Flows for non-smooth vector fields with subexponentially integrable divergence

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Abstract In this paper, we study flows associated to Sobolev vector fields with subexponentially integrable divergence. Our approach is based on the transport equation following DiPerna-Lions [DPL89]. A key ingredient is to use a quantitative estimate of solutions to the Cauchy problem of transport equation to obtain the regularity of density functions.

1 Introduction

Since the fundamental work by DiPerna-Lions [DPL89], the study of flows associated to non-smooth vector fields has attracted intensive interest, and has been found many applications in PDEs. The problem can be formulated as follows. Given a Sobolev (or more generally BV) vector field $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, does there exist a unique Borel map $X : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\frac{\partial}{\partial t} X(t, x) = b(t, X(t, x)) \tag{1.1}$$

for a.e. $x \in \mathbb{R}^n$? If this ODE is well-posed, then how about the regularity of the solution X?

In the seminal work by DiPerna and Lions [DPL89], the existence of flows for Sobolev velocity fields with bounded divergence was established. Their main ingredient was a careful analysis of the well posedness of the initial value problem for the linear transport equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases} \tag{1.2}$$

In their arguments, the notion of renormalized solution was shown to be essential. Later, Ambrosio [Am04] extended the renormalization property to the setting of bounded variation (BV) vector fields, and obtained the non-smooth flows by using some new tools from Probability and Calculus of Variations. Crippa and De Lellis [CDL08] used a direct approach to recover DiPerna-Lions' theory; see also Bouchut and Crippa [BC13]. Recently, in [ACF14], Ambrosio, Colombo and Figalli developed a purely local theory on flows for non-smooth vector fields as a natural analogy of the Cauchy-Lipschitz approach.

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Continuing our previous work about the transport equation [CJMO], in this paper we are concerned with the existence of flows for Sobolev vector fields having sub-exponentially integrable divergence. Let us review some developments in this spirit. In [D96], Desjardins showed existence and uniqueness of non-smooth flows for velocity fields having exponentially integrable divergence. Later, Cipriano and Cruzeiro [CiCr05] analyzed the flows for Sobolev vector fields with exponentially integrable divergence in the setting of Euclidean spaces equipped with Gaussian measures; see [AF09] for related progresses in Wiener spaces.

As already noticed in [CiCr05, AF09], when the divergence of the velocity field is not bounded, the solution $X(t,\cdot)$ of equation (1.1) still induces a quasi-invariant measure. This motivates the following definition.

Definition 1.1. *Let $b : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be a Borel vector field, and $X : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be a Borel map.*

(i) We say that X is a flow associated to b if for almost every $x \in \mathbb{R}^n$ the map $t \mapsto |b(t, X(t,x))|$ belongs to $L^1(0,T)$ and

$$X(t,x) = x + \int_0^t b(s, X(s,x))\, ds.$$

(ii) We say that X is a regular flow associated to b if:

1. *X is a flow associated to b;*
2. *for each $t \in [0,T]$ the image measure $X(t,\cdot)_{\#}\, dx$ is absolutely continuous with respect to the Lebesgue measure dx.*

(iii) We say that X has semigroup structure if for all $0 < s < t \le T$ it holds that $X(t,x) = X(s, X(t-s,x))$, a.e. $x \in \mathbb{R}^n$.

In this paper, we study regular flows as defined above. As in [CiCr05], in our arguments sometimes it will be convenient to replace the Lebesgue measure dx by the Gaussian measure μ on $\mathbb{R}^n$, i.e.,

$$d\mu(x) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{|x|^2}{2}\right\} dx.$$

The distributional divergence of a vector field b with respect to the measure μ is then defined via

$$\mathrm{div}_\mu b(x) = \mathrm{div}\, b(x) - x\cdot b(x), \quad \forall\, x \in \mathbb{R}^n,$$

that is, div_μ is the adjoint of the gradient operator with respect to the measure μ. This operator appears to be useful, among other reasons because it commutes with the Ornstein-Uhlenbeck smoothing semigroup [CiCr05, AF09].

Our main result concerns existence and uniqueness of a regular flow for non-smooth vector fields with subexponentially integrable divergence. Due to the scheme of the proof, we found it convenient to state it in two steps. First, we state the existence and uniqueness of a flow for which all t-advance maps $X(t,\cdot)$ leave the Gaussian measure quasi-invariant, together with a quantitative estimate of this fact. Secondly, we state that the Lebesgue measure is also quasi-invariant, so that the flow we have found is indeed a regular flow. Moreover, we also state the semigroup structure of the flow. The precise statement is as follows.

Main Theorem . *Let $b \in L^1(0,T;W^{1,1}_{\mathrm{loc}})$ satisfying*

$$\frac{|b(t,x)|}{1+|x|\log^+(|x|)} \in L^1(0,T;L^\infty), \tag{1.3}$$

and

$$\mathrm{div}_\mu b \in L^1\left(0,T;\mathrm{Exp}_\mu\left(\frac{L}{\log L}\right)\right). \tag{1.4}$$

Then the following statements hold.

(a) There exists a unique flow $X(t,x)$ associated to b. Moreover, for each $t \in [0,T]$ the image measure $X(t,\cdot)_\# d\mu$ is absolutely continuous with respect to $d\mu$, and

$$\frac{d}{d\mu}\left(X(t,\cdot)_\# d\mu\right) \in L^{\Phi_\alpha}(\mu) \qquad \textit{for every } 0<\alpha<\alpha_0(t)$$

where $\Phi_\alpha(s) = s\exp\{[\log^+(s)]^\alpha\}$ and $\alpha_0(t) = \exp\left\{-16e^2\int_0^t \|\mathrm{div}_\mu b(s,\cdot)\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}\,ds\right\}$.

(b) The flow $X(t,x)$ is regular and has semigroup structure.

It is worth mentioning here that, under condition (1.3), the assumption (1.4) is equivalent to

$$\mathrm{div} b \in L^1\left(0,T;\mathrm{Exp}_\mu\left(\frac{L}{\log L}\right)\right).$$

Concerning the optimality of (1.4), it was proven in [CJMO, Section 6] that for every $\gamma > 1$ there exists a velocity field b with

$$\mathrm{div} b \in L^1\left(0,T;\mathrm{Exp}_\mu\left(\frac{L}{\log^\gamma L}\right)\right) \tag{1.5}$$

for which (1.1) admits infinitely many solutions X satisfying (i) and (iii) in Definition 1.1. However, we do not know if (1.5) is sufficient or not to guarantee existence and uniqueness of solutions X satisfying (i), (ii) and (iii) in Definition 1.1.

Towards the proof of the Main Theorem, the main ingredient is an *a priori* quantitative estimate for the density function $\frac{d}{d\mu}(X(t,\cdot)_\# d\mu)$. Such estimate is established in the smooth case in Theorem 3.1, by means of a quantitative bound for solutions to a Cauchy problem for the transport equation; see Theorem 2.3 below. The use of this quantitative bound gives a natural estimate of the density function (see equation (3.1) below). Moreover, as a byproduct, our proof improves the integrability of the image measure $X(t,\cdot)_\# d\mu$ when $\mathrm{div}_\mu b$ is assumed to be exponentially integrable; see Theorem 3.2 below and [CiCr05, AF09].

As it was for DiPerna and Lions scheme, well-posedness of the Cauchy problem (1.2) is an essential tool in our arguments. For Sobolev vector fields b satisfying the classical growth condition $\frac{|b(t,x)|}{1+|x|} \in L^1(0,T;L^1) + L^1(0,T;L^\infty)$ and

$$\operatorname{div} b \in L^1(0,T;L^\infty) + L^1\Big(0,T;\operatorname{Exp}\Big(\frac{L}{\log L}\Big)\Big),$$

the well-posedness of (1.2) in L^∞ was established in [CJMO, Theorem 1]. Unfortunately, our Main Theorem does not cover the assumption $\frac{|b(t,x)|}{1+|x|} \in L^1(0,T;L^1)$, and indeed we do not know if a flow does exist in this case. However, the assumption on $\operatorname{div} b$ in the Main Theorem (also in Theorem 2.2 below) is less restrictive than it was in [CJMO, Theorem 1]. In other words, our Theorem 2.2 about the well-posedness of (1.2) in L^∞ slightly improves [CJMO, Theorem 1]. A similar situation is given in Theorem 2.4, see Section 2 for details.

From the result by Ambrosio-Figalli [AF09], it looks like our requirements on the growth condition on b are somehow natural, since the image measure $X(t,\cdot)_\# d\mu$ is only slightly beyond L^1 integrable, and to guarantee $b(t,X(t,x)) \in L^1(0,T;L^1_{\mathrm{loc}})$, we need to require that b has at least exponential integrability.

