The Plateau Problem in Hadamard Manifolds

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Abstract: Using a differential topological argument we prove the existence in Hadamard manifolds of hypersurfaces of constant Gaussian curvature which are solutions to certain Plateau problems.

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1 - Introduction.

In this paper we use a differential topological approach to construct solutions to the Plateau problem in Hadamard manifolds for hypersurfaces of constant Gaussian curvature. Explicitely, let M^{n+1} be an $(n+1)$-dimensional Hadamard manifold. An immersed hypersurface in M is a pair $\Sigma^n = (i, (S^n, \partial S^n))$ where $(S^n, \partial S^n)$ is a smooth, compact, n-dimensional manifold with boundary and $i : S \to M$ is a smooth immersion. An immersed hypersurface is said to be locally convex if and only if its shape operator is everywhere positive definite, and its boundary is said to be generic if and only if for any $p \neq q \in \partial S$ such that $i(p) = i(q)$:

$$Di \cdot T_p\partial S \neq Di \cdot T_q\partial S.$$

In other words, the two tangent spaces of $\partial\Sigma$ at these points do not coincide. Trivially, every smooth, locally convex immersion can be approximated arbitrarily closely by a smooth, locally convex immersion with generic boundary. We prove:

Theorem 1.1

Let $\hat{\Sigma}^n = (\hat{\imath}, (\hat{S}, \partial\hat{S}))$ be a locally strictly convex, immersed hypersurface in M with generic boundary. Let $\phi \in C^\infty(M)$ be a smooth, positive valued function such that, for every $p \in \hat{\Sigma}$, the Gaussian curvature of $\hat{\Sigma}$ at p is strictly greater than $\phi(p)$. Suppose that there exists a convex set, K, with smooth boundary and an open subset $\Omega \subset \partial K$ such that:

(i) $\partial\Omega$ is smooth;

(ii) Ω^c has finitely many connected components; and

(iii) Σ^n is isotopic by locally strictly convex, immersed hypersurfaces to a finite covering of Ω,

then there exists a locally strictly convex, immersed hypersurface Σ^n in M such that:

(a) $\partial\Sigma = \partial\hat{\Sigma}$;

(b) Σ is contained by $\hat{\Sigma}$; and

(c) for every point $p \in \Sigma$, the Gaussian curvature of Σ at p is equal to $\phi(p)$.

Remarks:

(a) this generalises the existence result [5] of Guan and Spruck;

(b) the concept of containment (condition (b)), is described explicitely in Section 3. Heuristically, if Σ is contained by $\hat{\Sigma}$, then $\hat{\Sigma}$ limits the geometry and, in particular, the extent of Σ: in fact, $\hat{\Sigma}$ is (more or less) a graph over Σ;

(c) when the ambient manifold is of dimension greater than 3, immersed hypersurfaces of constant Gaussian curvature typically do not behave well under passage to the limit. We thus do not expect that an approximation argument may be used to relax the condition of genericity along the boundary;

(d) if $(\hat{S}, \partial\hat{S})$ is diffeomorphic to the closed unit ball in $\mathbb{R}^n$, then the hypotheses of Theorem 1.1 are automatically satisfied for all ϕ sufficiently small;

(e) when $n = 2$, and thus when the dimension of the ambient manifold is equal to 3, Theorem 1.1 yields a stronger version of Proposition 5.0.3 of [8], which itself constitutes the analytic core of that paper; and

(f) in general manifolds (of arbitrary curvature), the situation is complicated by the possible existence of conjugate points along geodesics. However, most stages of the argument remain more or less intact, and the result may thus be adapted, albeit with stronger hypothesis, to the more general case.

Our approach uses the continuity method based on a marriage of various existing approaches to the study of immersions of constant Gaussian curvature. The continuity method itself divides into two stages: compactness and local deformation. The compactness stage is carried out using an adaptation of the now classical analysis of Caffarelli, Nirenberg and Spruck first described in [2] and first applied to constant curvature hypersurfaces by the same authors in [4]. These techniques were subsequently developed most notably by Guan and Spruck in [5] for hypersurfaces in $\mathbb{R}^{n+1}$ and Rosenberg and Spruck in [9] for hypersurfaces in $\mathbb{H}^{n+1}$, and were further refined by the author in [10] to treat the case of hypersurfaces in general manifolds (albeit with stronger hypotheses than those studied here). The analysis of [10] is used in the current paper to obtain second and higher order estimates on hypersurfaces of prescribed Gaussian curvature once the first order estimates have been established.

The first order estimates present, in our setting, a new challenge that is not so explicitely present in the cases hitherto studied. Typically, these estimates follow immediately from elementary properties of convex sets. In our setting, however, we require a stronger result, relating to the compactness of families of convex immersions, which generalises to the case of general manifolds the results [6] of Guan and Spruck and [14] of Trudinger and Wang (proven in these cases for convex immersions in $\mathbb{R}^{n+1}$). This, along with elementary geometric considerations, completes the compactness half of the argument, and it is this stage that requires the genericity assumption along the boundary.

The local deformation stage uses a further genericity argument based on Sard's Lemma and compactness (which has now already been established and which yields a finite dimensional reduction of the problem in a certain sense). Generalising the argument [8] of Labourie for three dimensional ambiant manifolds, we construct a smooth isotopy, $(\hat{\Sigma}_t)_{t\in[0,1]}$, of convex immersed balls and a smooth isotopy, $(\phi_t)_{t\in[0,1]}$, of smooth functions such that:

(i) $\hat{\Sigma}_1 = \hat{\Sigma}$, $\phi_1 = \phi$; and

(ii) there exists a unique solution for the data $(\hat{\Sigma}_0, \phi_0)$.

We then obtain (generically) an isotopy $(\Sigma_t)_{t\in[0,1]}$ of solutions. Σ_1 is then (generically) a solution to the problem, and existence then follows by taking limits (which is valid at this stage, in contrast to Remark (c) above).

Uniqueness presents an interesting problem. As shown by Labourie in [8], when the ambiant manifold is 3-dimensional, if its sectional curvature is bounded above by 1, and if the desired Gaussian curvature is less then 1, then the linearisation (derivative) of the Gauss Curvature Operator is always invertible. Consequently, any two distinct solutions for the data $(\hat{\Sigma}_1, \phi_1)$ can by interpolated to two distinct solutions for the data $(\hat{\Sigma}_0, \phi_0)$, for which, however, only one solution exists. They thus coincide, and uniqueness follows.

In the higher dimensional case, however, the linearisation of the Gauss Curvature Operator is no longer necessarily invertible, and this interpolation argument is therefore no longer valid. There is no obstacle to the apparition of multiple solutions. Our results are nonetheless of a differential topological nature, essentially proving a cobordism invariance of the number of zeroes of certain types of (generic) sections of certain bundles. In particular, the number of solutions modulo 2 is (generically) an invariant of isotopy classes of convex immersions. In fact, there should exist a natural orientation of solutions such that the number of solutions, counted algebraically (with orientation), is an invariant of isotopy classes of convex immersions. The resulting invariants are analogous to those arising from the theory of pseudo-holomorphic curves of Gromov: an analogy which becomes explicit when the ambient manifold is 3 dimensional (c.f. [8]). Nonetheless, it should be borne in mind that, firstly, the current arguments for Hadamard manifolds do not apply to compact manifolds of non-positive curvature (as an examination of the proof of Proposition 6.1 of [10] reveals), and, secondly, we so far know of no examples of locally rigid Plateau problems having more than one solution, which would be necessary for these invariants to ever take non-trivial values.

Another interesting associated problem is that of proving existence of solutions for other curvature functions, defined by $O(n)$-invariant functions of the shape operator of an immersed hypersurface (c.f., for example, [4]). Of these, perhaps the most interesting is $\sigma_2(A)$, where A is the shape operator and σ_2 is the second order symmetric polynomial of the eigenvalues. This curvature is equivalent to the scalar curvature of the immersed hypersurface, and thus describes its *intrinsic* geometry. In this case, it would be most interesting to prove the existence of hypersurfaces of prescribed curvature depending, not only on position, but also on the normal vector, since only this allows us to prescribe the scalar curvature of the solution. The techniques developed here and in [10] are so far not sufficiently strong to solve this problem.

This paper is structured as follows:

(i) in Sections 2 to 4 we introduce the concepts and notation used in the sequel: in Section 2, we introduce immersed hypersurfaces and describe the Banach manifold of immersed hypersurfaces modulo reparametrisation; in Section 3 we introduce locally convex immersions and describe the concept of containment; and in Section 4 we develope a higher codimensional concept of convexity which is required to understand the boundary conditions used in the sequel;

(ii) in Sections 5 to 9, which together constitute the most innovative part of the paper, we determine first order a-priori bounds near the boundary for generic, locally convex immersions of prescribed curvature: in Section 5, restricting to the case where the boundary is embedded, and using the notion of "semi-convexity", we obtain a compactness result for convex immersions which yields these a-priori bounds but requires various intuitive but technical propositions whose proofs are deferred to the subsequent two sections; in Section 6, we obtain technical results using the parabolic limit; in Section 7, we show that the limit of a sequence of semi-convex sets is also semi-convex; in Section 8, we show how, under a simple modification, the reasoning of Section 5 may be adapted to the case where the boundary is immersed and generic; and in Section 9, we obtain first order lower bounds along the boundary which are important in the sequel for the final (technical) step in obtaining second order bounds over the boundary;

(iii) in Sections 10 to 11, we recall the results of [10] to prove a conditional existence result: in Section 10, we prove compactness of families of immersions of prescribed curvature; and in Section 11, we show how Sard's Lemma may be used along with compactness to obtain (generically) solutions which interpolate between isotopic data; and

(iv) in Section 12, we prove the existence of isotopies between the given data and other data for which solutions are known to exist, and, using the concepts of local and global rigidity, we prove Theorem 1.1.

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2 - Immersed Submanifolds and Moduli Spaces.

Let M^{n+1} be a smooth Riemannian manifold. A (smooth, compact) **immersed submanifold** is a pair $\Sigma := (i,(S,\partial S))$ where:

(i) $(S,\partial S)$ is an oriented, compact, Riemannian manifold with boundary; and

(ii) $i:\Sigma\rightarrow M$ is a smooth immersion (i.e. Di is everywhere injective).

Remark: in the sequel, all submanifolds of M will be (relatively) compact. Likewise, unless stated otherwise, all submanifolds of M will be smooth.

Let $\Sigma=(i,(S,\partial S))$ and $\Sigma'=(i',(S',\partial S'))$ be two immersed hypersurfaces in M. We say that Σ and Σ' are **equivalent** if and only if there exists a diffeomorphism $\phi:(S,\partial S)\rightarrow(S',\partial S')$ such that:

$$i'\circ\phi=i.$$

Let Exp be the exponential map of M. Let N_Σ be the outward pointing normal vector field over Σ. We say that Σ' is a **graph** over Σ if and only if there exists $f\in C^\infty_0(S)$ and a diffeomorphism $\phi:(S,\partial S)\rightarrow(S',\partial S')$ such that:

$$i'\circ\phi=\mathrm{Exp}(f\mathsf{N}_\Sigma).$$

In particular, Σ and Σ' are equivalent if and only if Σ' is a trivial graph over Σ.

Let $(\Sigma_n)_{n\in\mathbb{N}}=(i_n,(S_n,\partial S_n)),\Sigma_0=(i_0,(S_0,\partial S_0))$ be immersed submanifolds in M. We say that $(\Sigma_n)_{n\in\mathbb{N}}$ **converges** to Σ_0 if and only if there exists $N\geqslant 0$ and, for all $n\geqslant N$ a diffeomorphism $\phi_n:(S_0,\partial S_0)\rightarrow(S_n,\partial S_n)$ such that $(i_n\circ\phi_n)_{n\geqslant N}$ converges to i_0 in the C^∞ sense.

Trivially, if $(\Sigma_n)_{n\in\mathbb{N}}$ converges to Σ_0, then there exists $N\geqslant 0$, and for all $n\geqslant N$ a vector field $X_n\in\Gamma(i_0^*TM)$ and a diffeomorphism $\phi_n:(S_0,\partial S_0)\rightarrow(S_n,\partial S_n)$ such that:

$$i_n\circ\phi_n=\mathrm{Exp}(X_n).$$

Moreover, $(X_n)_{n\geqslant N}$ tends to 0 in the C^∞ sense. If Σ_n and Σ_0 have the same boundary for all n, then, increasing N if necessary, X_n may always be chosen to be normal to Σ_0 and vanishing along ∂S_0. In other words, Σ_n is a graph over Σ_0 for sufficiently large n.

Let $(\Gamma_t)_{t\in[0,1]}=(j_t,(G_t,\partial G_t))_{t\in[0,1]}$ be a smooth family of (exact) immersed submanifolds in M. We denote by $\hat{\mathcal{M}}$ the family of all pairs (t,Σ) where $t\in I$ and Σ is an immersed submanifold in M such that $\partial\Sigma=\Gamma_t$. For all $t\in[0,1]$, let $\hat{\mathcal{M}}_t$ be the fibre of $\hat{\mathcal{M}}$ over t. We denote by $\mathcal{M}$ the family of all pairs $(t,[\Sigma])$ where $[\Sigma]$ denotes the equivalence class of Σ. Likewise, for all $t\in[0,1]$, we denote by $\mathcal{M}_t$ the fibre of $\mathcal{M}$ over t.

For all t, we interpret $\mathcal{M}_t$ as a smooth Banach manifold (strictly speaking, every relatively compact open subset is an intersection of an infinite family of nested Banach manifolds). Let $[\Sigma]$ be an element in $\mathcal{M}_t$. Let $V_\Sigma\subseteq\mathcal{M}_t$ be the set of those immersed hypersurfaces which are graphs over Σ. This is an open subset of $\mathcal{M}_t$, which we identify with an open subset U_Σ of $C_0^\infty(S)$. Let $\Phi_\Sigma:U_\Sigma\rightarrow V_\Sigma$ be the canonical identification. $(U_\Sigma,V_\Sigma,\Phi_\Sigma)$ consitutes a smooth chart of $\mathcal{M}_t$ which we call the **graph neighbourhood** of Σ.

We likewise interpret $\mathcal{M}$ also as a smooth Banach manifold. As before, let $(t,[\Sigma])$ be an element of $\mathcal{M}$, where $\Sigma=(i,(S,\partial S))$. We extend i to a smooth family $(i_s)_{s\in]t-\epsilon,t+\epsilon[}$ such that, for all s, $(i_s,\partial S)=\Gamma_s$. Thus, if, for all s, we define Σ_s by $\Sigma_s=(i_s,(S,\partial S))$, then $(s,[\Sigma_s])_{s\in]t-\epsilon,t+\epsilon[}$ is a smooth family in $\mathcal{M}$. Let $V_\Sigma\subseteq\mathcal{M}$ be the set of pairs $(s,[\Sigma'])$ where Σ' is a graph over Σ_s. V_Σ is an open subset of $\mathcal{M}$ which we identify with an open subset, U_Σ, of $]t-\epsilon,t+\epsilon[\times C_0^\infty(S)$. Let $\Phi_\Sigma:U_\Sigma\rightarrow V_\Sigma$ be the canonical identification. $(U_\Sigma,V_\Sigma,\Phi_\Sigma)$ consitutes a smooth chart of $\mathcal{M}$ which we likewise call the **graph neighbourhood** of Σ. Trivially, this does not depend canonically on Σ, but also on the choice of smooth family extending Σ.

