ON THE DEFINITION OF STRANGE NONCHAOTIC ATTRACTOR

LLUÍS ALSEDÀ AND SARA COSTA

Dedicated to Michał Misiurewicz on the occasion of his 60th Birthday

Abstract. The aim of this paper is twofold. From one hand we want to discuss some methodological issues related with the notion of Strange Nonchaotic Attractor. From another hand we want to formulate a precise definition of this kind of attractors, which is "observable" in the physical sense and that, in the two dimensional setting, it includes the well known models proposed by Grebogi et al. [7] by Keller [8], and a wide range of other examples proposed in the literature. Furthermore, we analytically prove that a whole family of two-dimensional quasiperiodic skew-products defined on $\mathbb{S}^1 \times \mathbb{R}$ has a strange nonchaotic attractor. As a corollary we show analytically that the system proposed in [7] has an a strange nonchaotic attractor.

1. Introduction

The notion of *Strange Nonchaotic Attractor* (briefly, and from now on, SNA) was introduced[1] by Grebogi et al. in [7] when studying attractors of quasiperiodically forced skew products of the form

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1 \\ x_{n+1} = \psi(\theta_n, x_n) \end{cases} \tag{1}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$. One of the two examples they considered consisted in taking $\psi(\theta, x) = 2\sigma \cos(2\pi\theta) \tanh(x)$ in the above system (see Figure 1 for a picture of the attractor of an instance of this system):

$$\begin{cases} \theta_{n+1} = \theta_n + \omega \pmod 1 \\ x_{n+1} = 2\sigma \cos(2\pi\theta_n) \tanh(x_n). \end{cases} \tag{2}$$

The authors of [7] defined an SNA as *an attractor whose Lyapunov exponents are non positive almost everywhere and it is not finite neither piecewise differentiable.* An attractor in this definition is a compact set with a neighbourhood such that, for almost every initial condition in this neighbourhood, the limit set of the orbit as time tends to $+\infty$ is the attractor. An attractor is piecewise differentiable if

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[1]This kind of attractors were already studied in the literature much before than the notion of SNA was coined — see [11, 12, 17] where constructions of flows containing Strange Nonchaotic Attractors and the one dimensional systems shown in the last part of this paper can be found.

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Departament de Matemàtiques.
http://www.uab.cat/matematiques
UAB Universitat Autònoma de Barcelona

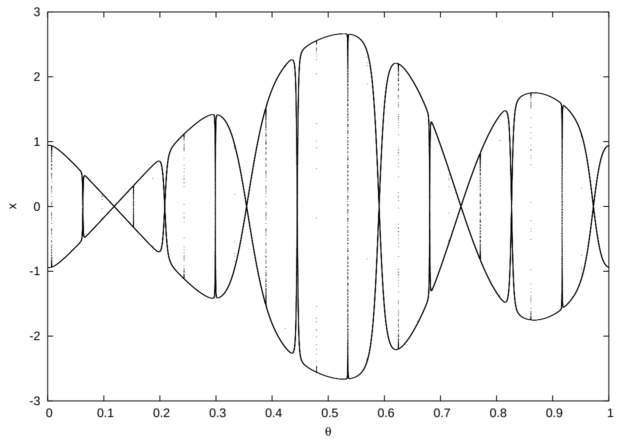

FIGURE 1. The attractor of (2) with $\sigma = 1.5$.

it is either a piecewise differentiable curve or surface, or a volume bounded by a piecewise differentiable closed surface. Later on, was proved by G. Keller [8] for systems similar to (2) that the attractor cuts the repeller $x = 0$ at an invariant set which is dense (in $x = 0$) and it is different from zero in a set whose projection onto $\mathbb{S}^1$ has full measure. This gives the strangeness such attractors.

As it has been already explained in [7], in this theory, the term *chaotic* refers to the dynamics on the attractor, while *strange* refers to the geometrical structure of the attractor.

There is a lot of controversy about which should be the precise definition of SNAand on whether the elements involved must be "observable" in a physical and computational sense. Indeed, this notion is not unified and often it is even not precisely formulated. Also, unfortunately, one can find much more empirical and rude numerical studies about SNA's than rigorous proofs of their existence[2]. There are also some theoretical issues about a number of the numerical studies.

The aim of this paper is to give a rigorous definition of the notion of SNA *in the topological setting*, while trying to clarify and fix the above mentioned theoretical issues. This will be done in the third section.

The second section is devoted to introduce the notation used along this paper and to show two paradigmatic examples in this field. In the third section, as we have said, we propose a rigorous definition and discuss some methodological aspects relative to the nonchaoticity part of the definition. This section is the core this paper. Finally, in the last section we check that the two examples shown in the second section are SNA's with the definition we propose. Also, we prove that a well known kind of attractor in one-dimensional dynamics, *the absorbing Cantor set*, is an SNA.

[2]Two remarkable examples of very nice and rigorous studies of the existence of such attractors can be found in [2, 8].

