Borel subset E of γ, $\lim_{\delta\to 0} \sum_{j\in II_2} \varphi_j(z) = \chi_E(z)$, $z \in \mathbb{C}$. Therefore

$$\sum_{j\in II_0} \left| \int_\gamma f_j(z)\, dz \right| \leq 2 \int_D \left(\sum_{j\in II_2} \varphi_j(z) \right) |\overline{\partial} f(z) \operatorname{Ind}(\gamma, z)|\, dA(z) \equiv \eta(\delta)$$

and, since $E \subset \gamma$, $\eta(\delta)$ tends to zero with δ. This completes the proof of (4) and of the Theorem.

3 Proof of the Main Lemma

We can assume , without loss of generality, that Γ intersects the circle $\partial\Delta$ in finitely many points. Indeed, by the Banach Indicatrix Theorem [Na, p. 225] applied to the function of bounded variation $|\gamma|$, there is a sequence of numbers $\lambda_n > 1$ with limit 1 such that Γ intersects $\partial(\lambda_n\Delta)$ in finitely many points. It is readily shown that the inequality (6) follows as soon as one knows it for the discs $\lambda_n\Delta$. Assume then that Γ intersects the circle $\partial\Delta$ in finitely many points.

We claim that there are finitely many subintervals (I_k) of the circle $\partial\Delta$, which are mutually disjoint, such that

$$\int_{\Gamma\cap(\overline{\Delta})^c} h(z)\,dz = \sum_k \epsilon_k \int_{I_k} h(z)\,dz, \tag{8}$$

where $\epsilon_k = \pm 1$ determines the orientation on the interval I_k ($\epsilon_k = 1$ corresponds to the counterclockwise orientation of $\partial\Delta$). It is plain that (8) completes the proof of the Main Lemma.

Take a connected component C of $\Gamma \cap (\overline{\Delta})^c$. The open Jordan arc C has two end points on $\partial\Delta$,which determine two complementary open intervals I_1 and I_2 in $\partial\Delta$. Each $I_j, j = 1, 2$, determines a closed Jordan curve C_j which is the union of C and the closure of I_j. We claim that one and only one of the two closed curves C_j has index zero with respect to the center z of Δ. This means that the domain enclosed by this Jordan curve lies completely outside the closed disc $\overline{\Delta}$. To show the claim take a path joining z with ∞ without touching the closure of C. Consider the point w where this path leaves the disc $\overline{\Delta}$ for the last time. Modifying the path by taking first the segment joining z with w, we may assume that it indeed intersects $\partial\Delta$ at only one point w, which belongs to I_1 or I_2. If it lies on I_2 then the index of C_1 with respect to z is 0 and the index of C_2 with respect z is ± 1. The case $w \in I_1$ is symmetric and so the claim is proven.

Given a connected component C of $\Gamma\cap(\overline{\Delta})^c$, we define $I(C)$ to be the interval I_j in the preceding discussion such that the domain enclosed by C_j lies in the complement of $\overline{\Delta}$.

