A NOTE ON HURWITZ'S INEQUALITY

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Abstract. Given a simple closed plane curve Γ of length L enclosing a compact convex set K of area F, Hurwitz found an upper bound for the isoperimetric deficit, namely $L^2 - 4\pi F \leq \pi|F_e|$, where F_e is the algebraic area enclosed by the evolute of Γ.

In this note we improve this inequality finding strictly positive lower bounds for the deficit $\pi|F_e| - \Delta$, where $\Delta = L^2 - 4\pi F$. These bounds involve wether the visual angle of Γ or the pedal curve associated to K with respect to the Steiner point of K or the $\mathcal{L}^2$ distance between K and the Steiner disk of K.

For each established inequality we study when equality holds. This occurs for those compact convex sets being bounded by a curve parallel to an hypocycloid of $3, 4$ or 5 cusps or the Minkowski sum of this kind of sets.

1. Introduction

Let Γ be a simple closed plane curve of length L enclosing a region of area F. The classical isoperimetric inequality states that

$$L^2 - 4\pi F \geq 0,$$

with equality attained only for a circle.

In the case that Γ bounds a convex set K, Hurwitz ([6]) established a kind of reverse isoperimetric inequality, namely

$$L^2 - 4\pi F \leq \pi|F_e|, \tag{1}$$

where F_e is the algebraic area ($F_e \leq 0$) enclosed by the evolute of Γ. We recall that the evolute of a curve is the envelope of its normal lines. Moreover equality holds in (1) if and only if Γ is a circle or a curve parallel to an astroid.

The goal of this note is to improve Hurwitz's inequality (1) finding strictly positive lower bounds for the *Hurwitz deficit* $\pi|F_e| - \Delta$, where $\Delta = L^2 - 4\pi F$. These bounds involve wether the visual angle of Γ or the pedal curve associated to K with respect to the Steiner point of K or the $\mathcal{L}^2$ distance between the support function of K and the support function of the Steiner disk of K.

Hurwitz's inequality (1) can be improved without introducing new quantities for some special compact sets. For instance, if K has constant width one gets

$$L^2 - 4\pi F \leq \frac{4}{9}\pi|F_e|,$$

as shown in Theorem 4.4.

Key words and phrases. Convex set, isoperimetric inequality, evolute, hypocycloid, pedal curve, visual angle.

The authors where partially supported by grants 2014SGR289 (Generalitat de Catalunya) and MTM2015-66165-P (FEDER/Mineco).

Prepublicació Núm 9, abril 2017.
Departament de Matemàtiques.
http://www.uab.cat/matematiques
UAB Universitat Autònoma de Barcelona

For the general case we prove in Theorem 5.1 the inequality

$$\pi|F_e| - \Delta \geq \frac{5}{4}L^2 + 5\int_{P\notin K}(\omega - \sin\omega - \frac{2}{3}\sin^3\omega)\,dP,$$

where ω is the visual angle of Γ from P, that is the angle between the tangents from P to Γ, and dP the area measure. For the case of constant width Theorem 5.3 asserts that

$$\frac{4}{9}\pi|F_e| - \Delta \geq \frac{64}{9}\int_{P\notin K}\left(\omega - 2\sin\omega + \sin 2\omega - \frac{1}{4}\sin 4\omega - \sin^3\omega\right)\,dP.$$

In both cases the quantities in the right hand side are strictly positive except when the left hand side vanishes.

In terms of the area A of the pedal curve associated to the compact strictly convex set K, with respect to its Steiner point, we prove (Theorem 5.4)

$$\pi|F_e| - \Delta \geq \frac{40}{9}\left(\pi(A - F) + \frac{2}{3}L^2 - \frac{8}{9}\int_{P\notin K}\sin^3\omega\,dP\right).$$

When K has constant width we obtain (Corollary 5.5)

$$\pi|F_e| - \Delta \geq \frac{40}{9}\pi(A - F).$$

In both cases the lower bounds for the positive Hurwitz deficit are strictly positive.

For each established inequality we study when equality holds. This occurs for those compact convex sets being bounded by a curve parallel to an hypocycloid of $3, 4$ or 5 cusps or the Minkowski sum of this kind of sets.

We base the proof of our results on a relationship between the Hurwitz deficit and the Wirtinger deficit (Proposition 4.1) and the corresponding expansion of these quantities in Fourier series. In order to deal with the area of the evolute of Γ we use a formula that relates this area, the area of K and the integral of the radius of curvature of Γ. We give a general version of this known formula which is valid for curves with multiple points (Proposition 3.1).

2. Preliminaries

2.1. **Convex sets and support function.** A set $K \subset \mathbb{R}^2$ is *convex* if it contains the complete segment joining every two points in the set. We shall consider nonempty compact convex sets. The *support function* of K is defined as

$$p_K(u) := \sup\{\langle x, u\rangle \,:\, x \in K\} \qquad \text{for} \quad u \in \mathbb{R}^2.$$

For a unit vector $u \in S^1$ the number $p_K(u)$ is the signed distance of the support line to K with outer normal vector u from the origin. The distance is negative if and only if u points into the open half-plane containing the origin (cf. [7]). We shall denote by $p(\varphi)$ the 2π-periodic function obtained by evaluating $p_K(u)$ on $u = (\cos\varphi, \sin\varphi)$. Note that ∂K is the envelope of the one parametric family of lines given by

$$x\cos\varphi \ + y\sin\varphi \ = p(\varphi).$$

If the support function $p(\varphi)$ is differentiable we can parametrize the boundary ∂K by

$$\gamma(\varphi) = p(\varphi)N(\varphi) + p'(\varphi)N'(\varphi), \tag{2}$$

where $N(\varphi) = (\cos\varphi, \sin\varphi)$. When p is a $\mathcal{C}^2$ function the radius of curvature $\rho(\varphi)$ of ∂K at the point $\gamma(\varphi)$ is given by $p(\varphi) + p''(\varphi)$. Then, convexity is equivalent to $p(\varphi) + p''(\varphi) \geq 0$. We say that a $\mathcal{C}^2$ support function p defines a *strictly convex* set if $p(\varphi) + p''(\varphi) > 0$ for every value of φ.