Let (t,Σ) be an element of $\hat{\mathcal{M}}$, where $\Sigma=(i,(S,\partial S))$. The group of smooth diffeomorphisms of $(S,\partial S)$ acts linearly on $C^\infty(S)$. $C^\infty(S)$ therefore defines a bundle $\mathcal{E}$ over $\mathcal{M}$, whose fibre at $(t,[\Sigma])$ is $C^\infty(S)$. Since the constant functions over S are preserved by the diffeomorphisms of $(S,\partial S)$, these generate a subbundle of $\mathcal{E}$ which we identify with $\mathcal{M}\times\mathbb{R}$. Likewise, if $(\phi_t)_{t\in[0,1]}\in C^\infty(M)$ is a smooth family of smooth functions, then it defines a section of $\mathcal{E}$, which we also denote by ϕ, given by:

$$\phi(t,[\Sigma])=\phi_t\circ i.$$

For all t, $\mathcal{E}$ restricts canonically to a bundle over $\mathcal{M}_t$, which we denote by $\mathcal{E}_t$. Let $(U_\Sigma,V_\Sigma,\Phi_\Sigma)$ be a graph neighbourhood of $\mathcal{M}_t$ about Σ. Trivially:

$$\mathcal{E}|_{V_\Sigma}=U_\Sigma\times C^\infty(S).$$

This yields a canonical splitting of $T\mathcal{E}_t$ over the fibre over Σ. Since every point in $\mathcal{M}_t$ has a canonical graph neighbourhood, we thus obtain a canonical splitting of $T\mathcal{E}_t$ which in turn generates a covariant derivative of $\mathcal{E}_t$. More explicitely, for every $\Sigma' = (i', (S', \partial S')) \in V_\Sigma$, let $\pi_{\Sigma'} : S' \to S$ be the canonical projection. A section, f, of $\mathcal{E}_t$ is covariant constant at Σ if and only if there exists a function $f_0 \in C^\infty(S)$ such that, up to second order around Σ:

$$f_{\Sigma'} = f_0 \circ \pi_{\Sigma'}.$$

We advise the reader unfamiliar with the theory of Banach manifolds not to trouble himself with the details of this construction. In the sequel, it suffices to know that, locally, $\mathcal{E}_t$ behaves like the constant bundle $U_\Sigma \times C^\infty(S)$ and it is not really necessary to have an explicit choice of splitting of $\mathcal{E}$.

We define the **Gauss curvature mapping**, K, to be the mapping that associates to every element $(t, [\Sigma])$, where $\Sigma = (i, (S, \partial S))$, the function $f \in C^\infty(S)$ whose value at the point $p \in S$ is the Gaussian curvature of Σ at p. K defines a smooth section of $\mathcal{E}$ over $\mathcal{M}$.

We determine a formula for the covariant derivative, ∇K of K with respect to the canonical splitting of $\mathcal{E}_t$. Let $\Sigma = (i, (S, \partial S))$ be an element of $\mathcal{M}_t$. Let N be the outward pointing unit normal vector field over Σ. Let R be the Riemann curvature tensor of M. We define the operator W acting on sections of TS by:

$$W \cdot X = R_{\mathsf{N}X}\mathsf{N}.$$

Lemma 2.1

With respect to the canonical splitting, identifying $T_{[\Sigma]}\mathcal{M}_t$ with $C_0^\infty(S)$:

$$\nabla_f K = K\mathrm{Tr}(A^{-1}(W - A^2))f - K\mathrm{Tr}(A^{-1}\mathrm{Hess}(f)),$$

where A is the shape operator of Σ.

Proof: See Proposition 3.1.1 of [8]. $\square$

This yields the following result, which will be of use in the sequel:

Corollary 2.2

∇K is a second order linear differential operator. Moreover:

(i) if Σ is strictly convex, then ∇K is elliptic; and

(ii) when $\mathrm{Tr}(A^{-1}(W - A^2)) > 0$, ∇K has trivial kernel.

Remark: In particular, if the sectional curvature of M is bounded above by -1 and if $A \leqslant \mathrm{Id}$, then $W - A^2 \geqslant 0$ and so, by (ii), ∇K is invertible.

Proof: (i) is immediate. (ii) follows by the Maximum Principal. $\square$

3 - Locally Convex Hypersurfaces.

Let M^{n+1} be a Riemannian manifold. A **locally convex hypersurface** in M is a pair $\Sigma = (i, S^n)$ where S is an n-dimensional topological manifold and $i : S \to M$ is a continuous map such that, for all $p \in S$, there exists a neighbourhood, U, of p in S, a convex subset $K \subseteq M$ with non-trivial interior, and an open subset $V \subseteq \partial K$ such that i restricts to a homeomorphism from U to V. We refer to such a triplet (U, V, K) as a **convex chart** of Σ. Pulling back the metric on M through i yields a natural length metric on Σ which we denote by d_Σ. Let $(\Sigma_n)_{n\in\mathbb{N}} = (i_n, S_n)_{n\in\mathbb{N}}$ and $S_0 = (i_0, S_0)$ be convex immersions. We say that $(\Sigma_n)_{n\in\mathbb{N}}$ **converges** to Σ_0 if and only if:

(i) $(S_n, d_{\Sigma_n})_{n\in\mathbb{N}}$ converges to (S_0, d_{Σ_0}) in the Gromov-Hausdorff sense; and

(ii) $(i_n)_{n\in\mathbb{N}}$ converges to i_0 locally uniformly.

Let $\Sigma = (i, S)$ and $\Sigma' = (i', S')$ be two locally convex hypersurfaces in M. We say that Σ and Σ' are equivalent if and only if there exists a homeomorphism $\phi : S \to S'$ such that:

$$i = i' \circ \phi.$$

Example: Let $K \subseteq M$ be a convex subset with non trivial interior. Then any open subset of ∂K is a locally convex hypersurface. $\square$

Example: Let Σ be a (smooth) hypersurface on M. Σ is a locally convex hypersurface if and only if its second fundamental form is everywhere non-negative definite. $\square$

Suppose now that M is a Hadamard manifold. Let $K \subseteq M$ be a convex set with non-trivial interior. Let K^o be the interior of K. We define $\pi_K : M \setminus K^o \to \partial K$ to be projection onto the closest point in ∂K. Let $V \subseteq \partial K$. We call the set $\pi_K^{-1}(V)$ the **end** of V, and we denote it by $\mathcal{E}(V)$. Trivially, $\mathcal{E}(V)$ is foliated by half geodesics leaving points in V in directions normal to K. Let Σ be a locally convex hypersurface. Let (U, V, K) and (U', V', K') be convex charts of Σ. Trivially:

$$\pi_K^{-1}(i(U \cap U')) = \pi_{K'}^{-1}(i(U \cap U')).$$

We thus define the **end** of Σ to be the manifold (with non-smooth, concave boundary) whose coordinate charts are the ends of the convex charts of Σ. We denote this manifold by $\mathcal{E}(\Sigma)$. $\mathcal{E}(\Sigma)$ has the following properties:

(i) Σ naturally embeds as the boundary of $\mathcal{E}(\Sigma)$;

(ii) in the complement of Σ, $\mathcal{E}(\Sigma)$ has the structure of a smooth Riemannian manifold with non-positive curvature;

(iii) $\mathcal{E}(\Sigma)$ is foliated by half geodesics leaving points in Σ in directions normal to Σ; and

(iv) there exists a natural embedding $I : \mathcal{E}(\Sigma) \to M$ which restricts to i over Σ and which is a local diffeomorphism over the complement of Σ.

Let $K \subseteq \mathcal{E}(\Sigma)$ be a subset of the end of Σ. Suppose moreover that K contains Σ and that K coincides with Σ outside a compact set. Let p be a point in $\mathcal{E}(\Sigma) \setminus \Sigma$ lying on the boundary of K. We say that K is **boundary convex** at p if and only there exists a neighbourhood, U, of p in $\mathcal{E}(\Sigma)$, a convex subset $K' \subseteq M$ with non trivial interior, and a neighbourhood V of $I(p)$ in M such that I restricts to a homeomorphism from U to V, and:

$$I(K \cap U) = K' \cap V.$$

Bearing in mind that, near any point $p \in \Sigma$, $\mathcal{E}(\Sigma)$ may always be extended over an open set containing p, we extend this definition to also include boundary points lying in Σ. We then say that K is **boundary convex** if and only if it is boundary convex at p for every $p \in \partial K$. Importantly, the image under I of the boundary of a boundary convex set is a locally convex hypersurface.

We say that a subset $K \subseteq \mathcal{E}(\Sigma)$ is **semi-convex** if and only if for every geodesic segment $\gamma : [0,1] \to \mathcal{E}(\Sigma)$ contained within $\mathcal{E}(\Sigma)$, if $\gamma(0), \gamma(1) \in K$, then the whole of γ is contained in K.

Proposition 3.1

Let K be a subset of the end of Σ which contains Σ and coincides with Σ outside a convex set. If K is semi-convex, then K is boundary convex.

Proof: Let $p \in \partial K$. If p lies in the interior of $\mathcal{E}(\Sigma)$, then K is trivially boundary convex at p. Suppose therefore that $p \in \Sigma$. Let (U, V, K') be a convex chart of Σ at p. Let $r > 0$ be such that $B_r(p) \subseteq \mathcal{E}(U)$. Consider $X = (K' \cap B_r(p)) \cup (K \cap B_r(p))$. Let $\gamma : [0,1] \to B_r(p)$ be a geodesic segment with endpoints in X. Let γ' be a maximal subsegment of γ lying outside $K' \cap B_r(p)$. Since $\Sigma \subseteq K$, the endpoints of γ' are contained in $K \cap B_r(p)$. Thus, by semi-convexity, γ' is contained in K, and therefore also in X. It follows that the whole of γ is contained in X. Since γ is arbitrary, X is convex and K is therefore boundary convex at p. This completes the proof. □

Let K be a semi-convex subset of the end of Σ which contains Σ and coincides with Σ outside a convex set. $(\partial K, I|_{\partial K})$ defines a convex immersion in M which, by abuse of notation, we simply denote by ∂K. Let Σ and Σ' be two locally convex hypersurfaces in M. We say that Σ is **contained by** $\hat{\Sigma}'$ (and Σ' **contains** Σ) if and only if there exists a semi-convex subset, $K \subseteq \mathcal{E}(\Sigma)$, which contains Σ and which coincides with Σ outside a convex set such that Σ' is equivalent to ∂K. In this case, we often identify Σ' with ∂K and thus view it as a subset of $\mathcal{E}(\Sigma)$.

Example: Let $K, K' \subseteq M$ be two convex sets. Then ∂K is contained by $\partial K'$ if and only if $K \subseteq K'$. □

Let $\Sigma = (i, S)$ be a locally convex hypersurface. For $p \in S$, let $N_p \subseteq UM$ be the set of supporting normals of Σ at S. We define $N\Sigma$ by:

$$N\Sigma = \bigcup_{p \in S} N_p.$$

$N\Sigma$ defines a C^0 immersed submanifold of UM which we call the **normal** of Σ.

If Σ' contains Σ, then there exists an upper semi-continuous function $f : N\Sigma \to [0, \infty[$ such that Σ' is the graph of f over Σ. Moreover, f vanishes outside a convex set. We call

f and $\mathrm{Supp}(f)$ respectively the **graph function** and **graph support** of Σ' with respect to Σ.

The property of containment is preserved by passage to limits:

Lemma 3.2

Let $(\Sigma_n)_{n\in\mathbb{N}}, \Sigma_0$ and $(\Sigma'_n)_{n\in\mathbb{N}}, \Sigma'_0$ be convex immersions in M. Suppose that, for all $n > 0$, Σ'_n contains Σ_n. For all $n > 0$, let f_n and $X_n = \mathrm{Supp}(f_n)$ be the graph function and graph support respectively of Σ'_n with respect to Σ_n. Suppose that there exists $R > 0$ and that, for all n, there exists a compact set $X'_n \subseteq \Sigma_n$ such that:

(i) $f_n \leqslant R$ for all $n > 0$;

(ii) for all $n > 0$, $X_n \subseteq X'_n$; and

(iii) $(X'_n)_{n\in\mathbb{N}}$ converges to X'_0 in the Hausdorff sense,

then Σ'_0 also contains Σ_0.

Proof: For all n, let $K_n \subseteq \mathcal{E}(\Sigma_n)$ be the semi-convex subset such that $\partial K_n = \Sigma'_n$. The hypotheses on $(f_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ imply that $(K_n)_{n\in\mathbb{N}}$ is uniformly bounded. By compactness of the family of semi-convex sets, $(K_n)_{n\in\mathbb{N}}$ subconverges in the Hausdorf sense to a semi-convex set $K_0 \subseteq \mathcal{E}(\Sigma_0)$, say. By Proposition 3.1, K_0 is boundary convex and so $(I|_{\partial K_0}, \partial K_0)$ is a locally convex hypersurface. Moreover $(I|_{\partial K_n}, \partial K_n)_{n\in\mathbb{N}}$ converges to $(I|_{\partial K_0}, \partial K_0)$ in the sense of convex immersions. Since $(I|_{\partial K_n}, \partial K_n) = \Sigma'_n$ for all n, and since $(\Sigma'_n)_{n\in\mathbb{N}}$ converges to Σ'_0 in the sense of convex immersions, $(I|_{\partial K_0}, \partial K_0)$ is equivalent to Σ'_0. Σ'_0 therefore contains Σ_0, and this completes the proof. $\square$

In the sequel, we require a slight variation of this definition. Let $\Sigma = (i, (S, \partial S))$ and $\Sigma' = (i', (S', \partial S'))$ be (smooth) immersed hypersurfaces which are also convex. Let N_Σ and $\mathsf{N}_{\Sigma'}$ be the outward pointing normal vector fields over Σ and Σ' respectively. Let $\mathsf{N}_{\partial\Sigma}$ be the normal vector field over $\partial\Sigma$ which is tangent to Σ and points outwards from Σ.

Suppose that $\partial\Sigma' = \partial\Sigma =: \Gamma$. We suppose moreover that Σ' lies "locally strictly above" Σ along Γ: i.e. for all $p \in \Gamma$:

$$\langle \mathsf{N}_{\Sigma'}, \mathsf{N}_{\partial\Sigma} \rangle > 0.$$

Since Σ' is smooth, it may be extended to a (smooth) convex, immersed hypersurface $\tilde{\Sigma}'$ strictly containing $\partial\Sigma'$ in its interior. Let Σ'_c denote the collar region of $\tilde{\Sigma}'$ lying outside Σ'. We define the piecewise smooth immersed hypersurface $\tilde{\Sigma}$ by:

$$\tilde{\Sigma} = \Sigma \cup \Sigma'_c.$$

Since $\tilde{\Sigma}'$ lies locally strictly above Σ along Γ, $\tilde{\Sigma}$ is also a locally convex hypersurface. We now say that Σ' **contains** Σ if and only if $\tilde{\Sigma}'$ contains $\tilde{\Sigma}$.

Suppose that Σ' lies locally strictly above Σ along $\partial\Sigma$ and contains Σ. Let f be the graph function of $\tilde{\Sigma}'$ with respect to $\tilde{\Sigma}$. Let $\pi : \tilde{\Sigma}' \to N\tilde{\Sigma}$ be the canonical projection. We say that Σ' **strictly contains** Σ if and only for all $p \in S' \setminus \partial S'$:

$$f \circ \pi(p) > 0.$$

In this case, the property of strict containment is preserved by small deformations:

Lemma 3.3

Let $(\Sigma_n)_{n\in\mathbb{N}}, \Sigma_0$ and $(\Sigma_n')_{n\in\mathbb{N}}, \Sigma_0'$ be smooth, convex, immersed hypersurfaces. Suppose that Σ_0' lies locally strictly above Σ_0 along $\partial\Sigma_0$ and strictly contains Σ_0. Suppose moreover that, for all n, $\partial\Sigma_n = \partial\Sigma_n'$ and that $(\Sigma_n)_{n\in\mathbb{N}}$ and $(\Sigma_n')_{n\in\mathbb{N}}$ converge to Σ_0 and Σ_0' respectively. Then, for sufficiently large n, Σ_n' lies locally strictly above Σ_n along $\partial\Sigma_n$ and contains Σ_n.

Proof: For all n, let $\Sigma_n' = (i_n', (S_n', \partial S_n'))$. For all n, $\mathcal{E}(\tilde{\Sigma}_n)$ may be extended beyond $\tilde{\Sigma}_n$ to contain a neighbourhood of $\tilde{\Sigma}_n$. Let $\mathcal{E}_{\text{ext}}(\tilde{\Sigma}_n)$ denote this extension. For sufficiently large N, $\hat{\Sigma}_n$ is contained in $\mathcal{E}_{\text{ext}}(\tilde{\Sigma}_n)$. Let $d_n : S_n' \to \mathbb{R}$ be the signed distance in $\mathcal{E}_{\text{ext}}(\tilde{\Sigma}_n)$ to $\tilde{\Sigma}_n$. For sufficently large n, d_n is smooth, and $(d_n)_{n\in\mathbb{N}}$ converges to d_0 in the C^∞ sense. However, $d_0 > 0$ and $\nabla d \neq 0$ along $\partial\Sigma_0$. Thus, for sufficiently large n, $d_n > 0$ and so $\Sigma_n' \subseteq \mathcal{E}(\tilde{\Sigma}_n)$. This completes the proof. $\square$

4 - Convexity in Higher Codimension.