2. Notation and Motivating examples

We are interested in two-dimensional quasiperiodically forced skew products. These are systems defined on the cartesian product of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and a metric space X, of the form:

$$\begin{cases} \theta_{n+1} = R_\omega(\theta_n) = \theta_n + \omega \pmod 1 \\ x_{n+1} = \psi(\theta_n, x_n) \end{cases} \tag{3}$$

where $(\theta_n, x_n) \in \mathbb{S}^1 \times X$ and and $\omega \in \mathbb{R} \setminus \mathbb{Q}$. The first component, which is independent of the second, is the *basis* of the skew product. These systems map each fibre $\{\theta\} \times X$ into another one $\{R_\omega(\theta)\} \times X$.

Recall that R_ω is minimal and has the Lebesgue measure as a unique invariant measure (thus being uniquely ergodic). Moreover, every invariant attracting set is the graph of a correspondence φ from Λ to X, where Λ is an invariant set of R_ω of full Lebesgue measure. Then, φ must satisfy the *invariance equation*:

$$\varphi(R_\omega(\theta)) = \psi(\theta, \varphi(\theta))$$

for every $\theta \in \Lambda$ or , equivalently, it must be a fixed point of the *transfer operator* defined by:

$$\mathcal{T}\varphi(\theta) := \psi(\theta, \varphi(R_\omega^{-1}(\theta))).$$

Note that since φ is a correspondence, the equality in the above formula is between sets.

Example 1 (Grebogi et al. [7])**.** In this example we discuss the study of System (2) done in [7], together with the notion of SNA they propose. We recall that the system considered is

$$\begin{cases} \theta_{n+1} = R_\omega(\theta_n) \\ x_{n+1} = 2\sigma \tanh(x_n) \cos(2\pi\theta_n) \end{cases} \tag{2}$$

$(\theta_n, x_n) \in \mathbb{S}^1 \times \mathbb{R}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$. For every (θ_0, x_0) and $n \in \mathbb{N}$ it follows that $(\theta_n, x_n) \in \mathbb{S}^1 \times [-2\sigma, 2\sigma]$. So, the system has an attractor in this region. As it has been said before, attractor in [7] means a compact set with a neighbourhood such that, for almost every initial condition in this neighbourhood, the limit set of the orbit as time tends to $+\infty$ is the attractor. Observe that this definition is very similar to the one proposed by Milnor ([13]).

The attractor is the closure of the graph of a map $\varphi\colon \mathbb{S}^1 \longrightarrow [-2\sigma, 2\sigma] \subset \mathbb{R}$ and, as we already know, it must satisfy the invariance equation

$$\varphi(R_\omega(\theta)) = 2\sigma \cos(\theta) \tanh(\varphi(\theta)). \tag{4}$$

Since $\tanh(0) = 0$, the circle $x \equiv 0$ is invariant. Thus, by Birkhoff Ergodic Theorem, it can be shown that the vertical Lyapunov exponent at $x \equiv 0$ is

$$\int_{\mathbb{S}^1} \log \left| \left. \frac{\partial(2\sigma \tanh(x) \cos(2\pi\theta))}{\partial x} \right|_{x=0} \right| d\theta = \log|\sigma|$$

for almost every $\theta \in \mathbb{S}^1$ with respect to the Lebesgue measure. So, if $|\sigma| > 1$, the vertical Lyapunov exponent at $x \equiv 0$ it is positive and the circle $x \equiv 0$ is a repeller. Consequently, the map φ is not identically zero. Moreover, since $\cos(2\pi\theta)$ vanishes for $\theta \in \{1/4, 3/4\}$ it follows that the set

$$D := \left\{ \left(\frac{i}{4} + n\omega \pmod 1, 0 \right)^t : n \in \mathbb{Z},\ i \in \{1, 3\} \right\}$$

is invariant and dense in $x \equiv 0$, and belongs at the same time to the repeller and to the attractor. The authors use numerical techniques to find that the graph of φ is positive almost everywhere and, hence, discontinuous almost everywhere. This gives the strangeness of the attractor.

The nonchaoticity is studied in terms of Lyapunov exponents. An attractor is defined to be *nonchaotic* whenever the Lyapunov exponents are non positive in almost every point in the basin of attraction of the attractor (that is, in the closure of the set of initial conditions which approach the attractor as time tends to $+\infty$). For this kind of systems the vertical Lyapunov exponent at $(\theta, x)^t$ is given by

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| \begin{pmatrix} \frac{\partial \theta_n}{\partial \theta} & \frac{\partial \theta_n}{\partial x} \\ \frac{\partial x_n}{\partial \theta} & \frac{\partial x_n}{\partial x} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \limsup_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right|. \tag{5}$$

By Oseledec's Theorem, $\lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right|$ exists for almost every point in the support of any invariant measure and, as we will see, the other Lyapunov exponent is zero.