A straightforward computation shows that the area F of K is given by

$$F = \frac{1}{2}\int_0^{2\pi} p(p + p'')\, d\varphi.$$

Since p is 2π-periodic, integrating by parts, we get

$$F = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\varphi. \tag{3}$$

Also it can be seen that the length L of ∂K is given by

$$L = \int_0^{2\pi} p\, d\varphi.$$

In general, a one parameter family of lines

$$x\cos t + y\sin t = f(t),$$

where f is a differentiable function, defines a curve in the plane. In this setting the curve is not necessarily closed nor convex. When a curve $\gamma(t)$ is defined as the envelope of a family of lines of this type, for a function f of class $\mathcal{C}^2$, we say that $f(t)$ is the *generalized support function* of the curve. The *area with multiplicities* swept by the radius vector of the curve is given by

$$F = \frac{1}{2}\int_a^b f(f + f'')\, dt. \tag{4}$$

Let $p(\varphi)$ be the support function of a strictly convex set K. Then $p_r(\varphi) = p(\varphi) + r$ defines for each real r a parallel curve to ∂K. If the origin is in the interior of K then p is a strictly positive function. If $r > 0$ the function p_r corresponds to the outer parallel set at distance r. When $r < 0$ the curve given by p_r is not necessarily convex (this is the case when $|r| > \min(\rho)$, ρ being the radius of curvature).

The Steiner formula (see for instance [7])

$$F_r = \pi r^2 + L\, r + F$$

gives the area F_r of the r-parallel set to K. The discriminant of this polynomial is the isoperimetric deficit $L^2 - 4\pi F$. It is always strictly positive except for a circle. Thus, for every convex set K there are interior parallel sets with negative area. The minimum area value is $F - L^2/4\pi$ and it is attained for the parallel set at distance $\xi = -L/2\pi$. Then

$$L^2 - 4\pi F = -4\pi F_\xi = 4\pi|F_\xi|. \tag{5}$$

A special type of convex sets are those of *constant width*, that is those convex sets whose orthogonal projection on any direction have the same length w. In terms of the support function p of K, constant width means that $p(\varphi) + p(\varphi + \pi) = w$. Expanding p in Fourier series

$$p(\varphi) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\varphi) + b_n \sin(n\varphi), \tag{6}$$

it follows that

$$p(\varphi) + p(\varphi + \pi) = 2\sum_{n=0}^{\infty}(a_{2n} \cos 2n\varphi + b_{2n} \sin 2n\varphi),$$

so constant width is equivalent to $a_n = b_n = 0$ for all even $n > 0$.

2.2. **Hypocycloids.** Consider a curve defined by the generalized support function

$$p(\theta) = A \sin(B\theta), \quad \theta \in \mathbb{R}$$

with B a positive rational number and $A > 0$. If we define $k = 2B/(B - 1)$ and $A = r(k - 2)$, then $p(\theta)$ can be written in the more convenient form

$$p(\theta) = r(k-2)\sin\left(\frac{k}{k-2}\theta\right), \ k > 2.$$

The envelope curve given by this generalized support function can be parametrized by by

$$\gamma(\theta) = r(k-2)\sin\left(\frac{k}{k-2}\theta\right) N(\theta) + rk\cos\left(\frac{k}{k-2}\theta\right) N'(\theta).$$

Putting $\theta = (k - 2)t/2$ the curve $\tilde{\gamma}(t) = \gamma(\frac{k-2}{2}t)$ has components

$$\left.\begin{aligned} x(t) &= r(k-2)\sin\left(\tfrac{k}{2}t\right)\cos\left(\tfrac{k-2}{2}t\right) - rk\cos\left(\tfrac{k}{2}t\right)\sin\left(\tfrac{k-2}{2}t\right) \\ y(t) &= r(k-2)\sin\left(\tfrac{k}{2}t\right)\sin\left(\tfrac{k-2}{2}t\right) + rk\cos\left(\tfrac{k}{2}t\right)\cos\left(\tfrac{k-2}{2}t\right) \end{aligned}\right\}.$$

Using known trigonometric identities we get

$$\left.\begin{aligned} x(t) &= r(k-1)\sin(t) - r\sin((k-1)t) \\ y(t) &= r(k-1)\cos(t) + r\cos((k-1)t) \end{aligned}\right\}.$$

This is just the parametrization of an *hypocycloid* obtained by rolling a circle of radius r inside a circle of radius $R = kr$.

Writing $k = m/n$ with m, n coprime numbers, in order to obtain a closed hypocycloid the parameter t has to vary in the interval $[0, 2n\pi]$ and the parameter θ has to vary in the interval $[0, (m - 2n)\pi]$. Note that for a generalized support function $\sin(B\theta)$ with B an integer greater or equal than two, the hypocycloid is traveled twice if B is odd and once if B is even.

When k is an integer the curve has k *cusps* (extremal points of the curvature). For $k = m/n$ with m, n coprime numbers the curve has m cusps. In the special case $k = 3$ the hypocycloid is called a *deltoid* or *Steiner curve*; for $k = 4$ it is called an *astroid*.

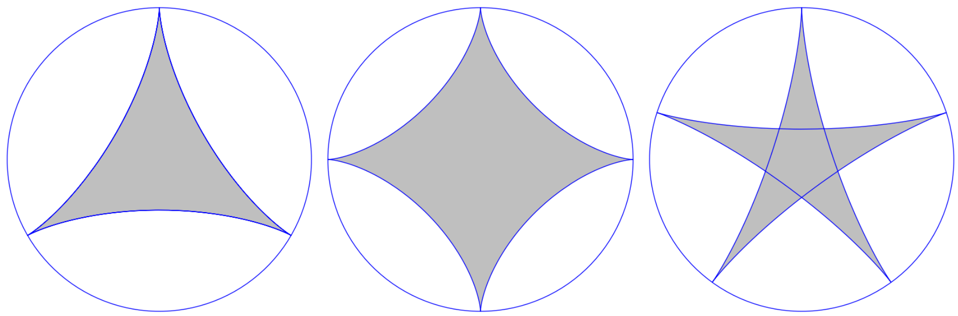

FIGURE 1. Hypocycloids with $B = 3, 2$ and 5.