The paper is organized as follows. In Section 2 we present the quantitative estimate of solutions to the transport equation (Theorem 2.3), and in Section 3, we use such estimate to deduce a priori estimate of the density function (Theorem 3.1). In section 4, we give the proof of part (a) of the Main Theorem. In the final section, we prove part (b) of the Main Theorem and give a stability result concerning the flows. Throughout the paper, we denote by C positive constants which are independent of the main parameters, but which may vary from line to line.

2 Well-posedness of the transport equation in the Gaussian setting

We will need to use some Orlicz spaces and their duals. For the reader's convenience, we recall here some definitions. See the monograph [RR91] for the general theory of Orlicz spaces. Let

$$P : [0,\infty) \mapsto [0,\infty),$$

be an increasing homeomorphism onto $[0,\infty)$, so that $P(0) = 0$ and $\lim_{t\to\infty} P(t) = \infty$. The Orlicz space L^P is the set of measurable functions f for which the Luxembourg norm

$$\|f\|_{L^P} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} P\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}$$

is finite. In this paper we will be mainly interested in two particular families of Orlicz spaces. Given $r, s \ge 0$, the first family corresponds to

$$P(t) = t\,(\log^+ t)^r\ (\log^+ \log^+ t)^s\,,$$

where $\log^+ t := \max\{1, \log t\}$. The obtained L^P spaces are known as *Zygmund spaces*, and will be denoted from now on by $L \log^r L \log^s \log L$. The second family is at the upper borderline. For $\gamma \geq 0$ we set

$$P(t) = \exp\left\{\frac{t}{(\log^+ t)^\gamma}\right\} - 1, \qquad t \geq 0. \tag{2.1}$$

Then we will denote the obtained L^P by $\mathrm{Exp}(\frac{L}{\log^\gamma L})$. If $\gamma = 0$ or $\gamma = 1$, we then simply write $\mathrm{Exp}L$ and $\mathrm{Exp}(\frac{L}{\log L})$, respectively. For each $\alpha > 0$, throughout the paper, we denote by Φ_α the Orlicz function

$$\Phi_\alpha(t) = t \exp\{(\log^+ t)^\alpha\}, \qquad t \geq 0. \tag{2.2}$$

When changing the reference measure from Lebesgue measure to the Gaussian measure, we will simply add μ to the notions of the spaces, as $L \log L \log\log L(\mu)$, $\mathrm{Exp}_\mu\left(\frac{L}{\log L}\right)$, etc.

The following lemma can be proved in the same way as [CJMO, Lemma 11].

Lemma 2.1. *If $f \in L \log L \log\log L(\mu)$ and $g \in \mathrm{Exp}_\mu(\frac{L}{\log L})$ then $fg \in L^1(\mu)$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, d\mu \leq 2\|f\|_{L \log L \log\log L(\mu)} \|g\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}.$$

Moreover, if $f \in L^\infty(\mu) \cap L \log L \log\log L(\mu)$ then

$$\begin{aligned}
&\|f\|_{L \log L \log\log L(\mu)} \\
&\leq 2e\|f\|_{L^1(\mu)}\Big(\log(e + \|f\|_{L^\infty(\mu)}) + |\log(\|f\|_{L^1(\mu)})|\Big)\Big(\log\log(e^e + \|f\|_{L^\infty(\mu)}) + |\log|\log(\|f\|_{L^1(\mu)})||\Big).
\end{aligned}$$

In this section we present a well-posedness result for the initial value problem for the transport equation in L^∞. This is a new result, which neither contains [CJMO, Theorem 1], nor is contained in it. In order to state it, we write the transport equation in the Lebesgue case as

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0, T) \times\, dx, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases} \tag{2.3}$$

and in the Gaussian case as

$$\begin{cases} \dfrac{\partial u}{\partial t} + b \cdot \nabla u = 0 & (0, T) \times\, d\mu, \\ u(0, \cdot) = u_0 & \mathbb{R}^n. \end{cases} \tag{2.4}$$

A function $u \in L^1(0, T; L^1_{\mathrm{loc}})$ is called a *weak solution* to (2.3) if for each $\varphi \in C^\infty([0, T) \times \mathbb{R}^n)$ with compact support in $[0, T) \times \mathbb{R}^n$ it holds that

$$-\int_0^T \int_{\mathbb{R}^n} u \frac{\partial \varphi}{\partial t}\, dx\, dt - \int_{\mathbb{R}^n} u_0\, \varphi(0, \cdot)\, dx - \int_0^T \int_{\mathbb{R}^n} u \operatorname{div}(b\, \varphi)\, dx\, dt = 0.$$

We also say that the problem (2.3) is *well-posed* in $L^\infty(0,T;L^\infty)$ if weak solutions exist and are unique, for any $u_0 \in L^\infty$.

Weak solutions of the transport equation (2.4) can be defined in a similar way. A simple observation is that a function $u \in L^\infty(0,T;L^\infty)$ is a weak solution of (2.3) if and only if it is a weak solution of (2.4). Indeed, if $u \in L^\infty(0,T;L^\infty)$ is a weak solution of (2.3), and $\varphi \in C_c^\infty([0,T)\times\mathbb{R}^n)$ is a test function, then $\frac{\varphi(x)}{(2\pi)^{n/2}}\exp(-|x|^2/2) \in C_c^\infty([0,T)\times\mathbb{R}^n)$, and so we can conclude that

$$-\int_0^T \int_{\mathbb{R}^n} u\,\frac{\partial\varphi}{\partial t}\,d\mu\,dt - \int_{\mathbb{R}^n} u_0\,\varphi(0,\cdot)\,d\mu - \int_0^T \int_{\mathbb{R}^n} u\left(\varphi\,\mathrm{div}_\mu b + b\cdot\nabla\varphi\right)\,d\mu(x)\,dt = 0.$$

For the converse, we only need to use $\frac{\varphi(x)}{(2\pi)^{n/2}}\exp(|x|^2/2) \in C_c^\infty([0,T)\times\mathbb{R}^n)$ as a test function.

We now prove our well posedness result for the transport equation in the Gaussian setting. Let us observe that under the assumption

$$\frac{|b(t,x)|}{1+|x|\log^+|x|} \in L^1(0,T;L^\infty),$$

conditions $\mathrm{div}\,b \in L^1(0,T;\mathrm{Exp}_\mu(\frac{L}{\log L}))$ and $\mathrm{div}_\mu b \in L^1(0,T;\mathrm{Exp}_\mu(\frac{L}{\log L}))$ are equivalent.

Theorem 2.2. *Let $T>0$. Assume that $b \in L^1(0,T;W^{1,1}_{\mathrm{loc}})$ satisfying* (1.3) *and* (1.4). *Then for each $u_0 \in L^\infty$ there exists a unique weak solution $u \in L^\infty(0,T;L^\infty)$ of the Cauchy problem of the transport problem* (2.4).

Proof. Existence of solution follows immediately from [DPL89, Proposition 2.1], while uniqueness will follow from the following stability estimate. □

The following theorem is similar to [CJMO, Theorem 5]. We report the proof here for completeness.

Theorem 2.3. *Let $T, M > 0$ and $1 \le p < \infty$. Suppose that $b \in L^1(0,T;W^{1,1}_{\mathrm{loc}})$ satisfies* (1.3) *and* (1.4). *Let $\epsilon \in (0,\frac{1}{2}\exp(-e^{e+M}))$ satisfy*

$$\exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\epsilon} - 32e\int_0^T \beta(s)\,ds\right\}\right\}\right\} < \frac{1}{2}\exp(-e^{e+M}),$$

where $\beta(t) = \|\mathrm{div}_\mu b(t,\cdot)\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}$. Then for each $u_0 \in L^\infty(\mu)$ with $\|u_0\|_{L^\infty(\mu)} \le M$ and $\|u_0\|^p_{L^p(\mu)} < \epsilon$, the transport problem (2.4) *has a unique solution $u \in L^\infty(0,T;L^\infty)$, moreover it holds that*

$$\left|\log\log\log\left(\frac{1}{\|u(T,\cdot)\|^p_{L^p(\mu)}}\right) - \log\log\log\left(\frac{1}{\|u_0\|^p_{L^p(\mu)}}\right)\right| \le 16e\int_0^T \beta(s)\,ds.$$

Proof. For each $R>0$, let $\psi_R \in C_c^\infty(\mathbb{R}^n)$ be a cutoff function, so that

$$0 \le \psi_R \le 1,\ \ \psi_R(x) = 1 \text{ whenever } |x| \le R,\qquad \psi_R(x) = 0 \text{ whenever } |x| \ge 2R,\ \ \text{and } |\nabla\psi_R(x)| \le \frac{C}{R}. \tag{2.5}$$

By the renormalization property of solutions to the transport equation (see [DPL89, Theorem 2.1] or [Am04, Theorem 3.5]), we find that for each $p \in [1, \infty)$, it holds that

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u(t,x)|^p\, \psi_R\, d\mu(x) = \int_{\mathbb{R}^n} \mathrm{div}_\mu b|u|^p\, \psi_R\, d\mu(x) + \int_{\mathbb{R}^n} b \cdot \nabla\psi_R\, |u|^p\, d\mu(x). \tag{2.6}$$

