A PRIORI ESTIMATES FOR SEMISTABLE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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1. Introduction

In this note we consider semistable solutions of the boundary value problem

$$\left\{ \begin{array}{rcll} Lu + f(u) & = & 0 & \text{in} \quad \Omega, \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with $n \geq 2$, $f \in C^2$, and $Lu := \partial_i(a^{ij}(x)u_j)$ is uniformly elliptic. More precisely, we assume that $(a^{ij}(x))$ is a symmetric $n \times n$ matrix with bounded measurable coefficients, *i.e.*, $a^{ij} = a^{ji} \in L^\infty(\Omega)$, for which there exist positive constants c_0 and C_0 satisfying

$$c_0|\xi|^2 \leq a^{ij}(x)\xi^i\xi^j \leq C_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \ x \in \Omega. \tag{1.2}$$

By semistability of the solution u, we mean that the lowest Dirichlet eigenvalue of the linearized operator at u is nonnegative. That is, we have the *semistability inequality*

$$\int_\Omega f'(u)\eta^2 \, dx \leq \int_\Omega a^{ij}(x)\eta_i\eta_j \, dx \quad \text{for all } \eta \in H_0^1(\Omega). \tag{1.3}$$

There is a large literature on a priori estimates, beginning with the seminal paper of Crandall and Rabinowitz [4]. In [4] and subsequent works, a basic and standard assumption is that u is positive in Ω and $f \in C^2$ is positive, nondecreasing, and superlinear at infinity:

$$f(0) > 0, \quad f' \geq 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{f(t)}{t} = \infty. \tag{1.4}$$

Note that, under these assumptions and with $f(u)$ replaced by $\lambda f(u)$ with $\lambda \geq 0$, semistable solutions do exist for an interval of parameters $\lambda \in (0, \lambda^*)$; see [4].

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In recent years there have been strong efforts to obtain a priori bounds under minimal assumptions on f (essentially (1.4)), mainly after Brezis and Vázquez [1] raised several open questions. The following are the main results in this direction. The important paper of Nedev [5] obtains the L^∞ bound for $n = 2$ and 3 if f satisfies (1.4) and in addition f is convex. Nedev states his result for $L = \Delta$ but it is equally valid for general L. When $2 \le n \le 4$ and $L = \Delta$, Cabré [2] established that the L^∞ bound holds for *arbitrary* f if in addition Ω convex. Villegas [9] replaced the condition that Ω is convex in Cabré's result assuming instead that f is convex. For the radial case, Cabré and Capella [3] proved the L^∞ bound when $n \le 9$. On the other hand, it is well known that there exist unbounded semistable solutions when $n \ge 10$ (for instance, for the exponential nonlinearity e^u).

For convex nonlinearities f and under extra assumptions involving the two numbers

$$\tau_- := \liminf_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} \le \tau_+ := \limsup_{t\to\infty} \frac{f(t)f''(t)}{f'(t)^2} \tag{1.5}$$

much more is known (see more detailed comments after Corollary 1.3). For instance, Sanchón [6] proved that $u \in L^\infty(\Omega)$ whenever $\tau_- = \tau_+ \ge 0$ and $n \le 9$. This hypothesis is satisfied by $f(u) = e^u$, as well as by $f(u) = (1+u)^m$, $m > 1$.

It is still an **open problem** to establish an L^∞ estimate in general domains Ω when $n \le 9$ under (1.4) as the only assumption on f.

Our purpose here is to prove the following results:

Theorem 1.1. *Let $f \in C^2$ be convex and satisfy* (1.4)*. Assume in addition that for every $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and $C = C(\varepsilon)$ such that*

$$f'(t) \le C f(t)^{1+\varepsilon} \quad \textit{for all } t > T. \tag{1.6}$$

Then if u is a positive semistable solution of (1.1)*, we have $f'(u) \in L^p(\Omega)$ for all $p < 3$ and $n \ge 2$, while $f(u) \in L^p(\Omega)$ for all $p < \frac{n}{n-4}$ and $n \ge 6$.*

As a consequence, we deduce respectively:

(a) If $n \le 5$, then $u \in L^\infty(\Omega)$.

(b) If $n \ge 6$, then $u \in W_0^{1,p}(\Omega)$ for all $p < \frac{n}{n-5}$ and $u \in L^p(\Omega)$ for all $p < \frac{n}{n-6}$. In particular, if $n \le 9$ then $u \in H_0^1(\Omega)$.

