Cubic systems with invariant lines of total multiplicity eight and with four distinct infinite singularities

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Abstract

In this article we prove a classification theorem (Main Theorem) of real planar cubic vector fields which possess four distinct infinite singularities and eight invariant straight lines, including the line at infinity and including their multiplicities. This classification, which is taken modulo the action of the group of real affine transformations and time rescaling, is given in terms of invariant polynomials. The algebraic invariants and comitants allow one to verify for any given real cubic system with four infinite distinct singularities whether or not it has invariant lines of total multiplicity eight, and to specify its configuration of lines endowed with their corresponding real singularities of this system. The calculations can be implemented on computer.

1 Introduction and the statement of the Main Theorem

We consider here real polynomial differential systems

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where $P,\ Q$ are polynomials in $x,\ y$ with real coefficients, i.e. $P,\ Q \in \mathbb{R}[x,y]$. We shall say that systems (1) are *cubic* if $\max(\deg(P), \deg(Q)) = 3$.

Let

$$\mathbf{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

be the polynomial vector field corresponding to systems (1).

A straight line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ satisfies

$$\mathbf{X}(f) = uP(x,y) + vQ(x,y) = (ux + vy + w)R(x,y)$$

for some polynomial $R(x,y)$ if and only if it is *invariant* under the flow of the systems. If some of the coefficients $u,\ v,\ w$ of an invariant straight line belongs to $\mathbb{C} \setminus \mathbb{R}$, then we say that *the straight line is complex*; otherwise *the straight line is real.* Note that, since systems (1) are real, if a system has a complex invariant straight line $ux + vy + w = 0$, then it also has its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$.

To a line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$ we associate its projective completion $F(X,Y,Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbf{P}_2(\mathbb{C})$, $(x,y) \mapsto [x:y:1]$. The line $Z = 0$ in $\mathbf{P}_2(\mathbb{C})$ is called the line at infinity of the affine plane $\mathbb{C}^2$. It follows from

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the work of Darboux (see, for instance, [5]) that each system of differential equations of the form (1) over $\mathbb{C}$ yields a differential equation on the complex projective plane $\mathbf{P}_2(\mathbb{C})$ which is the compactification of the differential equation $Qdx - Pdy = 0$ in $\mathbb{C}^2$. The line $Z = 0$ is an invariant manifold of this complex differential equation.

Definition 1.1. *We say that an invariant affine straight line $f(x, y) = ux + vy + w = 0$ (respectively the line at infinity $Z = 0$) for a cubic vector field $\mathbf{X}$ has multiplicity m if there exists a sequence of real cubic vector fields X_k converging to $\mathbf{X}$, such that each $\mathbf{X}_k$ has m (respectively $m - 1$) distinct invariant affine straight lines $f_i^j = u_i^j x + v_i^j y + w_i^j = 0$, $(u_i^j, v_i^j) \neq (0, 0)$, $(u_i^j, v_i^j, w_i^j) \in \mathbb{C}^3$, converging to $f = 0$ as $k \to \infty$ (with the topology of their coefficients), and this does not occur for $m + 1$ (respectively m).*

We give here some references on polynomial differential systems possessing invariant straight lines. For quadratic systems see [6, 19, 20, 22, 24–26] and [27]; for cubic systems see [10–13, 21, 30] and [31]; for quartic systems see [29] and [33]; for some more general systems see [8, 16, 17] and [18].

According to [2] the maximum number of invariant straight lines taking into account their multiplicities for a polynomial differential system of degree m is $3m$ when we also consider the infinite straight line. This bound is always reached if we consider the real and the complex invariant straight lines, see [4].

So the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems is 9. A classification of all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities have been made in [11]. We also remark that a subclass of the family of cubic systems with invariant lines was discussed in [30] and [31]. More precisely, in these articles authors consider the cubic systems with exactly 7 invariant affine line considered with their "parallel" multiplicity. They say that an invariant line $f(x, y) = ux + vy + w = 0$ of a cubic system (1) has the parallel multiplicity $1 \leq k \leq 3$ if the identity $\mathbf{X}(f) = f^k R(x, y)$ holds for some polynomial $R(x, y)$.

In this paper we classify the family of cubic systems with four distinct infinite singularities (real and/or complex), which possess eight invariant straight lines, including the line at infinity and taking into account their multiplicities.

It is well known that for a cubic system (1) there exist at most 4 different slopes for invariant affine straight lines, for more information about the slopes of invariant straight lines for polynomial vector fields, see [1].

Definition 1.2. *Consider a planar cubic system* (1). *We call configuration of invariant straight lines of this system, the set of (complex) invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.*

If a cubic system (1) possesses 8 distinct invariant straight lines (including the line at infinity) we say that these lines form a *configuration of type* $(3, 3, 1)$ if there exist two triplets of parallel lines and one additional line every set with different slopes. And we shall say that these lines form a *configuration of type* $(3, 2, 1, 1)$ if there exist one triplet and one couple of parallel lines and two additional lines every set with different slopes. Similarly are defined *configurations of types* $(3, 2, 2)$ and $(2, 2, 2, 1)$ and these four types of the configurations exhaust all possible configurations formed by 8 invariant lines for a cubic system.

Note that in all configurations the straight line which is omitted is the infinite one.

If a cubic system (1) possesses 8 invariant straight lines taking into account their multiplicities we shall say that these lines form a *potential configuration of type* $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$) if there exists a sequence of vector fields $\mathbf{X}_k$ as in the definition of geometric multiplicity having 8 distinct line of type $(3, 3, 1)$ (respectively, $(3, 2, 2)$; $(3, 2, 1, 1)$; $(2, 2, 2, 1)$).

Our main result is the following one.

Main Theorem. *Assume that a cubic system possesses invariant lines of total multiplicity 8, including the line at infinity with its own multiplicity. In addition we assume that this system has four distinct infinite singularities. Then:*

I. *The system possesses only one of the 17 possible configurations Config. 8.1 – Config. 8.17 of invariant lines given in* FIGURE 1. *If in a configuration an invariant straight line has multiplicity $k>1$, then the number k appears near the corresponding straight line and this line is in bold face. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. We indicate next to the real singular points of the system, located on the invariant lines, their corresponding multiplicities.*

II. *This system possesses the specific configuration Config. 8.j ($j\in\{1,2,\dots,17\}$ if and only if the corresponding conditions included below are fulfilled. Moreover the system could be brought via an affine transformation and time rescaling to the canonical forms, written below next to the configuration*

A) Four real distinct infinite singularities $\Leftrightarrow \mathcal{D}_1>0,\ \mathcal{D}_2>0,\ \mathcal{D}_3>0$;

A$_1$) Configuration of type $(3,3,1)$ $\Leftrightarrow \mathcal{V}_1=\mathcal{V}_2=\mathcal{U}_1=\mathcal{L}_1=\mathcal{L}_2=\mathcal{K}_1=0,\ \mathcal{K}_2\neq 0$;

- *Config. 8.1* $\Leftrightarrow$ $\mathcal{K}_3>0$: $\dot{x}=x(x+1)(x-a),\ \ \dot{y}=y(y+1)(y-a),\ \ 0<a\neq 1$;
- *Config. 8.2* $\Leftrightarrow$ $\mathcal{K}_3<0$: $\dot{x}=x\big[(x+a)^2+1\big],\ \ \dot{y}=y\big[(y+a)^2+1\big],\ \ a\neq 0$;
- *Config. 8.3* $\Leftrightarrow$ $\mathcal{K}_3=0$: $\dot{x}=x^2(1+x),\ \ \dot{y}=y^2(1+y)$.

A$_2$) Configuration of type $(3,2,1,1)$ $\Leftrightarrow \mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=0,\ \ \mathcal{D}_4\neq 0$;

- *Config. 8.4* $\Leftrightarrow$ $\mathcal{L}_1\neq 0$ *and* $\mathcal{K}_7>0$: $\begin{cases}\dot{x}=x(x-1)(x+r), & r>0,\\ \dot{y}=y(y-1)\big[(1-r)x+ry+r\big];\end{cases}$
- *Config. 8.5* $\Leftrightarrow$ $\mathcal{L}_1\neq 0$ *and* $\mathcal{K}_7<0$: $\begin{cases}\dot{x}=x(x-1)(x+r), & r<0,\\ \dot{y}=y(y-1)\big[(1-r)x+ry+r\big];\end{cases}$
- *Config. 8.6* $\Leftrightarrow$ $\mathcal{L}_1=0$: $\dot{x}=rx^3,\ \ \dot{y}=(r-1)xy^2+y^3,\ r\neq 0$.

A$_3$) Configuration of type $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3=\mathcal{K}_2=\mathcal{K}_4=\mathcal{K}_8=0,\ \ \mathcal{D}_4\neq 0$;

- *Config. 8.7* $\Leftrightarrow$ $\mathcal{K}_9>0$: $\begin{cases}\dot{x}=(x^2-1)(rx+2y+ry),\ r(r^2-1)\neq 0,\\ \dot{y}=(y^2-1)(x+2rx+y),\ (r+2)(2r+1)\neq 0;\end{cases}$
- *Config. 8.8* $\Leftrightarrow$ $\mathcal{K}_9<0$: $\begin{cases}\dot{x}=(x^2+1)(rx+2y+ry),\ r(r^2-1)\neq 0,\\ \dot{y}=(y^2+1)(x+2rx+y),\ (r+2)(2r+1)\neq 0;\end{cases}$
- *Config. 8.9* $\Leftrightarrow$ $\mathcal{K}_9=0$: $\begin{cases}\dot{x}=x^2(rx+2y+ry),\ r(r^2-1)\neq 0,\\ \dot{y}=y^2(x+2rx+y),\ (r+2)(2r+1)\neq 0;\end{cases}$

B) Two real and two complex distinct infinite singularities $\Leftrightarrow \mathcal{D}_1<0$;

B$_1$) Configuration of type $(3,3,1)$ $\Leftrightarrow \mathcal{V}_1=\mathcal{V}_2=\mathcal{U}_1=\mathcal{L}_1=\mathcal{L}_2=\mathcal{K}_1=0,\ \mathcal{K}_2\neq 0$;

- *Config. 8.10* $\Leftrightarrow$ $\mathcal{K}_3>0$: $\begin{cases}\dot{x}=(1-a^2)x/4+x^2-y^2+x^3-3xy^2,\\ \dot{y}=(1-a^2)y/4+2xy+3x^2y-y^3,\ a^2\neq 0,1/9,\ 1;\end{cases}$
- *Config. 8.11* $\Leftrightarrow$ $\mathcal{K}_3<0$: $\begin{cases}\dot{x}=(1+a^2)x/4+x^2-y^2+x^3-3xy^2,\\ \dot{y}=(1+a^2)y/4+2xy+3x^2y-y^3,\ \ a\neq 0;\end{cases}$
- *Config. 8.12* $\Leftrightarrow$ $\mathcal{K}_3=0$: $\begin{cases}\dot{x}=x^2-y^2+x^3-3xy^2,\\ \dot{y}=2xy+3x^2y-y^3.\end{cases}$

B$_2$) Configuration of type $(3,2,1,1)$ $\Leftrightarrow \mathcal{V}_5=\mathcal{U}_2=\mathcal{K}_4=\mathcal{K}_5=\mathcal{K}_6=0,\ \ \mathcal{D}_4\neq 0$;

- *Config. 8.13* $\Leftrightarrow$ $\mathcal{L}_1 \neq 0$: $\begin{cases} \dot{x} = (1+s^2)x\big[(x+s)^2+1\big], \quad s \neq 0, \\ \dot{y} = (1+s^2)^2 y + 2s(1+s^2)xy - sx^3 \\ \qquad +s^2x^2y - sxy^2 - y^3; \end{cases}$

- *Config. 8.14* $\Leftrightarrow$ $\mathcal{L}_1 = 0$: $\begin{cases} x = (1+s^2)x^3, \quad s \neq 0 \\ \dot{y} = -sx^3 + s^2x^2y - sxy^2 - y^3. \end{cases}$

B_3) *Configuration of type* $(2,2,2,1)$ $\Leftrightarrow \mathcal{V}_3 = \mathcal{K}_2 = \mathcal{K}_4 = \mathcal{K}_8 = 0, \quad \mathcal{D}_4 \neq 0$;

- *Config. 8.15* $\Leftrightarrow$ $\mathcal{K}_9 > 0$: $\begin{cases} \dot{x} = x(x-1)(1+s^2-2x+2sy), \ s \neq 0, \\ \dot{y} = -(1+s^2)y + (3+s^2)xy - sx^3 \\ \qquad -3x^2y - 2sy^2 + sxy^2 - y^3; \end{cases}$

- *Config. 8.16* $\Leftrightarrow$ $\mathcal{K}_9 < 0$: $\begin{cases} \dot{x} = 2(1+x^2)(sy-x-s), \ s \neq 0, \\ \dot{y} = s(s^2+3)x + (1-s^2)y - sx^3 \\ \qquad -3x^2y + sxy^2 - y^3; \end{cases}$

- *Config. 8.17* $\Leftrightarrow$ $\mathcal{K}_9 = 0$: $\begin{cases} \dot{x} = -2x^2(x-sy), \ s \neq 0, \\ \dot{y} = -sx^3 - 3x^2y - 2sy^2 + sxy^2 - y^3. \end{cases}$

III. *This system could not have exactly 8 invariant lines (including the line at infinity) in the configuration of the type* $(3,3,2)$ *neither could it have 4 infinite complex singularities.*

The symbols $(\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{K}_8, \mathcal{K}_9)$ *used above denote invariant polynomials defined in Section 2.1 of the paper.*

Corollary 1.1. *A cubic system with four distinct infinite singularities possesses the configuration or potential configuration of a given type if and only if the following conditions are satisfied, respectively:*

$$
\begin{array}{lcl}
(3,3,1) & \Leftrightarrow & \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0, \ \mathcal{K}_2 \neq 0; \\
(3,2,1,1) & \Leftrightarrow & \mathcal{V}_5 = \mathcal{U}_2 = \mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0, \quad \mathcal{D}_4 \neq 0; \\
(2,2,2,1) & \Leftrightarrow & \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0, \quad \mathcal{D}_4 \neq 0.
\end{array}
$$

The work is organized as follows. In Section 2 we define the main invariant polynomials associated to configurations of invariant straight lines of cubic systems and we give some preliminary results needed for this paper. In Section 3 we prove step by step our Main Theorem considering three subfamilies of cubic systems possessing four distinct infinite singularities, and namely: systems with four real, systems with two real and two imaginary and systems with four imaginary singularities.

2 Preliminaries

Consider real cubic systems, i.e. systems of the form:

$$
\begin{aligned}
\dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y), \\
\dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y)
\end{aligned} \tag{2}
$$

with real coefficients and variables x and y. The polynomials p_i and q_i $(i = 0,1,2,3)$ are homogeneous polynomials of degree i in x and y:

$$
\begin{aligned}
& p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3, \\
& p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\
& q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3, \\
& q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2.
\end{aligned}
$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (2) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y]$.

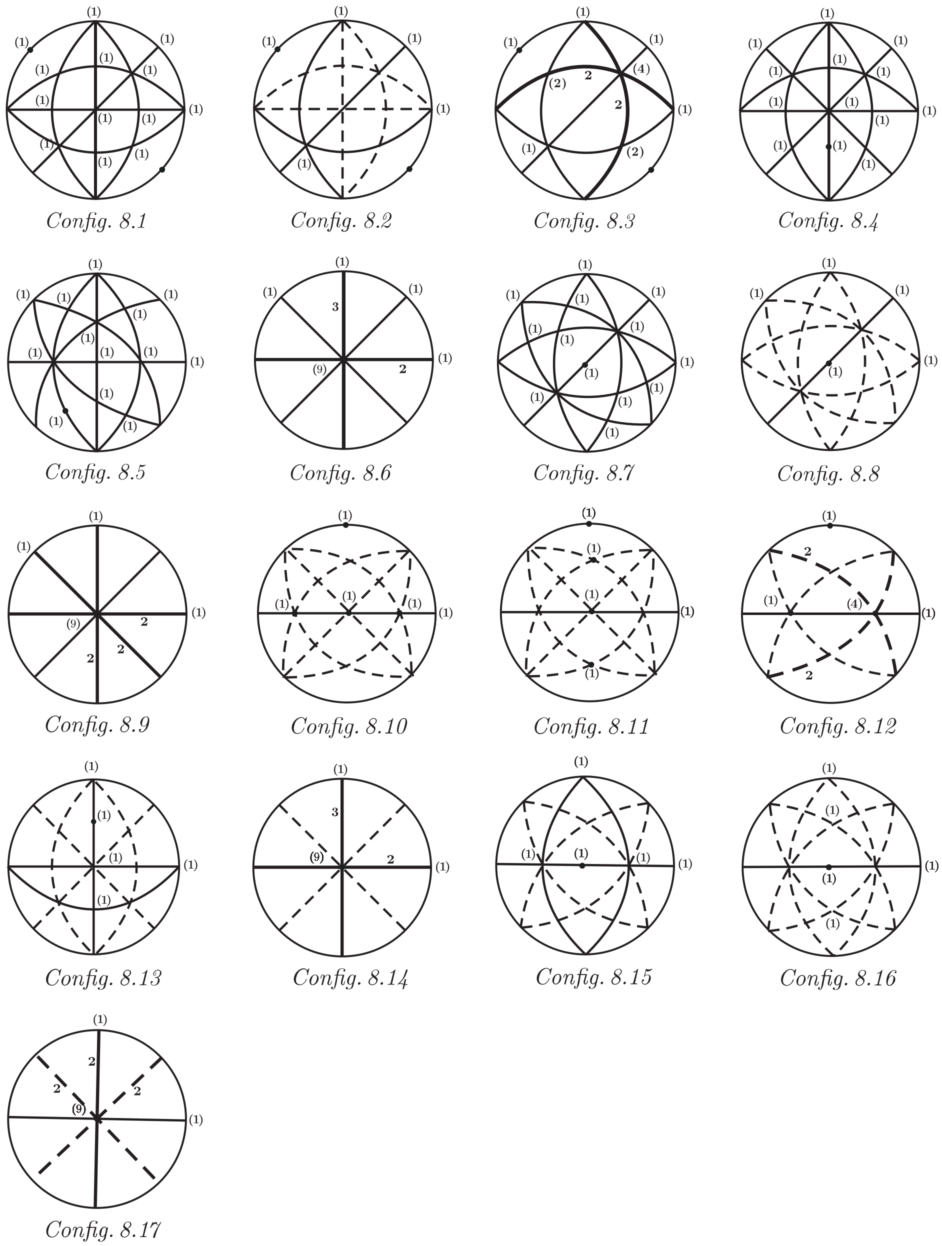

Figure 1: **The configurations of invariant lines for cubic systems with 4 distinct infinite singularities**

2.1 The main invariant polynomials associated to configurations of invariant lines

It is known that on the set **CS** of all cubic differential systems (2) acts the group $Aff(2,\mathbb{R})$ of affine transformation on the plane [22]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **CS**. We can identify the set **CS** of systems (2) with a subset of $\mathbb{R}^{20}$ via the map $\mathbf{CS} \longrightarrow \mathbb{R}^{20}$ which associates to each system (2) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03})$ of its coefficients.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-

comitant and CT-comitant we refer the reader to [22]. Here we shall only construct the necessary T-comitants associated to configurations of invariant lines for the class of cubic systems with four distinct infinite singularities and with exactly eight invariant lines including the line at infinity and including multiplicities.

Let us consider the polynomials

$$C_i(a,x,y) = yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 0,1,2,3,$$
$$D_i(a,x,y) = \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 1,2,3.$$

As it was shown in [28] the polynomials

$$\big\{C_0(a,x,y),\ C_1(a,x,y),\ C_2(a,x,y),\ C_3(a,x,y),\ D_1(a),\ D_2(a,x,y)\ D_3(a,x,y)\big\} \tag{3}$$

of degree one in the coefficients of systems (2) are GL-comitants of these systems.

Notation 2.1. *Let $f,\ g \in \mathbb{R}[a,x,y]$ and*

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{4}$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [7], [14])

Theorem 2.1. [32] *Any GL-comitant of systems (2) can be constructed from the elements of the set (3) by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f,g)^{(k)}$.*

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(a,x,y)$ and $q(a,x,y)$. We obtain $\tilde{p}(\tilde{a}(a,x_0,y_0),x',y') = p(a,x'+x_0,y'+y_0)$, $\quad \tilde{q}(\tilde{a}(a,x_0,y_0),x',y') = q(a,x'+x_0,y'+y_0)$. We construct the following polynomials

$$\Omega_i(a,x_0,y_0) \equiv \operatorname{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\Omega_i(a,x_0,y_0) \in \mathbb{R}[a,x_0,y_0],\ (i=1,2,3).$$

Notation 2.2.

$$\tilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)\big|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \quad (i=1,2,3). \tag{5}$$

Remark 2.1. *We note that the constructed polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ are affine comitants of systems (2) and are homogeneous polynomials in the coefficients $a_{00},\dots,b_{02}$ and non-homogeneous in x,y and*

$$\begin{array}{lll} \deg_a \mathcal{G}_1 = 3, & \deg_a \mathcal{G}_2 = 4, & \deg_a \mathcal{G}_3 = 5, \\ \deg_{(x,y)} \mathcal{G}_1 = 8, & \deg_{(x,y)} \mathcal{G}_2 = 10, & \deg_{(x,y)} \mathcal{G}_3 = 12. \end{array}$$

Notation 2.3. *Let $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ be the homogenization of $\tilde{\mathcal{G}}_i(a,x,y)$, i.e.*

$$\mathcal{G}_1(a,X,Y,Z) = Z^8\tilde{\mathcal{G}}_1(a,X/Z,Y/Z),$$
$$\mathcal{G}_2(a,X,Y,Z) = Z^{10}\tilde{\mathcal{G}}_2(a,X/Z,Y/Z),$$
$$\mathcal{G}_3(a,X,Y,Z) = Z^{12}\tilde{\mathcal{G}}_3(a,X/Z,Y/Z),$$

and $\quad \mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ *in* $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [11]):

Lemma 2.1. *The straight line $\mathcal{L}(x,y) \equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ is an invariant line for a quadratic system* (2) *if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{G}}_i(a,x,y)=(ux+vy+w)\widetilde{W}_i(x,y) \quad (i=1,2,3),$$

where $\widetilde{W}_i(x,y)\in\mathbb{C}[x,y]$.