Using the geometrical properties of the hyperbolic tangent function it can be easily proved that $|\tanh'(x)| < \frac{\tanh(x)}{x}$; which is used by the authors to check that, whenever the above limit exists, the vertical Lyapunov exponent is negative. This provides a rigorous proof of the nonpositivity of the Lyapunov exponent for almost every point in the support of an invariant measure but in general it does not control the value of the Lyapunov exponent for almost every point in the basin of attraction of the attractor.

Example 2 (G. Keller [8])**.** As far as we know [8] together with [2] contain the first rigorous proofs of the existence of such SNA.

Keller proved the existence of an SNA (with the same definition used in [7]) for two-dimensional quasiperiodically forced skew-products of the form

$$\begin{cases} \theta_{n+1} = R_\omega(\theta_n) \\ x_{n+1} = f(x_n) g(\theta_n) \end{cases} \tag{6}$$

where $(\theta_n, x_n) \in \mathbb{S}^1 \times [0, \infty)$, the map $g \colon \mathbb{S}^1 \longrightarrow [0, \infty)$ is continuous and bounded, and the map $f \colon [0, \infty) \longrightarrow [0, \infty)$ is $\mathcal{C}^1$, bounded, increasing, strictly concave and such that $f(0) = 0$. These kind of models are a "one sided" generalisation of (2). To see it and to fix ideas, notice that $f(x)$ can be taken equals to $\tanh(x)$ and $g(x)$ to $2|\sigma \cos(2\pi\theta)|$.

Set

$$\mathcal{I}(g) := \int_{\mathbb{S}^1} \log g(\theta) d\theta \geq -\infty$$

which is well defined because g is bounded. As in System (2), the circle $x \equiv 0$ is invariant and the vertical Lyapunov exponent on this invariant set it is the logarithm of

$$\sigma := \begin{cases} f'(0) \exp(\mathcal{I}(g)) & \text{if } \mathcal{I}(g) > -\infty, \\ 0 & \text{otherwise.} \end{cases}$$

for almost every $\theta \in \mathbb{S}^1$. So, if $\sigma > 1$, the circle $x \equiv 0$ is a repeller.

Then the attractor and its dynamics is described by the following

Theorem 2.1 (Keller). *There exists an upper semicontinuous map $\varphi\colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ whose graph is invariant under System* (6)*. Moreover,*

(a) The Lebesgue measure on the circle, lifted to the graph of φ is a Sinai-Ruelle-Bowen measure (that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \vartheta(F^k(\theta, x)) = \int_{\mathbb{S}^1} \vartheta(\theta, \varphi(\theta)) d\theta$$

for every $\vartheta \in \mathcal{C}^0(\mathbb{S}^1 \times \mathbb{R}^+, \mathbb{R})$ and Lebesgue almost every $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}^+$),

(b) if $\sigma \leq 1$ then $\varphi \equiv 0$,

(c) if $\sigma > 1$ then $\varphi(\theta) > 0$ for almost every θ,

(d) if $\sigma > 1$ and $g(\theta_0) = 0$ for some θ_0 then the set $\{\theta\colon \varphi(\theta) > 0\}$ is meager and φ is almost everywhere discontinuous,

(e) if $\sigma > 1$ and $g > 0$ then φ is positive and continuous; if g is $\mathcal{C}^1$ then so is φ,

(f) if $\sigma \neq 1$ then $|x_n - \varphi(\theta_n)| \to 0$ exponentially fast for almost every θ and every $x > 0$.

When $\sigma > 1$ it follows that the vertical Lyapunov exponent exists for almost every θ and for every $x > 0$ and it is equal to

$$\mathcal{I}(g) + \int_{\mathbb{S}^1} \log f'(\varphi(\theta)) d\theta < 0.$$

By using Oseledec's Theorem it can be shown that the second Lyapunov exponent is zero for every $(\theta, x)^t$ such that the vertical Lyapunov exponent exists. Hence, Case (d) of the above theorem shows the existence of SNA for such systems.

Let us point out that the graph of φ is not closed because it is discontinuous almost everywhere. Therefore the attractor must be the closure of this graph.

3. A definition of Strange Nonchaotic Attractor

As we have said in the introduction, the notion of SNA is not unified in the literature. Moreover, depending on the chosen definition, certain objects will be or will not be SNA's and the difficulty of its study can vary. The aim of this section is to give a rigorous definition of this kind of attractors which supports theoretically the SNA's found in the above models. We also look for a definition that agrees with the numerical approach in the quest for SNA's. Indeed, we want a definition whose elements are "observable" in the physical sense (that is, that occurs for a set of positive Lebesgue measure).

We divide this section into three subsections. In each of them we will define one of the terms involved in the definition of SNA: *attractor*, *nonchaotic* and *strange*.

