Sobolev regularity of quasiconformal mappings on domains. Part I

Martí Prats *

July 17, 2015

Abstract

Consider a Lipschitz domain Ω and a measurable function μ supported in $\overline{\Omega}$ with $\|\mu\|_{L^\infty} < 1$. Then the derivatives of a quasiconformal solution of the Beltrami equation $\overline{\partial} f = \mu\, \partial f$ inherit the Sobolev regularity $W^{n,p}(\Omega)$ of the Beltrami coefficient μ as long as Ω is regular enough. The condition obtained is that the outward unit normal vector N of the boundary of the domain is in the trace space, that is, $N \in B^{n-1/p}_{p,p}(\partial\Omega)$.

In this part we prove that this geometric condition implies that the Beurling transform $\mathcal{B}f = -\text{p.v.}\frac{1}{\pi z^2} * f$ satisfies that $\mathcal{B}\chi_\Omega \in W^{n,p}(\Omega)$.

1 Introduction

Let $\mu \in L^\infty$ supported in a certain ball $B \subset \mathbb{C}$ with $k := \|\mu\|_{L^\infty} < 1$ and consider $K := \frac{1+k}{1-k}$. We say that f is a K-quasiregular solution to the Beltrami equation

$$\overline{\partial} f = \mu\, \partial f \tag{1.1}$$

with Beltrami coefficient μ if $f \in W^{1,2}_{loc}$, that is, if f and ∇f are square integrable functions in any compact subset of $\mathbb{C}$, and $\overline{\partial} f(z) = \mu(z)\partial f(z)$ for almost every $z \in \mathbb{C}$. Such a function f is said to be a K-quasiconformal mapping if it is a homeomorphism of the complex plane. If, moreover, $f(z) = z + \mathcal{O}(\frac{1}{z})$ as $z \to \infty$, then we say that f is the principal solution to (1.1).

Given a compactly supported Beltrami coefficient μ, the existence and uniqueness of the principal solution is granted by the measurable Riemann mapping Theorem (see [AIM09, Theorem 5.1.2], for instance). The principal solution can be given by means of the Cauchy and the Beurling transforms. For $g \in L^p$ its Cauchy transform is defined as

$$\mathcal{C}g(z) := \frac{1}{\pi} \int \frac{g(w)}{z-w} dm(w) \qquad \text{for all } z \in \mathbb{C},$$

and its Beurling transform, as

$$\mathcal{B}g(z) := \lim_{\varepsilon \to 0} \frac{-1}{\pi} \int_{|w-z|>\varepsilon} \frac{g(w)}{(z-w)^2} dm(w) \qquad \text{for almost every } z \in \mathbb{C}.$$

The Beurling transform is a bounded operator in L^p for $1 < p < \infty$ and for $g \in W^{1,p}(\mathbb{C})$ we have that $\mathcal{B}(\overline{\partial} g) = \partial g$. Given a ball B, the Cauchy transform sends functions in $L^p(B)$ and vanishing

*MP (Departament de Matemàtiques, Universitat Autònoma de Barcelona, Catalonia): `mprats@mat.uab.cat`.

in the complement of B to $W^{1,p}(\mathbb{C})$. Furthermore, the operator $I-\mu\mathcal{B}$ is invertible in L^2 and, if we call

$$h:=(I-\mu\mathcal{B})^{-1}\mu, \tag{1.2}$$

then

$$f(z)=\mathcal{C}h(z)+z \tag{1.3}$$

is the principal solution of (1.1) because $\overline{\partial}f=h$ and $\partial f=\mathcal{B}h+1$.

A natural question is to what spaces h belongs. The key point to answer that question is inverting the operator $(I-\mu\mathcal{B})$ in some space. Astala showed in [Ast94] that $h\in L^p$ for $1+k<p<1+1/k$ (in fact, since h is also compactly supported, one can say the same for every $1\leq p\leq 1+k$ even though $(I-\mu\mathcal{B})$ may not be invertible in L^p for that values of p, as shown by Astala, Iwaniec and Saksman in [AIS01]). Clop et al. in [CFM$^+$09] and Cruz, Mateu and Orobitg in [CMO13] proved that if μ belongs to the Sobolev space $W^{s,p}(\mathbb{C})$ (in the Bessel potential sense when $s\notin\mathbb{N}$) with $sp>2$ then also $h\in W^{s,p}(\mathbb{C})$. One also finds some results in the same spirit for the critical case $sp=2$ and the subcritical case $sp<2$ in [CFM$^+$09] and [CFR10], but here the space to which h belongs is slightly worse than the space to which μ belongs, that is, either some integrability or some smoothness is lost.

When it comes to dealing with a Lipschitz domain Ω with $\operatorname{supp}(\mu)\subset\overline{\Omega}$, Mateu, Orobitg and Verdera showed in [MOV09] that, if the parameterizations of the boundary of Ω are in $C^{1,\varepsilon}$ with $0<\varepsilon<1$, then for every $0<\sigma<\varepsilon$ one has that

$$\mu\in C^{0,\varepsilon}(\Omega)\implies h\in C^{0,\sigma}(\Omega). \tag{1.4}$$

Furthermore, the principal solution to (1.1) is bilipschitz in that case. They allow the domain to have a finite number of holes with tangent boundaries. In [CF12], Giovanna Citti and Fausto Ferrari proved that, if one does not allow this situation, then (1.4) holds for $\sigma=\varepsilon$. In [CMO13] the authors study also the Sobolev spaces to conclude that for the same kind of domains, when $0<\sigma<\varepsilon<1$ and $1<p<\infty$ with $\sigma p>2$ one has that

$$\mu\in W^{\sigma,p}(\Omega)\implies h\in W^{\sigma,p}(\Omega). \tag{1.5}$$

A key point is proving the boundedness of the Beurling transform in $W^{\sigma,p}(\Omega)$. To do so, the authors note that $\mathcal{B}\chi_\Omega\in W^{\sigma,p}(\Omega)$ by means of some results from [MOV09] and then they prove a $T(1)$ theorem that grants the boundedness of $\mathcal{B}$ in $W^{\sigma,p}(\Omega)$ if $\mathcal{B}\chi_\Omega\in W^{\sigma,p}(\Omega)$. The other key point is the invertibility of $I-\mu\mathcal{B}$ in $W^{\sigma,p}(\Omega)$, which is shown using Fredholm theory.

Cruz and Tolsa proved in [CT12] that for $0<s\leq 1$, $1<p<\infty$ with $sp>1$, if the outward unit normal vector N is in the Besov space $B^{s-1/p}_{p,p}(\partial\Omega)$ then $\mathcal{B}\chi_\Omega\in W^{s,p}(\Omega)$. This condition is necessary for Lipschitz domains with small Lipschitz constant (see [Tol13]). Moreover, being $N\in B^{s-1/p}_{p,p}(\partial\Omega)$ implies the parameterizations of the boundary of Ω to be in $B^{s+1-1/p}_{p,p}$ and, for $sp>2$, the parameterizations are in $C^{1,s-2/p}$ by the Sobolev Embeding Theorem. In that situation, one can use the $T(1)$ result in [CMO13] to deduce the boundedness of the Beurling transform in $W^{s,p}(\Omega)$. However, their result on quasiconformal mappings only allows to infer that for every $2/p<\sigma<s-2/p$ we have that (1.5) holds. Note that the condition $2/p<\sigma<s-2/p$ may be too restrictive (when $sp=3$, for instance, we can't deduce (1.5) for any σ because $s-2/p=1/p<2/p$).

The goal of this paper and its sequel is to prove that for $s\in\mathbb{N}$, (1.5) holds for $\sigma=s$ as long as $p>2$.

Theorem 1.1 (See [Pra15])**.** *Let $n\in\mathbb{N}$, let Ω be a bounded domain with outward unit normal vector N in $B^{n-1/p}_{p,p}(\partial\Omega)$ for some $2<p<\infty$ and let $\mu\in W^{n,p}(\Omega)$ with $\|\mu\|_{L^\infty}=k<1$ and $\operatorname{supp}(\mu)\subset\overline{\Omega}$. Then, the principal solution f to (1.1) is in the Sobolev space $W^{n+1,p}(\Omega)$.*

Note that this theorem only deals with the natural values of s, but the restrictions $\sigma < s - 2/p$ and $s < 1$ are eliminated. For $n = 1$ the author expects this to be a sharp result in view of the result in [Tol13].

The proof is divided in two steps. In this article we perform the first one, that is, we prove that the Beurling transform is bounded in $W^{n,p}(\Omega)$. To do so we will see that if $N \in B^{n-1/p}_{p,p}(\partial\Omega)$, then $\mathcal{B}\chi_\Omega \in W^{n,p}(\Omega)$, in the same spirit of [CT12]. In the present paper, however, the proof will be slightly more tricky since we will need to approximate the boundary of the domain by polynomials instead of straight lines. The derivative of the Beurling transform of the characteristic function of a half-plane is zero (see [CT12]), but the derivative of the Beurling transform of the characteristic function of a domain bounded by a polynomial of degree greater than one is not zero anymore. Using the $T(P)$ Theorem of [PT15] this will suffice to see the boundedness of the Beurling transform.

Theorem 1.2. *Consider $p > 2$, and $n \in \mathbb{N}$ and let Ω be a Lipschitz domain with $N \in B^{n-1/p}_{p,p}(\partial\Omega)$. Then, for every $f \in W^{n,p}(\Omega)$ we have that*

$$\|\mathcal{B}(\chi_\Omega f)\|_{W^{n,p}(\Omega)} \leq C\|N\|_{B^{n-1/p}_{p,p}(\partial\Omega)}\|f\|_{W^{n,p}(\Omega)},$$

where C depends on p, n, $\mathrm{diam}(\Omega)$ *and the Lipschitz character of the domain.*

In [Pra15], the sequel of the present article, we will face the invertibility of $(I - \mu\mathcal{B})(\chi_\Omega\cdot)$ in $W^{n,p}(\Omega)$ and we will complete the proof of Theorem 1.1.

The plan of the paper is the following. In Section 2 some preliminary assumptions are stated. Subsection 2.1 explains the notation to be used and recalls some well-known facts. In Subsection 2.2 one finds the definition of some generalized β-coefficients related to Jones and David-Semmes' celebrated betas. In Subsection 2.3 the definition of the Besov spaces $B^s_{p,p}$ is given along with some related well-known facts and an equivalent norm in terms of the generalized β-coefficients using a result by Dorronsoro in [Dor85]. Subsection 2.4 is about some operators related to the Beurling transform, providing a standard notation for the whole article.

Section 3 is devoted to prove Theorem 1.2. The first step is to study the case of unbounded domains whose boundary can be expressed as the graph of a Lipschitz function. Subsection 3.1 contains the outline of the proof, reducing it to two lemmas. The first one studies the relation with the β-coefficients and is proven in Subsection 3.2. The second one, proven in Subsection 3.3, is about the case where the domain is bounded by the graph of a polynomial, and here one finds the exponential behavior of the bounds for the iterates of the Beurling transform, which entangles the more subtle details of the proof. Finally, in Subsections 3.4 and 3.5 one finds a more quantitative version of Theorem 1.2 for bounded Lipschitz domains using a localization principle and the T(P) Theorem.

2 Preliminaries

2.1 Some notation and well-known facts

On inequalities: When comparing two quantities x_1 and x_2 that depend on some parameters $p_1, \ldots, p_j$ we will write

$$x_1 \leq C_{p_{i_1},\ldots,p_{i_j}} x_2$$

if the constant $C_{p_{i_1},\ldots,p_{i_j}}$ depends on $p_{i_1}, \ldots, p_{i_j}$. We will also write $x_1 \lesssim_{p_{i_1},\ldots,p_{i_j}} x_2$ for short, or simply $x_1 \lesssim x_2$ if the dependence is clear from the context or if the constants are universal. We may omit some of these variables for the sake of simplicity. The notation $x_1 \approx_{p_{i_1},\ldots,p_{i_j}} x_2$ will mean that $x_1 \lesssim_{p_{i_1},\ldots,p_{i_j}} x_2$ and $x_2 \lesssim_{p_{i_1},\ldots,p_{i_j}} x_1$.

On polynomials: We write $\mathcal{P}^n(\mathbb{R}^d)$ for the vector space of real polynomials of degree smaller or equal than n with d real variables. If it is clear from the context we will just write $\mathcal{P}^n$. For any set $U \subset \mathbb{R}^d$ we will write $\mathcal{P}^n(U)$ for the vector space of polynomials in $\mathcal{P}^n$ restricted to U.

On sets: Given two sets A and B, their symmetric difference is $A\Delta B := (A \cup B) \setminus (A \cap B)$. Given $x \in \mathbb{R}^d$ and $r > 0$, we write $B(x, r)$ or $B_r(x)$ for the open ball centered at x with radius r and $Q(x, r)$ for the open cube centered at x with sides parallel to the axis and side-length $2r$. Given any cube Q, we write $\ell(Q)$ for its side-length, and rQ will stand for the cube with the same center but enlarged by a factor r. We will use the same notation for balls and one dimensional cubes, that is, intervals.