Lemma 2.2. *Consider a cubic system* (2) *and let $a\in\mathbb{R}^{20}$ be its 20-tuple of coefficients.*

1) If $\mathcal{L}(x,y)\equiv ux+vy+w=0$, $u,v,w\in\mathbb{C}$, $(u,v)\neq(0,0)$ is an invariant straight line of multiplicity k for this system then $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1,\tilde{\mathcal{G}}_2),\tilde{\mathcal{G}}_3)$ in $\mathbb{C}[x,y]$, i.e. there exist $W_i(a,x,y)\in\mathbb{C}[x,y]$ $(i=1,2,3)$ such that

$$\tilde{\mathcal{G}}_i(a,x,y)=(ux+vy+w)^k W_i(a,x,y), \quad i=1,2,3. \tag{6}$$

2) If the line $l_\infty: Z=0$ is of multiplicity $k>1$ then $Z^{k-1}\mid\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$, i.e. we have $Z^{k-1}\mid H(a,X,Y,Z)$.

In order to define the needed invariant polynomials we first construct the following comitants of second degree with respect to the coefficients of the initial system:

$$\begin{array}{lll}
S_1=(C_0,C_1)^{(1)}, & S_{10}=(C_1,C_3)^{(1)}, & S_{19}=(C_2,D_3)^{(1)},\\
S_2=(C_0,C_2)^{(1)}, & S_{11}=(C_1,C_3)^{(2)}, & S_{20}=(C_2,D_3)^{(2)},\\
S_3=(C_0,D_2)^{(1)}, & S_{12}=(C_1,D_3)^{(1)}, & S_{21}=(D_2,C_3)^{(1)},\\
S_4=(C_0,C_3)^{(1)}, & S_{13}=(C_1,D_3)^{(2)}, & S_{22}=(D_2,D_3)^{(1)},\\
S_5=(C_0,D_3)^{(1)}, & S_{14}=(C_2,C_2)^{(2)}, & S_{23}=(C_3,C_3)^{(2)},\\
S_6=(C_1,C_1)^{(2)}, & S_{15}=(C_2,D_2)^{(1)}, & S_{24}=(C_3,C_3)^{(4)},\\
S_7=(C_1,C_2)^{(1)}, & S_{16}=(C_2,C_3)^{(1)}, & S_{25}=(C_3,D_3)^{(1)},\\
S_8=(C_1,C_2)^{(2)}, & S_{17}=(C_2,C_3)^{(2)}, & S_{26}=(C_3,D_3)^{(2)},\\
S_9=(C_1,D_2)^{(1)}, & S_{18}=(C_2,C_3)^{(3)}, & S_{27}=(D_3,D_3)^{(2)}.
\end{array}$$

We shall use here the following invariant polynomials constructed in [11] to characterize the family of cubic systems possessing the maximal number of invariant straight lines:

$$\begin{array}{rcl}
\mathcal{D}_1(a) &=& 6S_{24}^3-\left[(C_3,S_{23})^{(4)}\right]^2,\\
\mathcal{D}_2(a,x,y) &=& -S_{23},\\
\mathcal{D}_3(a,x,y) &=& (S_{23},\,S_{23})^{(2)}-6C_3(C_3,\,S_{23})^{(4)},\\
\mathcal{D}_4(a) &=& (C_3,\,\mathcal{D}_2)^{(4)},\\
\mathcal{V}_1(a,x,y) &=& S_{23}+2D_3^2,\\
\mathcal{V}_2(a,x,y) &=& S_{26},\\
\mathcal{V}_3(a,x,y) &=& 6S_{25}-3S_{23}-2D_3^2,\\
\mathcal{V}_4(a,x,y) &=& C_3\left[(C_3,S_{23})^{(4)}+36\,(D_3,S_{26})^{(2)}\right],\\
\mathcal{L}_1(a,x,y) &=& 9C_2\,(S_{24}+24S_{27})-12D_3\,(S_{20}+8S_{22})-12\,(S_{16},D_3)^{(2)}\\
&& -3\,(S_{23},C_2)^{(2)}-16\,(S_{19},C_3)^{(2)}+12\,(5S_{20}+24S_{22},C_3)^{(1)},\\
\mathcal{L}_2(a,x,y) &=& 32\,(13S_{19}+33S_{21},D_2)^{(1)}+84\,(9S_{11}-2S_{14},D_3)^{(1)}\\
&& +8D_2\,(12S_{22}+35S_{18}-73S_{20})-448\,(S_{18},C_2)^{(1)}\\
&& -56\,(S_{17},C_2)^{(2)}-63\,(S_{23},C_1)^{(2)}+756D_3S_{13}-1944D_1S_{26}\\
&& +112\,(S_{17},D_2)^{(1)}-378\,(S_{26},C_1)^{(1)}+9C_1\,(48S_{27}-35S_{24}).
\end{array}$$

However these invariant polynomials are not sufficient to characterize the cubic systems with invariant lines of the total multiplicity 8. So we construct here the following new invariant polynomials:

$$
\begin{aligned}
\mathcal{V}_5(a,x,y) =& 6T_1(9A_3-7A_4)+2T_2(4T_5-T_6)-3T_3(3A_1+5A_2)+3A_2T_4+36T_5^2-3T_{29},\\
\mathcal{L}_5(a) =& 2A_1^3-A_3^2,\\
\mathcal{K}_1(a,x,y) =& \big(3223\,T_2^2\,T_{57}+2718\,T_4\,T_{57}-829\,T_2^2\,T_{58},\,T_{54}\big)^{(10)},\\
\mathcal{K}_2(a,x,y) =& T_{40},\\
\mathcal{K}_3(a,x,y) =& 9T_{57}T_{58}-100A_7^2T_1^2T_{117}-7T_{57}^2-2T_{58}^2,\\
\mathcal{K}_4(a,x,y) =& T_{61}-2T_{11},\\
\mathcal{K}_5(a,x,y) =& 45T_{127}-T_2T_{13}+2T_2T_{14}+12T_{26}+45T_{27}-45T_{28}+30T_{96},\\
\mathcal{K}_6(a,x,y) =& 4T_1T_8(2663T_{13}-8161T_{14})+6T_8(178T_{20}+70T_{21}+555T_{22})+18T_9(30T_2T_8-\\
&-488T_1T_{11}-119T_{19})+5T_2(25T_{55}+16T_{56})-15T_1(25T_{57}-11T_{58})-165T_{59},\\
\mathcal{K}_7(a) =& A_1+3A_2,\\
\mathcal{K}_8(a,x,y) =& 10A_8T_1-3T_2T_{14}+4T_{26}-8T_{27},\\
K_9(a,x,y) =& 3T_1(11T_{14}-8T_{13})-T_{20}+5T_{21},
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= S_{24}/288,\\
A_2 &= S_{27}/72,\\
A_3 &= \big(S_{23},C_3\big)^{(4)}/2^7/3^5,\\
A_4 &= \big(S_{26},D_3\big)^{(2)}/2^5/3^3,\\
A_7 &= \big[72D_1A_2+(S_{22},D_2)^{(1)}\big]/24,\\
A_8 &= \big[9D_1(S_{24}-288A_2)+4\big(9S_{11}-2S_{14},D_3\big)^{(2)}+8\big(3S_{18}-S_{20}-4S_{22},D_2\big)^{(1)}\big]/2^7/3^3,
\end{aligned}
$$

are affine invariants, whereas the polynomials

$$
\begin{aligned}
T_1 &= C_3,\\
T_2 &= D_3,\\
T_3 &= S_{23}/18,\\
T_4 &= S_{25}/6,\\
T_5 &= S_{26}/72,\\
T_6 &= \big[3C_1(D_3^2-9T_3+18T_4)-2C_2(2D_2D_3-S_{17}+2S_{19}-6S_{21})+\\
&\quad +2C_3(2D_2^2-S_{14}+8S_{15})\big]/2^4/3^2,\\
T_8 &= \big[5D_2(D_3^2+27T_3-18T_4)+20D_3S_{19}+12\big(S_{16},D_3\big)^{(1)}-8D_3S_{17}\big]/5/2^5/3^3,\\
T_9 &= \big[9D_1(9T_3-18T_4-D_3^2)+2D_2(D_2D_3-3S_{17}-S_{19}-9S_{21})+18\big(S_{15},C_3\big)^{(1)}-\\
&\quad -6C_2(2S_{20}-3S_{22})+18C_1S_{26}+2D_3S_{14}\big]/2^4/3^3,\\
T_{11} &= \big[\big(D_3^2-9T_3+18T_4,C_2\big)^{(2)}-6\big(D_3^2-9T_3+18T_4,D_2\big)^{(1)}-12\big(S_{26},C_2\big)^{(1)}+\\
&\quad +12D_2S_{26}+432(A_1-5A_2)C_2\big]/2^7/3^4,\\
T_{13} &= \big[\big(8S_{19}+9S_{21},D_2\big)^{(1)}-D_2(8S_{20}+3S_{22})+18D_1S_{26}+1296C_1A_2\big]/2^4/3^3,
\end{aligned}
$$

$$
\begin{aligned}
T_{14} = {} & 8\big(9S_{19}+2S_{21},D_2\big)^{(1)}+3\big(9T_3-18T_4-D_3^2,C_1\big)^{(2)}-4\big(S_{17},C_2\big)^{(2)}+\\
& +4\big(S_{14}-17S_{15},D_3\big)^{(1)}-8\big(S_{14}+S_{15},C_3\big)^{(2)}+432C_1(5A_1+11A_2)+\\
& +36D_1S_{26}-4D_2(S_{18}+4S_{22})\big]/2^6/3^3,\\
T_{19} = {} & \big(T_8,C_3\big)^{(1)},\\
T_{20} = {} & \big(T_6,C_3\big)^{(2)}/6,\\
T_{21} = {} & \big(T_6,D_3\big)^{(1)}/6,\\
T_{22} = {} & \big(T_9,C_3\big)^{(1)}/4,\\
T_{26} = {} & \big(T_6,D_3\big)^{(2)}/12,\\
T_{27} = {} & \big(T_9,C_3\big)^{(2)}/12,\\
T_{28} = {} & \big(T_9,D_3\big)^{(1)}/12,\\
T_{29} = {} & \big((S_{23},C_3)^{(1)}),\ D_3\big)^{(2)}/5/2^6/3^3,\\
T_{40} = {} & \big[27C_0(9T_3-18T_4-D_3^2)^2+C_1\big(-62208T_{11}C_3-3(9T_3-18T_4-D_3^2)\times\\
& \times(2D_2D_3-S_{17}+2S_{19}-6S_{21})\big)+20736T_{11}C_2^2+C_2(9T_3-18T_4-D_3^2)\times\\
& \times(8D_2^2+54D_1D_3-27S_{11}+27S_{12}-4S_{14}+32S_{15})-54C_3(9T_3-18T_4-D_3^2)\times\\
& \times(2D_1D_2-S_8+2S_9)-54D_1(9T_3-18T_4-D_3^2)S_{16}-\\
& -576T_6(2D_2D_3-S_{17}+2S_{19}-6S_{21})\big]/2^8/3^4,\\
T_{54} = {} & \big(T_{40},C_3\big)^{(1)},\\
T_{55} = {} & \big(T_{40},C_3\big)^{(2)}/24,\\
T_{56} = {} & \big(T_{40},D_3\big)^{(1)}/6,\\
T_{57} = {} & \big(T_{40},D_3\big)^{(2)}/12,\\
T_{58} = {} & \big(T_{40},C_3\big)^{(3)}/36,\\
T_{59} = {} & \big((T_{40},C_3)^{(2)},C_3\big)^{(1)}/72,\\
T_{61} = {} & \Big[27\big(T_3,C_2\big)^{(2)}-18\big(T_4,C_2\big)^{(2)}+48D_3S_{22}-216\big(T_4,D_2\big)^{(1)}+36D_2S_{26}-\\
& -432C_2(3A_1+17A_2)+\big(D_3^2,C_2\big)^{(2)}\Big]/2^7/3^4,\\
T_{96} = {} & \big(T_6,C_3\big)^{(3)}/2^4/3^2,\\
T_{117} = {} & \big(T_9,C_3\big)^{(1)}/4,\\
T_{127} = {} & \big(T_{13},C_3\big)^{(1)}/2
\end{aligned}
$$

are T-comitants of cubic systems (2) (see for details [22]). We note that these invariant polynomials are the elements of the polynomial basis of T-comitants up to degree six constructed by Iu. Calin [3].

2.2 Preliminary results

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a,x,y)$ for $i=1,2,3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [9], [14]).

Following [11] we consider two polynomials

$$
f(z) \;=\; a_0z^n+a_1z^{n-1}+\cdots+a_n, \qquad g(z)=b_0z^m+b_1z^{m-1}+\cdots+b_m,
$$

in the variable z of degree n and m, respectively.

We say that the k–th *subresultant* with respect to variable z of the two polynomials $f(z)$ and $g(z)$ is the $(m+n-2k)\times(m+n-2k)$ determinant

$$R_z^{(k)}(f,g)=\begin{vmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{m+n-2k-1}\\ 0 & a_0 & a_1 & \dots & \dots & a_{m+n-2k-2}\\ 0 & 0 & a_0 & \dots & \dots & a_{m+n-2k-3}\\ \dots & \dots & \dots & \dots & \dots & \dots\dots\dots\\ 0 & 0 & b_0 & \dots & \dots & b_{m+n-2k-3}\\ 0 & b_0 & b_1 & \dots & \dots & b_{m+n-2k-2}\\ b_0 & b_1 & b_2 & \dots & \dots & b_{m+n-2k-1}\end{vmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}(m-k)-times\\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}(n-k)-times\end{matrix} \tag{7}$$

in which there are $m-k$ rows of a's and $n-k$ rows of b's, and $a_i=0$ for $i>n$, and $b_j=0$ for $j>m$.

For $k=0$ we obtain the standard resultant of two polynomials. In other words we can say that the k–th subresultant with respect to the variable z of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last k rows and the first and the last k columns from its resultant written in the form (7) when $k=0$.

The geometrical meaning of the subresultants is based on the following lemma.

Lemma 2.3. (see [9], [14]). *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g)=R_z^{(1)}(f,g)=R_z^{(2)}(f,g)=\cdots=R_z^{(k-1)}(f,g)=0\neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 2.3 the following result.

Lemma 2.4. *Two polynomials $\tilde f(x_1,x_2,...,x_n)$ and $\tilde g(x_1,x_2,...,x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde f,\tilde g)=R_{x_j}^{(1)}(\tilde f,\tilde g)=R_{x_j}^{(2)}(\tilde f,\tilde g)=\cdots=R_{x_j}^{(k-1)}(\tilde f,\tilde g)=0\neq R_{x_j}^{(k)}(\tilde f,\tilde g),$$

where $R_{x_j}^{(i)}(\tilde f,\tilde g)=0$ in $\mathbb{R}[x_1,\dots x_{j-1},x_{j+1},\dots,x_n]$.

In paper [11] all the possible configurations of invariant lines are determined in the case, when the total multiplicity of these line (including the line at infinity) equals nine. For this propose in [11] there are proved some lemmas concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. Here we complete these results proving the following theorem.

Theorem 2.2. *If a cubic system* (2) *possess a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

(i)	*2 triplets*	$\Rightarrow$	$\mathcal{V}_1=\mathcal{V}_2=\mathcal{U}_1=0$;
(ii)	*1 triplet and 2 couples*	$\Rightarrow$	$\mathcal{V}_3=\mathcal{V}_4=\mathcal{U}_2=0$;
(iii)	*1 triplet and 1 couple*	$\Rightarrow$	$\mathcal{V}_4=\mathcal{V}_5=\mathcal{U}_2=0$;
(iv)	*one triplet*	$\Rightarrow$	$\mathcal{V}_4=\mathcal{U}_2=0$;
(v)	*3 couples*	$\Rightarrow$	$\mathcal{V}_3=0$;
(vi)	*2 couples*	$\Rightarrow$	$\mathcal{V}_5=0$.

Proof: The statements *(i), (ii), (iv)* and *(v)* follow directly from [11] (see Lemmas 1-3). It remains to prove the statement *(vi)* because the statement *(iii)* is a consequence of *(iv)* and *(vi)*.

So assume that a system (S) in the family (2) possesses two distinct couples of parallel invariant straight lines. According to [11] (see the proof of Lemma 3, page 1314) for the existence of two distinct couples of parallel invariant straight lines it is necessary the existence of two distinct solutions (α_i, β_i) $(i = 1, 2)$ of the system of equations

$$\begin{aligned} F_1(\alpha, \beta) &\equiv s\alpha^3 + (w - 2r)\alpha^2\beta + (q - 2v)\alpha\beta^2 + u\beta^3 = 0, \\ F_2(\alpha, \beta) &\equiv r\alpha^3 + (v - 2q)\alpha^2\beta + (p - 2u)\alpha\beta^2 + t\beta^3 = 0. \end{aligned} \tag{8}$$

On the other hand by Lemma 2.4 the homogeneous polynomials $F_1(\alpha, \beta)$ and $F_2(\alpha, \beta)$ has precisely 2 roots α_i/β_i or β_i/α_i $(i = 1, 2)$ if and only if

$$R^{(0)}_{\alpha/\beta}(F_1, F_2) = R^{(1)}_{\alpha/\beta}(F_1, F_2) = 0 \neq R^{(2)}_{\alpha/\beta}(F_1, F_2), \tag{9}$$

or (which is equivalent)

$$R^{(0)}_{\beta/\alpha}(F_1, F_2) = R^{(1)}_{\beta/\alpha}(F_1, F_2) = 0 \neq R^{(2)}_{\beta/\alpha}(F_1, F_2). \tag{10}$$

Remark 2.2. *We observe that as $F_1(\alpha, \beta)$ and $F_2(\alpha, \beta)$ are cubic forms in α and β then*

$$R^{(0)}_{\alpha/\beta}(F_1, F_2) = R^{(0)}_{\beta/\alpha}(F_1, F_2)$$

as this is the resultant of the polynomials F_1 and F_2, whereas for the subresultants we have

$$R^{(1)}_{\alpha/\beta}(F_1, F_2) \neq R^{(1)}_{\beta/\alpha}(F_1, F_2).$$

The straightforward computations yield for systems (2)

$$\mathcal{V}_5 = 96\big[\mathcal{S}_1x^4 + \mathcal{S}_2x^3y + \mathcal{S}_3x^2y^2 + \mathcal{S}_4xy^3 + \mathcal{S}_5y^4\big],$$

where

$$\begin{aligned} \mathcal{S}_1 =& -R^{(1)}_{\beta/\alpha}(F_1, F_2), \quad \mathcal{S}_5 = -R^{(1)}_{\alpha/\beta}(F_1, F_2), \\ \mathcal{S}_2 =& 4\big[4su^3 - u^2(4ps + 3rv) + u(p^2s - pqr + 2r^2t + 4qst + 2prv - 5stv - rtw) \\ &+ t(q^2r - pqs - 2rst - 4qrv + 2psv + 4rv^2 + stw)\big], \\ \mathcal{S}_3 =& 6\big[r^2(2qt - 2pu + 3u^2 - 4tv) + r(2stu - qtw + puw - 2u^2w + 2tvw) \\ &- s(2q^2t + st^2 - 2pqu + 4qu^2 - 5qtv + puv - 2u^2v + 2tv^2)\big], \\ \mathcal{S}_4 =& 4\big[4r^3t - r^2(3qu + 4tw) + r(2quw - 5qst - psu + 2su^2 + 4stv - uvw + tw^2) \\ &+ s(pst + 4q^2u - 2stu - 4quv + uv^2 + 2qtw - tvw)\big]. \end{aligned} \tag{11}$$

We claim that for the system (S) with two distinct couples of parallel invariant straight lines the conditions

$$R^{(0)}_{\alpha/\beta}(F_1, F_2) = \mathcal{V}_5 = 0$$

are necessary. Indeed, assume that for the system (S) the conditions (9) (or, equivalently, the conditions (10)) are satisfied. By (11) this implies $\mathcal{S}_1 = \mathcal{S}_5 = 0$. If we suppose the contrary, that the invariant polynomial $\mathcal{V}_5$ is not identically zero then the condition $\mathcal{S}_2^2 + \mathcal{S}_3^2 + \mathcal{S}_4^2 \neq 0$ holds. So the nonzero polynomial $\mathcal{V}_5$ has as factors x and y and clearly via a rotation of the phase plane of system (S) we could obtain a new invariant polynomial $\mathcal{V}_5'$ for which $\mathcal{S}_1'\mathcal{S}_5' \neq 0$. After rotation the system (S) will be transformed in the system (S') which also possesses two distinct couples of parallel invariant straight lines. So for this system according to (9) (and (10)) the conditions $\mathcal{S}_1' = \mathcal{S}_5' = 0$ must hold and the obtained contradiction proves our claim.

Remark 2.3. *The condition $\mathcal{V}_5 = 0$ implies*

$$R^{(0)}_{\alpha/\beta}(F_1, F_2) = R^{(0)}_{\beta/\alpha}(F_1, F_2) = 0,$$

i.e. the resultant of the polynomials F_1 and F_2 vanishes.

To show this it is sufficient to verify the following identity:

$$24R^{(0)}_{\alpha/\beta}(F_1,F_2) = 2(ru-st)\mathcal{S}_3 + 3(2rt-2qu+uv-tw)\mathcal{S}_4 - 12(qt-pu+2u^2-2tv)\mathcal{S}_5.$$

So the relations $\mathcal{S}_i = 0$ $(i=1,\ldots,5)$ imply $R^{(0)}_{\alpha/\beta}(F_1,F_2)=0$ and this completes the proof of the lemma. ∎

We rewrite the systems (2) in the coefficient form:

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3. \end{aligned} \tag{12}$$

Let $L(x,y) = Ux + Vy + W = 0$ be an invariant straight line of this family of cubic systems. Then, we have

$$UP(x,y) + VQ(x,y) = (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F),$$

and this identity provides the following 10 relations:

$$\begin{array}{ll} Eq_1 = (p-A)U + tV = 0, & Eq_6 = (2h-E)U + (2m-D)V - 2BW = 0, \\ Eq_2 = (3q-2B)U + (3u-A)V = 0, & Eq_7 = kU + (n-E)V - CW = 0, \\ Eq_3 = (3r-C)U + (3v-2B)V = 0, & Eq_8 = (c-F)U + eV - DW = 0 \\ Eq_4 = (s-C)U + Vw = 0, & Eq_9 = dU + (f-F)V - EW = 0, \\ Eq_5 = (g-D)U + lV - AW = 0, & Eq_{10} = aU + bV - FW = 0. \end{array} \tag{13}$$

It is well known that the infinite singularities (real or complex) of systems (12) are determined by the linear factors of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

So in the case of four distinct infinite singularities there can be three possibilities:

- all four infinite singularities are real;
- there are two real and two complex infinite singularities;
- all four infinite singularities are complex.