3.1. Attractor.

We adopt the definition of attractor proposed by Milnor in [13]:

Definition 3.1. Let (X, f) be a (semi-)dynamical system where X is a smooth compact manifold endowed with a measure μ equivalent to the Lebesgue one when it is restricted to any coordinate neighbourhood and f is a continuous map. A closed subset $\mathcal{A} \subset X$ will be called an *attractor* if it satisfies the following two conditions:

(I) The set $\rho(\mathcal{A}) := \{z \in X : \omega(z) \subset \mathcal{A}\}$ has strictly positive measure;

(II) there is no strictly smaller closed set $\mathcal{A}' \subset \mathcal{A}$ so that $\rho(\mathcal{A}')$ coincides with $\rho(\mathcal{A})$ up to a set of measure zero.

The set $\rho(\mathcal{A})$ is called *realm of attraction* of $\mathcal{A}$, and it can be defined for every subset of X. When it is open, it is called *basin of attraction* of $\mathcal{A}$. A set satisfying this definition is called a *metric attractor*.

Any attractor $\mathcal{A}$ must be a non-empty closed f-invariant (i.e. $f(\mathcal{A}) = \mathcal{A}$.) subset of the the nonwandering set of f.

Notice that realm of attraction of a repeller has measure zero. Thus, an attractor $\mathcal{A}$ may contain a repeller. Also, it may contain a smaller attractor $\mathcal{A}' \subsetneq \mathcal{A}$ as long as $\rho(\mathcal{A})$ and $\rho(\mathcal{A}')$ differ in a set of positive measure. The attractors for which this is not possible are the *minimal attractors*. They are defined by replacing Condition (II) of Definition 3.1 by

(II') There is no strictly smaller closed set $\mathcal{A}' \subset \mathcal{A}$ such that $\mu(\rho(\mathcal{A}'))$ is positive.

If the space X is not compact but there exists a compact set $\widetilde{X}$ with positive measure such that $f(\widetilde{X}) \subset \widetilde{X}$, then there exists at least an attractor in $\widetilde{X}$.

There is an important choice for the definition of attractor. Either we ask the attractor to be closed and then, in the considered examples, the repeller $x \equiv 0$ is contained in the attractor or we release this condition. In such a case, the ω-limit sets of points in the attractor need not be contained in it, and this gives invariance problems. We have chosen to define an attractor as a closed set.

3.2. **Strangeness.** Concerning SNA's we have found three different definitions of strangeness. In the literature, an attractor is considered strange in the following cases:

(1) It is not a finite set of points neither piecewise differentiable (see, for instance, [7]).
(2) It has fractal geometry (that is, its Hausdorff dimension is greater than its topological one — see, for instance, [15]).
(3) Its Hausdorff dimension is greater than one (see, for instance, [14]).

As far as we know, the three definitions are used in articles where two-dimensional systems are studied, while for higher-dimensional systems only the first one is used.

It is an exercise to show that the Hausdorff dimension of the graph of a one-dimensional piecewise differentiable map is one. Thus, one can show that, in the two-dimensional case, the third definition implies the second one and, in turn, this implies the first one. So we choose as a definition of strangeness the more general one:

Definition 3.2. An attractor is *strange* if it is neither a finite set of points nor a piecewise differentiable manifold. A manifold X is *piecewise differentiable* if there exists a finite set of disjoint differentiable submanifolds $A_1, \dots, A_k$ such that

$$X \subset \mathrm{Cl}(\cup_{i=1}^{k} A_i).$$

If $\partial X \neq \emptyset$, then the boundary is also required to be piecewise differentiable.

3.3. **Nonchaoticity.** As it is customary in the world of SNA's we define the notion of nonchaotic in terms of Lyapunov exponents.

Let (X, f) be a (semi-)dynamical system where X is a finite-dimensional, smooth, compact, Riemannian manifold. The *maximal upper Lyapunov exponent* of $x \in X$

is defined as:

$$\begin{aligned}\lambda_{\max}(x) &= \limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)| \\ &= \max\left\{ \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| : v \in T_xX \setminus \{0\} \right\}.\end{aligned} \tag{7}$$

Observe that the maximal upper Lyapunov exponent always exists. Moreover, since X is finite-dimensional, its value does not depend on the choice of the compatible matrix norm. Also, by Oseledec's Theorem, it exists in terms of limits for almost every point in the support of any invariant measure.

Definition 3.3. Let (X, f) be a (semi-)dynamical system where X is a finite-dimensional, smooth, compact, Riemannian manifold. An attractor $\mathcal{A}$ is called *nonchaotic* if the set of points $x \in \rho(\mathcal{A})$ for which $\lambda_{\max}(x) > 0$ has zero Lebesgue measure.

This definition implies that the maximal upper Lyapunov exponent is non-positive for almost every point in $\rho(\mathcal{A})$, which has positive Lebesgue measure. Thus, in contrast with other definitions of nonchaoticity used in the literature (that only take into account sets of Lebesgue measure zero) the above definition depends on quantities ($\limsup$) that are well defined for every point, and properties ($\lambda_{\max}(x) \leq 0$) that must be satisfied in sets of positive Lebesgue measure; thus being *observable*.