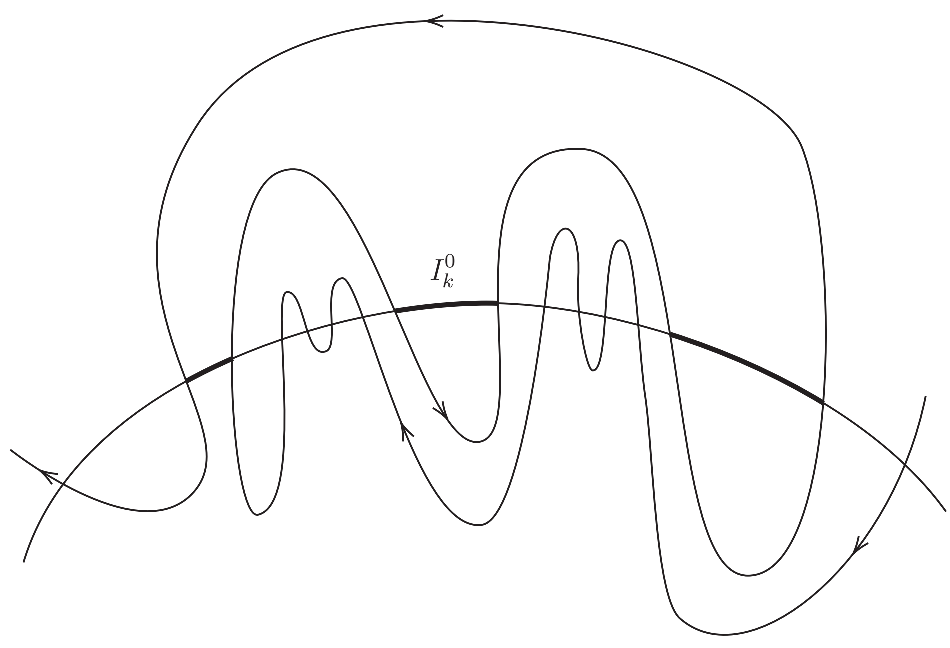

Figure 1

If C_1 and C_2 are two different components of $\Gamma \cap (\overline{\Delta})^c$ then $I(C_1) \subset I(C_2)$ or $I(C_2) \subset I(C_1)$ or the intervals $I(C_1)$ and $I(C_2)$ have disjoint interiors. This "dyadic" structure allows a classification of the intervals $I(C)$ in generations. Since there are finitely many components of $\Gamma \cap (\overline{\Delta})^c$ some of the intervals $I(C)$ are maximal with respect to inclusion. These are said to be of generation 0 and they form a set G_0. If an interval $I(C)$ is contained in exactly one interval of G_0 it is said to be of generation 1. The intervals $I(C)$ of generation 1 form a set G_1. In this way we define inductively intervals $I(C)$ of generation p and the corresponding set G_p of intervals of generation p. Clearly there are only finitely many generations (see Figure 1).

To construct the intervals (I_k) in (8) we proceed inductively as follows. Notice that the subset of $\partial\Delta$

$$\cup_{I \in G_0} \left(I \setminus \cup_{I \supset J \in G_1} J\right)$$

is a union of disjoint intervals. Call this intervals (I_k^0). We then have, for appropriately chosen $\epsilon_k = \pm 1$,

$$\sum_{I(C) \in G_0} \int_C h(z)\, dz + \sum_{I(C) \in G_1} \int_C h(z)\, dz = \sum_k \epsilon_k \int_{I_k^0} h(z)\, dz. \tag{9}$$

This is a consequence of Cauchy's integral theorem applied to the function h and the Jordan curves defined as follows. For each $I(C_0) \in G_0$ define the Jordan curve which consists of the arc C_0, the arcs C such that $I(C) \in G_1$ and $I(C) \subset I(C_0)$ and the intervals $I_k^0 \subset I(C_0)$. The subarcs of Γ keep the orientation of Γ and the orientation on the I_k^0 can be chosen so that (9) holds because of a topological fact that we discuss below.

We say that Γ enters the disc Δ at the point $\Gamma(t_0) \in \partial\Delta$ if there is $\epsilon > 0$ with the property that $\Gamma(t) \in (\overline{\Delta})^c$ for $t_0 - \epsilon < t < t_0$ and $\Gamma(t) \in \Delta$ for $t_0 < t < t_0 + \epsilon$. We say that Γ leaves the disc Δ at the point $\Gamma(t_0) \in \partial\Delta$ if there is $\epsilon > 0$ with the property that $\Gamma(t) \in \Delta$ for $t_0 - \epsilon < t < t_0$ and $\Gamma(t) \in (\overline{\Delta})^c$ for $t_0 < t < t_0 + \epsilon$. There is a third category of points in $\Gamma \cap \partial\Delta$, namely those with the property that the curve just before and just after the point stays either in the disc or in the complement of its closure. We will ignore these points. Consider now two points in $\Gamma \cap \partial\Delta$ at which Γ enters or leaves the disc and assume that in one of the complementary intervals in $\partial\Delta$ determined by these two points there is no other point at which Γ enters or leaves the disc. Then we claim that at one of the two points the curve enters the disc and at the other the curve leaves the disc. In other words, it is not possible that either the curve enters the disc at both points or that the curve leaves the disc at both points. Before embarking in the proof of this claim we remark that, with (9) at our disposition, and arguing inductively with the intervals of generation 2 and subsequent (if any), we finally get (8).

