

TOPOLOGICAL MODULI SPACE FOR GERMS OF HOLOMORPHIC FOLIATIONS

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ABSTRACT. This work deals with the topological classification of germs of singular foliations on $(\mathbb{C}^2, 0)$. Working in a suitable class of foliations we fix the topological invariants given by the separatrix set, the Camacho-Sad indices and the projective holonomy representations and we compute the moduli space of topological classes in terms of the cohomology of a new algebraic object that we call group-graph. This moduli space may be an infinite dimensional functional space but under generic conditions we prove that it has finite dimension and we describe its algebraic and topological structures.

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1. INTRODUCTION AND STATEMENTS OF RESULTS

This work deals with the **topological classification** of germs of **singular foliations** on $(\mathbb{C}^2, 0)$. To every (possibly dicritical) foliation \mathcal{F} we can associate the following analytical invariants: the separatrix set **Sep** $_{\mathcal{F}}$, that is the collection of all germs at $0 \in \mathbb{C}^2$ of invariant irreducible analytic curves, called **separatrices**, its **canonical reduction map** $E_{\mathcal{F}} : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \rightarrow (\mathbb{C}^2, 0)$,

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cf. [4], and the **marked exceptional divisor**

$$\mathcal{E}_{\mathcal{F}}^{\diamond} = (\mathcal{E}_{\mathcal{F}}, \Sigma_{\mathcal{F}}, \iota_{\mathcal{F}}),$$

where $\Sigma_{\mathcal{F}} := \text{Sing}(\mathcal{F}^{\sharp})$ is the finite set consisting of the singular points of $\mathcal{F}^{\sharp} := E_{\mathcal{F}}^* \mathcal{F}$ and $\iota_{\mathcal{F}}$ is the intersection pairing of $\mathcal{E}_{\mathcal{F}}$ in $M_{\mathcal{F}}$. The topological class of **Sep** $_{\mathcal{F}}$ is clearly a topological invariant of \mathcal{F} . In this paper we will assume that \mathcal{F} is a **generalized curve**, i.e. \mathcal{F}^{\sharp} has no saddle-node singularities. The topological class $[\mathcal{E}_{\mathcal{F}}^{\diamond}]$ of $\mathcal{E}_{\mathcal{F}}^{\diamond}$ (as a marked intrinsic curve) is then a topological invariant of \mathcal{F} because in this situation $E_{\mathcal{F}}$ is also the minimal desingularization map of **Sep** $_{\mathcal{F}}$, cf. [2].

We know [15] that under some assumptions the Camacho-Sad indices of \mathcal{F}^{\sharp} at the points of $\Sigma_{\mathcal{F}}$ and the holonomy representations (up to inner automorphisms) of every component of $\mathcal{E}_{\mathcal{F}}$ are also topological invariants of the germ \mathcal{F} at $0 \in \mathbb{C}^2$. Our purpose in this work is to highlight all the other topological invariants and to describe the geometric and algebraic structure of the set consisting of all topological classes of germs of foliations when we fix the topological class of the marked exceptional divisor, the Camacho-Sad indices and the holonomies. We shall prove that under generic conditions this set has finite dimension but may have an uncountable number of connected components. We give examples for which it can only be parametrized by a functional space of infinite dimension and we give optimal conditions of finiteness.

To give a precise sense to our problem let us call **marked divisor** any collection $\mathcal{E}^{\diamond} = (\mathcal{E}, \Sigma, \iota)$ consisting of a compact curve with normal crossings \mathcal{E} whose irreducible components are biholomorphic to \mathbb{P}^1 , a finite subset Σ of \mathcal{E} and a symmetric map $\iota : \text{Comp}(\mathcal{E})^2 \rightarrow \mathbb{Z}$, $\text{Comp}(\mathcal{E})$ denoting the set of irreducible components of \mathcal{E} . We will denote by $\mathcal{E}^d \subset \mathcal{E}$ the union of the irreducible components of \mathcal{E} , that do not contain any point of Σ ; we call them **dicritical components**.

A **marking** of a foliation \mathcal{F} by \mathcal{E}^{\diamond} will be a homeomorphism $f : \mathcal{E} \rightarrow \mathcal{E}_{\mathcal{F}}$ sending Σ onto $\Sigma_{\mathcal{F}}$ compatible with the intersection pairing:

$$\iota_{\mathcal{F}}(f(D), f(D')) = \iota(D, D').$$

In this way, the holonomy representations and the Camacho-Sad indices of all pairs $\mathcal{F}^{\diamond} := (\mathcal{F}, f)$ can be now associated to two common sets of indices: the set $C_{\mathcal{E}^{\diamond}} := \text{Comp}(\overline{\mathcal{E} \setminus \mathcal{E}^d})$ of irreducible components of $\overline{\mathcal{E} \setminus \mathcal{E}^d}$ and

$$I_{\mathcal{E}^{\diamond}} := \{(s, D) \in \Sigma \times C_{\mathcal{E}^{\diamond}} \mid s \in D\},$$

by defining

$$\text{CS}^{\mathcal{F}^{\diamond}} := (\text{CS}(\mathcal{F}^{\diamond}, D, s))_{(s, D) \in I_{\mathcal{E}^{\diamond}}}, \quad \text{CS}(\mathcal{F}^{\diamond}, D, s) := \text{CS}(\mathcal{F}^{\sharp}, f(D), f(s)),$$

$$\dot{\mathcal{H}}^{\mathcal{F}^{\diamond}} := (\dot{\mathcal{H}}_D^{\mathcal{F}^{\diamond}})_{D \in C_{\mathcal{E}^{\diamond}}}, \quad \mathcal{H}_D^{\mathcal{F}^{\diamond}} := \mathcal{H}_{f(D)}^{\mathcal{F}^{\sharp}} \circ f_* : \pi_1(D \setminus \Sigma, \cdot) \longrightarrow \text{Diff}(\mathbb{C}, 0),$$

where $\mathcal{H}_{f(D)}^{\mathcal{F}^{\sharp}}$ is the \mathcal{F}^{\sharp} -holonomy representation of $\pi_1(f(D) \setminus \Sigma_{\mathcal{F}}, \cdot)$ in the group $\text{Diff}(\mathbb{C}, 0)$ of germs of holomorphic automorphisms of $(\mathbb{C}, 0)$, $\dot{\mathcal{H}}_D^{\mathcal{F}^{\diamond}}$ is its class up to inner automorphisms, f_* is the isomorphism induced by f at

the fundamental groups level, $\text{CS}(\mathcal{F}^\sharp, f(D), f(s))$ is the Camacho-Sad index of \mathcal{F}^\sharp along $f(D)$ at $f(s)$. We shall assume that \mathcal{E}^\diamond satisfies the following technical condition:

- (TC) *The closure of each connected component of $\mathcal{E} \setminus \mathcal{E}^d$ contains an irreducible component D with $\text{card}(D \cap \Sigma) \neq 2$.*

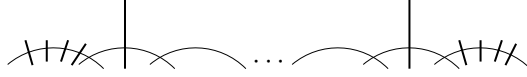


FIGURE 1. The only situation excluded by Condition (TC), the extremity divisors being dicritical components.

On the set $\text{Fol}(\mathcal{E}^\diamond)$ of germs of generalized curves \mathcal{F} at $0 \in \mathbb{C}^2$ for which there exists a marking $f : \mathcal{E} \rightarrow \mathcal{E}_\mathcal{F}$ of \mathcal{F} by \mathcal{E}^\diamond , we shall consider the two following equivalence relations:

- $\mathcal{F} \sim_{\mathcal{C}^0} \mathcal{G}$ if \mathcal{F} and \mathcal{G} are topologically equivalent as germs at $0 \in \mathbb{C}$;
- $\mathcal{F} \sim_\mathcal{E} \mathcal{G}$ if \mathcal{F}^\sharp and \mathcal{G}^\sharp are topologically conjugated, as germs along the exceptional divisors, by a germ of a homeomorphism $(M_\mathcal{F}, \mathcal{E}_\mathcal{F}) \rightarrow (M_\mathcal{G}, \mathcal{E}_\mathcal{G})$ which is holomorphic at each point of $\Sigma_\mathcal{F} \setminus \mathcal{N}_\mathcal{F}$,

$\mathcal{N}_\mathcal{F}$ denoting the subset of the singular points of $\mathcal{E}_\mathcal{F}$, called **nodal corners**, where the Camacho-Sad index of \mathcal{F}^\sharp is a strictly positive real number. Clearly relation $\sim_\mathcal{E}$ is stronger than $\sim_{\mathcal{C}^0}$, but they will coincide on a generic class of foliations only when \mathcal{E}^\diamond fulfills condition (TC). Specifically, let us call **cut-component** of $\mathcal{E}_\mathcal{F}$ any closure \mathcal{C} of a connected component of $\mathcal{E}_\mathcal{F} \setminus (\mathcal{E}_\mathcal{F}^d \cup \mathcal{N}_\mathcal{F})$; if $\text{card}(D \cap \Sigma) \leq 2$ for each $D \in \text{Comp}(\mathcal{C})$ we will say that \mathcal{C} is **exceptional**. Now consider the following transverse rigidity condition:

- (TR) *Any non exceptional cut-component of $\mathcal{E}_\mathcal{F}$ contains an irreducible component with topologically rigid¹ holonomy group.*

The Krull-open density in $\text{Fol}(\mathcal{E}^\diamond)$ of the subset $\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)$ consisting of the foliations \mathcal{F} fulfilling Condition (TR) is proven in [9]. An extended version of Main Theorem of [15] is given in Appendix (Theorem 10.4). It asserts that

Theorem A. *If \mathcal{E}^\diamond satisfies condition (TC) then the relations $\sim_\mathcal{E}$ and $\sim_{\mathcal{C}^0}$ are equal on $\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)$.*

This result implies that under the hypothesis (TC) and (TR) the collection of Camacho-Sad indices at the singular points of \mathcal{F}^\sharp is a topological invariant of the germ of \mathcal{F} at 0. In fact, the topological classification of logarithmic foliations given in [25, Théorème 3.5] shows that Condition (TR) is necessary. On the other hand, it is easy to construct counterexamples when Condition (TC) is not satisfied.

¹We recall that a subgroup G of the group $\text{Diff}(\mathbb{C}, 0)$ of germs of biholomorphisms of \mathbb{C} at 0 is called **topologically rigid** if every topological conjugation between G and another subgroup $G' \subset \text{Diff}(\mathbb{C}, 0)$ is necessarily conformal. This class contains the non-solvable groups [22] and the non-abelian groups with dense linear part [6, Théorème 2].

It follows from Theorem A that the quotient set $[\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)]_{\sim_{\mathcal{E}}}$ of $\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)$ by the relation $\sim_{\mathcal{E}}$ is a subset of the quotient set $[\text{Fol}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}$ of $\text{Fol}(\mathcal{E}^\diamond)$ by the relation $\sim_{\mathcal{C}^0}$:

$$[\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)]_{\sim_{\mathcal{E}}} = [\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0} \subset [\text{Fol}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}.$$

In order to describe $[\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)]_{\sim_{\mathcal{E}}}$ we will adapt the equivalence relation $\sim_{\mathcal{E}}$ to the set $\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)$ of **marked by \mathcal{E}^\diamond foliations** $\mathcal{F}^\diamond = (\mathcal{F}, f)$ such that $\mathcal{F} \in \text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)$:

- $(\mathcal{F}, f) \sim_{\diamond} (\mathcal{G}, g)$ if there is a germ of homeomorphism $\Phi : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \rightarrow (M_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}})$ that conjugates \mathcal{F}^\sharp and \mathcal{G}^\sharp , is holomorphic at each point of $\Sigma_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}}$ and its restriction to $\mathcal{E}_{\mathcal{F}}$ is isotopic to $g \circ f^{-1}$ by an isotopy fixing $\Sigma_{\mathcal{F}}$.

We will describe the **Teichmüller quotient space**

$$(1) \quad [\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)] := \text{MFol}_{\text{tr}}(\mathcal{E}^\diamond) / \sim_{\diamond}.$$

Indeed, thanks to Remark 3.3, the homeomorphism Φ is necessarily transversely holomorphic at each point of $\mathcal{E}_{\mathcal{F}} \setminus (\mathcal{E}_{\mathcal{F}}^d \cup \Sigma_{\mathcal{F}})$ and induces a holomorphic conjugacy of holonomies:

$$\mathcal{F}^\diamond \sim_{\diamond} \mathcal{G}^\diamond \implies (\text{CS}^{\mathcal{F}^\diamond}, \dot{\mathcal{H}}^{\mathcal{F}^\diamond}) = (\text{CS}^{\mathcal{G}^\diamond}, \dot{\mathcal{H}}^{\mathcal{G}^\diamond}).$$

We shall describe the fibers of the well defined map

$$(2) \quad \tilde{\mathcal{H}} := \text{CS} \times \dot{\mathcal{H}} : [\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)] \longrightarrow \mathbb{C}^{I_{\mathcal{E}^\diamond}} \times \dot{\mathcal{R}}, \quad \mathcal{F}^\diamond \mapsto (\text{CS}^{\mathcal{F}^\diamond}, \dot{\mathcal{H}}^{\mathcal{F}^\diamond}),$$

$\dot{\mathcal{R}}$ being the set of conjugacy classes of group morphisms from the free product of the groups $\pi_1(D \setminus \Sigma, \cdot)$ for all $D \in \mathcal{C}_{\mathcal{E}^\diamond}$ with values in $\text{Diff}(\mathbb{C}, 0)$.

Definition 1.1. We call **topological moduli space** of $[\mathcal{F}^\diamond] \in [\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)]$ the fiber of $\tilde{\mathcal{H}}$ above $\tilde{\mathcal{H}}([\mathcal{F}^\diamond])$ that is the set

$$\text{Mod}([\mathcal{F}^\diamond]) := \left\{ [\mathcal{G}^\diamond] \in [\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)] \mid \text{CS}^{\mathcal{F}^\diamond} = \text{CS}^{\mathcal{G}^\diamond}, \dot{\mathcal{H}}^{\mathcal{F}^\diamond} = \dot{\mathcal{H}}^{\mathcal{G}^\diamond} \right\}.$$

We consider in Section 2 a new algebraic notion, which we call **group-graph**, that will be the key tool in the whole paper. It allows us by combining Theorems 3.5 and 4.14 to obtain a bijection between this moduli space $\text{Mod}([\mathcal{F}^\diamond])$ and the cohomology of a suitable group-graph defined in Section 4, namely the **symmetry group-graph** $\text{Sym}^{\mathcal{F}^\diamond}$. If we call **\mathcal{F}^\diamond -cut-component** of \mathcal{E} any counter-image $f^{-1}(\mathcal{C})$ of a cut-component \mathcal{C} of $\mathcal{E}_{\mathcal{F}}$ and if we denote by $\mathbf{A}_{\mathcal{F}^\diamond}$ the dual graph of the disjoint union of all the **\mathcal{F}^\diamond -cut-components** of \mathcal{E} , then one can prove:

Theorem B. If \mathcal{E}^\diamond satisfies assumption (TC) and $[\mathcal{F}^\diamond] \in [\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)]$ we have a natural bijection

$$\text{Mod}([\mathcal{F}^\diamond]) \xrightarrow{\sim} H^1(\mathbf{A}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}).$$

Without any other assumption the computation of this cohomological space is difficult and the usefulness of this result is essentially theoretical; in particular it will allow us in Section 7 to construct examples for which $\text{Mod}([\mathcal{F}^\diamond])$ is an infinite dimensional functional space. To get finiteness we shall need to restrict to some Krull open dense subsets of $\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)$ by requiring conditions on $\mathcal{H}^{\mathcal{F}^\diamond}$ depending only on a finite jet of a 1-differential form defining \mathcal{F} .