Integrating over time and letting $R \to \infty$ yield that

$$\int_{\mathbb{R}^n} |u(t,x)|^p\, d\mu(x) \le \int_{\mathbb{R}^n} |u_0|^p\, d\mu + \int_0^t \int_{\mathbb{R}^n} |\mathrm{div}_\mu b||u|^p\, d\mu\, ds. \tag{2.7}$$

for each $t \in (0, T)$. For convenience, in what follows we denote by $\alpha(t)$, $\beta(t)$ the quantities $\|u(t,\cdot)\|^p_{L^p(\mu)}$, $\|\mathrm{div}_\mu\, b\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}$, respectively. For simplicity, let $\|u\|_{L^\infty(0,T;L^\infty)} = M$. From the first estimate of Lemma 2.1, we find that

$$\int_{\mathbb{R}^n} |\mathrm{div}_\mu b||u|^p\, d\mu \le 2\|\mathrm{div}_\mu b\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}\||u|^p\|_{L\log L\log\log L(\mu)}. \tag{2.8}$$

By the second estimate of Lemma 2.1, the factor $\||u|^p\|_{L\log L\log\log L}(\mu)$ is bounded by

$$2e\,\alpha(t)\Big(\log(e+M) + |\log\alpha(t)|\Big)\Big(\log\log(e^e+M) + |\log|\log(\alpha(t))||\Big). \tag{2.9}$$

By the absolute continuity of integral, we can choose $i \in \mathbb{N}$ and $0 = T_0 < T_1 < \cdots < T_i < T_{i+1} = T$ such that for each $0 \le j \le i$,

$$\int_{T_j}^{T_{j+1}} \int_{\mathbb{R}^n} |\mathrm{div}_\mu b||u|^p\, d\mu\, ds \le \frac{1}{2}\exp(-e^{e+M}). \tag{2.10}$$

Notice that by (2.7), (2.10) and the fact $\int_{\mathbb{R}^n} |u_0|^p\, d\mu < \epsilon$, for $t \in (0, T_1)$, we have

$$\alpha(t) < \frac{1}{2}\exp(-e^{e+M}) + \frac{1}{2}\exp(-e^{e+M}) = \exp(-e^{e+M}).$$

Therefore, we can conclude that for $t \in (0, T_1)$ it holds

$$\log(e+M) \le \left|\log(\alpha(t))\right| = \log\frac{1}{\alpha(t)}$$

and

$$\log\log(e^e+M) \le \left|\log\left(|\log\alpha(t)|\right)\right| = \log\log\frac{1}{\alpha(t)}.$$

This fact, together with the inequalities (2.8) and (2.9), gives

$$\alpha(t) \le \epsilon + 16e\int_0^t \beta(s)\alpha(s)\log\frac{1}{\alpha(s)}\log\log\frac{1}{\alpha(s)}\, ds. \tag{2.11}$$

Step 2. Let us introduce a continuous function as, for each $t \in (0, T]$,

$$\alpha^*(t) = \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\epsilon} - 16e\int_0^t \beta(s)\, ds\right\}\right\}\right\},$$

where by the assumption of ϵ, we see that for each $t \in (0, T)$, it holds

$$\alpha^*(t) < \frac{1}{2}\exp(-e^{e+M}).$$

From the definition, we see that α^* is Lipschitz smooth and increasing on $[0, T]$, and

$$\alpha^*(t) = \epsilon + 16e\int_0^t \beta(s)\alpha^*(s)\log\frac{1}{\alpha^*(s)}\log\log\frac{1}{\alpha^*(s)}\,ds.$$

Step 3. Using the assumption $\alpha(0) = \|u_0\|^p_{L^p(\mu)} < \epsilon = \alpha^*(0)$, we find by (2.11) that

$$\alpha(t) \le \alpha^*(t)$$

for all $t \in [0, T_1]$.

We now iterate the approach to get the desired estimates. By the choice of T_i, we see that for each $t \in (T_1, T_2]$,

$$\alpha(t) \le \alpha(T_1) + \int_{T_1}^{T_2}\int_{\mathbb{R}^n} |\mathrm{div}_\mu\, b||u|\,ds < \exp\{-\exp\{e + M\}\}.$$

Hence, for all $t \in (0, T_2]$, we have

$$\alpha(t) \le 16e\int_0^t \beta(s)\alpha(s)\log\frac{1}{\alpha(s)}\log\log\frac{1}{\alpha(s)}\,ds,$$

which implies that

$$\alpha(t) \le \alpha^*(t)$$

for all $t \in (0, T_2]$. Repeating this argument $i-1$ times more, we can conclude that for all $t \in (0, T]$, it holds

$$\|u\|^p_{L^\infty(0,T;L^p)} \le \alpha^*(T) = \exp\left\{-\exp\left\{\exp\left\{\log\log\log\frac{1}{\epsilon} - 16e\int_0^T \beta(s)\,ds\right\}\right\}\right\},$$

which implies that

$$\log\log\log\frac{1}{\|u(T,\cdot)\|^p_{L^p(\mu)}} - \log\log\log\frac{1}{\|u_0\|^p_{L^p(\mu)}} \ge -16e\int_0^T \beta(s)\,ds. \tag{2.12}$$

Step 4. Using the backward equation, it is easy to show that

$$\log\log\log\frac{1}{\|u(T,\cdot)\|^p_{L^p(\mu)}} - \log\log\log\frac{1}{\|u_0\|^p_{L^p(\mu)}} \le 16e\int_0^T \beta(s)\,ds, \tag{2.13}$$

which together with (2.12) completes the proof of the theorem. □

We have the following analogy for the divergence being exponentially integrable. The proof is similar to the above theorem and will be omitted.

Theorem 2.4. *Let $T, M > 0$ and $1 \le p < \infty$. Suppose that $b \in L^1(0, T; W^{1,1}_{\mathrm{loc}})$ satisfies*

$$\frac{|b(t,x)|}{1+|x|\log^+(|x|)} \in L^1(0,T;L^\infty) + L^1(0,T;L^1)$$

and

$$\mathrm{div}_\mu b \in L^1\left(0, T; \mathrm{Exp}_\mu(L)\right). \tag{2.14}$$

Let $\epsilon \in (0, 1/e)$ satisfy

$$\exp\left\{-\exp\left\{\log\log\frac{1}{\epsilon} - 8\int_0^T \beta(s)\,ds\right\}\right\} < \frac{1}{2(e+M)},$$

where $\beta(t) = \|\mathrm{div} b(t,\cdot)\|_{\mathrm{Exp}_\mu(L)}$. Then for each $u_0 \in L^\infty(\mu)$ with $\|u_0\|_{L^\infty(\mu)} \le M$ and $\|u_0\|^p_{L^p(\mu)} < \epsilon$, the transport problem (2.4) *has a unique solution $u \in L^\infty(0, T; L^\infty)$, moreover it holds that*

$$\left|\log\log\left(\frac{1}{\|u(T,\cdot)\|^p_{L^p(\mu)}}\right) - \log\log\left(\frac{1}{\|u_0\|^p_{L^p(\mu)}}\right)\right| \le 4\int_0^T \beta(s)\,ds.$$

3 A priori estimates of the Jacobian

In this section, we give a priori estimates of the density functions when we assume that the vector fields are smooth. Recall that $\Phi_\alpha(s) = s\exp\{[\log^+(s)]^\alpha\}$ is given in (2.2).

Theorem 3.1. *Let $b \in C^2(\mathbb{R}^n)$ satisfying* (1.3) *and* (1.4). *Then there exists a unique flow $X(t,x)$ satisfying*

$$\frac{\partial X}{\partial t} = b(t, X).$$

Moreover, if $t \in [0,T]$ and $0 < \alpha < \exp\left\{-16e\int_0^t \beta(s)\,ds\right\}$, $\beta(s) = \|\mathrm{div}_\mu b(s,\cdot)\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}$, then the density function $K_t(x) = \frac{d}{dt}(X(t)_\# d\mu)$ belongs to $L^{\Phi_\alpha}(\mu)$, and

$$\int_{\mathbb{R}^n} \Phi_\alpha(K_t(x))\,d\mu(x) \le C(\alpha, t, \|\mathrm{div}_\mu b\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}). \tag{3.1}$$

Proof. The existence and uniqueness of the flow is an immediate consequence of the assumption that $b \in C^2(\mathbb{R}^n)$ satisfies

$$\frac{|b(t,x)|}{1+|x|\log^+|x|} \in L^1(0,T;L^\infty).$$

Moreover, the flow $X(t,x)$ is locally Lipschitz for each $t \in [0,T]$. See Hale [Ha80] for instance.