Theorem 1.1 establishes the L^∞ bound up to dimension 5 when (1.6) holds. If we assume more about f we can obtain an L^∞ bound up to dimension $n = 6$.

Theorem 1.2. *Let $f \in C^2$ be convex and satisfy* (1.4). *Assume in addition that there exist $\varepsilon \in (0,1)$ and $T = T(\varepsilon)$ such that*

$$f'(t) \leq C f(t)^{1-\varepsilon} \quad \text{for all } t > T. \tag{1.7}$$

Then if u is a positive semistable solution of (1.1), *we have $f'(u) \in L^{\frac{3-\varepsilon}{1-\varepsilon}}(\Omega)$ for all $n \geq 2$, while $f(u) \in L^p(\Omega)$ for all $p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n-4+2\varepsilon}$ and $n \geq 6 + \frac{4\varepsilon}{1-\varepsilon}$.*

As a consequence, we deduce respectively:

(*a*) *If $n < 6 + \frac{4\varepsilon}{1-\varepsilon}$, then $u \in L^\infty(\Omega)$.*

(*b*) *If $n \geq 6 + \frac{4\varepsilon}{1-\varepsilon}$, then $u \in W_0^{1,p}(\Omega)$ for all $p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n-5+3\varepsilon}$ and $u \in L^p(\Omega)$ for all $p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n-6+4\varepsilon}$. In particular, if $n < 10 + \frac{4\varepsilon}{1-\varepsilon}$ then $u \in H_0^1(\Omega)$.*

The main novelty of our results are twofold. On the one hand, we do not assume any lower bound on f' to obtain our estimates, nor any bound on f'' as in [4] or [6] (as commented below). On the other hand, we obtain L^p estimates for $f'(u)$. To our knowledge such estimates do not exist in the literature. In fact, using the L^p estimate for $f(u)$ established in Theorem 1.2 and standard regularity results for uniformly elliptic equations, it follows that u is bounded in $L^\infty(\Omega)$ whenever $n < 6 + \frac{2\varepsilon}{1-\varepsilon}$. Note that the range of dimensions obtained in Theorem 1.2 (a), $n < 6 + \frac{4\varepsilon}{1-\varepsilon}$, is bigger than this one. This will follow from the L^p estimate on $f'(u)$. Of course, in both results (and also in the rest of the paper), $u \in L^p$ or $u \in W^{1,p}$ mean that u is bounded in L^p or in $W^{1,p}$ by a constant independent of u.

Our assumptions (1.6) and (1.7) in Theorems 1.1 and 1.2 are related to the hypothesis $\tau_+ \leq 1$ (recall (1.5)). Indeed, by the definition of τ_+, for every $\delta > 0$ there exists $T = T(\delta)$ such that $f(t)f''(t) \leq (\tau_+ + \delta) f'(t)^2$ for all $t > T$, or equivalently, $\frac{d}{dt}\frac{f'(t)}{f(t)^{\tau_+ + \delta}} \leq 0$ for all $t > T$. Thus,

$$\frac{f'(t)}{f(t)^{\tau_+ + \delta}} \leq \frac{f'(T)}{f(T)^{\tau_+ + \delta}} = C \quad \text{for all } t > T.$$

From this, it is clear that if $\tau_+ \leq 1$, then assumption (1.6) holds choosing $\delta = \varepsilon$. Note that $0 \leq \tau_- \leq 1$ always holds since f is a continuous function defined in $[0, +\infty)$. If instead $\tau_+ < 1$, then (1.7) is satisfied with $\varepsilon = 1 - \tau_+ - \delta$, where $\delta > 0$ is arbitrarily small. Therefore as an immediate consequence of part (i) of Theorems 1.1 and 1.2 we obtain the following.

Corollary 1.3. *Let $f \in C^2$ be convex and satisfy* (1.4). *Let u be a positive semistable solution of* (1.1). *The following assertions hold:*

(a) If $\tau_+ = 1$ *and* $n < 6$ *then* $u \in L^\infty(\Omega)$.
(b) If $\tau_+ < 1$ *and* $n < 2 + \frac{4}{\tau_+}$ *then* $u \in L^\infty(\Omega)$.