Another approach to the definition of nonchaoticity for systems of the form (1) is (see for instance [9], [10]) to consider the dynamical system in dimension one restricted to the attractor. Then the original system is called *nonchaotic* if the unique Lyapunov exponent of this reduced system is non positive. The Lyapunov exponent is defined (with lim) for almost every point in the attractor by the Birkhoff Ergodic Theorem since the dynamics on the attractor is driven by $\theta_{n+1} = R_\omega(\theta_n)$, which is uniquely ergodic with the unique ergodic measure being the Lebesgue measure. The drawback of this definition, in our opinion, is that the nonchaoticity condition is not observable in the above sense.

Rather often in the literature, Lyapunov exponents are computed (frequently in a very rude numerical way) for Lebesgue almost every point and with lim instead of lim sup without checking the existence of such numbers. To see the problems that can arise from this approach in the next section we discuss some theoretical issues about Lyapunov exponents and provide two examples showing that, in general, these problems have to be taken into account.

3.3.1. *Comments to the use of Lyapunov exponents in the definition of nonchaoticity.* Let X be an m-dimensional manifold, let F be a differentiable map from X to itself, and let μ be an F-invariant measure. Then, by Oseledec's Theorem there exist m *Lyapunov exponents*, $-\infty \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m < \infty$ such that for μ almost every $x \in X$ it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log \|DF^n(x)v\| \in \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$$

for every v in the tangent space to X at x, and

$$\lim_{n\to\infty} \frac{1}{n} \log |\det DF^n(x)| = \sum_{i=1}^{m} \lambda_i. \tag{8}$$

The points which satisfy the above conditions are called *regular points.*

Consequently, in the case of two-dimensional skew products of the form (1), for μ almost every (θ, x), the Lyapunov exponents are the *vertical Lyapunov exponent*:

$$\lambda_v = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right|, \tag{9}$$

and

$$\widehat{\lambda} = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial \theta_n}{\partial \theta} \right|. \tag{10}$$

Moreover, from (8) it follows that, for μ almost every (θ, x),

$$\begin{aligned} \lambda_v + \widehat{\lambda} &= \lim_{n\to\infty} \frac{1}{n} \log \left| \det \begin{pmatrix} 1 & 0 \\ \frac{\partial x_n}{\partial \theta} & \frac{\partial x_n}{\partial x} \end{pmatrix} \right| \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right| = \lambda_v. \end{aligned}$$

Thus,

Remark 3.4. the second Lyapunov exponent $\widehat{\lambda}$ is zero, and the maximal Lyapunov exponent $\lambda_{\max} = \max\{\lambda, \widehat{\lambda}(\theta, x)\}$ is nonpositive if and only if $\lambda_v \leq 0$.

It is important to point out that the above arguments do not work for arbitrary points outside the support of an invariant measure. Then the Lyapunov exponents are only defined in terms of lim sup. The following two examples show that in very simple systems the Lyapunov exponents may not be defined in terms of lim and Property (8) may not hold.

Example 3 (On the non-existence of Lyapunov exponents with lim)**.** Let us consider the system

$$\begin{cases} \theta_{n+1} &= R_\omega(\theta_n), \\ x_{n+1} &= \tau(x_n) + \varepsilon \cos(2\pi\theta_n); \end{cases} \tag{11}$$

where $\tau(x)$ is a tent-like map:

$$\tau(x) = \begin{cases} \alpha(x - \frac{1}{2}) + 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ -\beta(x - \frac{1}{2}) + 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

with $0 \leq \alpha, \beta \leq 2$.

The vertical Lyapunov exponent in this case is given by the sequence:

$$\frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right| = \frac{1}{n} \log \left| \frac{\partial \tau^n(x_0)}{\partial x} \right| = \frac{1}{n} \log \left(\alpha^{n_1} \cdot \beta^{n_2} \right) = \frac{n_1 \log(\alpha) + n_2 \log(\beta)}{n_1 + n_2}.$$

where $n = n_1 + n_2$ and n_1 (respectively n_2) is the number of times that the orbit $x_0, x_1, \ldots, x_{n-1}$ visits the the interval $[0, \frac{1}{2})$ (respectively $(\frac{1}{2}, 1]$).

By using elementary symbolic dynamics, we see that there exits an infinite set of points (with Lebesgue measure zero) so that the above sequence has no limit (even it can have the whole interval with endpoints $\log(\alpha)$ and $\log(\beta)$ as the set of accumulation points).

The next example is due to R. de la Llave [5] and shows that Property (8) may not hold. The system we consider is a nonsymmetric horseshoe. To define it with detail we follow Robinson's construction of the Horseshoe map [16].