To prove the claim, take two points A and B in $\Gamma \cap \partial\Delta$ at which Γ enters or leaves the disc and assume that in one of the complementary open intervals in $\partial\Delta$ determined by A and B there is no other point at which Γ enters or leaves the disc. Then we have to show that the curve enters the disc at one of the points A and B and leaves the disc at the other.

Proceeding by contradiction, we assume that at A and B the curve leaves the disc (the argument is similar for the case in which the curve enters the disc at A and B). Assume also that in the interval on the circle $\partial\Delta$ which joins A to B in the clockwise direction there are no points at which the curve enters or leaves the disc (see Figure 2).

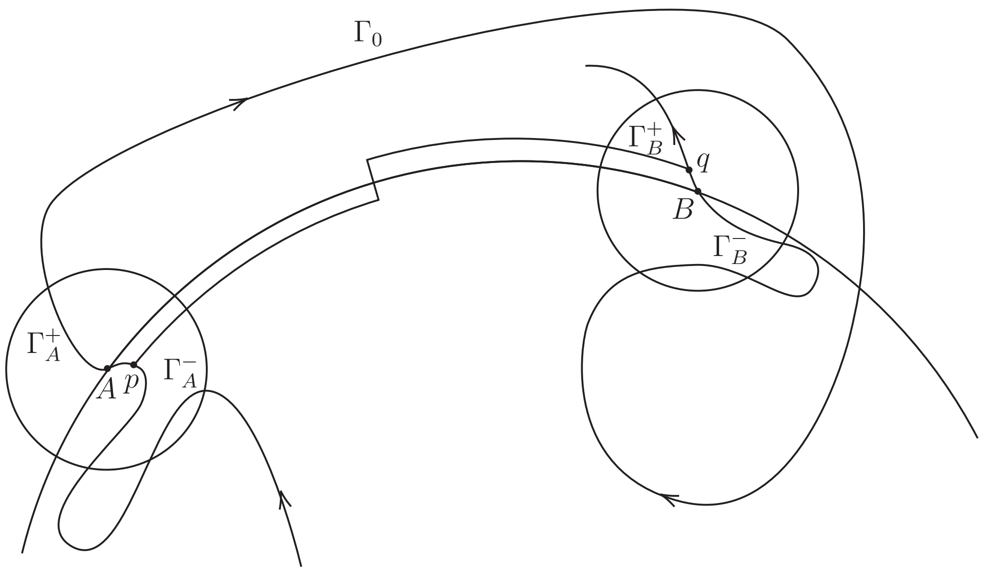

Figure 2

In view of the definition that the curve leaves the disc at A, there is a disc D_A centered at A and of radius small enough so that $\Gamma \cap D_A$ contains a Jordan arc Γ_A^- joining ∂D_A to A inside Δ (that is, $\Gamma_A^- \setminus \{A\} \subset \Delta$) and a Jordan arc Γ_A^+ joining A to ∂D_A inside $(\overline{\Delta})^c$ (that is, $\Gamma_A^+ \setminus \{A\} \subset (\overline{\Delta})^c$.) The same argument applies to B to produce a disc D_B centered at B and Jordan arcs Γ_B^- and Γ_B^+ joining respectively ∂D_B to B inside Δ and B to ∂D_B inside $(\overline{\Delta})^c$. Let Γ_0 be the Jordan arc contained in Γ joining the end point of Γ_A^+ to the initial point of Γ_B^-. Let Γ_{AB} be the closed Jordan curve formed by the union of the 4 arcs $\Gamma_A^+, \Gamma_0, \Gamma_B^-$ and $\widehat{AB}$, where $\widehat{AB}$ is the interval joining B to A in the circle $\partial\Delta$ with the counterclockwise orientation. The sub-arcs of Γ keep the orientation provided by the original parametrization of Γ.

Out goal is to find points $p \in \Gamma_A^-$ and $q \in \Gamma_B^+$ with different index with respect to the Jordan curve Γ_{AB}. This will provide a contradiction, because the sub-arc of Γ starting at B and ending at A joins p and q without intersecting Γ_{AB}, which means that p and q have the same index with respect to Γ_{AB}.