If $\tau_+ < 1$, then for every $\varepsilon \in (0, 1 - \tau_+)$, there exists a positive constant C such that $f(t) \le C(1+t)^{\frac{1}{1-\tau_+-\varepsilon}}$ for all $t \ge 0$ (this can be easily seen integrating twice in the definition of τ_+). Thus under this hypothesis, f has at most polynomial growth.

All the results in the literature considering τ_- and τ_+ (defined in (1.5)) assume $\tau_- > 0$. Instead in Corollary 1.3, no assumption is made on τ_-.

Crandall and Rabinowitz [4] proved an a priori L^∞ bound for semistable solutions when $0 < \tau_- \le \tau_+ < 2+\tau_-+2\sqrt{\tau_-}$ and $n < 4+2\tau_-+4\sqrt{\tau_-}$. Note that for nonlinearities f such that $\tau_- = 1$ and $\tau_+ < 5$ one obtains the L^∞ bound if $n \le 9$ (a dimension which is optimal). This is the case for many exponential type nonlinearities, as for instance $f(u) = e^{u^\alpha}$ for any $\alpha \in \mathbb{R}^+$. The results in [4] were improved in [6] establishing that $u \in L^\infty(\Omega)$ whenever $\tau_- > 0$ and $n < 6 + 4\sqrt{\tau_-}$ (remember that $\tau_- \le 1$). Moreover, if $0 < \tau_- \le \tau_+ < 1$, then using an iteration argument in [4], one has that $u \in L^\infty(\Omega)$ whenever $n < 2 + \frac{4}{\tau_+}\left(1 + \sqrt{\tau_-}\right)$. Note that Corollary 1.3 coincides with these results in the case where $\tau_- = 0$.

Let us make some further comments on conditions (1.6) and (1.7) in Theorems 1.1 and 1.2, respectively.

Remark 1.4. *(i)* Condition (1.6) is equivalent to

$$\limsup_{t\to+\infty} \frac{\log f'(t)}{\log f(t)} \le 1,$$

since (1.6) holds if and only if

$$\frac{\log f'(t)}{\log f(t)} \le (1+\varepsilon) + \frac{C}{\log f(t)} \quad \text{for all } t > T;$$

note that $f(t) \to +\infty$ as $t \to +\infty$ by (1.4). Many nonlinearities f satisfy this condition (like exponential or power type nonlinearities).

(ii) Setting $s = f(t)$ and $t = \gamma(s)$, (1.6) is equivalent to the condition $\gamma'(s) \ge \theta s^{-1-\varepsilon}$ for some $\theta > 0$ and for all s sufficiently large. This clearly shows that (1.6) does not follow from the convexity of f alone (which is equivalent to γ' being nonincreasing).

Instead, condition (1.7) is equivalent to $\gamma'(s) \ge \theta s^{-1+\varepsilon}$ for some $\theta > 0$ and for all s sufficiently large. In particular, $\gamma(s) \ge \theta s^\varepsilon$ for s large enough, or equivalently, $f(t) \le C(1+t)^{\frac{1}{\varepsilon}}$ for some constant $C > 0$ and for all t.

On the other hand, $f(t)f''(t)/f'(t)^2 = -s\gamma''(s)/\gamma'(s)$, a second derivative condition on γ, in contrast with the first derivative conditions of (1.6) and (1.7). Therefore, for most nonlinearities satisfying (1.6) (or (1.7)), the limit $ff''/(f')^2$ at infinity does not exist (*i.e.*, $\tau_- < \tau_+$) and in addition, it may happen that $\tau_- = 0$.

(*iii*) Note that by convexity, $\varepsilon f'(t) \le f(t+\varepsilon) - f(t) \le f(t+\varepsilon)$ for all t. Therefore, (1.6) holds if $f(t+\varepsilon) \le Cf(t)^{1+\varepsilon}$ for all t sufficiently large.

2. Preliminary estimates

We start by recalling the following standard regularity result for uniformly elliptic equations.

Proposition 2.1. *Let $a^{ij} = a^{ji}$, $1 \le i,j \le n$, be measurable functions on a bounded domain Ω. Assume that there exist positive constants c_0 and C_0 such that* (1.2) *holds. Let $u \in H_0^1(\Omega)$ be a weak solution of*

$$\left\{ \begin{array}{rcll} Lu + c(x)u & = & g(x) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

with c, $g \in L^p(\Omega)$ for some $p \ge 1$.

