A CONTINUOUS RATING METHOD
FOR PREFERENTIAL VOTING. THE COMPLETE CASE

Rosa Camps, Xavier Mora and Laia Saumell

Departament de Matemàtiques, Universitat Autònoma de Barcelona,
Catalonia, Spain

xmora @ mat.uab.cat

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Abstract

A method is given for quantitatively rating the social acceptance of different options which are the matter of a complete preferential vote. Completeness means that every voter expresses a comparison (a preference or a tie) about each pair of options. The proposed method is proved to have certain desirable properties, which include: the continuity of the rates with respect to the data, a decomposition property that characterizes certain situations opposite to a tie, the Condorcet-Smith principle, and clone consistency. One can view this rating method as a complement for the ranking method introduced in 1997 by Markus Schulze. It is also related to certain methods of one-dimensional scaling or cluster analysis.

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The outcome of a vote is commonly expected to specify not only a winner and an ordering of the candidates, but also a quantitative estimate of the social acceptance of each of them. Such a quantification is expected even when the individual votes give only qualitative information.

The simplest voting methods are clearly based upon such a quantification. This is indeed the case of the plurality count as well as that of the Borda count. However, it is well known that these methods do not comply with basic majority principles nor with other desirable conditions. In order to satisfy certain combinations of such principles and conditions, one must resort to other more elaborate methods, such as the celebrated rule of Condorcet, Kemény and Young [12; 20: p. 182–190], the method of ranked

pairs [$19, 21$; 20: p. 219–223], or the method introduced in 1997 by Markus Schulze [$15, 16$; 20: p. 228–232], which we will refer to as the method of paths. Now, as they stand, these methods rank the candidates in a purely ordinal way, without properly quantifying the social acceptance of each of them. So, it is natural to ask for a method that combines the above-mentioned principles and conditions with a quantitative rating of the candidates.

A quantitative rating should allow to sense the closeness between two candidates, such as the winner and the runner-up. For that purpose, it is essential that the rates vary in a *continuous* way, especially through situations where ties or multiple orders occur.

On the other hand, it should also allow to recognise certain situations that are opposite to a tie. For instance, a candidate should get the best possible rate *if and only if* it has been placed first by all voters. This is a particular case of a more general condition that we will call *decomposition.* This condition, that will be made precise later on, places sharp constraints on the rates that should be obtained when the candidates are partitioned in two classes X and Y such that each member of X is unanimously preferred to every member of Y.

In this article we will produce a rating method that combines such a quantitative character with other desirable properties of a qualitative nature. Among them we will be especially interested in the following extension of the Condorcet principle introduced in 1973 by John H. Smith [17]: Assume that the set of candidates is partitioned in two classes X and Y such that for each member of X and every member of Y there are more than half of the individual votes where the former is preferred to the latter; in that case, the social ranking should also prefer each member of X to any member of Y. This principle is quite pertinent when one is interested not only in choosing a winner but also in ranking all the alternatives (or in rating them).

To our knowledge, the existing literature does not offer any other rating method that combines this principle with the above-mentioned quantitative properties of continuity and decomposition. We will refer to our method as the **CLC rating method**, where the capital letters stand for "Continuous Llull Condorcet". The reader interested to try it can use the *CLC calculator* which has been made available at [13].

Of course, any rating automatically implies a ranking. In this connection, it should be noticed that the CLC rating method is built upon Schulze's method of paths as underlying ranking method. As we will remark in the concluding section, we doubt that any of the other ranking methods mentioned above could be extended to a rating method with the properties of

continuity and decomposition. Having said that, it should be clear that the present work is not aimed at saying anything new about ranking methods as such. Whatever we might say about them will always be in reference to the rating issue.

By reasons of space, this article is restricted to the complete case and to a particular class of rates that we call rank-like rates. We are in the complete case when every individual expresses a comparison (a preference or a tie) about each pair of options. The incomplete case requires some additional developments that are dealt with in a separate article [3]. In another separate article we deal with another class of rates that have a fraction-like character [4].

The present article is organized as follows: Section 1 gives a more precise statement of the problem and finishes with a general remark. Section 2 presents an heuristic outline of our proposal, ending with a summary of the procedure and an illustrative example. Section 3 introduces some mathematical language. Sections 4–10 give detailed mathematical proofs of the claimed properties. Sections 11–12 are devoted to other interesting properties of the concomitant social ranking, namely clone consistency and two weak forms of monotonicity. Finally, section 13 makes some concluding remarks and poses a few open questions.

1   Statement of the problem and a general remark

1.1   Let us consider a set of N options which are the matter of a vote. Let us assume that each voter expresses his preferences in a qualitative way, for instance by listing those options in order of preference. Our aim is to combine such individual preferences so as to rate the social acceptance of each option on a continuous scale. More specifically, we would like to do it in accordance with the following conditions:

A   *Scale invariance (homogeneity).* The rates depend only on the relative frequency of each possible content of an individual vote. In other words, if every individual vote is replaced by a fixed number of copies of it, the rates remain exactly the same.

B   *Permutation equivariance (neutrality).* Applying a certain permutation of the options to all of the individual votes has no other effect than getting the same permutation in the social rating.

C   *Continuity.* The rates depend continuously on the relative frequency of each possible content of an individual vote.

The next two conditions choose a specific form of rating. From now on we will refer to it as rank-like rating. In the complete case considered in the present paper, these two conditions take the following form:

D   *Rank-like form (complete case).* Each rank-like rate is a number, integer or fractional, between 1 and N. The best possible value is 1 and the worst possible one is N. The average rank-like rate is $(N+1)/2$.

E   *Rank-like decomposition (complete case).* Consider a splitting of the options in two classes X and Y. Consider the case where each member of X is unanimously preferred to every member of Y. This fact is equivalent to each of the following ones, where $|X|$ denotes the number of elements of X: (a) The rank-like rates of X coincide with those that one obtains when the individual votes are restricted to X. (b) After diminishing them by the number $|X|$, the rank-like rates of Y coincide with those that one obtains when the individual votes are restricted to Y. (c) The average rank-like rate of X is $(|X|+1)/2$.

In particular, an option will get a rank-like rate exactly equal to 1 if and only if it is unanimously preferred to any other. Similarly, an option will get a rank-like rate exactly equal to N if and only if it is unanimously considered worse than any other.

Finally, we require a condition that concerns only the concomitant social ranking, that is, the ordinal information contained in the social rating:

M   *Condorcet-Smith principle.* Consider a splitting of the options in two classes X and Y. Assume that for each member of X and every member of Y there are more than half of the individual votes where the former is preferred to the latter. In that case, the social ranking also prefers each member of X to every member of Y.

1.2 Let us emphasize that the individual votes that we are dealing with do *not* have a quantitative character (at least for the moment): each voter is allowed to express a preference for x rather than y, or vice versa, or maybe a tie between them, but he is not allowed to quantify such a preference.

This contrasts with "range voting" methods, where each individual vote is already a quantitative rating [20: p. 174–176; 1]. Such methods are free from many of the difficulties that lurk behind the present setting. However, they make sense only as long as all voters mean the same by each possible value of the rating variable. This hypothesis may be reasonable in some cases, but in many others it is hardly valid. In fact, voting is often used in connection

with moral, psychological or aesthetic qualities, whose appreciation may be as little quantifiable, but also as much "comparable", as, for instance, the feelings of pleasure or pain.

In our case, the quantitative character of the output will derive from the fact of having a certain number of qualitative preferences. The larger this number, the more meaningful will be the quantitative character of the social rating. This is especially applicable to the continuity property C, according to which a small variation in the proportion of votes with a given content produces only small variations in the rates. In fact, a few votes will be a small proportion only in the measure that the total number of votes is large enough.

2   Heuristic outline

This section presents our proposal as the result of a quest for the desired properties. Hopefully, this will communicate the main ideas that lie behind the formulas.

2.1   The aim of complying with condition M calls for the point of view of **paired comparisons**. So our starting point will be the numbers V_{xy} that count how many voters prefer x to y. In this connection we will take the view that each vote that ties x with y is equivalent to half a vote preferring x to y plus another half a vote preferring y to x. In order to achieve scale invariance, we will immediately switch to the corresponding fractions $v_{xy} = V_{xy}/V$, where V denotes the total number of votes. The numbers v_{xy} will be called the binary **scores** of the vote, and their collection will be called the **Llull matrix** of the vote. Since we are considering the case of complete votes, these numbers are assumed to satisfy

$$v_{xy} + v_{yx} \,=\, 1. \tag{1}$$

Besides the scores v_{xy}, in the sequel we will often deal with the **margins** m_{xy}, which are defined by

$$m_{xy} \,=\, v_{xy} - v_{yx}. \tag{2}$$

Obviously, their dependence on the pair xy is antisymmetric, that is

$$m_{yx} \,=\, -m_{xy}. \tag{3}$$

It is clear also that the equality (1) allows to recover the scores from the margins by means of the following formula:

$$v_{xy} \,=\, (1 + m_{xy})/2. \tag{4}$$

2.2   A natural candidate for defining the social preference is the following: x is socially preferred to y whenever $v_{xy} > v_{yx}$. Of course, it can happen that $v_{xy} = v_{yx}$, in which case one would consider that x is socially tied with y. The binary relation that includes all pairs xy for which $v_{xy} > v_{yx}$ will be denoted by $\mu(v)$ and will be called the **comparison relation**; together with it, we will consider also the relation $\hat{\mu}(v)$ defined by the non-strict inequality $v_{xy} \geq v_{yx}$.

As it is well known, the main problem with paired comparisons is that the comparison relations $\mu(v)$ and $\hat{\mu}(v)$ may lack transitivity even if the individual preferences are all of them transitive [11, 20].

2.3   The next developments rely upon an operation $(v_{xy}) \rightarrow (v_{xy}^*)$ that transforms the original system of binary scores into a new one. This operation is defined in the following way: for every pair xy, one considers all possible paths $x_0 x_1 \ldots x_n$ going from $x_0 = x$ to $x_n = y$; every such path is associated with the score of its weakest link, i.e. the smallest value of $v_{x_i x_{i+1}}$; finally, v_{xy}^* is defined as the maximum value of this associated score over all paths from x to y. In other words,

$$v_{xy}^* \quad = \quad \max_{\substack{x_0 = x \\ x_n = y}} \quad \min_{\substack{i \geq 0 \\ i < n}} \quad v_{x_i x_{i+1}}, \tag{5}$$

where the max operator considers all possible paths from x to y, and the min operator considers all the links of a particular path. The scores v_{xy}^* will be called the **indirect scores** associated with the (direct) scores v_{xy}.

If (v_{xy}) is the table of 0's and 1's associated with a binary relation ρ (by putting $v_{xy} = 1$ if and only if $xy \in \rho$), then (v_{xy}^*) is exactly the table associated with ρ^*, the transitive closure of ρ. So, the operation $(v_{xy}) \mapsto (v_{xy}^*)$ can be viewed as a quantitative analogue of the notion of transitive closure (see [6 : Ch. 25]).

The main point, remarked in 1998 by Markus Schulze [15 b], is that the comparison relation associated with a table of indirect scores is always transitive (Theorem 4.3). So, $\mu(v^*)$ is always transitive, no matter what the case is for $\mu(v)$. This is true in spite of the fact that $\mu(v^*)$ can easily differ from $\mu^*(v)$ (the transitive closure of $\mu(v)$).