Let M^{n+1} be a Riemannian manifold. Let $\Gamma^k = (i, (G^k, \partial G^k)) \subseteq M$ be a k-dimensional immersed submanifold. Let $N\Gamma \subseteq i^*(UM)$ be the bundle of unit normal vectors over Γ. $N\Gamma$ has spherical fibres of dimension $(n-k)$. For all $\mathsf{N}_p \in N\Gamma$, let $A_\Gamma(\mathsf{N}_p)$ be the shape operator of Γ with respect to N_p. In other words, for all vector fields X and Y tangent to Γ:

$$A_\Gamma(\mathsf{N}_p)(X,Y) = -\langle \nabla_X Y, \mathsf{N}_p \rangle.$$

For all $p \in \Gamma$, we define define $X_p \subseteq T_p\Gamma$ by:

$$X_p = \{\mathsf{N}_p \text{ s.t. } A_\Gamma(\mathsf{N}_p) > 0\},$$

where, for a matrix, M, we write $M > 0$ if and only if it is positive definite. Since the set of positive definite matrices is an open convex cone, X_p is a convex subset of $N_p\Gamma$. In particular, it is contained within a hemisphere. We say that Γ is **locally strictly convex** at p if and only if X_p is non-empty. We say that Γ is **locally strictly convex** if and only if it is locally strictly convex at every point $p \in \Gamma$.

We now consider the case where Γ is of codimension 2, in which case $N\Gamma$ is a circle bundle over Γ and, for all $p \in \Gamma$, X_p is an open interval of length at most π. We define a **convexity orientation** of Γ to be a continuous section, N^-, of $N\Gamma$ over Γ such that, for all $p \in \Gamma$:

$$\mathsf{N}^-(p) \in \partial X_p.$$

We say that Γ carries a convexity orientation when such a section exists. A convexity orientation defines an order over X_p in the following manner: we say that, given two vectors, $V_p, V_p' \in X_p$, V_p lies below V_p' if and only if it lies between $\mathsf{N}^-(p)$ and V_p'. Given a convexity orientation, N^-, we define the section N^+ such that, for all p:

$$\partial X_p = \left\{\mathsf{N}^+, \mathsf{N}^-\right\}.$$

We call this vector field the **convexity coorientation** of Γ.

Example: If $(\hat{\Sigma},\partial\hat{\Sigma})$ is a strictly convex immersed hypersurface in M, then $\Gamma := \partial\hat{\Sigma}$ is a locally strictly convex, codimension 2, immersed submanifold. Moreover, Γ inherits a convexity orientation from $\hat{\Sigma}$ in the following manner: For $p \in \Gamma$, we identify each unit vector in $N_p\Gamma$ with the (oriented) hyperplane in T_pM normal to that vector. $T_p\hat{\Sigma}$ defines a half-hyperplane with upward pointing unit normal in X_p. Let H_p be another (oriented) hyperplane in X_p that is close to $T_p\Sigma$. We say that H_p lies above (resp. below) $T_p\hat{\Sigma}$ if and only if it is a graph over (resp. beneath) $T_p\hat{\Sigma}$. We extend this to an order on X_p, and define $\mathsf{N}^-(p)$ to be the end point of X_p lying below $T_p\hat{\Sigma}$.

More formally, for $p \in \Gamma$, let $E = T_pM/T_p\Gamma$. E is a two dimensional vector space. Moreover, $N_p\Gamma$ projects down to a circle, S_p, in E. We consider X_p as a subinterval of S_p. Let $\mathsf{N}_p \in X_p$ be the outward pointing exterior normal to $\hat{\Sigma}$ at p. $T_p\hat{\Sigma}$ defines a half-hyperplane which projects down to a half line in E. This half-line is parallel to the tangent line to X_p at N_p, and thus defines an orientation on S_p at N_p. $\mathsf{N}^-(p)$ is then the boundary point of S_p towards which $T_p\hat{\Sigma}$ points. $\square$

Suppose that Γ is locally strictly convex with convexity orientation, and suppose that $\partial\Sigma$ is a strictly convex immersed hypersurface such that $\partial\Sigma = \Gamma$. We say that Σ is **compatible** with the orientation on Γ if and only if the convexity orientation induced on Γ by Σ coincides with the pre-existing convexity orientation on Γ.

5 - First Order Upper Bounds.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let $\Gamma^{n-1} \subseteq M$ be a strictly convex, codimension 2, embedded submanifold with convexity orientation. Let N^- and N^+ be the convexity orientation and coorientation respectively of Γ. Let Σ be a strictly convex immersed hypersurface in M such that $\partial\Sigma = \Gamma$. Suppose, moreover that Σ is compatible with the convexity orientation on Γ. We denote by N_Σ the outward pointing unit normal over Σ.

First order bounds near the boundary follow from the following result:

Lemma 5.1

Choose $\theta > 0$. There exists $r > 0$, which only depends on M, Γ and θ such that if the angle between N_Σ and N^+ is always greater than θ, then, for all $p \in \Gamma$, there exists a convex subset $K \subseteq B_r(p)$ such that the connected component of $\Sigma \cap B_r(p)$ containing p is embedded and is a subset of ∂K.

Proof: This follows immediately from Proposition 5.2 (below). $\square$

We establish the framework. Choose $p \in \Gamma$. Choose $r_1 > 0$, and denote the connected component of $\Gamma \cap B_{r_1}(p)$ containing p by Γ_0. Reducing r_1 if necessary, there exists a smooth, embedded, locally strictly convex hypersurface $\hat{\Sigma} \subseteq B_{r_1}(p)$ such that $\partial\hat{\Sigma} \subseteq \partial B_{r_1}(p)$ and $\Gamma \subseteq \hat{\Sigma}$. We may suppose, moreover, that $\hat{\Sigma}$ and $B_{r_1}(p)$ together bound a convex set, $\hat{K}$. In the sequel, we will identify M with $B_{r_1}(p)$, reducing r_1 at various stages whenever necessary. We may thus assume that Γ divides $\hat{\Sigma}$ into two connected components: $\hat{\Sigma}^+$ and $\hat{\Sigma}^-$ which correspond to the interior and exterior respectively of $\hat{\Sigma}$ with respect to Γ.

Let $\mathsf{N}_{\hat{\Sigma}}$ be the unit normal vector field over $\hat{\Sigma}$. We may suppose that $\mathsf{N}_{\hat{\Sigma}}$ makes an angle of less than $\theta/2$ with N_Γ^+.

Since $\hat{\Sigma}$ is strictly convex, there exists $\epsilon > 0$ such that the shape operator of $\hat{\Sigma}$ is no less than $\epsilon\mathrm{Id}$. Let H be a strictly convex embedded hypersurface tangent to $\hat{\Sigma}$ at p whose second fundamental form is strictly bounded above by $\delta\mathrm{Id}$, for $\delta < \epsilon/2$. Let $(H_t)_{t\in]-\tau,\tau[}$ be the foliation of M by hypersurfaces equidistant to H. We may assume that each leaf of this foliation is embedded, strictly convex and complete with second fundamental form strictly bounded above by $\delta\mathrm{Id}$. Moreover, we may assume that $H_0 = H$ meets $\hat{\Sigma}$ at a single point, p. Thus, the upward pointing normal of H_0 coincides with that of $\hat{\Sigma}$ at this point.

Each leaf of $(H_t)_{t\in]-\tau,\tau[}$ divides M into two connected components, one of which we say lies above the leaf, and the other of which we say lies below the leaf. Recalling section 3, we say that a subset K of M is **semi-convex** with respect to a leaf H_t if and only if:

(i) it lies above that leaf; and

(ii) if γ is a geodesic segment lying above H_t whose endpoints are elements of K, then the whole of γ is contained in K.

Remark: Importantly, in contrast the the situation considered in Section 3, H_t is not contained in K. Semi-convexity is therefore no longer necessarily preserved by taking limits. This is a delicate point which will be discussed presently.

We extend Σ to a (piecewise smooth) convex immersed hypersurface by adjoining to it $\hat{\Sigma}^-$ and denote the resulting immersed hypersurface by $\tilde{\Sigma}$. For all t, let $\tilde{\Sigma}_t$ be the connected component of $\tilde{\Sigma}$ lying above H_t and containing p.

Lemma 5.1 follows immediately from the following proposition by taking intersections with a small ball about p:

Proposition 5.2

There exists $t_0 < 0$ (which only depends on M, Γ, $\hat{\Sigma}$, θ and r_1) such that $\tilde{\Sigma}_{t_0}$ is embedded and (along with H_{t_0}) bounds a semi-convex set.

Proof: This follows immediately from Proposition 5.8 (below). □

Let T denote the set of all $t < 0$ such that, for all $s \in]-t, 0[$:

(i) $\tilde{\Sigma}_s$ is embedded;

(ii) $\tilde{\Sigma}_s \subseteq \hat{K}$;

(iii) $\tilde{\Sigma}_s$ bounds a semi-convex set above H_s; and

(iv) $\tilde{\Sigma}_s$ intersects H_s transversally along $\partial\tilde{\Sigma}_s$.

Proposition 5.3

T is non-empty.

Proof: Since $\tilde{\Sigma}$ is a piecewise smooth, strictly convex immersion, there exists $0 < r_2 < r_1$ (which *does* depend on Σ) such that the connected component of the intersection of $\tilde{\Sigma}$ with $B_{r_2}(p)$ containing p is embedded and bounds a convex set. The portion of this convex set lying above H_t for t small is trivially semi-convex, and (i), (ii) and (iii) are therefore satisfied for all small t less than 0, likewise so is (iv), and the result follows. $\square$

Let t_0 be the infimum of T. We will obtain upper bounds for t_0, from which Proposition 5.2 will follow. The first step involves proving that $\tilde{\Sigma}_{t_0}$ is transverse to H_{t_0}. The main geometric obstacle is the possibility that the outwards pointing normal to Σ_{t_0} points upwards from H_{t_0}. This is dealt with by the following observation:

Proposition 5.4

For all $\theta' < \theta$, there exists $t_1 < 0$ (which only depends on $\hat{\Sigma}$, Γ, M, θ and θ') such that, if d is the (signed) distance function in M to $H = H_0$, and if $t_0 > t_1$, then, throughout Σ_{t_0}:

$$\langle \mathsf{N}_\Sigma, \nabla d\rangle \leqslant \cos(\theta') < 1.$$

Proof: Let ∇ and ∇^Σ denote the Levi-Civita covariant derivatives over M and Σ respectively. Define the function $\phi : \Sigma \to \mathbb{R}$ by:

$$\phi = \langle \mathsf{N}, \nabla d\rangle.$$

Let A be the shape operator of Σ. If X is a vector field over Σ, then:

$$\begin{aligned} X\phi &= \langle \nabla_X \mathsf{N}_\Sigma, \nabla d\rangle + \langle \mathsf{N}_\Sigma, \nabla_X \nabla d\rangle \\ &= \langle A\cdot X, \nabla d\rangle + \mathrm{Hess}(d)(\mathsf{N}_\Sigma, X) \\ &= \langle A\cdot X, \nabla^\Sigma d\rangle + \mathrm{Hess}(d)(\mathsf{N}_\Sigma, X). \end{aligned}$$

The final line follows since the normal component of $A \cdot X$ vanishes. Now let $X = \nabla^\Sigma d / \|\nabla^\Sigma d\|^2$. Since A is positive definite:

$$X\phi \geqslant \mathrm{Hess}(d)(\mathsf{N}_\Sigma, X).$$

Since d is the distance to a hypersurface, $\|\nabla d\| = 1$ is constant, and so $\mathrm{Hess}(d)(\nabla d, \cdot)$ vanishes. Thus, if N_0 denotes the component of N_Σ tangent to the foliation, $(H_t)_{t\in]-\tau,\tau[}$, of level subsets of d, then:

$$X\phi \geqslant \mathrm{Hess}(d)(\mathsf{N}_0, X).$$

However:

$$\|\mathsf{N}_0\|^2 = 1 - \langle \mathsf{N}_\Sigma, \nabla d\rangle^2 = \|\nabla^\Sigma d\|^2.$$

Thus $\mathsf{N}_0/\|\nabla^\Sigma d\|$ has norm equal to 1. Since the shape operator of H_t is bounded above by $\delta\mathrm{Id}$ for all t, the norm of $\mathrm{Hess}(d)$ is also bounded above by δ. Thus:

$$X\phi \geqslant -\delta.$$

However:

$$Xd = \langle X, \nabla^\Sigma d\rangle = 1.$$

Thus, if $\gamma : [0, \tau] \to \Sigma$ is an integral curve of X starting at q, then $\gamma(s)$ meets Γ for some $s \leqslant |t_0|$. There therefore exists $q' \in \Gamma$ such that:

$$\begin{array}{rll} & \phi(q') & \geqslant \phi(q) - \epsilon\,|t_0|\,/2 \\ \Rightarrow & \phi(q) & \leqslant \phi(q') + \epsilon\,|t_0|\,/2, \\ & & \leqslant \phi(q') + \epsilon\,|t_1|\,/2. \end{array}$$

Choosing t_1 sufficiently small, for all $q' \in \Gamma_{t_1}$:

$$\phi(q') + \epsilon\,|t_1|\,/2 \leqslant \cos(\theta').$$

The result follows. □

Proposition 5.5

There exists $t_1 < 0$ (which only depends on M, $\hat{\Sigma}$, θ and r_1) such that, if $t_0 > t_1$, then $\tilde{\Sigma}_{t_0}$ intersects H_{t_0} transversally along $\partial\tilde{\Sigma}_{t_0}$.

Proof: Suppose the contrary. Choose $q \in \partial\tilde{\Sigma}_{t_0}$ such that $\tilde{\Sigma}$ is tangent to H_{t_0} at q. The normal to $\tilde{\Sigma}$ at q either points downwards into H_{t_0} or upwards from H_{t_0}. By reducing r_1 if necessary, we may assume that the normal does not point upwards over $\hat{\Sigma}^- \setminus \{p\}$. By Proposition 5.4, for t_1 sufficiently small, the normal doesn't point upwards over Σ either, and it therefore does not point upwards anywhere over $\tilde{\Sigma}_{t_0}$.

We now show that the normal cannot point downwards. Since $\tilde{\Sigma}$ and H_{t_0} are strictly convex with opposing normals, they meet at a single point. For $t > t_0$, let $\tilde{\Sigma}'_t$ denote the connected component of $\tilde{\Sigma}$ lying below H_t containing q. Since $\tilde{\Sigma}$ is piecewise smooth, for t sufficiently close to t_0, $\tilde{\Sigma}'_t$ is topologically a ball whose boundary is an embedded topological sphere in H_t. Moreover, $\partial\tilde{\Sigma}'_t$ is a subset of $\partial\tilde{\Sigma}_t$. However, for all $s \in]t_0, 0[$, $\tilde{\Sigma}_s$ is transverse to H_s and does not self intersect. $\partial\tilde{\Sigma}_t$ is thus also an embedded topological sphere. It follows that $\partial\tilde{\Sigma}_t$ and $\partial\tilde{\Sigma}'_t$ coincide, and $\tilde{\Sigma}$ is therefore an embedded topological sphere lying above H_{t_0}. Γ is therefore not contained in $\tilde{\Sigma}$, which is absurd, and thus the normal to $\tilde{\Sigma}'_{t_0}$ does not point downwards, and this completes the proof. □

The next step uses the fact that K_{t_0}, being the limit of a sequence of semi-convex sets, is also semi-convex. Despite being an intuitive result, its proof is rather technical, and is deferred to Section 7.

Proposition 5.6

There exists $t_1 < 0$ (which only depends on M, $\hat{\Sigma}$, θ and t_1), such that, if $t_0 > t_1$, then $\partial\tilde{\Sigma}_{t_0}$ is embedded in H_{t_0} and bounds an open set.

Proof: For $t > t_0$, let K_t be the semi-convex set bounded by $\tilde{\Sigma}_t$ and N_t. By Proposition 7.1, K_{t_0} is also semi-convex. By the preceeding proposition, $\tilde{\Sigma}_{t_0}$ is transverse to H_{t_0} along $\partial\tilde{\Sigma}_{t_0}$. It follows that $\partial\tilde{\Sigma}_{t_0}$ is a (piecewise smooth) immersed submanifold of H_{t_0}. Suppose it is not embedded. Since $\partial\Sigma_t$ is embedded for all $t > t_0$, there exist two open submanifolds $\Sigma'_1, \Sigma'_2 \subseteq \Sigma_{t_0}$ such that:

(i) Σ'_1 and Σ'_2 are embedded; and

(ii) $\Sigma_1' \cap H_{t_0}$ and $\Sigma_2' \cap H_{t_0}$ meet tangentially at some point p.