Let us estimate the density function. Obviously, it holds that

$$\int_{\mathbb{R}^n} K_t(x)\,d\mu(x) = \int_{\mathbb{R}^n} d\mu(x) = 1,$$

i.e., $\|K_t\|_{L^1(\mu)} = 1$ for each $t \in [0,T]$. As a consequence,

$$\mu(\{x : K_t(x) > \lambda\}) \le \frac{1}{\lambda}$$

for all $\lambda > 0$ and $t \in [0, T]$.

Fix a $t \in [0, T]$. Let $k_0 \in \mathbb{N}$ large enough such that

$$\exp\left\{-\exp\left\{\exp\left\{\log\log\log 2^{k_0} - 32e\int_0^t \beta(s)\,ds\right\}\right\}\right\} < \frac{1}{2}\exp(-e^{2e}),$$

where $\beta(s) = \|\mathrm{div}_\mu b(s,\cdot)\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}$. Obviously, k_0 only depends on $\|\mathrm{div}_\mu b(s,\cdot)\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}$ and t.

For each $k > k_0$, let

$$E_k = \{x \in \mathbb{R}^n : 2^{k-1} < K_t(x) \le 2^k\},$$

and $u_k(x) = \chi_{E_k}(x)$. Then $u_k \in L^1(\mu) \cap L^\infty(\mu)$ with $\|u_k\|_{L^\infty(\mu)} \le 1$ and $\|u_k\|_{L^1(\mu)} \le 2^{1-k}$.

By the assumption on b, it is readily seen that $u := u_k(X(t,x))$ is the unique solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} - b\cdot\nabla u = 0 & (0,T)\times\mathbb{R}^n, \\ u(0,\cdot) = u_k & \mathbb{R}^n. \end{cases}$$

By Theorem 2.3, and the choice of k_0, we find that

$$\left|\log\log\log\left(\frac{1}{\|u(t,\cdot)\|_{L^1(\mu)}}\right) - \log\log\log\left(\frac{1}{\|u_k\|_{L^1(\mu)}}\right)\right| \le 16e\int_0^t \beta(s)\,ds,$$

which implies that

$$\exp\left\{-16e\int_0^t \beta(s)\,ds\right\} \le \frac{\log\log\left(\frac{1}{\|u(t,\cdot)\|_{L^1(\mu)}}\right)}{\log\log\left(\frac{1}{\|u_k\|_{L^1(\mu)}}\right)} \le \exp\left\{16e\int_0^t \beta(s)\,ds\right\}.$$

Hence, we can conclude that

$$\left(\log\frac{1}{\|u_k\|_{L^1(\mu)}}\right)^{\exp\left\{-16e\int_0^t \beta(s)\,ds\right\}} \le \log\frac{1}{\|u(t,\cdot)\|_{L^1(\mu)}} \le \left(\log\frac{1}{\|u_k\|_{L^1(\mu)}}\right)^{\exp\left\{16e\int_0^t \beta(s)\,ds\right\}}.$$

The choose of u implies that

$$\|u(t,\cdot)\|_{L^1(\mu)} = \int_{E_k} K_t(x)\,d\mu(x),$$

and hence,

$$\left(\log\frac{1}{\mu(E_k)}\right)^{\exp\left\{-16e\int_0^t \beta(s)\,ds\right\}} \le \log\frac{1}{2^{k-1}\mu(E_k)} = \log\frac{1}{2^{k-1}} + \log\frac{1}{\mu(E_k)}.$$

A direct calculation gives

$$\log\frac{1}{\mu(E_k)} \ge \log 2^{k-1} + \left[\log 2^{k-1}\right]^{\exp\left\{-16e\int_0^t \beta(s)\,ds\right\}}$$

Therefore, we can conclude that,

$$\mu(E_k) \le \exp\left\{-\log 2^{k-1} - \left[\log 2^{k-1}\right]^{\exp\left\{-16e\int_0^t \beta(s)\,ds\right\}}\right\} \le \frac{1}{2^{k-1}}\exp\left\{-\left(\log 2^{k-1}\right)^{\exp\left\{-16e\int_0^t \beta(s)\,ds\right\}}\right\}.$$

For arbitrary $\alpha \in (0, \exp\left\{-16e\int_0^t \beta(s)\,ds\right\})$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} K_t(x)\exp\{[\log^+ K_t(x)]^\alpha\}\,d\mu(x)\\
&\le \int_{\mathbb{R}^n} 2^{k_0}\exp\{[\log^+ 2^{k_0}]^\alpha\}\,d\mu(x) + \sum_{k>k_0}\int_{E_k} 2^k\exp\{[\log^+(2^k)]^\alpha\}\,d\mu(x)\\
&\le 2^{k_0}\exp\{[\log^+ 2^{k_0}]^\alpha\} + \sum_{k>k_0}\mu(E_k)2^k\exp\{[\log^+(2^k)]^\alpha\}\\
&\le 2^{k_0}\exp\{[\log^+ 2^{k_0}]^\alpha\} + \sum_{k>k_0} 2\exp\left\{[\log^+(2^k)]^\alpha - \left(\log 2^{k-1}\right)^{\exp\left\{-16e\int_0^t\beta(s)\,ds\right\}}\right\}\\
&\le C(\alpha, t, \|\mathrm{div}_\mu b\|_{\mathrm{Exp}(\frac{L}{\log L})}).
\end{aligned}$$

This completes the proof. □

In the same way, using Theorem 2.4, we can prove the following quantitative estimate for vector fields with distributional divergence in $\mathrm{Exp}_\mu(L)$.

Theorem 3.2. *Let $b \in C^2(\mathbb{R}^n)$ satisfying*

$$\frac{|b(t,x)|}{1+|x|\log^+|x|} \in L^1(0,T;L^\infty),$$

and $\mathrm{div}_\mu b \in L^1(0,T;\mathrm{Exp}_\mu(L))$. Then there exists a unique flow $X(t,x)$ satisfying

$$\frac{\partial X}{\partial t} = b(t,X),$$

Moreover, for each $t \in [0,T]$ and each $p \in [1, \frac{1}{1-\exp(-4\int_0^t\beta(s)\,ds)})$, $\beta(s) = \|\mathrm{div}_\mu b(s,\cdot)\|_{\mathrm{Exp}_\mu(L)}$, the density function $K_t(x) = \frac{d}{dt}(X(t)_\# d\mu)$ belongs to $L^p(\mu)$ and

$$\int_{\mathbb{R}^n}[K_t(x)]^p\,d\mu(x) \le C(p,t,\|\mathrm{div}\, b\|_{\mathrm{Exp}_\mu(L)}).$$

Remark 3.3. Our method to prove the integrability of the density functions yields a sharper estimate than those from [AF09, Cr83, CiCr05]. It is worth to note that our proof yields that integrability of the density functions has some semigroup property, which is natural.

4 Flow in the Gaussian setting

In this section, we will prove part (a) of the Main Theorem. To do this, let us recall the Ornstein-Uhlenbeck semigroup P_s. For each $s > 0$ and $f \in L^1(\mu)$, $P_s f(x)$ is defined by

$$P_s f(x) = \int_{\mathbb{R}^n} f(e^{-s}x + \sqrt{1-e^{-2s}}y)\,d\mu(y).$$

Among other properties of the semigroup P_s, we will need the following:

(i) $\mathrm{div}_\mu (P_s b) = e^s P_s(\mathrm{div}_\mu b)$.

(ii) For each $p \in [1,\infty]$, it holds

$$\|P_s f\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)}.$$

(iii) For each convex function Φ on $[0,\infty)$, $\Phi(0) = 0$, $\lim_{s\to\infty} \frac{\Phi(s)}{s} = \infty$, it holds that

$$\|P_s f\|_{L^\Phi(\mu)} \le \|f\|_{L^\Phi(\mu)}.$$

The first two properties can be found from Bogachev [Bo98], and the third one is a consequence of (ii) and the Jensen inequality. Indeed, the Jensen inequality and L^1-boundedness of P_s imply

$$\int_{\mathbb{R}^n} \Phi\left(\frac{P_s f}{\lambda}\right) d\mu \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi\left(\frac{f(e^{-s}x + \sqrt{1-e^{-2s}}y)}{\lambda}\right) d\mu(y)\, d\mu(x) \le \int_{\mathbb{R}^n} \Phi\left(\frac{f(x)}{\lambda}\right) d\mu(x).$$

We will use the transport equation theory by DiPerna-Lions [DPL89] and follow some methods used by Cipriano-Cruzeiro [CiCr05]. Due to the fact that the divergence of the vector is only sub-exponentially integrable, we need to overcome some technical difficulties.

In what follows, we will always let $b \in L^1(0,T; W^{1,1}_{\mathrm{loc}})$ that satisfies

$$\frac{|b(t,x)|}{1+|x|\log^+|x|} \in L^1(0,T;L^\infty),$$

and $\mathrm{div}\, b \in L^1(0,T;\mathrm{Exp}_\mu(\frac{L}{\log L}))$. It follows by an easy calculation that

$$\mathrm{div}_\mu b = \mathrm{div}\, b - x\cdot b \in L^1\left(0,T;\mathrm{Exp}_\mu\left(\frac{L}{\log L}\right)\right).$$

For each $\epsilon > 0$, let $b_\epsilon = P_\epsilon b$.