2.4   In the following we put

$$\kappa = \mu(v^*), \qquad \hat{\kappa} = \hat{\mu}(v^*), \qquad m_{xy}^\kappa = v_{xy}^* - v_{yx}^*. \tag{6}$$

So, $xy \in \kappa$ if and only if $v_{xy}^* > v_{yx}^*$, i. e. $m_{xy}^\kappa > 0$, and $xy \in \hat{\kappa}$ if and only if $v_{xy}^* \geq v_{yx}^*$, i. e. $m_{xy}^\kappa \geq 0$. From now on we will refer to κ as the **indirect comparison relation**, and to m_{xy}^κ as the **indirect margin** associated with the pair xy.

As it has been stated above, the relation κ is transitive. Besides that, it is clearly asymmetric (one cannot have both $v_{xy}^* > v_{yx}^*$ and vice versa). On the other hand, it may be incomplete (one can have $v_{xy}^* = v_{yx}^*$). When it differs from κ, the complete relation $\hat{\kappa}$ is not asymmetric and —somewhat surprisingly— it may be not transitive either. However, one can always find a total order ξ that satisfies $\kappa \subseteq \xi \subseteq \hat{\kappa}$ (Theorem 5.1). From now on, any total order ξ that satisfies this condition will be called an **admissible order**. Let us remark that such a definition is redundant: in fact, one easily sees that each of the required inclusions implies the other one.

The rating that we are looking for will be based on such an order ξ. More specifically, it will be compatible with ξ in the sense that the rank-like rates R_x will satisfy the inequality $R_x \leq R_y$ whenever $xy \in \xi$. If κ is already a total order, so that $\xi = \kappa$, the preceding inequality will be satisfied in the strict form $R_x < R_y$, and this will happen if and only if $xy \in \kappa$.

The following steps assume that one has fixed an admissible order ξ. From now on the situation $xy \in \xi$ will be expressed also by $x \overset{\xi}{\succ} y$. According to the definitions, the inclusions $\kappa \subseteq \xi \subseteq \hat{\kappa}$ are equivalent to saying that $v_{xy}^* > v_{yx}^*$ implies $x \overset{\xi}{\succ} y$ and that the latter implies $v_{xy}^* \geq v_{yx}^*$. In other words, if the different options are ordered according to $\overset{\xi}{\succ}$, the matrix v_{xy}^* has then the property that each element above the diagonal is larger than or equal to its symmetric over the diagonal.

2.5   Rating the different options means positioning them on a line. Besides complying with the qualitative restriction of being compatible with ξ in the sense above, we want that the distances between items reflect the quantitative information provided by the binary scores. However, a rating is expressed by N numbers, whereas the binary scores are $N(N-1)$ numbers. So we are bound to do some sort of projection. Problems of this kind have a certain tradition in combinatorial data analysis and cluster analysis [10, 8]. In fact, some of the operations that will be used below can be viewed from that point of view.

Let us assume for a while that the votes are total orders, i. e. each vote lists all the options by order of preference, without any ties. This is the standard case for the application of Borda's method, which is linearly equivalent to rating each option by the mean value of its ranks, i. e. the ordinal numbers

that give its position in these different orders. As it was noticed by Borda himself (in his setting linearly related to ours), these mean ranks, which we will denote by $\bar{r}_x$, can be obtained from the Llull matrix by means of the following formula:

$$\bar{r}_x \;=\; N - \sum_{y \neq x} v_{xy}, \tag{7}$$

or equivalently,

$$\bar{r}_x \;=\; (N + 1 - \sum_{y \neq x} m_{xy})\,/\,2. \tag{8}$$

Let us look at the meaning of the margins m_{xy} in connection with the idea of projecting the Llull matrix into a rating: If there are no other items than x and y, we can certainly view the sign and magnitude of m_{xy} as giving respectively the qualitative and quantitative aspects of the relative positions of x and y on the rating line, that is, the order and the distance between them. When there are more than two items, however, we have several pieces of information of this kind, one for every pair, and these different pieces may be incompatible with each other, quantitatively or even qualitatively, which motivates indeed the problem that we are dealing with. In particular, the mean ranks $\bar{r}_x$ often violate the desired compatibility with the relation ξ.

In order to construct a rating compatible with ξ, we will use a formula analogous to (7) where the scores v_{xy} are replaced by certain **projected scores** v_{xy}^{π} to be defined in the following paragraphs. Together with them, we will make use of the corresponding **projected margins** $m_{xy}^{\pi} = v_{xy}^{\pi} - v_{yx}^{\pi}$. Like the original scores (but not necessarily the indirect ones) the projected scores will be required to satisfy the equality $v_{xy}^{\pi} + v_{yx}^{\pi} = 1$, from which it follows that $v_{xy}^{\pi} = (1 + m_{xy}^{\pi})/2$. So, the rates that we are looking for will be obtained in the following way:

$$R_x \;=\; N - \sum_{y \neq x} v_{xy}^{\pi}, \tag{9}$$

or equivalently,

$$R_x \;=\; (N + 1 - \sum_{y \neq x} m_{xy}^{\pi})\,/\,2. \tag{10}$$

Such formulas will be used not only in the case where the votes are total orders, but also in more general situations.

2.6   Our goals will be achieved by defining the projected margins in the following way, where we assume $x \overset{\xi}{\succ} y$ and x' denotes the item that imme-

diately follows x in the total order ξ:

$$m^\kappa_{xy} = v^*_{xy} - v^*_{yx}, \tag{11}$$

$$m^\sigma_{xy} = \min\{ m^\kappa_{pq} \mid p \stackrel{\xi}{\succeq} x, \; y \stackrel{\xi}{\succeq} q \}, \tag{12}$$

$$m^\pi_{xy} = \max\{ m^\sigma_{pp'} \mid x \stackrel{\xi}{\succeq} p \stackrel{\xi}{\succ} y \}, \tag{13}$$

$$m^\pi_{yx} = -m^\pi_{xy}. \tag{14}$$

As one can easily check, this construction ensures that

$$m^\pi_{xz} = \max(m^\pi_{xy}, m^\pi_{yz}), \qquad \text{whenever } x \stackrel{\xi}{\succ} y \stackrel{\xi}{\succ} z. \tag{15}$$

From this equality it follows that the absolute values $d_{xy} = |m^\pi_{xy}|$ satisfy the following inequality, which makes no reference to the relation ξ:

$$d_{xz} \le \max(d_{xy}, d_{yz}), \qquad \text{for any } x, y, z. \tag{16}$$

This condition, called the ultrametric inequality, is well known in cluster analysis, where it appears as a necessary and sufficient condition for the dissimilarities d_{xy} to define a hierarchical classification of the set under consideration [10: §7.2; 8: §3.2.1].

Remark. The operation $(m^\kappa_{xy}) \to (m^\pi_{xy})$ defined by (12–13) is akin to the single-link method of cluster analysis, which can be viewed as a continuous method for projecting a matrix of dissimilarities onto the set of ultrametric distances; such a continuous projection is achieved by taking the maximal ultrametric distance which is bounded by the given matrix of dissimilarities [10: §7.3, 7.4, 8.3, 9.3]. The operation $(m^\kappa_{xy}) \to (m^\pi_{xy})$ does the same kind of job under the constraint that the clusters be intervals of the total order ξ.

2.7 Summary of the procedure

0. Form the Llull matrix (v_{xy}) (§ 2.1).

1. Compute the indirect scores v^*_{xy} defined by (5). An efficient way to do it is the Floyd-Warshall algorithm [6: § 25.2]. Work out the indirect margins $m^\kappa_{xy} = v^*_{xy} - v^*_{yx}$.

2. Consider the indirect comparison relation $\kappa = \{xy \mid m^\kappa_{xy} > 0\}$. Fix an admissible order ξ, i.e. a total order that extends κ. For instance, it suffices to arrange the options by non-decreasing values of the "tie-splitting" Copeland scores $r_x = N - |\{ y \mid y \ne x, \; m^\kappa_{xy} > 0 \}| - \frac{1}{2}|\{ y \mid y \ne x, \; m^\kappa_{xy} = 0 \}|$ (Proposition 5.2).

3. Starting from the indirect margins m_{xy}^{κ}, work out the superdiagonal intermediate projected margins $m_{xx'}^{\sigma}$ as defined in (12).

4. Compute the projected margins m_{xy}^{π} according to (13–14). The projected scores are then determined by the formula $v_{xy}^{\pi} = (1 + m_{xy}^{\pi})/2$.

5. Compute the rank-like rates R_x according to (9) (here equivalent to (10)).

The computing time is of order N^3, where N is the number of options. The *CLC calculator* made available at [*13*] allows to follow the details of the procedure by choosing the option "Detailed mode".

2.8 Example. As an illustrative example we will consider the final round of a dancesport competition. Specifically, we have chosen the Professional Latin Rising Star section of the 2007 Blackpool Dance Festival (Blackpool, England, 25th May 2007). The data were taken from `http://www.scrutelle.info/results/estelle/2007/blackpool_2007/`. As usual, the final was contested by six couples, whose competition numbers were $3, 4, 31, 122, 264$ and 238. Eleven adjudicators ranked their simultaneous performances in four equivalent dances.

The all-round official result was $3 \succ 122 \succ 264 \succ 4 \succ 31 \succ 238$. This result comes from the so-called "Skating System", whose name reflects a prior use in figure-skating. The Skating System has a first part which produces a separate result for each dance. This is done mainly on the basis of the median rank obtained by each couple (by the way, this criterion underlies the "practical" method that Condorcet was proposing in 1792/93 [*11* : ch. 8]). However, the fine properties of this criterion are lost in the second part of the Skating System, where the all-round result is obtained by adding up the final ranks obtained in the different dances.

From the point of view of paired comparisons, it makes sense to base the all-round result on the Llull matrix which collects the 44 rankings produced by the 11 adjudicators over the 4 dances. As one can see below, in the present case this matrix exhibits several Condorcet cycles, like for instance $3 \succ 4 \succ 264 \succ 3$ and $3 \succ 122 \succ 264 \succ 3$, which means that the competition was closely contested. In consonance with it, the CLC rates obtained below are quite close to each other, particularly for the couples $3, 4, 122$ and 264. By the way, the CLC result orders the contestants differently than the Borda rule, whose associated ordering is $122 \succ 3 \succ 4 \succ 264 \succ 31 \succ 238$.

Instead of relative scores and margins, the following tables show their absolute counterparts, i.e. without dividing by the total number of votes. This has the virtue of staying with small integer numbers.