Let us call **singular chain**² of the dual graph $\mathcal{E}_{\mathcal{F}}$ any sequence D_0, \dots, D_{ℓ} , $\ell \geq 1$, of invariant irreducible components of $\mathcal{E}_{\mathcal{F}}$ such that:

- a) D_0 and D_{ℓ} contain at least 3 singular points of \mathcal{F}^{\sharp} ,
- b) $D_i \cap \Sigma_{\mathcal{F}} = \{s_i, s_{i+1}\}$ with $s_i = D_{i-1} \cap D_i$, if $0 < i < \ell - 1$.

Definition 1.2. A germ of a foliation \mathcal{F} is called **non-degenerate** if it satisfies the following properties:

- (i) \mathcal{F} fulfills condition (TR);
- (ii) the holonomy group $\text{Im}(\mathcal{H}_D^{\mathcal{F}^{\sharp}})$ of any invariant irreducible component D of $\mathcal{E}_{\mathcal{F}}$ with $\text{card}(D \cap \Sigma_{\mathcal{F}}) \geq 3$, is non-abelian;
- (iii) for any singular chain D_0, \dots, D_{ℓ} in $\mathcal{E}_{\mathcal{F}}$, the local holonomies of \mathcal{F}^{\sharp} at the singular points $s_i = D_{i-1} \cap D_i$, $i = 1, \dots, \ell$, are non-periodic.

The subset of $\text{Fol}_{\text{tr}}(\mathcal{E}^{\diamond})$ of all non-degenerate foliations will be denoted by $\text{Fol}_{\text{nd}}(\mathcal{E}^{\diamond})$.

The Krull-open-density of $\text{Fol}_{\text{nd}}(\mathcal{E}^{\diamond})$ in $\text{Fol}_{\text{tr}}(\mathcal{E}^{\diamond})$ is given by Theorem 6.1. We shall see that $\text{Mod}([\mathcal{F}^{\diamond}])$ is endowed with a very specific structure of topological group of finite dimension if $\mathcal{F} \in \text{Fol}_{\text{nd}}(\mathcal{E}^{\diamond})$. However this finiteness property continues to be valid for a larger class of foliations that we shall call **finite type foliations**. The set $\text{Fol}_{\text{ft}}(\mathcal{E}^{\diamond}) \supset \text{Fol}_{\text{nd}}(\mathcal{E}^{\diamond})$ of these foliations is defined in Section 5 and it is optimal for finiteness as Example 3 in Section 7 shows. Furthermore we obtain **complete families** of marked foliations parametrized by finite dimensional spaces, completeness meaning that the family contains all the topological types of \mathcal{E}^{\diamond} marked foliations with prescribed Camacho-Sad and \mathcal{H} invariants.

A marked foliation (\mathcal{F}, f) will be called non-degenerate or of finite type if \mathcal{F} is non-degenerate or of finite type respectively.

Theorem C. Suppose that \mathcal{E}^{\diamond} satisfies condition (TC) and let $\mathcal{F}^{\diamond} \in \text{MFol}_{\text{tr}}(\mathcal{E}^{\diamond})$ be a marked foliation of finite type. Then $\text{Mod}([\mathcal{F}^{\diamond}])$ admits an abelian group structure with identity element $[\mathcal{F}^{\diamond}]$ such that there is an exact sequence

$$(3) \quad \mathbb{Z}^p \rightarrow \mathbb{C}^{\tau_{\mathcal{F}}} \xrightarrow{\Lambda} \text{Mod}([\mathcal{F}^{\diamond}]) \xrightarrow{\Gamma} \mathbf{D} \rightarrow 0$$

where \mathbf{D} is a totally disconnected topological abelian group and $\tau_{\mathcal{F}}$ is the codimension of \mathcal{F} given by Definition 5.7. In addition given a section $i \mapsto [\mathcal{F}_i, f_i] \in \Gamma^{-1}(i)$ of Γ , there is a family parametrized by $i \in \mathbf{D}$ of SL -equisingular deformations $(\mathcal{F}_{i,t}^U)_{t \in \mathbb{C}^{\tau_{\mathcal{F}}}}$ of \mathcal{F}_i such that for all $t \in \mathbb{C}^{\tau_{\mathcal{F}}}$ we have $[\mathcal{F}_{i,t}^U, f_{i,t}^U] = \Lambda(t) \cdot [\mathcal{F}_i, f_i]$, $f_{i,t}^U$ being the marking induced by f_i and the dot \cdot denoting the operation in the group $\text{Mod}([\mathcal{F}^{\diamond}])$.

The group \mathbf{D} will be specified in the proof (Step (i) of Section 9): it is a quotient of a product of a finite family of totally discontinuous subgroups of $\mathbb{U}(1) := \{z \in \mathbb{C} \mid |z| = 1\}$, that can be uncountable. However let us highlight that \mathbf{D} is “generically finite” in the following sense:

²Notice that a singular chain may not correspond to a chain of the dual graph of \mathcal{E} , in the usual sense. Indeed the interior vertices D_i , $0 < i < \ell$, may meet dicritical components and the number of their adjacent edges can be greater than two, and also D_0 or D_{ℓ} may have only two adjacent edges. Conversely a chain of the dual graph of \mathcal{E} may not be a singular chain because they may exist points of Σ outside the singular locus of \mathcal{E} .

- there is a subset Z of zero measure in the algebraic subset $\overline{\text{CS}(\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond))}$ of $\mathbb{C}^{I_{\mathcal{E}^\diamond}}$ such that if $\text{CS}([\mathcal{F}^\diamond]) \notin Z$, the formally linearizable singularities of \mathcal{F}^\sharp are holomorphically linearizable, and in this case we can prove that \mathbf{D} is finite.

The notion of **SL-equisingular deformation** roughly means equireductibility and constancy of Camacho-Sad indices and holonomy, the markings f_i extending continuously without ambiguity. All this is made more precise in Definition 9.2, Step (vi) of Section 9.

As a direct consequence of the proof we can see that if $\widetilde{\mathcal{H}}([\mathcal{F}^\diamond]) = \widetilde{\mathcal{H}}([\mathcal{G}^\diamond])$ then the **sets** $\text{Mod}([\mathcal{F}^\diamond])$ and $\text{Mod}([\mathcal{G}^\diamond])$ coincide but their respective abelian group structures are related by the map $\mu \mapsto \gamma\mu$ where $\gamma = [\mathcal{G}^\diamond] \in \text{Mod}([\mathcal{F}^\diamond])$.

We shall say that a singular chain D_0, \dots, D_ℓ , $\ell \geq 1$, is linearizable, resonant normalizable or non-normalizable, non-resonant,... if at the points $s_i = D_{i-1} \cap D_i$ the foliation \mathcal{F}^\sharp possesses this property.

Theorem D. *Let $\mathcal{F}^\diamond \in \text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)$ and suppose that \mathcal{F} is non degenerate. Then*

$$\text{Mod}([\mathcal{F}^\diamond]) \simeq \left(F \oplus B \oplus_{j=1}^{\lambda} (\mathbb{C}^* / \alpha_j^{\mathbb{Z}}) \oplus (\mathbb{C}^*)^{\nu} \right) / Z,$$

where $\alpha_j \in \mathbb{C}^*$, F is a finite abelian group, Z is a finite subgroup, B is a direct sum of β totally disconnected subgroups of $\mathbb{U}(1)$ and λ, ν and β are respectively the number of linearizable, resonant normalizable and non-resonant non-linearizable singular chains contained in cut-components of $\mathcal{E}_{\mathcal{F}}$, the factor F corresponding to resonant non-normalizable chains. In particular, $\lambda + \nu$ is equal to the codimension $\tau_{\mathcal{F}}$ of \mathcal{F} defined in 5.7.

Now to recover the set $[\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}$ of topological classes of germs of foliations at $0 \in \mathbb{C}^2$, from the quotient space $[\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)]$ introduced in (1), it remains to understand the action of the **Mapping Class Group** $\text{Mcg}(\mathcal{E}^\diamond)$ of \mathcal{E}^\diamond on this quotient set. This group is defined as the isotopy class group of \mathcal{C}^0 -automorphisms of \mathcal{E} fixing Σ and leaving the intersection form \imath invariant; its action

$$(4) \quad \dot{\varphi} \star [\mathcal{F}, f] := [\mathcal{F}, f \circ \varphi^{-1}], \quad \dot{\varphi} \in \text{Mcg}(\mathcal{E}^\diamond), \quad (\mathcal{F}, f) \in \text{MFol}_{\text{tr}}(\mathcal{E}^\diamond);$$

is well defined. The **Forgetful-map**

$$[\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)] \longrightarrow [\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}, \quad [\mathcal{F}, f] \mapsto [\mathcal{F}]_{\mathcal{C}^0}$$

is clearly surjective and its fibers are the orbits of the $\text{Mcg}(\mathcal{E}^\diamond)$ -action. In other words we have:

$$(5) \quad [\text{Fol}_{\text{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0} \simeq [\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)] / \text{Mcg}(\mathcal{E}^\diamond).$$

Let us notice that the action (4) is compatible with the partition of $[\text{MFol}_{\text{tr}}(\mathcal{E}^\diamond)]$ in moduli sets:

$$\dot{\varphi} \star \text{Mod}([\mathcal{F}^\diamond]) = \text{Mod}([\dot{\varphi} \star \mathcal{F}^\diamond]).$$

Thus it induces an action of $\text{Mcg}(\mathcal{E}^\diamond)$ on $\mathbb{C}^{I_{\mathcal{E}^\diamond}} \times \dot{\mathcal{R}}$, easy to describe. However the subgroup that leaves $\text{Mod}([\mathcal{F}^\diamond])$ invariant and its action on $\text{Mod}([\mathcal{F}^\diamond])$ depend on interesting but not well understood group of germs of \mathcal{C}^0 -automorphisms of $(\mathbb{C}^2, 0)$ leaving \mathcal{F} invariant; and more precisely on its image in the mapping class group [14] of the curve formed by an appropriate union of separatrices, see Definition 10.1.

2. GROUP-GRAPHS

In this section we will introduce and study the algebraic notion of group-graph which differs in an essential way from the notion of graph of groups introduced by Serre in [31] and that will be a key tool of this work.

Let A be a finite graph with vertex set Ve_A and edge set Ed_A . Denote by

$$I_A := \{(v, e) \in \text{Ve}_A \times \text{Ed}_A \mid v \in \partial e\}$$

the set of oriented edges of A .

Definition 2.1. A **group-graph** G over A is the data of groups G_v and G_e for each vertex $v \in \text{Ve}_A$ and each edge $e \in \text{Ed}_A$, and of group morphisms $\rho_v^e : G_v \rightarrow G_e$ for each $(v, e) \in I_A$ which are called **restriction maps**. A **morphism** $\alpha : F \rightarrow G$ between group-graphs over the same graph A is given by group morphisms $\alpha_v : F_v \rightarrow G_v$ and $\alpha_e : F_e \rightarrow G_e$ such that the diagram

$$\begin{array}{ccc} F_v & \xrightarrow{\alpha_v} & G_v \\ \rho_v^e \downarrow & & \downarrow \rho_v^e \\ F_e & \xrightarrow{\alpha_e} & G_e \end{array}$$

commutes for each $(v, e) \in I_A$. A group-graph G is called **abelian** if all the groups G_v and G_e are abelian.

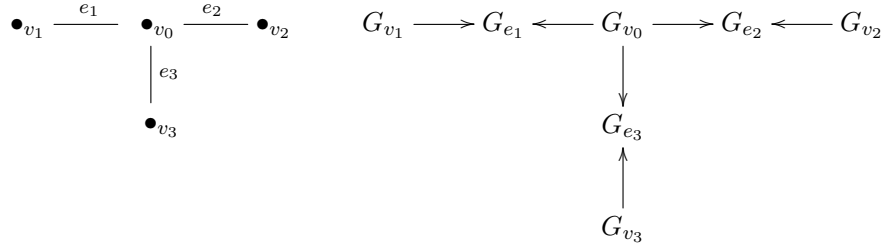


FIGURE 2. The dual tree of the simple cusp $y^2 - x^3 = 0$ and a group-graph G over this tree.

Remark 2.2. One can define in a natural way the notions of image and kernel of a group-graph morphism $\alpha : F \rightarrow G$, which are group-graphs over the same graph. In the abelian case the cokernel can also be defined as a group-graph. We also have an obvious notion of restriction of a group-graph over a graph to a subgraph. \square

Definition 2.3. Let G be a group-graph over a graph A . The **cochain complex** of G consists of

$$C^0(A, G) := \prod_{v \in \text{Ve}_A} G_v \quad \text{and} \quad C^1(A, G) := \prod_{(v, e) \in I_A} G_{v, e}, \quad G_{v, e} := G_e,$$

jointly with the right action $C^0(A, G) \times C^1(A, G) \rightarrow C^1(A, G)$ given by

$$(g_v) \star_G (g_{v, e}) := (\rho_v^e(g_v)^{-1} g_{v, e} \rho_{v'}^e(g_{v'}))$$

where $\partial e = \{v, v'\}$.

The set of **0-cocycles** is the subset $H^0(\mathbf{A}, G)$ of $C^0(\mathbf{A}, G)$ of all elements (g_v) satisfying the relations $\rho_v^e(g_v) = \rho_{v'}^e(g_{v'})$ whenever $\partial e = \{v, v'\}$. Let us consider the set of **1-cocycles**

$$Z^1(\mathbf{A}, G) := \{(g_{v,e}) \in C^1(\mathbf{A}, G) \mid g_{v,e} g_{v',e} = 1 \text{ when } \partial e = \{v, v'\}\} \subset C^1(\mathbf{A}, G)$$

which is invariant by the action of $C^0(\mathbf{A}, G)$ and its quotient, the 1-cohomology set:

$$H^1(\mathbf{A}, G) := Z^1(\mathbf{A}, G)/C^0(\mathbf{A}, G).$$

Remark 2.4. The cochains $C^0(\mathbf{A}, G)$ and $C^1(\mathbf{A}, G)$ are groups but in general $H^0(\mathbf{A}, G)$ and $H^1(\mathbf{A}, G)$ are merely sets (although $Z^1(\mathbf{A}, G)$ is in bijection with $\prod_{e \in \text{Ed}_\mathbf{A}} G_e$ which is a group). However, if the group-graph G is abelian

then we can consider the group $C^2(\mathbf{A}, G) := \prod_{e \in \text{Ed}_\mathbf{A}} G_e$ and the morphisms $\partial^0 : C^0(\mathbf{A}, G) \rightarrow C^1(\mathbf{A}, G)$ and $\partial^1 : C^1(\mathbf{A}, G) \rightarrow C^2(\mathbf{A}, G)$ given by

$$\partial^0(g_v) := (g_v) \star_G (1) = (\rho_v^e(g_v)^{-1} \rho_{v'}^e(g_{v'})) \quad \text{and} \quad \partial^1(g_{v,e}) = (g_{v,e} g_{v',e})$$

whenever $\partial e = \{v, v'\}$. It turns out that $\partial^1 \circ \partial^0 = 1$ and we obtain a complex

$$C^*(\mathbf{A}, G) : C^0(\mathbf{A}, G) \xrightarrow{\partial^0} C^1(\mathbf{A}, G) \xrightarrow{\partial^1} C^2(\mathbf{A}, G) \rightarrow 1$$

whose cohomology is $H^0(C^*(\mathbf{A}, G)) = H^0(\mathbf{A}, G) = \ker \partial^0$, $H^1(C^*(\mathbf{A}, G)) = H^1(\mathbf{A}, G) = Z^1(\mathbf{A}, G)/\partial^0 C^0(\mathbf{A}, G)$ and $H^2(C^*(\mathbf{A}, G)) = 1$ because $Z^1(\mathbf{A}, G) = \ker \partial^1$ and $\text{coker } \partial^1 = 1$. \square

The following result is straightforward.