2.3. **Steiner point and pedal curve.** Given a compact convex set K with support function $p(\varphi)$ the *Steiner point* of K is defined by the vector-valued integral

$$s(K) := \frac{1}{\pi}\int_0^{2\pi} p(\varphi)N(\varphi)\,d\varphi.$$

This functional on the space of convex sets is additive with respect to the Minkowski sum. The Steiner point is rigid motion equivariant; this means that $s(gK) = gs(K)$ for every rigid motion g. We remark that $s(K)$ can be considered, in the $\mathcal{C}^2$ case, as the centroid with respect to the curvature measure in the boundary ∂K; also we have that $s(K)$ lies in the interior of K (see [4]). In terms of the Fourier coefficients of $p(\varphi)$ given in (6) the Steiner point is

$$s(K) = (a_1, b_1).$$

The relation between the support function $p(\varphi)$ of a convex set K and the support function $q(\varphi)$ of the same convex set but with respect to a new reference with origin at the point (a, b), and axes parallel to the previous x and y-axes, is given by

$$q(\varphi) = p(\varphi) - a\cos\varphi - b\sin\varphi.$$

Hence, taking the Steiner point as a new origin, we have

$$q(\varphi) = a_0 + \sum_{n\geq 2} a_n \cos n\varphi + b_n \sin n\varphi.$$

We recall that the *Steiner disk* of K is the disk whose center is the Steiner point and whose diameter is the mean width of K.

The associated *pedal curve* to K is the curve that in polar coordinates with respect to the origin is given by $r = p(\varphi)$. Notice that this curve depends on the center point from which the support function is considered. In fact it is the geometrical locus of the orthogonal projection of the center on the tangents to the curve. The area enclosed by the pedal curve is

$$A = \frac{1}{2}\int_0^{2\pi} p(\varphi)^2\,d\varphi.$$

3. On the integral of the radius of curvature

We give here an elementary proof of a known formula that relates the integral of the radius of curvature of a plane closed curve with the areas enclosed by the curve and its evolute. This result is proved in [3] and [6] for the case of simple closed curves that bound a strictly convex domain, but our proof works also for curves with multiple points.

Proposition 3.1. *Let* Γ *be a plane closed* $\mathcal{C}^2$ *curve of length* L*, and positive curvature. Let* F *be the area enclosed by* Γ*, and let* F_e *be the area enclosed by the evolute of* Γ*, both counted with multiplicities. Then*

$$\int_0^L \rho(s)\,ds = 2(F - F_e),$$

where s *is the arc length parameter of* Γ *and* $\rho(s)$ *is the radius of curvature of* Γ*.*

Proof. Assume that Γ is parametrized by $\gamma(s) = (x(s), y(s))$, $0 \leq s \leq L$. By Green's formula, the area with multiplicities enclosed by Γ is given by

$$F = \frac{1}{2}\int_0^L (xy' - x'y)\,ds,$$

(see for instance [1]).

Denoting by J the operator $J(v) = iv$, $v \in \mathbb{C}$, the above formula can be written as

$$F = \frac{1}{2}\int_0^L \langle J(\gamma(s)), \gamma'(s)\rangle\,ds.$$

The evolute Γ_e of Γ can be parametrized as $\gamma_e(s) = \gamma(s) + \rho(s)N(s)$ where $\rho(s), N(s)$ are respectively the radius of curvature and the principal normal of Γ in the point $\gamma(s)$. Note that s is not the arc length parameter of Γ_e. The area with multiplicities enclosed by Γ_e is given by

$$F_e = \frac{1}{2}\int_0^L \langle J(\gamma_e(s)), \gamma_e'(s)\rangle\,ds.$$

But, omitting for brevity the reference to the parameter s,

$$\begin{aligned}\langle J\gamma_e, \gamma_e'\rangle &= \langle J\gamma + \rho JN, \gamma' + \rho' N + \rho N'\rangle \\ &= \langle J\gamma + \rho JN, \rho' N\rangle \\ &= \langle J\gamma, \rho' N\rangle\end{aligned}$$

since $N'(s) = -k(s)\gamma'(s)$ and $\langle J(v), v\rangle = 0$, for $v \in \mathbb{R}^2$.

Hence,

$$\begin{aligned}2F_e &= \int_0^L \rho'\langle J\gamma, N\rangle\,ds = -\int_0^L \rho\langle J\gamma, N\rangle'\,ds \\ &= -\int_0^L \rho\Big(\langle J\gamma', N\rangle + \langle J\gamma, N'\rangle\Big)\,ds = -\int_0^L \rho\Big(1 - k\langle J\gamma, \gamma'\rangle\Big)\,ds \\ &= -\int_0^L \rho\,ds + 2F.\end{aligned}$$

Equivalently,

$$\int_0^L \rho\,ds = 2(F - F_e).$$

□

Corollary 3.2. *Let Γ be a plane closed $\mathcal{C}^2$ curve with positive index of rotation r. Let F be the area enclosed by Γ and let F_e be the area enclosed by the evolute of Γ, both counted with multiplicities. Then*

$$F_e \leq F\left(1 - \frac{1}{r}\right). \tag{7}$$

In particular, if Γ is a positively oriented simple curve, then $F_e \leq 0$.