According to [11,] (see also [15]) we have the following result.

Lemma 2.5. *A cubic system* (12) *has 4 distinct infinite singularities if and only if* $\mathcal{D}_1 \neq 0$. *The types of these singularities are determined by the following conditions:*

(i) 4 *real iff* $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3 > 0$;

(ii) 2 *real and* 2 *imaginary iff* $\mathcal{D}_1 < 0$;

(iii) 4 *imaginary iff* $\mathcal{D}_1 > 0$ *and for every* (x,y) *where* $\mathcal{D}_2\mathcal{D}_3 \neq 0$ *either* $\mathcal{D}_2 < 0$ *or* $\mathcal{D}_3 < 0$.

Moreover in each one of these cases the respective homogeneous cubic parts of the system (12) *could be brought via a linear transformation to one of the canonical form* $(S_i)-(S_{iii})$, *respectively.*

$$(S_i)\quad \begin{cases} x' = (p+r)x^3 + (s+v)x^2y + qxy^2, & C_3 = xy(x-y)(rx+sy), \\ y' = px^2y + (r+v)xy^2 + (q+s)y^3, & rs(r+s) \neq 0 \end{cases}$$

$$(S_{ii})\quad \begin{cases} x' = (u+1)x^3 + (s+v)x^2y + rxy^2, & C_3 = x(sx+y)(x^2+y^2), \\ y' = -sx^3 + ux^2y + vxy^2 + (r-1)y^3, & \end{cases}$$

$$(S_{iii})\quad \begin{cases} x' = ux^3 + (p+q+v)x^2y + rxy^2 + qy^3, & C_3 = (px^2+qy^2)(x^2+y^2), \\ y' = -px^3 + ux^2y + vxy^2 + ry^3, & pq > 0 \end{cases}$$

In what follows we consider each one of three possibilities mentioned in the lemma above.

3 The proof of the Main Theorem

Following Lemma 2.5 we split the family of cubic systems having 4 distinct infinite singularities in three subfamilies depending on the types of these singularities and namely: systems with four real, systems with two real and two complex and systems with four complex infinite singularities.

For each one of this families (except the family with four complex infinite singularities, see page 48) the proof of the Main Theorem proceeds in 3 steps.

First we construct the cubic homogeneous parts $(\tilde{P}_3, \tilde{Q}_3)$ of systems for which the corresponding necessary conditions provided by Theorem 2.2 in order to have the a given number of triplets or/and couples of invariant parallel lines in the respective directions are satisfied.

Secondly keeping these cubic parts we perturb the quadratic and linear as well as the constant terms. Then using the equations (13) we determine these terms in order to get the required number of invariant lines in the required configuration. This leads us to the next remark.

Remark 3.1. *If the perturbed systems have a triplet (respectively a couple) of parallel lines in the direction $Ux + Vy = 0$ then the respective cubic homogeneous systems with right-hand parts $(\tilde{P}_3, \tilde{Q}_3)$ necessarily have the invariant line $Ux + Vy = 0$ of the multiplicity three (respectively two).*

Thus the second step ends with the construction of the canonical systems possessing the required configuration.

The third step consists in the determination of the affine invariant conditions necessary and sufficient for a cubic system to belong to the family of systems (constructed at the second step) which possess the corresponding configuration of invariant lines.

A. Cubic systems with four distinct real infinite singularities

Assuming that these systems possess four distinct real infinite singularities (i.e. the conditions $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3 > 0$ hold), according to Lemma 2.5 via a linear transformations they could be brought to the family of systems

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + (p+r)x^3 + (s+v)x^2y + qxy^2, \\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + px^2y + (r+v)xy^2 + (q+s)y^3 \end{aligned} \tag{14}$$

with $C_3 = xy(x-y)(rx+sy)$ and

$$rs(r+s) \neq 0. \tag{15}$$

As we have four real infinite singularities and the total multiplicity of the invariant lines (including the line at infinity) must be 8, then the systems above could have only one of the following four possible configurations (or potential configurations) of invariant lines:

$$(i)\ (3,\,3,\,1); \quad (ii)\ (3,\,2,\,2); \quad (iii)\ (3,\,2,\,1,\,1); \quad (iv)\ (2,\,2,\,2,\,1).$$

3.1 Systems with configuration (3, 3, 1)

Since we have two triplets of parallel invariant lines, according to Theorem 2.2 the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are necessary for systems (14). Moreover in [11, Section 5.1] it was proved that in this case via a linear transformation and time rescaling the cubic homogeneities of these systems could be brought to the forms:

$$\dot{x} = x^3, \quad \dot{y} = y^3. \tag{16}$$

So applying a translation we may assume $g = n = 0$ in the quadratic parts of systems (14) with the cubic homogeneities of the form (16). In such a way we get the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + y^3, \end{aligned} \tag{17}$$

for which we have $C_3(x, y) = xy(x - y)(x + y)$.

In order to find out the directions of two triplets according to Remark 3.1 we determine the multiplicity of the invariant lines of system (16). For this system we calculate (see the definition of the polynomial $H(X, Y, Z)$ on the page 6, Notation 2.3):

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 3X^3(X - Y)Y^3(X + Y).$$

So (16) possesses two triple invariant lines $x = 0$ and $y = 0$ and by Remark 3.1 systems (17) could have triplets of parallel invariant lines only in these two directions.

(i) The direction $x = 0$. Considering equations (13) we obtain

$$\begin{aligned} &U = 1,\ V = 0,\ A = 1,\ B = 0,\ C = 0,\ D = -W,\ E = 2h,\ F = c + W^2, \\ &Eq_7 = k,\ Eq_9 = d - 2hW,\ Eq_{10} = a - cW - W^3, \\ &Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0 \end{aligned} \tag{18}$$

and obviously we can have a triplet of parallel invariant line (which could coincide) in the direction $x = 0$ if and only if $k = d = h = 0$. Assuming that these conditions hold we consider the another direction for the second triplet.

(ii) The direction $y = 0$. In this case we have

$$\begin{aligned} &U = 0,\ V = 1,\ A = 0,\ B = 0,\ C = 1,\ D = 2m,\ E = -W,\ F = f + W^2, \\ &Eq_5 = l,\ Eq_8 = e - 2mW,\ Eq_{10} = b - fW - W^3, \\ &Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0 \end{aligned} \tag{19}$$

and again we conclude that for the existence of a triplet of parallel invariant lines for systems (17) the conditions $e = l = m = 0$ have to be satisfied.

It remains to examine the directions $y = x$ and $y = -x$ in order to determine the conditions for the existence of exactly one invariant line in one of these two directions.

For the *direction* $y = x$ we have

$$\begin{aligned} &U = 1,\ V = 1,\ A = 1,\ B = -1/2,\ C = 1,\ D = -W,\ E = 2W,\ F = c + W^2, \\ &Eq_7 = -3W;\quad Eq_9 = -c + f - 3W^2;\quad Eq_{10} = a + b - cW - W^3, \\ &Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0, \end{aligned} \tag{20}$$

whereas for the *direction* $y = -x$ we obtain

$$\begin{aligned} &U = 1,\ V = 1,\ A = 1,\ B = 1/2,\ C = 1,\ D = -W,\ E = -2W,\ F = c + W^2, \\ &Eq_7 = -3W;\quad Eq_9 = c - f + 3W^2;\quad Eq_{10} = a - b - cW - W^3, \\ &Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0. \end{aligned} \tag{21}$$

We observe that in each one of the cases we could have only one invariant line. Moreover the necessary and sufficient conditions for the existence of such a line are $c - f = a + b = 0$ in the first case and $c - f = a - b = 0$ in the second case.

Thus we conclude that for the existence of a single invariant line in one of the directions the following conditions are necessary and sufficient:

$$c - f = (a - b)(a + b) = 0, \quad a^2 + b^2 \neq 0.$$

Since the respective family of systems is of the form

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + cy + y^3 \tag{22}$$

we may assume $b = a \neq 0$ due to the rescaling $y \to -y$ in the case $b = -a$ and we arrive at the family of systems

$$\dot{x} = a + cx + x^3, \quad \dot{y} = a + cy + y^3. \tag{23}$$

These systems possess the invariant lines defined by the equations

$$x^3 + cx + a = 0, \quad x - y = 0, \quad y^3 + cy + a = 0.$$

Since the discriminant of the polynomial $x^3 + cx + c$ equals $\xi = -(27a^2 + 4c^3)$ (and clearly it coincides with the discriminant of the polynomial $y^3 + yx + a$) we conclude that the systems above possess 7 invariant affine lines which are as follows:

$$\begin{aligned}
\xi > 0 \quad &\Rightarrow \quad \text{7 real simple distinct;} \\
\xi < 0 \quad &\Rightarrow \quad \text{3 real and 4 complex all simple distinct;} \\
\xi = 0 \quad &\Rightarrow \quad \text{3 simple and 2 double all real distinct.}
\end{aligned}$$

As we have two triplets of parallel invariant lines it is clear that all 9 finite singularities (real and/or complex) are located at the intersections of these lines. It remains to observe that in the case of 4 complex lines we have only one real finite singularity: the intersection of the three real lines.

Thus we obtain the configuration *Config. 8.1* if $\xi > 0$; *Config. 8.2* if $\xi < 0$ and *Config. 8.3* if $\xi = 0$ (see Figure 1).

3.1.1 Invariant criteria for the realizations of the configurations *Config. 8.1, 8.2, 8.3*

First of all we recall that by Lemma 2.5 the conditions $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3 > 0$ are necessary and sufficient for a cubic systems to have four real distinct infinite singularities and via a linear transformation a cubic system could be brought to the form (14). Secondly by Theorem 2.2 for the cubic systems with two triplets of parallel invariant lines the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are satisfied and in this case a cubic system (14) via a linear transformation and a time rescaling could be brought to the form (17). And finally, it was proved in the previous subsection that a cubic system have the configurations *Config. 8.1-3-Config. 8.3-5* if and only if the conditions

$$k = d = h = l = e = m = c - f = a^2 - b^2 = 0, \quad a^2 + b^2 \neq 0 \tag{24}$$

are fulfilled for systems (17). Since for these systems the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ hold, according to the statement A_1) of the Main Theorem it remains to prove that the conditions (24) are equivalent to $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0$ and $\mathcal{K}_2 \neq 0$.

For systems (17) we calculate

$$\mathcal{L}_1 = -2^8 3^4 (lx^3 + 2mx^2y - 2hxy^2 - ky^3)$$

and hence the condition $\mathcal{L}_1 = 0$ gives $l = m = h = k = 0$. Then we obtain

$$\mathcal{L}_1 = 0, \quad \mathcal{L}_2 = 2^7 3^5 \big[-ex^2 - 6(c-f)xy + dy^2 \big]$$

and clearly the condition $\mathcal{L}_2 = 0$ implies $e = d = c - f = 0$ and this leads to the family of systems (22).

Next for systems (22) we calculate

$$\mathcal{K}_1 = 2^{18} 3^{15} 5^4 7^4 19 \cdot 41 (a^2 - b^2)(x^2 - y^2), \quad \mathcal{K}_2 = -27x^4y^4(bx - ay).$$

So clearly the condition $a^2 - b^2 = 0$ is equivalent to $\mathcal{K}_1 = 0$, whereas the condition $a^2 + b^2 \neq 0$ is equivalent to $\mathcal{K}_2 \neq 0$.

As it was mentioned above we could consider $b = a$ and then for systems (23) we need the expression $\text{Discrim}\,[a + cx + x^3, x] = -(27a^2 + 4c^3) = \xi$ which governs the type of the invariant lines (real, complex or coinciding) of these systems.

Finally for systems (23) we calculate

$$\mathcal{K}_3 = -5400(27a^2 + 4c^3)x^4(x-y)^2y^4(x+y)^2(x^2+y^2).$$

So clearly $27a^2 + 4c^3 = 0$ if and only if $\mathcal{K}_3 = 0$ and $\text{sign}\,(\mathcal{K}_3) = -\text{sign}\,(27a^2 + 4c^3) = \text{sign}\,(\xi)$.

To complete the proof of the statement A_1) of the Main Theorem we construct the respective canonical systems corresponding to each of the configurations. We consider systems (23). Since the equation $z^3 + cz + a = 0$ possesses at least one real solution, say $z = z_0$ then applying the translation $x = x_1 + z_0$, $y = y_1 + z_0$ to the systems (23) we get the family of systems

$$\dot{x} = x(f + gx + x^2), \quad \dot{y} = y(f + gy + y^2). \tag{25}$$

a) Assume first $\xi > 0$. Then the systems above possess three distinct real lines in the direction $x = 0$ as well as three such lines in the direction $y = 0$. Therefore $g^2 - 4f > 0$ and setting $g^2 - 4f = u^2 > 0$ we obtain $f = (g^2 - u^2)/4$. So this leads to the systems

$$\dot{x} = x(g - u + 2x)(g + u + 2x)/4, \quad \dot{y} = y(g - u + 2y)(g + u + 2y)/4,$$

where $g^2 - u^2 \neq 0$ because all the lines are distinct. Then via the rescaling

$$(x, y, t) \mapsto \big(2x/(g-u),\ 2y/(g-u),\ 4t/(g-u)^2\big)$$

we obtain the following 1-parameter family of systems

$$\dot{x} = x(x+1)(x-a), \quad \dot{y} = y(y+1)(y-a), \tag{26}$$

where $a = (g+u)/(u-g)$. These systems possess the invariant lines $x = 0$, $x = -1$, $x = a$ $y = 0$, $y = -1$, $y = a$, $y = x$.

We claim that the parameter $a \neq 0$ could be considered positive and different from 1. Indeed suppose that $a < 0$. If $a < -1$ then via the transformation $(x, y, t) \mapsto (-x-1,\ -y-1, t)$ we obtain the systems

$$\dot{x} = x(x+1)(x-a'), \quad \dot{y} = y(y+1)(y-a'),$$

where $a' = -(1+a) > 0$ as $a < -1$.

Assume now $-1 < a < 0$. Then applying the transformation $(x, y, t) \mapsto \big(a(x+1),\, a(y+1), t/a^2\big)$ we get the systems above with $a' = -(1+a)/a > 0$ as $-1 < a < 0$.

On the other hand considering the conditions provided by the statement A_1) of the Main Theorem we calculate for systems (26):

$$\mathcal{K}_2 = (a-1)(2+a)(1+2a)x^4(x-y)y^4.$$

So the condition $\mathcal{K}_2 \neq 0$ implies $a \neq 1$. Therefore our claim is proved and for the canonical systems (26) we assume $a > 0$ and $a \neq 1$.

b) Admitting $\xi < 0$ we have $g^2 - 4f < 0$ and we can set $g^2 - 4f = -u^2 < 0$. Then $f = (g^2 + u^2)/4$ and systems (25) becomes

$$\dot{x} = x\big[(2x+g)^2 + u^2\big]/4, \quad \dot{y} = y\big[(2y+g)^2 + u^2\big]/4,$$

and after the additional rescaling $(x, y, t) \mapsto \big(ux/2,\ uy/2,\ 4t/u^2\big)$ we arrive at the systems

$$\dot{x} = x\big[(x+a)^2 + 1\big], \quad \dot{y} = y\big[(y+a)^2 + 1\big], \tag{27}$$

where $a = g/u$. We remark that these systems possess the invariant lines $x = 0$, $x = -a \pm i$, $y = 0$, $y = -a \pm i$, $y = x$. For these systems we have

$$\mathcal{K}_2 = 2a(9 + a^2)x^4(x - y)y^4$$

and considering the condition $\mathcal{K}_2 \neq 0$ we obtain $a \neq 0$.

c) Suppose finally $\xi = 0$, i.e. the equation $z^3 + cz + a = 0$ possesses a real solution z_0 of the multiplicity at least two. Then applying the translation $x = x_1 + z_0$, $y = y_1 + z_0$ to the systems (23) we get the family of systems

$$\dot{x} = x^2(g + x), \quad \dot{y} = y^2(g + y).$$

For these systems we calculate

$$\mathcal{K}_2 = -2g^3x^4(x - y)y^4$$

and hence the condition $\mathcal{K}_2 \neq 0$ yields $g \neq 0$. Therefore via the rescaling $(x, y, t) \mapsto (gx,\ gy,\ t/g^2)$ we obtain the system

$$\dot{x} = x^2(1 + x), \quad \dot{y} = y^2(1 + y). \tag{28}$$

3.2 Systems with configuration $(3, 2, 2)$

Lemma 3.1. *A cubic system* (14) *could not have a configuration of invariant lines of the type* $(3, 2, 2)$.

Proof: For having the configuration (3, 2, 2) a cubic system has to possess three couples of parallel invariant lines and, moreover, one couple must increase up to a triplet. Thus according to Theorem 2.2, if a cubic system possesses 7 invariant affine straight lines in the configuration (3, 2, 2), then necessarily the conditions $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$ hold.

As it was proved in [11, Section 5.2] in this case via an affine transformation and time rescaling such systems could be brought to the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + 2x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + 3xy^2 - y^3. \end{aligned} \tag{29}$$

On the other hand for the respective homogeneous system $\dot{x} = 2x^3$, $\dot{y} = 3xy^2 - y^3$ we calculate

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 6X^3(X - Y)(2X - Y)^2Y^2.$$

So by Remark 3.1 the systems (29) could possess one triplet of invariant lines only in the direction $x = 0$. As regard the two couples of parallel lines we conclude that they could be only in the direction $y = 0$ and $y = 2x$. We shall examine each one of these directions.

(i) The direction $x = 0$. We obtain

$$\begin{gathered} U = 1,\ V = 0,\ A = 2,\ B = 0,\ C = 0,\ D = -2W,\ E = 2h,\ F = c + 2W^2, \\ Eq_7 = k,\ Eq_9 = d - 2hW,\ Eq_{10} = a - cW - 2W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0 \end{gathered} \tag{30}$$

and obviously we can have a triplet of parallel invariant line (which could coincide) in the direction $x = 0$ if and only if $k = d = h = 0$. Assuming that these conditions hold we consider the directions corresponding to the couples of parallel lines.

(ii) The direction $y = 0$. In this case we have

$$\begin{gathered} U = 0,\ V = 1,\ A = 0,\ B = 3/2,\ C = -1,\ D = 2m - 3W,\ E = W,\ F = f - W^2, \\ Eq_5 = l,\ Eq_8 = e - 2mW + 3W^2,\ Eq_{10} = b - fW + W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0 \end{gathered} \tag{31}$$

and again we conclude that for the existence of a couple of parallel invariant lines for systems (29) in this direction it is necessary and sufficient $l = 0$ and

$$R_W^{(0)}(Eq_8, Eq_{10}) = R_W^{(1)}(Eq_8, Eq_{10}) = 0.$$

We calculate

$$R_W^{(1)}(Eq_8, Eq_{10}) = -3e - 9f + 4m^2 = 0$$

and this implies $e = 4m^2/3 - 3f$. Then we obtain

$$R_W^{(0)}(Eq_8, Eq_{10}) = (27b + 18fm - 8m^3)^2/27 = 0$$

and hence we get $b = -2m(9f - 4m^2)/27$. Then we have

$$Eq_8 = (-9f + 4m^2 - 6mW + 9W^2)/3, \quad Eq_{10} = (2m + 3W)(-9f + 4m^2 - 6mW + 9W^2)/29$$

and therefore systems (29) possess two parallel lines in the direction $y = 0$ and these lines could be real (or complex) distinct or coinciding.

Considering the relations detected above among the coefficients of systems (29) we examine another direction in which could be a couple of parallel lines.

(iii) The direction $y = 2x$. In this case we have

$$\begin{gathered} U = 2,\ V = -1,\ A = 2,\ B = 1/2,\ C = -1,\ D = -W,\ E = -W,\ F = f - W^2, \\ Eq_6 = -2m,\ Eq_8 = (6c + 3f - 4m^2 + 9W^2)/3, \\ Eq_{10} = (54a + 18fm - 8m^3 - 27fW + 27W^3)/27, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = Eq_9 = 0 \end{gathered} \tag{32}$$

and we conclude that for the existence of a couple of parallel invariant lines for systems (29) in the direction $y = 2x$ it is necessary $m = 0$. Then we get the relations $g = n = k = d = h = l = m = 0,\ e = -3f$ and this leads to the equations

$$Eq_8 = 2c + f + 3W^2 = 0,\ Eq_{10} = 2a - fW + W^3 = 0.$$

By Lemma 2.4 these polynomials have two common roots W_1 and W_2 if and only if

$$R_W^{(0)}(Eq_8, Eq_{10}) = R_W^{(1)}(Eq_8, Eq_{10}) = 0.$$

We calculate

$$R_W^{(1)}(Eq_8, Eq_{10}) = -6(c + 2f) = 0,$$

i.e. $c = -2f$ and then we obtain

$$R_W^{(0)}(Eq_8, Eq_{10}) = 108a^2 = 0.$$

So $a = 0$ and we arrive at the family of systems

$$\dot{x} = 2x(x^2 - f), \quad \dot{y} = (y^2 - f)(3x - y),$$

which possess 8 invariant affine lines $x = 0$, $x^2 - f = 0$, $y^2 - f = 0$, $(2x - y)^2 - f = 0$ and $y = x$ in the configuration (3, 2, 2, 1). This completes the proof of Lemma 3.1.

3.3 Systems with configuration $(3, 2, 1, 1)$

In this subsection we construct the cubic systems with 4 real infinite singular points which possess 7 distinct invariant affine straight lines with configuration or potential configuration $(3, 2, 1, 1)$, having total multiplicity 8, as always the invariant straight line at the infinity is considered.

For having the configuration $(3, 2, 1, 1)$ a cubic system has to possess two couples of parallel invariant lines and, moreover, one couple must increase up to a triplet. Thus, according to Theorem 2.2, if a cubic system possesses 7 invariant straight lines in the configuration $(3, 2, 1, 1)$, then necessarily the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ hold.

3.3.1 Construction of the corresponding cubic homogeneities

As a first step we shall construct the cubic homogeneous parts of systems (14) for which the conditions above are fulfilled. Since we have 4 real infinite distinct singularities, according to Lemma 2.5 we consider the family of systems

$$\dot{x} = (p+r)x^3 + (s+v)x^2y + qxy^2, \quad \dot{y} = px^2y + (r+v)xy^2 + (q+s)y^3 \tag{33}$$

with $rs(r+s) \neq 0$ and we shall force the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ to be satisfied.