At some point we need to use intervals in $\mathbb{R}^d$: given $x, y \in \mathbb{R}^d$, we call the interval with endpoints x and y

$$[x, y] := \{(1-t)x + ty : t \in [0, 1]\}.$$

We may use the "open" interval $]x, y[:= [x, y] \setminus \{x, y\}$.

We call domain an open and connected subset of $\mathbb{R}^d$.

Definition 2.1. *Given $n \geq 1$, we say that $\Omega \subset \mathbb{C}$ is a $(\delta, R) - C^{n-1,1}$ domain if given any $z \in \partial\Omega$, there exists a function $A_z \in C^{n-1,1}(\mathbb{R})$ such that*

$$\left\| A_z^{(j)} \right\|_{L^\infty} \leq \frac{\delta}{R^{j-1}} \quad \text{for every } 0 \leq j \leq n, \tag{2.1}$$

and, possibly after a translation that sends z to the origin and a rotation that brings the tangent at z to the real line, we have that

$$\Omega \cap Q(0, R) = \{x + i\, y : y > A_z(x)\}.$$

In case $n = 1$ the assumption of the tangent is removed (we say that Ω is a (δ, R)-Lipschitz domain).

We call window *such a cube.*

On measure theory: We denote the d- dimensional Lebesgue measure in $\mathbb{R}^d$ by m_d, or simply m when the dimension is clear from the context. At some point we use m also to denote a natural number. We will write dz for the form $dx + i\, dy$ and analogously $d\bar{z} = dx - i\, dy$, where $z = x + i\, y$. Thus, when integrating a function with respect to the Lebesgue measure of a variable z we will always use $dm(z)$ to avoid confusion, or simply dm.

On indices: In this text $\mathbb{N}_0$ stands for the natural numbers including 0. Otherwise we will write $\mathbb{N}$. We will make wide use of the multiindex notation for exponents and derivatives. For $\alpha \in \mathbb{Z}^d$ its modulus is $|\alpha| = \sum_{i=1}^d |\alpha_i|$ and its factorial is $\alpha! = \prod_{i=1}^d \alpha_i!$. Given two multiindices $\alpha, \gamma \in \mathbb{Z}^d$ we write $\alpha \leq \gamma$ if $\alpha_i \leq \gamma_i$ for every i. We say $\alpha < \gamma$ if, in addition, $\alpha \neq \gamma$. Furthermore, we write

$$\binom{\alpha}{\gamma} := \prod_{i=1}^d \binom{\alpha_i}{\gamma_i} = \begin{cases} \prod_{i=1}^d \frac{\alpha_i!}{\gamma_i!(\alpha_i - \gamma_i)!} & \text{if } \alpha \in \mathbb{N}_0^d \text{ and } \vec{0} \leq \gamma \leq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}^d$ we write $x^\alpha := \prod x_i^{\alpha_i}$. Given any $\phi \in C_c^\infty$ (infintitely many times differentiable with compact support in $\mathbb{R}^d$) and $\alpha \in \mathbb{N}_0^d$ we write $D^\alpha \phi = \frac{\partial^{|\alpha|}}{\prod \partial_{x_i}^{\alpha_i}} \phi$.

At some point we will use also use roman letter for multiindices, and then, to avoid confusion, we will use the vector notation $\vec{i}, \vec{j}, \ldots$

On complex notation For $z = x + i\, y \in \mathbb{C}$ we write $\mathrm{Re}\,(z) := x$ and $\mathrm{Im}(z) := y$. Note that the symbol i will be used also widely as a index for summations without risk of confusion. The multiindex notation will change slightly: for $z \in \mathbb{C}$ and $\alpha \in \mathbb{Z}^2$ we write $z^\alpha := z^{\alpha_1} \bar{z}^{\alpha_2}$.

We also adopt the traditional Wirtinger notation for derivatives, that is, given any $\phi \in C_c^\infty$, then

$$\partial\phi(z) := \frac{\partial\phi}{\partial z}(z) = \frac{1}{2}(\partial_x \phi - i\, \partial_y \phi)(z)$$

and

$$\overline{\partial}\phi(z) := \frac{\partial \phi}{\partial \overline{z}}(z) = \frac{1}{2}(\partial_x \phi + i\,\partial_y \phi)(z),$$

Thus, given any $\phi \in C_c^\infty$ and $\alpha \in \mathbb{N}_0^2$, we write $D^\alpha \phi = \partial^{\alpha_1} \overline{\partial}^{\alpha_2} \phi$.

On Sobolev spaces: For any open set U, every distribution $f \in \mathcal{D}'(U)$ and $\alpha \in \mathbb{N}_0^d$, the *distributional derivative* $D_U^\alpha f$ is the distribution defined by

$$\langle D_U^\alpha f, \phi \rangle := (-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle \quad \text{for every } \phi \in C_c^\infty(U).$$

Abusing notation we will write D^α instead of D_U^α if it is clear from the context. If the distribution is regular, that is, if it coincides with an L^1_{loc} function acting on $\mathcal{D}(U)$, then we say that $D_U^\alpha f$ is a *weak derivative* of f in U. We write $|\nabla^n f| = \sum_{|\alpha|=n} |D^\alpha f|$.

Given numbers $n \in \mathbb{N}$, $1 \leq p \leq \infty$ an open set $U \subset \mathbb{R}^d$ and an $L^1_{loc}(U)$ function f, we say that f is in the Sobolev space $W^{n,p}(U)$ of smoothness n and order of integrability p if f has weak derivatives $D_U^\alpha f \in L^p$ for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq n$. When Ω is a Lipschitz domain, we will use the norm

$$\|f\|_{W^{n,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla^n f\|_{L^p(\Omega)},$$

which is equivalent to considering also the fewer order derivatives, that is,

$$\|f\|_{W^{n,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \sum_{|\alpha| \leq n} \|D^\alpha f\|_{L^p(\Omega)} \tag{2.2}$$

(see [Tri78, Theorem 4.2.4]) or, if Ω is an extension domain,

$$\|f\|_{W^{n,p}(\Omega)} \approx \inf_{F: F|_\Omega \equiv f} \|F\|_{W^{n,p}(\mathbb{R}^d)}.$$

From [Jon81], we know that uniform domains (and in particular, Lipschitz domains) are Sobolev extension domains for any indices $n \in \mathbb{N}$ and $1 \leq p \leq \infty$. One can find deeper results in that sense in [Shv10] and [KRZ15].

The reader can consider $n \in \mathbb{N}$ and $1 < p < \infty$ to be two given numbers along the whole text. At some point the restriction $2 < p$ will be needed.

On finite diferences: Given a function $f : \Omega \subset \mathbb{R}^d \to \mathbb{C}$ and two values $x, h \in \mathbb{R}^d$ such that $[x, x+h] \subset \Omega$, we call

$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x).$$

Moreover, for any natural number $i \geq 2$ we define the iterated difference

$$\Delta_h^i f(x) = \Delta_h^{i-1} f(x+h) - \Delta_h^{i-1} f(x) = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} f(x+jh)$$

whenever $[x, x+ih] \subset \Omega$.

On Whitney coverings: Given a domain Ω, we say that a collection of open dyadic cubes $\mathcal{W}$ is a Whitney covering of Ω if they are disjoint, the union of the cubes and their boundaries is Ω, there exists a constant $C_\mathcal{W}$ such that

$$C_\mathcal{W} \ell(Q) \leq \operatorname{dist}(Q, \partial\Omega) \leq 4 C_\mathcal{W} \ell(Q),$$

two neighbor cubes Q and R (i.e., $\overline{Q} \cap \overline{R} \neq \emptyset$) satisfy $\ell(Q) \leq 2\ell(R)$, and the family $\{20Q\}_{Q \in \mathcal{W}}$ has finite superposition. The existence of such a covering is granted for any open set different from $\mathbb{R}^d$ and in particular for any domain as long as $C_\mathcal{W}$ is big enough (see [Ste70, Chapter 1] for instance).

On the Leibniz rule: The Leibniz formula (see [Eva98, Section 5.2.3]) says that given a domain $\Omega \subset \mathbb{R}^d$, a function $f \in W^{n,p}(\Omega)$ and a multiindex $\alpha \in \mathbb{N}_0^d$, if $\phi \in C_c^\infty(\Omega)$, then $\phi \cdot f \in W^{n,p}(\Omega)$ and

$$D^\alpha(\phi \cdot f) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} D^\gamma \phi D^{\alpha-\gamma} f. \tag{2.3}$$

On Green's formula: The Green Theorem can be written in terms of complex derivatives (see [AIM09, Theorem 2.9.1]). Let Ω be a bounded Lipschitz domain. If $f, g \in W^{1,1}(\Omega) \cap C(\overline{\Omega})$, then

$$\int_\Omega \left(\partial f + \overline{\partial} g\right) \, dm = \frac{i}{2} \left(\int_{\partial\Omega} f(z) \, d\overline{z} - \int_{\partial\Omega} g(z) \, dz \right). \tag{2.4}$$

On the Rolle Theorem: We state here also a Complex Rolle Theorem for holomorphic functions [EJ92, Theorem 2.1] that will be a cornerstone of Section 3.3.

Theorem 2.2. *[see [EJ92]] Let f be a holomorphic function defined on an open convex set $U \subset \mathbb{C}$. Let $a, b \in U$ such that $f(a) = f(b) = 0$ and $a \neq b$. Then there exists z in the segment $]a, b[$ such that* $\mathrm{Im}(\partial f(z)) = 0$.

On the Sobolev Embedding Theorem: We state a reduced version of the Sobolev Embedding Theorem for Lipschitz domains (see [AF03, Theorem 4.12, Part II]). For each Lipschitz domain $\Omega \subset \mathbb{R}^d$ and every $p > d$, there is a continuous embedding of the Sobolev space $W^{1,p}(\Omega)$ into the Hölder space $C^{0,1-\frac{d}{p}}(\overline{\Omega})$. That is, writing

$$\|f\|_{C^{0,s}(\overline{\Omega})} = \|f\|_{L^\infty(\Omega)} + \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^s} \quad \text{for } 0 < s \leq 1,$$

we have that for every $f \in W^{1,p}(\Omega)$,

$$\|f\|_{C^{0,1-\frac{d}{p}}(\overline{\Omega})} \leq C_\Omega \|f\|_{W^{1,p}(\Omega)}. \tag{2.5}$$

On inequalities: We will use the Young's Inequality. It states that for measurable functions f and g, we have that

$$\|f * g\|_{L^q} \leq \|f\|_{L^r} \|g\|_{L^p} \tag{2.6}$$

for $1 \leq p, q, r \leq \infty$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ (see [Ste70, Appendix A2]).

2.2 Some generalized betas

In [Dor85], Dorronsoro introduces a characterization of Besov spaces in terms of the mean oscillation of the functions on cubes, and he uses approximating polynomials to do so. If the polynomials are of degree one, that is straight lines, this definition can be written in terms of a certain sum of David-Semmes betas (see [CT12] for instance). Following the ideas of Dorronsoro in our case we will use higher degree polynomials to approximate the Besov function that we want to consider, giving rise to some generalized betas. The following proposition comes from [Dor85], where it is not explicitly proven. We give a short proof of it for the sake of completeness.

Proposition 2.3. *Given a locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ and a cube $Q \subset \mathbb{R}^d$, there exists a unique polynomial $\mathbf{R}_Q^n f \in \mathcal{P}^n$ which we will call* approximating polynomial *of f on Q, such that given any multiindex γ with $|\gamma| \leq n$ one has that*

$$\int_Q (\mathbf{R}_Q^n f - f) \, x^\gamma = 0. \tag{2.7}$$

Remark 2.4. *In case of existence, the approximating polynomial verifies*

$$\sup_{x\in Q} |\mathbf{R}^n_Q f(x)| \leq C_{n,d} \frac{1}{|Q|}\int_Q |f|\, dm.$$

Proof. Indeed, since $\mathcal{P}^n$ is a finite dimensional vectorial space, all the norms are equivalent. In particular one can easily see that for any $P \in \mathcal{P}^n$

$$\|P\|^2_{L^\infty(Q)} \approx \frac{1}{|Q|}\|P\|^2_{L^2(Q)}.$$

Using the linearity of the integral in (2.7), one has

$$\frac{1}{|Q|}\int_Q |\mathbf{R}^n_Q f|^2\, dm = \frac{1}{|Q|}\int_Q \mathbf{R}^n_Q f \cdot f\, dm.$$

Combining both facts one gets

$$\left\|\mathbf{R}^n_Q f\right\|^2_{L^\infty(Q)} \lesssim \frac{1}{|Q|}\left\|\mathbf{R}^n_Q f\right\|_{L^\infty(Q)} \|f\|_{L^1(Q)}.$$

□

Proof of Proposition 2.3. By the Hilbert Projection Theorem, $L^2(Q) = \mathcal{P}^n(Q) \oplus (\mathcal{P}^n(Q))^\perp$. Thus, if $f \in L^2(Q)$, we can write $f = \mathbf{R}^n_Q f + (f - \mathbf{R}^n_Q f)$ satisfying (2.7).