ORIGINAL SCORES   INDIRECT SCORES   RANKS

x	V_{xy} 3	4	31	122	238	264	V^*_{xy} 3	4	31	122	238	264	r_x
3	*	**23**	**28**	**23**	**28**	20	*	23	**28**	23	**28**	23	4
4	21	*	**23**	20	**30**	**24**	24	*	24	23	**30**	24	2
31	16	21	*	15	**25**	18	21	21	*	21	**25**	21	5
122	21	**24**	**29**	*	**28**	**23**	24	24	29	*	**28**	24	1
238	16	14	19	16	*	19	19	19	19	19	*	19	6
264	**24**	20	**26**	21	**25**	*	24	23	**26**	23	**25**	*	3

INDIRECT MARGINS   PROJECTED MARGINS   RATES

x	M^{κ}_{xy} 122	4	264	3	31	238	M^{π}_{xy} 122	4	264	3	31	238	R_x
122	*	1	1	1	8	9	*	1	1	1	3	6	3.3636
4	*	*	1	1	3	11	*	*	1	1	3	6	3.3864
264	*	*	*	1	5	6	*	*	*	1	3	6	3.4091
3	*	*	*	*	7	9	*	*	*	*	3	6	3.4318
31	*	*	*	*	*	6	*	*	*	*	*	6	3.5682
238	*	*	*	*	*	*	*	*	*	*	*	*	3.8409

3   Mathematical setting

We consider a finite set A. Its elements represent the options which are the matter of a vote. The number of elements of A is N.

In order to deal with preferences we must consider (ordered) **pairs** of elements of A. The pair formed by a and b, in this order, will be denoted simply as ab. The pairs that consist of two copies of the same element, i.e. those of the form aa, are not relevant for our purposes. So, we will systematically exclude them from our considerations. This will help towards a more efficient language. The set of all proper pairs, i.e. the pairs ab with $a \neq b$, will be denoted as Π. Unless we say otherwise, from now on *any statement will be understood to imply the assumption that all the pairs that appear in it are proper pairs, i.e. they belong to Π*.

Besides pairs, we will be concerned also with longer sequences $a_0 a_1 \ldots a_n$, which will be referred to as **paths**.

3.1   As one could see in the heuristic outline, our developments make use of the mathematical concept of (binary) **relation**. Stating that two elements a and b are in a certain relation ρ is equivalent to saying that the pair ab is a member of a certain set ρ. Because of what has been said, unless we

say otherwise we will restrict our attention to strict relations, i. e. relations contained in Π. Under the convention made above, we can keep the following standard definitions.[1] A relation $\rho \subseteq \Pi$ will be called: **asymmetric** when $ab \in \rho$ implies $ba \notin \rho$; **total**, or **complete**, when $ab \notin \rho$ implies $ba \in \rho$; **transitive** when the simultaneous occurrence of $ab \in \rho$ and $bc \in \rho$ implies $ac \in \rho$; a **partial order** when it is transitive and asymmetric; a **total order** when it is transitive, asymmetric and total; a **total preorder** when it is transitive and total.

Given a relation ρ, we will often consider the relation $\hat{\rho}$ that consists of all pairs ab such that $ba \notin \rho$; $\hat{\rho}$ is called the **codual** of ρ. The following lemma collects several properties which are immediate consequences of the definitions:

Lemma 3.1.

    (a) $\hat{\hat{\rho}} = \rho$.

    (b) $\rho \subset \sigma \iff \hat{\sigma} \subset \hat{\rho}$.

    (c) ρ is asymmetric $\iff \rho \subseteq \hat{\rho} \iff \hat{\rho}$ is total.

    (d) ρ is total $\iff \hat{\rho} \subseteq \rho \iff \hat{\rho}$ is asymmetric.

The **transitive closure** of ρ, which we will denote as ρ^*, is defined as follows: $ab \in \rho^*$ if and only if there exists a path $a_0 a_1 \ldots a_n$ from $a_0 = a$ to $a_n = b$ such that $a_i a_{i+1} \in \rho$ for every i. ρ^* is the minimum transitive relation that contains ρ. The transitive-closure operator is easily seen to have the following properties: $\rho^* \subseteq \sigma^*$ whenever $\rho \subseteq \sigma$; $(\rho \cap \sigma)^* \subseteq (\rho^*) \cap (\sigma^*)$; $(\rho^*) \cup (\sigma^*) \subseteq (\rho \cup \sigma)^*$; $(\rho^*)^* = \rho^*$.

A subset $C \subseteq A$ is said to be **autonomous** for a relation ρ when, for any $x \notin C$, having $ax \in \rho$ for some $a \in C$ implies $bx \in \rho$ for any $b \in C$, and similarly, having $xa \in \rho$ for some $a \in C$ implies $xb \in \rho$ for any $b \in C$ (see for instance [2]). On the other hand, $C \subseteq A$ will be said to be an **interval** for a relation ρ when the simultaneous occurrence of $ax \in \rho$ and $xb \in \rho$ with $a, b \in C$ implies $x \in C$. The following facts are easy consequences of the definitions: If ρ is asymmetric and C is autonomous for ρ then C is an interval for ρ. If ρ is total and C is an interval for ρ then C is autonomous for ρ. As a corollary, if ρ is total and asymmetric, then C is autonomous for ρ if and only if it is an interval for that relation. Later on we will make use of the following fact, which is also an easy consequence of the definitions:

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[1]An equivalent way to put it is that we accept a relation $\rho \subseteq \Pi$ as total / transitive, whenever $\rho \cup \{xx \mid x \in A\}$ has the corresponding property in the standard sense.

Lemma 3.2. *C is autonomous for ρ $\Longleftrightarrow$ C is autonomous for $\hat{\rho}$.*

When C is an autonomous set for ρ, it is natural to consider a new set $\widetilde{A}$ and a new relation $\widetilde{\rho}$ by proceeding in the following way: $\widetilde{A}$ is obtained from A by replacing the set C by a single element $\widetilde{c}$, i.e. $\widetilde{A} = (A \setminus C) \cup \{\widetilde{c}\}$; for every $x \in A$, let us denote by $\widetilde{x}$ the element of $\widetilde{A}$ defined by $\widetilde{x} = \widetilde{c}$ if $x \in C$ and by $\widetilde{x} = x$ if $x \notin C$; with this notation, $\widetilde{\rho}$ is defined by putting $\widetilde{x}\widetilde{y} \in \widetilde{\rho}$ if and only if $xy \in \rho$ whenever $\widetilde{x} \neq \widetilde{y}$ (this definition is not ambiguous since C is autonomous for ρ). We will refer to this operation as the **contraction** of ρ by the autonomous set C.

Any relation can be interpreted as expressing a system of qualitative preferences: having $xy \in \rho$ and $yx \notin \rho$ means that x is preferred to y; having both $xy \in \rho$ and $yx \in \rho$ means that x is tied with y; having neither $xy \in \rho$ nor $yx \in \rho$ means that no information is given about the preference between x and y.

From this point of view, it is quite natural to rank the different $x \in A$ by taking into account the number of y such that x is preferred to y as well as the number of y such that x is tied with y. More precisely, it makes sense to define the **rank** of x in a relation ρ by the formula

$$r_x = N - \left| \{\, y \mid xy \in \rho, \; yx \notin \rho \} \right| - \tfrac{1}{2} \left| \{\, y \mid xy \in \rho, \; yx \in \rho \} \right|. \qquad (17)$$

Ranking by r_x is often considered in connection with tournaments. Such a method is known as the Copeland rule (see for instance $[20\colon \text{p. } 206\text{–}209]$, where the tie-splitting term is not present —since ties are not occurring— and an equivalent formulation is used). The next lemma is an easy consequence of the definitions. Its second part justifies using the term 'rank'.

Lemma 3.3. *If ρ is a partial order, then having $xy \in \rho$ implies $r_x < r_y$. If ρ is a total order, then having $xy \in \rho$ is equivalent to $r_x < r_y$; in fact, r_x coincides then with the ordinal number that gives the position of x in ρ.*

3.2 The Llull matrix of a vote, as well as the analogous matrices formed respectively by the indirect scores and by the projected scores, are all of them particular instances of the abstract notion of **valued relation** (also called 'fuzzy relation'). In fact, a valued relation on A means simply a mapping v whereby every pair $xy \in \Pi$ is assigned a score v_{xy} in the interval $[0, 1]$. The score v_{xy} measures "how much" is x related to y. Here and in the following we keep the notation and terminology introduced in the preceding sections.

Most of the notions that are associated with relations can be generalized to valued relations (sometimes in several different ways). Some of these generalized notions have already appeared in the heuristic outline of § 2. For instance, we have already remarked that the indirect scores v_{xy}^* generalize the notion of transitive closure. Another generalized notion, namely that of autonomous set, will appear in § 11.

Here we will only remark that for our purposes the valued analogues of asymmetry and totality are respectively the conditions $v_{xy} + v_{yx} \leq 1$ and $v_{xy} + v_{yx} \geq 1$.

3.3 In general terms, the problem of preference aggregation deals with valued relations whose scores satisfy the condition $v_{xy} + v_{yx} \leq 1$. The set of all such objects will be denoted by Ω. So, $\Omega = \{\, v \in [0, 1]^{\Pi} \mid v_{xy} + v_{yx} \leq 1 \,\}$. The complete case corresponds to the subset Γ determined by the equality $v_{xy} + v_{yx} = 1$. So, $\Gamma = \{\, v \in [0, 1]^{\Pi} \mid v_{xy} + v_{yx} = 1 \,\}$.

The sets Ω and Γ are respectively the fields of variation of the collective Llull matrix of a vote in the general case and in the complete one. On the other hand, the individual votes can also be viewed as belonging to these sets. In fact, any qualitative expression of preferences (not necesarily transitive) between the elements of A can be represented as a Llull matrix, i.e. an element of Ω, by putting

$$
v_{xy} = \begin{cases}
1, & \text{if } x \text{ is preferred to } y, \\
1/2, & \text{if } x \text{ is tied with } y, \\
0, & \text{if either } y \text{ is preferred to } x \text{ or no information} \\
& \text{is given about the preference between } x \text{ and } y.
\end{cases}
\tag{18}
$$

Such a mapping satisfies $v_{xy} + v_{yx} = 1$ whenever we are in the complete case, i.e. when either a preference or a tie is expressed about each pair of options. According to its definition, the collective Llull matrix is simply the center of gravity of the distribution of individual votes:

$$
v_{xy} = \sum_k \alpha_k \, v_{xy}^k,
\tag{19}
$$

where v^k are the Llull matrices associated with the individual votes, and α_k are the corresponding relative frequencies or weights.

Since the individual votes play no other role than contributing to the collective Llull matrix as described by the preceding equation, generally speaking there is no need to restrict them to express qualitative preferences only, but one can allow them to express valued preferences, i.e. to be

arbitrary elements of Ω (of Γ in the complete case). Such a possibility makes sense in that the individual opinions may already be the result of aggregating a variety of criteria.

The preceding idea of representing a system of valued preferences as an element of Ω appears already in [9].

4   The indirect scores and the associated comparison relation

Let us recall that the indirect scores v_{xy}^* are defined in the following way:

$$v_{xy}^* = \max\{v_\alpha \mid \alpha \text{ is a path } x_0 x_1 \ldots x_n \text{ from } x_0 = x \text{ to } x_n = y\},$$

where the score v_α of a path $\alpha = x_0 x_1 \ldots x_n$ is defined as

$$v_\alpha = \min\{v_{x_i x_{i+1}} \mid 0 \le i < n\}.$$

Remark. The matrix of indirect scores v^* can be viewed as a power of v (supplemented with $v_{xx} = 1$) for a matrix product defined in the following way: $(vw)_{xz} = \max_y \min(v_{xy}, w_{yz})$. More precisely, v^* coincides with such a power for any exponent greater than or equal to $N - 1$.