Lemma 4.1. *For each $\epsilon > 0$, $P_\epsilon b \in C^\infty(\mathbb{R}^n)$ satisfies*

$$\frac{|P_\epsilon b(t,x)|}{1+|x|\log^+|x|} \in L^1(0,T;L^\infty).$$

Proof. By making change of variables, we see that

$$\begin{aligned} P_\epsilon b(t,x) &= \int_{\mathbb{R}^n} b(t, e^{-\epsilon}x + \sqrt{1-e^{-2\epsilon}}y)\frac{1}{(2\pi)^{n/2}}\exp\left\{-\frac{|y|^2}{2}\right\} dy \\ &= \frac{1}{(2\pi)^{n/2}(1-e^{-2\epsilon})^{n/2}}\int_{\mathbb{R}^n} b(t,z)\exp\left\{-\frac{|z-e^{-\epsilon}x|^2}{2(1-e^{-2\epsilon})}\right\} dz. \end{aligned}$$

Then it is obvious that $P_\epsilon b(t,x) \in C^\infty(\mathbb{R}^n)$ for each $t > 0$. To see that

$$\frac{P_\epsilon b(t,x)}{1+|x|\log^+|x|} \in L^1(0,T;L^\infty),$$

it suffices to show that for each $t > 0$

$$\left\| \frac{P_\epsilon b(t,x)}{1+|x|\log^+|x|} \right\|_{L^\infty} \le C \left\| \frac{b(t,x)}{1+|x|\log^+|x|} \right\|_{L^\infty}.$$

By the fact $\log(a+b) \le \log a + \log b$ for $a, b \ge 2$, we see that

$$\begin{aligned} |P_\epsilon b(t,x)| &\le \int_{\mathbb{R}^n} |b(t, e^{-\epsilon}x + \sqrt{1-e^{-2\epsilon}}y)| \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{|y|^2}{2}\right\} dy \\ &\le \left\| \frac{b(t,\cdot)}{1+|\cdot|\log^+|\cdot|} \right\|_{L^\infty} \int_{\mathbb{R}^n} (1+|z|\log^+|z|)\Big|_{z=e^{-\epsilon}x+\sqrt{1-e^{-2\epsilon}}y} d\mu(y), \end{aligned}$$

where

$$\begin{aligned} \int_{\mathbb{R}^n} (1+|z|\log^+|z|)\Big|_{z=e^{-\epsilon}x+\sqrt{1-e^{-2\epsilon}}y} d\mu(y) &\le \int_{\mathbb{R}^n} C(1+|x|\log^+|x| + |y|\log^+|y|)\, d\mu(y) \\ &\le C(1+|x|\log^+|x|), \end{aligned}$$

where C does not depend on ϵ. The proof is completed. □

For each $\epsilon > 0$, it follows from Lemma 4.1 that b_ϵ satisfies the requirements from Theorem 3.1 uniformly in ϵ. Denote by $X_\epsilon(t,x)$ the unique flow arising from the equation

$$\frac{\partial X_\epsilon(t,x)}{\partial t} = b_\epsilon(t, X_\epsilon(t,x)).$$

Denote by $K_\epsilon(t,x)$ the density function of $X_\epsilon(t,\cdot)_\# \, d\mu$.

Given a sequence X_k of functions defined on some measurable space $(\mathscr{M}, \nu)$ with values in a Banach space $\mathscr{N}$ (endowed with the norm $\|\cdot\|$), we say that X_n converges to X in $L^0(\nu)$ if for each fixed $\gamma > 0$ it holds

$$\nu(\{x \in \mathscr{M} : \|X_k(x) - X(x)\| > \gamma\}) \to 0$$

as $k \to \infty$.

In what follows, let $\mathcal{L}^1$ be the one dimensional Lebesgue measure.

Lemma 4.2. *There exist a subsequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ and a Borel map $X(t,x)$ such that:*

(i) X_{ϵ_k} converges to X as $k \to \infty$, both in $L^0(\mathcal{L}^1 \times \mu)$ and almost everywhere.

(ii) For each fixed $t \in [0,T]$, $X_{\epsilon_k}(t,\cdot)$ converges to $X(t,\cdot)$ as $k \to \infty$, both in $L^0(\mu)$ and almost everywhere.

Proof. Let β be a continuous and bounded function on $\mathbb{R}$. Denote $X^i_\epsilon(t,x)$ the i-th component of $X_\epsilon(t,x)$. Then $\beta(X^i_\epsilon(t,x))$ and $\beta(X^i_\epsilon(t,x))^2$ are bounded sequences in $L^\infty(0,T;L^\infty)$. By the weak-star convergence of $L^\infty(0,T;L^\infty)$, we see that there exists a subsequence ϵ_k such that $\beta(X^i_{\epsilon_k}(t,x))$ and $\beta(X^i_{\epsilon_k}(t,x))^2$ converge in weak-star topology of $L^\infty(0,T;L^\infty)$ to v^i_β and w^i_β, respectively.

On the other hand, $\beta(X^i_\epsilon(t,x))$ and $\beta(X^i_\epsilon(t,x))^2$ are bounded solutions to the transport equation with vector fields b_{ϵ_k} corresponding to the initial values $\beta(x_i)$ and $\beta(x_i)^2$, respectively.

By using the well-posedness of the transport equation, Theorem 2.2, and the renormalization property of solutions in $L^\infty(0,T;L^\infty)$, we can conclude that v^i_β and w^i_β are bounded solutions to the transport equation with vector fields b corresponding to the initial values $\beta(x_i)$ and $\beta(x_i)^2$, respectively, and therefore $(v^i_\beta)^2 = w^i_\beta$.

Then, by the fact $1 \in L^1(\mu)$, we can conclude that for each $t \in [0,T]$ it holds

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} [\beta(X^i_\epsilon(t,x)) - v^i_\beta]^2 \, d\mu = 0. \tag{4.1}$$

Now we prove that the arbitrariness of β implies that $X^i_{\epsilon_k}(t,x)$ converges in measure to some function $X^i(t,x)$. Indeed, by Lemma 4.1 we see that

$$\frac{|b_\epsilon(t,x)|}{1+|x|\log^+|x|} \in L^1(0,T;L^\infty)$$

and hence, $b_\epsilon(t,x) \in L^1(0,T;\mathrm{Exp}_\mu(L))$, while by Theorem 3.1, we see that $K_\epsilon \in L^\infty(0,T;L\log L(\mu))$. These imply that

$$\begin{aligned}
\|X^i_{\epsilon_k}(t,\cdot)\|_{L^1(\mu)} &\le \int_{\mathbb{R}^n} \left| x_i + \int_0^t b_\epsilon(s, X_\epsilon(s,x))\, ds \right| d\mu(x) \\
&\le C + \int_{\mathbb{R}^n} \int_0^T |b_\epsilon(s,x)|\, K_\epsilon(t,x)\, d\mu\, ds \\
&\le C + 2\int_0^T \|b_\epsilon\|_{\mathrm{Exp}_\mu(L)} \|K_\epsilon\|_{L\log L(\mu)}\, ds \\
&\le C,
\end{aligned}$$

i.e., $X_{\epsilon_k}(t,\cdot) \in L^\infty(0,T;L^1(\mu))$, and $X_{\epsilon_k}(t,\cdot) \in L^1(\mu)$ for each t, uniformly in ϵ.

Denote by ν the product measure $\mathcal{L}^1 \times \mu$ on $[0,T] \times \mathbb{R}^n$. Given a fixed $\gamma > 0$, for each $\delta > 0$, there exists a $M > 0$ such that for all ϵ_k,

$$\nu(\{(t,x) : |X_{\epsilon_k}(t,x)| > M\}) < \delta.$$

On the other hand, let $\beta_M \in C^1(\mathbb{R},\mathbb{R})$ that $\beta_M : \mathbb{R} \mapsto [-2M, 2M]$ and $\beta_M(t) = t$ for all $|t| \le M$. Then from (4.1) we see that there exists $k_0 \in \mathbb{N}$, such that for all $k, j > k_0$, it holds that

$$\begin{aligned}
\nu(\{(t,x) : |X^i_{\epsilon_k}(t,x) - X^i_{\epsilon_j}(t,x)| > \gamma\}) \le{}& \nu(\{(t,x) : |X^i_{\epsilon_k}(t,x)| > M\}) \\
&+ \nu(\{(t,x) : |X^i_{\epsilon_j}(t,x)| > M\}) \\
&+ \nu(\{(t,x) : |\beta_M(X^i_{\epsilon_k}(t,x)) - \beta_M(X^i_{\epsilon_j}(t,x))| > \gamma\}) < 3\delta
\end{aligned}$$

and so we can conclude that $\{X^i_{\epsilon_k}\}_k$ is a Cauchy sequence in measure. Therefore, $X^i_{\epsilon_k}(t,x)$ converges in measure to some function $X^i(t,x)$.