Then, there exists a positive constant C independent of u such that the following assertions hold:

(*i*) *If $p > n/2$ then $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{L^1(\Omega)} + \|g\|_{L^p(\Omega)})$.*

(*ii*) *Assume $c \equiv 0$. If $1 \le p < n/2$ then $\|u\|_{L^r(\Omega)} \le C\|g\|_{L^p(\Omega)}$ for every $1 \le r < np/(n-2p)$. Moreover, $\|u\|_{W_0^{1,r}(\Omega)} \le C$ for every $1 \le r < np/(n-p)$.*

Part (i) of Proposition 2.1 is established in Theorem 3 of [7] with the L^2-norm of u instead of the L^1-norm. However, an immediate interpolation argument shows that the result also holds with $\|u\|_{L^1(\Omega)}$. Note also that in the right hand side of this estimate, $\|u\|_{L^\infty(\Omega)} \le C(\|u\|_{L^1(\Omega)} + \|g\|_{L^p(\Omega)})$, some dependence of u must appear (think on the equation with $g \equiv 0$ satisfied by the eigenfunctions of the Laplacian). For part (ii) we refer to Theorems 4.1 and 4.3 of [8].

As an easy consequence of Proposition 2.1 (i) we obtain the following:

Corollary 2.2. *Let $u \in H_0^1(\Omega)$ be a nonnegative weak solution of* (1.1) *with f nondecreasing and convex. Assume $p > n/2$. If there exists a positive constant C independent of u such that $\|u\|_{L^1(\Omega)} \le C$ and $\|f'(u)\|_{L^p(\Omega)} \le C$, then $\|u\|_{L^\infty(\Omega)} \le C$ for some positive constant C independent of u.*

Proof. Rewrite equation (1.1) as $Lu + c(x)u = -f(0)$ where $c(x) = (f(u) - f(0))/u$. Then by convexity, $0 \le c(x) \le f'(u)$ and the result follows by Proposition 2.1 (i). $\square$

The following estimates involving

$$\tilde{f}(u) := f(u) - f(0)$$

are due to Nedev [5] when $L = \Delta$. We give here a new proof of the estimates consistent with our own approach. Note that assumptions (1.6) and (1.7) in Theorems 1.1 and 1.2, respectively, also hold replacing f by $\tilde{f}$ on their right hand side, since $f(t) \le 2(f(t) - f(0)) = 2\tilde{f}(t)$ for t large enough. We will use this fact in the proof of both results.

Lemma 2.3. *Let $f \in C^2$ be convex and satisfy* (1.4)*. If u is a positive semistable solution of* (1.1)*, then there exists a positive constant C independent of u such that*

$$\int_\Omega \tilde{f}(u) f'(u)\, dx \le C \quad \text{and} \quad \int_\Omega \tilde{f}(u) f''(u) a^{ij}(x) u_i u_j\, dx \le C. \tag{2.1}$$

Proof. Let u be a semistable solution of (1.1) and $\tilde{f}(u) = f(u) - f(0)$. Note that $\tilde{f}(u)$ satisfies

$$L(\tilde{f}(u)) + f'(u)\tilde{f}(u) = -f(0)f'(u) + f''(u)a^{ij}(x)u_i u_j.$$

Multiplying the previous identity by $\tilde{f}(u)$ and using the semistability condition (1.3), we obtain

$$\begin{aligned} 0 &\le \int_\Omega (a^{ij}(\tilde{f}(u))_i(\tilde{f}(u))_j - f'(u)\tilde{f}(u)^2)\, dx \\ &= f(0)\int_\Omega f'(u)\tilde{f}(u)\, dx - \int_\Omega \tilde{f}(u) f''(u) a^{ij} u_i u_j\, dx, \end{aligned}$$

or equivalently,

$$\int_\Omega \tilde{f}(u) f''(u) a^{ij}(x) u_i u_j\, dx \le f(0) \int_\Omega \tilde{f}(u) f'(u)\, dx. \tag{2.2}$$

As a consequence, the second estimate in (2.1) follows by the first one.