For general $f \in L^1$, we can define a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subset L^2(Q)$ such that $|f_j| \leq |f|$ and $f_j \xrightarrow{a.e.} f$. By Remark 2.4 we have that the approximating polynomials $\mathbf{R}^n_Q f_j$ are uniformly bounded in Q by

$$\sup_{x\in Q} |\mathbf{R}^n_Q f_j(x)| \lesssim \frac{1}{|Q|}\int_Q |f_j|\, dm \leq \frac{1}{|Q|}\int_Q |f|\, dm.$$

Therefore there exists a convergent subsequence of $\{\mathbf{R}^n_Q f_j\}_j$ in L^1 (and in any other norm). We call $\mathbf{R}^n_Q f$ the limit of one such partial. By the Dominated Convergence Theorem we get (2.7).

To see uniqueness, we observe that if we find two polynomials P_1 and P_2 satisfying (2.7), then

$$\int_Q (P_1 - P_2)P = 0$$

for any $P \in \mathcal{P}^n$. In particular, if we take $P = P_1 - P_2$ we get that $\|P_1 - P_2\|_{L^2(Q)} = 0$. □

Remark 2.5. *Given $P \in \mathcal{P}^n$, a cube Q and $1 \leq p \leq \infty$ we have that*

$$\left\|f - \mathbf{R}^n_Q f\right\|_{L^p(Q)} \leq C_{d,n,p}\|f - P\|_{L^p(Q)}, \tag{2.8}$$

and given any cubes $Q \subset Q'$,

$$\left\|f - \mathbf{R}^n_Q f\right\|_{L^p(Q)} \leq C_{d,n,p}\left\|f - \mathbf{R}^n_{Q'} f\right\|_{L^p(Q')}. \tag{2.9}$$

Proof. By means of the Triangle Inequality and (2.7), we have that for any $P \in \mathcal{P}^n$

$$\left\|f - \mathbf{R}^n_Q f\right\|_{L^p(Q)} \leq \|f - P\|_{L^p(Q)} + \left\|P - \mathbf{R}^n_Q f\right\|_{L^p(Q)} = \|f - P\|_{L^p(Q)} + \left\|\mathbf{R}^n_Q (P - f)\right\|_{L^p(Q)}.$$

Therefore, we use twice Hölder's Inequality and Remark 2.4 to get

$$\begin{aligned}\left\|f - \mathbf{R}^n_Q f\right\|_{L^p(Q)} &\lesssim_{p,n,d} \|f - P\|_{L^p(Q)} + |Q|^{1/p}\left\|\mathbf{R}^n_Q (P - f)\right\|_{L^\infty(Q)} \\ &\lesssim_{n,d} \|f - P\|_{L^p(Q)} + \frac{|Q|^{1/p}}{|Q|}\|P - f\|_{L^1(Q)} \leq 2\|f - P\|_{L^p(Q)}.\end{aligned}$$

The inequality (2.9) is just a consequence of (2.8) replacing P by $\mathbf{R}^n_{Q'} f$. □

Remark 2.6. *In the one dimensional case, if f is continuous one can easily see that $f - \mathbf{R}_Q^n f$ has $n+1$ zeroes at least. Indeed, if it did not happen, one could find a polynomial $P \in \mathcal{P}^n$ with a simple zero at every point where $f - \mathbf{R}_Q^n f$ changes its sign, and no more. Therefore, $(f - \mathbf{R}_Q^n f) \cdot P$ would have constant sign and, thus, the integral in (2.7) would not vanish (see Figure 2.1).*

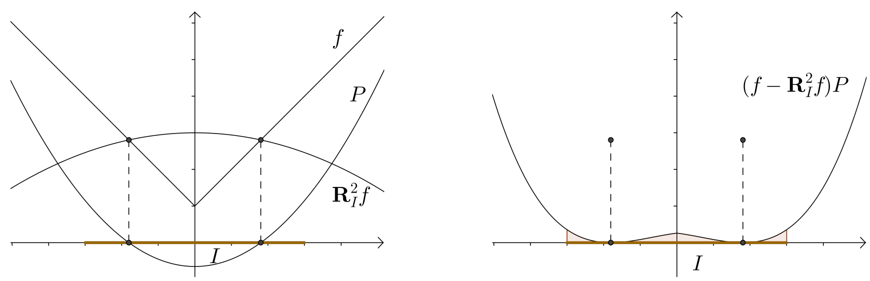

Figure 2.1: If $f - \mathbf{R}_Q^2 f$ had only 2 zeroes, there would exist $P \in \mathcal{P}^2$ with $\int (f - \mathbf{R}_Q^2 f) P \, dm > 0$.

Now we can define the generalized betas.

Definition 2.7. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function and $Q \subset \mathbb{R}^d$ a cube. Then we define*

$$\beta_{(n)}(f, Q) = \frac{1}{|Q|} \int_{3Q} \frac{|f(x) - \mathbf{R}_{3Q}^n f(x)|}{\ell(Q)} \, dm(x).$$

Remark 2.8. *Taking into account (2.8), we can conclude that*

$$\beta_{(n)}(f, Q) \approx \inf_{P \in \mathcal{P}^n} \frac{1}{|Q|} \int_{3Q} \frac{|f(x) - P(x)|}{\ell(Q)} dm(x).$$

This can be seen as a generalization of David and Semmes β_1 coefficient since $\beta_{(1)}$ and β_1 are comparable as long as some Lipschitz condition is assumed on f.

2.3 Function spaces

Next we recall some definitions and results on the function spaces that we will use. For a complete treatment we refer the reader to [Tri83] and [RS96].

Definition 2.9. *Let $\Phi(\mathbb{R}^d)$ be the collection of all the families $\Psi = \{\psi_j\}_{j=0}^\infty \subset C_c^\infty(\mathbb{R}^d)$ such that*

$$\begin{cases} \operatorname{supp} \psi_0 \subset \mathbb{D}(0, 2), \\ \operatorname{supp} \psi_j \subset \mathbb{D}(0, 2^{j+1}) \setminus \mathbb{D}(0, 2^{j-1}) \quad \text{if } j \geq 1, \end{cases} \tag{2.10}$$

for all multiindex $\alpha \in \mathbb{N}^d$ there exists a constant c_α such that

$$\|D^\alpha \psi_j\|_\infty \leq \frac{c_\alpha}{2^{j|\alpha|}} \quad \text{for every } j \geq 0 \tag{2.11}$$

and

$$\sum_{j=0}^\infty \psi_j(x) = 1 \quad \text{for every } x \in \mathbb{R}^d. \tag{2.12}$$

Definition 2.10. *Given any Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^d)$ one defines its Fourier transform*

$$F\psi(\zeta) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\zeta}\psi(x)dm(x).$$

One can extend this notion to the tempered distributions $\mathcal{S}(\mathbb{R}^d)'$ by duality.

Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\Psi \in \Phi(\mathbb{R}^n)$. For any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ we define the non-homogeneous Besov space

$$\|f\|^{\Psi}_{B^s_{p,q}} = \left\|\left\{2^{sj}F^{-1}\psi_j Ff\right\}\right\|_{l^q(L^p)} = \left\|\left\{2^{sj}\left\|F^{-1}\psi_j Ff\right\|_{L^p}\right\}\right\|_{l^q}, \tag{2.13}$$

and we call $B^s_{p,q} \subset \mathcal{S}'$ to the set of tempered distributions such that this norm is finite.

These norms are equivalent for diferent choices of Ψ. In general one works with radial ψ_j and such that $\psi_{j+1}(x) = \psi_j(x/2)$. Of course we will ommit Ψ in our notation since it plays no role.

Proposition 2.11 (See [Tri83, Sections 2.3.3 and 2.7.1])**.** *The following properties hold:*

1. *Let $1 \leq q_0, q_1 \leq \infty$ and $1 \leq p \leq \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. Then*

$$B^{s+\varepsilon}_{p,q_0} \subset B^s_{p,q_1}.$$

2. *Given $1 \leq p_0 \leq p_1 \leq \infty$ and $-\infty < s_1 \leq s_0 < \infty$. Then,*

$$B^{s_0}_{p_0,p_0} \subset B^{s_1}_{p_1,p_1} \quad \text{if } s_0 - \frac{d}{p_0} = s_1 - \frac{d}{p_1}. \tag{2.14}$$

If we set $j \in \mathbb{Z}$ instead of $j \in \mathbb{N}$ in Definition 2.9, then we get the homogeneous spaces of tempered distributions (modulo polynomials) $\dot{B}^s_{p,q}$. In particular, by [Tri92, Theorem 2.3.3] we have that

$$\|f\|_{B^s_{p,q}} \approx \|f\|_{\dot{B}^s_{p,q}} + \|f\|_{L^p} \quad \text{for any } f \in \mathcal{S}'. \tag{2.15}$$

In the particular case of homogeneous Besov spaces with $1 \leq p, q \leq \infty$ and $s > 0$, one can give an equivalent definition in terms of differences of order $M \geq [s] + 1$:

$$\|f\|_{\dot{B}^s_{p,q}} \approx \left(\int_{\mathbb{R}^d} \frac{\left\|\Delta^M_h f\right\|^q_{L^p}}{|h|^{s\,q}} \frac{dm(h)}{|h|^d}\right)^{\frac{1}{q}}. \tag{2.16}$$

In [CT12] the authors point out that the seminorm of the homogeneous Besov space $\dot{B}^s_{p,q}$ for $0 < s < 1$ can be defined in terms of the approximating polynomials of degree 1 from the previous section. In general, [Dor85, Theorem 1] together with (2.9) and Remark 2.8 can be used to prove without much effort that for any $s > 0$ and $n \geq [s]$,

$$\|f\|_{\dot{B}^s_{p,q}} \approx \left(\int_0^\infty \left(\frac{\left\|\beta_{(n)}(f, Q(\cdot, t))\right\|_{L^p}}{t^{s-d}}\right)^q \frac{dt}{t}\right)^{1/q}.$$

In the particular case when $p = q$, which is in fact the one we are interested on, via Fubini's Theorem one can conclude that

$$\|f\|^q_{\dot{B}^s_{p,p}} \approx \sum_{Q\in\mathcal{D}} \left(\frac{\beta_{(n)}(f, Q)}{\ell(Q)^{s-d}}\right)^p |Q|. \tag{2.17}$$

Consider the boundary of a Lipschitz domain $\Omega \subset \mathbb{C}$. When it comes to the Besov space $B^s_{p,q}(\partial\Omega)$ we can just define it using the arc parameter of the curve, $z : I \to \partial\Omega$ with $|z'(t)| = 1$ for all t. Then, if $1 \leq p, q < \infty$ and $n - 1 \leq s < n$, we define naturally the homogeneous Besov norm on the boundary of Ω as

$$\|f\|^q_{\dot{B}^s_{p,q}(\partial\Omega)} := \int_{\mathbb{R}} \left(\int_I \frac{|\Delta^n_h (f \circ z)(t)|^p}{|h|^{sp}} \, dt \right)^{\frac{q}{p}} \frac{dh}{h}.$$

Note that since the domain is bounded, then I is a finite interval with length equal to the length of the boundary of Ω and we need to extend z perodically to $\mathbb{R}$ in order to have a sensible definition.

Lemma 2.12. *Let Ω be a $(\delta, R) - C^{n-1,1}$ domain and consider R-windows $\{\mathcal{Q}_j\}_{j=1}^M$ such that $\{\frac{1}{\max\{10,10\delta\}}\mathcal{Q}_j\}$ cover the boundary of Ω with A_j the parameterizations of the boundary associated to each window with support in $[-2R, 2R]$. Consider $N : \partial\Omega \to \mathbb{R}^2$ to be the unitary outward normal vector. Then, for any $1 < p < \infty$*

$$\sum_{j=1}^M \sum_{I \in \mathcal{D}} \frac{\beta_{(n)}(A_j, I)^p}{\ell(I)^{n\,p-2}} \approx \sum_{j=1}^M \|A_j\|^p_{\dot{B}^{n+1-1/p}_{p,p}} \lesssim \|N\|^p_{\dot{B}^{n-1/p}_{p,p}(\partial\Omega)} + R^{2-np} \lesssim_R \|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)}$$

with constants depending on n, p, δ, the length of the boundary $\mathcal{H}^1(\partial\Omega)$ and M.

The proof of this lemma for $n = 1$ can be found in [CT12, Lemma 3.3]. The general case is quite technical but uses the same tools, so we leave it to the reader. Note that M can be chosen to be $M \approx \frac{\mathcal{H}^1(\partial\Omega)}{R}$.

2.4 Even Calderón-Zygmund convolution operators in the plane

Definition 2.13. *Consider a function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$. For any $f \in L^1_{loc}$ we define*

$$T^K f(y) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(x)} K(y - x) f(x) \, dm(x)$$

as long as the limit exists, for instance, when K is bounded away from 0, $f \in L^1$ and $y \notin \operatorname{supp}(f)$ or when $f = \chi_U$ for an open set U with $y \in U$, $\int_{B_\varepsilon(0)\setminus B_{\varepsilon'}(0)} K\, dm = 0$ for every $\epsilon > \varepsilon' > 0$ and K is integrable at infinity. We say that K is the kernel of T^K.

In this paper we are interested in the case $d = 2$ in particular, where we can identify the ambient space with the complex plane $\mathbb{C}$ so that we can use its complex structure. In this case, for any multiindex $\gamma \in \mathbb{Z}^2$, we will consider $K^\gamma(z) = z^\gamma = z^{\gamma_1}\overline{z}^{\gamma_2}$ and then we will put shortly $T^\gamma f := T^{K^\gamma} f$, that is,

$$T^\gamma f(z) = \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_\varepsilon(z)} (z - w)^\gamma f(w) \, dm(w) \tag{2.18}$$

as long as the limit exists.