Since $\tilde{\Sigma}_{t_0}$ bounds K_{t_0}, the hypersurfaces Σ_1' and Σ_2' divide a neighbourhood of p above H_{t_0} into three (roughly) wedge-shaped open sets. Consider the central one of these three wedges. It is either a subset of K_{t_0} or a subset of its complement. If it is a subset of K_{t_0}, then we say that $\Sigma_1' \cap H_{t_0}$ and $\Sigma_2' \cap H_{t_0}$ lie on each others interior. Otherwise, we say that they lie on each others exterior.

Suppose that $\Sigma_1' \cap H_{t_0}$ and $\Sigma_2' \cap H_{t_0}$ lie on each others interior. Let P_1 and P_2 be the respective tangent hyperplanes of Σ_1' and Σ_2' at p. We identify these with their images under the exponential map. P_1 and P_2 do not coincide. Indeed, suppose the contrary. By strict convexity, the interiors of Σ_1' and Σ_2' coincide in a single point. This point is contained in K_{t_0}. However, K_{t_0} is connected and also contains p, which is absurd and the assertion follows.

By convexity, near p, Σ_1' lies above P_1 and Σ_2' lies above P_2. However, the region lying above both P_1 and P_2 forms a wedge making an angle at p strictly greater than 0 and strictly less than π. In particular, Σ_1' and Σ_2' intersect transversally at p. They therefore also intersect over a hypersurface contained inside this wedge. However, since H_{t_0} is strictly convex, this wedge lies strictly above H_{t_0}, and therefore Σ_1' and Σ_2' also meet at some point above H_{t_0}. This contradicts the hypothesis that $\partial\tilde{\Sigma}_t$ is an embedded submanifold of H_t for all $t > t_0$. It follows that these two submanifolds do not lie on each others interior.

Suppose that $\Sigma_1' \cap H_{t_0}$ and $\Sigma_2' \cap H_{t_0}$ lie on each others exterior. Let γ be a geodesic arc, tangent to H_{t_0} at p and normal to the common tangent space of $\Sigma_1' \cap H_{t_0}$ and $\Sigma_2' \cap H_{t_0}$. Near p, γ lies above H_{t_0} and has endpoints inside K_{t_0}. Moving γ upwards slightly yields a geodesic arc lying above H_{t_0}, having endpoints inside K_{t_0} whilst itself not being contained within K_{t_0}. This contradicts semi-convexity. It follows that these two submanifolds do not lie on each others exterior, and this completes the proof. □

Proposition 5.7

There exists $t_1 < 0$ (which only depends on M, $\hat{\Sigma}$, θ and r_1) such that, if $t_0 > t_1$, then $\tilde{\Sigma}_t \subseteq \hat{K}$.

Proof: By Proposition 5.6, for t sufficiently small, Σ_t does not intersect $\hat{\Sigma}_t^-$. By Proposition 6.2, semi-convexity and the hypotheses on Σ along the boundary, for all sufficiently small t, Σ_t does not intersect $\hat{\Sigma}_t^+$. It is therefore contained within the set bounded by $\hat{\Sigma}_t$ and H_t, and the result follows. □

Proposition 5.8

There exists $t_1 < 0$ (which only depends on M, $\hat{\Sigma}$, θ and r_1) such that, if $t_0 > t_1$, then t_0 cannot be the infimum of T.

Proof: Let t_1 be as in Propositions 5.4, 5.5, 5.6 and 5.7 and suppose that $t_0 > t_1$. $\partial\tilde{\Sigma}_{t_0}$ is embedded, is transverse to H_{t_0}, and is bounded away from $B_{r_1}(p)$. Thus, for all $t < t_0$ sufficiently close to t_0, $\tilde{\Sigma}_t$ is embedded and, along with H_t bounds a subset of $B_r(p)$. For all t, let K_t be the closure of this subset.

Suppose that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ converging to t_0 such that, for all n, K_{t_n} is not semi-convex. Then, for all n, there exists $p_n, q_n \in K_n := K_{t_n}$ and a geodesic arc γ_n such that:

(i) p_n and q_n are the endpoints of γ_n;

(ii) γ_n lies above H_{t_n}; and

(iii) there exists a point $r_n \in \gamma_n$ which lies outside K_n.

Without loss of generality, $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$, $(\gamma_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ converge to p_0, q_0, γ_0 and r_0 respectively. Trivially, γ_0 lies above H_{t_0}. Suppose first that $p_0 \neq q_0$. Suppose that r_0 does not coincide with either of the endpoints. Since K_{t_0} is semi-convex, γ_0 lies inside K_{t_0}. r_0 therefore lies on the boundary of K_{t_0}, and γ_0 is therefore an interior tangent to $\tilde{\Sigma}_{t_0}$ at this point, which contradicts local strict convexity. Likewise, if r_0 coincides with an end point, p_0, say, then γ_0 is contained inside K_{t_0} and points outwards (or is tangent) to $\tilde{\Sigma}_{t_0}$ at p_0, which also contradicts local strict convexity and semi-convexity. It follows that p_0 and q_0 coincide.

If $p_0 = q_0$ is an interior point of K_{t_0}, then γ_n trivially lies inside K_{t_0} for all sufficiently large n. Suppose therefore that $p := p_0 = q_0$ is a boundary point of $\tilde{\Sigma}_{t_0}$. By local strict convexity, there exists a neighbourhood of $\tilde{\Sigma}$ about p which lies on the boundary of a convex set, X. For all n, the intersection of X with the region lying above H_{t_n} is a subset of K_n. However, for sufficiently large n, p_n and q_n both lie in X. For all such n, γ_n is contained within X and therefore within K_n, which is absurd.

There therefore exists ϵ such that, for all $t > t_0 - \epsilon$, $\tilde{\Sigma}_t$ satisfies the hypotheses defining T, and therefore $t \in T$. This is absurd and the result follows. $\square$

6 - Parabolic Limits.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let $\hat{\Sigma}^n$ be a locally strictly convex immersed hypersurface in M. Let $\Gamma \subseteq \hat{\Sigma}$ be an embedded hypersurface. Let $\epsilon > 0$ be such that the shape operator of $\hat{\Sigma}$ is everywhere bounded below by ϵ. Choose $p \in \Gamma$. Let H^n be a strictly convex embedded hypersurface of M which is an exterior tangent to $\hat{\Sigma}$ at p. Let $\delta > 0$ be such that the shape operator of H is everywhere bounded above by δ and suppose that $\delta < \epsilon/2$. For simplicity, we assume throughout the rest of this section that the shape operator of H at p is equal to $\delta\mathrm{Id}$. The general case is similar.

Let d be the signed distance function in M to H. In particular, for $q \in \hat{\Sigma}$ near p, $d(q) \leqslant 0$. For all t, let H_t be the level hypersurface at distance t from H. For small $t < 0$, let $\hat{\Sigma}_t$ and Γ_t be the connected components of $\hat{\Sigma}$ and Γ respectively lying above H_t and containing p, and let $\hat{K}_t$ denote the compact set bounded by $\hat{\Sigma}_t$ and H_t. For small t, Γ_t divides $\hat{\Sigma}_t$ into two components, which we denote by $\hat{\Sigma}_t^+$ and $\hat{\Sigma}_t^-$.

Choose $t_0 < 0$. Let $(p_n)_{n\in\mathbb{N}} \in \hat{K}_{t_0}$ be a sequence converging to p. We consider a geodesic chart for H about p, and thus identify a neighbourhood of p in H with a neighbourhood of

0 in T_pH. Let $(e_1, ..., e_n)$ be an orthonormal basis for T_pH. There exists $r > 0$ such that $\hat{\Sigma}$ is the graph of a function, f over $B_r(p)$. By Taylor's Theorem, with respect to $(e_1, ..., e_n)$:

$$f(x) = -\langle x|A|x\rangle + O(\|x\|^3),$$

where A is a positive definite matrix. With respect to these coordinates, for all n, $p_n = (q_n, s_n)$, where $q_n \in T_pH$ and $s_n < 0$. For all n, we define $\hat{f}_n : B_{r/\sqrt{|s_n|}}(p) \to]-\infty, 0[$ and $\hat{q}_n \in T_pH$ by:

$$\hat{f}_n(x) = f(\sqrt{|s_n|}x)/|s_n|, \qquad \hat{q}_n = q_n/\sqrt{|s_n|}.$$

Trivially, $(\hat{f}_n)_{n\in\mathbb{N}}$ converges in the C^∞_{loc} sense over T_pH to $\hat{f}_0$, where:

$$\hat{f}_0(x) = -\langle x|A|x\rangle.$$

Moreover, for all n, since $p_n \in \hat{K}_{s_n}$:

$$\begin{aligned} & |f(q_n)| && \leqslant |s_n| \\ \Rightarrow\quad & \text{LimSup}_{n\to\infty} \tfrac{\epsilon}{2}\|\hat{q}_n\|^2 && \leqslant \text{LimSup}_{n\to\infty} \tfrac{1}{|s_n|}|f(q_n)| \quad \leqslant 1. \end{aligned}$$

There thus exists $\hat{q}_0 \in H$ towards which $(\hat{q}_n)_{n\in\mathbb{N}}$ subconverges. In particular, $\hat{f}_0(q_0) \geqslant -1$. We call $(\hat{f}_0, \hat{q}_0)$ a **parabolic limit** of $(\hat{\Sigma}, q_n)_{n\in\mathbb{N}}$.

Likewise, if we suppose that $(e_1, ..., e_{n-1})$ is tangent to Γ at p, then, reducing r if necessary, the projection of Γ onto H is the graph of some function g, over the space spanned by $(e_1, ..., e_{n-1})$. For all n, we define $\hat{g}_n : B_{r/\sqrt{|s_n|}}(p) \to \mathbb{R}$ by:

$$\hat{g}_n(x') = g(\sqrt{|s_n|}x')/\sqrt{|s_n|}.$$

Trivially, $(\hat{g}_n)_{n\in\mathbb{N}}$ subconverges in the C^∞_{loc} sense over the space spanned by $(e_1, ..., e_{n-1})$ to $\hat{g}_0 := 0$. It follows that the parabolic limit of Γ_{s_n} is the intersection of the graph of $\hat{f}_0$ with a vertical hyperplane in $\mathbb{R}^n \times \mathbb{R}$.

For $p \in M$, we call a **geodesic hyperplane** at p an immersed hypersurface consisting of geodesics passing through p. Explicitely, $P \subseteq M$ is a geodesic hyperplane if and only if there exists a hyperplane $H \subseteq T_pM$ such that:

$$P = \{\text{Exp}(V_p) \text{ s.t. } V_p \in H\}.$$

For all n, let P_n be the geodesic hyperplane tangent to H_{s_n} at p_n. Reducing r if necessary, P_n is the graph of the function $\phi_n : B_r(q_n) \to \mathbb{R}$, where, by convexity:

$$-s_n \leqslant \phi_n(x) \leqslant -s_n + \langle x - q_n|B_n|x - q_n\rangle + O(\|x\|^3),$$

where $(B_n)_{n\in\mathbb{N}}$ converges to δId. For all n, we define $\hat{\phi}_n$ by:

$$\hat{\phi}_n(x) = \phi_n(\sqrt{|s_n|}x)/|s_n|.$$

$(\hat{\phi}_n)_{n\in\mathbb{N}}$ converges in the C^∞_{loc} sense over T_pH to $\hat{\phi}_0$ where:

$$\hat{\phi}_0(x) = \delta\|x - \hat{q}_0\|^2 - 1.$$

Thus, the parabolic limit of the geodesic hyperplanes tangent to H_{s_n} at p_n is a paraboloid on $(\hat{q}_0, -1)$. Finally, in like manner, the parabolic limit of a sequence of geodesics tangent to H_{s_n} at p_n is the intersection of this paraboloid with a vertical plane in $\mathbb{R}^n \times \mathbb{R}$.

Parabolic limits are of use in obtaining technical results concerning Σ.

Proposition 6.1

For δ sufficiently small, there exists $t_0 < 0$ (which only depends on $\hat{\Sigma}$, Γ, H and M) such that, for all $q \in K_{t_0} \setminus \{p\}$, if $t > t_0$ is such that $q \in H_t$, and if P is the geodesic hyperplane tangent to H_t at q, then:

(i) P intersects $\hat{\Sigma}_t$ transversally; and

(ii) P intersects Γ_t transverally.

Proof: (i) Suppose the contrary. Let $(p_n)_{n\in\mathbb{N}} \in \hat{K}_{t_0}$ be a sequence converging to p. For all n, let $s_n < 0$ be such that $p_n \in H_{s_n}$ and let P_n be the geodesic hyperplane tangent to H_{s_n} at p_n. Trivially, P_n intersects $\hat{\Sigma}_n$ non-trivially for all n. Suppose that, for all n, P_n is tangent to $\hat{\Sigma}_{s_n}$ at some point. It follows that the parabolic limit of $(P_n)_{n\in\mathbb{N}}$ is tangent to the parabolic limit of $(\hat{\Sigma}_{s_n})_{n\in\mathbb{N}}$ at some point. This is absurd, and the first assertion follows.

(ii) Suppose the contrary. Let $(p_n)_{n\in\mathbb{N}} \in \hat{\Sigma}$ be a sequence converging towards p. For all n, let $s_n < 0$ be such that $p_n \in H_{s_n}$, let $\Gamma_n = \Gamma_{s_n}$ and let P_n be the geodesic hyperplane tangent to H_{s_n} at p_n. We suppose that, for all n:

$$P_n \cap \Gamma_n = \emptyset.$$

The parabolic limit of P_n intersects the parabolic limit of Γ_n transversally. Thus, for sufficiently large n, $P_n \cap \Gamma_n \neq \emptyset$, which is absurd. It follows that, for t_0 sufficiently small, P intersects Γ_{t_0}. Transversality follows as in the proof of part (i), and this completes the proof. $\square$

Proposition 6.2

Choose $\theta \in]0, \pi/2[$. For δ sufficiently small, there exists $t_0 < 0$ (which only depends on $\hat{\Sigma}$, Γ, H, M and θ) such that for $t > t_0$ and for all $q \in \hat{\Sigma}_t \cap H_t$, there exists a geodesic segment, γ, joining q to Γ such that the hyperplane spanned by $\partial_t\gamma$ and $T\Gamma$ at the point of intersection of γ with Γ makes an angle strictly less than θ with $T\hat{\Sigma}$.

Proof: Suppose the contrary. Let $(p_n)_{n\in\mathbb{N}} \in \hat{\Sigma}$ be a sequence converging to p, and let $\hat{p}_0$ be its parabolic limit. For all n, let $s_n < 0$ be such that $p_n \in H_{s_n}$ and let γ_n be a geodesic segment tangent to H_{s_n} at p and terminating in Γ_{s_n}. Suppose that, for all n, the hyperplane spanned by $\partial_t\gamma_n$ and $T\Gamma_{s_n}$ at the point of intersection of γ_n with Γ_{s_n} makes an angle of at least θ with $T\hat{\Sigma}$. Let $\hat{\gamma}_0$ and $\hat{\Gamma}_0$ be the parabolic limits of $(\gamma_n)_{n\in\mathbb{N}}$ and $(\Gamma_{s_n})_{n\in\mathbb{N}}$ respectively. Then, at its point of intersection with $\hat{\Gamma}_0$, $\hat{\gamma}_0$ is tangent to the vertical hyperplane containing $\hat{\Gamma}_0$. $\hat{\gamma}_0$ is thus entirely contained in this vertical hyperplane. It follows that every parabolic limit of every sequence of geodesic segments joining $(p_n)_{n\in\mathbb{N}}$ to Γ is contained in the vertical hyperplane containing $\hat{\Gamma}_0$. When $\hat{p}_0 \notin \hat{\Gamma}_0$, this is trivially absurd. When $\hat{p}_0 \in \hat{\Gamma}_0$, there exists a parabolic limit of such geodesic segments which is normal to the hyperplane containing $\hat{\Gamma}_0$, which is also absurd. The result follows. □

Proposition 6.3

For δ sufficiently small, there exists $t_0 < 0$ (which only depends on $\hat{\Sigma}$, Γ, H and M) such that, for $t > t_0$, if γ is a geodesic segment lying in $\hat{K}_{t_0}$ such that:

(i) γ is tangent to H_t; and

(ii) the endpoints of γ both lie in Γ,

then there exists a sequence of geodesic segments $(\gamma_n)_{n\in\mathbb{N}}$ converging to γ such that, for all n:

(i) γ_n is tangent to H_t; and

(ii) the end points of γ_n lie in $\hat{\Sigma}^-_t$.