Lemma 2.5 (Functoriality). *Every morphism $\alpha : F \rightarrow G$ of group-graphs over the same graph \mathbf{A} induces well-defined maps $\alpha_i : H^i(\mathbf{A}, F) \rightarrow H^i(\mathbf{A}, G)$, for $i = 0, 1$, given by*

$$\alpha_0(g_v) = (\alpha_v(g_v)) \quad \text{and} \quad \alpha_1([g_{v,e}]) = [\alpha_e(g_{v,e})].$$

Moreover, if F and G are abelian then α_i are morphisms.

A short sequence $1 \rightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} J \rightarrow 1$ of morphisms of group-graphs over the same graph \mathbf{A} is exact if for all $a \in \text{Ve}_\mathbf{A} \cup \text{Ed}_\mathbf{A}$ the corresponding short sequence of groups $1 \rightarrow F_a \xrightarrow{\alpha_a} G_a \xrightarrow{\beta_a} J_a \rightarrow 1$ is exact. In the abelian case the complexes of abelian groups considered in Remark 2.4 fit into a short exact sequence $1 \rightarrow C^*(\mathbf{A}, F) \rightarrow C^*(\mathbf{A}, G) \rightarrow C^*(\mathbf{A}, J) \rightarrow 1$. The following results are classical.

Lemma 2.6 (Long exact sequence). *If $1 \rightarrow F \rightarrow G \rightarrow J \rightarrow 1$ is a short exact sequence of abelian group-graphs over the same graph \mathbf{A} , then there is a long exact sequence $1 \rightarrow H^0(\mathbf{A}, F) \rightarrow H^0(\mathbf{A}, G) \rightarrow H^0(\mathbf{A}, J) \rightarrow H^1(\mathbf{A}, F) \rightarrow H^1(\mathbf{A}, G) \rightarrow H^1(\mathbf{A}, J) \rightarrow 1$.*

Lemma 2.7 (Mayer-Vietoris). *Let G be an abelian group-graph over a graph \mathbf{A} . If \mathbf{A}_0 and \mathbf{A}_1 are subgraphs of \mathbf{A} such that $\mathbf{A} = \mathbf{A}_0 \cup \mathbf{A}_1$ then there is an exact sequence $1 \rightarrow H^0(\mathbf{A}, G) \rightarrow H^0(\mathbf{A}_0, G) \oplus H^0(\mathbf{A}_1, G) \rightarrow H^0(\mathbf{A}_0 \cap \mathbf{A}_1, G) \rightarrow H^1(\mathbf{A}, G) \rightarrow H^1(\mathbf{A}_0, G) \oplus H^1(\mathbf{A}_1, G) \rightarrow H^1(\mathbf{A}_0 \cap \mathbf{A}_1, G) \rightarrow 1$.*

Proof. We have a short exact sequence of complexes of abelian groups

$$1 \rightarrow C^*(A, G) \rightarrow C^*(A_0, G) \oplus C^*(A_1, G) \rightarrow C^*(A_0 \cap A_1, G) \rightarrow 1$$

and we consider the long exact sequence of cohomology. \square

Definition 2.8. The *valency in A* or the *A-valency* of a vertex v of A is the cardinality $\text{val}_A(v)$ of the set $\{e \in \text{Ed}_A ; v \in \partial e\}$. We say that v is an *extremity of A* if $\text{val}_A(v) = 1$; in that case we will write $v \in \partial A$.

A **partial dead branch** (M, v_0) of A is the data of a vertex v_0 of A called **attaching point** and a connected subgraph M of A such that:

- M contains an extremity v'_0 of A,
- all its vertices are of valency 2 in A, except possibly its extremities that are v'_0 and v_0 .

Notice that M is always a chain. When $M \neq A$ and $\text{val}_A(v_0) \geq 3$ one says that M is a **dead branch of A**. We define a **total order** $<_M$ on the sets of its vertices $\text{Ve}_M := \{v_0, \dots, v_\ell := v'_0\}$ and of its edges $\text{Ed}_M := \{e_1, \dots, e_\ell\}$ with $\partial e_j = \{v_{j-1}, v_j\}$, by setting $v_0 <_M \dots <_M v_\ell$ and $e_1 <_M \dots <_M e_\ell$ $j = 1, \dots, \ell$.

Definition 2.9. For a group-graph G , we say that a partial dead branch M is **G-repulsive** if the morphisms $\rho_v^e : G_v \rightarrow G_e$ are surjective for all $e \in \text{Ed}_M$ such that $\partial e = \{v, v'\}$ and $v' <_M v$.

Now we will give a process that will allow us to restrict a group-graph to a subgraph without changing its cohomology.

Definition 2.10. If (M, v_0) is a partial dead branch of A, the **pruning \check{A} of M in A**, at the attaching point v_0 , is the subgraph $\check{A} = (A \setminus M) \cup \{v_0\}$.

Theorem 2.11 (Pruning). *Let G be a (not necessarily abelian) group-graph over A and M a G-repulsive partial dead branch of A then there is a natural bijection $H^1(A, G) \xrightarrow{\sim} H^1(\check{A}, \check{G})$ where \check{A} is the pruning of M in A and \check{G} is the restriction of G to \check{A} . Moreover, if G is abelian, this bijection is an isomorphism of groups.*

Before giving the proof let us notice that the natural projections $\text{pr}^i : C^1(A, G) \rightarrow C^1(\check{A}, \check{G})$, $i = 0, 1$, are group morphisms commuting with the actions \star_G and $\star_{\check{G}}$ and inducing a natural map $\text{pr}_*^1 : H^1(A, G) \rightarrow H^1(\check{A}, \check{G})$.

On the other hand, we have an “extension by 1” map $\text{ext} : C^1(\check{A}, \check{G}) \rightarrow C^1(A, G)$ such that $\text{pr}^1 \circ \text{ext}$ is the identity map, $\text{ext}(Z^1(\check{A}, \check{G})) \subset Z^1(A, G)$ and

$$\text{ext} \circ \text{pr}^1((g_{v,e})) = (g'_{v,e}), \quad \text{with} \quad g'_{v,e} = \begin{cases} g_{v,e} & \text{if } (v, e) \in I_A \setminus I_M \\ 1 & \text{if } (v, e) \in I_M. \end{cases}$$

Proof of Theorem 2.11. We will see that the G-repulsivity of M implies that:

a) ext induces a map $\text{ext}_* : H^1(\check{A}, \check{G}) \rightarrow H^1(A, G)$ such that $\text{pr}_*^1 \circ \text{ext}_*$ is the identity of $H^1(\check{A}, \check{G})$,

b) $\text{ext}_* \circ \text{pr}_*^1$ is the identity of $H^1(A, G)$. When G is abelian the maps pr_*^1 and ext_* are group morphisms, one the inverse of the other, and the map $H^1(A, G) \xrightarrow{\sim} H^1(\check{A}, \check{G})$ induced by pr_*^1 is trivially an isomorphism.

a) By G -repulsivity we have the following diagram of groups and morphisms

$$G_{v_0} \xrightarrow{\rho_{v_0}^{e_1}} G_{e_1} \xleftarrow{\rho_{v_1}^{e_1}} G_{v_1} \xrightarrow{\rho_{v_1}^{e_2}} G_{e_2} \xleftarrow{\rho_{v_2}^{e_2}} G_{v_2} \rightarrow \dots \xleftarrow{\rho_{v_{\ell-1}}^{e_{\ell-1}}} G_{v_{\ell-1}} \xrightarrow{\rho_{v_{\ell-1}}^{e_{\ell}}} G_{e_{\ell}} \xleftarrow{\rho_{v_{\ell}}^{e_{\ell}}} G_{v_{\ell}}$$

Let $\check{h}^1 = (\check{h}_v, e)$ and $\check{g}^1 = (\check{g}_v, e)$ be two cohomologous elements if $Z^1(\check{A}, \check{G})$:

$$\check{h}_{v,e} = \rho_v^e(\check{h}_v)^{-1} \check{g}_{v,e} \rho_{v'}^e(\check{h}_{v'}), \quad \check{h}^0 := (\check{h}_v) \in C^0(\check{A}, \check{G}).$$

We will determine $h^0 := (h_v) \in C^0(A, G)$ such that $h^0 \star_G \text{ext}(\check{h}^1) = \text{ext}(\check{g}^1)$. We define $h_v = \check{h}_v$ if $v \in \text{Ve}_{\check{A}}$. For $v \notin \text{Ve}_{\check{A}}$ it is sufficient to solve the system of ℓ equations:

$$\begin{cases} 1 &= \rho_{v_0}^{e_1}(h_{v_0})^{-1} \cdot 1 \cdot \rho_{v_1}^{e_1}(h_{v_1}), & h_{v_1} \in G_{v_1} \\ \vdots & \\ 1 &= \rho_{v_{\ell-1}}^{e_{\ell-1}}(h_{v_{\ell-1}})^{-1} \cdot 1 \cdot \rho_{v_{\ell}}^{e_{\ell}}(h_{v_{\ell}}), & h_{v_{\ell}} \in G_{v_{\ell}} \end{cases}$$

This can be easily done using the surjectivity of the maps $\rho_{v_j}^{e_j}$, $j = 1, \dots, \ell$.

b) We have to prove that for each $(g_{v,e}) \in Z^1(A, G)$ there is $(g_v) \in C^0(A, G)$ such that:

$$\text{ext} \circ \text{pr}^1((g_{v,e})) = (\rho_v^e(g_v)^{-1} g_{v,e} \rho_{v'}^e(g_{v'})) .$$

We define $g_v = 1$ when $v \in \text{Ve}_{\check{A}}$, so that in particular $g_{v_0} = 1$ and therefore $\rho_{v_0}^{e_1}(g_{v_0}) = 1$. As the maps $\rho_{v_j}^{e_j}$, $1 \leq j \leq \ell$, are surjective, the following system of ℓ equations has a solution with $g_{v_1} \in G_{v_1}, \dots, g_{v_{\ell}} \in G_{v_{\ell}}$

$$\begin{cases} 1 &= \rho_{v_0}^{e_1}(g_{v_0})^{-1} g_{v_0,e_1} \rho_{v_1}^{e_1}(g_{v_1}) \\ 1 &= \rho_{v_1}^{e_2}(g_{v_1})^{-1} g_{v_1,e_2} \rho_{v_2}^{e_2}(g_{v_2}) \\ \vdots & \\ 1 &= \rho_{v_{\ell-1}}^{e_{\ell-1}}(g_{v_{\ell-1}})^{-1} g_{v_{\ell-1},e_{\ell-1}} \rho_{v_{\ell}}^{e_{\ell}}(g_{v_{\ell}}) . \end{cases}$$

□

Remark 2.12. By repeating this process we obtain a subtree A_{pr} of A such that the restriction G_{pr} of G to A_{pr} has no G_{pr} -repulsive partial dead branches and the map $\text{ext} : Z^1(A_{\text{pr}}, G_{\text{pr}}) \rightarrow Z^1(A, G)$ of extension by 1, induces an isomorphism $H^1(A_{\text{pr}}, G_{\text{pr}}) \xrightarrow{\sim} H^1(A, G)$. In particular if all the morphisms $\rho_v^e : G_v \rightarrow G_e$ are surjective, the subtree A_{pr} is reduced to a single vertex and $H^1(A, G)$ is trivial. □

3. AUTOMORPHISM GROUP-GRAPH

Let us fix once for all a marked divisor $\mathcal{E}^\diamond = (\mathcal{E}, \Sigma, \iota)$ and a marked foliation $\mathcal{F}^\diamond = (\mathcal{F}, f) \in \text{Fol}(\mathcal{E}^\diamond)$. We recall that \mathcal{E}^d is the union of **dicritical components** D of \mathcal{E}^\diamond , i.e. $D \cap \Sigma = \emptyset$, cf. Introduction.

The **dual tree** $A_{\mathcal{E}}$ of \mathcal{E} is the graph having $\text{Comp}(\mathcal{E})$ and $\text{Sing}(\mathcal{E})$ as sets $\text{Ve}_{A_{\mathcal{E}}}$ of vertices and $\text{Ed}_{A_{\mathcal{E}}}$ of edges respectively, with $\partial s = \{D, D'\}$ whenever $D \cap D' = s$.

Definition 3.1. *The subgraph of $A_{\mathcal{E}}$ obtained by removing the vertices associated to \mathcal{E}^d , the edges attached with these vertices and the edges corresponding by f to nodal singularities of \mathcal{F}^\sharp , is called **cut-graph of \mathcal{F}°** . We denote it by $A_{\mathcal{F}^\circ}$, or more simply by A when there is not ambiguity.*

This graph is a disjoint union of trees, denoted by $A_{\mathcal{F}^\circ}^i$, or more simply by A^i , that can be considered as the dual graphs of the \mathcal{F} -cut-components of $\mathcal{E}_{\mathcal{F}}$ defined in the introduction. It only depends on the class $[\mathcal{F}, f]$ and in fact it is constant on the fiber $\text{CS}^{-1}(\text{CS}([\mathcal{F}^\circ]))$ that contains $\text{Mod}([\mathcal{F}^\circ])$. Notice that if G is a group-graph on A then

$$H^1(A_{\mathcal{F}^\circ}, G) = \prod_i H^1(A_{\mathcal{F}^\circ}^i, G_i)$$

where G_i is the restriction of G to $A_{\mathcal{F}^\circ}^i$.

Definition 3.2 (The group-graph $\text{Aut}^{\mathcal{F}^\circ}$). *For $s \in \text{Ed}_{A_{\mathcal{F}^\circ}}$ and $D \in \text{Ve}_{A_{\mathcal{F}^\circ}}$, let us denote by*

- $\text{Aut}_s^{\mathcal{F}^\circ}$ *the group of germs at $f(s)$ of holomorphic automorphisms of \mathcal{F}^\sharp ,*
- $\text{Aut}_D^{\mathcal{F}^\circ}$ *the group of germs along $f(D)$ of continuous automorphisms of \mathcal{F}^\sharp preserving $\mathcal{E}_{\mathcal{F}}$, that are holomorphic at each singular point of \mathcal{F}^\sharp that is not a nodal corner (cf. Introduction) and whose restriction to $f(D) \setminus \text{Sing}(\mathcal{F}^\sharp)$ is homotopic to the identity.*

*We define by these data the **automorphism group-graph** $\text{Aut}^{\mathcal{F}^\circ}$ over $A_{\mathcal{F}^\circ}$, the morphisms ρ_D^s , $s \in D$, being just the restriction maps.*

Remark 3.3. If D is not dicritical, the elements of $\text{Aut}_D^{\mathcal{F}^\circ}$ are transversely holomorphic at each point of $f(D) \setminus \Sigma_{\mathcal{F}}$, with $\Sigma_{\mathcal{F}} := \text{Sing}(\mathcal{F}^\sharp)$, because they are holomorphic on an open set whose saturation by \mathcal{F}^\sharp is a neighborhood of $f(D) \setminus \Sigma_{\mathcal{F}}$, cf. [16, Theorem A] or [3, Theorem 2]. \square

Now we will assign to each topological class $\mathbf{g} \in \text{Mod}([\mathcal{F}^\circ])$ a cohomology class $\mathbf{i}_{\mathcal{F}^\circ}(\mathbf{g}) \in H^1(A_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ})$. To do that we fix a representative (\mathcal{G}, g) of \mathbf{g} .

Definition 3.4. *A **good fibration** along an invariant component $g(D)$ of $\mathcal{E}_{\mathcal{G}}$ is a germ along $g(D)$ of smooth map from a neighborhood of $g(D)$ to $g(D)$, that is holomorphic at each singular point of \mathcal{G}^\sharp , equal to the identity on $g(D)$ and constant on each component adjacent to $g(D)$.*

Clearly good fibrations along invariant components always exist.