For any operator T and any domain Ω, we can consider $T_\Omega f = \chi_\Omega T(\chi_\Omega f)$.

Example 2.14. *As the reader may have observed, the Beurling and the Cauchy transforms are in that family of operators. Namely, when $K(z) = z^{-2}$, that is, for $\gamma = (-2, 0)$, then $\frac{-1}{\pi} T^\gamma$ is the Beurling transform. The operator $\frac{1}{\pi} T^{(-1,0)}$ coincides with the Cauchy transform.*

Consider the iterates of the Beurling transform $\mathcal{B}^m$ for $m > 0$. For every $f \in L^p$ and $z \in \mathbb{C}$ we have

$$\mathcal{B}^m f(z) = \frac{(-1)^m m}{\pi} \lim_{\varepsilon \to 0} \int_{|z-\tau|>\varepsilon} \frac{(\overline{z-\tau})^{m-1}}{(z-\tau)^{m+1}} f(\tau) \, dm(\tau) = \frac{(-1)^m m}{\pi} T^{(-m-1,m-1)} f(z). \tag{2.19}$$

That is, for $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 + \gamma_2 = -2$ and $\gamma_1 \leq -2$, the operator T^γ is an iteration of the Beurling transform modulo constant (see [AIM09, Section 4.2]), and it maps $L^p(U)$ to itself for every open set U. If $\gamma_2 \leq -2$, then T^γ is an iterate of the conjugate Beurling transform and it is bounded in L^p as well.

3 The characteristic function

3.1 The case of unbounded domains $\Omega \subset \mathbb{C}$

Definition 3.1. *Given $\delta > 0$ and $R > 0$, we say that $\Omega = \{x + i\, y \in \mathbb{C} : y > A(x)\}$ is a (δ, R, n, p)-admissible domain with defining function A if*

- *The defining function $A \in B^{n+1-1/p}_{p,p} \cap C^{n-1,1}$.*
- *We have $A(0) = 0$ and, if $n \geq 2$, $A'(0) = 0$.*
- *We have Lipschitz bounds on on the function and its derivatives $\left\|A^{(j)}\right\|_{L^\infty} < \frac{\delta}{R^{j-1}}$ for $1 \leq j \leq n$.*

We associate a Whitney covering $\mathcal{W}$ with appropriate constants to Ω. The constants will be fixed along this section, depending on n and δ.

In this Section we will prove the next result for the operators T^γ defined in (2.18).

Theorem 3.2. *Consider $\delta, R, \epsilon > 0$, $p > 1$ and a natural number $n \geq 1$. There exists a radius $\rho_\epsilon < R$ such that for any (δ, R, n, p)-admissible domain Ω and any multiindex $\gamma \in \mathbb{Z}^2$ with $\gamma_1 + \gamma_2 = -n-2$ and $\gamma_1 \cdot \gamma_2 \leq 0$, we have that $T^\gamma \chi_\Omega \in L^p(\Omega \cap B(0, \rho_\epsilon))$. In particular, we have that*

$$\|T^\gamma \chi_\Omega\|^p_{L^p(\Omega \cap B(0,\rho_\epsilon))} \leq C|\gamma|^{np} \left(\|A\|^p_{\dot{B}^{n-1/p+1}_{p,p}} + \rho_\epsilon^{2-np}(1+\epsilon)^{|\gamma|p} \right),$$

where C depends on p, n and the Lipschitz character of Ω.

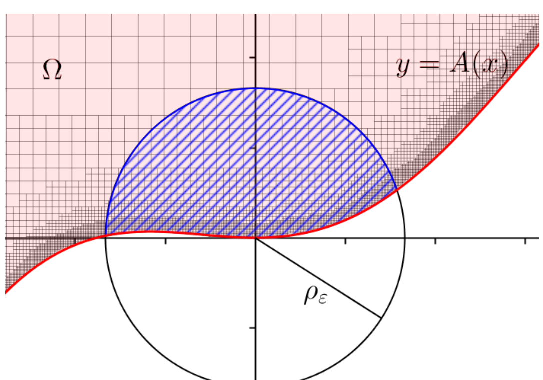

Figure 3.1: Disposition in Theorem 3.2.

Definition 3.3. *Consider a given (δ, R, n, p)-admissible domain with defining function A. Then, for every interval I we have an approximating polynomial $\mathbf{R}^n_{3I} := \mathbf{R}^n_{3I}A$, and*

$$\beta_{(n)}(I) = \frac{1}{\ell(I)} \int_{3I} \frac{|A(x) - \mathbf{R}^n_{3I}(x)|}{\ell(I)}\, dx.$$

We call

$$\Omega^n_I := \{x + i\, y : y > \mathbf{R}^n_{3I}(x)\}.$$

Let $\pi : \mathbb{C} \to \mathbb{R}$ be the vertical projection (to the real axis) and Q a cube in $\mathbb{C}$. If $\pi(Q) = I$ we will write $\Omega^n_Q := \Omega^n_I$.

Remark 3.4. *Note that π sends dyadic cubes of $\mathbb{C}$ to dyadic intervals of $\mathbb{R}$ and, in particular, any dyadic interval has a finite number of pre-images in the Whitney covering $\mathcal{W}$ of Ω uniformly bounded by a constant depending on δ and the Whitney constants of $\mathcal{W}$.*

Proof of Theorem 3.2. Assume that A is the defining function of Ω, that is, the graph of A coincides with the boundary of Ω. By (2.17) we have that

$$\sum_{I\in\mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-1/p}}\right)^p \ell(I) \approx \|A\|^p_{\dot{B}^{n-1/p+1}_{p,p}},$$

so it is enough to prove that

$$\|T^\gamma \chi_\Omega\|^p_{L^p(\Omega\cap B(0,\rho_\epsilon))} \leq C|\gamma|^{np} \left(\sum_{I\in\mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-1/p}}\right)^p \ell(I) + \rho_\epsilon^{2-np}(1+\epsilon)^{|\gamma|p}\right). \tag{3.1}$$

We begin the proof by some basic observation. Let $j_1, j_2 \in \mathbb{Z}$ such that $j_2 \neq j_1 + 1$. Then, the line integral

$$\int_{\partial\mathbb{D}} w^{j_1}\overline{w}^{j_2}\, dw = i \int_0^{2\pi} e^{i\theta(j_1-j_2+1)} d\theta = 0 \tag{3.2}$$

so, as long as $j_2 > 0$, given $0 < \varepsilon < 1$ Green's formula (2.4) says that

$$\int_{\mathbb{D}\setminus B(0,\varepsilon)} w^{j_1}\overline{w}^{j_2-1}\, dm(w) = \frac{i}{2j_2} \int_{\partial\mathbb{D}\cup\partial B(0,\varepsilon)} w^{j_1}\overline{w}^{j_2}\, dw = 0. \tag{3.3}$$

We may assume that $\gamma_2 \geq 0$. The case $\gamma_1 \geq 0$ can be proven mutatis mutandis. Consider a Whitney cube Q and $z \in B(0, \rho_\epsilon) \cap Q$. Then by (3.3) we have that

$$\begin{aligned} |T^\gamma \chi_\Omega(z)| &= \left|\int_{|z-w|>\ell(Q)} (w-z)^\gamma \chi_\Omega(w)\, dm(w)\right| \qquad (3.4)\\ &\leq \left|\int_{|z-w|>\ell(Q)} (w-z)^\gamma \chi_{\Omega^n_Q}(w)\, dm(w)\right| + \int_{|z-w|>\ell(Q)} \frac{|\chi_{\Omega^n_Q}(w) - \chi_\Omega(w)|}{|w-z|^{n+2}}\, dm(w). \end{aligned}$$

If we have taken appropriate Whitney constants, then we also have that $\ell(Q) < \text{dist}(Q, \partial\Omega^n_Q)$ (see Remark 2.4) and, thus, by (3.3) again, we have that

$$\int_{|z-w|>\ell(Q)} (w-z)^\gamma \chi_{\Omega^n_Q}(w)\, dm(w) = T^\gamma \chi_{\Omega^n_Q}(z). \tag{3.5}$$

We will see in Section 3.3 that the following claim holds.

Claim 3.5. *There exists a radius ρ_ϵ (depending on δ, R, n and ϵ) such that for every $z \in B(0,\rho_\epsilon)$ with $z \in Q \in \mathcal{W}$, we have that*

$$|T^\gamma \chi_{\Omega_Q^n}(z)| \lesssim_n \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n}. \tag{3.6}$$

The last term in (3.4) will bring the beta coefficients into play. Recall that we defined the symmetric difference of two sets A_1 and A_2 as $A_1 \Delta A_2 := (A_1 \cup A_2) \setminus (A_1 \cap A_2)$. For our choice of the Whitney constants we have that $Q \subset \Omega_Q^n \cap \Omega$ so

$$\int_{|z-w|>\ell(Q)} \frac{|\chi_{\Omega_Q^n}(w) - \chi_\Omega(w)|}{|w-z|^{n+2}}\, dm(w) = \int_{\Omega_Q^n \Delta \Omega} \frac{1}{|w-z|^{n+2}}\, dm(w). \tag{3.7}$$

Next we split the domain of integration in vertical strips. Namely, if we call $S_j = \{w \in \mathbb{C} : |\mathrm{Re}\,(w-z)| \leq 2^j \ell(Q)\}$ for $j \geq 0$ and $S_{-1} = \emptyset$, we have that

$$\begin{aligned}\int_{\Omega_Q^n \Delta \Omega} \frac{1}{|w-z|^{n+2}}\, dm(w) &= \sum_{j=0}^{\infty} \int_{(\Omega_Q^n \Delta \Omega)\cap S_j \setminus S_{j-1}} \frac{1}{|w-z|^{n+2}}\, dm(w) \\ &\leq \sum_{j=0}^{\infty} \left|(\Omega_Q^n \Delta \Omega) \cap S_j\right| \frac{1}{(2^{j-1}\ell(Q))^{n+2}}.\end{aligned} \tag{3.8}$$

We will see in Section 3.2 the following.

Claim 3.6. *We have*

$$\left|(\Omega_Q^n \Delta \Omega) \cap S_j\right| \lesssim_n \sum_{\substack{I \in \mathcal{D} \\ \pi(Q) \subset I \subset 2^j \pi(Q)}} \frac{\beta_{(n)}(I)}{\ell(I)^{n-1}} (2^j \ell(Q))^{n+1}. \tag{3.9}$$

Summing up, plugging (3.5) and (3.6) in the first term of the right-hand side of (3.4) and plugging (3.7), (3.8) and (3.9) in the other term, we get

$$|T^\gamma \chi_\Omega(z)| \lesssim_n \sum_{j=0}^{\infty} \sum_{\substack{I \in \mathcal{D} \\ \pi(Q) \subset I \subset 2^j \pi(Q)}} \frac{\beta_{(n)}(I)}{\ell(I)^{n-1}} (2^j \ell(Q))^{n+1} \frac{1}{(2^j \ell(Q))^{n+2}} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n}.$$

Reordering and computing,

$$|T^\gamma \chi_\Omega(z)| \lesssim_n \sum_{\substack{I \in \mathcal{D} \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^{n-1}} \sum_{\substack{j \in \mathbb{N}_0 \\ I \subset 2^j \pi(Q)}} \frac{1}{2^j \ell(Q)} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n} \lesssim \sum_{\substack{I \in \mathcal{D} \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^{n}} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n}. \tag{3.10}$$

Raising to power p, integrating in Q and adding we get that for ρ_ϵ small enough

$$\begin{aligned}\|T^\gamma \chi_\Omega\|^p_{L^p(\Omega \cap B(0,\rho_\epsilon))} &\lesssim_n \sum_{\substack{Q \in \mathcal{W} \\ Q \cap B(0,\rho_\epsilon) \neq \emptyset}} |Q| \left(\sum_{\substack{I \in \mathcal{D} \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^n} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n} \right)^p \\ &\lesssim_p \sum_{\substack{Q \in \mathcal{W} \\ Q \cap B(0,\rho_\epsilon) \neq \emptyset}} |Q| \left(\sum_{\substack{I \in \mathcal{D} \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^n} \right)^p + \rho_\epsilon^{2-np}(1+\epsilon)^{|\gamma|p}.\end{aligned} \tag{3.11}$$

Regarding the double sum, we use Hölder Inequality to find that

$$
\begin{aligned}
\sum_{\substack{Q\in\mathcal{W}\\ Q\cap B(0,\rho_\epsilon)\neq\emptyset}} |Q| \left(\sum_{\substack{I\in\mathcal{D}\\ \pi(Q)\subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^n} \right)^p &\leq \sum_{Q\in\mathcal{W}} |Q| \sum_{\substack{I\in\mathcal{D}\\ \pi(Q)\subset I}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{2p}}} \right)^p \left(\sum_{\substack{I\in\mathcal{D}\\ \pi(Q)\subset I}} \frac{1}{\ell(I)^{\frac{p'}{2p}}} \right)^{\frac{p}{p'}} \\
&\lesssim_p \sum_{Q\in\mathcal{W}} \ell(Q)^2 \sum_{\substack{I\in\mathcal{D}\\ \pi(Q)\subset I}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{2p}}} \right)^p \ell(Q)^{\frac{-1}{2}} \qquad (3.12)\\
&\leq \sum_{I\in\mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{2p}}} \right)^p \sum_{\substack{Q\in\mathcal{W}\\ \pi(Q)\subset I}} \ell(Q)^{\frac{3}{2}} \lesssim_{\mathcal{W}} \sum_{I\in\mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{p}}} \right)^p \ell(I),
\end{aligned}
$$

where the constant in the last inequality depends on the maximum number of Whitney cubes that can be projected to a given interval, depending only on δ and n.