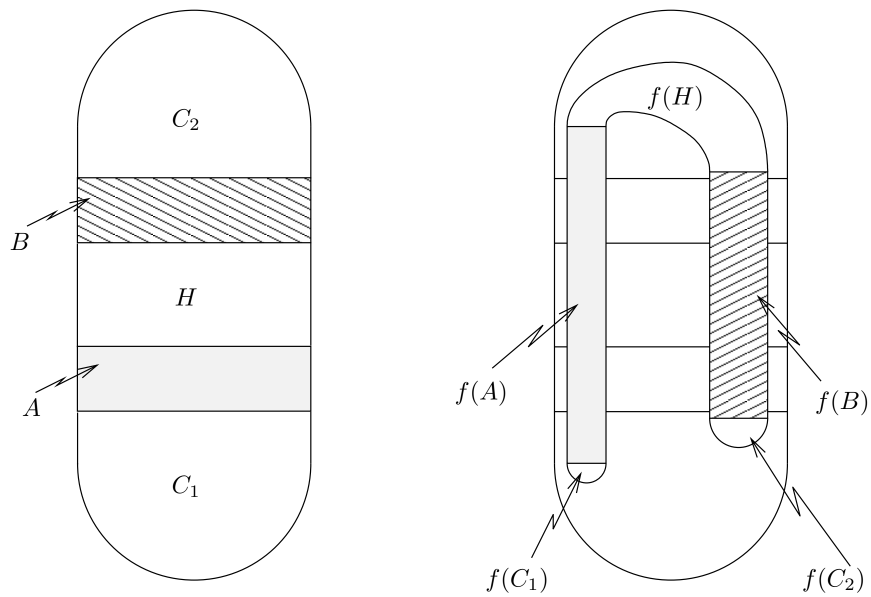

FIGURE 2. Construction of the sets N and $f(N)$

We start by taking $\lambda < 1/2$ and the unit square divided in three horizontal strips (see Figure 2):

$$[0,1] \times [0,1] = A \cup H \cup B,$$

where $A := [0,1] \times [0,\lambda]$ and $B := [0,1] \times [1-\lambda, 1]$. Let $f \in \mathcal{C}^\infty([0,1] \times [0,1])$ be such that

$$Df\big|_A = \begin{pmatrix} d/(\kappa\alpha) & 0 \\ 0 & \kappa\alpha \end{pmatrix} \qquad \text{and} \qquad Df\big|_B = \begin{pmatrix} -d/\alpha & 0 \\ 0 & -\alpha \end{pmatrix},$$

with $\kappa > 1,\ d, \alpha > 0,\ 1/\alpha < \lambda < 1/2$ and $0 < d/(\kappa\alpha) + d/\alpha < 1$. Therefore,

$$\kappa\alpha > \alpha \qquad \text{and} \qquad \frac{d}{\kappa\alpha} < \frac{d}{\alpha}. \tag{12}$$

Now we set $N := \big([0,1] \times [0,1]\big) \cup C_1 \cup C_2$, where

$$C_1 = \left([0,1] \times \left[-\frac{\kappa\alpha\lambda - 1}{2}, 0\right]\right) \cup S_1, \quad C_2 = \left([0,1] \times \left[1, 1 + \frac{\kappa\alpha\lambda - 1}{2}\right]\right) \cup S_2$$

and S_1 and S_2 are the semidiscs of radius $1/2$ on the bottom of $[0,1] \times [-(\kappa\alpha\lambda - 1)/2, 0]$ and on the top of $[0,1] \times [1, 1 + (\kappa\alpha\lambda - 1)/2]$, respectively.

Finally, we can extend f to a $\mathcal{C}^\infty$ diffeomorphism of N in such a way that $f(H) \subset C_2,\ f(C_2) \subset C_1,\ f(C_1) \subset C_1$ and $f\big|_{C_1}$ is contractive, so that f has a unique fixed point which is contained in $f(C_1)$.

Associated to this diffeomorphism we can consider a symbolic dynamics analogous to the one associated to Smale's horseshoe.

It is well known that the unique invariant set of this map contained in the unit square is:

$$\Lambda := \{x \in [0,1] \times [0,1] \colon f^i(x) \in A \cup B \text{ for all } i \in \mathbb{Z}\}.$$

We consider the set $\widetilde{\Lambda}$ of those points $x \in \Lambda$ such that, for some $l \in \mathbb{N} \cup \{0\}$, the itinerary of $f^l(x)$ is of the form

$$A^{p_1} B^{p_2-p_1} A^{p_3-p_2} B^{p_4-p_3} \dots A^{p_{2k+1}-p_{2k}} B^{p_{2k+2}-p_{2k+1}} \dots \tag{13}$$

where $p_k = \rho^k$ with $\rho \in \mathbb{N} \setminus \{1\}$. We set $p_0 = 0$. Then, for every $n \in [p_k, p_{k+1} - 1]$,

$$f^{l+n}(x) \in \begin{cases} A & \text{if } k \text{ is even} \\ B & \text{if } k \text{ is odd} \end{cases} \tag{13}$$

Clearly, $f(\widetilde{\Lambda}) \subset \widetilde{\Lambda}$.

The next result summarises the situation for the Lyapunov exponents of the above model and, in particular, shows that Condition (8) does not hold for this example.