Proof. By the Cauchy-Schwarz inequality, the length L of Γ verifies

$$\begin{aligned} L^2 &= \left(\int_0^L ds\right)^2 = \left(\int_0^L \frac{1}{\sqrt{\rho}}\sqrt{\rho}\, ds\right)^2 \\ &\leq \left(\int_0^L \frac{1}{\rho(s)}\, ds\right)\left(\int_0^L \rho(s)\, ds\right) = 2\pi r \int_0^L \rho\, ds. \end{aligned}$$

The isoperimetric inequality gives

$$F \leq \frac{L^2}{4\pi} \leq \frac{r}{2}\int_0^L \rho\, ds = r(F - F_e),$$

and we get (7). □

4. Hurwitz's inequality

For a $\mathcal{C}^1$ function $q(\varphi)$ of period 2π, let us introduce the *Wirtinger deficit* W_q of q by

$$W_q = \int_0^{2\pi} (q'^2 - q^2)\, d\varphi.$$

Recall that Wirtinger's inequality (see [5]) states that if

$$\int_0^{2\pi} q(\varphi)\, d\varphi = 0,$$

then

$$W_q \geq 0.$$

If p is the support function of a convex set of area F then $W_p = -2F$.

Now we use Proposition 3.1 to give a relationship between the Wirtinger deficit and Hurwitz's deficit.

Proposition 4.1. *Let K be a compact strictly convex set of area F bounded by a curve $\Gamma = \partial K$ of class $\mathcal{C}^2$ and length L. Let p be the support function of K anf let F_e be the area with multiplicities enclosed by the evolute of Γ. Then*

$$\pi|F_e| - \Delta = \frac{\pi}{2}(W_{q'} - 4W_q),$$

where $q(\varphi) = p(\varphi) - L/2\pi$ and $\Delta = L^2 - 4\pi F$.

Proof. By Proposition 3.1 we have

$$\pi|F_e| - \Delta = \frac{\pi}{2}\int_0^L \rho(s)\, ds - L^2 + 3\pi F.$$

In terms of p, and since $ds = \rho\, d\phi = (p + p'')\, d\phi$, it follows

$$\begin{aligned}\pi|F_e| - \Delta &= \frac{\pi}{2}\int_0^{2\pi}(p+p'')^2\,d\varphi - L^2 + \frac{3\pi}{2}\int_0^{2\pi}(p^2 - p'^2)\,d\varphi \\ &= \frac{\pi}{2}\left(4\int_0^{2\pi}p^2\,d\varphi - 5\int_0^{2\pi}p'^2\,d\varphi + \int_0^{2\pi}p''^2\,d\varphi\right) - L^2.\end{aligned}$$

Taking $q(\varphi) = p(\varphi) - L/2\pi$ we get

$$\pi|F_e| - \Delta = \frac{\pi}{2}\left(\int_0^{2\pi}(q''^2 - q'^2)\,d\varphi\right) - 4\int_0^{2\pi}(q'^2 - q^2)\,d\varphi = \frac{\pi}{2}(W_{q'} - 4W_q).$$

□

Remark. When p is the support function of a convex set K the function $q(\varphi) = p(\varphi) - L/2\pi$ is the support function of the interior parallel set to $\Gamma = \partial K$ at a distance $L/2\pi$. If the curve Γ is parametrized by $\gamma(\varphi)$ as in (2), its evolute can be parametrized by

$$\begin{aligned}\tilde{\gamma}(\varphi) &= \gamma(\varphi) - (p(\varphi) + p''(\varphi))N(\varphi) = p'(\varphi)N'(\varphi) - p''(\varphi)N(\varphi) \\ &= p'(\varphi)N\left(\varphi + \frac{\pi}{2}\right) + p''(\varphi)N'\left(\varphi + \frac{\pi}{2}\right).\end{aligned}$$

Therefore $p'(\varphi - \pi/2)$ is a generalized support function for the evolute of Γ. So by (4) we get $W_{q'} = -2F_e$. By Corollary 3.2 we know that $F_e \le 0$ and using equality (5) we conclude

$$\pi|F_e| - \Delta = \pi(|F_e| - 4|F_{-L/2\pi}|) = \frac{\pi}{2}(W_{q'} - 4W_q).$$

This gives an alternative proof to Proposition 4.1.

Next Lemma compares the Wirtinger deficit of a given function with that of its derivative. The proof follows the standard pattern of the proof of Wirtinger inequality using Fourier series.

Lemma 4.2. *Let $q = q(\varphi)$ a 2π-periodic $\mathcal{C}^2$ function. Then*

$$W_{q'} \ge 4W_q + \frac{2}{\pi}\left(\int_0^{2\pi} q\,d\varphi\right)^2 \ge 0.$$

Moreover the first inequality is an equality if and only if

$$q(\varphi) = a_0 + a_1\cos\varphi + b_1\sin\varphi + a_2\cos 2\varphi + b_2\sin 2\varphi,$$

for some constants $a_0, a_1, b_1, a_2, b_2 \in \mathbb{R}$.

Proof. Let

$$q(\varphi) = a_0 + \sum_{n=1}^{\infty} a_n\cos n\varphi + b_n\sin n\varphi,$$

be the Fourier series expansion of $q(\varphi)$. Using the Parseval identity we get

$$\begin{aligned}W_{q'} &= \pi\sum_{n=1}^{\infty} n^2(n^2-1)(a_n^2 + b_n^2) \\ &\ge 4\pi\sum_{n=1}^{\infty}(n^2-1)(a_n^2+b_n^2) = 4W_q + \frac{2}{\pi}\left(\int_0^{2\pi} q\,d\varphi\right)^2.\end{aligned}$$

Equality holds if and only if $a_n = b_n = 0$, if $n \ge 3$. □

Remark that Lemma 4.2, in case that

$$\int_0^{2\pi} q(\varphi)\, d\varphi = 0$$

gives Wirtinger's inequality but it improves this inequality for the derivative of 2π-periodic functions.

For reader's convenience we provide a simple proof of Hurwitz's inequality based on Proposition 4.1.