Thus, by (3.11) and (3.12) we have proven (3.1) when $\gamma_2 \geq 0$. The case $\gamma_2 \leq 0$ can be proven analogously. □

3.2 The interstitial region

Proof of Claim 3.6. Recall that we have a point $z \in Q \in \mathcal{W}$, and a vertical strip $S_j = \{w \in \mathbb{C} : |\mathrm{Re}\,(w - z)| \leq 2^j \ell(Q)\}$ for a certain $j \in \mathbb{N}_0$. For $j = 0$ the proof is quite simple. Assume $j > 0$. Let $J_0 = \pi(Q)$, $r = 2^{j-1}\ell(Q)$. Let J_{j-1} be the dyadic interval of length r containing J_0. Then it is enough to see that

$$
\left|(\Omega^n_{J_0} \Delta \Omega) \cap S_j\right| \lesssim_n \sum_{\substack{I\in\mathcal{D}\\ J_0\subset I\subset J_{j-1}}} \beta_{(n)}(I) \frac{r^{n-1}}{\ell(I)^{n-1}} r^2. \qquad (3.13)
$$

We write $N = j - 1$ for the sake of clarity. Then,

$$
\begin{aligned}
\left|(\Omega^n_{J_0} \Delta \Omega) \cap S\right| &= \int_{\mathrm{Re}\,(z)-r}^{\mathrm{Re}\,(z)+r} |A - \mathbf{R}^n_{3J_0}|\, dm_1 \qquad (3.14)\\
&\leq \int_{3J_N} |A - \mathbf{R}^n_{3J_N}|\, dm_1 + \int_{3J_N} |\mathbf{R}^n_{3J_N} - \mathbf{R}^n_{3J_0}|\, dm_1 = \boxed{1} + \boxed{2}.
\end{aligned}
$$

Trivially,

$$
\boxed{1} = \beta_{(n)}(J_N)\ell(J_N)^2. \qquad (3.15)
$$

To deal with the second term, we consider the chain of dyadic intervals

$$
J_0 \subset \cdots \subset J_k \subset J_{k+1} \subset \cdots \subset J_N,
$$

with $0 < k < N$ and $\ell(J_k) = 2^k \ell(J_0)$. We use the Triangle Inequality in the chain of intervals:

$$
\boxed{2} \leq \sum_{k=0}^{N-1} \int_{3J_N} |\mathbf{R}^n_{3J_{k+1}} - \mathbf{R}^n_{3J_k}|\, dm_1 = \sum_{k=0}^{N-1} \left\| \mathbf{R}^n_{3J_{k+1}} - \mathbf{R}^n_{3J_k} \right\|_{L^1(3J_N)}. \qquad (3.16)
$$

For any polynomial $P(x) = \sum_{i=1}^n a_i x^i$ of degree n and any interval J centered at 0, using the linear map ϕ that sends the interval $(-1, 1)$ to J as a change of coordinates, we have that

$$
\|P\|_{L^1(J)} \approx \ell(J) \|P \circ \phi\|_{L^1(-1,1)},
$$

and using the fact that all norms in a finite dimensional vector space are equivalent (in particular the $L^1(-1,1)$ norm and the sum of coefficients) we have that

$$\|P\|_{L^1(J)} \approx_n \ell(J) \sum_{i=1}^{n} \ell(J)^i \, |a_i| \, .$$

By the same token, for any $k_0 \in \mathbb{N}$,

$$\|P\|_{L^1(2^{k_0}J)} \approx_n 2^{k_0}\ell(J) \sum_{i=1}^{n} \left(2^{k_0}\ell(J)\right)^i |a_i| \lesssim_n 2^{k_0(n+1)} \|P\|_{L^1(J)}.$$

Fix $0 \leq k < N$. Then,

$$\left\| \mathbf{R}^n_{3J_{k+1}} - \mathbf{R}^n_{3J_k} \right\|_{L^1(3J_N)} \lesssim_n \left\| \mathbf{R}^n_{3J_{k+1}} - \mathbf{R}^n_{3J_k} \right\|_{L^1(3J_k)} \frac{\ell(J_N)^{n+1}}{\ell(J_k)^{n+1}},$$

with constants depending only on n. Thus, by Remark 2.5

$$\left\| \mathbf{R}^n_{3J_{k+1}} - \mathbf{R}^n_{3J_k} \right\|_{L^1(3J_N)} \lesssim_n \left(\beta_{(n)}(J_{k+1}) + \beta_{(n)}(J_k)\right) \frac{\ell(J_N)^{n+1}}{\ell(J_k)^{n+1}} \ell(J_k)^2. \tag{3.17}$$

Thus, combining (3.14), (3.15), (3.16) and (3.17) we get (3.13). □

3.3 Domain bounded by a polynomial graph

We will consider only very "flat" polynomials. Let us see what we can say about their coefficients.

Lemma 3.7. *Let $n \geq 2$, $A \in C^{n-1,1}(\mathbb{R})$ with $A(0) = 0$, $A'(0) = 0$, $\left\|A^{(j)}\right\|_{L^\infty} < \frac{\delta}{R^{j-1}}$ for $j \leq n$ and consider two intervals $J \subset I = [-R, R]$. Then we have the following bounds for the approximating polynomial $P = \mathbf{R}^n_J A$ in the interval I:*

$$\left\|P^{(j)}\right\|_{L^\infty(I)} \leq \frac{3^{n-j}\delta}{R^{j-1}} \quad \text{for } j \leq n.$$

Furthermore, if $\rho > 0$ and $J \subset [-\rho, \rho]$, then

$$\|P\|_{L^\infty(-\rho,\rho)} \leq \frac{3^n \delta \rho^2}{R} \qquad \text{and} \qquad \|P'\|_{L^\infty(-\rho,\rho)} \leq \frac{3^{n-1}\delta\rho}{R}. \tag{3.18}$$

Proof. By Remark 2.6 we know that there are at least $n+1$ common points $\tau_0^0, \cdots, \tau_n^0 \in 3J$ for A and P, that is, $A(\tau_j^0) = P(\tau_j^0)$ for every j. By the Mean Value Theorem, there are n common points $\tau_0^1, \cdots, \tau_{n-1}^1 \in 3J$ for their derivatives. By induction we find points $\tau_0^k \cdots \tau_{n-k}^k \in 3J$ where the k-th derivatives coincide for $0 \leq k \leq n-1$, that is, $A^{(k)}(\tau_j^k) = P^{(k)}(\tau_j^k)$ for every $0 \leq j \leq n-k$.

Note that the polynomial derivative $P^{(n)}$, which is in fact a constant, coincides with the differential quotient of $P^{(n-1)}$ evaluated at any pair of points. In particular given $x \in \mathbb{R}$, for the points τ_0^{n-1} and τ_1^{n-1} we have that

$$\left|P^{(n)}(x)\right| = \left|\frac{P^{(n-1)}(\tau_0^{n-1}) - P^{(n-1)}(\tau_1^{n-1})}{\tau_0^{n-1} - \tau_1^{n-1}}\right| = \left|\frac{A^{(n-1)}(\tau_0^{n-1}) - A^{(n-1)}(\tau_1^{n-1})}{\tau_0^{n-1} - \tau_1^{n-1}}\right| \leq \frac{\delta}{R^{n-1}}.$$

Now we argue by induction again. Assume that $\left\|P^{(j+1)}\right\|_{L^\infty(I)} \leq 3^{n-j-1}\delta/R^j$ for a certain $j \leq n-1$. Consider $x \in I$ and, by the Mean Value Theorem, we have a point ξ such that $|P^{(j)}(x) - P^{(j)}(\tau_0^j)| = |P^{(j+1)}(\xi)||x - \tau_0^j|$. Thus, since $P^{(j)}(\tau_0^j) = A^{(j)}(\tau_0^j)$ we have that

$$|P^{(j)}(x)| \leq |P^{(j+1)}(\xi)||x - \tau_0^j| + |A^{(j)}(\tau_0^j)| \leq \frac{3^{n-j-1}\delta}{R^j} 2R + \frac{\delta}{R^{j-1}} = \frac{3^{n-j}\delta}{R^{j-1}}.$$

We have not used yet the fact that $A'(0) = A(0) = 0$. Let us fix $\rho \leq R$ and assume that $I \subset [-\rho, \rho]$. Then for $x \in [-\rho, \rho]$, we can write $P'(x) = P'(x) - P'(\tau_0^1) + A'(\tau_0^1) - A'(0)$, so

$$|P'(x)| \leq \|P''\|_{L^\infty(I)} |x - \tau_0^1| + \|A''\|_{L^\infty(I)} |\tau_0^1| \leq \frac{3^{n-2}\delta}{R}\, 2\rho + \frac{\delta}{R}\,\rho \leq \frac{3^{n-1}\delta\rho}{R}.$$

By the same token,

$$|P(x)| \leq \|P'\|_{L^\infty([-\rho,\rho])} |x - \tau_0^0| + \|A'\|_{L^\infty([-\rho,\rho])} |\tau_0^0| \leq \frac{3^{n-1}\delta\rho}{R}\, 2\rho + \delta\,\rho \leq \frac{3^n \delta \rho^2}{R}.$$

□

Now we can prove Claim 3.5. Recall that we want to find a radius $\rho_{int} < R$ depending on ϵ such that every point z contained in a Whitney cube $Q \subset B(0, \rho_{int})$ satisfies (3.6), that is,

$$|T^\gamma \chi_{\Omega_Q^n}(z)| \lesssim_n \frac{(1+\epsilon)^{|\gamma|}}{\rho_{int}^n},$$

where $\gamma \in \{(-j_1, j_2) : j_1, j_2 \in \mathbb{N}_0 \text{ and } j_1 - j_2 = n+2\}$ (recall that we assumed that $\gamma_2 \geq 0$). According to the previous lemma, when $n \geq 2$ we are dealing with a domain Ω_Q^n whose boundary is the graph of a polynomial $P(x) = \sum_{j=0}^n a_j x^j$ such that

$$\begin{aligned} |a_0| = |P(0)| &\leq \frac{3^n \delta \rho_{int}^2}{R}, && \\ |a_1| = |P'(0)| &\leq \frac{3^{n-1}\delta\rho_{int}}{R} && \text{and} \\ |a_j| = \frac{|P^{(j)}(0)|}{j!} &\leq \frac{3^{n-j}\delta}{j! R^{j-1}} && \text{for } 2 \leq j < n. \end{aligned} \tag{3.19}$$

We call $\Omega_P := \{x + i\,y : y > P(x)\}$ to such a domain. Note that (3.18) implies that for ρ_{int} small enough the polynomial P is "flat", namely $|P(x)| < \frac{\rho_{int}}{4}$ for $|x| < \rho_{int}$.

One can think of the "exterior" radius ρ_{ext} below as a geometric version of ϵ, namely $\rho_{ext} = (\varepsilon/16)^2$. Further, we can assume that $\rho_{ext} < R$.

Proposition 3.8. *Consider two real numbers $\delta, R > 0$ and $n \geq 2$. For ρ_{ext} small enough, there exists $0 < \rho_{int} < \rho_{ext}$ depending also on n, δ and R such that for all $j_1, j_2 \in \mathbb{N}_0$ with $j_1 - j_2 = n+2$, all $P \in \mathcal{P}^n$ such that, if $n \geq 2$, P satisfies (3.19), all $z \in Q(0, \rho_{int}) \cap \Omega_P$ and $0 < \varepsilon < \operatorname{dist}(z, \partial\Omega_P)$ we have*

$$\left| \int_{\Omega_P \setminus B(z,\varepsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}}\, dm(w) \right| \leq \frac{C_n}{\rho_{int}^n} \left(1 + 16\rho_{ext}^{1/2}\right)^{j_2}, \tag{3.20}$$

with C_n depending only on n.