Passing to a further subsequence if necessary, we can conclude that $X_{\epsilon_k}(t,x)$ converges in $L^0(\mathcal{L}^1 \times \mu)$ and almost everywhere to $X(t,x)$. Moreover, it follows that for each $t \in [0,T]$, $X_{\epsilon_k}(t,x)$ converges in $L^0(\mu)$ and almost everywhere to $X(t,x)$. □

In what follows, let $X(t,x)$ be the limit function found in Lemma 4.2.

Lemma 4.3. *For each $t \in [0, T]$ and under the assumptions of the Main Theorem, the image measure $X(t, \cdot)_\# d\mu$ is absolutely continuous with respect to μ. Moreover, the density function $K_t(x) = \frac{d}{dt}(X(t)_\# d\mu)$ belongs to the Orlicz space $L^{\Phi_\alpha}(\mu)$ for each $0 < \alpha < \exp\left\{-16e^2 \int_0^t \beta(s)\, ds\right\}$, where $\beta(s) = \|\mathrm{div}_\mu b(s, \cdot)\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}$.*

Proof. Since $b_\epsilon = P_\epsilon b$, by the property of the Ornstein-Uhlenbeck semigroup, we see that for each $\epsilon < 1$, it holds

$$\|\mathrm{div}_\mu b_\epsilon\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})} \le e\|\mathrm{div}_\mu b\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})}.$$

By Theorem 3.1, for each $t \in [0, T]$ and each $0 < \alpha < \exp\left\{-16e^2 \int_0^t \beta(s)\, ds\right\}$, $\mathrm{div}_\mu b_\epsilon$ is uniformly bounded in the space $\mathrm{Exp}_\mu(\frac{L}{\log L})$. Therefore, we see that the density function of $K_\epsilon(t, x) = \frac{d}{d\mu}(X_\epsilon(t)_\# d\mu)$ is uniformly bounded in $L^{\Phi_\alpha}(\mu)$. Therefore, there exists a subsequence $\{\epsilon_k\}$ and $K_t \in L^{\Phi_\alpha}(\mu)$ such that

$$K_{\epsilon_k}(t, x) \rightharpoonup K_t(x) \text{ in } L^{\Phi_\alpha}(\mu).$$

Finally, for each compactly supported continuous function ψ, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} \psi(X(t, x))\, d\mu(x) &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \psi(X_{\epsilon_k}(t, x))\, d\mu(x)\\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \psi(x) K_\epsilon(t, x)\, d\mu(x) = \int_{\mathbb{R}^n} \psi(x) K_t(x)\, d\mu(x),\end{aligned}$$

as desired. □

Lemma 4.4. *Under the assumptions of the Main Theorem, for each open set E with sufficient small μ-measure, it holds that*

$$\log\log\log \frac{1}{\int_{\mathbb{R}^n} \chi_E(X(t, x))\, d\mu} \ge \log\log\log \frac{1}{\mu(E)} - 16e^2 \int_0^T \beta(s)\, ds.$$

Proof. Since E is an open set, by the a.e. convergence of $X_{\epsilon_k}(t, x)$, it is easy to see that

$$\liminf_{k\to\infty} \chi_E(X_{\epsilon_k}(t, x)) \ge \chi_E(X(t, x)), \quad \mu - a.e.$$

Therefore,

$$\int_{\mathbb{R}^n} \chi_E(X(t, x))\, d\mu \le \int_{\mathbb{R}^n} \liminf_{k\to\infty} \chi_E(X_{\epsilon_k}(t, x))\, d\mu \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} \chi_E(X_{\epsilon_k}(t, x))\, d\mu.$$

Since

$$\|\mathrm{div}_\mu b_\epsilon\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})} \le e\|\mathrm{div}_\mu b\|_{\mathrm{Exp}_\mu(\frac{L}{\log L})},$$

by Theorem 2.3, we know that for each k, it holds

$$\left|\log\log\log \frac{1}{\int_{\mathbb{R}^n} \chi_E(X_{\epsilon_k}(t, x))\, d\mu} - \log\log\log \frac{1}{\mu(E)}\right| \le 16e^2 \int_0^t \beta(s)\, ds,$$

which together with the last estimate completes the proof. □

Lemma 4.5. *Under the assumptions of the Main Theorem, for each measurable vector field $F : [0,T]\times\mathbb{R}^n \mapsto \mathbb{R}^n$, it holds*

$$F(t, X_{\epsilon_k}(t,x)) \to F(t, X(t,x)) \text{ in } L^0(\mathcal{L}^1\times\mu);$$

and for measurable function $F : \mathbb{R}^n \mapsto \mathbb{R}^n$, it holds for each $t\in[0,T]$ that

$$F(X_{\epsilon_k}(t,\cdot)) \to F(X(t,\cdot)) \text{ in } L^0(\mu).$$

Proof. We only prove the second statement, since the first one can be proved in the same way. By the Egorov Theorem, for each $\delta > 0$, there exists a measurable set E_δ such that $\mu(\mathbb{R}^n\setminus E_\delta) < \delta$ and F is uniformly continuous on E_δ.

On the other hand, by using the Egorov Theorem again and the fact $X_{\epsilon_k}(t,x)$ converges in measure to $X(t,x)$, we find that there exists $\widetilde{E}_\delta$ such that $\mu(\mathbb{R}^n\setminus\widetilde{E}_\delta) < \delta$ and $X_{\epsilon_k}(t,x)$ converges uniformly to $X(t,x)$ on $\widetilde{E}_\delta$.

Therefore, for a fixed constant c,

$$\begin{aligned}
&\mu\left(\{x : |F(X_{\epsilon_k}(t,x)) - F(X(t,x))| > c\}\right)\\
&\quad\le \mu(\mathbb{R}^n\setminus\widetilde{E}_\delta) + \mu(\{x : X(t,x)\in\mathbb{R}^n\setminus E_\delta\}) + \mu(\{x : X_{\epsilon_k}(t,x)\in\mathbb{R}^n\setminus E_\delta\})\\
&\qquad+\mu\left(\left\{x\in\widetilde{E}_\delta, X_{\epsilon_k}(t,x), X(t,x)\in E_\delta : |F(X_{\epsilon_k}(t,x)) - F(X(t,x))| > c\right\}\right).
\end{aligned}$$

Notice that by Theorem 2.2, we have that

$$\mu(\{x : X_{\epsilon_k}(t,x)\in\mathbb{R}^n\setminus E_\delta\}) \le \exp\left\{-\left(\log\frac{1}{\delta}\right)^{\exp\left\{C\int_0^t\beta(s)\,ds\right\}}\right\}$$

uniformly in k, and by Lemma 4.4

$$\mu(\{x : X(t,x)\in\mathbb{R}^n\setminus E_\delta\}) \le \mu\left(\left\{x : X(t,x)\in\widetilde{\mathbb{R}^n\setminus E_\delta}\right\}\right) \le \exp\left\{-\left(\log\frac{2}{\delta}\right)^{\exp\left\{C\int_0^t\beta(s)\,ds\right\}}\right\},$$

where $\widetilde{\mathbb{R}^n\setminus E_\delta}$ is an open set containing $\mathbb{R}^n\setminus E_\delta$ satisfying

$$\mu(\widetilde{\mathbb{R}^n\setminus E_\delta}) \le 2\mu(\mathbb{R}^n\setminus E_\delta).$$

By choosing large enough k, we have

$$\mu\left(\left\{x\in\widetilde{E}_\delta, X_{\epsilon_k}(t,x), X(t,x)\in E_\delta : |F(X_{\epsilon_k}(t,x)) - F(X(t,x))| > c\right\}\right) = 0.$$

Therefore, for each $\gamma > 0$, by choosing sufficiently small δ, we see that there exists k_γ, such that for each $k > k_\gamma$, it holds

$$\mu\left(\{x : |F(X_{\epsilon_k}(t,x)) - F(X(t,x))| > c\}\right) < \gamma,$$

which completes the proof. $\square$

Lemma 4.6. *Under the assumptions of the Main Theorem, for each $t \in (0, T]$, we have*

$$X(t, x) = x + \int_0^t b(s, X(s, x))\, ds$$

for μ-a.e. $x \in \mathbb{R}^n$.