Multiplying the equation (1.1) by the test functions $\zeta = f'(u) - f'(0)$ and

$$\zeta = \begin{cases} 0 & \text{if } \ u \le M \\ f'(u) - f'(M) & \text{if } \ u > M, \end{cases}$$

we find

$$\int_\Omega f''(u) a^{ij}(x) u_i u_j\, dx = \int_\Omega f(u)(f'(u) - f'(0))\, dx \tag{2.3}$$

and

$$\int_{\{u>M\}} f''(u)a^{ij}(x)u_iu_j\ dx = \int_{\{u>M\}} f(u)(f'(u)-f'(M))\ dx, \tag{2.4}$$

respectively.

Combining (2.2) and (2.3), we obtain

$$\int_\Omega (f(u)-2f(0))f''(u)a^{ij}(x)u_iu_j\ dx \le f(0)f'(0)\int_\Omega f(u)\ dx - f(0)^2\int_\Omega f'(u)\ dx. \tag{2.5}$$

Choose M (depending on f) such that $f(t) > 2f(0)+2$ for all $t \ge M$. On the one hand, using (2.4), the convexity of f, and that (a^{ij}) is a positive definite matrix, we obtain

$$\begin{aligned} 2\int_{\{u>M\}} f(u)(f'(u)-f'(M))\ dx &= 2\int_{\{u>M\}} f''(u)a^{ij}(x)u_iu_j\ dx \\ &\le \int_{\{u>M\}} (f(u)-2f(0))f''(u)a^{ij}(x)u_iu_j\ dx. \end{aligned}$$

On the other hand, for some constant C depending only on f (and M), there holds

$$\begin{aligned} -\int_{\{u\le M\}} (f(u)-2f(0))f''(u)a^{ij}(x)u_iu_j\ dx &\le C\int_{\{u\le M\}} a^{ij}(x)u_iu_j\ dx \\ &\le CM\int_\Omega f(u)\ dx, \end{aligned}$$

where the last inequality follows from multiplying equation (1.1) by $\min\{u,M\}$. Combining the previous bounds with (2.5), it follows that

$$2\int_{\{u>M\}} f(u)(f'(u)-f'(M))\ dx \le f(0)f'(0)\int_\Omega f(u)\ dx + CM\int_\Omega f(u)\ dx. \tag{2.6}$$

Finally, choose $\overline{M} > M$ (depending only on f) such that $f'(M) < \frac{f'(t)}{2}$ if $t > \overline{M}$. Then (2.6) implies

$$\int_{\{u>\overline{M}\}} f(u)f'(u)\ dx \le C\int_\Omega f(u)\ dx,$$

and using that $f'(t)\to+\infty$ at infinity (see Remark 2.4 below), we conclude

$$\int_\Omega f(u)f'(u)\ dx \le C\ ,$$

where C is independent of u. □

Remark 2.4. Note that $\tilde{f}(t)/t \le f'(t)$ for all $t \ge 0$ since f is convex. In particular, by condition (1.4), we obtain $\lim_{t\to\infty} f'(t) = \infty$. Therefore, as a consequence of estimate (2.1) we obtain

$$\int_\Omega f(u)dx \le C, \tag{2.7}$$

where C is a constant independent of u. As in [5], from this and Proposition 2.1 (ii), one deduces

$$u \text{ in } L^q(\Omega) \text{ for all } q < n/(n-2). \tag{2.8}$$

Our results improve this estimate under the additional assumptions on f of Theorems 1.1 and 1.2.

The following is a sufficient condition on f to guarantee $u \in H_0^1(\Omega)$. Note that by convexity of f, $tf'(t) - \tilde{f}(t) \ge 0$ for all $t \ge 0$. If we further assume that for some $\varepsilon > 0$, $tf'(t) - \tilde{f}(t) \ge \varepsilon t$ for $t > T(\varepsilon)$, then u is bounded in $H_0^1(\Omega)$ by a constant independent of u. Indeed, noting that

$$\begin{aligned}
\varepsilon \int_\Omega a^{ij}(x) u_i u_j \, dx &= \varepsilon \int_\Omega f(u) u \, dx \le C + \int_\Omega f(u)(uf'(u) - \tilde{f}(u)) \, dx \\
&= C + \int_\Omega a^{ij}(x) u_i (uf'(u) - \tilde{f}(u))_j \, dx \\
&= C + \int_\Omega uf''(u) a^{ij}(x) u_i u_j \, dx \le C,
\end{aligned}$$

where in the last inequality we used the superlinearity of f and the second estimate in (2.1).

3. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Assume (1.6). In fact, as we said before Lemma 2.3 we may assume that (1.6) holds replacing f by $\tilde{f}$: for every $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and $C = C(\varepsilon)$ such that

$$f'(t) \le C \tilde{f}(t)^{1+\varepsilon} \quad \text{for all } t > T. \tag{3.1}$$

In the following, the constants C may depend on ε and T but are independent of u.

We start by proving that $f'(u) \in L^p(\Omega)$ for all $p < 3$ and as a consequence the statement in part (a). Let $\alpha = \frac{3+\varepsilon}{1+\varepsilon}$ (with ε as in (3.1)). Multiplying (1.1) by

$\frac{(f'(u)-f'(0))^\alpha}{1+f(u)}$ and integrating by parts we obtain

(3.2)
$$\begin{aligned}
&\int_\Omega \frac{f(u)}{1+f(u)}(f'(u)-f'(0))^\alpha\ dx + \int_\Omega \frac{f'(u)}{(1+f(u))^2}(f'(u)-f'(0))^\alpha a^{ij}(x)u_iu_j\ dx \\
&= \alpha \int_\Omega \frac{f''(u)}{1+f(u)}(f'(u)-f'(0))^{\alpha-1}a^{ij}(x)u_iu_j\ dx \\
&\le \alpha \int_{\{u\le T\}} \frac{f'(u)^{\alpha-1}f''(u)}{1+f(u)}a^{ij}(x)u_iu_j\ dx \\
&\qquad + C\int_{\{u>T\}} \tilde{f}(u)^{(1+\varepsilon)(\alpha-1)-1}f''(u)a^{ij}(x)u_iu_j\ dx \\
&\le C\left\{\int_{\{u<T\}} a^{ij}(x)u_iu_j\ dx + \int_\Omega \tilde{f}(u)f''(u)a^{ij}(x)u_iu_jdx\right\} \\
&\le C\left\{T\int_\Omega f(u)\ dx + \int_\Omega \tilde{f}(u)f''(u)a^{ij}(x)u_iu_jdx\right\}.
\end{aligned}$$

In particular, by Lemma 2.3 and the bound (2.7), we obtain

$$\int_\Omega f'(u)^\alpha\ dx \le C \quad \text{where } \alpha = \frac{3+\varepsilon}{1+\varepsilon}.$$

Therefore, by the arbitrariness of $\varepsilon > 0$, we obtain $f'(u) \in L^p(\Omega)$ for all $p < 3$. As a consequence, by Corollary 2.2 and since $u \in L^1(\Omega)$ (see Remark 2.4), we obtain the L^∞ estimate established in part (a), *i.e.*, if $n < 6$ then $\|u\|_{L^\infty(\Omega)} \le C$.

In the following, we may assume $n \ge 6$. Let us prove now that $f(u) \in L^p(\Omega)$ for all $p < n/(n-4)$, and as a consequence, the statement in part (b). Now we take $\alpha = 1 + \frac{1}{1+\varepsilon}$. Multiplying (1.1) by $(f'(u)-f'(0))^\alpha$ and using (3.1) and Lemma 2.3, we obtain

$$\begin{aligned}
\int_\Omega f(u)(f'(u)-f'(0))^\alpha\ dx &= \alpha\int_\Omega (f'(u)-f'(0))^{\alpha-1}f''(u)a^{ij}u_iu_j\ dx \\
&\le C\int_\Omega \tilde{f}(u)f''(u)a^{ij}u_iu_j\ dx \le C.
\end{aligned} \tag{3.3}$$

Hence, using the convexity of f and that $f'(0) \le f'(t)/2$ for t large, we obtain

$$\int_\Omega \frac{\tilde{f}(u)^{\alpha+1}}{u^\alpha}dx \le C \quad \text{for all } \alpha \in (1,2). \tag{3.4}$$

We now repeat the iteration argument of Nedev [5]. Assume that $u \in L^p(\Omega)$ for all $1 \le p < p_0$. Given any positive number β, set

$$\Omega_1 := \{x \in \Omega : \frac{\tilde{f}(u)^{\alpha+1}}{u^\alpha} > \tilde{f}(u)^{\alpha+1-\beta}\},\ \Omega_2 := \Omega \setminus \overline{\Omega}_1 = \{x \in \Omega : \tilde{f}(u) \le u^{\frac{\alpha}{\beta}}\}\ .$$