If $P \in \mathcal{P}^1$, that is, if $n = 1$, then for all $j_1, j_2 \in \mathbb{N}_0$ with $j_1 - j_2 = 3$ we have that

$$\int_{\Omega_P \setminus B(z,\varepsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}}\, dm(w) = 0.$$

Proof. First consider $n = 1$. In that case, Ω_P is a half plane. By rotation and dilation, we can assume $\Omega_P = \mathbb{R}^2_+ := \{w = x + i\,y : y > 0\}$. Note that $\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}$ is infinitely many times

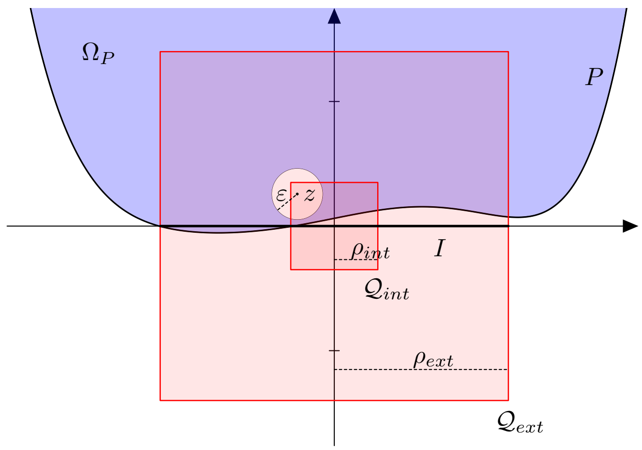

Figure 3.2: Disposition in Proposition 3.8.

differentiable with respect to w in any ring centered in z. Then we can apply Green's formula (2.4) and use the decay at infinity of the integrand and (3.2) to see that for $\varepsilon > 0$ small enough

$$\int_{\mathbb{R}^2_+\setminus B(z,\varepsilon)} \frac{\overline{(z-w)}^{j_1-3}}{(z-w)^{j_1}}\,dm(w) = c_{j_1}\int_{\mathbb{R}} \frac{\overline{(z-w)}^{j_1-3}}{(z-w)^{j_1-1}}\,d\overline{w} = c_{j_1}\int_{\mathbb{R}} \frac{\overline{(z-w)}^{j_1-3}}{(z-w)^{j_1-1}}\,dw$$
$$= c_{j_1}\int_{\mathbb{R}^2_+\setminus B(z,\varepsilon)} \frac{\overline{(z-w)}^{j_1-4}}{(z-w)^{j_1-1}}\,dm(w). \tag{3.21}$$

When $j_1 = 3$ the last constant is zero. By induction, all these integrals equal zero.

So we can assume $n \geq 2$. Consider a given $\rho_{ext} > 0$. We define the interval $I := [-\rho_{ext}, \rho_{ext}]$, the *exterior window* $\mathcal{Q}_{ext} := Q(0, \rho_{ext})$, and the *interior window* $\mathcal{Q}_{int} := Q(0, \rho_{int})$. Note that (3.19) implies that for ρ_{ext} small enough, the set $\{x + i\,P(x) : x \in I\} \subset \mathcal{Q}_{ext}$, that is, the boundary $\partial\Omega_P$, intersects the vertical sides of the window $\mathcal{Q}_{ext}$ but does not intersect the horizontal ones. The same can be said for the sides of $\mathcal{Q}_{int}$.

Fix $z \in \mathcal{Q}_{int}$ and $\varepsilon < \operatorname{dist}(z, \partial\Omega)$. Splitting the domain of integration in two regions we get

$$\int_{\Omega_P\setminus B(z,\varepsilon)} \frac{\overline{(z-w)}^{j_2}}{(z-w)^{j_1}}\,dm(w) = \int_{\Omega_P\setminus \mathcal{Q}_{ext}} \frac{\overline{(z-w)}^{j_2}}{(z-w)^{j_1}}\,dm(w) + \int_{\Omega_P\cap\mathcal{Q}_{ext}\setminus B(z,\varepsilon)} \frac{\overline{(z-w)}^{j_2}}{(z-w)^{j_1}}\,dm(w). \tag{3.22}$$

We bound the non-local part trivially by taking absolute values and using polar coordinates. Choosing $\rho_{int} < \rho_{ext}/2$, we have that

$$\int_{\Omega_P\setminus\mathcal{Q}_{ext}} \frac{1}{|z-w|^{j_1-j_2}}\,dm(w) \leq \int_{\frac{\rho_{ext}}{2}}^{\infty} \frac{1}{r^{j_1-j_2}} \int_0^1 dm_1\, 2\pi r\,dr = \frac{2\pi}{j_1-j_2-2}\,\frac{2^{j_1-j_2-2}}{(\rho_{ext})^{j_1-j_2-2}}, \tag{3.23}$$

where dm_1 stands for the Lebesgue length measure. Note that $j_1 - j_2 - 2 = n$.

To bound the local part, we can apply Green's Theorem again and we get

$$\begin{aligned}\frac{2(j_1-1)}{i}\int_{\Omega_P\cap\mathcal{Q}_{ext}\setminus B(z,\varepsilon)}\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}}\,dm(w) = &-\int_{|z-w|=\varepsilon}\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}\,d\overline{w}\\ &-\int_{\Omega_P\cap\partial\mathcal{Q}_{ext}}\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}\,d\overline{w}\\ &+\int_{\partial\Omega_P\cap\mathcal{Q}_{ext}}\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}\,d\overline{w}.\end{aligned}\tag{3.24}$$

The first term in the right-hand side of (3.24) is zero by (3.2). For the second term we note that $z\in\mathcal{Q}_{int}$, and every w in the integration domain is in $\partial\mathcal{Q}_{ext}$, so $|z-w|>\rho_{ext}-\rho_{int}$. Thus,

$$\int_{\Omega_P\cap\partial\mathcal{Q}_{ext}}\frac{1}{|z-w|^{j_1-j_2-1}}\,d\overline{w}\le\frac{1}{|\rho_{ext}-\rho_{int}|^{j_1-j_2-1}}6\rho_{ext}.\tag{3.25}$$

Summing up, by (3.22), (3.23), (3.24) and (3.25), since $\rho_{int}<\frac{\rho_{ext}}{2}$, we get that

$$\left|\int_{\Omega_P\setminus B(z,\varepsilon)}\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}}\,dm(w)\right|\le\left|\int_{\partial\Omega_P\cap\mathcal{Q}_{ext}}\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}\,d\overline{w}\right|+\frac{C_n}{\rho_{ext}^n},\tag{3.26}$$

with C_n depending only on n.

It remains to bound the first term in the right-hand side of (3.26). We begin by using the change of coordinates $w=x+i\,P(x)$ to get a real variable integral:

$$\int_{\partial\Omega_P\cap\mathcal{Q}_{ext}}\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}\,d\overline{w}=\int_I\frac{(\overline{z}-(x-i\,P(x)))^{j_2}}{(z-(x+i\,P(x)))^{j_1-1}}\,(1-i\,P'(x))\,dx.\tag{3.27}$$

Note that the denominator on the right-hand side never vanishes because $z\notin\partial\Omega_P$. Now we take a closer look to the fraction in order to take as much advantage of cancellation as we can, namely

$$\begin{aligned}\frac{(\overline{z}-(x-i\,P(x)))^{j_2}}{(z-(x+i\,P(x)))^{j_1-1}}&=\frac{((\overline{z}-z+2i\,P(x))+(z-(x+i\,P(x))))^{j_2}}{(z-(x+i\,P(x)))^{j_1-1}}\\&=\sum_{j=0}^{j_2}\binom{j_2}{j}(\overline{z}-z+2i\,P(x))^j(z-(x+i\,P(x)))^{j_2-j-j_1+1}\\&=\sum_{j=0}^{j_2}\binom{j_2}{j}\frac{(-2i\,\mathrm{Im}(z)+2i\,P(x))^j}{(z-(x+i\,P(x)))^{n+1+j}}.\end{aligned}\tag{3.28}$$

Next, we complexify the right-hand side of (3.28) so that we have a holomorphic function in a certain neighborhood of I to be able to change the integration path. To do this change we need a key observation. If $\tau\in\mathcal{Q}_{ext}$, then $|\tau|<\sqrt{2}\rho_{ext}$ and by (3.19) writing $\widetilde{\delta}=3^n\delta$ we have that

$$|P'(\tau)|\le|a_1|+2|a_2||\tau|+\cdots\le\widetilde{\delta}\left(\frac{\rho_{int}}{R}+\frac{2}{R^{-1}}2\rho_{ext}+\frac{3}{R^{-2}}(2\rho_{ext})^2+\cdots\right)<1/2\tag{3.29}$$

if ρ_{ext} is small enough. Thus, we have that $\mathrm{Re}\,(1+i\,P'(\tau))>\frac{1}{2}$ in $\mathcal{Q}_{ext}$ and, by the Complex Rolle Theorem 2.2, we can conclude that $\tau\mapsto\tau+i\,P(\tau)$ is injective in $\mathcal{Q}_{ext}$. In particular, $z-(\tau+i\,P(\tau))$ has one zero at most in $\mathcal{Q}_{ext}$, and this zero is not real because $z\notin\partial\Omega_P$. Therefore, since the real line divides $\mathcal{Q}_{ext}$ in two congruent open rectangles, there is one of them whose closure has a neighborhood containing no zeros of this function. We call this open rectangle $\mathcal{R}$. Now, for any $j \geq 0$ we have that $\tau \mapsto \frac{(P(\tau)-\mathrm{Im}(z))^j}{(z-(\tau+i\,P(\tau)))^{n+1+j}}\,(1-i\,P'(\tau))$ is holomorphic in $\mathcal{R}$, so we can change the path of integration and get

$$\int_I \frac{2^j (P(x)-\mathrm{Im}(z))^j}{(z-(x+i\,P(x)))^{n+1+j}}\,(1-i\,P'(x))\,dx = -\int_{\partial\mathcal{R}\setminus I} \frac{2^j (P(\tau)-\mathrm{Im}(z))^j}{(z-(\tau+i\,P(\tau)))^{n+1+j}}\,(1-i\,P'(\tau))\,d\tau. \tag{3.30}$$

On the other hand, if $|\tau| < \sqrt{2}\rho_{ext}$, then we have that

$$\begin{aligned} |P(\tau)| &\leq |a_0| + |a_1||\tau| + |a_2||\tau|^2 + |a_3||\tau|^3 + \cdots \\ &\leq \widetilde{\delta}\left(\frac{\rho_{int}^2}{R} + \frac{\rho_{int}}{R}\,2\rho_{ext} + \frac{1}{R}(2\rho_{ext})^2 + \frac{1}{R^2}(2\rho_{ext})^3 + \cdots\right) \leq \rho_{ext}^{3/2} \end{aligned} \tag{3.31}$$

for ρ_{ext} small enough. Then, taking absolute values inside the last integral in (3.30) and using (3.29) and (3.31) we get

$$\int_{\partial\mathcal{R}\setminus I} \frac{2^j |P(\tau)-\mathrm{Im}(z)|^j}{|z-(\tau+i\,P(\tau))|^{n+1+j}}\,|1-i\,P'(\tau)|\,|d\tau| \leq \frac{3}{2}\int_{\partial\mathcal{R}\setminus I} \frac{2^j (\rho_{ext}^{3/2}+\rho_{int})^j}{|z-(\tau+i\,P(\tau))|^{n+1+j}}\,|d\tau|. \tag{3.32}$$

Finally, we have that for any $\tau \in \partial R \setminus I \subset \partial\mathcal{Q}_{ext}$,

$$|z-(\tau+i\,P(\tau))| \geq |\tau| - |z| - |P(\tau)| \geq \rho_{ext} - \sqrt{2}\rho_{int} - \rho_{ext}^{\frac{3}{2}} \geq \frac{\rho_{ext}}{2} - 2\rho_{int} \tag{3.33}$$

for ρ_{ext} small enough. Using this fact we rewrite (3.32) as

$$\int_{\partial\mathcal{R}\setminus I} \frac{2^j |P(\tau)-\mathrm{Im}(z)|^j}{|z-(\tau+i\,P(\tau))|^{n+1+j}}\,|1-i\,P'(\tau)|\,|d\tau| \leq \frac{3}{2}\,\frac{2^j (\rho_{ext}^{3/2}+\rho_{int})^j}{(\rho_{ext}/2-2\rho_{int})^{n+1+j}}\int_{\partial\mathcal{R}\setminus I} |d\tau|. \tag{3.34}$$

Putting together (3.27), (3.28), (3.30) and (3.34) we can write

$$\begin{aligned} \left|\int_{\partial\Omega_P\cap\mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}\,d\overline{w}\right| &\leq \frac{3}{2\,(\rho_{ext}/2-2\rho_{int})^{n+1}}\sum_{j=0}^{j_2}\left(2\cdot\frac{\rho_{ext}^{3/2}+\rho_{int}}{\rho_{ext}/2-2\rho_{int}}\right)^j \binom{j_2}{j} 4\rho_{ext} \\ &= \frac{6\rho_{ext}}{(\rho_{ext}/2-2\rho_{int})^{n+1}}\left(1+2\cdot\frac{\rho_{ext}^{3/2}+\rho_{int}}{\rho_{ext}/2-2\rho_{int}}\right)^{j_2}, \end{aligned}$$

and, choosing $\rho_{int} = \min\{\rho_{ext}/8, \rho_{ext}^{3/2}\}$,

$$\left|\int_{\partial\Omega_P\cap\mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}\,d\overline{w}\right| \leq \frac{C_n}{\rho_{ext}^n}\left(1+16\rho_{ext}^{1/2}\right)^{j_2}, \tag{3.35}$$

where the constant C_n depends only on n.

Now, (3.26) together with (3.35) prove (3.20). □

Remark 3.9. *Note that we have assumed $\gamma_2 \geq 0$ in the proof Theorem 3.2. When proving the case $\gamma_2 \leq 0$, we would have to prove Proposition 3.8 with $\gamma \in \{(j_1, -j_2) : j_1, j_2 \in \mathbb{N}_0$ and $j_2 - j_1 = n+2\}$. The proof is analogous to the one shown above with slight modifications, and it is left to the reader to complete the details.*

3.4 Bounded domains: a localization principle

We are going to follow a standard localization argument, so we will give a sketch, leaving some details for the reader.