Lemma 3.5. *For every point $x \in \widetilde{\Lambda}$, the upper Lyapunov exponents are*

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| Df^n x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \log(d/\alpha), \text{ and}$$
$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| Df^n x \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \log(\kappa\alpha).$$

Moreover,

$$\limsup_{n\to\infty} \frac{1}{n} \log |\det(Df^n x)| = \lim_{n\to\infty} \frac{1}{n} \log |\det(Df^n x)| = \log(d).$$

Therefore, since $\kappa > 1$, Formula (8) does not hold.

Proof. Since we have to take limits, we can skip the first iterates of x and assume that x already has an itinerary of the form (13) (that is, we can assume that $l = 0$).

We start by proving that the upper Lyapunov exponent in the direction $(1, 0)^t$ is $\log(d/\alpha)$. Without loss of generality we can work with $\|\cdot\|_\infty$, because the value of the Lyapunov exponents does not depend on the chosen norm. Given $n \in \mathbb{N}$ we set

$$K(n) := \frac{1}{n} \log \left\| Df^n x \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_\infty = \frac{m_1 \log(d/(\kappa\alpha)) + m_2 \log(d/\alpha)}{m_1 + m_2},$$

where $n = m_1 + m_2$ and $m_1 = m_1(n)$ and $m_2 = m_2(n)$ are the number of iterates of x which are in A and B, respectively. Since $m_1 \geq 0$ and $m_2 \geq 0$ we have that,

$$\log(d/\kappa\alpha) \leq K(n) \leq \log(d/\alpha).$$

On the other hand,

$$\begin{aligned}
\limsup_{n\to\infty} K(n) &= \limsup_{n\to\infty} \frac{m_1 \log(d/(\kappa\alpha)) + m_2 \log(d/\alpha)}{m_1 + m_2} \\
&= \limsup_{n\to\infty} \frac{m_2(\log(d/\alpha) - \log(d/\kappa\alpha)) + (m_1 + m_2)\log(d/\kappa\alpha)}{n} \\
&= \log(d/\kappa\alpha) + (\log(d/\alpha) - \log(d/\kappa\alpha)) \limsup_{n\to\infty} \frac{m_2}{n} \leq \log(d/\alpha).
\end{aligned}$$

Thus, to prove that $\limsup_{n\to\infty} K(n) = \log(d/\alpha)$ it is enough to show that there exists a sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{m_2(n_k)}{n_k} = 1.$$

We set $n_k = p_{2k} - 1 = \rho^{2k} - 1$. Then, in view of (13),

$$m_2(n_k) = \sum_{i=1}^{k} (p_{2i} - p_{2i-1}) = \sum_{i=1}^{k} (\rho^{2i} - \rho^{2i-1}) = \frac{\rho - 1}{\rho} \frac{\rho^2 - \rho^{2k}}{1 - \rho^2} = \frac{\rho^{2k+1} - \rho}{(\rho + 1)}.$$

Hence

$$\frac{m_2(n_k)}{n_k} = \frac{1}{\rho^{2k} - 1} \frac{\rho^{2k+1} - \rho}{(\rho + 1)}$$

which tends to 1 when $k \to \infty$.

The proof that the vertical Lyapunov exponent is $\log(\kappa\alpha)$ follows in a similar way as before.

To prove the last statement of the lemma note that

$$|\det(Df\big|_A)| = \frac{d}{\kappa\alpha}\kappa\alpha = d \qquad \text{and} \qquad |\det(Df\big|_B)| = \frac{d}{\alpha}\alpha = d.$$

Then, for every $x \in \Lambda$ (and in particular for every $x \in \widetilde{\Lambda}$),

$$\frac{1}{n} \log|\det(Df^n x)| = \log(d).$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log|\det(Df^n x)| = \lim_{n\to\infty} \frac{1}{n} \log|\det(Df^n x)| = \log(d).$$

□

4. Some examples of SNA's

First of all notice that Keller's [8] model (see Example 1 and Theorem 2.1) has an SNA when $\sigma > 1$ and g vanishes at some point. Indeed, since φ (the map giving the invariant graph) is discontinuous almost everywhere, its graph is not closed. So, a reasonable choice for the attractor is the closure of the graph of φ. This set is an attractor in view of Theorem 2.1(f), and the strangeness follows from Theorem 2.1(d) (see also statement (c)). Finally, the system in nonchaotic by Remark 3.4 and because the vertical Lyapunov exponent exists for almost every θ and for every $x > 0$, and it is negative.

Next we want to see that the Grebogi et al. [7] model (see Example 1) has an SNA. To do it we will prove that a larger class of models which are a "two sided" generalisation of the Keller model have an SNA, and we will obtain the desired result as a corollary.

The systems we will consider are of the form

$$\begin{cases} \theta_{n+1} = R_\omega(\theta_n) \\ x_{n+1} = f(x_n)g(\theta_n) \end{cases} \tag{14}$$

where $(\theta_n, x_n) \in \mathbb{S}^1 \times \mathbb{R}$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$, $g\colon \mathbb{S}^1 \longrightarrow \mathbb{R}$ is bounded, differentiable and vanishes at some point, and $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is odd, bounded, $\mathcal{C}^1$ and verifies that $f(0) = 0$. Moreover, f is either increasing and $f\big|_{(0,\infty)}$ is strictly concave or decreasing and $f\big|_{(0,\infty)}$ is strictly convex.