Lemma 4.1.   *The indirect scores satisfy the following inequalities:*

$$v_{xz}^* \ge \min(v_{xy}^*, v_{yz}^*) \quad \text{for any } x, y, z. \tag{20}$$

Proof.   Let α be a path from x to y such that $v_{xy}^* = v_\alpha$; let β be a path from y to z such that $v_{yz}^* = v_\beta$. Consider now their concatenation $\alpha\beta$. Since $\alpha\beta$ goes from x to z, one has $v_{xz}^* \ge v_{\alpha\beta}$. On the other hand, the definition of the score of a path ensures that $v_{\alpha\beta} = \min(v_\alpha, v_\beta)$. Putting these things together gives the desired result. $\square$

Lemma 4.2.   *Assume that the original scores satisfy the following inequalities:*

$$v_{xz} \ge \min(v_{xy}, v_{yz}) \quad \text{for any } x, y, z. \tag{21}$$

In that case, the indirect scores coincide with the original ones.

Proof.   The inequality $v_{xz}^* \ge v_{xz}$ is an immediate consequence of the definition of v_{xz}^*. The converse inequality can be obtained in the following way:

Let $\gamma = x_0 x_1 x_2 \ldots x_n$ be a path from x to z such that $v^*_{xz} = v_\gamma$. By virtue of (21), we have

$$\min \left(v_{x_0 x_1}, v_{x_1 x_2}, v_{x_2 x_3}, \ldots, v_{x_{n-1} x_n} \right) \;\leq\; \min \left(v_{x_0 x_2}, v_{x_2 x_3}, \ldots, v_{x_{n-1} x_n} \right).$$

So, $v^*_{xz} \leq v_{\gamma'}$ where $\gamma' = x_0 x_2 \ldots x_n$. By iteration, one eventually gets $v^*_{xz} \leq v_{xz}$. $\hfill\square$

Remark. On the basis of the preceding results it makes sense to take condition (21) as the definition of transitivity for a valued relation (v_{xy}). The matrix of indirect scores v^* can be characterized as the lowest one that lies above v and satisfies such a notion of transitivity; a proof of this fact —in a more general setting— will be found in $[5\colon\text{Theorem } 3.3]$.

Theorem 4.3 (Schulze, 1998 $[15\,b]$; see also $[20\colon\text{p. } 228\text{–}229]$)**.** *The indirect comparison relation $\kappa = \mu(v^*)$ is a partial order.*

Proof. Since $\mu(v^*)$ is clearly asymmetric, it is only a matter of showing its transitivity. We will argue by contradiction. Let us assume that $xy \in \mu(v^*)$ and $yz \in \mu(v^*)$, but $xz \notin \mu(v^*)$. This means respectively that (a) $v^*_{xy} > v^*_{yx}$ and (b) $v^*_{yz} > v^*_{zy}$, but (c) $v^*_{zx} \geq v^*_{xz}$. On the other hand, Lemma 4.1 ensures also that (d) $v^*_{xz} \geq \min(v^*_{xy}, v^*_{yz})$. We will distinguish two cases depending on which of the last two quantities is smaller: (i) $v^*_{yz} \geq v^*_{xy}$; (ii) $v^*_{xy} \geq v^*_{yz}$.

Case (i): $v^*_{yz} \geq v^*_{xy}$. We will see that in this case (c) and (d) entail a contradiction with (a). In fact, we have the following chain of inequalities: $v^*_{yx} \geq \min(v^*_{yz}, v^*_{zx}) \geq \min(v^*_{yz}, v^*_{xz}) \geq \min(v^*_{yz}, v^*_{xy}) = v^*_{xy}$, where we are using successively: Lemma 4.1, (c), (d) and (i).

Case (ii): $v^*_{xy} \geq v^*_{yz}$. An entirely analogous argument shows that in this case (c) and (d) entail a contradiction with (b). In fact, we have $v^*_{zy} \geq \min(v^*_{zx}, v^*_{xy}) \geq \min(v^*_{xz}, v^*_{xy}) \geq \min(v^*_{yz}, v^*_{xy}) = v^*_{yz}$, where we are using successively: Lemma 4.1, (c), (d) and (ii). $\hfill\square$

5   Admissible orders

Let us recall that an admissible order is a total order ξ such that $\kappa \subseteq \xi \subseteq \hat{\kappa}$. Let us recall also that this definition is redundant since each of the two inclusions implies the other one. So an admissible order is simply a total order that extends the partial order κ. The results that follow deal with the existence and efficient finding of such extensions.

Theorem 5.1 (Szpilrajn, 1930 [*18*])**.** *Given a partial order ρ on a finite set A, one can always find a total order ξ such that $\rho \subseteq \xi \subseteq \hat{\rho}$. If ρ contains neither xy nor yx, one can constrain ξ to include the pair xy.*

Proposition 5.2. *Let ρ be a partial order. Let r_x denote the rank of x in ρ as defined by equation* (17) *of* § *3.1. Any total ordering of the elements of A by non-decreasing values of r_x is an extension of ρ.*

Proof. Let ξ be a total order of A for which $x \mapsto r_x$ does not decrease. This means that $xy \in \xi$ implies $r_x \leq r_y$. Now, the contrapositive of the first statement in Lemma 3.3 ensures that $r_x \leq r_y$ implies $yx \notin \rho$, i.e. $xy \in \hat{\rho}$. So $\xi \subseteq \hat{\rho}$, from which it follows that also $\rho \subseteq \xi$.   □

Remark. The preceding proposition replaces the problem of finding a total order that contains ρ by the similar problem of finding a total order contained in the total preorder $\hat{\omega} = \{xy \in \Pi \mid r_x \leq r_y\}$. However, from a practical point of view the latter is a much easier thing to do, since one is guided by the function $x \mapsto r_x$.

6   The projection

Let us recall that our rating method is based upon certain projected scores v_{xy}^{π}. These quantities are obtained through the corresponding margins m_{xy}^{π} by means of the procedure (11–14). That procedure makes use of an admissible order ξ, whose existence has been dealt with in the preceding section, and it assumes $xy \in \xi$.

Lemma 6.1. *The projected margins m_{xy}^{π} have the following properties:*

$$0 \leq m_{xy}^{\pi} \leq 1, \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y. \tag{22}$$

$$(15) \qquad m_{xz}^{\pi} = \max\left(m_{xy}^{\pi}, m_{yz}^{\pi}\right), \qquad \text{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z. \tag{23}$$

Proof. Both properties are immediate consequences of (11–14) and the fact that $0 \leq m_{xy}^{\kappa} \leq 1$.   □

Theorem 6.2. *The projected scores do not depend on the admissible order ξ used for their calculation, i.e. the value of v_{xy}^{π} is independent of ξ for every $xy \in \Pi$. On the other hand, the matrix of the projected scores in an admissible order ξ is also independent of ξ; i.e. if x_i denotes the element of rank i in ξ, the value of $v_{x_i x_j}^{\pi}$ is independent of ξ for every pair of indices i, j.*

Remark. The two statements say different things since the identity of x_i and x_j may depend on the admissible order ξ.

Proof. Let us consider the effect of replacing ξ by another admissible order $\widetilde{\xi}$. In the following, the tilde is systematically used to distinguish between homologous objects which are associated respectively with ξ and $\widetilde{\xi}$; in particular, such a notation will be used in connection with the labels of the equations which are formulated in terms of the assumed admissible order.

With this terminology, we will prove the two following equalities, which amount to the two statements of the theorem. First,

$$m^{\pi}_{xy} \ = \ \widetilde{m}^{\pi}_{xy}, \qquad \text{for any pair } xy \ (x \neq y). \tag{24}$$

Secondly, we will see also that

$$m^{\pi}_{x_i x_j} \ = \ \widetilde{m}^{\pi}_{\tilde{x}_i \tilde{x}_j}, \qquad \text{for any pair of indices } ij \ (i \neq j), \tag{25}$$

where x_i denotes the element of rank i in ξ, and analogously for $\tilde{x}_i$ in $\widetilde{\xi}$.

Now, it is well known that the set of total order extensions of a given partial order is always connected through transpositions of consecutive elements (see for instance [7: p. 30]). Therefore, it suffices to deal with the case of two admissible orders ξ and $\widetilde{\xi}$ which differ from each other only by the transposition of two consecutive elements. So, we will assume that there are two elements a and b such that the only difference between ξ and $\widetilde{\xi}$ is that ξ contains ab whereas $\widetilde{\xi}$ contains ba. According to the definition of an admissible order, this implies that $m^{\kappa}_{ab} = m^{\kappa}_{ba} = 0$.

In order to control the effect of the differences between ξ and $\widetilde{\xi}$, we will make use of the following notation: p will denote the immediate predecessor of a in ξ; in this connection, any statement about p will be understood to imply the assumption that the set of predecessors of a is not empty. Similarly, q will denote the immediate successor of b in ξ; here too, any statement about q will be understood to imply the assumption that the set of successors of b is not empty. So, ξ and $\widetilde{\xi}$ contain respectively the paths *pabq* and *pbaq*. Finally, x' means here the immediate successor of x in ξ, which is the same as in $\widetilde{\xi}$ if $x \neq p, a, b$.

Let us look first at the superdiagonal intermediate projected margins $m^{\sigma}_{hh'}$. According to their definition, namely equation (12), $m^{\sigma}_{hh'}$ is the minimum of a certain set of values of m^{κ}_{xy}. In a table where x and y are ordered according to ξ, this set is an upper-right rectangle with lower-left vertex at hh'. Using $\widetilde{\xi}$ instead of ξ amounts to interchanging two consecutive columns and the corresponding rows of that table, namely those labeled by a and b.

In spite of such a rearrangement, in all cases but one the underlying set from which the minimum is taken is exactly the same, so the mininum is the same. The only case where the underlying set is not the same occurs for $h = a$ in the order ξ, or $h = b$ in the order $\widetilde{\xi}$; but then the minimum is still the same because the underlying set includes $m_{ab}^{\kappa} = m_{ba}^{\kappa} = 0$. So,

$$m_{x_i x_{i+1}}^{\sigma} = \widetilde{m}_{\tilde{x}_i \tilde{x}_{i+1}}^{\sigma}, \qquad \text{for any } i = 1, 2, \ldots N{-}1, \tag{26}$$

$$m_{ab}^{\sigma} = \widetilde{m}_{ba}^{\sigma} = 0. \tag{27}$$

On account of the definition of $m_{x_i x_j}^{\pi}$ and $\widetilde{m}_{\tilde{x}_i \tilde{x}_j}^{\pi}$, (26) results in (25).