Let us start with some remarks. First we make some general observations on admissible domains. In these first two remarks we assume $n \geq 2$ since the case $n = 1$ is simpler (there is no need for rotations) and it is fully covered in [CT12].

Remark 3.10. *If Ω is a (δ, R, n, p)-admissible domain with defining function A, then for every $\tau \in \partial\Omega$ one can perform a translation of the domain that sends τ to the origin and a rotation in the same spirit of Definition 2.1, so that $\partial\Omega$ coincides with the graph of a new function $\widetilde{A} \in C^{n-1,1}(\mathbb{R})$ in a certain ball $B(0, \widetilde{R})$ with fixed radius $\widetilde{R}$ (depending on δ and R) with $\widetilde{A}(0) = 0$, $\widetilde{A}'(0) = 0$, $\left\|\widetilde{A}'\right\|_{L^\infty} \leq \widetilde{\delta}$ and $\operatorname{supp}(\widetilde{A}) \subset [-2\widetilde{R}, 2\widetilde{R}]$. One can see that $\left\|\widetilde{A}\right\|_{\dot{B}^s_{p,p}} \lesssim \|A\|_{B^s_{p,p}}$ for $s < n+1$. Therefore $\widetilde{A}$ determines a $(\widetilde{\delta}, \widetilde{R}, n, p)$-admissible domain $\widetilde{\Omega}$ with compactly supported defining function (see Figure 3.10).*

Consider $\gamma \in \mathbb{Z}^2$ with $\gamma_1 + \gamma_2 = -n-2$ and $\gamma_1 \cdot \gamma_2 \leq 0$. Note that $\chi_\Omega(z) = \chi_{\widetilde{\Omega}}(z)$ for $z \in B(0, \widetilde{R})$. For every $z \in \Omega \cap B\left(0, \frac{\widetilde{R}}{2}\right)$ we use the decomposition $T^\gamma \chi_\Omega(z) = T^\gamma \chi_{\widetilde{\Omega}}(z) + T^\gamma(\chi_\Omega - \chi_{\widetilde{\Omega}})(z)$:

$$|T^\gamma \chi_\Omega(z)| \leq |T^\gamma \chi_{\widetilde{\Omega}}(z)| + \int_{|w|>\widetilde{R}} \frac{|\chi_\Omega(w) - \chi_{\widetilde{\Omega}}(w)|}{|w-z|^{n+2}}\, dm(w) \lesssim |T^\gamma \chi_{\widetilde{\Omega}}(z)| + \frac{1}{\widetilde{R}^n}. \tag{3.36}$$

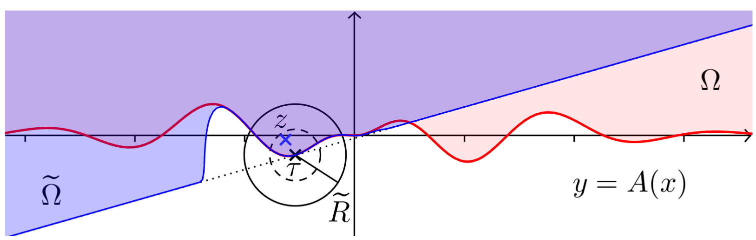

Figure 3.3: Disposition in Remark 3.10 before the rotation and the translation.

Next we take a look at admissible domains with compact support.

Remark 3.11. *Let Ω be a (δ, R, n, p)-admissible domain with defining function A supported in $I = [-2R, 2R]$. For a given $\epsilon > 0$ small enough, take ρ to be the radius ρ_ϵ from Theorem 3.2 associated to the parameters $\widetilde{\delta}, \widetilde{R}, n, p$ of the previous remark. We assume $\rho < \widetilde{R}/2$.*

Since A is supported in I, we can cover the area close to the graph $\mathcal{G} = \{x + i\, A(x) : x \in I\}$ by a finite number of balls of radius ρ (see Figure 3.4). In each ball we can apply (3.36) for the corresponding domain $\widetilde{\Omega}$. Thus, given $\gamma \in \mathbb{Z}^2$ with $\gamma_1 + \gamma_2 = -n-2$ and $\gamma_1 \cdot \gamma_2 \leq 0$, writing $U_\rho \mathcal{G} = \bigcup_{z \in \mathcal{G}} B\left(z, \frac{\rho}{2}\right)$ and using Theorem 3.2 we have that

$$\|T^\gamma \chi_\Omega\|^p_{L^p(\Omega \cap U_\rho \mathcal{G})} \leq C \left(\|A\|^p_{B^{n+1-1/p}_{p,p}(\partial\Omega)} + (1+\epsilon)^{|\gamma| p} \right),$$

with C depending on n, p, δ, R and ϵ (but not on $|\gamma|$).

Finally, for $z \notin U_\rho \mathcal{G}$ we can use the same argument of (3.36) replacing the domain $\widetilde{\Omega}$ by the half plane $\mathbb{R}^2_+$. Namely,

$$|T^\gamma \chi_\Omega(z)| \leq \left| T^\gamma \chi_{\mathbb{R}^2_+}(z) \right| + \int_{\Omega \Delta R^2_+} \frac{1}{|w-z|^{n+2}}\, dm(w).$$

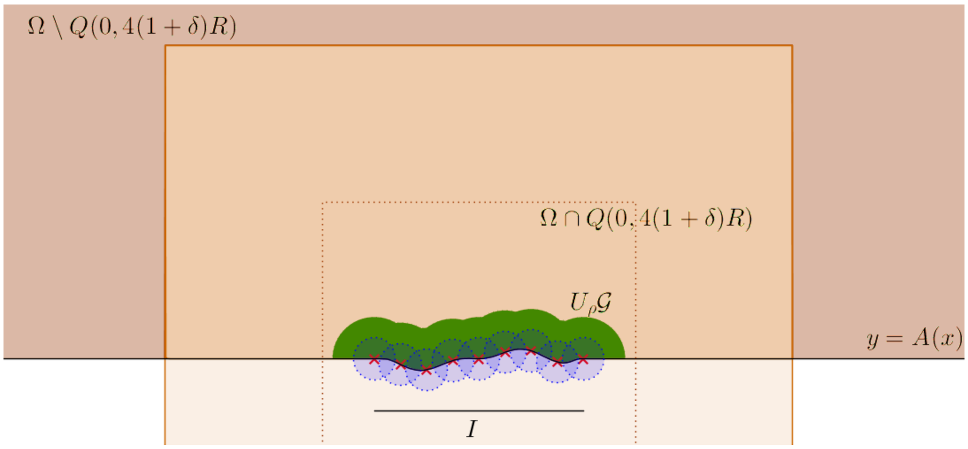

Figure 3.4: Decomposition of a (δ, R, n, p)-admissible domain Ω with defining function A supported in $I = [-2R, 2R]$ considered in Remark 3.11.

In that case, the first term is zero just by (3.21). Since A is compactly supported in $[-2R, 2R]$ and it is Lipschitz with constant δ, the domain of integration of the second term is contained in $Q(0, 2(1+\delta)R)$. Thus, when $z \in \Omega \setminus Q(0, 4(1+\delta)R)$ then $|T^\gamma \chi_\Omega(z)|$ is bounded by a constant times $\frac{R^2}{|z|^{n+2}}$. When $z \in \Omega \cap Q(0, 4(1+\delta)R) \setminus U_\rho \mathcal{G}$ then $|T^\gamma \chi_\Omega(z)|$ is bounded by $\frac{C}{\rho^n}$. Summing up, we have a global bound

$$\|T^\gamma \chi_\Omega\|^p_{L^p(\Omega)} \leq C \left(\|A\|^p_{B^{n+1-1/p}_{p,p}} + (1+\epsilon)^{|\gamma|p} + \frac{R^2}{\rho^{np}} \right) \lesssim \|A\|^p_{B^{n+1-1/p}_{p,p}} + (1+\epsilon)^{|\gamma|p},$$

with constants depending on n, p, δ, R and ϵ.

Now we turn to the case of bounded domains. First we note how differentiation works for $T^\gamma \chi_\Omega$.

Remark 3.12. *Consider a $(\delta, R) - C^{(n-1,1)}$ domain Ω and let us fix $\gamma \in \mathbb{Z}^2$ with either $\gamma_1 \geq 0$ or $\gamma_2 \geq 0$, and $\alpha \in \mathbb{N}_0^2$ with modulus $|\alpha| = n$. Then for $z \in \Omega$ we have*

$$D^\alpha T^\gamma_\Omega 1(z) = \begin{cases} C_n \chi_\Omega(z) & \text{if } \gamma = (n-1, -1) \text{ and } \alpha = (n, 0) \\ & \quad \text{or } \gamma = (-1, n-1) \text{ and } \alpha = (0, n) \\ 0 & \text{if } \alpha_1 > \gamma_1 \geq 0 \text{ or } \alpha_2 > \gamma_2 \geq 0 \text{ except in the previous case,} \\ C_{\gamma,\alpha} T^{\gamma-\alpha}_\Omega 1(z) & \text{otherwise,} \end{cases}$$

where D^α stands for the weak derivative in Ω. The constants satisfy $|C_{\gamma,\alpha}| \lesssim (|\gamma| + n)^n$.

Proof. Let us assume that $\gamma_2 \geq 0$. If $\gamma_1 \geq 0$ as well, differenciating a polynomial under the integral sign makes the proof trivial, so we assume $\gamma_1 \leq -1$. Recall that we write $w^\gamma = w^{\gamma_1} \overline{w}^{\gamma_2}$. For every $z \in \Omega$ choose $\varepsilon_z := \mathrm{dist}(z, \partial\Omega)/2$. By (3.3), Green's formula and (3.2) we get that

$$T^\gamma \chi_\Omega(z) = \int_{\Omega \setminus B(z, \varepsilon_z)} (z-w)^\gamma \, dm(w) = \frac{i}{2(\gamma_2+1)} \int_{\partial\Omega} (z-w)^{\gamma+(0,1)} \, dw \qquad (3.37)$$

and we can differenciate under the integral sign.

If $\gamma_2 \geq \alpha_2$, then we have

$$D^\alpha T^\gamma \chi_\Omega(z) = \frac{i}{2(\gamma_2+1)}(-1)^{\alpha_1}\frac{(\gamma_2+1)!}{(\gamma_2-\alpha_2+1)!}\frac{(-\gamma_1+\alpha_1-1)!}{(-\gamma_1-1)!}\int_{\partial\Omega}(z-w)^{\gamma-\alpha+(0,1)}\,dw.$$

Since $\gamma_2-\alpha_2 \geq 0$, we can apply (3.37) to $\gamma-\alpha$ instead of γ and, thus,

$$D^\alpha T^\gamma \chi_\Omega(z) = (-1)^{\alpha_1}\frac{(\gamma_2)!}{(\gamma_2-\alpha_2)!}\frac{(-\gamma_1+\alpha_1-1)!}{(-\gamma_1-1)!}T^{\gamma-\alpha}\chi_\Omega(z).$$

If $\gamma_2+1=\alpha_2$ we must pay special attention. In that case differentiating under the integral sign in (3.37) we get

$$\begin{aligned} D^\alpha T^\gamma \chi_\Omega(z) &= \frac{i}{2}(-1)^{\alpha_1}\frac{(\gamma_2)!}{(\gamma_2-\alpha_2+1)!}\frac{(-\gamma_1+\alpha_1-1)!}{(-\gamma_1-1)!}\int_{\partial\Omega}(z-w)^{\gamma-\alpha+(0,1)}\,dw \\ &= C_{\gamma,\alpha}\int_{\partial\Omega}\frac{1}{(z-w)^{-\gamma_1+\alpha_1}}\,dw, \end{aligned}$$

where $|C_{\gamma,\alpha}| \lesssim (|\gamma|+n)^n$. If, moreover, $\gamma_1-\alpha_1 \leq -2$, we can use (3.2) to write

$$D^\alpha T^\gamma \chi_\Omega(z) = C_{\gamma,\alpha}\int_{\partial\Omega\cup\partial B(0,\varepsilon_z)}\frac{1}{(z-w)^{-\gamma_1+\alpha_1}}\,dw = C_{\gamma,\alpha}\int_{\Omega\setminus\partial B(0,\varepsilon_z)}0\,dm(w) = 0 \tag{3.38}$$

Otherwise, that is, if $\gamma_2+1=\alpha_2$ and $\gamma_1-\alpha_1=-1$, then $\alpha=(0,n)$ and $\gamma=(-1,n-1)$. This implies that

$$D^\alpha T^\gamma \chi_\Omega(z) = C_n\int_{\partial\Omega}\frac{1}{(z-w)}\,dw = C_n\chi_\Omega(z), \tag{3.39}$$

with $|C_n| \leq (n-1)!$. Let us remark the fact that $\gamma=(-1,0)$ together with $\alpha=(0,1)$ is the case of the $\overline{\partial}$-derivative of the Cauchy transform, which is the identity.