Theorem 4.3 (**Hurwitz**). *Let K be a compact strictly convex set of area F bounded by a curve $\Gamma = \partial K$ of class $\mathcal{C}^2$ and length L and let F_e be the area with multiplicities enclosed by the evolute of Γ. Then*

$$\Delta = L^2 - 4\pi F \leq \pi |F_e|. \tag{8}$$

Equality holds if and only if Γ is a circle or it is a curve parallel to an astroid at distance $L/2\pi$.

Proof. The inequality follows from Proposition 4.1 and Lemma 4.2.

Since $q(\varphi) = p(\varphi) - L/2\pi$ it is

$$\int_0^{2\pi} q(\varphi)\, d\varphi = 0,$$

and so equality in (8) is equivalent to equality in the first inequality of Lemma 4.2. This implies

$$p(\varphi) = a_0 + a_1 \cos\varphi + b_1 \sin\varphi + a_2 \cos 2\varphi + b_2 \sin 2\varphi.$$

Taking the Steiner point (a_1, b_1) as a new origin of coordinates the new support function of K becomes

$$\tilde{p}(\varphi) = a_0 + a_2 \cos 2\varphi + b_2 \sin 2\varphi.$$

If $a_2 = b_2 = 0$ we get a circle. Otherwise we put $u = \varphi - \varphi_0 + \pi/4$, where

$$\tan 2\varphi_0 = \frac{b_2}{a_2}$$

and in terms of u the support function of K is

$$\tilde{p}(u) = a_0 + a \sin 2u$$

with $a = \sqrt{a_2^2 + b_2^2} > 0$. Notice that, since $\tilde{p} + \tilde{p}'' > 0$, one has $a < a_0/3 = L/6\pi$.

From section 2.2 it follows that Γ is parallel to an astroid at distance $a_0 = L/2\pi$. □

Although inequality (8) can not be improved for general convex domains, it is possible to obtain a stronger inequality for convex sets of *constant width*, that is those convex sets whose orthogonal projection in any direction have the same length. In this case we have the following result.

Theorem 4.4. *Let K be a compact strictly convex set of constant width and area F bounded by a curve $\Gamma = \partial K$ of class $\mathcal{C}^2$ and length L. Let F_e be the area with multiplicities of the evolute of Γ. Then*

$$L^2 - 4\pi F \leq \frac{4}{9}\pi |F_e|. \tag{9}$$

Equality holds if and only if Γ is a circle or a curve parallel to a Steiner curve at distance $L/2\pi$.

Proof. Since K is of constant width, the Fourier series of its support function has only odd terms. Following the pattern of the proof of Proposition 4.1 one gets

$$4\pi|F_e| - 9\Delta = 2\pi(W_{q'} - 9W_q)$$

with $q(\varphi) = p(\varphi) - L/2\pi$.

Following the proof of Lemma 4.2 for the special case that the Fourier series of $q(\varphi)$ has only odd terms, one gets

$$W_{q'} \geq 9W_q + \frac{9}{2\pi}\left(\int_0^{2\pi} q(\varphi)\,d\varphi\right)^2 = 9W_q.$$

As here $\int_0^{2\pi} q(\varphi)\,d\varphi = 0$ the inequality (9) follows.

Equality in (9) holds if and only if $a_n = b_n = 0$, for $n \geq 5$. This implies

$$p(\varphi) = a_0 + a_1\cos\varphi + b_1\sin\varphi + a_3\cos 3\varphi + b_3\sin 3\varphi.$$

Taking the Steiner point (a_1, b_1) as a new origin of coordinates the new support function of K becomes

$$\tilde{p}(\varphi) = a_0 + a_3\cos 3\varphi + b_3\sin 3\varphi.$$

We make the change of variable given by $u = \varphi - \varphi_0/3$, where $\tan\varphi_0 = b_3/a_3$. Then

$$p(u) = a_0 + a\cos(3u),$$

with $a = a_3/\cos\psi_0$. Notice that $a < a_0/8 = L/16\pi$ because p represents the support function of a strictly convex set K.

From section 2.2 it follows that Γ is a Steiner curve. □

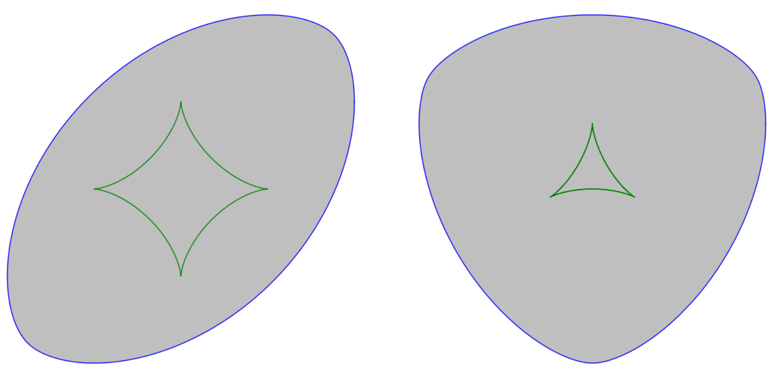

FIGURE 2. Convex curves parallel to an astroid and to a Steiner curve at distance $L/2\pi$.

Corollary 4.5. *Under the same hypothesis as in Theorem 4.4 one has*

$$(A - F) \leq \frac{1}{8}|F_e|,$$

where A is the area enclosed by the associated pedal curve to K with respect to its Steiner point.

Equality holds if and only if Γ is a circle or a curve parallel to a Steiner curve at distance $L/2\pi$.

Proof. By Proposition 3.2 of [2] one has

$$\frac{32}{9}\pi(A-F) \leq \Delta.$$

This inequality combined with (9) gives the result. The characterization of equality follows from Corollary 4.4 of [2] and Theorem 4.4. □

Remark. In [8] it is considered the area A_w of the Wigner caustic of Γ. If this area is counted with multiplicities it is proved that

$$L^2 - 4\pi F \geq 4\pi|A_w|$$

with equality if and only if K is of constant width.