For each vertex D of $A_{\mathcal{F}^\circ}$ we fix a regular point $o_D \in D$ and good fibrations along $f(D)$ and $g(D)$. Up to isotopy we can suppose that f and g are holomorphic at the singular points of \mathcal{E} . Since $\tilde{\mathcal{H}}([\mathcal{F}, f]) = \tilde{\mathcal{H}}([\mathcal{G}, g])$, cf. (2), for each $D \in C_{\mathcal{E}^\circ}$ the conjugating map of these holonomies between fibers of the good fibrations over $f(o_D)$ and $g(o_D)$ extends by the lifting path method to a unique germ of a transversely holomorphic homeomorphism

$$\psi_D : (M_{\mathcal{F}}, f(D)) \rightarrow (M_{\mathcal{G}}, g(D)), \quad \psi_D(\mathcal{F}^\sharp) = \mathcal{G}^\sharp.$$

that conjugates the fibrations and the foliations and such that the restriction to $f(D)$ is $f^{-1} \circ g$. Classically, since the good fibrations are holomorphic

near the singular points, ψ_D is also holomorphic at these points. However, a precision must be given for the extension to the nodal singularities. The good fibrations at the nodal singularities can be chosen to coincide with the projections given by linearizing holomorphic coordinates of the node. In fact, we can choose linearizing holomorphic coordinates for the nodes $f(s)$ and $g(s)$ such that $g^{-1} \circ f$ writes as the identity. Thus, on the linear model $yx^{-\lambda}$, $\lambda \in \mathbb{R}^+ \setminus \mathbb{Q}$, we have an automorphism which is the identity on $|x| = 1$. Since it commutes with the linear holonomy $y \mapsto ye^{2i\pi\lambda}$ we deduce that it is linear on the fibers and it extends to $|x| \leq 1$ by linearity.

Thus, for each edge $s \in \text{Ed}_{\mathbf{A}_{\mathcal{F}^\circ}}$, the germs at $f(s)$

$$\varphi_{D,s} = \psi_D^{-1} \circ \psi_{D'}, \quad \varphi_{D',s} = \psi_{D'}^{-1} \circ \psi_D, \quad D \cap D' = s,$$

are holomorphic automorphisms of \mathcal{F}^\sharp and the 1-cocycle $\mathbf{c}^1 := (\varphi_{D,s})$, with $(D, s) \in I_{\mathbf{A}_{\mathcal{F}^\circ}}$, is an element of $Z^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ})$. If we choose another element $(\check{\mathcal{G}}, \check{g})$ in \mathfrak{g} , and good fibrations for $\check{\mathcal{G}}$, taking in the same way homeomorphisms

$$\check{\psi}_D : (M_{\mathcal{F}}, f(D)) \rightarrow (M_{\check{\mathcal{G}}}, \check{g}(D)), \quad \check{\psi}_D(\mathcal{F}^\sharp) = \check{\mathcal{G}}^\sharp,$$

we obtain another 1-cocycle

$$\check{\mathbf{c}}^1 = (\check{\varphi}_{D,s}) = (\check{\psi}_D^{-1} \circ \check{\psi}_{D'}) \in H^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ}).$$

Since $(\mathcal{G}, g) \sim_\diamond (\check{\mathcal{G}}, \check{g})$ there is a homeomorphism Φ between neighborhoods of the exceptional divisors $\mathcal{E}_{\mathcal{G}}$ and $\mathcal{E}_{\check{\mathcal{G}}}$ of the reductions of these foliations that conjugates \mathcal{G}^\sharp and $\check{\mathcal{G}}^\sharp$ and is holomorphic at the singular points of \mathcal{G}^\sharp , except perhaps at the nodal corners; moreover, when restricted to $\mathcal{E}_{\mathcal{G}}$, Φ is isotopic to $\check{g} \circ g^{-1}$.

Let us denote by Φ_D the germs of Φ along the invariant components D of $\mathcal{E}_{\mathcal{G}}$. One checks easily that the 0-cochain $\mathbf{c}^0 = (\psi_D^{-1} \circ \Phi_D^{-1} \circ \check{\psi}_D) \in C^0(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ})$ fulfills $\mathbf{c}^0 \star \mathbf{c}^1 = \check{\mathbf{c}}^1$. This proves that the cohomology class of \mathbf{c}^1 does not depend on the choice of the representative of the class \mathfrak{g} neither on the good fibrations used to define it. We denote this cohomology class by $\mathbf{i}_{\mathcal{F}^\circ}(\mathfrak{g})$.

Theorem 3.5. *The map*

$$(6) \quad \mathbf{i}_{\mathcal{F}^\circ} : \text{Mod}([\mathcal{F}^\circ]) \xrightarrow{\sim} H^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ})$$

is bijective. Moreover, $\mathbf{i}_{\mathcal{F}^\circ}([\mathcal{F}^\circ]) = [(\text{id})]$.

Proof. Let us recall that $\mathbf{A}_{\mathcal{F}^\circ}$ is the common cut-graph to all marked foliations (\mathcal{G}, g) with $[\mathcal{G}, g] \in \text{Mod}([\mathcal{F}^\circ])$; in this proof we will denote it by \mathbf{A} and by \mathbf{A}^i its connected components. Let us show first the injectivity of $\mathbf{i}_{\mathcal{F}^\circ}$. If

$$\mathbf{i}_{\mathcal{F}^\circ}([\mathcal{G}, g]) = [(\varphi_{D,s})] = [(\check{\varphi}_{D,s})] = \mathbf{i}_{\mathcal{F}^\circ}([\check{\mathcal{G}}, \check{g}])$$

then for each $D \in C_{\mathcal{E}^\circ}$ there exists $\xi_D \in \text{Aut}_D^{\mathcal{F}^\circ}$ such that

$$\xi_D^{-1} \circ \varphi_{D,s} \circ \xi_{D'} = \check{\varphi}_{D,s}.$$

Writing $\varphi_{D,s} = \psi_D^{-1} \circ \psi_{D'}$ and $\check{\varphi}_{D,s} = \check{\psi}_D^{-1} \circ \check{\psi}_{D'}$, with $s = D \cap D'$, we deduce that

$$\check{\psi}_D \circ \xi_D^{-1} \circ \psi_D^{-1} = \check{\psi}_{D'} \circ \xi_{D'}^{-1} \circ \psi_{D'}^{-1}$$

defines conjugations $\Phi_i : W_i \rightarrow \check{W}_i$ between the foliations \mathcal{G}^\sharp and $\check{\mathcal{G}}^\sharp$ restricted to some tubular neighborhoods W_i and \check{W}_i of $\mathcal{E}_\mathcal{G}^i := \bigcup_{D \in \mathbf{ve}_{\mathcal{A}_i}} g(D) \subset \mathcal{E}_\mathcal{G}$ and $\bigcup_{D \in \mathbf{ve}_{\mathcal{A}_i}} \check{g}(D) \subset \mathcal{E}_{\check{\mathcal{G}}}$ respectively. By composing Φ_i with suitable automorphisms of \mathcal{G}^\sharp isotopic to the identity along the leaves and whose supports are disjoint from the singular locus of \mathcal{G}^\sharp , we can assume that Φ_i conjugates the attaching points of the adjacent dicritical components.

On the other hand, since the self-intersection of $g(D)$ and $\check{g}(D)$ coincides for each $D \in \text{Comp}(\mathcal{E}^d)$, there is a conjugation Φ_D between the foliations \mathcal{G}^\sharp and $\check{\mathcal{G}}^\sharp$ restricted to some tubular neighborhoods W_D and \check{W}_D of $g(D)$ and $\check{g}(D)$ whose restriction to $g(D)$ is $\check{g} \circ g^{-1}$.

In order to glue the conjugations Φ_i and Φ_D we use the following trick:

For each $0 < \varepsilon < 1$, any germ of biholomorphism of $(\mathbb{C}^2, 0)$ preserving the fibration $(x, y) \mapsto x$ and the curve $\{y = 0\}$ can be represented by a \mathcal{C}^1 diffeomorphism F from $\mathbb{D}_1 \times \mathbb{D}_1$ onto a neighborhood of $(0, 0)$ satisfying the same properties with support in $\{|x| < \varepsilon\}$.

This implies that there is an automorphism F_D on a neighborhood of $g(D)$ preserving $g(D)$ and \mathcal{G}^\sharp , which is equal to $\Phi_D^{-1} \circ \Phi_i$ in a neighborhood of the attaching point $g(D) \cap \mathcal{E}_\mathcal{G}^i$ with support a polydisk centered at this point. Shrinking the neighborhood of definition of $\Phi_D \circ F_D$ we obtain a conjugation of pairs $(\mathcal{G}^\sharp, g(D))$ and $(\check{\mathcal{G}}^\sharp, \check{g}(D))$ which can be glued with Φ_i .

The gluing of Φ_i and Φ_j at the nodal singularities is made by using linearizing coordinates for $(\mathcal{G}, g(s))$ and $(\check{\mathcal{G}}, \check{g}(s))$ as in [15, §8.5]. In this way we obtain a global conjugation $\Phi : M_\mathcal{G} \rightarrow M_{\check{\mathcal{G}}}$ between the foliations \mathcal{G}^\sharp and $\check{\mathcal{G}}^\sharp$ which is holomorphic at the singular points.

By definition of $\text{Aut}_D^{\mathcal{F}^\diamond}$, the restrictions of ξ_D to the divisor are isotopic to the identity. Hence the restriction of Φ to $\mathcal{E}_\mathcal{G}^\diamond$ is isotopic to $\check{g} \circ g^{-1}$. Therefore $[\mathcal{G}, g] = [\check{\mathcal{G}}, \check{g}]$.

To prove the surjectivity of $\mathbf{i}_{\mathcal{F}^\diamond}$ we consider a cocycle $\mathbf{c} = (\varphi_{D,s})$ in a given class of $H^1(\mathbf{A}_{\mathcal{F}^\diamond}, \text{Aut}^{\mathcal{F}^\diamond})$. We define $\varphi_{D,s} = \text{id}$ when $f(s)$ is a nodal singularity or an attaching point of a dicritical component. By gluing open neighborhoods U_D of $f(D)$ using the local biholomorphisms $\varphi_{D,s}$ we obtain a complex manifold $M_\mathbf{c}$ endowed with a foliation $\mathcal{F}_\mathbf{c}$, a divisor $\mathcal{E}_\mathbf{c}$ and a biholomorphism between $\mathcal{E}_\mathbf{c}$ and $\mathcal{E}_\mathcal{F}$ sending the singular locus of $\mathcal{F}_\mathbf{c}$ onto $\Sigma_\mathcal{F}$. There is a composition of blow-ups $E' : M' \rightarrow (\mathbb{C}^2, 0)$ and a biholomorphism $g : M_\mathbf{c} \rightarrow M'$ sending $\mathcal{E}_\mathbf{c}$ onto the exceptional divisor $E'^{-1}(0)$, see for instance [17, p. 306]. We obtain a foliation $\mathcal{F}' = (E' \circ g)(\mathcal{F}_\mathbf{c})$ on $(\mathbb{C}^2, 0)$ and a biholomorphism $h : \mathcal{E}_\mathcal{F} \rightarrow \mathcal{E}_{\mathcal{F}'}$ satisfying $h(\Sigma_\mathcal{F}) = \Sigma_{\mathcal{F}'}$. We define $f' := h \circ f : \mathcal{E} \rightarrow \mathcal{E}_{\mathcal{F}'}$. By construction $\mathbf{i}_{\mathcal{F}^\diamond}([\mathcal{F}', f']) = [\mathbf{c}]$. \square

4. SYMMETRY GROUP-GRAPH

We keep the notations and the fixed data of the previous section. In order to define the remaining group-graph associated to \mathcal{F} , we moreover fix for each $D \in C_{\mathcal{E}^\diamond}$ a regular point $o_D \in D$ and a transverse section Δ_D to $f(D)$ passing through $f(o_D)$.

Definition 4.1. *For $s \in \text{Ed}_{\mathbf{A}_{\mathcal{F}^\diamond}}$ we say that $\phi \in \text{Aut}_s^{\mathcal{F}^\diamond}$ fixes the leaves of \mathcal{F}^\sharp if for every neighborhood V of $f(s)$ there is a neighborhood V' of $f(s)$*

such that $\phi(V') \subset V$ and for all $p \in V'$ the points p and $\phi(p)$ belong to the same leaf of $\mathcal{F}|_V$. We denote by $\text{Fix}_s^{\mathcal{F}^\diamond}$ the (normal) subgroup of $\text{Aut}_s^{\mathcal{F}^\diamond}$ of these automorphisms.

Remark 4.2. It is easy to see that an example of element of $\text{Fix}_s^{\mathcal{F}^\diamond}$ is provided by $\phi \in \text{Aut}_s^{\mathcal{F}^\diamond}$ such that $\phi|_{f(D)} = \text{id}_{f(D)}$ and $F_p \circ \phi = F_p$ for any local first integral F_p at every point $p \in f(D \setminus \Sigma)$ in a neighborhood of $f(s)$. The diffeomorphisms of the flow of a vector field tangent to the foliation fulfill this property for small times and, by composition, all the diffeomorphisms of the flow are in $\text{Fix}_s^{\mathcal{F}^\diamond}$. \square

Remark 4.3. For a fundamental system (V_α) of open neighborhoods of $f(s)$ let us denote by $\mathcal{Q}_{V_\alpha}^{\mathcal{F}}$ the leaf space of the restriction of the foliation \mathcal{F}^\sharp to $V_\alpha \setminus \mathcal{E}_{\mathcal{F}}$. The inclusion relation on the leaves induces an inverse system of continuous maps $\mathcal{Q}^{\mathcal{F}^\diamond}(s) := (\mathcal{Q}_{V_\alpha}^{\mathcal{F}} \leftarrow \mathcal{Q}_{V_\beta}^{\mathcal{F}})_{V_\beta \subset V_\alpha}$. Every $\psi \in \text{Aut}_s^{\mathcal{F}^\diamond}$ defines an automorphism³ of this inverse system $\tilde{\psi} \in \text{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s))$ and the map $\zeta : \psi \mapsto \tilde{\psi}$ is a group morphism. It turns out that ψ is the identity if and only if $\psi \in \text{Fix}_s^{\mathcal{F}^\diamond}$, i.e. we have an exact sequence:

$$1 \rightarrow \text{Fix}_s^{\mathcal{F}^\diamond} \longrightarrow \text{Aut}_s^{\mathcal{F}^\diamond} \xrightarrow{\zeta} \text{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s)).$$

\square

Definition 4.4. For $s \in \text{Ed}_{\mathcal{A}^{\mathcal{F}^\diamond}}$ and $D \in \text{Ve}_{\mathcal{A}^{\mathcal{F}^\diamond}}$ we consider the groups

$$\text{Sym}_s^{\mathcal{F}^\diamond} := \text{Aut}_s^{\mathcal{F}^\diamond} / \text{Fix}_s^{\mathcal{F}^\diamond}$$

and

$$\text{Sym}_D^{\mathcal{F}^\diamond} := \begin{cases} C(H_D) & \text{if } \text{val}_\Sigma(D) \geq 3, \\ C(H_D)/H_D & \text{if } \text{val}_\Sigma(D) \leq 2, \end{cases}$$

where $H_D \subset \text{Diff}(\Delta_D, f(o_D))$ is the holonomy group of \mathcal{F}^\sharp along $f(D)$, $C(H_D)$ is its **centralizer** inside $\text{Diff}(\Delta_D, f(o_D))$ and $\text{val}_\Sigma(D)$, called here **singular valency of D** , is the number of elements of $D \cap \Sigma$.