Finally, if $\gamma_2 < \alpha_2-1$, then differentiating (3.38) or (3.39) we get

$$D^\alpha T^\gamma \chi_\Omega(z) = 0.$$

One can argue analogously if $\gamma_1 \geq 0$. □

Remark 3.13. *If Ω is a $(\delta,R)-C^{(n-1,1)}$ domain, by Definition 2.1 we have that for every $z\in\partial\Omega$, the parameterization A_z of the boundary around z satisfies $A_z(0)=0$ and $A_z'(0)=0$. Multiplying by an appropriate bump function, we can see that there is no loss of generality in assuming that the parameterizations A_z are compactly supported in $[-2R,2R]$ (possibly increasing the value for δ).*

By the preceeding remarks, Lemma 2.12 and other standard arguments, one gets the following theorem, which is the keystone of this paper.

Theorem 3.14. *Let Ω be a (δ,R)-$C^{n-1,1}$ domain with parameterizations in $B^{n+1-1/p}_{p,p}$. Then, for any $\gamma\in\mathbb{Z}^2\setminus\{(-1,-1)\}$ with $\gamma_1+\gamma_2=-2$, we have that $T^\gamma\chi_\Omega\in W^{n,p}(\Omega)$ and, in particular, for any $\epsilon>0$, we have that*

$$\|\nabla^n T^\gamma \chi_\Omega\|^p_{L^p(\Omega)} \lesssim C_\epsilon|\gamma|^{np}\left(\|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)}+(1+\epsilon)^{|\gamma|p}\right),$$

where C_ϵ depends on p, n, δ, R, the length of the boundary $\mathcal{H}^1(\partial\Omega)$ and ϵ but not on $|\gamma|$.

3.5 The case $p > 2$

From [PT15, Theorem 1.1], we have the following corollary.

Corollary. *Let $\Omega \subset \mathbb{C}$ be a Lipschitz domain, let $\gamma \in \mathbb{Z}^2 \setminus (-1,-1)$ with $\gamma_1 + \gamma_2 = -2$ and let $p > 2$. Then the following statements are equivalent:*

a) The truncated operator T^γ_Ω is bounded in $W^{n,p}(\Omega)$.

b) For every polynomial P of degree at most $n-1$, we have that $T^\gamma_\Omega(P) \in W^{n,p}(\Omega)$.

We will use a quantitative version of this corollary. We state it below without proof. We refer the reader to [PT15, pages 2965–2969] for the details.

Let us fix some notation. Given a multiindex $\lambda \in \mathbb{N}_0^2$, we write $P^\lambda(z) = z^{\lambda_1}\bar{z}^{\lambda_2}$, that is, $P^\lambda(z) = z^\lambda$.

Corollary 3.15. *Let $\Omega \subset \mathbb{C}$ be a Lipschitz domain, let $\gamma \in \mathbb{Z}^2 \setminus (-1,-1)$ with $\gamma_1 + \gamma_2 = -2$ and let $p > 2$. Then*

$$\|\nabla^n T^\gamma_\Omega f\|_{L^p(\Omega)} \lesssim_n \left(\|T^\gamma\|_{L^p \to L^p} + C_\delta \|K_\gamma\|_{CZ} + \sum_{|\lambda|<n} \|\nabla^n T^\gamma_\Omega P_\lambda\|_{L^p(\Omega)} \right) \|f\|_{W^{n,p}(\Omega)}, \tag{3.40}$$

where we wrote

$$\|K_\gamma\|_{CZ} = \sup_{j \le n, z \in \mathbb{C}} |\nabla^j K_\gamma(z)| |z|^{j+2}.$$

Using Theorem 3.14 and Corollary 3.15, we will prove the following theorem.

Theorem 3.16. *Consider $p > 2$, $n \ge 1$ and let Ω be a Lipschitz domain with parameterizations in $B^{n+1-1/p}_{p,p}$. Then, for every $\epsilon > 0$ there exists a constant C_ϵ such that for every multiindex $\gamma \in \mathbb{Z}^2 \setminus \{(-1,-1)\}$ with $\gamma_1 + \gamma_2 \ge -2$, one has*

$$\|T^\gamma_\Omega\|_{W^{n,p}(\Omega) \to W^{n+\gamma_1+\gamma_2+2,p}(\Omega)} \le C_\epsilon |\gamma|^{n+\gamma_1+\gamma_2+2} \left(\|N\|_{B^{n-1/p}_{p,p}(\partial\Omega)} + (1+\epsilon)^{|\gamma|} \right) + \operatorname{diam}(\Omega)^{\gamma_1+\gamma_2+2}. \tag{3.41}$$

In particular, for every $m \in \mathbb{N}$ we have that the iteration of the Beurling transform $(\mathcal{B}^m)_\Omega$ is bounded in $W^{n,p}(\Omega)$, with norm

$$\|(\mathcal{B}^m)_\Omega\|_{W^{n,p}(\Omega) \to W^{n,p}(\Omega)} \le C_\epsilon m^{n+1} \left(\|N\|_{B^{n-1/p}_{p,p}(\partial\Omega)} + (1+\epsilon)^m \right). \tag{3.42}$$

Proof. Note that by (2.14), we have that $B^{n+1-1/p}_{p,p} \subset B^{n+1-2/p}_{\infty,\infty}$ and, since $1 - 2/p > 0$, we also have that $B^{n+1-2/p}_{\infty,\infty} = C^{n,1-2/p}$ (see [Tri83, Section 2.5.7]) so Ω is in fact a (δ, R)-$C^{n-1,1}$-domain, where δ and R depend on the size of the local parameterizations of the boundary and on $\|N\|_{\dot{B}^{n-1/p}(\partial\Omega)} + \mathcal{H}^1(\partial\Omega)$, and we can use Theorem 3.14.

First we study the case $\gamma_1 + \gamma_2 + 2 = 0$. Consider a given $\gamma \in \mathbb{Z}^2 \setminus \{(-1,-1)\}$ with $\gamma_1 + \gamma_2 = -2$. Recall that $\mathcal{B}^m = \frac{(-1)^m m}{\pi} T^{(-m-1,m-1)}$ by (2.19). The proof of the L^p boundedness of these operators with norm smaller than Cm^2 can be found in [AIM09, Section 4]. Thus,

$$\|T^\gamma\|_{L^p \to L^p} = \frac{\pi}{m} \|\mathcal{B}^m\|_{L^p \to L^p} \lesssim m = \frac{|\gamma|}{2}. \tag{3.43}$$

On the other hand, a short computation shows that

$$\|K_\gamma\|_{CZ} = \sup_{j \le n, z \in \mathbb{C}} |\nabla^j K_\gamma(z)| |z|^{j+2} \lesssim |\gamma|^n, \tag{3.44}$$

with constant depending on n.

In order to use Corollary 3.15, it only remains to check the bounds for $\left\|D^\alpha T^\gamma_\Omega P^\lambda\right\|_{L^p(\Omega)}$ for all multiindices $\alpha,\lambda\in\mathbb{N}_0^2$ with $|\alpha|=n$ and $|\lambda|<n$. Using the binomial expansion $w^\lambda=\sum_{\nu\leq\lambda}(-1)^{|\nu|}\binom{\lambda}{\nu}(z-w)^\nu z^{\lambda-\nu}$, we can write

$$T^\gamma_\Omega P_\lambda(z)=\lim_{\varepsilon\to 0}\int_{\mathbb{C}\setminus B_\varepsilon(z)}(z-w)^\gamma w^\lambda\,dm(w)=\sum_{\vec{0}\leq\nu\leq\lambda}(-1)^{|\nu|}\binom{\lambda}{\nu}z^{\lambda-\nu}T^{\gamma+\nu}\chi_\Omega(z).$$

Differentiating (and assuming that $0\in\Omega$) we find that

$$|\nabla^n T^\gamma_\Omega P_\lambda(z)|\lesssim 2^n\sum_{\vec{0}\leq\nu\leq\lambda}\sum_{j=0}^n(1+\operatorname{diam}(\Omega))^n|\nabla^j T^{\gamma+\nu}\chi_\Omega(z)|$$

and, thus, by the equivalence of norms in the Sobolev space (2.2), we have that

$$\|\nabla^n T^\gamma_\Omega P_\lambda\|^p_{L^p(\Omega)}\lesssim_\Omega\sum_{\vec{0}\leq\nu\leq\lambda}\left(\left\|\nabla^{n+|\nu|}T^{\gamma+\nu}\chi_\Omega\right\|^p_{L^p(\Omega)}+\left\|T^{\gamma+\nu}\chi_\Omega\right\|^p_{L^p(\Omega)}\right),$$

with constants depending on n, p and the diameter and the Sobolev embedding constant of Ω. By Remark 3.12, Theorem 3.14 and (3.43), we have that

$$\|\nabla^n T^\gamma_\Omega P_\lambda\|^p_{L^p(\Omega)}\lesssim\sum_{\gamma\leq\nu\leq\gamma+\lambda}|\nu|^{np}\left(\|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)}+(1+\epsilon)^{|\nu|p}\right)+\sum_{\gamma<\nu\leq\gamma+\lambda}\|T^\nu\chi_\Omega\|^p_{L^p(\Omega)}.\tag{3.45}$$

The Young Inequality says that for all functions $f\in L^p$ and $g\in L^1$, $\|f*g\|_{L^p}\leq\|f\|_{L^p}\|g\|_{L^1}$. Thus, since $\chi_\Omega K_\nu\in L^1$, for $\gamma<\nu\leq\gamma+\lambda$ we have that

$$\|T^\nu_\Omega f\|_{L^p}\leq\operatorname{diam}(\Omega)^{\nu_1+\nu_2+2}\|f\|_{L^p},\tag{3.46}$$

and taking $f=\chi_\Omega$, $\|T^\nu\chi_\Omega\|^p_{L^p}\leq\operatorname{diam}(\Omega)^{(n-1)p+2}$.

Since $p>2$, putting (3.40), (3.43), (3.44), (3.45) and (3.46) together, we get

$$\begin{aligned}\|\nabla^n T^\gamma_\Omega\|_{W^{n,p}(\Omega)\to L^p(\Omega)}&\lesssim|\gamma|+|\gamma|^n+|\gamma|^n\left(\|N\|_{B^{n-1/p}_{p,p}(\partial\Omega)}+(1+\epsilon)^{|\gamma|}\right)\\&\lesssim|\gamma|^n\left(\|N\|_{B^{n-1/p}_{p,p}(\partial\Omega)}+(1+\epsilon)^{|\gamma|}\right),\end{aligned}$$

with constants depending on n, p, δ, the diameter of Ω, its Sobolev embedding constant and ϵ, but not on γ. This proves (3.41) when $\gamma_1+\gamma_2=-2$ and (3.42) for every $m>0$.

It remains to study the operators of homogeneity greater than -2. In that case we will see that we can differentiate under the integral sign to recover the previous situation. Fix $\gamma\in\mathbb{Z}^2$ such that $\gamma_1+\gamma_2+2>0$. By (3.46) we have that $\|T^\gamma_\Omega f\|_{L^p}\leq\operatorname{diam}(\Omega)^{\gamma_1+\gamma_2+2}\|f\|_{L^p}$. Thus, to prove (3.41), one only needs to see that for $f\in W^{n,p}(\Omega)$ one has

$$\left\|\nabla^{n+\gamma_1+\gamma_2+2}T^\gamma_\Omega f\right\|_{L^p(\Omega)}\leq C_\epsilon|\gamma|^{np}\left(\|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)}+(1+\epsilon)^{|\gamma|p}\right)\|f\|_{W^{n,p}(\Omega)}.$$

By (3.40) it is enough to check that for any $\nu\in\mathbb{N}_0^2$ with $|\nu|=\gamma_1+\gamma_2+2$, one has

$$D^\nu T^\gamma_\Omega f(z)=\begin{cases}C_n\chi_\Omega f(z) & \text{if } \gamma=(|\nu|-1,-1)\text{ and }\nu=(|\nu|,0)\\ & \quad\text{or }\gamma=(-1,|\nu|-1)\text{ and }\nu=(0,|\nu|)\\ 0 & \text{if }\nu_1>\gamma_1>0\text{ or }\nu_2>\gamma_2>0\text{ except in the previous case,}\\ C_{\nu,\gamma}T^{\gamma-\nu}_\Omega f(z) & \text{otherwise.}\end{cases}\tag{3.47}$$

To prove this statement, one can express the partial derivative as

$$\partial_x T^\gamma_\Omega f(z) = \lim_{h\to 0} \frac{T^\gamma_\Omega(f - f(z))(z+h) - T^\gamma_\Omega(f - f(z))(z)}{h} + \partial_x T^\gamma_\Omega \chi_\Omega(z) f(z)$$

and use that $f \in C^{0,\varepsilon}$ for a certain $\varepsilon > 0$ by the Sobolev Embedding Theorem, allowing one to apply the Dominated Convergence Theorem to differenciate under the integral sign. The same happens for ∂_y. The details are left to the reader (see Remark 3.12). □

Acknowledgements

The author was funded by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement 320501. Also, partially supported by grants 2014-SGR-75 (Generalitat de Catalunya), MTM-2010-16232 and MTM-2013-44304-P (Spanish government) and by a FI-DGR grant from the Generalitat de Catalunya, (2014FI-B2 00107).

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