Proof: Suppose the contrary. Let $(p_n)_{n\in\mathbb{N}} \in \hat{K}_{t_0}$ be a sequence converging to p. For all n, let $s_n < 0$ be such that $p_n \in H_{s_n}$ and let γ_n be a geodesic segment tangent to H_{s_n} at p_n with both end points in Γ_{s_n}. We suppose that, for all n, there exists $\epsilon_n > 0$ such that if $q_n \in H_{s_n}$ is such that $d(q_n, p_n) < \epsilon_n$, then no geodesic segment tangent to H_{s_n} at q_n has both endpoints in $\hat{\Sigma}^-_{s_n}$. Let $\hat{\gamma}_0$, $\hat{p}_0$, $\hat{\Gamma}_0$ and $\hat{\Sigma}^-_{s_n}$ be the parabolic limits of $(\gamma_n)_{n\in\mathbb{N}}$, $(p_n)_{n\in\mathbb{N}}$, $(\Gamma_n)_{n\in\mathbb{N}}$ and $(\hat{\Sigma}^-_{s_n})_{n\in\mathbb{N}}$ respectively. Let $\hat{V}_0$ be the horizontal unit vector at $(p_0, -1)$ normal to the vertical hyperplane containing $\hat{\Gamma}_0$ and pointing towards $\hat{\Sigma}^-_0$. For all n, let V_n be a unit vector tangent to H_{s_n} at p_n and suppose that $\hat{V}_0$ is the parabolic limit of $(V_n)_{n\in\mathbb{N}}$. For all n, let $\eta_n : \mathbb{R} \to H_{s_n}$ be the geodesic in H_{s_n} such that:

$$\partial_t\eta_n(0) = V_n,$$

and let X_n be the parallel transport of $\partial_t\gamma_n(0)$ along η_n (with respect to the Levi-Civita covariant derivative of H_{s_n}). Let Exp be the exponential map of M and for all n define:

$$\phi_{n,t}(s) = \mathrm{Exp}(sX_n(t)).$$

If $\hat{X}_0$ is the unit tangent vector to $\hat{\gamma}_0$ at $\hat{p}_0$, then the parabolic limit of $(\phi_n)_{n\in\mathbb{N}}$ is $\hat{\phi}_{0,t}(s)$, where:

$$\hat{\phi}_{0,t}(s) = (s\hat{X}_0 + t\hat{V}_0, \delta s^2 - 1).$$

The intersection of this family with $\hat{\Sigma}^-_0$ is transverse to $\hat{\Gamma}_0$ at the intersection of $\hat{\gamma}_0$ with $\hat{\Gamma}_0$. Thus, for sufficiently large n, and sufficiently small t, the two endpoints of the geodesic segment $s \mapsto \phi_{n,t}(s)$ both lie in $\hat{\Sigma}^-_0$. This is absurd, and the result follows. □

7 - Semi-Convexity.

In this section we show that the property of being semi-convex is preserved after taking limits. Using the same notation as in the Section 5, we show:

Proposition 7.1

There exists $t_1 < 0$ (which only depends on M, $\hat{\Sigma}$, θ and r_1) such that, if $t_0 \geqslant t_1$, then $\tilde{\Sigma}_{t_0}$ bounds a semi-convex set above H_{t_0}.

If P is a geodesic hyperplane (see Section 6), we say that two points $q_1, q_2 \in P$ are **coaxial** if and only if they both lie on the same radial geodesic on opposite sides of p. We require the following technical result:

Lemma 7.2

Choose $\varphi > 0$. Let $K \subseteq M$ be compact. There exists $r > 0$ (which only depends on φ and K) such that, if P is a geodesic hyperplane at $p \in K$, if $q_1, q_2 \in P$ are coaxial points and if X is a Jacobi field over the geodesic joining q_1 to q_2 such that:

(i) $d(q_1, p), d(q_2, p) < r$;

(ii) $\|X(q_0)\| \leqslant 1$ and X lies strictly above TP at q_0; and

(iii) $\|X(q_1)\| = 1$ and X lies strictly above TP at q_1, making an angle of at least φ with TP at that point.

Then X lies strictly above TP at every point of the geodesic joining q_0 to q_1.

Proof: Assume the contrary. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence converging to 0. For all n, let $p_n \in K$ be a point, P_n a geodesic hyperplane at p_n, $q_{1,n}, q_{2,n}$ two coaxial points in P_n and X_n a Jacobi field over the geodesic joining $q_{1,n}$ to $q_{2,n}$ such that:

(i) $\mathrm{Max}(d(q_{1,n}, p_n), d(q_{2,n}, p_n)) = r_n$;

(ii) $\|X_n(q_{1,n})\| \leqslant 1$ and X_n lies strictly above TP_n at $q_{1,n}$; and

(iii) $\|X_n(q_{2,n})\| = 1$ and X_n lies strictly above TP_n at $q_{2,n}$, making an angle of at least φ with TP_n at this point.

Suppose, moreover, that, for all n, X_n is tangent to TP_n at some point lying between $q_{1,n}$ and $q_{2,n}$, x_n, say. By compactness, there exists $p_0 \in K$ towards which $(p_n)_{n\in\mathbb{N}}$ subconverges. Let g be the Riemannian metric of M. For all n, define $g_n = r_n^{-2} g$. The sequence of pointed manifolds $(M, g_n, p_n)_{n\in\mathbb{N}}$ converges towards $(\mathbb{R}^{n+1}, g_{\mathrm{Euc}}, 0)$ in the C^∞ Cheeger/Gromov sense, where g_{Euc} is the Euclidean metric over $\mathbb{R}^{n+1}$. For all n, P_n is also a geodesic hyperplane of (M, g_n) and so $(P_n, p_n)_{n\in\mathbb{N}}$ subconverges in the C^∞ Cheeger/Gromov sense for pointed, immersed submanifolds to a pointed, affine hyperplane $(P_0, 0)$. Likewise, there exist coaxial points $q_{1,0}, q_{2,0} \in P_0$, a Jacobi field X_0, and a point x_0 lying between $q_{1,0}$ and $q_{2,0}$ towards which $(q_{1,n})_{n\in\mathbb{N}}$, $(q_{2,n})_{n\in\mathbb{N}}$, $(r_n X_N)_{n\in\mathbb{N}}$ and x_0 subconverge respectively. Moreover:

(i) $\text{Max}(d(q_{1,0},0),d(d_{2,0},0))=1$;

(ii) $\|X_0(q_{1,0})\|\leqslant 1$ and X_0 lies (not necessarily strictly) above TP at $q_{1,0}$; and

(iii) $\|X_0(q_{2,0})\|=1$ and X_0 lies strictly above TP at $q_{2,0}$.

It follows that X_0 is not tangent to P at any point along the closed geodesic joining $q_{1,0}$ to $q_{2,0}$, except possibly at $q_{1,0}$. Moreover, if X_0 is tangent to P at $q_{1,0}$, then its derivative in the direction normal to P at this point is non vanishing. However, X_0 is tangent to TP at x_0. It follows from the first assertion that $x_0=q_{1,0}$, but then the derivative of X_0 in the direction normal to P at $q_{1,0}$ vanishes, and this contradicts the second assertion. This is absurd and the result follows. □

This lemma allows us to prove Propostion 7.1:

Proof of Proposition 7.1: Let K_{t_0} be the set bounded by $\tilde{\Sigma}_{t_0}$ and H_{t_0}. Let $\gamma:[0,1]\rightarrow M$ be a geodesic above H_{t_0} with endpoints in K_{t_0}. We aim to show that the whole of γ is contained in K_{t_0}. It suffices to consider the case where both endpoints of γ lie in $\tilde{\Sigma}_{t_0}$. The remaining cases are similar and much simpler. Recall that $\tilde{\Sigma}_{t_0}$ divides into two components, $\hat{\Sigma}^-_{t_0}$ and Σ_{t_0}. These components have different properties and we thus consider the various resulting cases seperately. Let $\hat{K}_{t_0}$ be the set bounded by $\hat{\Sigma}_{t_0}$ and H_{t_0}. We may assume that $\hat{K}_{t_0}$ is semi-convex. Since the endpoints of γ lie in $\tilde{\Sigma}_{t_0}\subseteq\hat{K}_{t_0}$, the whole of γ therefore lies in $\hat{K}_{t_0}$. Thus, by choosing t_1 sufficiently small, we may assume that γ is sufficiently short to satisfy the hypotheses of Proposition 7.1 with $\varphi=\theta/2$.

Suppose that γ lies strictly above H_{t_0}. Then there exists $\epsilon>0$ such that γ lies above $H_{t_0+\epsilon}$. Since $\tilde{\Sigma}_{t_0+\epsilon}$ is semi-convex, γ lies in $K_{t_0+\epsilon}\subseteq K_{t_0}$ and the result follows in this case. We thus assume that γ meets H_{t_0} at some point, $s\in[0,1]$.

Suppose that γ is transverse to H_{t_0} at s. Then, s is an endpoint of $[0,1]$ and, without loss of generality, $s=0$. By strict convexity of H_{t_0}, $\gamma(]0,1])$ lies strictly above H_{t_0}. Suppose that $\gamma(0)$ lies in $\hat{\Sigma}^-_{t_0}$. By Proposition 6.1, both $\hat{\Sigma}$ and Γ are transverse to H_{t_0} at this point. There thus exists a smooth curve $\eta:[0,\epsilon[\rightarrow M$ such that:

(i) $\eta(0)=\gamma(0)$;

(ii) $\partial_t\eta(0)$ is transverse to TH_{t_0};

(iii) for $s>0$, $\eta(s)$ lies strictly above H_{t_0}; and

(iv) for all s, $\eta(s)$ lies in $\hat{\Sigma}^-$.

For all $s\in[0,\epsilon[$, let γ_s be the unique geodesic joining $\eta(s)$ to $\gamma(1)$. For sufficiently small s, γ_s lies strictly above H_{t_0}. Since, for all $t>t_0$, $\tilde{\Sigma}_t$ is semi-convex, for all sufficiently small s, γ_s is contained in K_{t_0}. The result follows in this case by taking limits.

Suppose that $\gamma(0)$ lies in $\Sigma\setminus\Gamma$. By Proposition 5.4, after reducing t_1 if necessary, the outward pointing normal to Σ makes an angle of at least $\theta/2$ with H_{t_0} at $\gamma(0)$. There therefore exists a smooth curve $\eta:[0,\epsilon[\rightarrow M$ such that:

(i) $\eta(0)=\gamma(0)$;

(ii) $\partial_t\eta(0)$ is transverse to TH_{t_0};

(iii) for $s > 0$, $\eta(s)$ lies strictly above H_{t_0}; and

(iv) for all s, $\eta(s)$ lies in K_{t_0}.

For all $s \in [0, \epsilon[$, let γ_s be the unique geodesic joining $\eta(s)$ to $\gamma(1)$. For sufficiently small s, γ_s lies strictly above H_{t_0}, and the result follows in this case as before. This completes the case where γ is transverse to H_{t_0} at s, and we thus suppose that γ is tangent to H_{t_0} at s.

Let P be the geodesic hyperplane tangent to H_{t_0} at $\gamma(s)$. Suppose that $\gamma(0)$ and $\gamma(1)$ both lie in $\hat{\Sigma}^-_{t_0} \setminus \Gamma$. Since $\hat{\Sigma}$ bounds a strictly convex set, K, γ is transverse to $\hat{\Sigma}^-$ at $\gamma(0)$ and $\gamma(1)$ (for otherwise, by strict convexity, it could only intersect $\hat{\Sigma}^-$ at one point, which is absurd). Let X be a Jacobi field over the geodesic joining $\gamma(0)$ and $\gamma(1)$ such that X equals the unit upward pointing normal to P at both endpoints. By Lemma 7.2, X lies everywhere above TP. Thus, if γ_t is a geodesic variation of γ with Jacobi field X, then, for sufficiently small t, γ_t lies strictly above P and therefore also above H_{t_0}. Moreover, by transversality, for sufficiently small t, γ_t intersects $\hat{\Sigma}^-_{t_0}$ at two points near $\gamma(0)$ and $\gamma(1)$. We thus obtain a family of geodesic segments lying strictly above H_{t_0} with endpoints in $\tilde{\Sigma}_{t_0}$ converging towards γ. By semi-convexity, all these geodesic segments are contained within K_{t_0}, and thus, taking limits, γ is contained within K_{t_0}. This proves the result in this case.

Suppose that $\gamma(0)$ lies in $\hat{\Sigma}^-_{t_0} \setminus \Gamma$ and $\gamma(1)$ lies in $\Sigma \setminus \Gamma$. As before, γ is transverse to $\hat{\Sigma}$ at $\gamma(0)$. By Proposition 5.4, after reducing t_1 if necessary, the outward pointing normal to Σ makes an angle of at least $\theta/2$ with TP at $\gamma(1)$. Let X be a Jacobi field over γ such that $X(0)$ is the upward pointing normal vector over P at $\gamma(0)$ and $X(1)$ points into K_{t_0} making an angle of at least $\theta/2$ with TP at $\gamma(1)$. By Lemma 7.2, X lies everywhere above TP. Thus, if γ_t is a geodesic variation of γ with Jacobi field X, then, for sufficiently small t, γ_t lies strictly above P and therefore also above H_{t_0}. Moreover, for small t, $\gamma_t(1)$ lies inside K_{t_0}, and, by transversality, γ_t intersects $\hat{\Sigma}^-_{t_0}$ at some point near $\gamma(0)$. We thus obtain a family of geodesic segments lying strictly above H_{t_0} with endpoints in K_{t_0} converging towards γ. By semi-convexity, all these geodesic segments are contained within K_{t_0}, and thus, taking limits, γ is contained within K_{t_0}. This proves the result in this case.

Suppose that both $\gamma(0)$ and $\gamma(1)$ lie in $\Sigma \setminus \Gamma$. By Proposition 5.4, after reducing t_1 if necessary, the outward pointing normal to Σ makes an angle of at least $\theta/2$ with P at both these points. Let X be a Jacobi field over γ such that both $X(0)$ and $X(1)$ point into K_{t_0} at $\gamma(0)$ and $\gamma(1)$ respectively, making an angle of at least $\theta/2$ with TP at these points. By Lemma 7.2, X lies everywhere above TP, and the result follows in this case as before.

We now consider the case where at least one end point of γ lies on Γ. Suppose that $\gamma(0)$ lies on Γ but $\gamma(1)$ doesn't. By Proposition 6.1, Γ is transverse to P at $\gamma(0)$. Let X be a Jacobi field over γ such that $X(0)$ is tangent to Γ and points strictly upwards from P at $\gamma(0)$. If $\gamma(1)$ lies in $\hat{\Sigma}^-_{t_0}$, then we suppose that $X(1)$ is the upward pointing unit normal over P at $\gamma(1)$. If $\gamma(1)$ lies in Σ_{t_0}, then we assume that $X(1)$ points into K_{t_0} at $\gamma(1)$, making an angle of at least $\theta/2$ with TP at this point. By Lemma 7.2, X lies everywhere above TP, and the result follows in this case as before.

Finally suppose that both $\gamma(0)$ and $\gamma(1)$ lie on Γ. It follows by Proposition 6.3 that, after increasing t_1 if necessary, there exists a small deformation of γ whose end points both lie on $\hat{\Sigma}^-_{t_0} \setminus \Gamma$. We thus reduce this case to an earlier case, and this completes the proof. $\square$

8 - Immersed Boundaries.

Let M^{n+1} be an $(n+1)$-dimensional manifold. We recall that the reasoning of Section 5 is only valid when the boundary is embedded. We now show how this reasoning may be adapted by a simple modification to also treat the case where the boundary is permitted to have self intersections.

Let $\Gamma^{n-1} = (i, (G^{n-1}, \partial G^{n-1}))$ be a compact, codimension 2, immersed submanifold in M. We say that Γ is **generic** if and only if, for all $p \neq q$ such that $i(p) = i(q)$:

$$T_p\Gamma \neq T_q\Gamma.$$

This definition is motivated by the following elementary result:

Proposition 8.1

(i) Let $\Gamma \subseteq M$ be a compact, codimension 2, immersed submanifold. There exists a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of generic, compact, codimension 2, immersed submanifolds which converges to Γ in the C^∞ sense.