In order to define maps $\rho_D^s : \text{Sym}_D^{\mathcal{F}^\diamond} \rightarrow \text{Sym}_s^{\mathcal{F}^\diamond}$, $s \in D$ we will need the following result:

Lemma 4.5. If $\psi \in \text{Aut}_D^{\mathcal{F}^\diamond}$ satisfies $\psi|_{f(D)} = \text{id}_{f(D)}$ and $\psi|_{\Delta_D} = \text{id}_{\Delta_D}$ then the germ of ψ at $f(s)$ belongs to $\text{Fix}_s^{\mathcal{F}^\diamond}$.

Proof. For each $p \in f(D) \setminus \text{Sing}(\mathcal{F}^\sharp)$ we choose a local holomorphic first integral F_p of \mathcal{F} defined in a neighborhood of p . The set

$$\Omega := \{p \in f(D) \setminus \text{Sing}(\mathcal{F}^\sharp) \mid F_p \circ \psi = F_p\}$$

is open and closed in $f(D) \setminus \text{Sing}(\mathcal{F}^\sharp)$ and it contains $f(o_D) = \Delta_D \cap f(D)$. Hence $\Omega = f(D) \setminus \text{Sing}(\mathcal{F}^\sharp)$ and we conclude thanks to Remark 4.2. \square

³ The system $\mathcal{Q}^{\mathcal{F}^\diamond}(s)$ is an element of the category $\underline{\text{Top}}$ of the pro-objects associated to the category of the topological spaces and continuous maps. The objects of this category are the inverse families of topological spaces and $\text{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s))$ is the group of the invertible elements of $\varprojlim_{\beta} \varinjlim_{\alpha} C^0(\mathcal{Q}_{V_\alpha}^{\mathcal{F}}, \mathcal{Q}_{V_\beta}^{\mathcal{F}})$, cf. [7, §2.8] or [15, §3.1].

Using good fibrations (cf. Definition 3.4), each element ϕ of $C(H_D)$ can be extended to an element of $\text{Aut}_D^{\mathcal{F}^\circ}$. Thanks to Lemma 4.5, the class modulo $\text{Fix}_s^{\mathcal{F}^\circ}$ of the germ at $f(s)$ of this extension does not depend on the way of extending and hence on the choice of the good fibrations. We define $\rho_D^s(\phi)$ as this class in case $\text{val}_\Sigma(D) \geq 3$.

Before defining ρ_D^s for D with $\text{val}_\Sigma(D) \leq 2$, we must make some preliminary considerations. Let us fix for each point $s \in \Sigma \cap D$ a conformal compact disc $\mathcal{K}_s \subset \mathcal{E}$ such that $s \in \mathring{\mathcal{K}}_s$, $o_D \in \partial \mathcal{K}_s$ and $\Sigma \cap \mathcal{K}_s = \{s\}$. The simple loops γ_s that parametrize $\partial \mathcal{K}_s$ with the conformal orientation, form a generator system of the fundamental group $\pi_1(D \setminus \Sigma, o_D)$. Hence the holonomies $h_{D,s} \in \text{Diff}(\Delta_D, f(o_D))$ of the foliation \mathcal{F}^\sharp along $f \circ \gamma_s$, $s \in \Sigma \cap D$, generate H_D .

Definition 4.6. We call $h_{D,s}$ the *local holonomies* given by the *appropriate compact discs system* $(\mathcal{K}_s)_{s \in D \cap \Sigma}$.

Let us denote $K_s := f(\mathcal{K}_s)$. If we fix a good fibration, any element ϕ of the **centralizer** $C(h_{D,s})$ of $h_{D,s}$ in $\text{Diff}(\Delta_D, f(o_D))$ has a unique extension to a neighborhood of K_s , by a homeomorphism ϕ^{ext} that leaves invariant the foliation \mathcal{F}^\sharp and each fiber of the fibration; moreover ϕ^{ext} is necessarily holomorphic at $f(s)$. Taking its germ at $f(s)$ we obtain a map

$$(7) \quad \text{ext} : C(h_{D,s}) \longrightarrow \text{Aut}_s^{\mathcal{F}^\circ}, \quad \phi \mapsto \phi^{\text{ext}}$$

and the property of uniqueness of the extensions imply that this map is a group morphism.

Let us consider the inverse system $\mathcal{Q}^{\mathcal{F}^\circ}(\mathcal{K}_s) = (\mathcal{Q}_{W_\alpha}^{\mathcal{F}} \leftarrow \mathcal{Q}_{W_\beta}^{\mathcal{F}})_{W_\beta \subset W_\alpha}$, where $(W_\alpha)_\alpha$ is the fundamental system of neighborhoods of K_s and $\mathcal{Q}_{W_\alpha}^{\mathcal{F}}$ is the leaf space of the restriction of the foliation to $W_\alpha \setminus \mathcal{E}_{\mathcal{F}}$. Let $V_s \subset K_s$ be a small open disc centered at $f(s)$. Over $K_s \setminus V_s$ the foliation is a collar; thus we have an isomorphism of inverse systems (i.e. an isomorphism of the category $\underline{\text{Top}}$)

$$(8) \quad \mathcal{Q}^{\mathcal{F}^\circ}(s) \xrightarrow{\sim} \mathcal{Q}^{\mathcal{F}^\circ}(\mathcal{K}_s).$$

We also consider the orbit spaces $\mathcal{Q}_\alpha^{h_{D,s}}$ of the pseudogroup defined by the restriction of $h_{D,s}$ to $\Delta_D \cap W_\alpha$; they form an inverse system $\mathcal{Q}^{h_{D,s}} = (\mathcal{Q}_\alpha^{h_{D,s}} \leftarrow \mathcal{Q}_\beta^{h_{D,s}})_{W_\beta \subset W_\alpha}$. We can choose each W_α such that there are retractions along the leaves from $W_\alpha \setminus \mathcal{E}_{\mathcal{F}}$ on $(W_\alpha \setminus \mathcal{E}_{\mathcal{F}}) \cap \pi_D^{-1}(\partial K_s)$, π_D being the good fibration; moreover we can require that $W_\alpha \cap \pi_D^{-1}(\partial K_\alpha)$ is a set of suspension type, cf. [13, Definition 3.1.1]. This property implies that the leaf space of the restriction of the foliation to this set can be identified to the orbit space $\mathcal{Q}_\alpha^{h_{D,s}}$ of the restriction of $h_{D,s}$ to $\Delta_D \cap W_\alpha$. Hence using (8), the retractions induce isomorphisms

$$\tau : \mathcal{Q}^{\mathcal{F}^\circ}(s) \xrightarrow{\sim} \mathcal{Q}^{h_{D,s}} \text{ and } \tau_* : \text{Aut}(\mathcal{Q}^{\mathcal{F}^\circ}(s)) \xrightarrow{\sim} \text{Aut}(\mathcal{Q}^{h_{D,s}}), \quad \tau_*(\varphi) := \tau \circ \varphi \circ \tau^{-1},$$

the inverse of τ being given by the inclusion relations of the orbits of $h_{D,s}$ in the leaves of the foliation on neighborhoods of K_s .

Each element ϕ of $C(h_{D,s})$ induces an automorphism $\xi(\phi) := \tau_*(\xi(\phi^{\text{ext}}))$ of $\mathcal{Q}^{h_{D,s}}$ and the map $\xi : C(h_{D,s}) \rightarrow \text{Aut}(\mathcal{Q}^{h_{D,s}})$ is a group morphism.

Lemma 4.7. *The kernel of the morphism ξ is the cyclic group generated by $h_{D,s}$.*

Proof. Let us take $\phi \in \ker(\xi)$. This means that for each open neighborhood U of $f(o_D)$ in Δ_D there is an open set $V \supset U$ such that for each $z \in U$, z and $\phi(z)$ are in the same V -orbit of $h_{D,s}$. The V -orbit of z being the set of points z' of V such that either there exists $n \in \mathbb{N}$ fulfilling either $\phi(z), \dots, \phi^n(z) \in V$ and $\phi^n(z) = z'$, or $\phi^{-1}(z), \dots, \phi^{-n}(z) \in V$ and $\phi^{-n}(z) = z'$. If $|h'_{D,s}(f(o_D))| \neq 1$ then $h_{D,s}(z) = \lambda z$ and $\phi(z) = \mu z$. Hence $\phi(z) = \mu z = \lambda^{\nu(z)} z = h_{D,s}^{\nu(z)}(z)$ implies $\nu(z)$ constant.

Otherwise, let V_n be the set of points $z \in U$ such that $h_{D,s}^k(z) \in V$ for each $k = 0, \dots, n$. There is an uncountable set K invariant by $h_{D,s}$ such that for all $n \in \mathbb{Z}$ it is contained in the connected component of V_n containing $f(o_D)$. If $h_{D,s}$ is linearizable (conjugated to a rotation) then we can take an invariant conformal disc as K . If $h_{D,s}$ is resonant non linearizable then K is a union of petals contained in U . If $h_{D,s}$ is non resonant non linearizable we take as K the hedgehog associated to U , cf. [27].

For each $z \in K$ there is an integer $\nu(z)$ such that $\phi(z) = h_{D,s}^{\nu(z)}(z)$. Thus, there is $n \in \mathbb{Z}$ such that ϕ and $h_{D,s}^n$ coincide on an uncountable subset of K and by isolated zeros principle they coincide on the connected component of V_n containing $f(o_D)$. Then the germs of ϕ and $h_{D,s}^n$ at $f(o_D)$ are equal, that achieves the proof. \square

Corollary 4.8. *The extension map (7) induces an isomorphism:*

$$[\text{ext}] : C(h_{D,s}) / \langle h_{D,s} \rangle \xrightarrow{\sim} \text{Sym}_s^{\mathcal{F}^\circ}.$$

Proof. By construction the following diagram is commutative

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Fix}_s^{\mathcal{F}^\circ} & \longrightarrow & \text{Aut}_s^{\mathcal{F}^\circ} & \xrightarrow{\xi} & \text{Aut}(\mathcal{Q}^{\mathcal{F}}(s)) \\ & & & & \uparrow \text{ext} & & \downarrow \tau \\ 1 & \longrightarrow & \langle h_{D,s} \rangle & \longrightarrow & C(h_{D,s}) & \xrightarrow{\xi} & \text{Aut}(\mathcal{Q}^{h_{D,s}}) \end{array}$$

Thanks to Remark (4.3) and Lemma (4.7) the lines are exact. Because τ is an isomorphism, ext induces an isomorphism between $C(h_{D,s}) / \langle h_{D,s} \rangle$ and $\text{Aut}_s^{\mathcal{F}^\circ} / \text{Fix}_s^{\mathcal{F}^\circ} = \text{Sym}_s^{\mathcal{F}^\circ}$. \square

Remark 4.9. Suppose that there are local coordinates u_1, u_2 at s for which \mathcal{F}^\sharp is defined by a linear differential 1-form $\omega = \mu u_2 du_1 - u_1 du_2$ and $D = \{u_2 = 0\}$, $D' = \{u_1 = 0\}$ are the components of $\mathcal{E}_{\mathcal{F}}$. On the transversals $\{u_i = 1\}$ the local holonomies are $h_{D_1,s}(u_2) = e^{2\pi i \mu} u_2$ and $h_{D_2,s}(u_1) = e^{2\pi i \frac{1}{\mu}} u_1$ and their centralizers are formed by the linear automorphisms in the coordinates u_i , $C(h_{D_i,s}) = \mathbb{C}^* u_i$. Therefore we have isomorphisms

$$\frac{C(h_{D_1,s})}{\langle h_{D_1,s} \rangle} \simeq \frac{\mathbb{C}}{2\pi i(\mathbb{Z} + \mu \mathbb{Z})} \xrightarrow{\tau_1} \text{Sym}_s^{\mathcal{F}^\circ} \xleftarrow{\tau_2} \frac{\mathbb{C}}{2\pi i(\mathbb{Z} + \frac{1}{\mu} \mathbb{Z})} \simeq \frac{C(h_{D_2,s})}{\langle h_{D_2,s} \rangle}$$

To describe $\tau_2^{-1} \circ \tau_1$ let us remark that the automorphisms $(u_1, u_2) \mapsto (e^t u_1, e^{\mu t} u_2)$, $t \in \mathbb{C}$ are elements of $\text{Fix}_s^{\mathcal{F}^\circ}$. Thus the automorphisms $(u_1, u_2) \mapsto (e^t u_1, u_2)$ and $(u_1, u_2) \mapsto (u_1, e^{\mu t} u_2)$ in $\text{Aut}_s^{\mathcal{F}^\circ}$ that extend $h_{D_1, s}$ and $h_{D_2, s}$ respectively, define the same element of $\text{Sym}_s^{\mathcal{F}^\circ}$. It follows:

$$\tau_2^{-1} \circ \tau_1 : \mathbb{C}/2\pi i(\mathbb{Z} + \mu\mathbb{Z}) \longrightarrow \mathbb{C}/2\pi i(\mathbb{Z} + \frac{1}{\mu}\mathbb{Z}), \quad t \mapsto -\frac{1}{\mu}t.$$

□

Remark 4.10. If $\text{val}_\Sigma(D) \geq 3$, then $[\text{ext}]^{-1} \circ \rho_D^s$ is the quotient map

$$\text{Sym}_D^{\mathcal{F}^\circ} = C(H_D) \hookrightarrow C(h_{D, s}) \rightarrow C(h_{D, s})/\langle h_{D, s} \rangle.$$

□

For D containing at most two singular points of $\mathcal{F}^\#$ we define

$$\rho_D^s := [\text{ext}] : \text{Sym}_D^{\mathcal{F}^\circ} = C(h_{D, s})/\langle h_{D, s} \rangle \rightarrow \text{Sym}_s^{\mathcal{F}^\circ}.$$

Definition 4.11. We call **symmetry group-graph** and we denote by $\text{Sym}^{\mathcal{F}^\circ}$ the group-graph consisting of the groups $\text{Sym}_D^{\mathcal{F}^\circ}$, $\text{Sym}_s^{\mathcal{F}^\circ}$, with $D \in \text{Ve}_{\mathcal{A}_{\mathcal{F}^\circ}}$, $s \in \text{Ed}_{\mathcal{A}_{\mathcal{F}^\circ}}$ and the morphisms ρ_D^s , $s \in D$.

Now, we are going to define a group-graph morphism $\alpha : \text{Aut}^{\mathcal{F}^\circ} \rightarrow \text{Sym}^{\mathcal{F}^\circ}$ which will induce an isomorphism on the 1-cohomology. If $s \in \text{Ed}_{\mathcal{A}_{\mathcal{F}^\circ}}$, we define α_s as the quotient map $\text{Aut}_s^{\mathcal{F}^\circ} \rightarrow \text{Sym}_s^{\mathcal{F}^\circ}$. If $D \in \text{Ve}_{\mathcal{A}_{\mathcal{F}^\circ}}$, we define $\alpha_D : \text{Aut}_D^{\mathcal{F}^\circ} \rightarrow \text{Sym}_D^{\mathcal{F}^\circ}$ as follows. Fix $\Phi \in \text{Aut}_D^{\mathcal{F}^\circ}$ and take an homotopy $\phi_t : f(D \setminus \Sigma) \rightarrow f(D \setminus \Sigma)$, $t \in [0, 1]$, between $\phi_0 := \Phi|_{f(D \setminus \Sigma)}$ and $\phi_1 := \text{id}_{f(D \setminus \Sigma)}$. Consider the path $\beta(t) = \phi_t(o_D)$ and the holonomy map $h_\beta : (\Phi(\Delta_D), \Phi(o_D)) \rightarrow (\Delta_D, o_D)$ associated to it. It is easy to see that $h_\beta \circ \Phi|_{\Delta_D}$ belongs to $C(H_D)$. If D has singular valency $\text{val}_\Sigma(D) \geq 3$, the group consisting of the homeomorphisms of $f(D \setminus \Sigma)$ which are homotopic to the identity is simply connected [33]; consequently h_β does not depend on the chosen isotopy ϕ_t and we can put $\alpha_D(\Phi) := h_\beta \circ \Phi$. Finally, if $v(D) \leq 2$ then only the class $[h_\beta \circ \Phi|_{\Delta_D}]$ of $h_\beta \circ \Phi|_{\Delta_D}$ modulo H_D is well-defined and we put $\alpha_D(\Phi) := [h_\beta \circ \Phi|_{\Delta_D}]$.