By (3.4), we have

$$\int_{\Omega_1} \tilde{f}(u)^{\alpha+1-\beta}\ dx \le C. \tag{3.5}$$

Moreover,

$$\int_{\Omega_2} \tilde{f}(u)^p\ dx \le \int_{\Omega_2} u^{\frac{\alpha}{\beta}p} dx \le C \quad \text{for all } p < \frac{\beta}{\alpha} p_0. \tag{3.6}$$

Choose β such that $\alpha + 1 - \beta = \frac{\beta}{\alpha} p_0$, i.e., $\beta = (\alpha+1)/(1+\frac{p_0}{\alpha})$. Then, by (3.5), (3.6), and letting $\alpha \uparrow 2$, we obtain $\tilde{f}(u) \in L^p(\Omega)$ for all $1 < p < \frac{3p_0}{2+p_0}$. Hence, by elliptic regularity theory (see Proposition 2.1 (ii)),

$$u \in L^p(\Omega) \quad \text{for all } 1 < p < p_1 := \frac{n\frac{3p_0}{2+p_0}}{n - 2\frac{3p_0}{2+p_0}} = \frac{3np_0}{2n + (n-6)p_0}.$$

By (2.8) we can start the iteration process with $p_0 = n/(n-2)$. Set $p_{k+1} := \frac{3np_k}{2n+(n-6)p_k}$ for $k \ge 1$. Note that $p_k \le n/(n-6)$ for all $k \ge 1$ (by induction and since $p_0 = n/(n-2) \le n/(n-6)$). Moreover $p_{k+1} > p_k$ (in fact, this is equivalent to $p_k < n/(n-6)$), and hence, $\lim_{k\to\infty} p_k = n/(n-6) =: p_\infty$.

Therefore, we obtain $f(u) \in L^p(\Omega)$ for all $1 < p < \frac{3p_\infty}{2+p_\infty} = \frac{n}{n-4}$. The remainder of the statements of Theorem 1.1 follow from standard elliptic regularity theory (see Proposition 2.1 (ii)). □

Proof of Theorem 1.2. The proof of Theorem 1.2 is essentially the same. Using assumption (1.7) in (3.2), the first part of the proof gives $f'(u) \in L^\alpha(\Omega)$ with $\alpha = \frac{3-\varepsilon}{1-\varepsilon}$. Therefore, by Corollary 2.2 and since $u \in L^1(\Omega)$ (see Remark 2.4), $u \in L^\infty(\Omega)$ when $\frac{3-\varepsilon}{1-\varepsilon} > \frac{n}{2}$, or equivalently, when $n < 6 + \frac{4\varepsilon}{1-\varepsilon}$.

Assume $n \ge 6 + \frac{4\varepsilon}{1-\varepsilon}$. To obtain the estimate on $f(u)$, we deduce (3.3) with $\alpha = 1 + \frac{1}{1-\varepsilon}$ using now (1.7) instead of (1.6). In particular,

$$\int_\Omega \frac{\tilde{f}(u)^{\alpha+1}}{u^\alpha}\ dx \le C \quad \text{for } \alpha = 1 + \frac{1}{1-\varepsilon}.$$

At this point, we repeat the previous iteration argument to obtain the increasing sequence

$$p_0 = \frac{n}{n-2}, \quad p_{k+1} = \frac{(3-2\varepsilon)np_k}{(2-\varepsilon+(1-\varepsilon)p_k)n - 2(3-2\varepsilon)p_k}, \quad \text{for all } k \geq 0,$$

with limit $p_\infty = \frac{(1-\varepsilon)n}{(1-\varepsilon)n-6+4\varepsilon}$. As a consequence, $f(u) \in L^p(\Omega)$ for all $p < \frac{\beta}{\alpha}p_\infty$ where $\beta = (\alpha+1)/(1+\frac{p_\infty}{\alpha})$, i.e., $f(u) \in L^p(\Omega)$ for all $p < \frac{(1-\varepsilon)n}{(1-\varepsilon)n-4+2\varepsilon}$. The remainder of the statements of Theorem 1.2 follow from standard elliptic regularity theory (see Proposition 2.1 (ii)). □

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