We observe that the invariant polynomials $\mathcal{V}_5$, $\mathcal{U}_2$ and $\mathcal{V}_4$ are homogeneous polynomials of degree four in x and y. So we shall use the following notations:

$$\mathcal{V}_5 = \sum_{j=0}^{4} \mathcal{V}_{5j} x^{4-j} y^j, \quad \mathcal{U}_2 = \sum_{j=0}^{4} \mathcal{U}_{2j} x^{4-j} y^j.$$

Regarding the polynomial $\mathcal{V}_4$, a straightforward computation of its value for systems (33) yields

$$\begin{aligned} &\mathcal{V}_4 = -9216\, \widehat{\mathcal{V}}_4\, C_3(x,y), \quad \text{where} \\ &\widehat{\mathcal{V}}_4 = r^2(3q+s+v) + r(2pq - s^2 + 3qv + v^2) - s(2pq + 3ps + 3pv + sv + v^2). \end{aligned} \tag{34}$$

As for systems (33) we have $C_3 = xy(x-y)(rx+sy) \neq 0$, we conclude that the condition $\mathcal{V}_4 = 0$ for these systems is equivalent to $\widehat{\mathcal{V}}_4 = 0$.

Calculating the value of the polynomial $\mathcal{V}_5$ for systems (33) we obtain

$$\mathcal{V}_{52} = 16pq(qr + ps + 3rs)/3, \quad \mathcal{V}_{53} = 108pq(qs - qr - rs + 2s^2 + sv)/9$$

and we consider two cases: $pq \neq 0$ and $pq = 0$.

3.3.1.1 The case $pq \neq 0$ Then the condition $\mathcal{V}_5 = 0$ gives

$$qr + ps + 3rs = qs - qr - rs + 2s^2 + sv = 0$$

and considering the condition (15) we get $p = -r(q+3s)/s$ and $v = (qr - qs + rs - 2s^2)/s$. In this case for systems (33) we obtain $\mathcal{V}_5 = 0$ and we calculate

$$\mathcal{U}_{24} = -3 \cdot 2^{12} q(q+3s)(qr + qs + 2rs + s^2)(qr + qs + rs + 2s^2)/s^2.$$

Therefore, as $q(q+3s) \neq 0$ (due to $pq \neq 0$) we obtain

$$\left[q(r+s) + 2rs + s^2\right]\left[q(r+s) + rs + 2s^2\right] = 0$$

and we shall examine these two subcases taking into consideration the condition (15).

3.3.1.1.1 The subcase $q = -s(2r+s)/(r+s)$. Then we obtain the relations

$$p = -\frac{r(r+2s)}{r+s}, \quad q = -\frac{s(2r+s)}{r+s}, \quad v = -\frac{r^2+s^2}{r+s}$$

and in this case all three polynomials $\mathcal{V}_5$, $\mathcal{U}_2$ and $\widehat{\mathcal{V}}_4$ vanish. On the other hand due to the additional time rescaling $t \to (r+s)t/s$, systems (33) become the systems

$$\begin{aligned} \dot{x} &= rx^3 + r(r-1)x^2y + (1+2r)xy^2, \\ \dot{y} &= r(2+r)x^2y - (r-1)xy^2 + ry^3. \end{aligned} \tag{35}$$

3.3.1.1.2 The subcase $q = -s(r+2s)/(r+s)$. This leads to the relations

$$p = -\frac{r(2r+s)}{r+s}, \quad q = -\frac{s(r+2s)}{r+s}, \quad v = -\frac{2rs}{r+s}$$

and we again get $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. In this case after the additional time rescaling $t \to (r+s)t/r$ systems (33) become the systems

$$\begin{aligned} \dot{x} &= x^3 + s(1-s)x^2y + s(1+2s)xy^2, \\ \dot{y} &= (2+s)x^2y + (s-1)xy^2 + s^2y^3. \end{aligned} \tag{36}$$

Remark 3.2. *We observe, that via the change* $(x, y, t, r) \mapsto (-x/r, y, rt, s)$ *the systems* (35) *could be written as systems* (36).

3.3.1.2 The case $pq = 0$ Then without loss of generality we may assume $p = 0$ for systems (33) due to the change

$$(x, y, p, q, r, s) \ \mapsto \ (y, x, q, p, s, r) \tag{37}$$

which preserves these systems. Then we have

$$\mathcal{V}_{55} = -32q(q+2r+2s+v)(qr+rs+s^2-sv)/9$$

and we consider the three subcases given by these three factors.

3.3.1.2.1 The subcase $q = 0$. For systems (33) in this case we have

$$\mathcal{U}_{21} = 2^{12}3^2r(r-s)(r+v)(s+v) = 0$$

and due to the condition (15) (i.e. $r \neq 0$) and to the change (37) (as we have $p = q = 0$) we arrive at the condition $(r-s)(r+v) = 0$. So we examine the following three possibilities:
1) $r - s = 0 \neq r + v$; 2) $r - s = 0 = r + v$; 3) $r - s \neq 0 = r + v$.

1) *The possibility* $r - s = 0 \neq r + v$. We set $s = r$ and we calculate

$$\mathcal{U}_{22} = 3 \cdot 2^{12}(2r - v)(r+v)^2(4r+v) = 0.$$

Hence due to $r + v \neq 0$ we have either $v = 2r$ or $v = -4r$.

a) *Assume first* $v = 2r$. In this case we get $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$ and due to the additional time rescaling $t \to t/r$ we obtain the system

$$\dot{x} = x^3 + 3x^2y, \quad \dot{y} = 3xy^2 + y^3. \tag{38}$$

We observe that this system belongs to the family of systems (36) for $s = 1$.

b) *Suppose now* $v = -4r$. Then we have $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$ and after the the rescaling $(x, y, t) \mapsto (x, -y, t/r)$ we arrive at the system (38).

2) *The possibility* $r - s = 0 = r + v$. Then we have $s = r$ and $v = -r$ and this implies $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. So due to a time rescaling we get the system $\dot{x} = x^3$, $\dot{y} = y^3$, which could be brought to the system (38) via the transformation

$$x_1 = x + y, \quad y_1 = x - y, \quad t_1 = t/4.$$

3) *The possibility* $r - s \neq 0 = r + v$. Then we have $v = -r$ and we again obtain $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. On the other hand after the time rescaling $t \to t/r$ we arrive at the 1-parameter family of systems

$$\dot{x} = x^3 + (s-1)x^2y, \quad \dot{y} = sy^3, \tag{39}$$

for which $C_3 = x(x-y)y(x+sy)$ with $rs(r+s) \neq 0$. We claim that for any fixed value s_0 of the parameter s satisfying the condition $s_0(s_0+1) \neq 0$ there exist a linear transformation and time rescaling such that this system can be brought to the system (35) corresponding to the parameter $r = s_0$. To prove this it is sufficient to present the respective transformation:

$$x_1 = x - y, \quad y_1 = x + s_0y, \quad t_1 = t/(s_0+1)^2.$$

3.3.1.2.2 The subcase $q \neq 0 = (q + 2r + 2s + v)$. Then $v = -(q + 2r + 2s)$ and for systems (33) we calculate

$$\mathcal{U}_{21} = -2^{12}3^2 r(q - r + s)(2r + s)(q + r + 2s) = 0.$$

Since $r \neq 0$ we arrive at the three possibilities given by the other three factors above.

1) *The possibility $q - r + s = 0$.* We set $q = r - s$ and we get $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. On the other hand after the time rescaling $t \to t/r$ we arrive at the 1-parameter family of systems

$$\dot{x} = x^3 - 3x^2y + (1 - s)xy^2, \quad \dot{y} = -(2 + s)xy^2 + y^3. \tag{40}$$

We observe that for any fixed value s_0 (here $s_0(s_0+1) \neq 0$) of the parameter s the transformation

$$x_1 = (1 + s_0)x, \quad y_1 = x + s_0y, \quad t_1 = -(1 + s_0)t/s_0^2$$

will brought the system above with $s = s_0$ to the system (35) with $r = -1/(1 + s_0)$.

2) *The possibility $q - r + s \neq 0 = 2r + s$.* Then $s = -2r$ (this implies $q - r + s = q - 3r \neq 0$) and we obtain

$$\mathcal{U}_{22} = -3 \cdot 2^{12} q(q - 6r)(q - 3r)^2 = 0.$$

Therefore due to $q(q - 3r) \neq 0$ we get $q = 6r$. Then via a time rescaling we arrive at the system

$$\dot{x} = x^3 - 6x^2y + 6xy^2, \quad \dot{y} = -3xy^2 + 4y^3,$$

which belongs to the family of systems (36) for $s = -2$.

3) *The possibility $(q - r + s)(2r + s) \neq 0 = q + r + 2s$.* We set $q = -(r + 2s)$ and this gives $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. On the other hand applying a time rescaling we arrive at the 1-parameter family of systems

$$\dot{x} = x^3 + (s - 1)x^2y - (2s + 1)xy^2, \quad \dot{y} = -(s + 1)y^3.$$

We observe that for any fixed value s_0 (here $s_0(s_0+1) \neq 0$) of the parameter s the transformation

$$x_1 = x, \quad y_1 = x + s_0y, \quad t_1 = t/s_0^2$$

will brought the system above with $s = s_0$ to the system (35) with $r = -(1 + s_0)$.

3.3.1.2.3 The subcase $q(q + 2r + 2s + v) \neq 0 = (qr + rs + s^2 - sv)$. We obtain $v = r + s + qr/s$ and this implies $q + 2r + 2s + v = (r + s)(q + 3s)/s \neq 0$. In this case for systems (33) we calculate

$$\mathcal{U}_{24} = -3 \cdot 2^{12} q(q + 3s)(qr + rs - s^2)(qr + 2rs + s^2)/s^2 = 0$$

and due to the condition $q(q + 3s) \neq 0$ we obtain either $qr + rs - s^2 = 0$, or $qr + 2rs + s^2 = 0$.

1) *The possibility $qr+rs-s^2 = 0$.* We obtain $q = s(s-r)/r$ and this yields $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. On the other hand applying a time rescaling we arrive at the 1-parameter family of systems

$$\dot{x} = x^3 + 3sx^2y - (1 - s)sxy^2, \quad \dot{y} = (1 + 2s)xy^2 + s^2y^3.$$

We observe that for any fixed value s_0 of the parameter s (where $s_0(s_0 + 1) \neq 0$) the transformation

$$x_1 = (1 + s_0)x/s_0, \quad y_1 = x - y, \quad t_1 = -s_0(1 + s_0)t$$

will brought the system above to the system (35) with $r = -s_0/(1 + s_0)$.

2) *The possibility $qr + 2rs + s^2 = 0$.* Then we have $q = -s(s + 2r)/r$ and this again implies $\mathcal{V}_5 = \mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. In this case applying a time rescaling we arrive at the 1-parameter family of systems

$$\dot{x} = x^3 + (s - 1)x^2y - s(2 + s)xy^2, \quad \dot{y} = -s(1 + s)y^3.$$

It remains to observe that for any fixed value s_0 (where $s_0(s_0+1)\neq 0$) of the parameter s the systems above can be brought to the system (35) with $r=-s_0/(1+s_0)$ via the transformation

$$x_1 = x, \quad y_1 = x - y, \quad t_1 = s_0^2 t.$$

Observation 3.1. *Examining all the possibilities above we conclude, that the condition* $\mathcal{V}_5 = \mathcal{U}_2 = 0$ *implies* $\mathcal{V}_4 = 0$.

So it remains to consider the cubic systems with cubic homogeneities (35). However we prefer to pass to simpler ones as it is mentioned by the next remark.

Remark 3.3. *In order to construct the whole class of systems possessing the configuration or potential configuration* $(3,2,1,1)$ *it is sufficient to consider the family of cubic systems with the homogeneous cubic parts of the forms* (35). *However we shall consider the simpler cubic parts*

$$\dot{x} = rx^3, \qquad \dot{y} = (r-1)xy^2 + y^3, \tag{41}$$

which we obtain from (35) *applying the transformation* $x_1 = y - x$, $y_1 = rx + y$.

Since $r\neq 0$, due to a translation we may assume $g = n = 0$ in the quadratic parts of systems (12) with the cubic homogeneities of the form (41).

Thus considering the remark above we get the next result.

Lemma 3.2. *Assume that a cubic system* (14) *possesses* 7 *invariant affine straight lines with configuration or potential configuration* (3, 2, 1, 1). *Then via a linear transformation and a time rescaling this system could be brought to a system belonging to the following family:*

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + rx^3, \quad r(r+1)\neq 0,\\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + (r-1)xy^2 + y^3. \end{aligned} \tag{42}$$

3.3.2 Construction of the cubic systems possessing configuration or potential configuration (3, 2, 1, 1)

In what follows we shall determine necessary and sufficient conditions for a system (42) to have a configuration or potential configuration (3, 2, 1, 1).

Considering Remark 3.1 for the homogeneous systems (41) corresponding to (42) we calculate

$$H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^3(X-Y)Y^2(rX+Y). \tag{43}$$

So the invariant line $x=0$ (respectively $y=0$ of systems (41) is of multiplicity three (respectively two). Hence by Remark 3.1 the systems (42) could possess one triplet (respectively one couple) of invariant lines in the direction $x=0$ (respectively $y=0$). However for some values of the parameter r the common divisor $\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$ could contain additional factors. To detect them we calculate:

$$\begin{aligned} &Res_X(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)Y^3,\\ &Res_X(\mathcal{G}_3/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)(4-r)(4r-1)Y^5;\\ &Res_Y(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)X^3,\\ &Res_Y(\mathcal{G}_3/H,\, \mathcal{G}_1/H) = (r-1)(2+r)(1+2r)(r-4)(4r-1)X^3. \end{aligned}$$

Therefore in order to have a nonconstant common factor of the polynomials $\mathcal{G}_1/H$, $\mathcal{G}_2/H$ and $\mathcal{G}_3/H$ the condition $(r-1)(2+r)(1+2r)=0$ has to be satisfied. And in this case we obtain

$$\begin{aligned} &H(X,Y,Z) = \gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3) = X^3Y^3(X-Y)(X+Y) \quad \text{if} \quad r=1;\\ &H(X,Y,Z) = \gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3) = X^3Y^2(X-Y)(2X-Y)^2 \quad \text{if} \quad r=-2;\\ &H(X,Y,Z) = \gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3) = X^3Y^2(X-Y)^2(X-2Y) \quad \text{if} \quad r=-1/2. \end{aligned}$$

Thus considering Remark 3.1 we arrive at the next remark.

Remark 3.4. *A cubic system* (42) *could possess: (i) a triplet of parallel invariant lines in the direction* $x = 0$ *or* $y = 0$ *and in the second case the condition* $r = 1$ *holds; (ii) a couple of parallel invariant lines either in the direction* $y = 0$*, or* $y = x$ *(if* $r = -1/2$*), or* $y = 2x$ *(if* $r = -2$*).*

3.3.2.1 The case $(r-1)(1+2r)(2+r) \neq 0$**.** Then by the remark above systems (42) could have a triplet (respectively a couple) of parallel invariant lines only in the direction $x = 0$ (respectively $y = 0$). We shall examine these directions.

(i) The direction $x = 0$. Considering the equations (13) we obtain

$$\begin{gathered} U = 1,\ V = 0,\ A = r,\ B = C = 0,\ D = -rW,\ E = 2h,\ F = c + rW^2, \\ Eq_7 = k,\ Eq_9 = d - 2hW,\ Eq_{10} = a - cW - rW^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0. \end{gathered} \tag{44}$$

and obviously we can have a triplet of parallel invariant line (which could coincide) in the direction $x = 0$ if and only if $k = d = h = 0$.

(ii) The direction $y = 0$. In this case considering the equations (13) and conditions above we have

$$\begin{gathered} U = 0,\ V = 1,\ A = 0,\ B = (r-1)/2,\ C = 1, \\ D = 2m + W - rW,\ E = -W,\ F = f + W^2,\ Eq_5 = l, \\ Eq_8 = e - 2mW + (r-1)W^2,\ Eq_{10} = b - fW - W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0. \end{gathered} \tag{45}$$

So we conclude that for the existence of a couple of parallel invariant lines for systems (42) in this direction it is necessary and sufficient $l = 0$ and

$$R_W^{(0)}(Eq_8, Eq_{10}) = R_W^{(1)}(Eq_8, Eq_{10}) = 0.$$

We calculate

$$R_W^{(1)}(Eq_8, Eq_{10}) = -4m^2 + e(r-1) - f(r-1)^2 = 0$$

and as $r \neq 1$ we have $e = \big[4m^2 + f(r-1)^2\big]/(r-1)$. Then we obtain

$$R_W^{(0)}(Eq_8, Eq_{10}) = \frac{8m^3 + 2fm(r-1)^2 + b(r-1)^3}{(r-1)^3} = 0$$

and hence we get $b = -2m\big[4m^2 + f(r-1)^2\big]/(r-1)^3$. Herein we have

$$\begin{gathered} Eq_8 = \frac{4m^2 - f(r-1)^2 - 2m(r-1)W + (r-1)^2W^2}{r-1} \equiv \frac{1}{r-1}Z(f,m,r,W), \\ Eq_{10} = -\frac{1}{(r-1)^3}(2m - W + rW)Z(f,m,r,W) \end{gathered}$$

and therefore systems (42) possess in the direction $y = 0$ two parallel lines, which could be real (or complex) distinct or coinciding.

Thus in order to have a triplet in the direction $x = 0$ and a couple in the direction $y = 0$ in the case $r - 1 \neq 0$ the following conditions are necessary and sufficient for systems (42):

$$\begin{gathered} k = d = h = l = 0, \quad e = \big[4m^2 + f(r-1)^2\big]/(r-1), \\ b = -2m\big[4m^2 + f(r-1)^2\big]/(r-1)^3. \end{gathered} \tag{46}$$

(iii) The direction $y = x$. In this case we have

$$\begin{gathered} U = -1,\ V = 1,\ A = r,\ B = r/2,\ C = 1,\ D = rW - l,\ E = 2h - l - 2m + 2rW, \\ F = c - e - lW + rW^2,\quad Eq_7 = l - 2h - k + 2m - (1+2r)W, \\ Eq_9 = -c - d + e + f + 2(l - h + m)W - 3rW^2, \\ Eq_{10} = -a + b + (e - c)W + lW^2 - rW^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0. \end{gathered} \tag{47}$$

Since $2r+1\neq 0$ these equations could have only one solution. Considering the conditions (46) the equation $Eq_7=0$ gives $W=2m/(2r+1)$. Then the equations $Eq_9=0$ and $Eq_{10}=0$ yield

$$c=fr+\frac{12m^2r(2+r)}{(-1+r)(1+2r)^2},\quad a=-\frac{6fmr}{(r-1)(1+2r)}-\frac{72m^3r(1+r+r^2)}{(r-1)^3(1+2r)^3}. \tag{48}$$

(iv) The direction $rx+y=0$. In this case we have

$$\begin{gathered}U=r,\ V=1,\ A=r,\ B=-1/2,\ C=1,\ D=2m+2hr-kr^2+(1+r)W,\\ E=kr-W,\ F=f+dr-krW+W^2,\\ Eq_5=l-2mr-2hr^2+kr^3-r(2+r)W,\\ Eq_{10}=b+ar-(f+dr)W+krW^2-W^3,\\ Eq_8=e+cr-fr-dr^2-2(m+hr-kr^2)W-(1+2r)W^2,\\ Eq_1=Eq_2=Eq_3=Eq_4=Eq_6=Eq_7=Eq_9=0.\end{gathered} \tag{49}$$

Since $r(r+2)\neq 0$, considering the conditions (46) and (48) the equation $Eq_5=0$ gives $W=-2m/(r+2)$. Then the equations $Eq_8=0$ and $Eq_{10}=0$ take the form

$$Eq_8=(r+1)U(f,m,r)=0,\quad Eq_{10}=-\frac{6m(1+r)(1+r+r^2)U(f,m,r)}{(-1+r)(2+r)(1+2r)}=0$$

where

$$U(f,m,r)=f(r-1)+\frac{12m^2(1+5r+15r^2+5r^3+r^4)}{(-1+r)(2+r)^2(1+2r)^2}.$$

Since $r+1\neq 0$ the condition $Eq_8=0$ gives $U(f,m,r)=0$ and then $Eq_{10}=0$. The condition $U(f,m,r)=0$ implies $f=\dfrac{12m^2(1+5r+15r^2+5r^3+r^4)}{(r-1)^2(2+r)^2(1+2r)^2}$ and taking into account (46) and (48) we arrive at the following relations among the parameters of systems (42) in the case $(r-1)(2+r)(1+2r)\neq 0$:

$$\begin{gathered}k=d=h=l=0,\quad f=-\frac{12m^2(1+5r+15r^2+5r^3+r^4)}{(r-1)^2(2+r)^2(1+2r)^2},\\ b=-\frac{8m^3(1+7r+r^2)}{(r-1)(2+r)^2(1+2r)^2},\quad c=-\frac{108m^2r(1+r+r^2)}{(r-1)^2(2+r)^2(1+2r)^2},\\ a=\frac{216m^3r}{(r-1)^2(2+r)^2(1+2r)^2},\quad e=\frac{4m^2(r-1)(1+7r+r^2)}{(2+r)^2(1+2r)^2}.\end{gathered} \tag{50}$$

So we obtain the following 2-parameter family of systems

$$\begin{aligned}\dot x&=r\Big[x+\frac{6m}{(r-1)(2+r)}\Big]\Big[x-\frac{6m}{(r-1)(1+2r)}\Big]\Big[x-\frac{6m}{(1+2r)(2+r)}\Big],\\ \dot y&=\Big[y+\frac{2m(r-1)}{(2+r)(1+2r)}\Big]\Big[y+\frac{2m(1+7r+r^2)}{(r-1)(2+r)(1+2r)}\Big]\Big[y+(r-1)x-\frac{2m}{(r-1)}\Big].\end{aligned}$$

Since $(r-1)(2+r)(1+2r)\neq 0$ we set a new parameter u as follows: $m=\dfrac{u}{6}(r-1)(2+r)(1+2r)$ and this leads to the systems

$$\begin{aligned}\dot x&=r\big[x+u(1+2r)\big]\big[x-u(2+r)\big]\big[x-u(r-1)\big],\\ \dot y&=\big[y+u(r-1)^2/3\big]\big[y+u(1+7r+r^2)/3\big]\big[y+(r-1)x-u(2+r)(1+2r)/3\big].\end{aligned} \tag{51}$$

Assume first that $u\neq 0$. Since $r\neq 0$ by means of the transformation

$$x_1=\frac{x}{3u}+\frac{1-r}{3},\quad y_1=-\frac{y}{3ru}-\frac{(r-1)^2}{9r},\quad t_1=9ru^2t$$

systems (51) become the systems (we keep the old notations of variables)

$$\dot{x} = x(x-1)(x+r),$$
$$\dot{y} = y(y-1)\big[(1-r)x + ry + r\big]. \tag{52}$$

We observe that the systems above possess seven invariant affine lines

$$L_1 = x,\ \ L_2 = x-1,\ \ L_3 = x+r,\ \ L_4 = y,\ \ L_5 = y-1,\ \ L_6 = x-y,\ \ L_7 = x+ry$$

in the configuration $(3,2,1,1)$. Since $r(r+1) \neq 0$ we conclude that we could not have coinciding invariant lines.