Lemma 4.12. $\alpha : \text{Aut}^{\mathcal{F}^\circ} \rightarrow \text{Sym}^{\mathcal{F}^\circ}$ is a group-graph morphism.

Proof. We must see that the following diagram is commutative:

$$\begin{array}{ccc} \text{Aut}_D^{\mathcal{F}^\circ} & \xrightarrow{\alpha_D} & \text{Sym}_D^{\mathcal{F}^\circ} \\ \rho_D^s \downarrow & & \downarrow \rho_D^s \\ \text{Aut}_s^{\mathcal{F}^\circ} & \xrightarrow{\alpha_s} & \text{Sym}_s^{\mathcal{F}^\circ} \end{array}$$

where $\check{\rho}_D^s$, resp. ρ_D^s , denote the “restriction morphisms” of the group-graphs $\text{Aut}^{\mathcal{F}^\circ}$, resp. $\text{Sym}^{\mathcal{F}^\circ}$. Let us consider two cases, depending on the singular valency $\text{val}_\Sigma(D)$. First let us assume $\text{val}_\Sigma(D) \geq 3$ and let us fix $\phi \in \text{Aut}_D^{\mathcal{F}^\circ}$. Then $\alpha_D(\phi) = h_\beta \circ \phi|_{\Delta_D}$ with $h_\beta : \phi(\Delta_D) \rightarrow \Delta_D$ the holonomy along a path β . There exists $\phi' \in \text{Aut}_D^{\mathcal{F}^\circ}$ with compact support outside $f(D) \cap \text{Sing}(\mathcal{F}^\#)$ whose restriction to Δ_D coincides with h_β . Indeed ϕ' can be constructed by composition of flows of tangent vector fields whose

supports intersect the divisor D in conformal disks disjoint from the singularities and which cover the image of β . Thus $\alpha_D(\phi) = \phi' \circ \phi|_{\Delta_D}$ and $\rho_D^s(\alpha_D(\phi))$ coincides with the class modulo $\text{Fix}_s^{\mathcal{F}^\circ}$ of the germ of $\phi' \circ \phi$ at s thanks to Lemma 4.5. This germ is just the germ of ϕ at s because the support of ϕ' does not intersect the singularities. This achieves the proof in the case $\text{val}_\Sigma(D) \geq 3$. If $\text{val}_\Sigma(D) \leq 2$, the only difference is that only the class of $h_\beta \circ \phi|_{\Delta_D}$ modulo $H_D = \langle h_{D,s} \rangle$ is well-defined; but we can proceed analogously choosing arbitrarily h_β . \square

Proposition 4.13 (Extension). *Let W be a neighborhood of $f(s)$, $s \in \text{Ed}_{\mathcal{A}^{\mathcal{F}^\circ}}$, then each germ $\phi \in \text{Fix}_s^{\mathcal{F}^\circ}$ can be extended to a germ $\Phi \in \text{Aut}_D^{\mathcal{F}^\circ}$ along $f(D)$, whose support fulfills $\text{supp}(\Phi) \cap f(D) \subset W$.*

Proof. At the point $f(s)$ let us fix local holomorphic coordinates (u, v) , $u(f(s)) = v(f(s)) = 0$ such that the axes are invariant by the foliation, and $v = 0$ is a local equation of $f(D)$. We denote by $Z = u \frac{\partial}{\partial u} + v B(u, v) \frac{\partial}{\partial v}$ the holomorphic vector field tangent to the foliation.

First, we will see that each germ of biholomorphism $\zeta : (f(D), f(s)) \rightarrow (f(D), f(s))$ can be extended as an element g of $\text{Aut}_D^{\mathcal{F}^\circ}$ whose germ at $f(s)$ belongs to $\text{Fix}_s^{\mathcal{F}^\circ}$ and whose support intersect $f(D)$ inside W . This is easy to prove when ζ is embedded in the flow $(\psi_t)_t$ of a vector field $a(u)u \frac{\partial}{\partial u}$, i.e. $\zeta = \psi_1$. Indeed in this case, let us consider the real vector field Y whose flow is the flow of aZ , but with real times. Let us take a real smooth function ρ equal to 1 on an open neighborhood of $f(s)$, such that $\text{supp}(\rho) \cap f(D)$ is contained in W and in a definition domain of Y . Then ρY extends by zero along $f(D)$ and the elements Ψ_t of its flow induce homeomorphisms defined on neighborhoods of $f(D)$. Their supports are contained in the support of ρ and their germs at $f(s)$ are element of $\text{Fix}_s^{\mathcal{F}^\circ}$, cf. Example (4.2). Clearly the restriction of Ψ_1 to $f(D)$ is equal to ζ near $f(s)$. Now, when ζ is not embedded in a flow, we decompose $\zeta = \zeta_1 \circ \zeta_2$, with $|\zeta_1'(0)|, |\zeta_2'(0)| \neq 1$. Both ζ_1 and ζ_2 are linearizable. Thus they can both be embedded in a flow and have convenient extensions. Their composition extends ϕ along $f(D)$, and fulfils the required properties.

Now, up to composition we can suppose that the restriction of the germ ϕ to $f(D)$ is the identity. Let us choose $\varepsilon > 0$ such that the compact disc $\overline{\mathbb{D}_{2\varepsilon}} \subset f(D)$ defined by $|u| \leq 2\varepsilon$, is contained in W and in a definition domain of ϕ . Denote by C the compact annulus contained in $\overline{\mathbb{D}_{2\varepsilon}}$ given by $\varepsilon \leq |u| \leq 2\varepsilon$. By implicit function theorem, there is a holomorphic function τ defined in an open neighborhood Ω of C , that verifies:

$$(u \circ \phi)(m) = u \circ \Phi_{\tau(m)}^Z(m) \quad \text{and} \quad \tau|_{f(D)} = 0,$$

Φ_t^Z being the flow of the previous vector field Z . Let us take a \mathcal{C}^∞ function $\alpha : f(D) \rightarrow \mathbb{R}$ with compact support in $\Omega \cap f(D)$, that is equal to 1 on a neighborhood of C . The map

$$\xi : m \mapsto \xi(m) := \Phi_{\alpha(u(m))\tau(m)}^Z(m)$$

is a \mathcal{C}^∞ diffeomorphism, because its restriction to $f(D)$ is the identity and moreover it is a local diffeomorphism. Indeed using coordinates (u, z) at each point of $f(D)$, with z a local first integral of the foliation, we easily see

that the jacobian matrix of ξ is the identity. Clearly $\chi := \phi \circ \xi^{-1}$ coincides with ϕ near $f(s)$, it preserves the foliation and it leaves invariant each fiber of u :

$$\Delta_c := \{u = c\}, \quad \varepsilon \leq |c| \leq 2\varepsilon.$$

Thus the restriction $\chi|_{\Delta_c}$ of χ to Δ_c leaves invariant the orbits of the holonomy map of \mathcal{F}^\sharp around $f(s)$ represented on Δ_c -which is equal to the restriction of $\Phi_{2\pi}^Z$ to Δ_c . By Lemma 4.7, $\chi|_{\Delta_c}$ is an iterated of this holonomy map. We deduce of it the existence of an integer $p \in \mathbb{Z}$ such that near C , χ coincides with $\Phi_{2i\pi p}^Z$.

Let us take now a function $\sigma : [0, 2\varepsilon] \rightarrow \mathbb{R}$ vanishing on $[0, \varepsilon]$ and being equal to 1 on $[\frac{3}{2}\varepsilon, 2\varepsilon]$. The homeomorphism $\Theta : m \mapsto \Phi_{2i\pi p\sigma(|u(m)|)}^Z(m)$ is the identity near $f(s)$, it coincides with Ψ for $\frac{3}{2}\varepsilon \leq |u| \leq 2\varepsilon$ and it leaves \mathcal{F} invariant. To end the proof, we define the required diffeomorphism Φ as the germ along $f(D)$ of the diffeomorphism equal to $\Theta^{-1} \circ \Psi$ when $|u| \leq 2\varepsilon$ and equal to the identity otherwise. \square

Theorem 4.14. *The morphism of group-graphs $\alpha : \text{Aut}^{\mathcal{F}^\circ} \rightarrow \text{Sym}^{\mathcal{F}^\circ}$ induces a natural bijection*

$$(9) \quad \alpha_1 : H^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ}) \xrightarrow{\sim} H^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Sym}^{\mathcal{F}^\circ}).$$

Proof. The surjectivity of α^1 follows easily from the surjectivity of $\alpha_s : \text{Aut}_s^{\mathcal{F}^\circ} \rightarrow \text{Sym}_s^{\mathcal{F}^\circ}$. For this we fix an orientation \prec of the tree and for $(c_{D,s}) \in Z^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Sym}^{\mathcal{F}^\circ})$, we choose for each edge s with $\partial s = \{D, D'\}$, $D \prec D'$, an element $\varphi_{D,s}$ such that $\alpha_s(\varphi_{D,s}) = c_{D,s}$ and we set $\varphi_{D',s} := \varphi_{D,s}^{-1}$. Clearly the family $(\varphi_{D,s})$ is an element of $Z^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ})$ defining a lift of $(c_{D,s})$.

To prove the injectivity of α^1 we consider $[\phi_{D,s}], [\tilde{\phi}_{D,s}] \in H^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ})$ such that $\alpha^1([\phi_{D,s}]) = [\alpha_s(\phi_{D,s})] = [\alpha_s(\tilde{\phi}_{D,s})] = \alpha^1([\tilde{\phi}_{D,s}])$. Then there is $(g_D) \in C^0(\mathbf{A}_{\mathcal{F}^\circ}, \text{Sym}^{\mathcal{F}^\circ})$ such that

$$\alpha_s(\tilde{\phi}_{D,s}) = \rho_D^s(g_D)^{-1} \circ \alpha_s(\phi_{D,s}) \circ \rho_{D'}^s(g_{D'}) \in \text{Sym}_s^{\mathcal{F}^\circ}$$

where $s = D \cap D'$. Let $\varphi_D \in \text{Aut}_D^{\mathcal{F}^\circ}$ be extensions of $g_D \in \text{Sym}_D^{\mathcal{F}^\circ}$ and let us denote by $(\varphi_D)_s \in \text{Aut}_s^{\mathcal{F}^\circ}$ its germ at s . Then

$$\alpha_s(\tilde{\phi}_{D,s}) = \alpha_s((\varphi_D^{-1})_s) \circ \alpha_s(\phi_{D,s}) \circ \alpha_s((\varphi_{D'})_s)$$

and there is $F_s \in \text{Fix}_s^{\mathcal{F}^\circ}$ such that $\tilde{\phi}_{D,s} = (\varphi_D^{-1})_s \circ \phi_{D,s} \circ (\varphi_{D'})_s \circ F_s$. Now we choose a map $\delta : \text{Ed}_{\mathbf{A}_{\mathcal{F}^\circ}} \rightarrow \text{Ve}_{\mathbf{A}_{\mathcal{F}^\circ}}$ such that $s \in \delta(s)$ for each $s \in \text{Ed}_{\mathbf{A}_{\mathcal{F}^\circ}}$ and we define \bar{F}_D as the composition over the set $\{s \in \text{Ed}_{\mathbf{A}_{\mathcal{F}^\circ}} \mid \delta(s) = D\}$ of extensions of F_s to a neighborhood of $\delta(s)$ with disjoint supports given by Proposition 4.13. Finally putting $\bar{\varphi}_D = \varphi_D \circ \bar{F}_D \in \text{Aut}_D^{\mathcal{F}^\circ}$ we have that

$$\tilde{\phi}_{D,s} = \bar{\varphi}_D^{-1} \circ \phi_{D,s} \circ \bar{\varphi}_{D'}^{-1} \in \text{Aut}_s^{\mathcal{F}^\circ},$$

i.e. $[\phi_{D,s}] = [\tilde{\phi}_{D,s}]$ in $H^1(\mathbf{A}_{\mathcal{F}^\circ}, \text{Aut}^{\mathcal{F}^\circ})$. \square

Proof of Theorem A. It follows immediately from Theorem 3.5 and Theorem 4.14. \square

5. FOLIATIONS OF FINITE TYPE

In this section we introduce the optimal condition on a singular germ of foliation \mathcal{F} in order to have a finite dimensional moduli space $\text{Mod}([\mathcal{F}^\diamond])$. We keep all the notations of previous sections.

Given a marked foliation $\mathcal{F}^\diamond = (\mathcal{F}, f)$ and a sheaf Q defined on a neighborhood of $\mathcal{E}_{\mathcal{F}}$ in the ambient space $M_{\mathcal{F}}$ of \mathcal{F}^\sharp , we can associate a group-graph, denoted $Q^{\mathcal{F}^\diamond}$, over the cut-graph $A_{\mathcal{F}^\diamond}$ as follows: if $s \in \text{Ed}_{A_{\mathcal{F}^\diamond}}$ then $Q_s^{\mathcal{F}^\diamond}$ is the stalk of Q at $f(s)$ and if $D \in \text{Ve}_{A_{\mathcal{F}^\diamond}}$, then

$$Q_D^{\mathcal{F}^\diamond} := H^0(f(D), \iota_{f(D)}^{-1} Q),$$

$\iota_{f(D)}$ being the inclusion map of $f(D)$ in $M_{\mathcal{F}}$ and for $s \in D$ the morphism $\rho_D^s : Q_D^{\mathcal{F}^\diamond} \rightarrow Q_s^{\mathcal{F}^\diamond}$ being the canonical restrictions.

Definition 5.1. We call **group-graph of transverse infinitesimal symmetries** of \mathcal{F} the group-graph $\mathcal{T}^{\mathcal{F}^\diamond}$ associated to the sheaf $\mathcal{T}^{\mathcal{F}^\diamond} := \mathcal{B}^{\mathcal{F}^\sharp} / \mathcal{X}^{\mathcal{F}^\sharp}$ on $M_{\mathcal{F}}$ equal to the quotient of the sheaf $\mathcal{B}^{\mathcal{F}^\sharp}$ of \mathcal{F}^\sharp -basic⁴ holomorphic vector fields tangent to the \mathcal{F}^\sharp -invariant components of $\mathcal{E}_{\mathcal{F}}$, by the sheaf $\mathcal{X}^{\mathcal{F}^\sharp}$ of holomorphic vector fields tangent to \mathcal{F}^\sharp .