Finally, let us see that (24) holds too. To this effect, we begin by noticing that (27) is saying that

$$m_{ab}^{\pi} = \widetilde{m}_{ba}^{\pi} = 0 \tag{28}$$

Let us consider now the equation $m_{pa}^{\pi} = \widetilde{m}_{pb}^{\pi}$, which is contained in (25). On account of (23), these equalities entail

$$m_{pb}^{\pi} = m_{pa}^{\pi} = \widetilde{m}_{pb}^{\pi} = \widetilde{m}_{pa}^{\pi}. \tag{29}$$

By means of an analogous argument, one obtains also that

$$m_{aq}^{\pi} = m_{bq}^{\pi} = \widetilde{m}_{aq}^{\pi} = \widetilde{m}_{bq}^{\pi}. \tag{30}$$

On the other hand, (25) ensures also that

$$m_{xx'}^{\pi} = \widetilde{m}_{xx'}^{\pi}, \qquad \text{whenever } x \neq p, a, b. \tag{31}$$

Finally, (23) allows to go from (28–31) to the desired general equality (24).
$\square$

Theorem 6.3. *The projected scores and their asssociated margins satisfy the following properties with respect to any admissible order ξ:*

(a) *The following inequalities hold whenever $x \overset{\xi}{\succ} y$ and $z \notin \{x, y\}$:*

$$v_{xy}^{\pi} \geq v_{yx}^{\pi}, \qquad m_{xy}^{\pi} \geq 0, \tag{32}$$

$$v_{xz}^{\pi} \geq v_{yz}^{\pi}, \qquad v_{zx}^{\pi} \leq v_{zy}^{\pi}, \tag{33}$$

$$m_{xz}^{\pi} \geq m_{yz}^{\pi}, \qquad m_{zx}^{\pi} \leq m_{zy}^{\pi}, \tag{34}$$

(b) *If $v_{xy}^{\pi} = v_{yx}^{\pi}$, or equivalently $m_{xy}^{\pi} = 0$, then (33) and (34) are satisfied all of them with an equality sign.*

Proof. Part (a). Let us begin by noticing that (32) reduces to (22). Notice also that (33) follows from (34), and that (34.1) and (34.2) are equivalent to each other. So, it suffices to prove either (34.1) or (34.2). We will distinguish three cases, namely: (i) $x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z$; (ii) $z \overset{\xi}{\succcurlyeq} x \overset{\xi}{\succcurlyeq} y$; (iii) $x \overset{\xi}{\succcurlyeq} z \overset{\xi}{\succcurlyeq} y$. In case (i), (34.1) follows from (23). In case (ii), (34.2) follows from (23). Finally, in case (iii) it suffices to use (22) to see that $m_{xz}^\pi \geq 0 \geq m_{yz}^\pi$.

Part (b). Similarly to part (a), it suffices to prove the statement corresponding to (34.1), i.e. that $m_{xy}^\pi = 0$ implies $m_{xz}^\pi = m_{yz}^\pi$. This follows immediately from (23) in cases (i) and (ii). In case (iii), (23) allows to derive that $m_{xz}^\pi = m_{zy}^\pi = 0$, and therefore also the equality $m_{xz}^\pi = m_{yz}^\pi$. □

Proposition 6.4. *Assume that there exists a total order ξ such that the original scores and the associated margins satisfy the following conditions:*

$$v_{xy} \geq v_{yx}, \ \ i.e. \ m_{xy} \geq 0, \qquad \textit{whenever } x \overset{\xi}{\succcurlyeq} y, \tag{35}$$

$$m_{xz} = \max(m_{xy}, m_{yz}), \qquad \textit{whenever } x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z. \tag{36}$$

In that case, the projected scores coincide with the original ones.

Proof. We will begin by showing that

$$m_{xz} \geq \min(m_{xy}, m_{yz}), \qquad \text{for any } x, y, z. \tag{37}$$

In order to prove this inequality we will distinguish six cases depending on the relative position of x, y, z according to ξ: (a) If $x \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} z$, then (37) is an immediate consequence of (36). (b) If $z \overset{\xi}{\succcurlyeq} y \overset{\xi}{\succcurlyeq} x$, then (36) (with x and z interchanged with each other) gives $m_{zx} = \max(m_{zy}, m_{yx})$, which owing to the antisymmetric character of the margins is equivalent to (37) with an equality sign. (c) If $x \overset{\xi}{\succcurlyeq} z \overset{\xi}{\succcurlyeq} y$, then condition (35) guarantees that $m_{xz} \geq 0 \geq m_{yz} = \min(m_{xy}, m_{yz})$. (d) If $z \overset{\xi}{\succcurlyeq} x \overset{\xi}{\succcurlyeq} y$, then we have $m_{xz} \geq m_{yz} = \min(m_{xy}, m_{yz})$, where the inequality holds because (36) ensures that $m_{zy} \geq m_{zx}$, and the equality derives from the hypothesis upon ξ. (e,f) The two remaining cases, namely $y \overset{\xi}{\succcurlyeq} x \overset{\xi}{\succcurlyeq} z$ and $y \overset{\xi}{\succcurlyeq} z \overset{\xi}{\succcurlyeq} x$, are analogous respectively to (c) and (d).

Now, since we are in the complete case, the scores v_{xy} and the margins m_{xy} are related to each other by the monotone increasing transformation $v_{xy} = (1+m_{xy})/2$. Therefore, the inequality (37) on the margins is equivalent to the following one on the scores:

$$v_{xz} \geq \min(v_{xy}, v_{yz}), \qquad \text{for any } x, y, z. \tag{38}$$

According to Lemma 4.2, this inequality implies that $v^*_{xy} = v_{xy}$ and therefore $m^\kappa_{xy} = m_{xy}$. In particular, ξ is ensured to be an admissible order.

Let us now consider any pair xy contained in ξ. By making use of (36) we see that $m^\sigma_{xy} = m^\kappa_{xy} = m_{xy}$. As a consequence, the equality $m^\pi_{xy} = \max\{m^\sigma_{pp'} \mid x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y\}$ becomes $m^\pi_{xy} = \max\{m_{pp'} \mid x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y\}$. From here, a second application of (36) allows to derive that $m^\pi_{xy} = m_{xy}$, and therefore $v^\pi_{xy} = v_{xy}$. □

Since conditions (35–36) of Proposition 6.4 are included among the properties of the projected Llull matrix according to Lemma 6.1, one can conclude that they fully characterize the projected Llull matrices, and that the operator $(v_{xy}) \mapsto (v^\pi_{xy})$ really deserves being called a projection:

Theorem 6.5. *The operator $P : \Gamma \ni (v_{xy}) \mapsto (v^\pi_{xy}) \in \Gamma$ is idempotent, i. e. $P^2 = P$. Its image $P\Gamma$ consists of the complete Llull matrices (v_{xy}) that satisfy (35–36) for some total order ξ.*

7   The rank-like rates

Let us recall that the rank-like rates R_x are given by the formula (9), or equivalently by (10). From these formulas one easily checks that they satisfy condition D.

Lemma 7.1.

(a)   *If $x \overset{\xi}{\succ} y$ in an admissible order ξ, then $R_x \leq R_y$.*

(b) $R_x = R_y$ *if and only if* $v^\pi_{xy} = v^\pi_{yx}$.

(c) $R_x \leq R_y$ *implies   the inequalities* (32 – 34).

(d) $R_x < R_y$ *if and only if* $v^\pi_{xy} > v^\pi_{yx}$.

(e) $v^\pi_{xy} > v^\pi_{yx}$ *implies $x \overset{\xi}{\succ} y$ in any admissible order ξ.*

Proof.   Part (a). It is an immediate consequence of formula (9) together with the inequalities (32) and (33.1) ensured by Theorem 6.3.

Part (b). From (9) it follows that

$$R_y - R_x = (v^\pi_{xy} - v^\pi_{yx}) + \sum_{\substack{z \neq x \\ z \neq y}} (v^\pi_{xz} - v^\pi_{yz}). \tag{39}$$

Let ξ be an admissible order. By symmetry we can assume $xy \in \xi$. As a consequence, Theorem 6.3 ensures that the terms of (39) which appear in

parentheses are all of them greater than or equal to zero. So the only possibility for their sum to vanish is that each of them vanishes separately, i. e. $v_{xy}^\pi = v_{yx}^\pi$ and $v_{xz}^\pi = v_{yz}^\pi$ for any $z \notin \{x, y\}$. Finally, part (b) of Theorem 6.3 ensures that all of these equalities hold as soon as the first one is satisfied.

Part (c). When the hypothesis is satisfied as a strict inequality, the result follows by combining the contrapositive of (a) with part (a) of Theorem 6.3. In the case of equality, it suffices to combine (b) with part (b) of that theorem.

Part (d). It follows from (c) and its contrapositive on account of (b).

Part (e). It follows from (d) and the contrapositive of (a). □

The next theorem characterizes the preference relation determined by the rank-like rates in terms of the indirect comparison relation κ defined in § 2.4:

Theorem 7.2 ([2]). *The rank-like rating given by* (9) *is related to the indirect comparison relation* $\kappa = \mu(v^*)$ *in the following way:*

$$R_x < R_y \iff yx \notin (\hat\kappa)^*, \tag{40}$$

$$R_x \leq R_y \iff xy \in (\hat\kappa)^*. \tag{41}$$

Proof. The statements (40) and (41) are equivalent to each other (via the contrapositive of each implication plus a swap between x and y). So it suffices to prove (40). On the other hand, to establish the latter it suffices to prove the two following statements:

$$xy \in (\hat\kappa)^* \implies R_x \leq R_y, \tag{42}$$

$$yx \notin (\hat\kappa)^* \implies R_x < R_y. \tag{43}$$

Proof of (42). By transitivity, it suffices to consider the case $xy \in \hat\kappa$. Now, from Theorem 5.1 one easily sees that $xy \in \hat\kappa$ implies that xy belongs to some admissible order ξ. The conclusion that $R_x \leq R_y$ is then ensured by Lemma 7.1.(a).

Proof of (43). Since $(\hat\kappa)^*$ is complete, $yx \notin (\hat\kappa)^*$ implies $xy \in (\hat\kappa)^*$ and therefore, according to (42), $R_x \leq R_y$. So, (43) will follow if we show that

$$R_x = R_y \implies yx \in (\hat\kappa)^* \qquad \text{whenever } x \neq y. \tag{44}$$

Let ξ be an admissible order. Since $\xi \subseteq \hat\kappa$, the right-hand side of (44) is automatically true if $yx \in \xi$; so, it remains to consider the case where $xy \in \xi$.

Let us begin by assuming that $y = x'$. According to Lemma 7.1.(b), the equality $R_x = R_{x'}$ implies that $m^\pi_{xx'} = 0$, that is $m^\sigma_{xx'} = 0$, which means that there exist a, b such that $a \overset{\xi}{\succeq} x \overset{\xi}{\succ} b$ and $m^\kappa_{ab} = 0$. Now, the latter implies that $ba \in \hat{\kappa}$, which can be combined with the fact that $x'b, ax \in \xi \subseteq \hat{\kappa}$ to derive that $x'x \in (\hat{\kappa})^*$ (with the obvious modifications if $x{=}a$ or $x'{=}b$). Finally, if we only know that $xy \in \xi$, we can use Lemma 7.1.(a) to see that the equality $R_x = R_y$ implies $R_p = R_{p'}$ for any p such that $x \overset{\xi}{\succeq} p \overset{\xi}{\succ} y$, which reduces the problem to the preceding case. $\square$

Corollary 7.3.
  (a) $R_x < R_y \Rightarrow xy \in \kappa$.
  (b) *If $\hat{\kappa}$ is transitive (which is ensured whenever κ is total),*
       then $R_x < R_y \Leftrightarrow xy \in \kappa$.
  (c) *If κ contains a set of the form $X \times Y$ with $X \cup Y = A$,*
       then $R_x < R_y$ for any $x \in X$ and $y \in Y$.