On the other hand systems (52) possess 9 finite singularities:

$$(0,0),\ \ (0,1),\ \ (0,-1),\ \ (1,0),\ \ (-r,0),\ \ (1,1),\ (1,-1/r),\ \ (-r,1),\ \ (-r,-r).$$

We observe that 8 singular point are located at the intersections of the invariant lines, whereas the ninth one (and namely, (0,-1)) is located on the invariant line $L_1 = 0$. Moreover, the positions of the invariant lines which form the triplet depends on the parameter r. More precisely, if $r > 0$ then the line L_1 is placed between the parallel invariant lines L_2 and L_3 and in the case $r < 0$ the lines L_2 and L_3 are located on the right with respect to L_1. Thus taking into consideration that on the line L_1 it is located the unique point of the intersection of 4 invariant lines L_1, L_4, L_6 and L_7 (the origin of coordinate), we arrive at two different configurations. Namely we obtain the configuration *Config. 8.4* if $r > 0$ and *Config. 8.5* if $r < 0$ (see FIGURE 1).

Assume now $u = 0$. Then systems (51) become the homogeneous systems (41) possessing invariant lines $x = 0$ (triple), $y = 0$ (double), $y = x$ and $y = -rx$ (see (43)). This leads to the configuration *Config. 8.6* (see FIGURE 1).

3.3.2.2 The case $(r-1)(1+2r)(2+r) = 0$.

We examine each one of the three cases given by the three factors. However we observe the subcase $r = -2$ could be brought to the subcase $r = -1/2$ via the rescaling $x \to x/2$. Therefore we consider only two subcases: $r = 1$ and $r = -1/2$.

3.3.2.2.1 The subcase $r = 1$.

According to Remark 3.4 systems (42) could have the following parallel invariant lines: *(i)* a triplet in the direction $x = 0$ and a couple in the direction $y = 0$; *(ii)* a triplet in the direction $y = 0$ and a couple in the direction $x = 0$.

However for $r = 1$ from (44) and (45) it follows that in each one of these directions we could have either one, or three parallel invariant lines. So in the case $r = 1$ we could not have a configuration (or potential configuration) of the type $(3,2,1,1)$.

3.3.2.2.2 The subcase $r = -1/2$.

According to Remark 3.4 systems (42) could have a triplet of parallel invariant lines in the direction $x = 0$ and a couple either in the direction $y = 0$ or $y = x$.

As we have a triplet in the direction $x = 0$ then according to (44) for systems (42) the conditions $k = d = h = 0$ must be satisfied. On the other hand as these systems must have at least one line in the direction $y = 0$ then considering (45) the equation $Eq_5 = l = 0$ gives $l = 0$. Similarly as in the direction $y = x$ there must exist at least one invariant line, from (47) due to $r = -1/2$ and $k = d = h = l = 0$ we have $Eq_7 = 2m = 0$, i.e. we get $m = 0$.

Considering the conditions $k = d = h = l = m = 0$ we shall examine the direction $y = -rx = x/2$. Taking into account (49) and $r = -1/2$ we obtain

$$U = -1/2,\ V = 1,\ A = -1/2,\ B = -1/2,\ C = 1,\ D = W/2,$$
$$E = -W,\ F = f + W^2,\ Eq_5 = 3W/4,\ Eq_8 = (-c + 2e + f)/2,$$
$$Eq_{10} = (-a + 2b - 2fW - 2W^3)/2,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0.$$

So from $Eq_5 = 0$ we get the unique solution $W = 0$ and thenwe arrive at the conditions

$$-c + 2e + f = 0 = -a + 2b.$$

So we get the relations $c = 2e + f$ and $a = 2b$. Joining all the obtained conditions we have

$$r = -1/2, \quad k = d = h = l = m = 0, \quad c = 2e + f, \quad a = 2b \tag{53}$$

and considering them we examine simultaneously the directions $y = 0$ and $y = x$.

By (45) and (53) for the direction $y = 0$ we obtain $Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0$ and

$$Eq_8' = (2e - 3W^2)/2, \quad Eq_{10}' = b - fW - W^3,$$

On the other hand considering (47) and (53) for the direction $y = x$ we have $Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_8 = 0$ and

$$Eq_9'' = (-2e + 3W^2)/2, \quad Eq_{10}'' = \big[-2b - 2(e + 2f)W + W^3 \big]/2.$$

Now we need to have in these two directions exactly three invariant lines. We claim, that if in one of the directions we have a couple of parallel lines, that immediately in the second direction we also obtain a couple of invariant lines.

Indeed, considering Lemma 2.3 we calculate

$$\begin{aligned} &R_W^{(0)}(Eq_8', Eq_{10}') = -\big[27b^2 - 2e(2e + 3f)^2\big]/8, \quad R_W^{(1)}(Eq_8', Eq_{10}') = -3(2e + 3f)/4; \\ &R_W^{(0)}(Eq_9'', Eq_{10}'') = \big[27b^2 - 2e(2e + 3f)^2\big]/8, \quad R_W^{(1)}(Eq_9'', Eq_{10}'') = -3(2e + 3f)/4. \end{aligned}$$

Therefore by Lemma 2.3 if the polynomials Eq_8' and Eq_{10}' have two common roots them immediately the polynomials Eq_9'' and Eq_{10}'' have also two common roots, and conversely. So our claim is proved.

Thus we conclude that in the case $r = -1/2$ (and clearly in the case $r = -2$) we could not have a configurations of the type $(3, 2, 1, 1)$.

Thus we arrive at the next result.

Lemma 3.3. *A system* (42) *possesses the configuration or potential configuration of invariant lines of the type* $(3, 2, 1, 1)$ *only in the case* $(r - 1)(1 + 2r)(2 + r) \neq 0$. *Moreover if this system possesses such a configuration then it could be written either in the form* (52) *or* (41).

3.3.3 Invariant conditions for the configurations *Config. 8.4, 8.5, 8.6*

According to Lemma 3.3 for the existence of the configuration $(3, 2, 1, 1)$ the condition $(r-1)(2+r)(1 + 2r) \neq 0$ is necessary. On the other hand for systems (42) we have

$$\mathcal{D}_4 = -1152(r - 1)(2 + r)(1 + 2r)$$

and hence the condition above is equivalent to $\mathcal{D}_4 \neq 0$.

Now we concentrate our attention on the conditions (50) and according to the statement A_2) of the Main Theorem we shall prove that these conditions are equivalent to $\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0$. For systems (42) we calculate

$$\mathcal{K}_4 = l(2 + r)(1 + 2r)x^3/9 - 2h(-1 + r)rx^2y/3 + 2(r - 1)(-h + hr - 3kr)xy^2/9 - kry^3$$

and due to the condition $r(r-1)(2+r)(1+2r) \neq 0$ clearly that the condition $\mathcal{K}_4 = 0$ is equivalent to $k = l = h = 0$. Considering these conditions we calculate

$$\mathcal{K}_5 = Z_1x^4 + Z_2x^3y + Z_3x^2y^2 + Z_4xy^3 + Z_5y^4$$

where

$$\begin{aligned}
Z_1 =& -30r\Big[3r(1-r)(c-f)+e(4+r+4r^2)-4m^2(r-1)\Big],\\
Z_2 =& -10\Big[54fr^2-27dr^2(r-1)+8e(r-1)(2+r)(1+2r)-18cr(1+r+r^2)-\\
& -8m^2(2+r)(1+2r)\Big]/3,\\
Z_3 =& 10(r-1)\Big[5dr(r-1)+2c(1+7r+r^2)-2f(4+r+4r^2)-24m^2\Big],\\
Z_4 =& 10\Big[6c(1+7r+r^2)-d(r-1)(4-53r+4r^2)-6f(4+r+4r^2)-72m^2\Big]/3,\\
Z_5 =& 30d(1+7r+r^2).
\end{aligned}$$

We observe that the following relation holds:

$$Z_3-(r-1)Z_4=20d(r-1)^2(2-19r+2r^2)/3.$$

Therefore it is clear that the conditions $Z_3=Z_4=Z_5=0$ imply $d=0$ and then from the relations $Z_1=Z_2=Z_3=0$ we obtain:

$$\begin{gathered}
c=-\frac{108m^2r(1+r+r^2)}{(r-1)^2(2+r)^2(1+2r)^2},\quad e=\frac{4m^2(r-1)(1+7r+r^2)}{(2+r)^2(1+2r)^2},\\
f=-\frac{12m^2(1+5r+15r^2+5r^3+r^4)}{(r-1)^2(2+r)^2(1+2r)^2}.
\end{gathered}\tag{54}$$

Thus we obtain the respective conditions from (50) and it remains to find the invariant conditions corresponding to the expressions for the parameters a and b. For systems (42) with the conditions $k=l=h=0$ and (54) we calculate:

$$\text{Coefficient}[\mathcal{K}_6,x^9y^2]=-\frac{6480(r-1)^2r^3}{(2+r)^2(1+2r)^2}\big[b(r-1)(2+r)^2(1+2r)^2+8m^3(1+7r+r^2)\big].$$

So due to the condition $r(r-1)(2+r)(1+2r)\neq0$ the condition $\mathcal{K}_6=0$ implies

$$b=-\frac{8m^3(1+7r+r^2)}{(r-1)(2+r)^2(1+2r)^2}$$

and then we calculate:

$$\text{Coefficient}[\mathcal{K}_6,x^8y^3]=-\frac{7560(r-1)r^3}{(2+r)^2(1+2r)^2}\big[216m^3r-a(-1+r)^2(2+r)^2(1+2r)^2\big].$$

Therefore the condition $\mathcal{K}_6=0$ implies

$$a=\frac{216m^3r}{(r-1)^2(2+r)^2(1+2r)^2}$$

and we arrive at the conditions (50). It remains to note that for these expressions of the parameters a and b we have $\mathcal{K}_6=0$ identically.

On the other hand the conditions $u\neq0$ and $u=0$ (see the notation for the parameter u on the page 24) lead to different configurations of invariant lines for systems (51). So we need an invariant polynomial which govern this condition. For these systems we calculate

$$\mathcal{L}_1=-6912u(r-1)r(2+r)(1+2r)x^2y$$

and due to the condition $r(r-1)(2+r)(1+2r)\neq0$ the condition $u=0$ is equivalent to $\mathcal{L}_1=0$.

As it was shown above in the case $u\neq0$ we obtain the family of systems (52) which possess two distinct configurations (*Config. 8.1-4* and *Config. 8.5*) depending on the sign of the parameter r. On the other hand for systems (52) we have $\mathcal{K}_7=4r$ and hence this invariant polynomial distinguishes the mentioned configurations of invariant lines.

Thus the statement A_2) of the Main Theorem is proved.

3.4 Systems with configuration $(2,2,2,1)$

In this subsection we construct the cubic systems with 4 real infinite singular points which possess 7 invariant affine straight lines with configuration or potential configuration $(2,2,2,1)$, having total multiplicity 8, as always the invariant straight line of the infinity is considered.

For having the mentioned configuration a cubic system has to possess three couples of parallel invariant lines. Thus, according to Theorem 2.2 if a cubic system possesses 7 invariant straight lines in the configuration (2, 2, 2, 1), then necessarily the condition $\mathcal{V}_3 = 0$ holds.

3.4.1 Construction of the corresponding cubic homogeneities

As a first step we shall construct the cubic homogeneous parts of systems (14) for which the condition above is fulfilled. So we shall consider the family of systems (33) and we shall force the condition $\mathcal{V}_3 = 0$ to be satisfied.

We observe that the invariant polynomial $\mathcal{V}_3$ is a homogeneous polynomial of degree four in x and y. A straightforward computation of the value of $\mathcal{V}_3$ for systems (33) yields: $\mathcal{V}_3 = 32\sum_{j=0}^{4}\mathcal{V}_{3j}x^{4-j}y^j$, where

$$\begin{aligned}&\mathcal{V}_{30} = -p(p+3r), \quad \mathcal{V}_{31} = 2p(r-2s-v),\\&\mathcal{V}_{32} = 4rs+3rq-sv-vr+3ps-2pq+2s^2+2r^2-v^2,\\&\mathcal{V}_{33} = -2q(2r-s+v), \quad \mathcal{V}_{34} = -q(q+3s).\end{aligned} \tag{55}$$

So we shall consider two cases: $pq \neq 0$ and $pq = 0$.

3.4.1.1 The case $pq \neq 0$. Then by (55), the conditions $\mathcal{V}_{3i} = 0$, $i = 0,1,3,4$ yield $p = q = -3r = -3s = 3v \neq 0$, and then the condition $\mathcal{V}_{32} = -27v^2 = 0$ implies $v = 0$, a contradiction.

3.4.1.2 The case $pq = 0$. Then we can suppose $q = 0$, otherwise we interchange $x \leftrightarrow y$, $p \leftrightarrow q$ and $r \leftrightarrow s$. We consider two subcases: $p = 0$ and $p \neq 0$.

3.4.1.2.1 *The subcase $p = 0$.* Then we have $\mathcal{V}_{30} = \mathcal{V}_{31} = 0$, and from (55) we obtain $\mathcal{V}_{32} = (2s+2r+v)(s+r-v) = 0$.

1) *The possibility $v = r+s$.* Then $\mathcal{V}_3 = 0$ and as $s \neq 0$ via the time rescaling $t \to t/s$ we arrive at the family of cubic homogeneous systems

$$\dot{x} = rx^3 + (2+r)x^2y, \quad \dot{y} = (1+2r)xy^2 + y^3. \tag{56}$$

2) *The possibility $v = -2(r+s)$.* Then $\mathcal{V}_3 = 0$ and as $r \neq 0$ via the time rescaling $t \to t/r$ we arrive at the family of cubic homogeneous systems

$$\dot{x} = x^3 - (2+s)x^2y, \quad \dot{y} = -(1+2s)xy^2 + sy^3.$$

We observe that these systems could be brought to the systems (56) via the change $(x,y,t,s) \mapsto (y,-rx,t,-1/r)$.

3.4.1.2.2 *The subcase $p \neq 0$.* Then by (55) we obtain $p = -3r$ and $v = r-2s$ and this leads to the systems

$$\dot{x} = -2x^3 + (1-s)x^2y, \quad \dot{y} = -3x^2y + 2(1-s)xy^2 + sy^3.$$

It is not too difficult to observe that via the change $(x,y,t,s) \mapsto (x, x-y, -(1+r)t, -1/(1+r)$ the systems above can be brought to systems (56).

Since in systems (12) with the homogenous cubic parts of the form (56) due to a translation we may assume $n = 0$, we arrive at the next result.

Lemma 3.4. *Assume that a cubic system* (14) *possesses* 7 *invariant affine straight lines with configuration or potential configuration* $(2,2,2,1)$. *Then via an affine transformation and a time rescaling this system could be brought to a system belonging to the following family:*

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + rx^3 + (2+r)x^2y, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + (1+2r)xy^2 + y^3, \quad r(r+1) \neq 0. \end{aligned} \tag{57}$$

3.4.2 Construction of the cubic systems possessing configuration or potential configuration $(2,2,2,1)$

In what follows we shall determine necessary and sufficient conditions for a system (57) to have a configuration or potential configuration $(2,2,2,1)$.

Considering Remark 3.1 for the homogeneous systems (56), corresponding to systems (57) we calculate

$$H(X,Y,Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = X^2(X-Y)Y^2(rX+Y)^2. \tag{58}$$

So each one of the invariant lines $x = 0$, $y = 0$ and $rx + y = 0$ of systems (56) is of multiplicity two and in the direction $y = x$ there exists one line.

We claim that in order to have exactly three couples of invariant straight lines, for systems (57) the condition $(r+2)(2r+1)(r-1) \neq 0$ must hold. Indeed using the equations (13) we evaluate them for each one of the four directions.

(i) For the direction $x = 0$ we obtain

$$\begin{gathered} U = 1,\ V = 0,\ A = r,\ B = (2+r)/2,\ C = 0,\ D = g - rW, \\ E = 2h - 2W - rW,\ F = c - gW + rW^2, \quad Eq_7 = k, \\ Eq_9 = d - 2hW + (2+r)W^2,\ Eq_{10} = a - cW + gW^2 - rW^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0 \end{gathered} \tag{59}$$

and to have exactly two parallel invariant lines in this direction the condition $r + 2 \neq 0$ is necessary.

(ii) For the direction $y = 0$ we have

$$\begin{gathered} U = 0,\ V = 1,\ A = 0,\ B = (2r+1)/2,\ C = 1,\ D = 2m - W - 2rW,\ E = -W, \\ F = f + W^2,\ Eq_5 = l,\ Eq_8 = e - 2mW + (1+2r)W^2,\ Eq_{10} = b - fW - W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0. \end{gathered} \tag{60}$$

and clearly to have exactly two parallel invariant lines in this direction the condition $1 + 2r \neq 0$ must be satisfied.

(iii) For the direction $y = x$ we calculate

$$\begin{gathered} U = -1,\ V = 1,\ A = r,\ B = r+1,\ C = 1,\ D = g - l + rW,\ E = -k - W, \\ F = -d + f + kW + W^2, \quad Eq_6 = l - g - 2h - k + 2m - 3(1+r)W, \\ Eq_8 = e - c - d + f + (l - g + k)W + (1-r)W^2, \\ Eq_{10} = -a + b + dW - fW - kW^2 - W^3 \end{gathered} \tag{61}$$

and as $r + 1 \neq 0$ in this direction could be at most one invariant line.

(iv) For the direction $y = -rx$ we obtain

$$\begin{gathered}
U = r,\ V = 1,\ A = r,\ B = (r+1)/2,\ C = 1,\ D = 2m + 2hr - kr^2 - W,\quad E = kr - W,\\
F = f + dr - krW + W^2,\quad Eq_5 = l + gr - 2mr - 2hr^2 + kr^3,\\
Eq_8 = e + cr - fr - dr^2 - 2mW - 2hrW + 2kr^2W + (1-r)W^2,\\
Eq_{10} = b + ar - fW - drW + krW^2 - W^3,\\
Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0.
\end{gathered} \tag{62}$$

We observe, that to have exactly two invariant lines in this direction it is necessary $r - 1 \neq 0$.

Thus we conclude, that the three needed couples of parallel invariant lines could be only in the directions $x = 0$, $y = 0$ and $y = -rx$ and for this the condition $(r+2)(2r+1)(r-1) \neq 0$ must hold. So our claim is proved.

Since $r + 2 \neq 0$ without loss of generality in systems (57) we may assume $h = 0$ due to the translation

$$x = x_1 + h(1+2r)/(3(2+r)),\quad y = y_1 - h/(2+r),$$

which conserves the previous relation $n = 0$. So we have to force the existence of parallel lines in the mentioned above directions. Considering (59), (60), (62) and $h = 0$ we obtain: $k = l = 0$, $r(g - 2m) = 0$ and this implies $g = 2m$.

Now we look for the sufficient conditions under the parameters of systems (57) for the existence of three couples of parallel lines, assuming that the following conditions hold:

$$k = l = h = 0,\quad g = 2m. \tag{63}$$

(i) Direction $x = 0$. Considering (59) we get

$$Eq_9 = d + (2+r)W^2 = 0,\ Eq_{10} = a - cW + 2mW^2 - rW^3 = 0$$

and by Lemma 2.3 in order to have two common solutions the following conditions are necessary and sufficient:

$$R_W^{(0)}(Eq_9, Eq_{10}) = R_W^{(1)}(Eq_9, Eq_{10}) = 0.$$

Since $R_W^{(1)}(Eq_9, Eq_{10}) = -(2+r)(2c + cr - dr)$ and $r + 2 \neq 0$ we obtain $d = c(2+r)/r$ and we calculate

$$R_W^{(0)}(Eq_9, Eq_{10}) = (2+r)^3(2cm - ar)^2/r^2 = 0.$$

Therefore we get $a = 2cm/r$ and this implies

$$Eq_9 = (2+r)(c + rW^2)/r,\quad Eq_{10} = (2m - rW)(c + rW^2)/r$$

and hence, we have two common solutions, which could be real or complex, distinct or coinciding. On the other hand for the parameters of systems (57) we obtain the following relations:

$$\begin{gathered}
(r-1)(r+2)(2r+1) \neq 0,\quad k = l = h = 0,\\
g = 2m,\quad d = c(2+r)/r,\quad a = 2cm/r.
\end{gathered} \tag{64}$$

(ii) Direction $y = 0$. Taking into account (60) and (64) in this case we obtain

$$\begin{gathered}
Eq_8 = e - 2mW + (1+2r)W^2 = 0,\ Eq_{10} = b - fW - W^3 = 0,\\
R_W^{(1)}(Eq_8, Eq_{10}) = e(1+2r) - f(1+2r)^2 - 4m^2 = 0
\end{gathered}$$

and this implies $e = f(1+2r) + 4m^2/(1+2r)$. Therefore we calculate

$$R_W^{(0)}(Eq_8, Eq_{10}) = \left[8m^3 + 2fm(1+2r)^2 + b(1+2r)^3\right]^2/(1+2r)^3 = 0$$

and hence we get $b = -2\big[4m^3 + fm(1+2r)^2\big]/(1+2r)^3$. Then we obtain

$$Eq_9 = \frac{4m^2 + f(1+2r)^2 - 2m(1+2r)W + (1+2r)^2W^2}{1+2r},$$
$$Eq_{10} = -\frac{(2m+W+2rW)\big[4m^2 + f(1+2r)^2 - 2m(1+2r)W + (1+2r)^2W^2\big]}{(1+2r)^3}$$

and hence, we have two common solutions, which could be real or complex, distinct or coinciding.