Remark 5.2. For each $(D, s) \in I_{\mathcal{E}^\diamond}$ let us consider the local holonomies $h_{D,s}$ as in Definition 4.6. There are linear isomorphisms, depending on the choice of an appropriate disc system,

$$\mathcal{T}_D^{\mathcal{F}^\diamond} \xrightarrow{\sim} \mathcal{T}_{H_D}, \quad \mathcal{T}_s^{\mathcal{F}^\diamond} \xrightarrow{\sim} \mathcal{T}_{h_{D,s}},$$

where \mathcal{T}_{H_D} (resp. $\mathcal{T}_{h_{D,s}}$) is the vector space of germs at $f(o_D)$ of vector fields on the transversal disc Δ_D which are invariant by the holonomy group H_D of \mathcal{F}^\sharp along $f(D)$ (resp. invariant by $h_{D,s}$), see [11, 21]. Moreover, if $\mathcal{X}_{\Delta_D}^0$ denotes the set of germs of vector fields on $(\Delta_D, f(o_D))$ vanishing at $f(o_D)$, then the exponential map $\exp : \mathcal{X}_{\Delta_D}^0 \rightarrow \text{Diff}(\Delta_D, f(o_D))$ sends $\mathcal{T}_{h_{D,s}}$ into $C(h_{D,s})$ and \mathcal{T}_{H_D} into $C(H_D)$. \square

Now we define a **coloring on $A_{\mathcal{F}^\diamond}$** by saying:

- (1) $D \in \text{Ve}_{A_{\mathcal{F}^\diamond}}$ is **green** if the holonomy group H_D is finite,
- (2) $s \in \text{Ed}_{A_{\mathcal{F}^\diamond}}$ is **green** if for each $D \in \partial s$ the holonomy map $h_{D,s}$ is periodic.
- (3) $D \in \text{Ve}_{A_{\mathcal{F}^\diamond}}$ or $s \in \text{Ed}_{A_{\mathcal{F}^\diamond}}$ are **red** otherwise.

Let us denote by $\mathcal{J}^{\mathcal{F}^\diamond}$ the **group-graph of holomorphic first integrals** associated to the sheaf of germs of holomorphic first integrals of \mathcal{F}^\sharp . Because \mathcal{F}^\sharp does not have saddle-node singularities, an element $a \in \text{Ve}_{A_{\mathcal{F}^\diamond}} \cup \text{Ed}_{A_{\mathcal{F}^\diamond}}$ is green iff $\mathcal{J}_a^{\mathcal{F}^\diamond} \neq \mathbb{C}$, see [18]. Notice that if an edge $s = D \cap D'$ of $A_{\mathcal{F}^\diamond}$ is red then the vertices D and D' are also red. Hence the set of red elements of $A_{\mathcal{F}^\diamond}$ is a subgraph called **red graph of \mathcal{F}^\diamond** and denoted by $R_{\mathcal{F}^\diamond}$.

Proposition 5.3. Let s and $D \in \partial s$ be a green edge and a green vertex of $A_{\mathcal{F}^\diamond}$. Then the following properties are equivalent:

- (1) the holonomy group H_D is generated by $h_{D,s}$;

⁴i.e. whose flow leaves \mathcal{F}^\sharp invariant.

- (2) $\mathcal{J}_D^{\mathcal{F}^\diamond} \rightarrow \mathcal{J}_s^{\mathcal{F}^\diamond}$ is surjective;
- (3) $\mathcal{T}_D^{\mathcal{F}^\diamond} \rightarrow \mathcal{T}_s^{\mathcal{F}^\diamond}$ is surjective;
- (4) $\text{Sym}_D^{\mathcal{F}^\diamond} \rightarrow \text{Sym}_s^{\mathcal{F}^\diamond}$ is surjective.

Proof. Let D be a green vertex of $\mathcal{A}_{\mathcal{F}^\diamond}$ and let $z : (\Delta_D, f(o_D)) \rightarrow \mathbb{C}$ be a linearizing coordinate of the holonomy group $H_D \subset \text{Diff}(\mathbb{C}, 0)$ which is finite. For each singular point s of D (necessarily a green edge of $\mathcal{A}_{\mathcal{F}^\diamond}$) there is $n_{D,s} \in \mathbb{N}$ such that the local holonomies $h_{D,s}$ given in Definition 4.6 are $h_{D,s}(z) = \zeta_{D,s} z$ for some primitive $n_{D,s}$ -root of unity. Let us denote by $n_D \in \mathbb{N}$ the least common multiple of $\{n_{D,s}, s \in D \cap \Sigma\}$ and by ζ_D a primitive n_D -root of unity. Because a first integral is completely determined by its restriction to the transversal Δ_D , we can consider $\mathcal{J}_D^{\mathcal{F}^\diamond}$ as subrings of $\mathbb{C}\{z\}$. In the same way, by extending the elements of $\mathcal{J}_s^{\mathcal{F}^\diamond}$ along the compact discs used in Definition 4.6 to define $h_{D,s}$, we can also consider $\mathcal{J}_s^{\mathcal{F}^\diamond}$ as a subring of $\mathbb{C}\{z\}$. With these identifications and using Remark 5.2, we have the following well known equalities and isomorphisms:

$$\begin{aligned} \mathcal{J}_D^{\mathcal{F}^\diamond} &= \mathbb{C}\{z^{n_D}\}, \quad \mathcal{T}_D^{\mathcal{F}^\diamond} \simeq \mathcal{T}_{H_D} = \mathcal{J}_D^{\mathcal{F}^\diamond} z \partial_z, \quad C(H_D) = \{z(\alpha + \mathfrak{N}_D) \mid \alpha \in \mathbb{C}^*\}, \\ \mathcal{J}_s^{\mathcal{F}^\diamond} &= \mathbb{C}\{z^{n_{D,s}}\}, \quad \mathcal{T}_s^{\mathcal{F}^\diamond} \simeq \mathcal{T}_{h_{D,s}} = \mathcal{J}_s^{\mathcal{F}^\diamond} z \partial_z, \quad C(h_{D,s}) = \{z(\alpha + \mathfrak{N}_{D,s}) \mid \alpha \in \mathbb{C}^*\}, \end{aligned}$$

with $\mathfrak{N}_D \subset \mathcal{J}_D^{\mathcal{F}^\diamond}$ and $\mathfrak{N}_{D,s} \subset \mathcal{J}_{D,s}^{\mathcal{F}^\diamond}$ being the maximal ideals. Furthermore H_D is cyclic, generated by $\zeta_D z$. The required equivalences follow immediately. \square

If B is a nonempty connected subgraph of a connected component $\mathcal{A}_{\mathcal{F}^\diamond}^i$ of $\mathcal{A}_{\mathcal{F}^\diamond}$, then for every vertex $D \notin B$ of $\mathcal{A}_{\mathcal{F}^\diamond}^i$ or $D \in \partial B$ there is a unique geodesic $[D, B] \subset \mathcal{A}_{\mathcal{F}^\diamond}^i$ joining D and ∂B . We define the following **pre-order relation** on the set of vertices of the closure of $\mathcal{A}_{\mathcal{F}^\diamond}^i \setminus B$ by means of

$$D' \leq D \iff D' \in [D, B].$$

Definition 5.4. We say that $B \subset \mathcal{A}_{\mathcal{F}^\diamond}^i$ is **repulsive** if for each edge $s = D \cap D'$ of $(\mathcal{A}_{\mathcal{F}^\diamond}^i \setminus B)$ with $D' \leq D$, $\text{Sym}_D^{\mathcal{F}^\diamond} \rightarrow \text{Sym}_s^{\mathcal{F}^\diamond}$ is surjective.

A change of marking induces an isomorphism of colored graphs compatible with the repulsiveness property that gives sense to the following definition:

Definition 5.5. The foliation \mathcal{F} is of **finite type** if for each connected component $\mathcal{A}_{\mathcal{F}^\diamond}^i$ of $\mathcal{A}_{\mathcal{F}^\diamond}$ we have: either the subgraph $\mathcal{A}_{\mathcal{F}^\diamond}^i \cap \mathcal{R}_{\mathcal{F}^\diamond}$ is nonempty, connected and repulsive, or it is empty and there exists a green repulsive vertex in $\mathcal{A}_{\mathcal{F}^\diamond}^i$.

This finiteness property does not depend on the marking. In fact thanks to Theorem 10.4 of Appendix, it depends only on the topological class of the germ \mathcal{F} at $0 \in \mathbb{C}^2$ and it is fulfilled by all the foliations \mathcal{G} with $[\mathcal{G}^\diamond] \in \text{Mod}([\mathcal{F}^\diamond])$ as soon as it holds for one of them.

Theorem 5.6. If \mathcal{F} is of finite type then the extension by identity map $Z^1(\mathcal{R}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}) \rightarrow Z^1(\mathcal{A}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond})$ induces a bijection :

$$(10) \quad H^1(\mathcal{A}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}) \simeq H^1(\mathcal{R}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}).$$

Proof. It is a direct consequence of Pruning Theorem 2.11 and Proposition 5.3. \square

When the foliation \mathcal{F} has finite type let us now give the precise value of the integer $\tau_{\mathcal{F}}$ in the statement of Theorem C in the introduction. Let us consider the following subgraph $R_{\mathcal{F}^\diamond}^0$ whose

- vertices correspond by f to the invariant irreducible components of $\mathcal{E}_{\mathcal{F}}$ whose holonomy groups do not leave invariant a non-trivial vector field,
- edges correspond by f to the resonant non-normalizable or non-resonant non-linearizable singularities of \mathcal{F}^\sharp .

As before changes of marking induce isomorphisms between the graphs $R_{\mathcal{F}^\diamond}^0$; that allows us to put:

Definition 5.7. *If \mathcal{F} is of finite type we call **codimension of \mathcal{F}** the integer*

$$\tau_{\mathcal{F}} := \text{rank } H_1(R_{\mathcal{F}^\diamond}/R_{\mathcal{F}^\diamond}^0, \mathbb{Z}),$$

where $R_{\mathcal{F}^\diamond}/R_{\mathcal{F}^\diamond}^0$ is the quotient graph obtained from $R_{\mathcal{F}^\diamond}$ by collapsing $R_{\mathcal{F}^\diamond}^0$ to a single vertex.

We will highlight now a group structure on $H^1(R_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond})$ when \mathcal{F} has finite type. Let us choose an arbitrary map $s \mapsto D_s$ from $\text{Ed}_{R_{\mathcal{F}^\diamond}}$ to $\text{Ver}_{R_{\mathcal{F}^\diamond}}$ with $D_s \in \partial s$. Since $H^1(R_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond})$ is the quotient of

$$Z^1(R_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}) \simeq \bigoplus_{s \in \text{Ed}_{R_{\mathcal{F}^\diamond}}} \text{Sym}_s^{\mathcal{F}^\diamond} \simeq \bigoplus_{s \in \text{Ed}_{R_{\mathcal{F}^\diamond}}} \frac{C(h_{D_s, s})}{\langle h_{D_s, s} \rangle}$$

by $C^0(R_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}) = \bigoplus_{D \in \text{Ver}_{R_{\mathcal{F}^\diamond}}} \text{Sym}_D^{\mathcal{F}^\diamond}$, we must pay attention to the central-

izers $C(h)$ of the local holonomy transformations $h = h_{D, s} \in \text{Diff}(\Delta_D, f(o_D))$.

Trivially h is of one and only one following type:

- (P) periodic;
- (L^1) linearizable and non-periodic;
- (L^0) formally linearizable but non-linearizable;
- (R^1) resonant non-linearizable but normalizable;
- (R^0) resonant non-linearizable and non-normalizable.

Classically, in the first three cases there exists a (only formal, in the case (L^0)) local coordinate u on $(\Delta_D, f(o_D))$, such that $u \circ h = \alpha u$, with $\alpha \in \mathbb{C}^*$. In these situations $h = \exp X$, with $X := \log(\alpha)u\partial_u$. In the resonant cases (R^0) and (R^1) there exists a coordinate u on Δ_D , only formal in the case (R^0), such that $h = \ell^r \circ \exp t_0 X$, $X := \frac{u^{p+1}}{1+\lambda u^p} \partial_u$, where $p+1$ is the contact order of h^k with the identity when $h'(0)^k = 1$, ℓ is the formal diffeomorphism defined by $u \circ \ell := e^{2i\pi/p}u$, $h'(0) = e^{2i\pi r/p}$, $t_0 \in \mathbb{C}^*$ and we can choose $t_0 = 1$, (remark that ℓ and $\exp X$ commute). In all cases u is unique up to multiplication by an element of \mathbb{C}^* . Let us denote by $\widehat{C}(h)$ the centralizer of h inside the group $\widehat{\text{Diff}}(\Delta_D, f(o_D))$ of formal diffeomorphisms of $(\Delta, f(o_D))$. Clearly $C(h) = \widehat{C}(h) \cap \text{Diff}(\Delta_D, f(o_D))$. As in Remark 5.2 let us denote by \mathcal{T}_h the space of germs of holomorphic vector fields on $(\Delta, f(o_D))$ invariant by h . The following result contains several well-known facts.

Proposition 5.8. *According to the type of $h \in \text{Diff}(\Delta_D, f(o_D))$ we have:*

- (P) $C(h) = \{g \in \text{Diff}(\Delta_D, f(o_D)) \mid u \circ g = u(\alpha + F(u^q)), \alpha \in \mathbb{C}^*, F \in u\mathbb{C}\{u\}\}; C(h)/\langle h \rangle \simeq \text{Diff}(\mathbb{C}, 0); \mathcal{T}_h = \mathbb{C}\{u^q\}u\partial_u;$
- (L) $\widehat{C}(h) = \{g \in \widehat{\text{Diff}}(\Delta_D, f(o_D)) \mid u \circ g = \lambda g, \lambda \in \mathbb{C}^*\} = \exp \mathbb{C}X \simeq \mathbb{C}^*;$
- (R) $\widehat{C}(h) = \{\ell^n \circ \exp tX, n \in \mathbb{Z}/p\mathbb{Z}, t \in \mathbb{C}\} = \langle \ell \rangle \oplus \exp \mathbb{C}X \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{C}.$

Moreover:

- (L¹) $\mathcal{T}_h = \mathbb{C}X$ and $C(h)/\exp \mathcal{T}_h = \{1\};$
- (R¹) $\mathcal{T}_h = \mathbb{C}X, C(h)/\exp \mathcal{T}_h \simeq \mathbb{Z}/p\mathbb{Z}$ and $C(h)/\langle h, \exp \mathcal{T}_h \rangle \simeq \mathbb{Z}/(p, r);$
- (L⁰) $\mathcal{T}_h = \{0\}; C(h) = \{g \in \text{Diff}(\Delta_D, f(o_D)) \mid u \circ g = \lambda u, \lambda \in \mathbf{D}\} \simeq \mathbf{D},$ where $\mathbf{D} := \{\lambda \in \mathbb{C}^* \mid u^{-1} \circ (\lambda u) \text{ is convergent}\}$ is a totally discontinuous subgroup of $\mathbb{U}(1)$, that can be uncountable [26];
- (R⁰) $\mathcal{T}_h = \{0\}$, there exists $m \in \mathbb{N}^*$ such that the sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\alpha} C(h) \xrightarrow{\beta} \mathbb{Z}/p\mathbb{Z}, \quad \alpha(t) = \exp \frac{t}{m} X, \quad \beta(g) = \frac{1}{2i\pi} \log g'(0)$$

is exact and $C(h)/\langle h \rangle$ is finite.