(ii) Let $(\Gamma_t)_{t\in[0,1]} \subseteq M$ be a smooth family of compact, codimension 2, immersed submanifolds such that Γ_0 and Γ_1 are generic. There exists a sequence $(\Gamma_{n,t})_{n\in\mathbb{N}}$ of smooth families of generic, compact, codimension 2, immersed submanifolds such that:

(a) for all n, $\Gamma_{n,0} = \Gamma_0$ and $\Gamma_{n,1} = \Gamma_1$; and

(b) $(\Gamma_{n,t})_{n\in\mathbb{N}}$ converges to (Γ_t) in the C^∞ sense.

Proof: This follows from Sard's Lemma in the usual manner. Explicitely, a generic codimension 2 immersion self-intersects over a submanifold of codimension 4, from which (i) follows, and every immersion in a generic isotopy of codimension 2 immersions self-intersects over a submanifold of codimension 3, from which (ii) follows. See [7] for details. $\square$

Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of strictly convex, codimension 2, immersed submanifolds with convexity orientation. For all $n \in \mathbb{N}\cup\{0\}$, let N^+_n be the convexity coorientation of Γ_n. Suppose that $(\Gamma_n)_{n\in\mathbb{N}}$ converges in the C^∞ sense to a strictly convex, codimension 2, immersed submanifold, Γ_0 and suppose, moreover, that Γ_0 is generic. In particular, by taking a subsequence, we may suppose that Γ_n is also generic for all n.

Lemma 8.2

Choose $\theta > 0$. There exists $r > 0$ such that if $(\Sigma_n)_{n\in\mathbb{N}}$ is a sequence of strictly convex, immersed hypersurfaces such that, for all n:

(i) $\partial\Sigma_n = \Gamma_n$; and

(ii) the outward pointing unit normal over Σ_n makes an angle of at least θ with N_n^+ along Γ_n,

then, for all n, and for all $p \in \Gamma_n$:

(i) the connected component of $\Sigma_n \cap B_r(p)$ is embedded and lies on the boundary of a convex subset of $B_r(p)$; and

(ii) this connected component only meets one connected component of $\Gamma_n \cap B_r(p)$.

Remark: Using this result in conjunction with the compactness of the family of bounded convex sets, we obtain $C^{0,\alpha}$ compactness near the boundary for families of locally convex immersed hypersurfaces. In particular, this result may be used to extend the conclusions of [11] to the case of compact hypersurfaces with non-trivial boundary (see [13]).

Proof: For all $n \in \mathbb{N} \cup \{0\}$, choose $p_n \in \Gamma_n$ and suppose that $(p_n)_{n\in\mathbb{N}}$ converges to p_0. For all $n \in \mathbb{N} \cup \{0\}$, let $q_n \in M$ be the image of p_n. Choose $r > 0$ such that, for all $n \in \mathbb{N} \cup \{0\}$, the connected component of $\Gamma_n \cap B_r(q_n)$ containing p_n is embedded, and denote this component by $\Gamma_{n,0}$. For all n, we identify M with $B_r(p_n)$, reducing r whenever necessary.

As in Section 5, for all $n \in \mathbb{N} \cup \{0\}$, let H_n be a strictly convex, embedded hypersurface tangent to Γ_n at p_n such that:

(i) the outward pointing normal to H_n at p_n makes an angle of no more than $\theta/2$ with N_n^+ at p_n; and

(ii) the shape operator of H_n is everywhere strictly above by $\delta\mathrm{Id}$, where δ is small.

We suppose, moreover, that $(H_n)_{n\in\mathbb{N}}$ converges to H_0 in the C^∞ sense. Likewise, as in Section 5, for all $n \in \mathbb{N}$, we extend H_n to a foliation $(H_{n,t})_{t\in\mathbb{R}}$.

Since Γ_0 is generic, we may suppose that H_0 is transverse at q_0 to every connected component of $\Gamma_0 \cap B_r(q_0)$ not equal to $\Gamma_{0,0}$ which passes through q_0. Thus, reducing r if necessary, for all n, if $\Gamma'_{n,0}$ is a connected component of $\Gamma_n \cap B_r(q_n)$ which is different from $\Gamma_{n,0}$, then $\Gamma'_{n,0}$ is transverse to $H_{n,t}$, for all t.

Let $t_0 < 0$ be as in Section 5, and, for all $t \in]t_0, 0[$, let $\Sigma_{n,t}$ be the connected component of Σ_n containing p_n which lies above $H_{n,t}$. Define T to be the set of all $t \in]-t_0, 0[$ such that $\Gamma_{n,0}$ is the only connected component of $\Gamma_n \cap B_r(q_n)$ which intersects $\Sigma_{n,t}$. Trivially, T is non-empty. Let $t_1 = \mathrm{Inf}\, T$ and suppose that $t_1 > t_0$. Let $\Gamma'_{n,0} \neq \Gamma_{n,0}$ be the connected component of $\Gamma_n \cap B_r(q_n)$ which intersects Σ_{n,t_1}. For $t > t_1$, the reasoning of Section 5 proceeds as in the case where the boundary is embedded, and it follows that Σ_{n,t_1} is embedded, is transverse to H_{t_1} and bounds a semi-convex set above H_{t_1}. $\Gamma'_{n,0}$ is therefore tangent to H_{n,t_1} at the point of intersection, since, otherwise $\Gamma'_{n,0}$ would intersect $\Sigma_{n,t}$ non trivially at some point lying above H_{n,t_1}, which is absurd. However, this contradicts the definition of r. It follows that $t_1 = t_0$, and the result now follows as in the case of Lemma 5.1 by taking intersections with a ball of radius less than t_0. $\square$

9 - First Order Lower Bounds.

Let M^{n+1} be an $(n+1)$-dimensional Riemannian manifold. Let $\Gamma^{n-1} \subseteq M$ be a generic, strictly convex, codimension 2, immersed submanifold with convexity orientation. Let A_Γ be the shape operator of Γ and let N^- and N^+ be the convexity orientation and coorientation respectively of Γ. As in [2], second order bounds require uniform lower bounds on the angle between N^- and the normal to any hypersurface of constant Gaussian curvature with boundary equal to Γ. This is guaranteed by the following result:

Proposition 9.1

For all $k > 0$, there exists $\phi > 0$ (which only depends on M, Γ and θ) such that if $(\Sigma^n, \partial\Sigma^n)$ is a smooth, convex immersed hypersurface such that:

(i) $\partial\Sigma = \Gamma$;

(ii) the Gaussian curvature of Σ is at least k; and

(iii) the outward pointing normal to Σ over Γ makes an angle of at least θ with $\mathsf{N}^+(p)$,

then the outward pointing normal to Σ over Γ also makes an angle of at least ϕ with $\mathsf{N}^-(p)$.

Let $r > 0$ and let Σ be a $C^{0,1}$ locally convex hypersurface in M such that:

(i) $\partial\Sigma \subseteq \Gamma \cup B_r(p)$;

(ii) Σ is compatible with the orientation on Γ; and

(iii) $\mathsf{N}^-(p)$ coincides with the outward pointing normal of Σ at p.

Let $\mathrm{Symm}(\mathbb{R}^n)$ denote the set of positive definite, symmetric matrices over $\mathbb{R}^n$. For $t > 0$, we define $F_t \subseteq \mathrm{Symm}(\mathbb{R}^n)$ by:

$$F_t = \{A \in \mathrm{Symm}(\mathbb{R}^n) \text{ s.t. } A \geqslant 0 \ \& \ \mathrm{Det}(A) \geqslant t\}.$$

Observe that if $A \in F_t$ and if $M \geqslant 0$, then $A + M \in F_t$. In the language of [1], this implies that F_t is a Dirichlet set. In particular, if $A \notin F_t$ and $M \geqslant 0$, then $A - M \notin F_t$. Proposition 9.1 is proven using barriers, which are constructed using the following result:

Proposition 9.2

Choose $\delta > 0$. There exists a neighbourhood U of p and a smooth function $f : U \to \mathbb{R}$ such that:

(i) $f \geqslant 0$ along $\partial(U \cap \Sigma)$;

(ii) there exists $q \in U \cap \Sigma$ such that $f(q) < 0$; and

(iii) for all $q \in B_r(p)$, the shape operator of the level subset of f passing through q with respect to ∇f is conjugate to an element of F_δ^c.

Let S be a smooth, immersed hypersurface in M such that:

(i) $\partial S = \partial\Gamma$;

(ii) the upward pointing normal to S at p is equal to $\mathsf{N}^-(p)$; and

(iii) the shape operator of S at p is supported along the subspace $T_p\Gamma$.

Let H be a strictly concave immersed hypersurface in M such that:

(i) the upward pointing normal to H at p lies in the interior of $-X_p$; and

(ii) Γ, Σ and S locally lie strictly above H.

Let d_p, d_S and d_H denote the (signed) distance in M to p, S and H respectively. Observe that $(\nabla d_S, \nabla d_H)$ is a linearly independant pair which spans the space of normal vectors to Γ at p. For any two functions, f and g, we define the $(n-2)$-dimensional distribution, $E(f,g)$, near p by:

$$E(f,g) = \langle \nabla f, \nabla g\rangle^\perp,$$

where $\langle U, V\rangle$ here represents the subspace spanned by the vectors U and V. Let $e_1, ..., e_{n-1}$ be an orthonormal basis for $T_p\Gamma$ with respect to which $A_\Gamma(\mathsf{N}^-)$ is diagonal. Let $\lambda_1, ..., \lambda_{n-1}$ be the corresponding eigenvalues. We may suppose that $0 = \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_{n-1}$. We extend $(e_1, ..., e_{n-1})$ to a local frame in TM such that, for all vectors, X, at p:

$$\langle \nabla_X e_i, \nabla d_S\rangle = -\mathrm{Hess}(d_S)(X, e_i),$$
$$\langle \nabla_X e_i, \nabla d_H\rangle = -\mathrm{Hess}(d_H)(X, e_i).$$

Define the distribution E near P to be the span of $e_1, ..., e_{n-1}$.

Proposition 9.3

If D represents the Grassmannian distance between two $(n-1)$-dimensional subspaces, then:

$$D(E, E(d_S, d_H)) = O(d_p^2).$$

Proof: By definition of e_i, for all vectors X at p:

$$X\langle e_i, \nabla d_S\rangle = X\langle e_i, \nabla d_H\rangle = 0.$$

The result follows. $\square$

For any smooth function, f, we define $D(f, E)$ by:

$$D(f, E) = \mathrm{Det}(\mathrm{Hess}(f)|_E),$$

where $\mathrm{Hess}(f)|_E$ is the restriction of the Hessian of f to E.

Proposition 9.4

Let f be such that $f(p), \nabla f(p) = 0$ and the restriction of $\mathrm{Hess}(f)$ to H at p is positive definite. There exists a function x such that $x(p), \mathrm{Hess}(x)(p) = 0$ and:

$$D(d_S + x(d_H - f), E) = O(d_p)^2.$$

Proof: The Hessian of xf vanishes at p. Likewise, the Hessian of the second order term xd_H vanishes over $(\nabla d_H)^\perp$ and therefore over E at p. It follows that the term $x(d_H - f)$ does not affect the restriction of the Hessian of the function to E at p. Thus:

$$\nabla D = \mathrm{Tr}(\mathrm{Adj}(\mathrm{Hess}(d_S)|_E)\nabla(\mathrm{Hess}(d_S + x(d_H - f))(e_i, e_j))),$$

where $\mathrm{Adj}(\mathrm{Hess}(d_S)|_E)$ is the adjugate matrix of $\mathrm{Hess}(d_S)|_E$. If more than one of the eigenvalues of $\mathrm{Hess}(d_S)|_E$ vanishes, then $\mathrm{Adj}(\mathrm{Hess}(d_S)|_E)$ also vanishes, and the result follows trivially by taking $x = 0$. Suppose therefore that only one eigenvalue of $\mathrm{Hess}(d_S)|_E$ vanishes. Let $\mu_1, ..., \mu_{n-1}$ be the eigenvalues of the adjugate matrix, then $\mu_1 = \lambda_2 ... \lambda_{n-1}$ and $\mu_2 = ... = \mu_{n-1} = 0$. Define the vectors U and V at p by:

$$\begin{aligned} U &= \nabla D(d_S, E), \\ V &= \nabla D(d_S + x(d_H - f), E). \end{aligned}$$

Denote $P = x(d_H - f)$. At p:

$$\mathrm{Hess}(P) = \nabla x \otimes \nabla d_H + \nabla d_H \otimes \nabla x.$$

At p, for all i, by definition, $\langle e_i, \nabla d_H\rangle = 0$. Thus, recalling the formula for ∇e_i:

$$\begin{aligned} X\mathrm{Hess}(P)(e_i, e_j) &= (\nabla_X \mathrm{Hess}(P))(e_i, e_j) + \mathrm{Hess}(P)(\nabla_X e_i, e_j) + \mathrm{Hess}(P)(e_i, \nabla_X e_j) \\ &= (\nabla_X \mathrm{Hess}(P))(e_i, e_j) \\ &\qquad + \langle \nabla x, e_j\rangle\langle \nabla_X e_i, \nabla d_H\rangle + \langle \nabla x, e_i\rangle\langle \nabla_X e_j, \nabla d_H\rangle \\ &= (\nabla_X \mathrm{Hess}(P))(e_i, e_j) - \mathrm{Hess}(d_H)(X, e_i)x_{;j} - \mathrm{Hess}(d_H)(X, e_j)x_{;i}. \end{aligned}$$

We extend $(e_i)_{1\leqslant i\leqslant n-1}$ to an orthonormal basis $(e_i)_{0\leqslant i\leqslant n}$ for T_pM. With respect to this basis, for all k:

$$\frac{1}{\mu_1}\langle V - U, e_k\rangle = (d_{H;11} - f_{;11})x_{;k} - 2f_{;1k}x_{;1}.$$

Consider the linear map, M, given by:

$$(M\xi)_{;k} = (d_{H;11} - f_{;11})\xi_k - 2f_{;1k}\xi_1.$$

Suppose that $M\xi = 0$. Then, in particular, bearing in mind that $d_{H;11} \leqslant 0$ and $f_{;11} \geqslant 0$:

$$\begin{aligned} & (d_{H;11} - 3f_{;11})\xi_1 &&= 0 \\ \Rightarrow\ & \xi_1 &&= 0 \\ \Rightarrow\ & \xi &&= 0. \end{aligned}$$

M is therefore invertible, and there exists ξ such that:

$$M\xi = -U.$$

If we define x such that:

$$x(p) = 0, \qquad \nabla x(p) = \xi, \qquad \mathrm{Hess}(x)(p) = 0,$$

then:

$$\nabla D(d_S + x(d_H - f), E) = 0.$$

This completes the proof. □

Define Φ_0 by:

$$\Phi_0 = d_S + x(d_H - f).$$

For $M > 0$, define Φ by:

$$\Phi = d_S + x(d_H - f) + M d_H^2.$$

Proposition 9.5

If D represents the Grassmannian distance between two $(n-2)$-dimensional subspaces then:

$$D(E(d_S, d_H), E(\Phi, d_H)) = O(d_p^2) + O(d_H).$$

Proof: Since xf is of order 3 at p:

$$\nabla\Phi = \nabla d_S + (x + 2Md_H)\nabla d_H + O(d_p^2) + O(d_H).$$

Thus:

$$\langle\nabla\Phi, \nabla d_H\rangle = \langle\nabla d_S + O(d_p^2) + O(d_H), \nabla d_H\rangle,$$

where $\langle\cdot,\cdot\rangle$ here represents the subspace generated by two vectors. The result follows. □

Corollary 9.6

If D represents the Grassmannian distance between two $(n-2)$-dimensional subspaces, then:

$$D(E, E(\Phi, d_H)) = O(d_p^2) + O(d_H).$$

Proof: This follows from the triangle inequality and Proposition 9.3. □

We now prove Proposition 9.2:

Proof of Proposition 9.2: For $\epsilon > 0$, define the open set $U_\epsilon \subseteq M$ by:

$$U_\epsilon = \left\{p \in M \text{ s.t. } d_p(x) < \epsilon \text{ and } d_H(x) < \epsilon^2\right\}.$$

$\partial(\Sigma\cap U_\epsilon)$ consists of two components: $\partial\Sigma\cap U_\epsilon = \Gamma\cap U_\epsilon$ and $\partial U_\epsilon\cap\Sigma$. We first obtain lower estimates for Φ_0 along these two components.

We choose f such that, along Γ, $(f - d_H) = O(d_p^3)$. Consequently, $x(f - d_H) = O(d_p^4)$ along Γ. Thus, since $O(d_p^2) = O(d_H)$ along Γ, and since d_S vanishes along Γ, there exists $K_1 > 0$ such that, along Γ:

$$|d_S + x(d_H - f)| \leqslant K_1 d_H^2.$$

This yields lower bounds for Φ_0 along $\partial\Sigma\cap U_\epsilon$.