Proof. It suffices to prove that

$$\int_{\mathbb{R}^n} \int_0^T |b_{\epsilon_k}(s, X_{\epsilon_k}(s, x)) - b(s, X(s, x))|\, ds\, d\mu \to 0 \text{ as } \epsilon_k \to 0.$$

Write

$$\begin{aligned}\int_{\mathbb{R}^n} \int_0^T |b_{\epsilon_k}(s, X_{\epsilon_k}(s, x)) - b(s, X(s, x))|\, ds\, d\mu &\le \int_{\mathbb{R}^n} \int_0^T |b_{\epsilon_k}(s, X_{\epsilon_k}(s, x)) - b(s, X_{\epsilon_k}(s, x))|\, ds\, d\mu \\ &\quad + \int_{\mathbb{R}^n} \int_0^T |b(s, X_{\epsilon_k}(s, x)) - b(s, X(s, x))|\, ds\, d\mu \\ &=: I + II.\end{aligned}$$

By Theorem 3.1, we see that

$$\begin{aligned} I &\le \int_{\mathbb{R}^n} \int_0^T |b_{\epsilon_k}(s, x) - b(s, x)| K_{\epsilon_k}(s, x)\, ds\, d\mu \\ &\le \int_0^T 2\|b_{\epsilon_k}(s, \cdot) - b(s, \cdot)\|_{\mathrm{Exp}_\mu(L)} \|K_{\epsilon_k}(s, \cdot)\|_{LlogL(\mu)}\, ds \to 0, \text{ as } k \to \infty.\end{aligned}$$

On the other hand, by applying Lemma 4.5, we find that

$$b(s, X_{\epsilon_k}(s, x)) \to b(s, X(s, x))$$

a.e. in $(0, T) \times \mathbb{R}^n$. Let $b_M := \min\{\max\{b, -M\}, M\}$. Notice that

$$\begin{aligned}\int_{\mathbb{R}^n} \int_0^T |b_M(s, X(s, x))|\, ds\, d\mu &\le \int_{\mathbb{R}^n} \int_0^T |b_M(s, X_{\epsilon_k}(s, x)) - b_M(s, X(s, x))|\, ds\, d\mu \\ &\quad + \int_{\mathbb{R}^n} \int_0^T |b_M(s, X_{\epsilon_k}(s, x))|\, ds\, d\mu.\end{aligned}$$

By using the fact $X_{\epsilon_k}(s, x)$ converges to $X(s, t)$ a.e. on $[0, T] \times \mathbb{R}^n$, we apply the dominated convergence theorem to conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} \int_0^T |b_M(s, X(s, x))|\, ds\, d\mu &\le \liminf_{k\to\infty} \int_{\mathbb{R}^n} \int_0^T |b_M(s, X_{\epsilon_k}(s, x))|\, ds\, d\mu \\ &= \liminf_{k\to\infty} \int_{\mathbb{R}^n} \int_0^T |b(s, x)| K_{\epsilon_k}(t, x)\, ds\, d\mu \\ &\le C(b) < \infty,\end{aligned}$$

where $C(b)$ is independent of M. We therefore see that $b(s, X(s, x)) \in L^1(0, T; \mu)$, and

$$\begin{aligned}
II &\le \int_{\mathbb{R}^n} \int_0^T |b(s, X_{\epsilon_k}(s, x)) - b_M(s, X_{\epsilon_k}(s, x))|\, ds\, d\mu \\
&+ \int_{\mathbb{R}^n} \int_0^T |b(s, X(s, x)) - b_M(s, X(s, x))|\, ds\, d\mu \\
&+ \int_{\mathbb{R}^n} \int_0^T \left(|b_M(s, X_{\epsilon_k}(s, x)) - b_M(s, X(s, x))|\right)\, ds\, d\mu \\
&=: II_1 + II_2 + II_3.
\end{aligned}$$

For each $\gamma > 0$, we can choose M sufficient large such that $II_1 + II_2 < \gamma/2$. Applying the dominated convergence theorem to II_3, we see that

$$II_3 \to 0 \quad \text{as } k \to \infty.$$

Hence, we obtain that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \int_0^T |b(s, X_{\epsilon_k}(s, x)) - b(s, X(s, x))|\, ds\, d\mu = 0,$$

which together with the fact $X_{\epsilon_k}(s, x) \to X(s, x)$ a.e., implies that

$$X(t, x) = x + \int_0^T b(s, X(s, x))\, ds,\ \mu - a.e.$$

The proof is completed. □

Uniqueness of the flow will follow as a corollary of Theorem 2.2.

Proposition 4.7. *Under the assumptions of the Main Theorem, the flow $X(t, x)$ satisfying*

$$\frac{\partial X}{\partial t} = b(t, X),\ \mu - a.e.$$

is unique.

Proof. By the well-posedness of the transport equation (Theorem 2.2), it suffices to show that for each $u_0 \in C_c^\infty(\mathbb{R}^n)$, $u(t, x) := u_0(X(t, x))$ is a distributional solution to the transport equation

$$\frac{\partial}{\partial t} u(X(t, x)) - b(t, x) \cdot \nabla u(X(t, x)) = 0,$$

i.e., for each $\varphi \in C^\infty([0, T) \times \mathbb{R}^n)$ with compact support in $[0, T) \times \mathbb{R}^n$, it holds

$$\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^n} u(t, x) \frac{\partial \varphi(t, x)}{\partial t}\, dt\, d\mu(x) - \int_{\mathbb{R}^n} u(0, x)\, \varphi(0, x)\, d\mu(x) \\
&\quad = - \int_0^T \int_{\mathbb{R}^n} \left[u(t, x)\varphi\, \mathrm{div}_\mu(b)(t, x) + u(t, x) b(t, x) \cdot \nabla\varphi\right] d\mu(x)\, dt
\end{aligned}$$

Let $X_{\epsilon_k}(t,x)$ be flows generated from b_{ϵ_k} that converge to $X(t,x)$ in measure and μ-a.e. By the smoothness of b_{ϵ_k}, we know that $u_0(X_{\epsilon_k}(t,x))$ satisfies

$$\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^n} u_0(X_{\epsilon_k}(t,x)) \frac{\partial\varphi(t,x)}{\partial t}\, dt\, d\mu(x) - \int_{\mathbb{R}^n} u_0(x)\,\varphi(0,x)\, d\mu(x) \\
&\quad = -\int_0^T \int_{\mathbb{R}^n} \Big[u_0(X_{\epsilon_k}(t,x))\varphi \operatorname{div}_\mu(b_{\epsilon_k})(t,x) - u_0(X_{\epsilon_k}(t,x)) b_{\epsilon_k}(t,x)\cdot\nabla\varphi\Big]\, d\mu\, dt.
\end{aligned}$$

It is readily seen that first term converges to

$$-\int_0^T \int_{\mathbb{R}^n} u_0(X(t,x)) \frac{\partial\varphi(t,x)}{\partial t}\, dt\, d\mu(x) = -\int_0^T \int_{\mathbb{R}^n} u(t,x) \frac{\partial\varphi(t,x)}{\partial t}\, dt\, d\mu(x).$$

as $k \to \infty$. By the facts $\operatorname{div}_\mu(b_{\epsilon_k}) \to \operatorname{div}_\mu(b)$ and $b_{\epsilon_k} \to b$ in $\operatorname{Exp}_\mu(\frac{L}{\log L})$, and $u_0(X_{\epsilon_k}(t,x)) \to u_0(X(t,x))$ in $L^p(\mu)$ for any $p < \infty$, we can conclude that

$$-\int_0^T \int_{\mathbb{R}^n} \Big[u_0(X_{\epsilon_k}(t,x))\varphi \operatorname{div}_\mu(b_{\epsilon_k})(t,x) - u_0(X_{\epsilon_k}(t,x)) b_{\epsilon_k}(t,x)\cdot\nabla\varphi\Big]\, d\mu\, dt$$

converges to

$$-\int_0^T \int_{\mathbb{R}^n} \Big[u_0(X(t,x))\varphi \operatorname{div}_\mu(b)(t,x) - u_0(X(t,x)) b(t,x)\cdot\nabla\varphi\Big]\, d\mu\, dt$$

as $k \to \infty$. Therefore, $u(t,x) := u_0(X(t,x))$ is a distributional solution to the transport equation

$$\frac{\partial}{\partial t} u(X(t,x)) - b(t,x)\cdot\nabla u(X(t,x)) = 0.$$

By the well-posedness of the transport equation, Theorem 2.2, we see that the flow is unique. □

Proof of Main Theorem (a). The existence of a solution X to the equation $\frac{\partial}{\partial t}X = b(t,X)$ follows from Lemma 4.6. The uniqueness follows from Proposition 4.7. The estimate of the density function follows from Lemma 4.3. □

5 Regularity, semigroup structure and stability

In this section, we prove part (b) of the Main Theorem, and give a stability result. To do this, we start by stating the semigroup structure of our flow.

Lemma 5.1. *Let b be as in the Main Theorem, and let X be the flow associated to b. Then*

$$X(s, X(t-s,x)) = X(t,x)$$

for almost every x and each $0 < s < t \leq T$.