In the case of constant width the Wigner caustic and the interior parallel curve at distance $L/2\pi$ coincide. So using Theorem 4.4 and (5) one obtains, in the case of constant width, the estimate

$$|A_w| \leq \frac{1}{9}|F_e|$$

with equality if and only if Γ is a circle or a curve parallel to a Steiner curve at distance $L/2\pi$.

5. A lower bound for Hurwitz's deficit in terms of the visual angle

We proceed now to find a lower bound for the Hurwitz deficit $\pi|F_e| - \Delta$ so improving Theorem 4.3. If

$$p(\varphi) = a_0 + \sum_{n\geq 1} a_n \cos n\varphi + b_n \sin n\varphi$$

is the Fourier series of the support function of a compact convex set K, it is known that the quantities $c_n^2 = a_n^2 + b_n^2$, for $n \geq 2$, are invariants under the group of plane motions. This invariance will be clear through formula (10) due to Hurwitz.

Consider ω the *visual angle* of Γ from P, that is the angle between the tangents from P to Γ, and let dP be the area measure. Writing

$$I_n = \int_{P\notin K} \left(-2\sin(\omega) + \frac{n+1}{n-1}\sin(n-1)\omega - \frac{n-1}{n+1}\sin(n+1)\omega\right) dP,$$

it is proved in [6][1] that

$$I_n = L^2 + (-1)^n \pi^2 (n^2-1) c_n^2, \tag{10}$$

L being the length of the boundary of K.

For instance, if $n = 2$ one gets

$$\frac{4}{3}\int_{P\notin K} \sin^3\omega\, dP = L^2 + 3\pi^2 c_2^2. \tag{11}$$

Moreover, this visual angle also verifies the Crofton formula (see [6])

$$\frac{L^2}{2} - \pi\, F = \int_{P\notin K} (\omega - \sin\omega)\, dP. \tag{12}$$

[1] There is a misprint with the sign in Hurwitz's paper. Moreover the c_n coefficients appearing in (10) are different from those in Hurwitz's paper because the latter correspond to the Fourier series of the curvature radius function.

We can prove now the following result.

Theorem 5.1. *Let K be a compact strictly convex set of area F bounded by a curve $\Gamma = \partial K$ of class $\mathcal{C}^2$ and length L. Let F_e be the area with multiplicities enclosed by the evolute of Γ and let Δ be the isoperimetric deficit. Then*

$$\pi|F_e| - \Delta \geq \frac{5}{4}L^2 + 5\int_{P\notin K}(\omega - \sin\omega - \frac{2}{3}\sin^3\omega)\,dP. \tag{13}$$

The right hand side of this inequality is a strictly positive quantity except when $\pi|F_e| - \Delta = 0$ in which case it also vanishes.

Proof. As we have seen in the proof of Proposition 4.1 we have

$$\pi|F_e| - \Delta = \frac{\pi}{2}(W_{q'} - 4W_q) = \frac{\pi}{2}\left(4\int_0^{2\pi} q^2\,d\varphi - 5\int_0^{2\pi} q'^2\,d\varphi + \int_0^{2\pi} q''^2\,d\varphi\right),$$

where $q(\varphi) = p(\varphi) - L/2\pi$, and $p(\varphi)$ is the support function of K with respect to the Steiner point.

In terms of the Fourier coefficients of p

$$\pi|F_e| - \Delta = \frac{\pi^2}{2}\sum_{n\geq 3}(n^4 - 5n^2 + 4)c_n^2. \tag{14}$$

Observe now that, for $n \geq 3$, we have $n^4 - 5n^2 + 4 \geq 5(n^2 - 1)$, with equality only for $n = 3$. Therefore

$$\begin{aligned}\pi|F_e| - \Delta &\geq \frac{5\pi^2}{2}\sum_{n\geq 3}(n^2 - 1)c_n^2 = \frac{5\pi^2}{2}\left(\sum_{n\geq 2}(n^2 - 1)c_n^2 - 3c_2^2\right)\\ &= \frac{5}{4}L^2 - 5\pi F - \frac{15\pi^2}{2}c_2^2 = \frac{15}{4}L^2 - 5\pi F - \frac{10}{3}\int_{P\notin K}\sin^3\omega\,dP.\end{aligned} \tag{15}$$

Using Crofton's formula (12), the last expression can be written as

$$\frac{5}{4}L^2 + 5\int_{P\notin K}(\omega - \sin\omega - \frac{2}{3}\sin^3\omega)\,dP$$

and the inequality in the theorem is proved. Moreover, the first term in the right hand side of (15) vanishes if and only if $c_n = 0$ for $n \geq 3$ as well as $\pi|F_e| - \Delta$. □

We study now when equality holds in Theorem 5.1.

Proposition 5.2. *Equality in* (13) *holds if and only if for the compact strictly convex set K one of the following assertions holds:*

a) *K is a disk or it is bounded by a curve parallel to an astroid.*
b) *K is bounded by a curve parallel to a Steiner curve.*
c) *K is the Minkowski sum of compact sets of the above types.*

Proof. We have seen that equality in (13) holds if and only if the support function of the domain with respect to the Steiner point is of the form

$$p(\varphi) = a_0 + a_2\cos 2\varphi + b_2\sin 2\varphi + a_3\cos 3\varphi + b_3\sin 3\varphi.$$

If we put $p_1(\varphi) = a_0 + a_2 \cos 2\varphi + b_2 \sin 2\varphi$ and $p_2(\varphi) = a_3 \cos 3\varphi + b_3 \sin 3\varphi$, we have $p(\varphi) = p_1(\varphi) + p_2(\varphi)$ and so K is the Minkowski sum of the domains D_1 and D_2 with generalized support functions $p_1(\varphi)$ and $p_2(\varphi)$ respectively.

We know, by the proof of Theorem 4.3, that D_1 is the interior of a curve parallel to an astroid or a disc and, by the proof of Theorem 4.4, that D_2 is the interior of a Steiner curve. □

For the case of curves of constant width we can improve inequality (9) in terms of the visual angle.