Since Σ is a convex immersion, and since $\partial\Sigma = \Gamma$ is smooth, Σ has a unique supporting normal at p, which concides with ∇d_S. Thus, by the continuity of supporting normals to convex sets, there exists an increasing, continuous function $\delta : [0,\infty[\rightarrow [0,\infty[$ such that:

(i) $\delta(0) = 0$; and

(ii) for all $q \in \Sigma$:

$$\|\mathsf{N}_\Sigma(q) - \nabla d_S(q)\| \leqslant \delta(d_p(q)).$$

Moreover d_S vanishes along Γ. There thus exists K_2 which only depends on the angle between T_pH and T_pS such that throughout $\Sigma\cap U_\epsilon$, for sufficiently small ϵ:

$$\begin{aligned} |d_S(q)| &\leqslant \delta(\epsilon)d_\Sigma(q,\partial\Sigma) \\ &\leqslant K_2\delta(\epsilon)d(q,H) \\ &\leqslant K_2\delta(\epsilon)\epsilon^2. \end{aligned}$$

Moreover, along $\Sigma\cap\partial U_\epsilon$:

$$\begin{aligned} |xf| &= O(d_p^3) &&= O(\epsilon^3), \\ |xd_H| &= O(d_p)O(d_H) &&= O(\epsilon^3). \end{aligned}$$

There thus exists $\delta_1 > 0$ such that, along $\Sigma\cap\partial U_\epsilon$:

$$|\Phi_0| < \delta_1 d_H.$$

Moreover, δ_1 tends to 0 as ϵ tends to 0.

Thus, if we choose $M = \mathrm{Max}(\delta_1\epsilon^{-2}, K_1)$, then $\Phi \geqslant 0$ along $\partial(\Sigma\cap U)$. Since $\mathrm{Hess}(\Phi_0)$ is bounded, by Proposition 9.4 and Corollary 9.6:

$$D(\Phi_0, E(\Phi, d_H)) = O(\epsilon^2).$$

However:

$$\mathrm{Hess}(\Phi) = \mathrm{Hess}(\Phi_0) + 2M\nabla d_H\otimes\nabla d_H + 2Md_H\mathrm{Hess}(d_H).$$

Since $M\epsilon^2$ tends to 0 as ϵ tends to 0, and since $\|\nabla\Phi\|$ remains uniformly bounded away from 0, for sufficiently small ϵ:

$$\begin{aligned} &\|\nabla\Phi\|^{-n}\mathrm{Det}((\mathrm{Hess}(\Phi_0) + 2M\nabla d_H\otimes\nabla d_H)|_{\nabla\Phi^\perp}) &&< \delta \\ \Rightarrow\ &\tfrac{1}{\|\nabla\Phi\|}(\mathrm{Hess}(\Phi_0) + 2M\nabla d_H\otimes\nabla d_H)|_{\nabla\Phi^\perp} &&\notin F_\delta. \end{aligned}$$

However, since H is concave, $2Md_H\mathrm{Hess}(d_H)$ is negative definite, and therefore:

$$\frac{1}{\|\nabla\Phi\|}\mathrm{Hess}(\Phi)|_{\nabla\Phi^\perp} \notin F_\delta.$$

Property (iii) now follows. Since f is non-negative over $\partial(\Sigma\cap U_\epsilon)$, property (i) also follows. Since $f(p) = 0$ and $(\nabla f)(p) = \mathsf{N}^+(p)$, deforming f slightly yields a function which still satisfies conditions (i) and (iii) but also satisfies condition (ii). This completes the proof. □

We now obtain Proposition 9.1:

Proof of Proposition 9.1: Assume the contrary. Let $(\Sigma_n, \partial\Sigma_n)_{n\in\mathbb{N}}$ be a sequence of convex immersed hypersurfaces such that:

(i) $\partial\Sigma_n = \Gamma$; and

(ii) the Gaussian curvature of Σ is at least k.

Suppose, moreover, that there exists $(p_n)_{n\in\mathbb{N}}, p_0 \in \Gamma$ such that $(p_n)_{n\in\mathbb{N}}$ converges to p_0 and the angle that the exterior normal of Σ_n makes with $\mathsf{N}^-(p_n)$ at p_n tends to 0.

By Lemma 8.2, there exists $r > 0$ such that, for all n, the connected component of $\Sigma_n \cap B_r(p_n)$ containing p_n is embedded and bounds a convex set. For all n, we denote this connected component by $\Sigma_{n,0}$. By compactness of the family of convex sets, there exists a convex immersion Σ_0 to which $(\Sigma_{n,0})_{n\in\mathbb{N}}$ converges in the $C^{0,\alpha}$ sense for all α. Let f be as in Proposition 9.2 with $\delta < k$. For sufficiently large n, f achieves a strict local minimum at some point $q_n \in \Sigma_{n,0}$.

Let $\mathrm{Hess}^0(f)$ be the Hessian of f over M, and, for all n, let $\mathrm{Hess}^n(f)$ be the Hessian of the restriction of f to Σ_n. At q_n:

$$\mathrm{Hess}^n(f) = \mathrm{Hess}^0(f)|_{\nabla f^\perp} - \|\nabla f\| A_{n,0},$$

where $A_{n,0}$ is the shape operator of $\Sigma_{n,0}$ at q_n. By the Maximum Principal, at q_n:

$$\begin{array}{rll} & \mathrm{Hess}^0(f)|_{\nabla f^\perp} - \|\nabla f\| A_{n,0} & \geqslant 0 \\ \Rightarrow & \mathrm{Hess}^0(f)|_{\nabla f^\perp} & \geqslant \|\nabla f\| A_{n,0} \\ \Rightarrow & \frac{1}{\|\nabla f\|}\mathrm{Hess}^0(f)|_{\nabla f^\perp} & \in F_k. \end{array}$$

This is absurd by definition of f, and the result follows. □

10 - Compactness.

Let M^{n+1} be a Hadamard manifold. Let $(\Gamma_m^{n-1})_{m\in\mathbb{N}}, \Gamma_0^{n-1} \subseteq M$ be generic, locally strictly convex, codimension 2, immersed submanifolds with convexity orientation such that $(\Gamma_m)_{m\in\mathbb{N}}$ converges to Γ_0. For all m, let N_m^- and N_m^+ be the convexity orientation and coorientation respectively of Γ_m. Let $(\phi_m)_{m\in\mathbb{N}}, \phi_0 : M \to]0,\infty[$ be smooth, positive functions such that $(\phi_m)_{m\in\mathbb{N}}$ converges to ϕ_0 in the C^∞_{loc} sense. Let $(\Sigma_m^n)_{m\in\mathbb{N}} \subseteq M$ be smooth, immersed, strictly convex, compact hypersurfaces such that, for all m:

(i) $\partial\Sigma_m = \Gamma_m$;

(ii) Σ_m is compatible with the orientation of Γ_m; and

(iii) the Gaussian curvature of Σ_m at any point $p \in \Sigma_m$ is equal to $\phi_m(p)$.

We obtain the folllowing precompactness result:

Lemma 10.1

Let $\theta \in]0, \pi[$ be an angle and let $D > 0$ be a positive real number. Suppose that, for all m:

(i) the outward pointing normal to Σ_m makes an angle of at least θ with N_m^+ at every point of Γ_m; and

(ii) the diameter of Σ_m is no greater than D.

Then there exists a strictly convex, immersed hypersurface, $(\Sigma_0, \partial\Sigma_0) \subseteq M$ towards which $(\Sigma_m)_{m\in\mathbb{N}}$ subconverges. Moreover:

(i) $\partial\Sigma_0 = \Gamma_0$; and

(ii) the Gaussian curvature of Σ_0 at any point $p \in \Sigma_0$ is equal to $\phi_0(p)$.

Proof: By the Arzela-Ascoli Theorem of [12], it suffices to obtain a-priori bounds for all the derivatives of the shape operators of the hypersurfaces $(\Sigma_m)_{m\in\mathbb{N}}$. For all m, let A_m be the shape operator of Σ_m. Let $(p_m)_{m\in\mathbb{N}}, p_0$ be points such that:

(i) for all m, $p_m \in \Gamma_m$; and

(ii) $(p_m)_{m\in\mathbb{N}}$ converges to p_0.

Choose $\epsilon > 0$. There exists $r_1 > 0$ and, for all m, a smooth, embedded, strictly locally convex hypersurface $\hat{\Sigma}_m$ such that:

(i) $p_m \in \hat{\Sigma}_m$;

(ii) $\hat{\Sigma}_m$ is complete with respect to $B_{r_1}(p_m)$, and along with $\partial B_{r_1}(p_m)$ bounds a convex set;

(iii) the connected component of $\Gamma_m \cap B_{r_1}(p_m)$ containing p_m, which we denote by $\Gamma_{m,0}$, is itself contained in $\hat{\Sigma}_m$;

(iv) the outward pointing normal over $\hat{\Sigma}_m$ makes an angle of no more than $\theta/2$ with N_m^+ along $\Gamma_{m,0}$; and

(v) the Gaussian curvature of $\hat{\Sigma}_m$ at the point q is at least $\phi_m(q) + \epsilon$.

Moreover, we may assume that $(\hat{\Sigma}_m)_{m\in\mathbb{N}}$ converges towards $\hat{\Sigma}_0$.

By Lemma 8.2, reducing r_1 if necessary we may assume that, for all m, the connected component of the intersection of Σ_m with $B_{r_1}(p_m)$ containing p_m, which we denote by $\Sigma_{m,0}$, is embedded and lies on the boundary of a convex set, K_m. By compactness of the family of compact sets, there exists a convex set K_0 to which $(K_m)_{m\in\mathbb{N}}$ converges in the Haussdorf sense. The angle that the normal to K_0 makes with $T\Sigma_0$ at p_0 is strictly less than π. Thus, for all m, Σ_m is a graph over some (almost) fixed hypersurface over a uniform radius about p: formally, reducing r_1 further if necessary, for all m, there exists a smooth embedded hypersurface $S_m \subseteq M$ and an open subset $\Omega_m \subseteq S_m$ with smooth boundary such that:

(i) $p_m \in S_m$ and S_m is complete with respect to $B_{r_1}(p_m)$;

(ii) the shape operator of S_m vanishes at p_m;

(iii) Γ_m is a graph over $\partial\Omega_m$; and

(iv) $\Sigma_{m,0}$ and $\hat{\Sigma}_m$ are graphs of functions f_m and $\hat{f}_m$ respectively over Ω_m such that $\hat{f}_m \geqslant f_m$.

Moreover, we may suppose that $(S_m)_{m\in\mathbb{N}}$ converges to S_0 and that $(\hat{f}_m)_{m\in\mathbb{N}}$ converges to $\hat{f}_0$ in the C^∞_{loc} sense. Using this construction in conjunction with Proposition 5.1 of [10] and Proposition 9.1, we obtain $K_1 > 0$ such that, for all m and for all $p \in \Gamma_m$:

$$\|A_m(p)\| \leqslant K_1.$$

Since the diameter of Σ_m is uniformly bounded above, by Proposition 6.1 of [10], we obtain $K_2 > 0$ such that, for all m, and for all $p \in \Sigma_m$:

$$\|A_m(p)\| \leqslant K_2.$$

Again, using the above construction along with Theorem 1 of [3], we show that there exists $\epsilon > 0$ and uniform $C^{0,\alpha}$ bounds for $(A_m)_{m\in\mathbb{N}}$. The Schauder estimates then yield uniform C^k bounds for $(A_m)_{m\in\mathbb{N}}$. The result now follows by the Arzela-Ascoli Theorem of [12]. □

Let $(\hat{\Sigma}_m)_{m\in\mathbb{N}}, \Sigma_0 \subseteq M$ be locally strictly convex, immersed hypersurfaces in M with generic boundaries such that $(\hat{\Sigma}_m)_{m\in\mathbb{N}}$ converges to Σ_0. Let $(\phi_m)_{m\in\mathbb{N}}, \phi_0 : M \to]0,\infty[$ be smooth, positive functions such that $(\phi_m)_{m\in\mathbb{N}}$ converges to ϕ_0 in the C^∞_{loc} sense.

Lemma 10.1 can be refined to the following result:

Lemma 10.2

Let $(\Sigma_m)_{m\in\mathbb{N}}$ be strictly convex immersed hypersurfaces in M such that, for all m:

(i) Σ_m is contained by $\hat{\Sigma}_m$; and

(ii) for all $p \in \Sigma_m$, the Gaussian curvature of Σ_m at p is equal to $\phi_m(p)$.

There exists a strictly convex immersed hypersurface, Σ_0 in M to which $(\Sigma_m)_{m\in\mathbb{N}}$ subconverges. Moreover:

(i) Σ_0 is contained by $\hat{\Sigma}_0$; and

(ii) for all $p \in \Sigma_0$, the Gaussian curvature of Σ_0 at p is equal to $\phi_0(p)$.

Proof: Since $(\hat{\Sigma}_m)_{m\in\mathbb{N}}$ converges to $\hat{\Sigma}_0$, there exists $D > 0$ such that, for all m, the diameter of $\hat{\Sigma}_m$ is bounded above D. Likewise, for all m, $\Gamma_m := \partial\hat{\Sigma}_m$ is locally strictly convex and, if N^-_m and N^+_m denote the convexity orientation and coorientation respectively of Γ_m, then there exists $\theta > 0$ such that the angle that the outward pointing unit normal to $\hat{\Sigma}_m$ makes with N^+_m along Γ_m is everywhere bounded below by θ.

For all m, let $\pi_m : \hat{\Sigma}_m \to \Sigma_m$ be the canonical projection. Since M has non-positive curvature, for all m, π_m is distance decreasing, and the diameter of Σ_m is thus bounded above by D. Moreover, for all m, since $\hat{\Sigma}_m$ contains Σ_m, the angle that the outward pointing unit normal to Σ_m makes with N^+_m along Γ_m is everywhere bounded below by θ. It follows by Lemma 10.1 that there exists a strictly convex immersed hypersurface, Σ_0 towards which $(\Sigma_m)_{m\in\mathbb{N}}$ subconverges such that, for all $p \in \Sigma_0$, the Gaussian curvature of Σ_0 at p is equal to $\phi_0(p)$. By Lemma 3.2, $\hat{\Sigma}_0$ contains Σ_0 and this completes the proof. □

11 - Local Deformation.

Let M^{n+1} be a Hadamard manifold. Let $(\hat{\Sigma}_t)_{t\in[0,1]}$ be a smooth family of locally convex immersed hypersurfaces in M with generic boundary. For all t, denote $\Gamma_t = \partial\hat{\Sigma}_t$. Let $\epsilon > 0$ and let $(\phi_t)_{t\in[0,1]} \in C^\infty(M,]0,\infty[)$ be a smooth family such that, for all t, the Gaussian curvature of $\hat{\Sigma}_t$ is everywhere greater than $\phi_t + \epsilon$.

For all $t \in [0,1]$ let $\mathcal{M}_t$ be as in Section 2 and let $\mathcal{N}_t$ be the family of (equivalence classes) of convex immersed hypersurfaces, $[\Sigma]$ in M such that $\partial\Sigma = \partial\hat{\Sigma}_t$ and Σ is strictly contained by $\hat{\Sigma}_t$. By Lemma 3.3, $\mathcal{N}_t$ is an open subset of $\mathcal{M}_t$ and is therefore interpreted as a smooth Banach manifold. Let $\mathcal{M}$ be as in Section 2 and let $\mathcal{N}$ be the family of all pairs $(t,[\Sigma])$ where $t \in [0,1]$ and $[\Sigma] \in \mathcal{N}_t$. $\mathcal{N}$ is likewise an open subset of $\mathcal{M}$.

Let $X_0 \subseteq \mathcal{N}$ be the set of all pairs $(t,[\Sigma])$ in $\mathcal{N}$ such that the Gaussian curvature of Σ is equal to ϕ_t. By Lemma 10.2, X_0 is compact. Let $P = (t_0,[\Sigma])$ be a point in X_0, where $\Sigma = (i,(S,\partial S))$. Let $(i_t)_{t\in]t_0-\epsilon,t_0+\epsilon[}$ be a smooth family of immersions such that $i_0 = i$ and, for all t, $\Gamma_t = (i_t,\partial S)$. We define the family $(\Sigma_t)_{t\in]t_0-\epsilon,t_0+\epsilon[}$ by:

$$\Sigma_s = (i_s,(S,\partial S)).$$

Let (U_P,V_P,Φ_P) be the resulting graph neighbourhood of $\mathcal{N}$ about Σ.