As we have done before, we define the parameter:

$$\sigma := f'(0) \exp\left(\int_{\mathbb{S}^1} \log|g(\theta)| d\theta \right)$$

which is the Lyapunov exponent of the invariant circle $x \equiv 0$.

Then we have,

Proposition 4.1. *Assume that model* (14) *verifies that* $\sigma > 1$. *Then, it has an* SNA.

Note that, from Proposition 4.1 it follows immediately that the closure of the attractor of the the Grebogi et al. [7] model is an SNA.

Proof of Proposition 4.1. Let $T(\theta, x) := (R_\omega(\theta), f(x)g(\theta))$. Since f and g are bounded, there exists Q such that $|f(x)g(\theta)| \leq Q$ for every $(\theta, x)^t \in \mathbb{S}^1 \times \mathbb{R}$. Then $T(\mathbb{S}^1 \times [-Q, Q]) \subset \mathbb{S}^1 \times [-Q, Q]$ and, since $\mathbb{S}^1 \times [-Q, Q]$ is compact, there is an attractor of the system (14) in in $\mathbb{S}^1 \times [-Q, Q]$.

Now we consider the following related model:

$$\begin{cases} \theta_{n+1} = R_\omega(\theta_n) \\ z_{n+1} = \left| f\big|_{\mathbb{R}^+}(z_n) \right| |g(\theta_n)| \end{cases} \tag{15}$$

where $z_n \in \mathbb{R}^+$. This model verifies the assumptions of the Keller one and thus, Theorem 2.1 hold. In particular, by using the notation of Theorem 2.1,

$$\varphi(\theta) = \left| f\big|_{\mathbb{R}^+}(\varphi(R_\omega^{-1}(\theta))) \right| |g(R_\omega^{-1}(\theta))|.$$

Then, since f is odd,

$$\begin{aligned} \pm\varphi(\theta) &= \pm \left| f\big|_{\mathbb{R}^+}(\varphi(R_\omega^{-1}(\theta))) \right| |g(R_\omega^{-1}(\theta))| \\ &= (\pm s) f(\varphi(R_\omega^{-1}(\theta))) g(R_\omega^{-1}(\theta)) \\ &= f\left((\pm s)\varphi(R_\omega^{-1}(\theta))\right) g(R_\omega^{-1}(\theta)), \end{aligned}$$

where $s := \operatorname{sign}(g(R_\omega^{-1}(\theta)) \operatorname{sign}(f(\varphi(R_\omega^{-1}(\theta))))$. Hence,

$$\{\varphi(\theta), -\varphi(\theta)\} = f\left(\{\varphi(R_\omega^{-1}(\theta)), -\varphi(R_\omega^{-1}(\theta))\}\right) g(R_\omega^{-1}(\theta))$$

and, the closure of the graph of $\Phi(\theta) = \{\varphi(\theta), -\varphi(\theta)\}$ is an attractor for System 14. Moreover, since φ is strange so is Φ.

Finally, since f is odd and the differences between Systems 14 and System 15 are essentially changes of sign in the coordinate x it follows that the vertical Lyapunov exponent of both systems is the same. Thus, Φ in nonchaotic if and only if φ is nonchaotic. Hence, Systems 14 has an SNA for $\sigma > 1$ by Theorem 2.1. □

To end this section let us display a very different kind of SNA. Indeed, we will remark that absorbing Cantor sets are, indeed, SNA's.

Let $f\colon [-1, 1] \longrightarrow [-1, 1]$ be an $\mathcal{S}$-unimodal map (i.e. its Schwarzian derivative is negative wherever it is well defined) with a non-flat critical point. By the topological classification of the attractors of these maps (see for example, [6, Theorem 1.3]), they have a unique attractor, which satisfies Milnor's definition. Moreover, if f is finitely often renormalisable, then the attractor is either a union of transitive intervals or is a Cantor set which is the ω-limit set of the critical point. In this last situation the attractor is called an *absorbing Cantor set* (and *wild Cantor attractors* in [4]). An absorbing Cantor set does not contain intervals so it is minimal and has zero Lebesgue measure. The existence of this kind of attractors was proved in [3] and [4] under certain conditions (a particular case of maps satisfying these conditions are the *Fibonacci maps*).

Remark 4.2. *Absorbing Cantor sets of $\mathcal{S}$-unimodal maps with non-flat critical point are* SNA*'s.* To see it, in view of what it has been said before, we only have to see that

$$\lambda_f := \limsup_{n\to\infty} \frac{1}{n} \log |Df^n(x)|$$

exists for almost every x and is non-negative. But this follows directly from Statement 2 of the Corollary in Page 366 of [6].

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Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain

E-mail address: alseda@mat.uab.es

Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain

E-mail address: scosta@mat.uab.es