Theorem 5.3. *Let K be a compact strictly convex set of constant width and area F bounded by a curve $\Gamma = \partial K$ of class $\mathcal{C}^2$ and length L. Let F_e be the area with multiplicities of the evolute of Γ and Δ be the isoperimetric deficit of Γ. Then*

$$\frac{4}{9}\pi|F_e| - \Delta \geq \frac{64}{9} \int_{P\notin K} \left(\omega - 2\sin\omega + \sin 2\omega - \frac{1}{4}\sin 4\omega - \sin^3\omega\right) dP.$$

The right hand side of this inequality is a strictly positive quantity except when $\frac{4}{9}\pi|F_e| - \Delta = 0$ in which case it also vanishes.

Equality holds if and only if K is a disk or it is bounded by a curve parallel to a Steiner curve or it is bounded by a curve parallel to an hypocycloid of five cusps or the Minkowski sum of compact sets of the previous types.

Proof. If p is the support function of K we have (see the proof of Theorem 4.4)

$$\frac{4}{9}\pi|F_e| - \Delta = \frac{2\pi}{9}(W_{q'} - 9W_q) = \frac{2\pi^2}{9}\sum_{n\geq 5}(n^2-1)(n^2-9)c_n^2$$

with $q(\varphi) = p(\varphi) - L/2\pi$ and $c_n^2 = a_n^2 + b_n^2$, being a_n, b_n the Fourier coefficients of the support function of K. Recall that since K has constant width we have $c_n = 0$, for n even, $n \neq 0$.

Since $(n^2-1)(n^2-9) \geq 16(n^2-1)$ for $n \geq 5$ it follows that

$$\frac{4}{9}\pi|F_e| - \Delta \geq \frac{32}{9}\pi^2 \cdot \sum_{n\geq 5}(n^2-1)c_n^2 = \frac{32}{9}\pi^2\left(\sum_{n\geq 2}(n^2-1)c_n^2 - 8c_3^2\right).$$

But

$$\pi\sum_{n\geq 2}(n^2-1)c_n^2 = \int_0^{2\pi}(p'^2 - p^2)\,d\varphi + 2\pi a_0^2 = -2F + \frac{L^2}{2\pi}.$$

So we get

$$\frac{4}{9}\pi|F_e| - \Delta \geq \frac{32}{9}\left(-2\pi F + \frac{L^2}{2} - 8\pi^2 c_3^2\right). \tag{16}$$

Now using Crofton's formula (12), the formula (10) for $n = 3$ and the fact that $L^2 = I_2$ it follows that the second member of (16) can be written as

$$-\frac{16}{3}L^2 + \frac{64}{9}\int_{P\notin K}(\omega - \sin\omega)\,dP + \frac{32}{9}I_3$$
$$= \frac{64}{9}\int_{P\notin K}(\omega - 2\sin\omega + \sin 2\omega - \frac{1}{2}\sin 4\omega - \sin^3\omega)\,dP.$$

The right hand side of (16) vanishes if and only if $c_n = 0$ for $n \geq 5$, as well as $\frac{4}{9}\pi|F_e| - \Delta$.

Moreover equality in (16) holds if and only if $c_n = 0$, $n \geq 7$. If we put $p_1(\varphi) = a_0 + a_3 \cos 3\varphi + b_3 \sin 3\varphi$ and $p_2(\varphi) = a_5 \cos 5\varphi + b_5 \sin 5\varphi$, we have $p(\varphi) = p_1(\varphi) + p_2(\varphi)$ and so K is the Minkowski sum of the domains D_1 and D_2 with generalized support functions $p_1(\varphi)$ and $p_2(\varphi)$ respectively. As seen before D_1 is parallel to a Steiner curve. For D_2 we can write

$$p_2(\varphi) = \sqrt{a_5^2 + b_5^2}\, \sin(5(\varphi_0/5 + \varphi)),$$

where $\tan \varphi_0 = b_5/a_5$. Then D_2 corresponds to the curve with support function

$$q(u) = \sqrt{a_5^2 + b_5^2} \sin(5u),$$

which by section 2.2 is the interior of an hypocycloid of five cusps. □

Relationship with the pedal curve. If F is the area of K and A is the area enclosed by the pedal curve associated to K with respect to its Steiner point we obviously have $A \geq F$, with equality if and only if K is a disk, and

$$A - F = \frac{1}{2} \int_0^{2\pi} p'^2 \, d\varphi.$$

Theorem 5.4. *Let K be a compact strictly convex set of area F bounded by a curve $\Gamma = \partial K$ of class $\mathcal{C}^2$ and length L. Let F_e be the area with multiplicities enclosed by the evolute of Γ. Let A be the area enclosed by the pedal curve associated to K with respect to its Steiner point. Then*

$$\pi |F_e| - \Delta \geq \frac{40}{9} \left(\pi(A - F) + \frac{2}{3} L^2 - \frac{8}{9} \int_{P \notin K} \sin^3 \omega \, dP \right).$$

The right hand side of this inequality is a strictly positive quantity except when $\pi|F_e| - \Delta = 0$ in which case it also vanishes.

Equality holds if and only if K is a disk or it is bounded by a curve parallel to an astroid or it is bounded by a curve parallel to a Steiner curve or the Minkowski sum of compact sets of the previous types.

Proof. From (14) and (11) it follows

$$\begin{aligned}
\pi|F_e| - \Delta &= \frac{\pi^2}{2} \sum_{n \geq 3} (n^2 - 1)(n^2 - 4) c_n^2 \geq \frac{20}{9} \pi^2 \sum_{n \geq 3} n^2 c_n^2 \\
&= \frac{20}{9} \pi^2 \left(\sum_{n \geq 2} n^2 c_n^2 - 4 c_2^2 \right) = \frac{20}{9} \pi \left(\int p'^2 \, d\varphi - 4\pi c_2^2 \right) \\
&= \frac{20}{9} \pi \left(\int p'^2 \, d\varphi - 4\pi c_2^2 \right) = \frac{40}{9} \left[\pi(A - F) + \frac{2}{3} L^2 - \frac{8}{9} \int_{P \notin K} \sin^3 \omega \, dP \right].
\end{aligned}$$

Moreover the right hand side vanishes if and only if $c_n = 0$ for $n \geq 3$ as well as $\pi|F_e| - \Delta$.