Considering the new obtained conditions we arrive at the following relations among the parameters of systems (57):

$$\begin{gathered}(r-1)(r+2)(2r+1)\neq 0, \quad k=l=h=0, \quad g=2m,\\ d=c(2+r)/r, \quad a=2cm/r, \quad e=f(1+2r)+4m^2/(1+2r),\\ b=-2\big[4m^3+fm(1+2r)^2\big]/(1+2r)^3\end{gathered} \tag{65}$$

(iii) Direction $y=-rx$. Considering (62) we get

$$Eq_8 = (1+r)(f-cr) + \frac{4m^2}{1+2r} - 2mW - (r-1)W^2 = 0,$$
$$Eq_{10} = 2cm - \frac{8m^3}{(1+2r)^3} - \frac{2fm}{1+2r} - (2c+f+cr)W - W^3 = 0,$$
$$R_W^{(1)}(Eq_8, Eq_{10}) = 2c(r-1) - 2fr(r-1) - \frac{12m^2r}{1+2r} = 0$$

and therefore we obtain $c = fr + \dfrac{6m^2r}{(r-1)(1+2r)}$. Then we calculate

$$R_W^{(0)}(Eq_8, Eq_{10}) = -\frac{144m^2r^2(1+r)^2\big[f(r-1)^2(1+2r)^2 + 3m^2(1-2r+4r^2)\big]^2}{(2+r)^6(1+2r)^4} = 0$$

and clearly we have either $m=0$ or $m\neq 0$ and $f = -\dfrac{3m^2(1-2r+4r^2)}{(-1+r)^2(1+2r)^2}$

1) *The case $m=0$.* Then by (65) we get the conditions

$$\begin{gathered}(r-1)(r+2)(2r+1)\neq 0, \quad k=l=h=g=m=a=b=0,\\ d=f(2+r), \quad e=f(1+2r), \quad c=fr\end{gathered} \tag{66}$$

and for the direction $y=-rx$ we obtain

$$Eq_8 = (1-r)(f+2fr+fr^2+W^2), \quad Eq_{10} = -W(f+2fr+fr^2+W^2).$$

So we have two common solutions, which could be real or complex, distinct or coinciding.

In this case for the last direction ($y=x$) we calculate

$$\begin{gathered}Eq_6 = -3(1+r)W, \quad Eq_8 = (1-r)W^2,\\ Eq_{10} = W(f+fr-W^2)\end{gathered}$$

and the common solution is $W=0$. Thus we get the family of systems

$$\dot{x} = (f+x^2)(rx+2y+ry), \quad \dot{y} = (f+y^2)(x+2rx+y) \tag{67}$$

with the condition $r(r^2-1)(r+2)(2r+1)\neq 0$ and $f\in\{-1,0,1\}$ due to the rescaling $(x,y,t)\mapsto(|f|^{1/2}x,\ |f|^{1/2}y,\ t/|f|)$. These systems possess the invariant lines

$$x^2+f=0, \quad y^2+f=0, \quad y-x=0, \quad (rx+y)^2+f(1+r)^2=0.$$

We observe that the line $y = x$ is real and all other lines are distinct real (respectively complex) if $f < 0$ (respectively $f > 0$) and we have three double invariant lines in the case $f = 0$.

Thus we obtain the configuration *Config. 8.7* if $f < 0$; *Config. 8.8* if $f > 0$ and *Config. 8.9* if $f = 0$ (see FIGURE 1).

2) *The case* $f = -\dfrac{3m^2(1-2r+4r^2)}{(-1+r)^2(1+2r)^2}$ *and* $m \neq 0$. We claim that in this case in the direction $y = x$ we could not have any invariant line. Indeed, considering (61) we obtain

$$Eq_6 = -3(1+r)W = 0, \quad Eq_8 = -\frac{8m^2(2+r)}{(r-1)(1+2r)} - 2mW + (1-r)W^2 = 0$$

and we observe that these equations could have only the common solution $W = 0$. However in this case we must have $m(2+r) = 0$ which contradicts $m(2+r) \neq 0$. So our claim is proved.

3.4.3 Invariant conditions for the configurations *Config. 8.7, 8.8, 8.9*

It was proved in the previous subsection that a system (57) with $h = 0$ possesses the configuration or potential configuration of invariant lines $(2, 2, 2, 1)$ if and only if the following conditions are satisfied:

$$\begin{gathered}(r-1)(r+2)(2r+1) \neq 0, \quad k = l = g = m = a = b = 0,\\ d = f(2+r), \quad e = f(1+2r), \quad c = fr.\end{gathered} \tag{68}$$

Following the statement A_3) of the Main Theorem we shall prove that these conditions are equivalent to

$$\mathcal{D}_4 \neq 0, \quad \mathcal{V}_3 = \mathcal{K}_4 = \mathcal{K}_2 = \mathcal{K}_8 = 0.$$

First of all we observe that for systems (57) with $h = 0$, i.e. for systems

$$\begin{aligned}\dot{x} &= a + cx + dy + gx^2 + ky^2 + rx^3 + (2+r)x^2y,\\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + (1+2r)xy^2 + y^3,\end{aligned} \tag{69}$$

we have $\mathcal{D}_4 = -1152(r-1)(2+r)(1+2r)$. Hence the first condition (68) is equivalent to $\mathcal{D}_4 \neq 0$.

For systems (69) we calculate

$$\begin{aligned}\mathcal{K}_4 = \frac{2}{9}\big[&l(r-1)(2+r)x^3 + (2+r)(3l + g - 2m + 2gr - mr)x^2y+\\ &+3(2m - kr + mr - 2kr^2)xy^2 + k(r-1)(1+2r)y^3\big]\end{aligned}$$

and it is obvious to detect that due to $(r-1)(r+2)(2r+1) \neq 0$ the condition $\mathcal{K}_4 = 0$ is equivalent to $k = l = g = m = 0$.

Next we examine the conditions for the coefficients of linear terms given in (68). Considering the conditions above we calculate

$$\mathcal{K}_8 = Z_1x^4 + Z_2x^3y + Z_3x^2y^2 + Z_4xy^3 + Z_5y^4$$

where

$$\begin{aligned}Z_1 =& -5r^2(4c + e - f + 8cr - 4er - 2fr),\\ Z_2 =& r(16c + 40e - 40f + 89cr - 3dr - 7er - 83fr + 6cr^2 - 6dr^2 - 60fr^2),\\ Z_3 =& -16c - 40e + 40f - 134cr + 28dr - 49er + 89fr - 89cr^2 + 49dr^2 - 28er^2+\\ &+ 134fr^2 - 40cr^3 + 40dr^3 + 16fr^3,\\ Z_4 =& -60c - 6e + 6f - 83cr - 7dr - 3er + 89fr - 40cr^2 + 40dr^2 + 16fr^2,\\ Z_5 =& 5(2c + 4d - 8f + cr - dr - 4fr).\end{aligned}$$

It is not too difficult to detect, that the relations $Z_1 = Z_2 = Z_5 = 0$ yield $c = fr$, $d = f(2 + r)$ and $e = f(1 + 2r)$ and then we get $\mathcal{K}_8 = 0$. Thus we obtain the respective conditions from (68) and it remains to find out the invariant conditions equivalent to $a = b = 0$. We observe that for systems (69) in this case we have

$$\mathcal{K}_2 = -3x^2y^2(rx + y)^2(bx - ay)(x + 2rx + 2y + ry)^2$$

and evidently the condition $\mathcal{K}_2 = 0$ is equivalent to $a = b = 0$.

In such a way we get the 2-parameter family of systems (67) possessing the configuration *Config. 8.7* if $f < 0$; *Config. 8.8* if $f > 0$ and *Config. 8.9* if $f = 0$ (see FIGURE 1).

On the other hand for these systems we calculate

$$K_9 = -180f(1 + r)^2x^2y^2(rx + y)^2$$

and as $r(r + 1) \neq 0$ we conclude that $f = 0$ if and only if $K_9 = 0$ and $\text{sign}\,(K_9) = -\text{sign}\,(f)$. This completes the proof of the statement A_3) of the Main Theorem.

B. Cubic systems with 2 real and 2 complex infinite singularities

According to Lemma 2.5 in this case the condition $\mathcal{D}_1 < 0$ holds and the systems (12) due to a linear transformation and time rescaling could be brought to the systems

$$\begin{aligned} \dot{x} &= p_0 + p_1(x, y) + p_2(x, y) + (u + 1)x^3 + (s + v)x^2y + rxy^2, \\ \dot{y} &= q_0 + q_1(x, y) + q_2(x, y) - sx^3 + ux^2y + vxy^2 + (r - 1)y^3. \end{aligned} \tag{70}$$

For these systems we have $C_3 = x(sx+y)(x^2+y^2)$ and hence, infinite singular points are situated at the "ends" of the straight lines: $x = 0$, $y = -sx$ and $y = \pm ix$.

As we have two real and two complex infinite singularities and the total multiplicity of the invariant line (including the line at infinity) must be 8, then the systems above could have only one of the following four possible configurations of invariant straight lines:

(i) (3, 3, 1); (ii) (3, 2, 2); (iii) (3, 2, 1, 1); (iv) (2, 2, 2, 1).

3.5 Systems with configuration (3, 3, 1)

Since we have two triplets of parallel invariant lines, according to Theorem 2.2 the conditions $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$ are necessary for systems (14). Moreover in [11, Section 6.1] it was proved that providing the conditions above, a cubic homogenous system with two real and two complex (all distinct) infinite singularities via a linear transformation and time rescaling could be brought either to the system

$$\dot{x} = x^3, \quad \dot{y} = -y^3$$

if $\mathcal{L}_4 < 0$, or to the system

$$\dot{x} = x^3 - 3xy^2, \quad \dot{y} = 3x^2y - y^3$$

if $\mathcal{L}_4 > 0$. For the first system we calculate

$$H(\tilde{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 3X^3Y^3(X^2 + Y^2), \quad C_3(x, y) = xy(x^2 + y^2),$$

whereas for the second one we have

$$H(\tilde{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 6XY(X^2 + Y^2)^3, \quad C_3(x, y) = -2xy(x^2 + y^2).$$

In the first case we must have two triplets of parallel lines in the real directions and hence, forcing the existence of a line in the complex direction we get 8 invariant affine lines.

Thus it remains to consider only the systems with cubic homogeneities of the second type. We observe that due to a translation we may assume $g = n = 0$ in the quadratic parts of the systems (70) and so we examine the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3 - 3xy^2, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + 3x^2y - y^3. \end{aligned} \tag{71}$$

(i) The direction $x + iy = 0$. In this case we obtain

$$\begin{gathered} U = 1,\ V = i,\ A = 1,\ B = i,\ C = -1,\ D = il - W,\ E = -i(k + W), \\ F = c + W^2 + i(e - lW),\ Eq_{10} = a + ib - (c + ie)W + ilW^3 - W^3, \\ Eq_6 = l + 2h + i(k + 2m),\ Eq_9 = d + e + i(f - c) - (l - ik)W, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = Eq_8 = 0. \end{gathered} \tag{72}$$

As all the parameters of systems (71) are real we conclude, that to have exactly three parallel invariant lines (which could coincide) in this direction it is necessary and sufficient to be satisfied the conditions $l = k = h = m = 0$, $d = -e$ and $f = c$.

Thus we arrive to the family of systems

$$\begin{aligned} \dot{x} &= a + cx - ey + x^3 - 3xy^2, \\ \dot{y} &= b + ex + cy + 3x^2y - y^3 \end{aligned} \tag{73}$$

for which we shall examine simultaneously the real directions: $x = 0$ and $y = 0$.

For the direction $x = 0$ we calculate

$$\begin{gathered} U = 1,\ V = 0,\ A = 1,\ B = 0,\ C = -3,\ D = -W,\ E = 0,\ F = c + W^2, \\ Eq_7 = 3W,\ Eq_9 = -e,\ Eq_{10} = a - cW - W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0, \end{gathered} \tag{74}$$

whereas for the direction $y = 0$ we have

$$\begin{gathered} U = 0,\ V = 1,\ A = 3,\ B = 0,\ C = -1,\ D = 0,\ E = W,\ F = c - W^2, \\ Eq_5 = -3W,\ Eq_8 = e,\ Eq_{10} = b - cW + W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0. \end{gathered} \tag{75}$$

We observe that in each one of the cases we could have only one invariant line (which is either $x = 0$ or $y = 0$). Moreover the necessary and sufficient conditions for the existence of such line are $e = a = 0$ in the first case and $e = b = 0$ in the second case.

Thus we conclude, that for the existence for systems (73) of exactly one invariant line in one of the real directions, the following conditions are necessary and sufficient:

$$e = ab = 0, \quad a^2 + b^2 \neq 0.$$

Since the respective family of systems is of the form

$$\dot{x} = a + cx + x^3 - 3xy^2, \quad \dot{y} = b + cy + 3x^2y - y^3 \tag{76}$$

we may assume $b = 0$ due to the change $(x, y, t) \mapsto (y, x, -t)$ in the case $a = 0$.

Thus we arrive at the family of systems

$$\dot{x} = a + cx + x^3 - 3xy^2, \quad \dot{y} = cy + 3x^2y - y^3 \tag{77}$$

possessing the invariant lines

$$y = 0, \quad (x + iy)^3 + c(x + iy) + a = 0, \quad (x - iy)^3 + c(x - iy) + a = 0.$$

Since the discriminant of the cubic polynomial $\phi(z) = z^3 + cz + a$, where $z = x \pm iy$ equals $\xi = -(27a^2 + 4c^3)$, we conclude that systems above possess 7 invariant affine lines (considered with their multiplicity), which are as follows:

$\xi \neq 0 \quad \Rightarrow \quad$ one real simple and 6 complex distinct simple;

$\xi = 0 \quad \Rightarrow \quad$ one real simple, two complex simple and 2 complex double, all distinct.

As we have two triplets of parallel complex invariant lines it is clear that all 9 finite singularities (real and/or complex) are located at the intersections of these lines. Moreover, as there exist three pair of complex conjugate lines we have three real finite singularities, which are distinct if $\xi \neq 0$ and two of them coincide if $\xi = 0$.

We observe that the singular points $(x_i, 0)$, $i = 1, 2, 3$, where x_i are the solutions of the cubic equation $x^3 + cx + a = 0$ are located on the real invariant line $y = 0$. As the discriminant of this equation is also ξ, we deduce that all the real singularities are located on the real line $y = 0$ if $\xi \geq 0$ and there are one real and two complex singularities on this line if $\xi < 0$.

Thus we obtain the configuration *Config. 8.10* if $\xi > 0$, *Config. 8.11* if $\xi < 0$ and *Config. 8.12* if $\xi = 0$ (see FIGURE 1).

3.5.1 Invariant conditions for the configurations *Config. 8.10, 8.11, 8.12*

It was shown above that for having the configurations of the type $(3, 3, 1)$ the conditions

$$k = h = l = m = e = d = c - f = ab = 0, \quad a^2 + b^2 \neq 0$$

must hold for systems (71). According to the statement B_1) of the Main Theorem we shall prove theta these conditions are equivalent to the affine invariant conditions

$$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \mathcal{L}_1 = \mathcal{L}_2 = \mathcal{K}_1 = 0, \ \mathcal{K}_2 \neq 0.$$

For systems (71) we calculate

$$\mathcal{L}_1 = 2^8 3^4 \left[(3l + 2h)x^3 + (k + 6m)x^2 y - (l + 6h)xy^2 - (3k + 2m)y^3\right]$$

and hence the condition $\mathcal{L}_1 = 0$ gives $l = h = k = m = 0$. Then we obtain

$$\mathcal{L}_1 = 0, \quad \mathcal{L}_2 = 2^8 3^5 \left[(7d - 5e)x^2 - 2(c - f)xy + (5d - 7e)y^2\right]$$

and clearly the condition $\mathcal{L}_2 = 0$ implies $e = d = c - f = 0$ and we arrive at the family of systems (73) with $e = 0$. So it remains to determine the invariant polynomials which govern the conditions $ab = 0$ and $a^2 + b^2 \neq 0$ for these systems. We calculate

$$\mathcal{K}_1 = 2^{32} 3^{15} 5^4 7^4 19 \cdot 43abxy, \quad \mathcal{K}_2 = -27(bx - ay)(x^2 + y^2)^4.$$

So clearly the condition $ab = 0$ is equivalent to $\mathcal{K}_1 = 0$, whereas the condition $a^2 + b^2 \neq 0$ is equivalent to $\mathcal{K}_2 \neq 0$.

As it was mentioned above we could consider $b = 0$ and then for systems (76) we need the expression $\text{Discrim}\left[a + cz + z^3, z\right] = -(27a^2 + 4c^3) = \xi$ which governs the type of the invariant lines (distinct or coinciding) of these systems. We calculate

$$\mathcal{K}_3 = 2^{10} 3^3 5^2 (27a^2 + 4c^3) x^2 y^2 (x - y)^2 (x^2 + y^2)^4.$$

So clearly $4a^2 + 27c^3 = 0$ if and only if $\mathcal{K}_3 = 0$ and $\text{sign}\,(\mathcal{K}_3) = -\text{sign}\,(27a^2 + 4c^3) = \text{sign}\,(\xi)$.

To complete the proof of the statement B_1) of the Main Theorem it remains to construct the respective canonical systems for each one of the configurations. Since on the line $y = 0$ of systems (77) there exist a real solution x_0 of the cubic equation $x^3 + cx + a = 0$, then via the translation $(x, y) \mapsto (x + x_0,\, y)$ we get the family of systems

$$\dot{x} = c_1 x + d_1 x^2 - d_1 y^2 + x^3 - 3xy^2, \quad \dot{y} = y(c_1 + 2d_1 x + 3x^2 - y^2), \tag{78}$$

where $c_1 = c + 3x_0^2$ and $d_1 = 3x_0$. Then on the invariant line $y = 0$ besides the singular point $(0, 0)$ there are locate two more singularities: $(x_{1,2}, 0)$, where $x_{1,2}$ are the solution of the quadratic equation $c_1 + d_1 x + x^2 = 0$. The discriminant of this equation equals $\delta = d_1^2 - 4c_1$ and obviously we must have $\operatorname{sign}(\delta) = \operatorname{sign}(\xi)$ and $\delta = 0$ if and only if $\xi = 0$.

For systems (78) we calculate

$$\mathcal{K}_2 = d_1(2d_1^2 - 9c)y(x^2 + y^2)^4$$

and by the statement B_1) of the Main Theorem the condition $d_1 \neq 0$ must be satisfied.

a) Assume first $\xi \neq 0$. Then we can set $d_1^2 - 4c_1 = u^2 \operatorname{sign}(\xi)$ and we have $c_1 = (d_1^2 - u^2 \operatorname{sign}(\xi))/4$. Since $d_1 \neq 0$ and $\operatorname{sign}(\xi) = \operatorname{sign}(K_6)$ the systems (78) after the rescaling $(x, y, t) \mapsto (dx,\, dy, t/d^2)$ could be brought to the systems

$$\begin{aligned} \dot{x} &= gx + x^2 - y^2 + x^3 - 3xy^2, \\ \dot{y} &= gy + 2xy + 3x^2 y - y^3, \end{aligned} \tag{79}$$

where $g = (1 - a^2 \operatorname{sign}(K_6))/4 \neq 0$. So depending of the sign of the invariant polynomial K_6 we get the corresponding canonical systems given by the Main Theorem (see the statement B_1)).

b) Suppose now $\xi = 0$. Then we have $c_1 = d_1^2/4$ and due to the same rescaling above we obtain systems (79) with $a = 0$.

3.6 Systems with configuration (3, 2, 2)

For having the configuration (3, 2, 2) a cubic system has to possess three couples of parallel invariant lines and in addition, one couple must increase up to a triplet. Thus, according to Theorem 2.2, if a cubic system possesses 7 invariant straight lines in the configuration (3, 2, 2), then necessarily the conditions $\mathcal{V}_3 = \mathcal{V}_4 = \mathcal{U}_2 = 0$ hold.

As it was proved in [11, Subsection 6.2] in this case via an affine transformation and time rescaling such systems could be brought to the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + 2x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + 3x^2 y + y^3, \end{aligned} \tag{80}$$

for which $C_3(x, y) = -xy(x^2 + y^2)$. Considering Remark 3.1 for the respective homogeneous system

$$\dot{x} = 2x^3, \quad \dot{y} = 3x^2 y + y^3$$

we calculate

$$H(X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 6X^3 Y(X^2 + Y^2)^2.$$

So by Remark 3.1 the systems (80) could possess one triplet of invariant lines only in the direction $x = 0$. As regard the two couples of parallel lines we conclude that they must be complex and in complex directions.

(i) The direction $x = 0$. In this case we obtain

$$\begin{gathered} U = 1,\ V = 0,\ A = 2,\ B = 0,\ C = 0,\ D = -2W,\ E = 2h,\ F = c + 2W^2, \\ Eq_7 = k,\ Eq_9 = d - 2hW,\ Eq_{10} = a - cW - 2W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0 \end{gathered} \tag{81}$$

and to have exactly three parallel invariant lines in this direction the condition $k = d = h = 0$ is necessary and sufficient. Therefore we get the family of systems

$$\dot{x} = a + cx + 2x^3, \quad \dot{y} = \quad b + ex + fy + lx^2 + 2mxy + 3x^2y + y^3. \tag{82}$$

(ii) The direction $x + iy = 0$. In this case we have

$$\begin{gathered} U = 1,\ V = i,\ A = 2,\ B = i/2,\ C = 1,\ D = 2(m - W),\ E = iW,\ F = f - W^2, \\ Eq_5 = il - 2m,\ Eq_8 = c - f + ie - 2mW + 3W^2, \\ Eq_{10} = a + ib - fW - W^3, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0 \end{gathered} \tag{83}$$

and to have exactly two parallel invariant lines in this direction it is necessary and sufficient $l = m = 0$ and

$$R_W^{(0)}(Eq_8, Eq_{10}) = R_W^{(1)}(Eq_8, Eq_{10}) = 0.$$

Assuming $l = m = 0$ (then $Eq_5 = 0$) we calculate

$$R_W^{(1)}(Eq_8, Eq_{10}) = -3(c + 2f + ie) = 0$$

and as the parameters c, f and e are real we obtain $c = -2f$ and $e = 0$. Herein we calculate $R_W^{(0)}(Eq_8, Eq_{10}) = 27(a + ib)^2 = 0$ which implies $a = b = 0$. Thus we arrive at the family of systems

$$\dot{x} = 2x(x^2 - f), \quad \dot{y} = y(f + 3x^2 + y^2), \tag{84}$$

which possess 8 invariant lines, defined by the factors

$$x(x^2 - f)y(-f + x^2 - 2ixy - y^2)(-f + x^2 + 2ixy - y^2) = 0.$$

So in the case of two complex and two real infinite singularities we also could not have cubic systems with the configuration (3, 2, 2).