Proof. Part (a). This is an immediate consequence of (40) since $yx \notin (\hat{\kappa})^* \Rightarrow yx \notin \hat{\kappa} \Leftrightarrow xy \in \kappa$.

Part (b). It is just a matter of noticing that under the hypothesis that $\hat{\kappa}$ is transitive the right-hand side of (40) reduces to $xy \in \kappa$.

Part (c). Let $x \in X$ and $y \in Y$. Since $X \times Y \subset \kappa \subseteq \hat{\kappa}$, part (a) ensures that $R_x \le R_y$. So, it suffices to exclude the possibility that $R_x = R_y$. By using (40) one easily sees that this equality would imply $yx \in (\hat{\kappa})^*$. In other words, there would be a path from $y \in Y$ to $x \in X$ entirely contained in $\hat{\kappa}$. Such a path would have to include a pair $ab \in \hat{\kappa}$ with $a \in Y$ and $b \in X$, which is not possible since $ab \in \hat{\kappa}$ means $ba \notin \kappa$. $\square$

Proposition 7.4.   *Assume that the votes are total orders. Assume also that the Llull matrix satisfies the hypothesis of Proposition 6.4. In that case, the rank-like rates R_x coincide exactly with the mean ranks $\bar{r}_x$.*

Proof.   Recall that the rank-like rates are related to the projected scores in the same way as the mean ranks are related to the original scores when the votes are total orders (§ 2.5). So, the result follows since Proposition 6.4 ensures that the projected scores coincide with the original ones. $\square$

8 Continuity

We claim that the rank-like rates R_x are continuous functions of the binary scores v_{xy}. The main difficulty in proving this statement lies in the admissible order ξ, which plays a central role in the computations. Since ξ varies in a discrete set, its dependence on the data cannot be continuous at all. Even so, we claim that the final result is still a continuous function of the data.

In this connection, one can consider as data the normalized Llull matrix (v_{xy}), its domain of variation being the set Γ introduced in § 3.3. Alternatively, one can consider as data the relative frequencies of the possible votes, i. e. the coefficients α_k mentioned also in § 3.3.

Theorem 8.1. *The projected scores v_{xy}^π and the rank-like rates R_x depend continuously on the Llull matrix (v_{xy}).*

Proof. The dependence of the rank-like rates on the projected scores is given by formula (9), which is not only continuous but even linear (non-homogeneous). So we are left with the problem of showing that the projection $P : (v_{xy}) \mapsto (v_{xy}^\pi)$ is continuous. As it has been mentioned above, this is not so clear, since the projected margins are the result of certain operations which are based upon an admissible order ξ which is determined separately. However, we will see, on the one hand, that P is continuous as long as ξ remains unchanged, and on the other hand, that the results of § 5–6 allow to conclude that P is continuous on the whole of Γ in spite of the fact that ξ can change. In the following we will use the following notation: for every total order ξ, we denote by Γ_ξ the subset of Γ which consists of the Llull matrices for which ξ is an admissible order, and we denote by P_ξ the restriction of P to Γ_ξ.

We claim that the mapping P_ξ is continuous for every total order ξ. In order to check the truth of this statement, one has to go over the different mappings whose composition defines P_ξ (see § 2.6), namely: $(v_{xy}) \mapsto (m_{xy}^\kappa)$, $(m_{xy}^\kappa) \mapsto (m_{xy}^\sigma)$, and finally $(m_{xx'}^\sigma) \mapsto (m_{xy}^\pi) \mapsto (v_{xy}^\pi)$. All of these mappings are certainly continuous since they involve only additions and substractions as well as the max and min operations.

Finally, the continuity of P (and the fact that it is well-defined) is a consequence of the following facts (see for instance [14 : § 2-7]): (a) $\Gamma = \bigcup_\xi \Gamma_\xi$; this is true because of the existence of ξ (Theorem 5.1). (b) Γ_ξ is a closed subset of Γ; this is true because Γ_ξ is described by a set of non-strict inequalities which concern quantities that are continuous functions of (v_{xy}) (namely the inequalities $m_{xy}^\kappa \geq 0$ whenever $xy \in \xi$). (c) ξ varies over a finite set. (d) P_ξ coincides with P_η at $\Gamma_\xi \cap \Gamma_\eta$, as it is proved in Theorem 6.2. $\square$

Corollary 8.2.   *The rank-like rates depend continuously on the relative frequency of each possible content of an individual vote.*

Proof.   It suffices to recall that the Llull matrix (v_{xy}) is simply the center of gravity of the distribution specified by these relative frequencies (formula (19) of § 3.3).   □

9   Decomposition

Property E is concerned with having a partition of A in two sets X and Y such that each member of X is unanimously preferred to any member of Y, that is:

$$v_{xy} \; = \; 1 \quad \text{(and therefore } v_{yx} = 0) \quad \text{whenever } \; xy \in X \times Y. \qquad (45)$$

According to property E, to be proved in the present section, in the complete case considered in this article such a situation is characterized by the following equalities:

$$R_x \; = \; \widetilde{R}_x, \qquad\qquad \text{for all } x \in X, \qquad\qquad (46)$$

$$R_y \; = \; \widetilde{R}_y + |X|, \qquad \text{for all } y \in Y, \qquad\qquad (47)$$

$$\sum_{x \in X} R_x \; = \; |X|\,(|X|+1)/2, \qquad\qquad\qquad\qquad (48)$$

where $\widetilde{R}_x$ and $\widetilde{R}_y$ denote the rank-like rates which are determined respectively from the submatrices associated with X and Y. More specifically, each of preceding equalities is separately equivalent to (45).

In the following we will continue using a tilde to distinguish between homologous objects associated respectively with the whole matrix and with its submatrices associated with X and Y.

Lemma 9.1.   *Given a partition $A = X \cup Y$ in two disjoint nonempty sets, one has the following equivalences:*

$$\left.\begin{array}{c} v_{xy} = 1 \\ \forall\, xy \in X \times Y \end{array}\right\} \; \Longleftrightarrow \; \left\{\begin{array}{c} m^{\kappa}_{xy} = 1 \\ \forall\, xy \in X \times Y \end{array}\right\} \; \Longleftrightarrow \; \left\{\begin{array}{c} v^{\pi}_{xy} = 1 \\ \forall\, xy \in X \times Y \end{array}\right. \qquad (49)$$

Proof.   Assume that $v_{xy} = 1$ for all $xy \in X \times Y$. Then $v_{yx} = 0$, for all such pairs, which implies that v_γ vanishes for any path γ which goes from Y to X. This fact, together with the inequality $v^*_{xy} \geq v_{xy}$, entails the following

equalities for all $x \in X$ and $y \in Y$: $v_{yx}^* = 0$, $v_{xy}^* = 1$, and consequently $m_{xy}^\kappa = 1$.

Assume now that $m_{xy}^\kappa = 1$ for all $xy \in X \times Y$. Let ξ be an admissible order. As an immediate consequence of the definition, it includes the set $X \times Y$. Let ℓ be the last element of X according to ξ. From the present hypothesis it is clear that $m_{\ell\ell'}^\sigma = 1$, which entails that $v_{xy}^\pi = 1$ for every $xy \in X \times Y$.

Assume now that $v_{xy}^\pi = 1$ for all $xy \in X \times Y$. Let ξ be an admissible order. Here too, we are ensured that it includes the set $X \times Y$; this is so by virtue of Theorem 6.3.(a). Let ℓ be the last element of X according to ξ. From the fact that $m_{\ell\ell'}^\sigma = m_{\ell\ell'}^\pi = 1$, one infers that $m_{xy}^\kappa = 1$ for all $xy \in X \times Y$.

Finally, let us assume again that $m_{xy}^\kappa = 1$ for all $xy \in X \times Y$. Since $m_{xy}^\kappa = v_{xy}^* - v_{yx}^*$ and both terms of this difference belong to $[0, 1]$, the only possibility is $v_{xy}^* = 1$ and $v_{yx}^* = 0$, which implies that $v_{yx} = 0$. This equality is equivalent to $v_{xy} = 1$. $\qquad\square$

Lemma 9.2. *Condition* (45) *implies, for any admissible order, the following equalities:*

$$m_{xx'}^\sigma = \widetilde{m}_{xx'}^\sigma, \qquad \text{whenever } x, x' \in X, \tag{50}$$

$$m_{yy'}^\sigma = \widetilde{m}_{yy'}^\sigma, \qquad \text{whenever } y, y' \in Y, \tag{51}$$

Proof. As we saw in the proof of Lemma 9.1, condition (45) implies the vanishing of v_γ for any path γ which goes from Y to X. Besides the conclusions obtained in that lemma, this implies also the following equalities:

$$v_{x\bar{x}}^* = \widetilde{v}_{x\bar{x}}^*, \qquad m_{x\bar{x}}^\kappa = \widetilde{m}_{x\bar{x}}^\kappa, \qquad \text{for all } x, \bar{x} \in X, \tag{52}$$

$$v_{y\bar{y}}^* = \widetilde{v}_{y\bar{y}}^*, \qquad m_{y\bar{y}}^\kappa = \widetilde{m}_{y\bar{y}}^\kappa, \qquad \text{for all } y, \bar{y} \in Y. \tag{53}$$

Let us fix an admissible order ξ. The second equality of (49) not only ensures that ξ includes the set $X \times Y$, but it can also be combined with (52) and (53) to obtain respectively (50) and (51). $\qquad\square$

Theorem 9.3. *Conditions* (45), (46), (47) *and* (48) *are equivalent to each other.*

Proof. Part (a): (45) $\implies$ (46), (47) and (48). As a consequence of (50) and (51) we get the following equalities:

$$v_{x\bar{x}}^\pi = \widetilde{v}_{x\bar{x}}^\pi, \qquad \text{for all } x, \bar{x} \in X, \tag{54}$$

$$v_{y\bar{y}}^\pi = \widetilde{v}_{y\bar{y}}^\pi, \qquad \text{for all } y, \bar{y} \in Y. \tag{55}$$

On the other hand, Lemma 9.1 ensures that

$$v_{xy}^{\pi} = 1, \qquad \text{for all } xy \in X \times Y. \tag{56}$$

When the projected scores are introduced in (9) these equalities result in (46) and (47). Finally, (48) is an immediate consequence of (46).

Part (b): (46) $\Rightarrow$ (45); (47) $\Rightarrow$ (45). On account of formula (9), conditions (46) and (47) are easily seen to be respectively equivalent to the following equalities:

$$\sum_{\substack{y \in A \\ y \neq x}} v_{xy}^{\pi} = \sum_{\substack{\bar{x} \in X \\ \bar{x} \neq x}} \widetilde{v}_{x\bar{x}}^{\pi} + |Y|, \qquad \text{for all } x \in X, \tag{57}$$

$$\sum_{\substack{x \in A \\ x \neq y}} v_{yx}^{\pi} = \sum_{\substack{\bar{y} \in Y \\ \bar{y} \neq y}} \widetilde{v}_{y\bar{y}}^{\pi} \qquad \text{for all } y \in Y. \tag{58}$$

Let us add up respectively the equalities (57) over $x \in X$ and the equalities (58) over $y \in Y$. Since $v_{pq}^{\pi} + v_{qp}^{\pi} = \widetilde{v}_{pq}^{\pi} + \widetilde{v}_{qp}^{\pi} = 1$, we obtain

$$\sum_{\substack{x \in X \\ y \in Y}} v_{xy}^{\pi} = |X|\,|Y|, \tag{59}$$

$$\sum_{\substack{y \in Y \\ x \in X}} v_{yx}^{\pi} = 0. \tag{60}$$

Since the projected scores belong to $[0, 1]$, the preceding equalities imply respectively

$$v_{xy}^{\pi} = 1, \qquad \text{for all } xy \in X \times Y, \tag{61}$$

$$v_{yx}^{\pi} = 0, \qquad \text{for all } xy \in X \times Y, \tag{62}$$

which are equivalent to each other since $v_{xy}^{\pi} + v_{yx}^{\pi} = 1$. Finally, Lemma 9.1 allows to arrive at (45).