Equality holds if and only if $c_n = 0$, $n \geq 4$ and the result follows as in Proposition 5.2. □

Corollary 5.5. *Under the hypothesis of Theorem 5.4 and assuming moreover that K has constant width one has*

$$\pi|F_e| - \Delta \geq \frac{40}{9}\pi(A-F).$$

Equality holds if and only if K is a disk or it is bounded by a curve parallel to a Steiner curve.

Proof. Just note that in the constant width case, (11) gives

$$\frac{4}{3}\int_{P\notin K} \sin^3\omega\, dP = L^2$$

and apply Theorem 5.4. □

Remark. If p is the support function of K, the support function of the Wigner caustic (see [8]) is given by $q(\varphi) = \frac{1}{2}(p(\varphi) - p(\varphi+\pi))$. A straightforward calculation involving the Fourier series of $p(\varphi)$ and $q(\varphi)$ shows that

$$A - F \geq |A_w|,$$

where A_w is the area of the Wigner caustic counted with multiplicities, with equality in the case of constant width. So Theorem 5.4 and Corollary 5.5 give lower bounds for the Hurwitz deficit $\pi|F_e| - \Delta$ in terms of $|A_w|$.

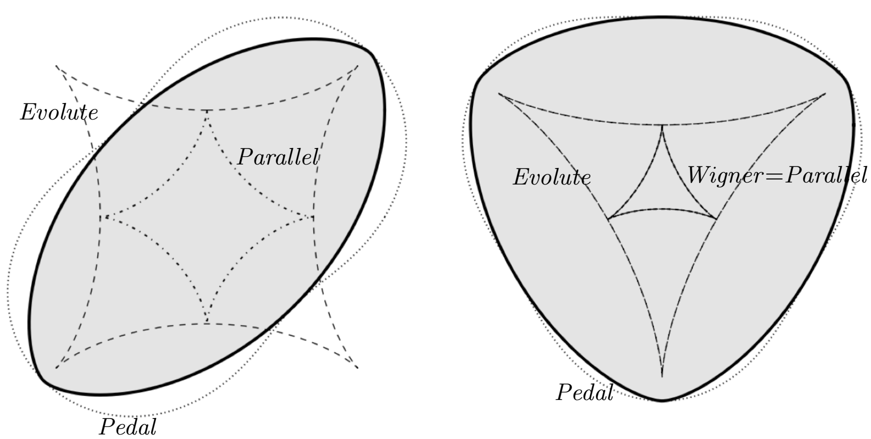

FIGURE 3. Different curves related to convex sets with central symmetry on the left and constant width on the right.

Relationship with the $\mathcal{L}^2$ metric. Consider now the quantity $\delta_2(K)$ equal to the the distance in $\mathcal{L}^2(S^1)$, where S^1 is the unit circle, between the support function of K and the support function of the Steiner disk of K. We have that

$$\delta_2(K)^2 = \pi \sum_{n\geq 2} c_n^2,$$

where $c_n^2 = a_n^2 + b_n^2$ being a_n, b_n the Fourier coefficients of the support function of K with respect to its Steiner point ([4]). Clearly the quantity $\delta_2(K)$ vanishes only when K is a disk.

Theorem 5.6. *Let K be a compact strictly convex set of area F bounded by a curve $\Gamma = \partial K$ of class $\mathcal{C}^2$ and length L. Let F_e be the area with multiplicities of the evolute of Γ. Then*

$$\pi|F_e| - \Delta \geq 20 \left(\pi\delta_2(K)^2 + \frac{L^2}{3} - \frac{4}{9} \int_{P \notin K} \sin^3 \omega \, dP \right).$$

The right hand side of this inequality is a strictly positive quantity except when $\pi|F_e| - \Delta = 0$ in which case it also vanishes. Equality holds for the same domains as in Theorem 5.4.

Proof. According to (14) and (11) we have

$$\begin{aligned} \pi|F_e| - \Delta &= \frac{\pi^2}{2} \sum_{n \geq 3} (n^2-1)(n^2-4)c_n^2 \geq 20\pi^2 \sum_{n\geq 3} c_n^2 \\ &= 20\pi \left(\pi \sum_{n \geq 2} c_n^2 - \pi c_2^2 \right) \\ &= 20\pi\delta_2(K)^2 - 20 \left(\frac{4}{9} \int_{P \notin K} \sin^3 \omega \, dP - \frac{L^2}{3} \right) \end{aligned}$$

as required. Equality holds if and only if $c_n = 0$, $n \geq 4$. □

Corollary 5.7. *Under the hypothesis of Theorem 5.6 and assuming moreover that K has constant width one has*

$$\pi|F_e| - \Delta \geq 20\,\pi\,\delta_2(K)^2.$$

As a consequence

$$|F_e| \geq 36\,\delta_2(K)^2.$$

Equality holds in both inequalities if and only if K is a disk or it is bounded by a curve parallel to a Steiner curve.

Proof. When K has constant width by (11) one has $L^2 = \frac{4}{3} \int_{P \notin K} \sin^3 \omega \, dP$ and the first inequality follows from Theorem 5.6.

Then we have

$$\begin{aligned} \pi|F_e| &\geq 20\,\pi\,\delta_2(K)^2 + \Delta \\ &= 20\,\pi\,\delta_2(K)^2 + 2\pi^2 \sum_{n \geq 3} (n^2-1)c_n^2 \\ &\geq 20\,\pi\,\delta_2(K)^2 + 16\pi^2 \sum_{n \geq 3} c_n^2 = 36\pi\delta_2(K)^2, \end{aligned}$$

which gives the second inequality.

Equalities hold if and only $c_n = 0$ for $n \geq 5$. □

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