3.7 Systems with configuration $(3, 2, 1, 1)$

For having the configuration (3, 2, 1, 1) a cubic system has to possess two couples of parallel invariant lines and, in addition, one couple must increase up to a triplet. Thus, according to Theorem 2.2, if a cubic system possesses 7 invariant straight lines in the configuration $(3, 2, 1, 1)$, then necessarily the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$ hold.

3.7.1 Construction of the corresponding cubic homogeneities

As a first step we shall construct the cubic homogeneous parts of systems (70) for which the conditions above are fulfilled. So we consider the family homogeneous cubic of systems

$$\dot{x} = (u + 1)x^3 + (s + v)x^2y + rxy^2, \quad \dot{y} = -sx^3 + ux^2y + vxy^2 + (r - 1)y^3 \tag{85}$$

and we shall force the conditions $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$.

Similarly as before (see page 19) we use the following notations:

$$\mathcal{V}_5 = \sum_{j=0}^{4} \mathcal{V}_{5j} x^{4-j} y^j, \quad \mathcal{U}_2 = \sum_{j=0}^{4} \mathcal{U}_{2j} x^{4-j} y^j.$$

Consider first the polynomial $\mathcal{V}_4$. A straightforward computation of its value for systems (85) yields

$$\begin{aligned} &\mathcal{V}_4 = 9216\, \widehat{\mathcal{V}}_4\, C_3(x, y), \quad \text{where} \\ &\widehat{\mathcal{V}}_4 = 6r^2s - r(9s - 2su + 3v) + (s + v)(sv - 3u). \end{aligned} \tag{86}$$

As for systems (85) we have $C_3 = x(sx+y)(x^2+y^2) \neq 0$, we conclude that the condition $\mathcal{V}_4 = 0$ for these systems is equivalent to $\widehat{\mathcal{V}}_4 = 0$.

Calculating the value of $\mathcal{V}_5$ for systems (85) we obtain

$$\mathcal{V}_{53} = -128r(9s - 6rs + r^2 s + 2su - rsu + uv)/9$$

and we shall consider two cases: $r \neq 0$ and $r = 0$.

3.7.1.1 The case $r \neq 0$. Then the condition $\mathcal{V}_5 = 0$ gives $s(9 - 6r + r^2 + 2u - ru) + uv = 0$ and we consider two subcases: $u \neq 0$ and $u = 0$.

3.7.1.1.1 The subcase $u \neq 0$. In this case we get $v = -s(9 - 6r + r^2 + 2u - ru)/u$ and we calculate

$$\mathcal{V}_{52} = 64r(r - 3 - u)\big[s^2(r-3)^2 + u^2/(3u) = 0.$$

So due to $ru \neq 0$ we obtain $r - 3 - u = 0$, i.e. $r = u + 3 \neq 0$ and we calculate $\mathcal{V}_5 = 0$ and

$$\mathcal{U}_2 = -12288\big[(3+2u)^2 + s^2\big](x^2+y^2)\big[(6s^2 - 3u - u^2)x^2 + 6sxy - u(3+u)y^2\big].$$

Since $ur = u(u+3) \neq 0$ the condition $\mathcal{U}_2 = 0$ is equivalent to $s = 0$ and $u = -3/2$. In this case $\widehat{\mathcal{V}}_4 = 0$ and after a time rescaling we arrive at the system

$$\dot{x} = 2x^3, \quad \dot{y} = 3x^2 y + y^3. \tag{87}$$

3.7.1.1.2 The subcase $u = 0$. Then we get $\mathcal{V}_{53} = -128rs(r-3)^2/9 = 0$ and therefore we have either $s = 0$ or $r = 3$. We claim that in this case the condition $r = 3$ must be satisfied. Indeed, supposing $s = 0$ we get $\mathcal{V}_{54} = -32r\big[(r-3)^2 + v^2\big]/9 = 0$ and since $r \neq 0$ this implies $r = 3$. Then $\mathcal{V}_{54} = -32(s-v)^2/3$ and therefore the condition $\mathcal{V}_5 = 0$ implies $v = s$. Therefore for $u = 0$, $r = 3$ and $v = s$ we calculate

$$\mathcal{V}_5 = 0, \;\; \mathcal{U}_2 = -73728s(9+s^2)x(sx+y)(x^2+y^2), \;\; \widehat{\mathcal{V}}_4 = 2s(9+s^2).$$

Thus the condition $\mathcal{U}_2 = 0$ gives $s = 0$ (this implies $\widehat{\mathcal{V}}_4 = 0$) and we arrive at the following system

$$\dot{x} = x^3 + 3xy^2, \quad \dot{y} = 2y^3.$$

which could be brought to (87) via the change $(x, y, t) \mapsto (y, x, t)$.

3.7.1.2 The case $r = 0$. Then we have

$$\mathcal{V}_{50} = -32(2s^2 - u + sv)\big[(s-v)^2 + (3+u)^2\big]/9$$

and hence the condition $\mathcal{V}_5 = 0$ implies either $u = s(v + 2s)$ or $s - v = u + 3 = 0$.

3.7.1.2.1 The subcase $u = s(v + 2s)$. Then we calculate $\mathcal{V}_5 = 0$ and

$$\begin{aligned}\mathcal{U}_2 = &- 12288(s+v)(3s+v)x(sx+y)\big[s(3 + (2s+v)^2)x^2 + \\ &+ (9 + (2s+v)^2)xy - 6sy^2\big], \quad \widehat{\mathcal{V}}_4 = -2s(s+v)(3s+v).\end{aligned}$$

We observe that the condition $\mathcal{U}_2 = 0$ is equivalent to $(s+v)(3s+v) = 0$ and this implies $\widehat{\mathcal{V}}_4 = 0$. So we consider two possibilities: $v = -s$ and $v = -3s$.

1) *The possibility $v = -s$.* In this case we arrive at the following family of systems

$$\dot{x} = (1+s^2)x^3, \quad \dot{y} = -sx^3 + s^2x^2y - sxy^2 - y^3. \tag{88}$$

2) *The possibility* $v=-3s$. In this case we obtain the following family of systems

$$\dot{x}=(s^2-1)x^3+2sx^2y,\quad \dot{y}=sx^3+s^2x^2y+3sxy^2+y^3.$$

We remark that these systems could be brought to the systems (88) with the same parameter s via the transformation

$$x_1=sx+y,\quad y_1=x-sy,\quad t_1=t/(1+s^2).$$

3.7.1.2.2 The subcase $s-v=u+3=0$**.** Then we have $v=s$, $u=-3$ and calculation yields

$$\mathcal{V}_5=0,\quad \widehat{\mathcal{V}}_4=2s(9+s^2)=0,$$
$$\mathcal{U}_2=-73728s(9+s^2)x(sx+y)(x^2+y^2)=0.$$

So the condition $\mathcal{U}_2=0$ gives $s=0$ (then $\widehat{\mathcal{V}}_4=0$) and this leads to the system (87).

Thus for the further examination it remains two family of systems: with cubic homogeneities (87) and (88).

Remark 3.5. *We note that for system* (87) *we have* $\mathcal{V}_3=0$*, whereas for systems* (88) *we have* $\mathcal{V}_3=-32(9+s^2)x^2(sx+y)^2\neq 0$*. So for* $\mathcal{V}_4=\mathcal{V}_5=\mathcal{U}_2=0$ *we get system* (87) *if* $\mathcal{V}_3=0$ *and the family of systems* (88) *if* $\mathcal{V}_3\neq 0$*. We observe also that for system* (87) *we have* $\mathcal{D}_4=0$*.*

3.7.2 Construction of the cubic systems possessing configuration or potential configuration $(3,2,1,1)$

3.7.2.1 The family of systems with cubic homogeneities (87). In order to determine the possible directions for a triplet and couples of invariant straight lines we shall use the affine comitants $\mathcal{G}_i$ $(i=1,2,3)$. Considering Remark 3.1 for system (87) we calculate

$$H(X,Y,Z)=\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)=6X^3Y(X^2+Y^2)^2.$$

So by Remark 3.1 the cubic systems with homogeneous parts (87) could possess one triplet of invariant lines only in the direction $x=0$. As regard the two couples of parallel lines we conclude that they must be complex and in complex directions $y=\pm ix$. Therefore it is clear that in this case we could not have the configuration or potential configuration of the type (3, 2, 1, 1).

3.7.2.2 The family of systems with cubic homogeneities (88). For homogeneous cubic systems (88) we have

$$H(\tilde{a},X,Y,Z)=\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)=(1+s^2)X^3(sX+Y)^2(X^2+Y^2). \tag{89}$$

Hence systems (88) possess one triple ($x=0$) and one double ($sx+y=0$) real lines as well as two complex invariant lines $y=\pm ix$. So by Remark 3.1 we conclude, that cubic systems

$$\begin{aligned}\dot{x}=&a+cx+dy+2hxy+ky^2+(1+s^2)x^3,\\ \dot{y}=&b+ex+fy+lx^2+2mxy-sx^3+s^2x^2y-sxy^2-y^3\end{aligned} \tag{90}$$

with the cubic homogeneities (88) (here we assume $g=n=0$ due to a rescaling) could have a triplet only in the direction $x=0$ and a couple of parallel lines only in the direction $y=-sx$. Moreover these systems could have two simple complex conjugate invariant lines. Using the equations (13) we evaluate them for each one of these directions.

(i) The direction $x = 0$. In this case we obtain

$$\begin{gathered}U = 1,\ V = 0,\ A = 1 + s^2,\ B = 0,\ C = 0,\ D = -(1+s^2)W,\\ E = 2h,\ F = c + (1+s^2)W^2,\quad Eq_7 = k,\\ Eq_9 = d - 2hW,\ Eq_{10} = a - cW - (1+s^2)W^3,\\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0\end{gathered} \tag{91}$$

and to have exactly three parallel invariant lines in this direction the condition $k = d = h = 0$ is necessary and sufficient. Therefore we get the family of systems

$$\begin{aligned}\dot{x} =& a + cx + (1+s^2)x^3,\\ \dot{y} =& b + ex + fy + lx^2 + 2mxy - sx^3 + s^2x^2y - sxy^2 - y^3.\end{aligned} \tag{92}$$

(ii) The direction $sx + y = 0$. Then for the systems above we calculate

$$\begin{gathered}U = s,\ V = 1,\ A = s^2,\ B = 0,\ C = -1,\ D = 2m - sW,\\ E = W,\ F = f - W^2,\quad Eq_5 = l - 2ms,\\ Eq_8 = e + s(c-f) - 2mW + 2sW^2,\ Eq_{10} = b + as - fW + W^3,\\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0\end{gathered} \tag{93}$$

and to have exactly two parallel invariant lines in this direction the conditions $l = 2ms$ and $s \neq 0$ must hold. Moreover in order to have two invariant lines in the direction $sx + y = 0$ the following conditions are necessary and sufficient:

$$R_W^{(0)}(Eq_8, Eq_{10}) = R_W^{(1)}(Eq_8, Eq_{10}) = 0.$$

Assuming $l = 2ms$ (then $Eq_5 = 0$) we calculate $R_W^{(1)}(Eq_8, Eq_{10}) = -2\big[es - 2m^2 + (c+f)s^2\big] = 0$ and as $s \neq 0$ we obtain $e = \big[2m^2 - (c+f)s^2\big]/s$. Then we calculate

$$R_W^{(0)}(Eq_8, Eq_{10}) = 8(fms^2 + bs^3 + as^4 - m^3)^2/s^3 = 0,$$

and this yields $b = -\big[fms^2 + as^4 - m^3\big]/s^3$. So we obtain

$$Eq_9 = -\frac{2}{s}(fs^2 - m^2 + msW - s^2W^2),\quad Eq_{10} = -\frac{1}{s^3}(m + sW)(fs^2 - m^2 + msW - s^2W^2)$$

and hence we have two common solutions which could be real or complex, distinct or coinciding.

(iii) The direction $x + iy = 0$. In this case we obtain

$$\begin{gathered}U = 1,\ V = i,\ A = 1 - is + s^2,\ B = -(i+s)/2,\ C = -1,\\ D = 2m + (2 - is)W,\ E = -iW,\ F = f + W^2,\\ Eq_5 = i(i+s)\big[2m + (3+is)W\big],\\ Eq_8 = c - f + i(2m^2/s - cs - fs) - 2mW + (is - 3)W^2,\\ Eq_{10} = a + i(m^3/s^3 - fm/s - as) - fW - W^3,\\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_6 = Eq_7 = Eq_9 = 0.\end{gathered} \tag{94}$$

So for the existence of an invariant line in this complex direction we must have $W = -2m/(3+is)$ and then we calculate

$$\begin{gathered}Eq_8 = (c-f) + \frac{48m^2s^2}{(9+s^2)^2} - i\bigg[(c+f)s + \frac{6m^2(s^4 - 18s^2 - 27)}{s(9+s^2)^2}\bigg],\\ Eq_{10} = a + \frac{6fm}{9+s^2} - \frac{72m^3(s^2-3)}{(9+s^2)^3} - i\bigg[as + \frac{3fm(3+s^2)}{s(9+s^2)} - \frac{9m^3(s^2-3)(s^4 - 18s^2 - 27)}{s^3(9+s^2)^3}\bigg].\end{gathered}$$

Since the coefficients of systems (92) are real, the conditions $Eq_8 = Eq_{10} = 0$ lead to the following four equalities:

$$(c-f)+\frac{48m^2s^2}{(9+s^2)^2}=(c+f)s+\frac{6m^2(s^4-18s^2-27)}{s(9+s^2)^2}=0,$$
$$a+\frac{6fm}{9+s^2}-\frac{72m^3(s^2-3)}{(9+s^2)^3}=as+\frac{3fm(3+s^2)}{s(9+s^2)}-\frac{9m^3(s^2-3)(s^4-18s^2-27)}{s^3(9+s^2)^3}=0.$$

Herein we obtain the following relations

$$c=-\frac{27m^2(s^2-3)(1+s^2)}{s^2(9+s^2)^2},\quad f=\frac{3m^2(27+18s^2+7s^4)}{s^2(9+s^2)^2},\quad a=-\frac{54m^3(1+s^2)}{s^2(9+s^2)^2}$$

and therefore we get the following dependencies among the parameters of systems (90) with $g = n = 0$ (due to a translation):

$$\begin{aligned} &k=d=h=0,\ l=2ms,\ c=-\frac{27m^2(-3+s^2)(1+s^2)}{s^2(9+s^2)^2},\ e=\frac{8m^2s(s^2-9)}{(9+s^2)^2}\\ &f=\frac{3m^2(27+18s^2+7s^4)}{s^2(9+s^2)^2},\ a=-\frac{54m^3(1+s^2)}{s^2(9+s^2)^2},\quad b=\frac{2m^3(9+17s^2)}{s(9+s^2)^2}. \end{aligned}\tag{95}$$

Thus we arrive to the family of systems

$$\begin{aligned} \dot{x}=&(1+s^2)\left(x-\frac{6m}{9+s^2}\right)\left[x^2+\frac{6m}{9+s^2}\,x+\frac{9m^2}{s^2(9+s^2)}\right],\\ \dot{y}=&b+ex+fy+2msx^2+2mxy-sx^3+s^2x^2y-sxy^2-y^3, \end{aligned}\tag{96}$$

where the parameters b, e and f have the values indicated above.

Assume first $m \neq 0$. Since $s \neq 0$ it is easy to find out that via the transformation

$$x_1=\frac{s(9+s^2)}{9m}x-\frac{2s}{3},\quad y_1=\frac{s(9+s^2)}{9m}+\frac{2s^2}{9},\quad t_1=\frac{81m^2}{s^2(9+s^2)^2}t$$

the systems above could be brought to the 1-parameter family of systems (we keep the old notations of variables)

$$\begin{aligned} \dot{x}=&(1+s^2)x\big[(x+s)^2+1\big],\\ \dot{y}=&(1+s^2)^2y+2s(1+s^2)xy-sx^3+s^2x^2y-sxy^2-y^3. \end{aligned}\tag{97}$$

These systems possess the invariant lines

$$\begin{gathered} L_1=x,\quad L_2=x+s+i,\quad L_3=x+s-i,\ L_4=sx+y,\\ L_5=sx+y+1+s^2,\quad L_6=y+ix,\quad L_7=y-ix \end{gathered}$$

and it is clear that all these lines are distinct. Systems (97) possess the following 3 real and 6 complex finite singularities:

$$\begin{gathered} (0,0),\ (0,\pm(1+s^2)),\ \big(i-s,\pm(1+is)\big),\ \big(-i-s,\pm(1-is)\big),\\ \big(i-s,s(s-i)\big),\ \big(-i-s,s(s+i)\big). \end{gathered}$$

We observe that all singular points except $(0, 1 + s^2)$ are located at the intersections of the invariant lines and this leads to the configuration *Config. 8.13* (see FIGURE 1).

Assume now $m = 0$. Then systems (96) become the homogeneous systems (88) and considering (89) we deduce that these systems possess two real invariant straight lines $x = 0$ (triple) and $y + sx = 0$ (double), as well as two complex lines $y = \pm ix$. Therefore in this we obtain the configuration *Config. 8.14* from FIGURE 1.

3.7.3 Invariant conditions for the configurations *Config. 8.13, 8.14*

As it was proved in the previous subsection for the existence of the configuration $(3,2,1,1)$ the condition $s \neq 0$ is necessary. On the other hand for systems (90) we have $\mathcal{D}_4 = 2304s(9+s^2)$ and hence the condition above is equivalent to $\mathcal{D}_4 \neq 0$. Considering Remark 3.5 we conclude that the condition $\mathcal{V}_3 \neq 0$ to distinguish the systems (87) and (88) could be substituted by $\mathcal{D}_4 \neq 0$.

Now we concentrate our attention on the conditions (95). Following the statement B_2) of the Main Theorem we shall prove that these conditions for systems (90) are equivalent to the affine invariant conditions

$$\mathcal{K}_4 = \mathcal{K}_5 = \mathcal{K}_6 = 0.$$

First we claim that the condition $\mathcal{K}_4 = 0$ is equivalent to $k = h = 0$ and $l = 2ms$. Indeed, for systems (90) we calculate

$$\text{Coefficient}[\mathcal{K}_4,\ y^3] = k(s^2+1)$$

and clearly the condition $\mathcal{K}_4 = 0$ implies $k = 0$. Then we have Coefficient$[\mathcal{K}_4,\ xy^2] = 8hs^2/9 = 0$, i.e. $h = 0$ and in this case we calculate

$$\mathcal{K}_4 = (2ms - l)(9+s^2)x^3/9 = 0$$

and this implies $l = 2ms$. So our claim is proved.

Thus it remains to prove that for the family of systems

$$\begin{aligned}\dot{x} =& a + cx + dy + (1+s^2)x^3,\\ \dot{y} =& b + ex + fy + 2msx^2 + 2mxy + ny^2 - sx^3 + s^2x^2y - sxy^2 - y^3\end{aligned} \tag{98}$$

the conditions

$$\begin{aligned} d = 0,\ c = -\frac{27m^2(-3+s^2)(1+s^2)}{s^2(9+s^2)^2},\ e = \frac{8m^2s(s^2-9)}{(9+s^2)^2},\\ f = \frac{3m^2(27+18s^2+7s^4)}{s^2(9+s^2)^2},\ a = -\frac{54m^3(1+s^2)}{s^2(9+s^2)^2},\ \ b = \frac{2m^3(9+17s^2)}{s(9+s^2)^2}.\end{aligned} \tag{99}$$

are equivalent to the conditions $\mathcal{K}_5 = \mathcal{K}_6 = 0$.

Indeed, if for the systems above the conditions (99) are fulfilled then $\mathcal{K}_5 = \mathcal{K}_6 = 0$.

Conversely, assume that $\mathcal{K}_5 = \mathcal{K}_6 = 0$ for systems (98). Calculations yield

$$\text{Coefficient}[\mathcal{K}_5,\ y^4] = 30d(9+5s^2) = 0$$

and this implies $d = 0$. Then we obtain

$$\mathcal{K}_5 = 10Z_1x^4/3 + 20Z_2x^3y/3 + 20Z_3xy^2(sx+y),$$

where

$$\begin{aligned} Z_1 =& cs(81+180s^2-5s^4) + fs(5s^4-108s^2-81) + e(81+162s^2-47s^4) + 136m^2s^3,\\ Z_2 =& c(2s^4+27s^2-27) + f(27+9s^2-2s^4) + 8es(9+s^2) + 4m^2(8s^2-9),\\ Z_3 =& c(9+5s^2) + f(7s^2-9) - 12m^2.\end{aligned}$$

So solving the system of equations $Z_1 = Z_2 = Z_3 = 0$ with respect to the parameters c, f and e we get the respective expressions from (99). Considering these values of the parameters we calculate

$$\text{Coefficient}[\mathcal{K}_6,\ x^5y^6] = \frac{40(9+5s^2)^2(225+493s^2)}{9s^2(9+s^2)^2}\left[as^2(9+s^2)^2 + 54m^3(1+s^2)\right] = 0$$

and this implies $a = -\dfrac{54m^3(1+s^2)}{s^2(9+s^2)^2}$. Then we obtain

$$\mathcal{K}_6 = -\frac{40}{9s(9+s^2)^2}\big[bs(9+s^2)^2 - 2m^3(9+17s^2)\big]x^6(sx+y)^2\big[(s^2-3)sx-(9+5s^2)y\big]\times$$
$$\big[s^2(2781+5718s^2+409s^4)x^2+4s(9+s^2)(125s^2-33)xy+(9+5s^2)(407s^2-225)y^2\big]$$

and we observe that the condition $\mathcal{K}_6 = 0$ implies $b = \dfrac{2m^3(9+17s^2)}{s(9+s^2)^2}$. So we get for the parameters a and b the expressions given in (99) and this completes the proof of the fact, that the conditions (99) are equivalent to the conditions $\mathcal{K}_5 = \mathcal{K}_6 = 0$.

It remains to find out the invariant polynomial which governs the condition $m = 0$ for systems (96). For these systems we calculate

$$\mathcal{L}_1 = 41472m(1+s^2)x^2(sx+y)$$

and it is clear that the condition $\mathcal{L}_1 = 0$ is equivalent to $m = 0$. Thus the statement B_2) of the Main Theorem is proved.