Part (c): (48) $\Rightarrow$ (45). From the definition of R_x and the fact that $v_{xy}^{\pi} \leq v_{xy}^{\pi} + v_{yx}^{\pi} = 1$, one easily derives the inequality $\sum_{x \in X} R_x \geq |X|\,(|X| + 1)/2$, with equality if and only if (61) holds. So, the result follows again by Lemma 9.1. □

Corollary 9.4.
  (a) $R_x = 1$ *if and only if* $v_{xy} = 1$ *for all* $y \neq x$.
  (b) $R_x = N$ *if and only if* $v_{xy} = 0$ *for all* $y \neq x$.

Proof. It suffices to apply Theorem 9.3 to the special cases $X = \{x\}$ and $X = A \setminus \{x\}$. The result can also be obtained directly from Lemma 9.1. □

10   The Condorcet-Smith principle

Theorem 10.1.   *Both the indirect majority relation $\kappa = \mu(v^*)$ and the preference relation determined by the rank-like rates comply with the Condorcet-Smith principle: If A is partitioned in two sets X and Y with the property that $v_{xy} > 1/2$ for any $x \in X$ and $y \in Y$, then one has also $xy \in \kappa$ and $R_x < R_y$ for any such x and y.*

Proof.   Assume that $x \in X$ and $y \in Y$. Since $v^*_{xy} \geq v_{xy}$, the hypothesis of the theorem entails that $v^*_{xy} > 1/2$. On the other hand, let γ be a path from y to x such that $v^*_{yx} = v_\gamma$; since it goes from Y to X, this path must contain at least one link $y_i y_{i+1}$ with $y_i \in Y$ and $y_{i+1} \in X$; now, for this link we have $v_{y_i y_{i+1}} \leq 1 - v_{y_{i+1} y_i} < 1/2$, which entails that $v^*_{yx} = v_\gamma < 1/2$. Therefore, we get $v^*_{yx} < 1/2 < v^*_{xy}$, i.e. $xy \in \kappa$. Finally, the fact that this holds for any $x \in X$ and $y \in Y$ implies, by Corollary 7.3.(c), that one has also $R_x < R_y$ for any such x and y. $\square$

11   Clone consistency

Clone consistency (also known as independence of clones) refers to the effect of adding or deleting similar options. For many voting methods, this may change the outcome in a substantial way. For instance, replacing a single option c by a set C of several options similar to c may change the result from c being the winner to giving the victory to some option outside C. This does not seem right: if c deserves being chosen when going alone, then in the second situation the right choice should be some member of C.

The notion of similarity that is relevant here can be formalized by the concept of autonomous set that was introduced in §3.1. Recall that C being autonomous for a given binary relation means that each element from outside C relates to all elements of C in the same way. In the context of voting theory, autonomous sets are often called sets of clones. So, it makes sense to ask for the following property, which we call clone consistency: If a set of options is autonomous for each of the individual votes, then: (a) this set is also autonomous for the social ranking; and (b) contracting it to a single option in all of the individual votes has no other effect in the social ranking than getting the same contraction.

This requirement was introduced in 1986–87 by Thomas M. Zavist and T. Nicolaus Tideman, who also devised a method that satisfies it, namely the rule of ranked pairs [19, 21].

This section is aimed at proving this property for both the indirect comparison relation κ as well as the preference relation determined by the rank-like rates. The core results were obtained by Markus Schulze [$15c$, 16].

11.1 In order to prepare the ground, we need to deal first with certain generalities. To begin with, the notion of an autonomous set will be extended to apply not only to a relation, as defined in § 3.1, but also to any valued relation (v_{xy}): A subset $C \subseteq A$ will be said to be autonomous for (v_{xy}) when

$$v_{ax} = v_{bx}, \quad v_{xa} = v_{xb}, \qquad \text{whenever } a, b \in C \text{ and } x \notin C. \qquad (63)$$

This definition can be viewed as an extension of that given in § 3.1 because of the following fact, which follows easily from the definitions:

Lemma 11.1. *Given a binary relation ρ, let u_{xy} and v_{xy} be the binary scores defined respectively by*

$$u_{xy} = \begin{cases} 1, & \text{if } xy \in \rho, \\ 0, & \text{if } xy \notin \rho; \end{cases} \qquad v_{xy} = \begin{cases} 1, & \text{if } xy \in \rho \text{ and } yx \notin \rho, \\ 1/2, & \text{if } xy \in \rho \text{ and } yx \in \rho, \\ 0, & \text{if } xy \notin \rho. \end{cases} \qquad (64)$$

One has the following equivalences:
  (a)   *C is autonomous for ρ if and only if C is autonomous for (u_{xy}).*
  (b)   *C is autonomous for ρ if and only if C is autonomous for (v_{xy}).*

Lemma 11.2. *Assume that $C \subset A$ is autonomous for (v_{xy}). Assume also that either x or y, or both, lie outside C. In this case*

$$v_{xy}^* = \max \{ v_\gamma \mid \gamma \text{ contains no more than one element of } C \}$$

Proof. It suffices to see that any path $\gamma = x_0 \ldots x_n$ from $x_0 = x$ to $x_n = y$ which contains more than one element of C can be replaced by another one $\widetilde{\gamma}$ which contains only one such element and satisfies $v_{\widetilde{\gamma}} \geq v_\gamma$. Consider first the case where $x, y \notin C$. In this case it will suffice to take $\widetilde{\gamma} = x_0 \ldots x_{j-1} x_k \ldots x_n$, where $j = \min \{ i \mid x_i \in C \}$ and $k = \max \{ i \mid x_i \in C \}$, which obviously satisfy $0 < j < k < n$. Since $x_{j-1} \notin C$ and $x_j, x_k \in C$, we have $v_{x_{j-1} x_j} = v_{x_{j-1} x_k}$, so that

$$\begin{aligned}
v_\gamma &= \min \left(v_{x_0 x_1}, \ldots, v_{x_{n-1} x_n} \right) \\
&\leq \min \left(v_{x_0 x_1}, \ldots, v_{x_{j-1} x_j}, v_{x_k x_{k+1}}, \ldots, v_{x_{n-1} x_n} \right) \\
&= \min \left(v_{x_0 x_1}, \ldots, v_{x_{j-1} x_k}, v_{x_k x_{k+1}}, \ldots, v_{x_{n-1} x_n} \right) = v_{\widetilde{\gamma}}.
\end{aligned}$$

The case where $x \notin C$ but $y \in C$ can be dealt with in a similar way by taking $\widetilde{\gamma} = x_0 \ldots x_{j-1} x_n$, and analogously, in the case where $x \in C$ and $y \notin C$ it suffices to take $\widetilde{\gamma} = x_0 x_{k+1} \ldots x_n$. □

Proposition 11.3. *If $C \subset A$ is autonomous for the scores (v_{xy}), then C is autonomous also for the indirect scores (v_{xy}^*).*

Proof. Consider $a, b \in C$ and $x \notin C$. Let $\gamma = x_0 x_1 x_2 \ldots x_n$ be a path from a to x such that $v_{ax}^* = v_\gamma$. By Lemma 11.2, we can assume that a is the only element of γ that belongs to C. In particular, $x_1 \notin C$, so that $v_{ax_1} = v_{bx_1}$, which allows to write

$$
\begin{aligned}
v_{ax}^* \;=\; v_\gamma \;&=\; \min\,\big(v_{ax_1}, v_{x_1 x_2}, \ldots, v_{x_{n-1}x}\big) \\
&=\; \min\,\big(v_{bx_1}, v_{x_1 x_2}, \ldots, v_{x_{n-1}x}\big) \;\leq\; v_{bx}^*.
\end{aligned}
$$

By interchanging a and b, one gets the reverse inequality $v_{bx}^* \leq v_{ax}^*$ and therefore the equality $v_{ax}^* = v_{bx}^*$. An analogous argument shows that $v_{xa}^* = v_{xb}^*$. □

Corollary 11.4. *If $C \subset A$ is autonomous for a relation ρ, then C is autonomous also for the transitive closure ρ^*.*

Proof. Because of Proposition 11.3 and Lemma 11.1.(a). □

11.2 The next results assume that $C \subset A$ is autonomous for the Llull matrix of a vote. Obviously, this assumption is satisfied whenever C is autonomous for all of the individual votes (which can be allowed to be arbitrary elements of Γ as mentioned in § 3.3).

Theorem 11.5. *Assume that $C \subset A$ is autonomous for the Llull matrix (v_{xy}). Then C is autonomous for the indirect comparison relation $\kappa = \mu(v^*)$ as well as for the total preorder determined by the rank-like rates (i. e. for the relation $\hat{\omega} = \{xy \in \Pi \mid R_x \leq R_y\}$).*

Proof. Proposition 11.3 ensures that C is autonomous for the indirect scores (v_{xy}^*), from which one easily derives that C is autonomous for the relation κ. Now, according to Theorem 7.2, $\hat{\omega} = (\hat{\kappa})^*$. So the statement about $\hat{\omega}$ follows by virtue of Lemma 3.2 and Corollary 11.4. □

Theorem 11.6. *Assume that $C \subset A$ is autonomous for the Llull matrix (v_{xy}). Then C is autonomous also for the projected scores (v_{xy}^π).*

Proof. Since $v_{xy}^{\pi} = (1 + m_{xy}^{\pi})/2$, it suffices to show that C is autonomous for the projected margins (m_{xy}^{π}). By Theorem 11.3, we know that C is autonomous for the indirect scores (v_{xy}^{*}), which immediately implies its being autonomous also for the indirect margins (m_{xy}^{κ}). So the problem lies at showing that the autonomy of C is maintained when going from (m_{xy}^{κ}) to (m_{xy}^{π}) via the procedure (11–14). In the following we let ξ be an admissible order and we distinguish two cases depending on whether C is or not an interval for ξ.