Let us proceed to the definition of p and q. Since Γ_0 and $\widehat{AB}$ are disjoint compact sets there is $\delta > 0$ such that $U_\delta \cap \Gamma_0 = \emptyset$, where $U_\delta = \{z : \mathrm{dist}(z, \widehat{AB}) < \delta\}$. Take $p \in U_\delta \cap \Gamma_A^-$ and $q \in U_\delta \cap \Gamma_B^+$. The next step is to construct a Jordan arc joining p and q, which intersects Γ_{AB} only once, so that p and q have different index with respect to Γ_{AB}. Start at p and follow the circle concentric with $\partial\Delta$ which contains

p in the clockwise direction until we are under the middle point of the interval $\widehat{AB}$. Continue along the ray emanating at the center of Δ towards $\partial\Delta$, cross $\partial\Delta$ and proceed until you touch the circle concentric with $\partial\Delta$ containing q. Then follow that circle until you get to q. Obviously you cross Γ_{AB} once through $\widehat{AB}$, but there is no other intersection with Γ_{AB}. The proof is now complete.

4 Appendix

In this section we prove the following.

Lemma 3. *Assume that f is a continuous function on an open set Ω such that its partial derivatives exist almost everywhere in Ω and $\overline{\partial} f \in L^1_{loc}(\Omega)$. Assume further that*

$$\int_{\partial Q} f(z)\, dz = 2i \int_Q \overline{\partial} f(z)\, dA(z), \tag{10}$$

for each closed square $Q \subset \Omega$ with sides parallel to the coordinates axis. Then the pointwise $\overline{\partial}$ derivative of f on Ω is indeed the distributional derivative of f on Ω.

Proof. One has to show that

$$-\int f(z)\, \overline{\partial}\varphi(z)\, dA(z) = \int \overline{\partial} f(z)\, \varphi(z)\, dA(z), \tag{11}$$

for each $\varphi \in C_0^\infty(\Omega)$. Take $\rho \in C_0^\infty(\mathbb{C})$ with support contained in $\{z : |z| \le 1\}$ and $\int \rho(z)\, dA(z) = 1$. Set $\rho_\epsilon(z) = \frac{1}{\epsilon^2}\rho(\frac{z}{\epsilon})$ and $f_\epsilon = f * \rho_\epsilon$. Then $\overline{\partial}(f * \rho_\epsilon) = f * \overline{\partial}\rho_\epsilon$, but it is not clear that this coincides with $\overline{\partial} f * \rho_\epsilon$. Given $\delta > 0$ set $\Omega_\delta = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \delta\}$ and take δ small enough so that the support of φ is contained in Ω_δ. Let $\epsilon < \delta$. Then

$$\begin{aligned} -\int f_\epsilon(z)\, \overline{\partial}\varphi(z)\, dA(z) &= -\int (f * \rho_\epsilon)(z)\, \overline{\partial}\varphi(z)\, dA(z), \\ &= \int \overline{\partial}(f * \rho_\epsilon)(z)\, \varphi(z)\, dA(z) \\ &= \int (f * \overline{\partial}\rho_\epsilon)(z)\, \varphi(z)\, dA(z). \end{aligned} \tag{12}$$

We show now that

$$(f * \overline{\partial}\rho_\epsilon)(z) = (\overline{\partial} f * \rho_\epsilon)(z), \quad z \in \Omega_\delta. \tag{13}$$

Inserting this in (12) and letting $\epsilon \to 0$ yields (11). To prove (13) let Q be the closed square with sides parallel to the coordinate axis with center z and side length 2ϵ. Then $Q \subset \Omega$. The function $g(w) = f(w)\rho_\epsilon(z - w)$ vanishes on the boundary of Q. Applying (10) to Q and g we get (13). □

Acknowledgements. The authors are grateful to J.Bruna and M.Melnikov for some useful conversations on the subject.

This work was partially supported by the grants 2009SGR420 (Generalitat de Catalunya) and MTM2010-15657 (Ministerio de Educación y Ciencia).

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