Proof. Notice that by the semigroup structure of X_{ϵ_k}, we have

$$X(t,x) = \lim_{k\to\infty} X_{\epsilon_k}(t,x) = \lim_{k\to\infty} X_{\epsilon_k}(s, X_{\epsilon_k}(t-s,x)),\ \mu - a.e.$$

Therefore, to prove the semigroup structure, it suffices to show that

$$\lim_{k\to\infty} X_{\epsilon_k}(s, X_{\epsilon_k}(t-s,x)) = X(s, X(t-s,x)),\ \mu - a.e.$$

Write

$$\begin{aligned}
&\left|X_{\epsilon_k}(s, X_{\epsilon_k}(t-s,x)) - X(s, X(t-s,x))\right| \\
&\le \left|X_{\epsilon_k}(s, X_{\epsilon_k}(t-s,x)) - X(s, X_{\epsilon_k}(t-s,x))\right| + \left|X(s, X(t-s,x)) - X(s, X_{\epsilon_k}(t-s,x))\right| =: I + II.
\end{aligned}$$

By Lemma 4.2, we see that $X_{\epsilon_k}(s,\cdot)$ converges to $X(s,\cdot)$ in measure. Let $c > 0$ be fixed. Then for each $\gamma > 0$, there exists k_γ, such that for $k > k_\gamma$, it holds

$$\mu\left(\{x : |X_{\epsilon_k}(s,x) - X(s,x)| > c\}\right) < \gamma.$$

Let $E_{k,c} = \{x : |X_{\epsilon_k}(s,x) - X(s,x)| > c\}$. Recall that by Theorem 2.2, for any measurable set E with sufficient small measure, it holds

$$\left|\log\log\log\left(\frac{1}{\int_{\mathbb{R}^n} \chi_E(X_{\epsilon_k}(t,x))\,d\mu}\right) - \log\log\log\left(\frac{1}{\mu(E)}\right)\right| \le C\int_0^T \beta(s)\,ds,$$

since $\mathrm{div}_\mu b_{\epsilon_k}$ has uniform bound in $\mathrm{Exp}_\mu(\frac{L}{\log L})$. We then can conclude that

$$\begin{aligned}
\mu\left(\{x : |X_{\epsilon_k}(s, X_{\epsilon_k}(t,x)) - X(s, X_{\epsilon_k}(t,x))| > c\}\right) &= \int_{\mathbb{R}^n} \chi_{E_{k,c}}(X_{\epsilon_k}(t,x))\,d\mu \\
&\le \exp\left\{-\left(\log\left(\frac{1}{\mu(E_{k,c})}\right)\right)^{-\exp\{C\int_0^T \beta(s)\,ds\}}\right\} \\
&\le \exp\left\{-\left(\log\left(\frac{1}{\gamma}\right)\right)^{-\exp\{C\int_0^T \beta(s)\,ds\}}\right\},
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \mu\left(\{x : |X_{\epsilon_k}(s, X_{\epsilon_k}(t,x)) - X(s, X_{\epsilon_k}(t,x))| > c\}\right) = 0.$$

On the other hand, using Lemma 4.5, we see that $X(s, X_{\epsilon_k}(t-s,x))$ converges to $X(s, X(t-s,x))$ in measure. Therefore, we see that $X_{\epsilon_k}(s, X_{\epsilon_k}(t-s,x))$ converges in measure to $X(s, X(t-s,x))$, up to a subsequence, we can conclude that

$$X(t,x) = \lim_{k\to\infty} X_{\epsilon_k}(t,x) = \lim_{k\to\infty} X_{\epsilon_k}(s, X_{\epsilon_k}(t-s,x)).$$

The proof is completed. □

By combining part (a) of the Main Theorem with Lemma 5.1, we can find a unique flow $X(t, x)$ for $t \in [-T, 0]$ such that

$$X(t, x) = x - \int_t^0 b(-s, X(s, x))\, ds = x - \int_0^{-t} b(s, X(-s, x))\, ds$$

for μ-a.e. $x \in \mathbb{R}^n$. Then by using the semigroup property we can conclude that

$$X(t, X(-t, x)) = x \tag{5.1}$$

for each $t \in [-T, T]$ and μ-a.e. $x \in \mathbb{R}^n$. Indeed, this identity holds trivially when b is smooth in the spatial direction, since

$$\frac{\partial}{\partial t} X(t, X(-t, x)) = b(t, X(-t, x)) \cdot \nabla X(t, X(-t, x)) - b(t, X(-t, x)) \cdot \nabla X(t, X(-t, x)) = 0.$$

For the non-smooth case, after regularization, we can conclude the thesis by following the proofs of Lemma 4.2, Lemma 4.5 and Lemma 5.1.

We are now in position to complete the proof of our Main Theorem.

Proof of Main Theorem (b). We already know that a flow X associated to b exists and is unique. Further, by Lemma 5.1 we also know it has semigroup structure. Thus, in order to prove that X is a regular flow it just remains to show that $X(t, \cdot)_\# \, dx \ll dx$. For each $\psi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\psi(X(t, x))|\, dx &= \int_{\mathbb{R}^n} (2\pi)^{n/2} |\psi(X(t, x))| \exp\left\{\frac{|x|^2}{2}\right\} d\mu(x) \\ &= \int_{\mathbb{R}^n} (2\pi)^{n/2} |\psi(y)| \exp\left\{\frac{|X(-t, y)|^2}{2}\right\} K_t(y)\, d\mu(y), \end{aligned}$$

where

$$X(-t, y) = y - \int_0^t b(s, X(-s, y))\, ds$$

is the inverse map of $X(t, y)$ as indicated in (5.1). From the assumption $\frac{|b|}{1+|x|\log^+|x|} \in L^1(0, T; L^\infty)$, we can see that $\{X(-t, y) : y \in \operatorname{supp}\psi\}$ is bounded in $[0, T] \times \mathbb{R}^n$. Therefore,

$$\int_{\mathbb{R}^n} |\psi(X(t, x))|\, dx \le C(b, t, \psi) \int_{\mathbb{R}^n} |\psi(y)| K_t(y)\, dy,$$

and hence, $X(t, \cdot)_\# \, dx \ll dx$. The proof is completed. □

As a result of the techniques we have used throughout this work we get the following result about stability.

Theorem 5.2. *Let $b, \{b_k\} \in L^1(0, T; W^{1,1}_{\mathrm{loc}})$ satisfying*

$$\frac{|b(t, x)|}{1 + |x| \log^+ |x|}, \frac{|b_k(t, x)|}{1 + |x| \log^+ |x|} \in L^1(0, T; L^\infty)$$

and

$$b_k \to b \quad \textit{in } \mathrm{Exp}_\mu(L).$$

Assume that $\mathrm{div}_\mu b, \mathrm{div}_\mu b_k$ *are uniformly bounded in* $L^1(0,T;\mathrm{Exp}_\mu(\frac{L}{\log L}))$ *and* $\mathrm{div}_\mu b_k$ *converges to* $\mathrm{div}_\mu b$ *in* $L^1(0,T;L^1_{\mathrm{loc}}(\mu))$ *. Let* $X(t,x), \{X_k(t,x)\}$ *be the flows generated from* $b, \{b_k\}$ *respectively. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \sup_{t\in[0,T]} |X(t,x) - X_k(t,x)| \, d\mu \to 0. \tag{5.2}$$

Proof. For each bounded function $\beta \in C^1(\mathbb{R},\mathbb{R})$, $\beta(X_k^i(t,x))$, $1 \le i \le n$, is the solution to the Cauchy problem of the transport equation associated to the vector field b_k, with the initial value $\beta(x^i)$. By weak star compactness in $L^\infty(0,T;L^\infty)$, we see that there exists a subsequence $\{\beta(X^i_{k_j})\}_j$, converges to a function $\widetilde{X}$, which is a solution to the Cauchy problem of the transport equation associated to the vector field b, with the initial value $\beta(x^i)$. By the uniqueness, we see that $\widetilde{X} = \beta(X^i(t,x))$.

By the well-posedness and the renormalization property of the transport equation , we see that, indeed, $\beta(X^i_k)$ converges in measure to $\beta(X(t,x))$. Following the same argument as in Lemma 4.2, we see that X_k converges in measure to $X(t,x)$.

Observing this, and the fact

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup_{t\in[0,T]} |X(t,x) - X_k(t,x)| \, d\mu \\
&\quad \le \int_{\mathbb{R}^n} \int_0^T |b(s,X(s,x)) - b_k(s,X_k(s,x))| \, ds \, d\mu \\
&\quad \le \int_{\mathbb{R}^n} \int_0^T |b(s,X(s,x)) - b(s,X_k(s,x))| \, ds \, d\mu + \int_{\mathbb{R}^n} \int_0^T |b(s,X_k(s,x)) - b_k(s,X_k(s,x))| \, ds \, d\mu,
\end{aligned}$$

we can follow the proof of Lemma 4.6 to conclude that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \sup_{t\in[0,T]} |X(t,x) - X_k(t,x)| \, d\mu \to 0.$$

The proof is completed. □

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