Assume first that C is an interval of ξ. Since margins are antisymmetric, in order to prove that C is autonomous for (m_{xy}^{π}) it suffices to show that

$$m_{xa}^{\pi} = m_{xb}^{\pi}, \qquad \text{for any } a, b \in C \text{ and } x \overset{\xi}{\succ} C, \tag{65}$$

$$m_{ay}^{\pi} = m_{by}^{\pi}, \qquad \text{for any } a, b \in C \text{ and } C \overset{\xi}{\succ} y. \tag{66}$$

In the following we prove (65), the proof of (66) being entirely analogous. If there are no $x \overset{\xi}{\succ} C$ there is nothing to prove. Otherwise, let k be the immediate predecessor of the first element of C. By using (13), one easily sees that (65) will follow if we show that

$$m_{cc'}^{\sigma} \leq m_{kk'}^{\sigma}, \qquad \text{for any } c \in C. \tag{67}$$

Now, this inequality holds because

$$m_{cc'}^{\sigma} = \min\left\{ m_{pq}^{\kappa} \mid p \overset{\xi}{\succeq} c, \ c' \overset{\xi}{\succeq} q \right\} \leq \min\left\{ m_{pq}^{\kappa} \mid p \overset{\xi}{\succeq} k, \ c' \overset{\xi}{\succeq} q \right\}$$
$$= \min\left\{ m_{pq}^{\kappa} \mid p \overset{\xi}{\succeq} k, \ k' \overset{\xi}{\succeq} q \right\} = m_{kk'}^{\sigma},$$

where the inequality is due to the fact that we pass to a smaller set, and the equality that starts the second line holds because $k' \overset{\xi}{\succeq} q \overset{\xi}{\succeq} c$ implies $q \in C$, whereas $p \overset{\xi}{\succeq} k$ implies $p \notin C$, so that $m_{pq}^{\kappa} = m_{pk'}^{\kappa}$ for such p and q.

Assume now that C is not an interval of ξ. That is, there exist $a, b \in C$ and $x \notin C$ such that $a \overset{\xi}{\succ} x \overset{\xi}{\succ} b$. This implies that $ax, xb \in \hat{\kappa}$. Since we know that C is autonomous for κ, it follows that $cx, xc \in \hat{\kappa}$ for all $c \in C$, that is, $m_{cx}^{\kappa} = m_{xc}^{\kappa} = 0$ for all $c \in C$. From this fact one easily derives, using (12), that $m_{pp'}^{\sigma} = 0$ for all $p \in A$ such that $p, p' \in \bar{C}$, where $\bar{C}$ means the minimum interval of ξ that contains C. Finally, this entails, using (13), that $m_{xy}^{\pi} = 0$ for all $x, y \in \bar{C}$, and that any subset of $\bar{C}$, in particular the set C, is autonomous for (m_{xy}^{π}). $\qquad\square$

11.3 Finally, we consider the effect of contracting C to a single element. In this connection we will make use of the notation and definitions of § 3.1, together with the following natural extension to a system of binary scores:

if (v_{xy}) admits C as an autonomous set, the contracted binary scores $(\widetilde{v}_{\widetilde{x}\widetilde{y}})$ are characterized by the equality $\widetilde{v}_{\widetilde{x}\widetilde{y}} = v_{xy}$ whenever $\widetilde{x} \neq \widetilde{y}$. In the following, a tilde is systematically used to distinguish between homologous objects associated respectively with (A, v) and $(\widetilde{A}, \widetilde{v})$.

Theorem 11.7. *Assume that $C \subset A$ is autonomous for the Llull matrix (v_{xy}). Then the relation $\widetilde{\kappa}$ coincides with the contraction of κ by the autonomous set C. Similarly, the relation $\widehat{\widetilde{\omega}} = \{xy \in \widetilde{\Pi} \mid \widetilde{R}_x \leq \widetilde{R}_y\}$ coincides with the contraction of $\hat{\omega} = \{xy \in \Pi \mid R_x \leq R_y\}$ by C.*

Proof. We begin by noticing that the operation of taking indirect scores commutes with that of contraction by the autonomous set C, i. e. $\widetilde{v}^*_{\widetilde{x}\widetilde{y}} = v^*_{xy}$ whenever $\widetilde{x} \neq \widetilde{y}$. This is a consequence of Lemma 11.2. As a consequence, $\widetilde{\kappa} = \mu(\widetilde{v}^*)$ coincides with the contraction of $\kappa = \mu(v^*)$ by C, and similarly for $\widehat{\widetilde{\kappa}}$ and $\hat{\kappa}$. On the other hand, a second application of Lemma 11.2 —to the binary scores associated with the relation $\hat{\kappa}$ by formula (64.1)— ensures that $(\widehat{\widetilde{\kappa}})^*$ is the contraction of $(\hat{\kappa})^*$. Finally, in order to see that $\widehat{\widetilde{\omega}}$ is the contraction of $\hat{\omega}$ it is just a matter of applying Theorem 7.2: For any $x, y \in A$ such that $\widetilde{x} \neq \widetilde{y}$, we have $\widetilde{R}_{\widetilde{x}} \leq \widetilde{R}_{\widetilde{y}} \Leftrightarrow \widetilde{x}\widetilde{y} \in (\widehat{\widetilde{\kappa}})^* \Leftrightarrow xy \in (\hat{\kappa})^* \Leftrightarrow R_x \leq R_y$. $\square$

12 About monotonicity

In this section we consider the effect of raising a particular option a to a more preferred status in the individual ballots *without any change in the preferences about the other options*. More generally, we consider the case where the scores v_{xy} are modified into new values $\widetilde{v}_{xy}$ such that

$$\widetilde{v}_{ay} \geq v_{ay}, \quad \widetilde{v}_{xa} \leq v_{xa}, \quad \widetilde{v}_{xy} = v_{xy}, \qquad \forall x, y \neq a. \tag{68}$$

In such a situation, one would expect the social rates to behave in the following way, where y is an arbitrary element of $A \setminus \{a\}$:

$$\widetilde{R}_a \leq R_a, \tag{69}$$

$$R_a < R_y \implies \widetilde{R}_a < \widetilde{R}_y, \qquad R_a \leq R_y \implies \widetilde{R}_a \leq \widetilde{R}_y, \tag{70}$$

where the tilde indicates the objects associated with the modified scores. Unfortunately, the rating method proposed in this paper does not satisfy these conditions, but generally speaking it satisfies only the following weaker ones:

$$R_a < R_y \implies \widetilde{R}_a \leq \widetilde{R}_y. \tag{71}$$

$$(R_a < R_y, \ \forall y \neq a) \implies (\widetilde{R}_a < \widetilde{R}_y, \ \forall y \neq a). \tag{72}$$

In plain words, (72) is saying that if a was the only winner for the scores v_{xy}, then it is still the only winner for the scores $\widetilde{v}_{xy}$.

An example exhibiting the lack of property (70) can be found in [13: number 10 of "Example inputs"].

In this connection, it is interesting to remark that the method of ranked pairs enjoys also property (72) [20: p. 221–222], but it fails at (71). A profile which exhibits this failure of (71) for the method of ranked pairs is given in [13: number 9 of "Example inputs"].

The remainder of this section is devoted to proving properties (71) and (72).

Theorem 12.1. *Assume that (v_{xy}) and $(\widetilde{v}_{xy})$ are related to each other in accordance with (68). In this case, the following properties are satisfied for any $x, y \neq a$:*

$$\widetilde{v}_{ay}^* \geq v_{ay}^*, \qquad \widetilde{v}_{xa}^* \leq v_{xa}^*, \tag{73}$$

(71) $$R_a < R_y \implies \widetilde{R}_a \leq \widetilde{R}_y. \tag{74}$$

Proof. Let $\kappa = \mu(v^*)$ and $\widetilde{\kappa} = \mu(\widetilde{v}^*)$. Observe that (73) implies

$$\widetilde{m}_{ay}^\kappa \geq m_{ay}^\kappa, \quad \widetilde{m}_{ya}^\kappa \leq m_{ya}^\kappa, \quad \forall y \neq a. \tag{75}$$

We begin by seeing that (74) will be a consequence of (73). In fact, we have the following chain of implications: $R_a < R_y \Rightarrow ay \in \kappa \Leftrightarrow m_{ay}^\kappa > 0 \Rightarrow \widetilde{m}_{ay}^\kappa > 0 \Leftrightarrow ay \in \widetilde{\kappa} \Rightarrow \widetilde{R}_a \leq \widetilde{R}_y$, where the central implication is provided by (75) and the other two strict implications are guaranteed by Corollary 7.3.(a). So, the problem has been reduced to proving (73).

Now, in order to obtain the indirect score for a pair of the form ay it is useless to consider paths involving xa for some $x \neq a$, since such paths contain cycles whose omission results in paths having a larger or equal score. So, the maximum which defines v_{ay}^* is realized by a path which does not involve any pair xa. For such a path γ we have

$$v_{ay}^* = v_\gamma \leq \widetilde{v}_\gamma \leq \widetilde{v}_{ay}^*,$$

where the first inequality follows directly by (68). An analogous argument gives $\widetilde{v}_{xa}^* \leq v_{xa}^*$. $\qquad \square$

Corollary 12.2 ([3]). *Under the hypothesis of Theorem 12.1 one has also the property (72).*

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[3]We thank Markus Schulze for pointing out this fact.

Proof. According to Corollary 7.3.(a), the left-hand side of (72) implies the strict inequality $v_{ay}^* > v_{ya}^*$ for all $y \neq a$. Now, this inequality can be combined with (73) to derive that $\widetilde{v}_{ay}^* > \widetilde{v}_{ya}^*$ for all $y \neq a$. Finally, Corollary 7.3.(c) with $X = \{a\}$ and $Y = A \setminus \{a\}$ guarantees that the right-hand side of (72) is satisfied.   □

13   Concluding remarks and open questions

To our knowledge, the existing literature does not offer any other rating method that combines the quantitative properties of continuity and decomposition with the Condorcet-Smith principle. As it has been pointed out in the introduction, the latter is quite pertinent when one is interested not only in choosing a winner but in ranking all the alternatives (or in rating them). In particular, the Borda rule (linearly equivalent to rating the options by their mean ranks) satisfies those two quantitative properties, but it does not comply with the Condorcet principle. Quite interestingly, the maximin rates, namely $\sigma_x = \min_{y \neq x} v_{xy}$, combine continuity and a certain form of decomposition with the standard Condorcet principle, that is, condition M restricted to the case where the subset X reduces to a single option.[4] However, they easily fail at this condition when the subset X contains several options; a counterexample is given for instance in [20: p. 212–213].

One may ask whether the CLC rating method is the only one that satisfies properties A–E and M. The answer to this question is surely negative: Although we are imposing sharp constraints on the rates to be obtained at certain special points of the Llull matrix space, in between these points there is still some degree of freedom (in particular, because rates vary in a continuum). This remaining freedom might allow for imposing some additional property. In this connection, we find especially interesting the following

Open question 1. *Can one give a rating method that satisfies* A–E *and* M *together with quantitative monotonicity in the sense that new Llull scores satisfying* (68) *implies new rates satisfying* (69–70)*?*

It is also natural to ask whether an alternative rating method with the same properties could be given where the underlying ranking method were not Schulze's method of paths, but some other one. In this connection, we consider especially interesting, because of their properties, the celebrated rule of Condorcet, Kemény and Young [12; 20: p. 182–190], as well as the ranked-pairs one [19, 21; 20: p. 219–223]:

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[4]We thank Salvador Barberà for having called our attention to the maximin rating.

Open question 2. *Is the rule of Condorcet, Kemény and Young compatible with a continuous rating method satisfying* A–E*?*

Open question 3. *Is the ranked-pairs rule compatible with a continuous rating method satisfying* A–E*?*

On the basis of a previous exploratory work, we believe that the answer to questions 2 and 3 is in both cases negative. More deeply into the question, one can ask:

Open question 4 ([5])**.** *Which properties characterize those ranking methods that can be extended into rating methods satisfying properties* A–E*?*

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