Consider the Gauss curvature mapping K. This is a smooth section of $\mathcal{E}$. If we identify $T_P\mathcal{N}_t$ with $C_0^\infty(S)$, then its covariant derivative, ∇K, defines a mapping from $C_0^\infty(S)$ to $C^\infty(S)$. By Corollary 2.2, ∇K is a second order elliptic linear differential operator. It is therefore Fredholm. Since it maps from $C_0^\infty(S)$ to $C^\infty(S)$, it is of index 0. There therefore exists a finite dimensional vector subspace $E \subseteq C^\infty(S)$ such that if M is defined by:

$$M : E \oplus C_0^\infty(S) \to C^\infty(S); (f,\phi) \mapsto \nabla K\cdot\phi + f,$$

then M is surjective. Since M differs from ∇K by a compact (in fact, finite rank) operator, it is Fredholm of index m, where m is the dimension of E. Let $f_1,...,f_n$ be a basis of E. For $Q := (t_Q,\Sigma_Q) \in U_P$, where $\Sigma_Q = (i_Q,(S_Q,\partial S_Q))$, let $\pi_Q : (S_Q,\partial S_Q) \to (S,\partial S)$ be the canonical projection (recall that Σ_Q is a graph over Σ_{t_Q}). For all i, we define $f_{i,Q} \in C^\infty(S_Q)$ by:

$$f_{i,Q} = f_i \circ \pi_Q.$$

For all i, $Q \mapsto f_{i,Q}$ defines a section of $\mathcal{E}|_{U_P}$, which we denote by F_i. We now define $\hat{K}_P : \mathbb{R}^m \times U_P \to \mathcal{E}|_{U_P}$ by:

$$\hat{K}_P(\sum_{i=1}^n \lambda_i e_i,(t,[\Sigma])) = K(\Sigma) + \sum_{i=1}^n \lambda_i F_i(t,[\Sigma]).$$

By reducing U_P if necessary, we may assume that $\nabla\hat{K}$ is Fredholm and surjective at every point of $\mathbb{R}^m \times U_P$. Since $\hat{K}$ is now a function over an open subset of $\mathcal{M}$ (as opposed to $\mathcal{M}_t$), it has index $(m+1)$. Let $\psi : U_P \to [0,\infty[$ be a smooth function such that:

(i) $\psi = 1$ near $(t_0, [\Sigma])$; and

(ii) the support of ψ is contained in U_P.

Let $U'_P \subseteq U_P$ be a neighbourhood of $(t_0, [\Sigma])$ such that $\psi = 1$ over U'_P. We define $\Psi_P : \mathbb{R}^m \to \Gamma(\mathcal{E})$ by:

$$\Psi_P(\sum_{i=1}^n \lambda_i e_i) = \sum_{i=1}^n \lambda_i \psi F_i.$$

By compactness of X_0, there exist finitely many points $P_1, ..., P_n \in X_0$ such that:

$$X_0 \subseteq \bigcup_{i=1}^n U'_{P_i} =: \Omega.$$

Denote $m = m_1 + ... + m_n$ and define $\Psi : \mathbb{R}^m \to \Gamma(\mathcal{E})$ by:

$$\Psi = \Psi_{P_1} \oplus ... \oplus \Psi_{P_n}.$$

Define $\hat{K} : \mathbb{R}^m \times \mathcal{N} \to \mathcal{E}$ by:

$$\hat{K}(v, (t, [\Sigma])) = K([\Sigma]) + \Psi(v).$$

For $v \in \mathbb{R}^m$, define X_v by:

$$X_v = \left\{(t, [\Sigma]) \in \mathcal{N} \text{ s.t. } \hat{K}(v, (t, [\Sigma])) = \phi_t\right\}.$$

Proposition 11.1

There exists $r > 0$ such that:

(i) for $\|v\| < r$, X_v is compact; and

(ii) for $\|v\| < r$, $X_v \subseteq \Omega$.

Proof: (i) Let $(t_m, [\Sigma_m])_{m\in\mathbb{N}}$ be a sequence in X_v. Let $(\Sigma'_m)_{m\in\mathbb{N}}$ be a sequence of smooth, immersed, compact hypersurfaces in M such that, for all m, Σ_m is a graph over Σ'_m. Suppose, moreover, that $(\Sigma'_m)_{m\in\mathbb{N}}$ converges to Σ'_0. For all $m \in \mathbb{N} \cup \{0\}$, choose $f_m \in C^\infty(\Sigma'_m)$ and suppose that $(f_m)_{m\in\mathbb{N}}$ converges in the C^∞ sense to f_0. For all m, let π_m be the canonical projection onto Σ'_m. With small modifications, Lemma 10.2 adapts to the case where $\phi_m = f_m \circ \pi_m$ for all m, and likewise to the case where ϕ_m is a finite linear combination of such functions. It follows that the closure of X_v in $\mathcal{M}$ is relatively compact.

Let $(t, [\Sigma])$ be a limit point of X_v. By Lemma 3.3, Σ is contained by $\hat{\Sigma}_t$. Suppose that $\Sigma \notin \mathcal{N}_t$. Then $\hat{\Sigma}_t$ does not strictly contain Σ, and Σ is thus an interior tangent to $\hat{\Sigma}_t$ at some point, p, say (possibly in $\partial\hat{\Sigma}$). However, for v sufficiently small, $\|\Psi(v)\| \leqslant \epsilon$ and so the Gaussian curvature of $\hat{\Sigma}_t$ at p is strictly greater than that of Σ at p. This contradicts the Geometric Maximum Principal (see, for example, [10]). There thus exists $r > 0$ such that for $\|v\| < r$, the closure of X_v is contained in $\mathcal{N}$ and so X_v is compact. (i) follows.

(ii) Suppose the contrary. There exists $(v_n)_{n\in\mathbb{N}}$ which converges to 0 and $(t_n, [\Sigma_n])_{n\in\mathbb{N}}$ such that, for all n:

$$(t_n, [\Sigma_n]) \in X_{v_n}, \qquad (t_n, [\Sigma_n]) \notin \Omega.$$

As in the previous paragraph, by Lemma 10.2, $(t_n, [\Sigma_n])_{n\in\mathbb{N}}$ subconverges to $(t_0, [\Sigma_0]) \in X_0$. Thus, for sufficiently large n, $(t_n, (\Sigma_n))_{n\in\mathbb{N}} \in \Omega$, which is absurd. *(ii)* follows, and this completes the proof. □

Define $X \subseteq \mathbb{R}^m \times \Omega$ by:

$$X = \left\{(v, (t, [\Sigma])) \in \mathbb{R}^m \times \Omega \text{ s.t. } \hat{K}(v, (t, [\Sigma])) = \phi_t\right\}.$$

Proposition 11.2

X is an $(m+1)$-dimensional smooth, embedded submanifold of $\mathbb{R}^m \times \Omega$.

Proof: By construction, $\hat{K}$ is everywhere Fredholm of index $(m+1)$ and surjective. The result now follows by the Implicit Function Theorem for Banach manifolds. □

Proposition 11.3

There exists $(v_n)_{n\in\mathbb{N}} \in \mathbb{R}^m$ such that:

(i) $(v_n)_{n\in\mathbb{N}}$ converges to 0;

(ii) for all n, X_{v_n} is a (potentially empty) 1-dimensional, smooth, compact, embedded submanifold of Ω; and

(iii) $\partial X_{v_n} \subseteq \mathcal{N}_0 \cup \mathcal{N}_1$.

Proof: Let $\pi : \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ be projection onto the first factor. Let π_X be the restriction of π to X. By Sard's Lemma, the set of critical values of π_X has Lebesgue measure 0. Let $(v_n)_{n\in\mathbb{N}} \in \mathbb{R}^m$ be a sequence of non-critical values of π_X converging to 0. By the Submersion Theorem, for all n, X_{v_n} is a 1-dimensional, smooth, embedded submanifold of Ω. By Proposition 11.1 we may suppose moreover that, for all n, $X_{v_n} \subset \Omega$ and that X_{v_n} is compact. *(i)* and *(ii)* follow. For all n, the end points of X_{v_n} lie in the (manifold) boundary of X. Since the (manifold) boundary of X is contained in $\mathcal{N}_0 \cup \mathcal{N}_1$, *(iii)* follows. This completes the proof. □

12 - Local and Global Rigidity.

Let M^{n+1} be an $(n+1)$-dimensional Hadamard manifold. Let $\hat{\Sigma} \subseteq M$ be a convex immersed hypersurface. Choose $\phi \in C^\infty(M)$. Let $\Sigma = (i, (S, \partial S))$ be another convex immersed hypersurface. We say that Σ is a **solution** to the problem $(\hat{\Sigma}, \phi)$ if and only if:

(i) $\partial\Sigma = \partial\hat{\Sigma}$;

(ii) Σ is contained by $\hat{\Sigma}$; and

(iii) for all $p \in S$, the Gaussian curvature of Σ at p is equal to $(\phi \circ i)(p)$.

Definition 12.1

(i) We say that $(\hat{\Sigma},\phi)$ is locally rigid if and only if, for all solutions, Σ to $(\hat{\Sigma},\phi)$, the linearisation, DK, of the Gauss Curvature Operator, K, over Σ is invertible.

(ii) We say that $(\hat{\Sigma},\phi)$ is globaly rigid if and only if there exists at most one solution, Σ to $(\hat{\Sigma},\phi)$.

We recall the following properties of local and global rigidity:

Proposition 12.2

(i) If $(\hat{\Sigma},\phi)$ is locally rigid, then $(\hat{\Sigma},\phi')$ is also locally rigid for all ϕ' sufficiently close to ϕ.

(ii) If $(\hat{\Sigma},\phi)$ is locally and globally rigid, then $(\hat{\Sigma},\phi')$ is globally rigid for all ϕ' sufficiently close to ϕ.

Proof: See [10]. $\square$

Now let $(\hat{\Sigma}_t)_{t\in[0,1]}$ be a smooth family of locally strictly convex, immersed hypersurfaces in M with generic boundaries. Let $\epsilon>0$ and let $(\phi_t)_{t\in[0,1]}\in C^\infty(M,]0,\infty[)$ be a smooth family of smooth, positive functions such that, for all t, the Gaussian curvature of Σ_t at any point p is no less than $\phi_t(p)+\epsilon$. Using local and global rigidity, we obtain existence:

Lemma 12.3

Suppose that $(\hat{\Sigma}_0,\phi_0)$ is both locally and globally rigid. If there exists a solution Σ_0 to $(\hat{\Sigma}_0,\phi_0)$, then there exists a solution to $(\hat{\Sigma}_1,\phi_1)$.

Remark: It follows that proving existence of solutions for a given problem reduces to showing the existence of a smooth isotopy by locally strictly convex immersions to a locally and globally rigid problem for which solutions are known to exist.

Proof: Let $\mathcal{N}$, $m\in\mathbb{N}$ and $\Psi:\mathbb{R}^m\rightarrow\Gamma(\mathcal{E})$ be as in Section 11 and, for all $v\in\mathbb{R}^m$, define $X_v\subseteq\mathcal{N}$ by:

$$X_v=\{(t,[\Sigma])\in\mathcal{N}\text{ s.t. }K([\Sigma])+\Psi(v)=\psi_t\}.$$

Let $(v_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}^m$ be as in Proposition 11.3. Since $(\hat{\Sigma}_0,\phi_0)$ is locally rigid, there exists $N>0$ such that, for all $n\geqslant N$, $X_{v_n}\cap\mathcal{N}_0$ is non-empty, and thus, in particular, X_{v_n} is non-empty. Since $(\hat{\Sigma}_0,\phi_0)$ is also globally rigid, it follows by Proposition 12.2 that, for sufficiently large n, $\Psi(v_n)+\phi_0$ is too, and therefore that $X_{v_n}\cap\mathcal{N}_0$ consists of a single point.

Let $\pi:\mathcal{N}\rightarrow[0,1]$ be the canonical projection. For all $n\geqslant N$, X_{v_n} is a smooth, embedded, compact, 1-dimensional submanifold of $\mathcal{N}$. It is thus homeomorphic, either to a compact interval or to a circle. By local and global rigidity, the restriction of π to X_{v_n} is a local diffeomorphism near the unique point lying in $\pi^{-1}(\{0\})$. It follows that X_{v_n} has non-trivial (manifold) boundary, and is therefore not a circle. It is thus a compact interval. By Proposition 11.3, the endpoints of X_{v_n} lie in $\mathcal{N}_0\cup\mathcal{N}_1$. By global rigidity, only one endpoint of X_{v_n} lies in $\mathcal{N}_0$, and the other therefore lies in $\mathcal{N}_1$.

For all n, let Σ_n be such that $(1,[\Sigma_n])$ is the unique endpoint of X_{v_n} in $\mathcal{N}_1$. By Lemma 10.2, there exists Σ_0 to which $(\Sigma_n)_{n\in\mathbb{N}}$ subconverges and Σ_0 is a solution of $(\hat{\Sigma}_1,\psi_1)$. This completes the proof. □

Lemma 12.3 may be easily adapted to treat the case where the metric of the underlying manifold also varies, and we obtain Theorem 1.1:

Proof of Theorem 1.1: Let $(\hat{\Sigma}_t)_{t\in[0,1]}$ be an isotopy by convex, immersed hypersurfaces such that $\hat{\Sigma}_0=\hat{\Sigma}$ and $\hat{\Sigma}_1$ is a finite covering of Ω. For ease of presentation, we will assume that the covering is of order one: the general case is almost identical. Let $p\in K$ be an interior point. Let $d_0,d_1:M\rightarrow\mathbb{R}$ be given by:

$$d_0(x)=d(x,K),\qquad d_1(x)=d(x,p).$$

Both d_0 and d_1 are smooth outside K. For $t\in[0,1]$, define d_t by:

$$d_t=td_1+(1-t)d_0.$$

Trivially, ∂K is isotopic by smooth convex immersions to $d_0^{-1}(\{r\})$ for all $r\geqslant 0$. Choose r_0 such that $K\subseteq B_{r_0}(p)$. For all t, $d_t^{-1}(\{r_0\})$ is a convex, embedded hypersurface and we thus obtain an isotopy by smooth convex immersions between $d_0^{-1}(\{r_0\})$ and $d_1^{-1}(\{r_0\})$. We may thus define $(\hat{\Sigma}_t)_{t\in[1,2]}$ such that $\hat{\Sigma}_2$ is a geodesic sphere with a finite number of open sets removed. Let g be the Riemannian metric on M. Define $(g_t)_{t\in[0,2]}$ such that $g_t=g$ for all t.

We may assume that $\hat{\Sigma}_2$ is as small as we wish. Define $(\hat{\Sigma}_t)_{t\in[2,3]}$ and $(g_t)_{t\in[2,3]}$ such that:

(i) $g_2=g$;

(ii) g_3 is complete with constant curvature equal to 1;

(iii) for all t, $\hat{\Sigma}_t$ is a geodesic sphere with respect to g_t with a finite number of open sets removed.

Define $(\hat{\Sigma}_t)_{t\in[3,4]}$ and $(g_t)_{t\in[3,4]}$ such that:

(i) for all t, $g_t=g_3$ is the complete hyperbolic metric;

(ii) for all t, $(\hat{\Sigma}_t)$ is a geodesic sphere with a finite number of open sets removed; and

(iii) $\hat{\Sigma}_4$ is a horosphere with a finite number of open sets (including a neighbourhood of the infinite point) removed.

Let $(\psi_t)_{t\in[0,4]}\in C^\infty(M)$ be a smooth family of smooth, positive valued functions such that:

(i) $\psi_0=\psi$;

(ii) for all t and for all $p\in\hat{\Sigma}_t$, the Gaussian curvature of $\hat{\Sigma}_t$ at p is greater than $\psi_t(p)$; and

(iii) ψ_4 is constant and equal to $1-\delta$ for some $\delta<1$.

The problem (Σ_4, ψ_4) in $(M, g_4) = \mathbb{H}^{n+1}$ is locally and globally rigid and has a non-trivial solution (see [10]). By Proposition 8.1, this isotopy by locally strictly convex, immersed hypersurfaces may be deformed to an isotopy by locally strictly convex, immersed hypersurfaces whose boundaries are generic. Existence therefore follows by (an appropriately modified version of) Lemma 12.3, and this completes the proof. □

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