3.8 Systems with configuration (2, 2, 2, 1)

In this subsection we determine the normal form of the family of cubic systems with 2 real and two complex infinite distinct singular points which possess 7 invariant affine straight lines with the configuration or potential configuration (2, 2, 2, 1).

For having this configuration a cubic system has to possess three couples of parallel invariant lines. Thus, according to Theorem 2.2 in the considered case the condition $\mathcal{V}_3 = 0$ necessarily holds.

3.8.1 Construction of the corresponding cubic homogeneities

As a first step we shall construct the cubic homogeneous parts of systems (70) for which the condition above is fulfilled. So we shall consider the family of systems (85) and we shall force the condition $\mathcal{V}_3 = 0$ to be satisfied.

As in the Subsection 3.4 we shall use here the notation $\mathcal{V}_3 = \sum_{j=0}^{4} \mathcal{V}_{3j}x^{4-j}y^j$. A straightforward computation of the value of $\mathcal{V}_3$ for systems (85) yields:

$$\mathcal{V}_{33} = 64r(s-v), \quad \mathcal{V}_{34} = -32r(r-3). \tag{100}$$

So we shall consider two cases: $r \neq 0$ and $r = 0$.

3.8.1.1 The case $r \neq 0$. Then by (100) we get $r = 3$, $v = s$ and then we calculate

$$\mathcal{V}_{30} = -32u(3+u), \quad \mathcal{V}_{31} = -192su, \quad \mathcal{V}_{32} = -288u.$$

So in this case the condition $\mathcal{V}_3 = 0$ yields $r = 3$, $v = s$ and $u = 0$ and we arrive at the family of systems

$$\dot{x} = x^3 + 2sx^2y + 3xy^2, \quad \dot{y} = -sx^3 + sxy^2 + 2y^3. \tag{101}$$

3.8.1.2 The case $r = 0$. Then we have $\mathcal{V}_{33} = \mathcal{V}_{34} = 0$ and we calculate $\mathcal{V}_{32} = 32(-9+2s^2-3u-sv-v^2) = 0$. So we get $u = (-9+2s^2-sv-v^2)/3$ and we obtain

$$\mathcal{V}_{30} = -32/9(s-v)^2\big[9+(2s+v)^2\big], \quad \mathcal{V}_{31} = -64/3(s-v)\big[9+(2s+v)^2\big].$$

Thus in this case the condition $\mathcal{V}_3 = 0$ implies $r = 0$, $v = s$ $u = -3$ and we get the family of systems

$$\dot{x} = -2x^3 + 2sx^2y, \quad \dot{y} = -sx^3 - 3x^2y + sxy^2 - y^3. \tag{102}$$

We remark that a system (101) could be brought to a system above with the same parameter s via the transformation

$$x_1 = sx + y, \quad y_1 = x - sy, \quad t_1 = -t/(1 + s^2).$$

So the systems (102) are affine equivalent to systems (101) and in what follows we shall consider the cubic systems with cubic non-linearities defined by systems (102). Moreover, as due to a translation we can consider $g = n = 0$ we arrive at the next result.

Lemma 3.5. *Assume that a cubic system* (70) *possesses* 7 *invariant affine straight lines with configuration or potential configuration* (2, 2, 2, 1)*. Then via an affine transformation and a time rescaling this system could be brought to a system belonging to the following family of systems:*

$$\begin{aligned}\dot{x} =& a + cx + dy + 2hxy + ky^2 - 2x^3 + 2sx^2y,\\ \dot{y} =& b + ex + fy + lx^2 + 2mxy - sx^3 - 3x^2y + sxy^2 - y^3.\end{aligned} \tag{103}$$

Remark 3.6. *We observe that due to the change $y \to -y$ for* (103) *we may assume $s \geq 0$.*

3.8.2 Construction of the cubic systems possessing configuration or potential configuration (2, 2, 2, 1)

For homogeneous cubic system (102) we have

$$H(\tilde{a}, X, Y, Z) = \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3) = 2X^2(sX + Y)(X^2 + Y^2)^2. \tag{104}$$

So systems (102) possess three double invariant lines: the real line $x = 0$ and two complex invariant lines $y = \pm ix$. So by Remark 3.1 we conclude, that cubic systems (103) could have three couples of parallel lines only in these directions. Moreover these systems could have one simple real invariant line in the direction $y = -sx$.

(i) The direction $x = 0$. In this case we obtain

$$\begin{gathered}U = 1,\ V = 1,\ A = -2,\ B = s,\ C = 0,\ D = 2W,\ E = 2(h - sW),\ F = c - 2W^2,\\ Eq_7 = k,\ Eq_9 = d - 2hW + 2sW^2,\ Eq_{10} = a - cW + 2W^3,\\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_6 = Eq_8 = 0.\end{gathered} \tag{105}$$

Therefore to have exactly two parallel invariant lines in this direction it is necessary and sufficient $Eq_7 = 0$, $s \neq 0$ and

$$R_W^{(0)}(Eq_9, Eq_{10}) = R_W^{(1)}(Eq_9, Eq_{10}) = 0.$$

The equality $Eq_7 = 0$ gives $k = 0$ and then we calculate

$$R_W^{(1)}(Eq_9, Eq_{10}) = 4(2h^2 - ds - cs^2) = 0.$$

So we get $d = (2h^2 - cs^2)/s$ and we have

$$R_W^{(0)}(Eq_9, Eq_{10}) = 8(-2h^3 + chs^2 + as^3)^2/s^3 = 0.$$

Therefore we obtain $a = h(2h^2 - cs^2)/s^3$ and this implies

$$\begin{gathered}Eq_9 = (2h^2 - cs^2 - 2hsW + 2s^2W^2)/s,\\ Eq_{10} = (h + sW)(2h^2 - cs^2 - 2hsW + 2s^2W^2)/s^3.\end{gathered}$$

So we have two common solutions, which could be real or complex, distinct or coinciding.

(ii) The direction $x+iy=0$. We obtain

$$
\begin{gathered}
U=1,\ V=i,\ A=-(2+is),\ B=(s-i)/2,\ C=-1,\ D=2W+i(l+sW),\\
E=l+2h+i(2m-W),\ F=c-2W^2+i(e-lW-sW^2),\ Eq_7=2m-i(l+2h),\\
Eq_{10}=(2h^3-chs^2)/s^3+ib-(c+ie)W+ilW^2+(2+is)W^3,\\
Eq_9=(2h^2)/s-cs+e+i(f-c)-2(l+h+im)W+(3i-s)W^2,\\
Eq_1=Eq_2=Eq_3=Eq_4=Eq_5=Eq_6=Eq_8=0
\end{gathered}
\tag{106}
$$

and to have exactly two parallel invariant lines in this direction it is necessary $Eq_7=0$. As the parameters of cubic systems are real the equality $Eq_7=0$ gives $m=0$, $l=-2h$. So considering the relations

$$
k=0,\quad d=(2h^2-cs^2)/s,\quad a=h(2h^2-cs^2)/s^3,\quad m=0,\quad l=-2h,
\tag{107}
$$

determined at this moment among the parameters of systems (103), we examine the fourth direction: $sx+y=0$.

(iii) The direction $sx+y=0$. Considering the conditions (107) we find out

$$
\begin{gathered}
U=s,\ V=1,\ A=-3,\ B=s,\ C=-1,\ D=2hs-3sW,\\
E=W,\ F=f+2h^2-cs^2-W^2,\\
Eq_5=(1+s^2)(3W-2h),\quad Eq_8=e+(c-f-2h^2)s+cs^3-2hsW+4sW^2,\\
Eq_{10}=b-ch+2h^3/s^2-(f+2h^2-cs^2)W+W^3,\\
Eq_1=Eq_2=Eq_3=Eq_4=Eq_6=Eq_7=Eq_9=0.
\end{gathered}
\tag{108}
$$

Hence the unique value for the parameter W given by the equality $Eq_5=0$ is $W=2h/3$. Then calculations yield:

$$
\begin{aligned}
&Eq_8\big|_{W=2h/3}=e+(c-f)s+cs^3-14h^2s/9=0,\\
&Eq_{10}\big|_{W=2h/3}=b-h(3c+2f-2cs^2)/3+2h^3(27-14s^2)/(27s^2)=0
\end{aligned}
$$

and we get

$$
e=(f-c)s-cs^3+14h^2s/9,\quad b=\frac{h\big[(27c+18f+28h^2)s^2-18cs^4-54h^2\big]}{27s^2}
$$

Therefore, considering (106) for the direction $x+iy=0$ we obtain

$$
\begin{aligned}
Eq_9=&2h^2(9+7s^2))/(9s)-2cs+fs-cs^3+i(f-c)+2hW+(3i-s)W^2,\\
Eq_{10}=&\frac{h(2h^2-cs^2)}{s^3}-\frac{ih(54h^2-27cs^2-18fs^2-28h^2s^2+18cs^4)}{27s^2}-\\
&-\big[c+i(fs-cs-cs^2+14h^2s/9\big]W-2ihW^2+(2+is)W^3
\end{aligned}
$$

and the conditions

$$
R_W^{(0)}(Eq_9,Eq_{10})=R_W^{(1)}(Eq_9,Eq_{10})=0
\tag{109}
$$

have to be satisfied. We calculate

$$
R_W^{(1)}(Eq_9,Eq_{10})=\Phi_1(c,f,h,s)+i\Phi_2(c,f,h,s)=0,
$$

where

$$
\begin{aligned}
&\Phi_1=(3-s^2)(c+2f-cs^2)+2h^2(27-7s^2)/9,\\
&\Phi_2=8fs-4cs(s^2-1)+4h^2(5s^2-9)/(3s).
\end{aligned}
$$

Therefore the relation $\Phi_2 = 0$ gives $f = \Big[3cs^2(s^2-1) + h^2(9-5s^2)\Big]/(6c^2)$ and then we obtain

$$\Phi_1 = h^2(s^2-9)^2/9s^2 = 0. \tag{110}$$

So we have either $s = \pm 3$ or $h = 0$ and in both cases the conditions (109) are fulfilled.

In the case $s = \pm 3$ by Remark 3.6 we may assume $s = 3$ and we get $f = 2(6c - h^2)/3$. Then we obtain $\Phi_1(c, f, h, s) = \Phi_2(c, f, h, s) = 0$ and $R_W^{(0)}(Eq_9, Eq_{10}) = 0$. In such a way we arrive at the following relations among the parameters of systems (103):

$$\begin{gathered} s = 3, \quad k = m = 0, \; l = -2h, \quad d = (2h^2 - 9c)/3, \quad e = 2(4h^2 - 27c)/3, \\ f = 2(6c - h^2)/3, \quad a = h(2h^2 - 9c)/27, \quad b = h(10h^2 - 63c)/27 \end{gathered} \tag{111}$$

and this leads to the family of systems (for the further necessity we set here $h = h_1$ and $c = c_1$)

$$\begin{aligned} \dot{x} =& \big[(h_1 + 6x)^2 + 3(h_1^2 - 6c_1)\big](h_1 - 3x + 9y)/54, \\ \dot{y} =& \frac{h_1}{27}(10h_1^2 - 63c) + \frac{2}{3}(4h_1^2 - 27c_1)x + \frac{2}{3}(6c_1 - h_1^2)y - 2h_1x^2 - \\ &- 3x^3 - 3x^2y + 3xy^2 - y^3. \end{aligned} \tag{112}$$

In the case $h = 0$ we arrive at the following relations among the parameters of systems (103):

$$\begin{gathered} k = m = h = l = 0, \quad d = -cs, \quad e = -cs(3 + s^2)/2, \\ f = c(s^2 - 1)/2, \quad a = b = 0 \end{gathered} \tag{113}$$

and this leads to the family of systems

$$\begin{aligned} \dot{x} =& (c - 2x^2)(x - sy), \\ \dot{y} =& -cs(3 + s^2)x/2 + c(s - 1)y/2 - sx^3 - 3x^2y + sxy^2 - y^3. \end{aligned} \tag{114}$$

We observe that systems (112) (respectively systems (114)) possess two parallel invariant lines in the direction $x = 0$, which are real if $6c_1 - h_1^2 > 0$ (respectively $c > 0$); complex if $6c_1 - h_1^2 < 0$ (respectively $c < 0$) and they coincide if $6c_1 - h_1^2 = 0$ (respectively $c = 0$).

It is easy to check that in the case $(6c_1 - h_1^2)c \neq 0$ as well as in the case $6c_1 - h_1^2 = c = 0$ systems (112) could be brought to the systems (114) with $s = 3$ via the transformation

$$x_1 = \alpha x + h_1\alpha/6, \quad y_1 = \alpha y + h_1\alpha/6, \quad t_1 = t/\alpha^2,$$

where $\alpha = \sqrt{6c/(6c_1 - h_1^2)}$ if $(6c_1 - h_1^2)c > 0$ and $\alpha = 1$ if $6c_1 - h_1^2 = c = 0$.

Thus it was proved the next lemma.

Lemma 3.6. *A system* (103) *possesses the configuration or potential configuration of invariant lines* $(2, 2, 2, 1)$ *if and only this system via an affine transformation and time rescaling could be brought to a cubic system belonging to the subfamily* (114)*, which is defined in the family* (103) *by the conditions*

$$\begin{gathered} s \neq 0, \quad k = m = h = l = 0, \quad d = -cs, \quad e = -cs(3 + s^2)/2, \\ f = c(s^2 - 1)/2, \quad a = b = 0. \end{gathered} \tag{115}$$

Next we examine systems (114), considering each one of of the cases: $c > 0$, $c < 0$ and $c = 0$.

3.8.2.1 The subcase $c > 0$.

Then we may assume $c = 2u^2 \neq 0$ and via the transformation

$$\big(x, y, t\big) \mapsto \big(-(2x + 1)u,\; (s - 2y)u,\; t/(4u^2)\big)$$

systems (114) can be brought to the systems

$$\begin{aligned}\dot{x} =& x(x-1)(1+s^2-2x+2sy),\\ \dot{y} =& -sx^3-y-s^2y+3xy+s^2xy-3x^2y-2sy^2+sxy^2-y^3.\end{aligned} \tag{116}$$

These systems possesses the invariant lines

$$x=0,\quad x=1,\quad y=-sx,\quad y=\pm ix,\quad y\pm i(x-1)+s=0$$

and considering its nine finite singularities

$$\begin{gathered}(0,0),\quad (1,-s),\quad (1/2,-s/2),\quad (1,\pm i),\quad (0,-s\pm i),\\ \big((1+is)/2,(i-s)/2\big),\quad \big((1-is)/2,(-i-s)/2\big)\end{gathered}$$

we arrive at the*Config. 8.15* (see Figure 1).

3.8.2.2 The subcase $c<0$. Then we may assume $c=-2u^2\neq 0$ and due to the rescaling $(x,y,t)\mapsto(ux,uy,t/u^2)$ we obtain the systems

$$\begin{aligned}\dot{x} =& 2(1+x^2)(sy-x-s),\\ \dot{y} =& s(s^2+3)x+(1-s^2)y-sx^3-3x^2y+sxy^2-y^3.\end{aligned} \tag{117}$$

These systems possess the invariant lines

$$y=-sx,\quad x=\pm i,\quad y-ix\pm(1-is)=0,\quad y+ix\pm(1+is).$$

So considering the nine finite singularities

$$(0,0),\quad (-s,\,-1),\quad (s,\,1),\quad \big(i,\,is\pm 2\big),\quad \big(-i,\,-is\pm 2\big),\quad \big(i,\,-is\big),\quad \big(-i,\,is\big)$$

of the above systems, we arrive at the configuration given by *Config. 8.16* from Figure 1.

3.8.2.3 The subcase $c=0$. Then systems (114) become the homogeneous systems (102), which possess the real invariant lines $x=0$ (double) and $y=-sx$ (simple) as well as the complex invariant lines $y=\pm ix$ (both doubles). As a result we get the configuration of invariant lines given by *Config. 8.17* in Figure 1.

3.8.3 Invariant conditions for the configurations *Config. 8.15, 8.16, 8.17*

Considering the statement B_3) of the Main Theorem and Lemma 3.6 we shall prove we shall prove that the affine invariant conditions

$$\mathcal{D}_4\neq 0,\quad \mathcal{V}_3=\mathcal{K}_4=\mathcal{K}_2=\mathcal{K}_8=0. \tag{118}$$

applied to a cubic system (103) force this system to be from the class determined by Lemma 3.6.

For the family of systems (103) we have $\mathcal{D}_4=2304s(9+s^2)$ and hence the condition $s\neq 0$ is equivalent to $\mathcal{D}_4\neq 0$. Assume that for a system (103) the condition $\mathcal{K}_4=0$ is satisfied. Then we obtain Coefficient$[\mathcal{K}_4,y^3]=-2k(9+s^2)/9$ and hence the condition $\mathcal{K}_4=0$ implies $k=0$. In this case we obtain

$$\mathcal{K}_4=\frac{2}{9}\big[(6ms-9h-2ls^2-3hs^2)x^3-2s(3l+6h+2ms)x^2y-(9h+6ms-hs^2)xy^2\big]$$

and equalizing with zero the first two coefficients of the polynomial $\mathcal{K}_4$ due to $s\neq 0$ we get

$$l=-\frac{3h(9+s^2)}{9+2s^2},\quad m=-\frac{3h(s^2-9)}{2s(9+2s^2)},\quad \mathcal{K}_4=-\frac{4h(s^2-9)(s^2+9)xy^2}{9(9+2s^2)}. \tag{119}$$

Therefore the condition $\mathcal{K}_4=0$ gives either $h=0$, or $s=\pm 3$.

3.8.3.1 The case $h = 0$. Then we obtain $k = l = m = h = 0$ and we calculate

$$\mathcal{K}_2 = -3x^2(bx - ay)(3x + s^2x - 2sy)^2(x^2 + y^2)^2,$$
$$\mathcal{K}_8 = Z_1x^4 + Z_2x^3y + Z_3x^2y^2 + 3Z_4xy^3 + 5Z_5y^4,$$

where

$$\begin{aligned}
Z_1 =& -2(99d - 63e - 9fs + 15ds^2 - 10es^2 - 32cs^3 - 10fs^3),\\
Z_2 =& 2(3c + 6f + 159ds - 103es - 3fs^2 + 50ds^3),\\
Z_3 =& 2(54d + 27e + 105cs + 21fs + 45ds^2 - 20es^2 + 8cs^3 - 20fs^3),\\
Z_4 =& 6(-9c - 18f + 3ds - es + 10cs^2 - fs^2),\\
Z_5 =& 10(-3d - cs + 4fs + 2ds^2).
\end{aligned}$$

So setting $Z_2 = Z_4 = Z_5 = 0$ we find out that $d = -cs$, $e = -cs(3 + s^2)/2$, $f = c(s^2 - 1)/2$ and then $\mathcal{K}_8 = 0$.

On the other hand the condition $\mathcal{K}_2 = 0$ obviously gives $a = b = 0$ and hence in this case we arrive at the conditions (115).

3.8.3.2 The case $s = \pm 3$. By Remark 3.6 we may assume $s = 3$. Then $\mathcal{K}_4 = 0$ and we calculate

$$\mathcal{K}_8 = Z_1'x^4 + Z_2'x^3y + Z_3'x^2y^2 + 3Z_4'xy^3 + 5Z_5'y^4,$$

where

$$\begin{aligned}
Z_1' =& 18(96c - 26d + 17e + 33f - 6h^2),\\
Z_2' =& 6(c + 609d - 103e - 7f - 136h^2),\\
Z_3' =& -18(59c + 51d - 17e - 53f - 24h^2),\\
Z_4' =& 6(81c - 4l^2 + 9d - 3e - 27f),\\
Z_5' =& -10(3c - 15d - 12f + 2h^2).
\end{aligned}$$

Setting $Z_2' = Z_4' = Z_5' = 0$ we obtain $d = (2h^2 - 9c)/3$, $e = 2(4h^2 - 27c)/3$, $f = 2(6c - h^2)/3$ and then $\mathcal{K}_8 = 0$ and

$$\mathcal{K}_2 = -4x^2(2x - y)^2(x^2 + y^2)^2\big[(27b + 63ch - 10h^3)x + (-27a - 9ch + 2h^3)y\big],$$

Therefore the condition $\mathcal{K}_2 = 0$ yields

$$a = h(2h^2 - 9c)/27, \quad b = h(10h^2 - 63c)/27$$

and we arrive at the conditions (111) corresponding to the case $s = 3$. These conditions lead to the systems (112), which via an affine transformation and time rescaling could be brought to systems (114) as it was shown on the page 46.

Next we consider the necessary and sufficient conditions to distinguish the configurations *Config. 8.15* – *Config. 8.17*. For systems (114) we calculate

$$K_9 = 90c(1 + s^2)^2x^2(x^2 + y^2)^2.$$

Therefore we obtain that $K_9 = 0$ if and only if $c = 0$. Moreover if $K_9 \neq 0$ then $\operatorname{sign}(K_9) = \operatorname{sign}(c)$. Thus we get *Config. 8.15* if $K_9 > 0$, *Config. 8.16* if $K_9 < 0$ and *Config. 8.17* if $K_9 = 0$. This completes the proof of the statement B_3) of the Main Theorem.

C. Cubic systems with four complex infinite singularities

If a cubic system has 4 imaginary infinite singular points via a linear transformation can written into the form (see Lemma 2.5):

$$\begin{aligned}
x' &= p_0 + p_1(x,y) + p_2(x,y) + ux^3 + (p + q + v)x^2y + rxy^2 + qy^3,\\
y' &= q_0 + q_1(x,y) + q_2(x,y) - px^3 + ux^2y + vxy^2 + ry^3,
\end{aligned} \tag{120}$$

for which $C_3(x, y) = (px^2 + qy^2)(x^2 + y^2)$, $pq > 0$.

We claim that systems above could not have invariant lines of total multiplicity 8 (including the line at infinity). Indeed as it was mentioned on the page 13, a cubic system with exactly 7 affine distinct invariant lines could possess only one of the following configurations:

$$(i)\ (3,\,3,\,1);\quad (ii)\ (3,\,2,\,2);\quad (iii)\ (3,\,2,\,1,\,1);\quad (iv)\ (2,\,2,\,2,\,1).$$

Moreover if it possesses a potential configuration of one of the types above, then via a small perturbation, which conserve the 4 infinite complex singularities, we arrive at one of the same configurations above with 7 distinct invariant affine lines.

On the other hand as the number (seven) of affine invariant straight lines is odd we conclude that at least one of the invariant lines must be real. However this contradicts to the existence of only complex infinite singular points of systems (120). Hence our claim is proved.

As all the cases are examined the Main Theorem is proved.

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