Proof. The periodic case has been already described in the proof of Proposition 5.3 unless the isomorphism $C(h)/\langle h \rangle \simeq \text{Diff}(\mathbb{C}, 0)$ which follows easily from the fact that every $g \in \text{Diff}(\mathbb{C}, 0)$ commuting with a rotation $z \mapsto e^{2i\pi/q}z$ writes as $(g^b(z^q))^{\frac{1}{q}}$ for a unique $g^b \in \text{Diff}(\mathbb{C}, 0)$. The description of the formal centralizers is given for instance in [11, Proposition 1.3.2] and [4, p. 150], where it is also shown that $C(h) = \widehat{C}(h)$ in the normalizable cases (L¹) and (R¹). In addition, in the case (L¹), $\mathcal{T}_h = \mathbb{C}u\frac{\partial}{\partial u}$ and $\exp : \mathcal{T}_h \rightarrow C(h)$ can be canonically identified to the surjective morphism $\mathbb{C} \rightarrow \mathbb{C}^*$ given by $\mu \mapsto e^\mu$. In the case (R¹), \mathcal{T}_h is equal to $\mathbb{C}X$ and

$$C(h)/\langle h, \exp \mathbb{C}X \rangle \simeq (\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{C})/\langle i \oplus 1, 0 \oplus \mathbb{C} \rangle \simeq \mathbb{Z}/\langle p, r \rangle.$$

Thanks to the description of $\widehat{C}(h)$, in the case (R⁰) the kernel of β consists of the convergent elements of the flow of X and by Écalle-Liverpool Theorem [11, Corollary 2.8.2] it is equal to $\alpha(\mathbb{Z})$ for a suitable $m \in \mathbb{N}^*$. Finally in the case (L⁰), let g be an element of $C(h) = \widehat{C}(h) \cap \text{Diff}(\mathbb{C}, 0)$. Then $u \circ g = \lambda u$ with $|\lambda| = 1$. Indeed if $|\lambda| \neq 1$, g would be linearizable in a convergent coordinate and h , that commutes with g , would be also linearizable in the same coordinate, contradicting the assumption (R⁰). On the other hand, \mathbf{D} is a totally discontinuous subgroup of $\mathbb{U}(1)$ because otherwise $\mathbf{D} = \mathbb{U}(1)$ and h would be linearizable. \square

It follows from this proposition and from Remark 2.4,

Theorem 5.9. *Given $a \in \text{Ve}_{\mathbf{A}_{\mathcal{F}^\diamond}} \cup \text{Ed}_{\mathbf{A}_{\mathcal{F}^\diamond}}$ we have the equivalences:*

$$a \text{ is red} \iff \text{Sym}_a^{\mathcal{F}^\diamond} \text{ is abelian} \iff \dim_{\mathbb{C}} \mathcal{T}_a^{\mathcal{F}^\diamond} < \infty$$

$$a \in \text{Ve}_{\mathbf{R}_{\mathcal{F}^\diamond}^0} \cup \text{Ed}_{\mathbf{R}_{\mathcal{F}^\diamond}^0} \iff \mathcal{T}_a^{\mathcal{F}^\diamond} = 0.$$

Furthermore the abelian structure of the group-graph $\text{Sym}^{\mathcal{F}^\diamond}$ over $\mathbf{R}_{\mathcal{F}^\diamond}$ induces an abelian group structure on $H^1(\mathbf{R}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond})$.

Remark 5.10. By following the natural bijections (6), (9) and (10) provided by Theorems 3.5, 4.14 and 5.6 respectively, one can check that if $\mathcal{H}([\mathcal{F}^\diamond]) = \mathcal{H}([\mathcal{G}^\diamond])$ then $\text{Mod}([\mathcal{F}^\diamond])$ and $\text{Mod}([\mathcal{G}^\diamond])$ coincide as sets, but their respective

abelian group structures are related by the map $\mu \mapsto \gamma\mu$ where $\gamma = [\mathcal{G}^\diamond] \in \text{Mod}([\mathcal{F}^\diamond])$. \square

We end this section by proving some properties of centralizers which will be useful in the sequel.

Lemma 5.11. *If g and h are non-periodic and $g \in C(h)$, then $C(g) = C(h)$.*

Proof. The group $C(h)$ in case (L) only depends on the formal coordinate u that linearize h . Similarly in case (R) all the non-periodic elements of $C(h)$ have the same invariants p, λ and if we fix these invariants, the centralizers of resonant diffeomorphisms only depend on the normalizing coordinate u . Thus the lemma follows from the fact that in both cases all the non-periodic elements of a centralizer can be linearized or normalized by using the same coordinate. \square

Lemma 5.12. *Let H be a finitely generated subgroup of $\text{Diff}(\mathbb{C}, 0)$. Suppose that H and its centralizer are infinite. Then H is abelian, it contains a non-periodic element h and $H \subset C(H) = C(h)$.*

Proof. If H is abelian, we can take as h a non-periodic element: these can not all be periodic, otherwise H would be finite. If H is not abelian, any non trivial commutator is non-periodic. On the other hand, the description of centralizers given by Proposition 5.8 allows us to see that each infinite subgroup of $C(h)$ contains an element of infinite order. Then there is a non-periodic element g in $C(H) \subset C(h)$. Because $C(g)$ is abelian and clearly contains H , we have $C(g) \subset C(H) \subset C(h)$. By applying Lemma 5.11 to $g \in C(h)$, we obtain the equality $C(H) = C(h)$. The inclusion $H \subset C(H)$ follows from the abelianity of H . \square

It follows immediately:

Proposition 5.13. *Under the hypothesis of the previous lemma, all the non-periodic elements of H are of the same type (L^1) , (L^0) , (R^1) or (R^0) .*

6. NON-DEGENERATE FOLIATIONS

Before proving Theorem D stated in Introduction, we are going to look at the genericity of non-degenerate foliations. In [21] one finds results of this type but in a more restricted framework. Theorem 6.2.1 of this paper claims the Krull open density, in the set of 1-forms defining foliations of second kind (in particular generalized curves), of the set of 1-forms defining foliations fulfilling conditions (ii) and (iii) of Definition 1.2. We can resume the proof of this theorem by using Theorem A of [19] instead of [21, Lemma 6.2.7]. We obtain in this way:

Theorem 6.1. *If η is germ at $0 \in \mathbb{C}^2$ of a holomorphic differential 1-form with isolated singularity defining a generalized curve foliation \mathcal{G} , then for all $p \in \mathbb{N}$ there exists a 1-form η' with same p -jet as η and an integer $n \geq p$ such that any foliation \mathcal{F} defined by a differential form ω with same n -jet at 0 as η' , fulfills*

- (1) *the holonomy group $\text{Im}(\mathcal{H}_D^{\mathcal{F}^\sharp})$ of any component D of $\mathcal{E}_{\mathcal{F}}$ with $v := \text{card}(D \cap \Sigma_{\mathcal{F}}) \geq 3$, is a free product of $v - 1$ cyclic subgroups of $\text{Diff}(\mathbb{C}, 0)$; in particular it is non solvable hence topologically rigid;*

(2) for any singular chain D_0, \dots, D_ℓ in $\mathcal{E}_\mathcal{F}$, the local holonomies of $\mathcal{F}^\#$ at the singular points $s_i = D_{i-1} \cap D_i$, $i = 1, \dots, \ell$, are non-periodic; and in particular \mathcal{F} is non-degenerate.

It is well known that for p large enough all the foliations \mathcal{F} have the same reduction map, the same singular points on the exceptional divisor, the same Camacho-Sad indices and the same dicritical components as \mathcal{G} .

Let us call \mathcal{F}^\diamond -**singular chain** of \mathcal{E} any sequence D_0, \dots, D_{ℓ_C} of irreducible components of \mathcal{E} defining a connected subgraph

$$(11) \quad (\mathcal{C}) \quad \bullet_{D_0} \xrightarrow{s_1} \bullet_{D_1} \cdots \cdots \xrightarrow{s_{\ell_C}} \bullet_{D_{\ell_C}}$$

of $\mathbf{A}_{\mathcal{F}^\diamond}$ such that the singular valency (cf. Definition 4.4) of its **extremities** D_0 and D_{ℓ_C} is at least three, and that of the others, called **interior vertices**, being exactly two. If $\ell_C = 1$, then \mathcal{C} consists only of two adjacent divisors of singular valency at least three: $\bullet_{D_0} \xrightarrow{s_1} \bullet_{D_1}$. The image by the marking map f of a \mathcal{F}^\diamond -singular chain of \mathcal{E} is a singular chain of $\mathcal{E}_\mathcal{F}$ as considered in Introduction.

Proposition 6.2. *Let \mathcal{F}^\diamond be a non-degenerate marked foliation. Then the union $\check{\mathbf{R}}_{\mathcal{F}^\diamond}$ of all \mathcal{F}^\diamond -singular chains is a connected repulsive subgraph of $\mathbf{A}_{\mathcal{F}^\diamond}$ contained in $\mathbf{R}_{\mathcal{F}^\diamond}$; hence \mathcal{F} is of finite type and*

$$H^1(\mathbf{R}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}) \simeq H^1(\check{\mathbf{R}}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond}).$$

In order to simplify the notations in the two proofs below, we will write \mathbf{R} , $\check{\mathbf{R}}$, Sym , instead of $\mathbf{R}_{\mathcal{F}^\diamond}$, $\check{\mathbf{R}}_{\mathcal{F}^\diamond}$, $\text{Sym}^{\mathcal{F}^\diamond}$.

Proof of Proposition 6.2. Clearly $\check{\mathbf{R}}$ is connected and the closure of $\mathbf{R} \setminus \check{\mathbf{R}}$ in \mathbf{R} is exactly the union of all connected subgraphs \mathcal{C} denoted as in (11) but with singular valencies satisfying $\text{val}_\Sigma(D_0) \geq 3$, $\text{val}_\Sigma(D_j) = 2$ for $0 < j < \ell_C$ and $\text{val}_\Sigma(D_{\ell_C}) = 1$ or 2 . By definition of the group-graph Sym , for $j \geq 1$ the morphisms $\rho_{D_j}^{s_j} : \text{Sym}_{D_j} \rightarrow \text{Sym}_{s_j}$ are bijective and $\check{\mathbf{R}}$ is repulsive in \mathbf{R} .

Because \mathcal{F} is non degenerate all the vertices and edges of $\check{\mathbf{R}}$ are red; thus \mathbf{R} is also repulsive and connected. By using Pruning Theorem 2.11 we obtain the group isomorphism $H^1(\mathbf{R}, \text{Sym}) \simeq H^1(\check{\mathbf{R}}, \text{Sym})$. \square

Proof of Theorem D. To have uniqueness of the numbering in the notation (11) of a singular chain \mathcal{C} , we fix in the sequel an extremity \check{D} of $\check{\mathbf{R}}$ and we prescript that D_0 belongs to the geodesic joining D_{ℓ_C} to \check{D} . We will say that D_0 , resp. s_{ℓ_C} , is the **initial vertex**, resp. **terminal edge** of \mathcal{C} .

For an interior vertex D_j of \mathcal{C} the morphisms $\rho_{D_j}^{s_j}$ and $\rho_{D_j}^{s_{j+1}}$ are bijective and by composition they induce isomorphisms

$$\xi_{D_j} : \text{Sym}_{s_1} \simeq \text{Sym}_{D_j}, \quad \xi_{s_j} : \text{Sym}_{s_1} \simeq \text{Sym}_{s_j}, \quad 0 < j < \ell_C.$$

Let us consider the subgroups:

- $\tilde{Z}^1(\check{\mathbf{R}}, \text{Sym}) \subset Z^1(\check{\mathbf{R}}, \text{Sym})$ of the 1-cocycles $(\phi_{D,a})_{(D,a) \in I_{\check{\mathbf{R}}}}$ such that $\phi_{D,a} = 1$ if a is not the terminal edge of some singular chain,

- $\tilde{C}^0(\check{R}, \text{Sym}) \subset C^0(\check{R}, \text{Sym})$ of the 0-cochains $(\phi_D)_{D \in \text{Ve}_{\check{R}}}$ such that $\phi_D = \xi_D \circ \rho_{D_0}^{s_1}(\phi_{D_0})$ for all the interior vertices D of any singular chain, D_0 denoting its initial vertex.

Notice that the coboundary morphism ∂^0 defined in Remark 2.4 maps $\tilde{C}^0(\check{R}, \text{Sym})$ in $\tilde{Z}^1(\check{R}, \text{Sym})$, allowing us to define the group

$$\tilde{H}^1(\check{R}, \text{Sym}) := \text{coker}(\partial^0 : \tilde{Z}^0(\check{R}, \text{Sym}) \longrightarrow \tilde{Z}^1(\check{R}, \text{Sym})).$$

We easily see that each element of $H^1(\check{R}, \text{Sym})$ can be represented by a cocycle belonging to $\tilde{Z}^1(\check{R}, \text{Sym})$. We deduce that the morphism

$$\tau : \tilde{H}^1(\check{R}, \text{Sym}) \longrightarrow H^1(\check{R}, \text{Sym})$$

is surjective. On the other hand if a cocycle $\mathbf{c}^0 := (\phi_D)_{D \in \text{Ve}_{\check{R}}} \in Z^1(\check{R}, \text{Sym})$ satisfies $\partial^0(\mathbf{c}^0) \in \tilde{Z}^1(\check{R}, \text{Sym})$, then for each singular chain \mathcal{C} of length $\ell_{\mathcal{C}} \geq 2$ denoted as in (11) we have the equalities

$$\rho_{D_j}^{s_j}(\phi_{D_j}) = \rho_{D_{j-1}}^{s_j}(\phi_{D_{j-1}}), \quad j < \ell_{\mathcal{C}}.$$

It follows that $\mathbf{c}^0 \in \tilde{C}^0(\check{R}, \text{Sym})$. Therefore $\ker(\tau)$ is trivial and τ is an isomorphism. To achieve the proof of Theorem D, let us first notice that the group $\tilde{C}^0(\check{R}, \text{Sym})$ is finite. Indeed it is isomorphic to the product of all centralizers of holonomy groups associated to the vertices of \check{R} having singular valency at least three. These holonomy groups being non-abelian by non-degenerate assumption, thanks to Lemma 5.12 their centralizers are finite. On the other hand, Proposition 5.8 gives a decomposition of $\tilde{Z}^1(\check{R}, \text{Sym})$ as $F \oplus B \oplus_{j=1}^{\lambda} (\mathbb{C}^*/\alpha_j^{\mathbb{Z}}) \oplus (\mathbb{C}^*)^{\nu}$; that completes the proof of Theorem D. \square

7. EXAMPLES

Before proving Theorem C in full generality let us motivate its statement by computing the moduli space of some non-trivial examples using the identification $\text{Mod}([\mathcal{F}^{\diamond}]) \simeq H^1(\mathcal{R}_{\mathcal{F}^{\diamond}}, \text{Sym}^{\mathcal{F}^{\diamond}})$.

• Example 0: a logarithmic generic multicusp

Let \mathcal{L} be the logarithmic germ foliation at $0 \in \mathbb{C}^2$ defined by the meromorphic form

$$\omega := \sum_{i=1}^p \alpha_i \frac{d(y^2 + a_i x^3)}{y^2 + a_i x^3} + \delta \frac{d(y - x)}{y - x} + \sum_{i=1}^p \beta_i \frac{d(x^2 - b_i y^3)}{x^2 - a_i y^3},$$

with $a_i, b_i \in \mathbb{C}$ mutually distincts. We normalize the coefficients $\alpha_i, \beta_i, \delta \in \mathbb{C}^*$ by requiring $\delta + 2 \sum_{i=1}^p (\alpha_i + \beta_i) = 1$. To simplify the exposition we suppose that \mathcal{E} is equal to the exceptional divisor $\mathcal{E}_{\mathcal{L}}$, the marking being the identity map and $\mathcal{L}^{\diamond} = (\mathcal{L}, \text{id}_{\mathcal{E}_{\mathcal{L}}})$. Clearly \mathcal{E} is formed by five irreducible components, its dual graph is equal to $A_{\mathcal{L}^{\diamond}}$

$$(A_{\mathcal{L}^{\diamond}}) \quad \bullet_{C'} \xrightarrow{s'_{\infty}} \bullet_{D'} \xrightarrow{s'_0} \bullet_D \xrightarrow{s''_0} \bullet_{D''} \xrightarrow{s''_{\infty}} \bullet_{C''}$$

$$\sigma^{\diamond}(C') = \sigma^{\diamond}(C'') = 1, \quad \sigma^{\diamond}(D') = \sigma^{\diamond}(D'') = p + 2, \quad \sigma^{\diamond}(D) = 3,$$

and it decomposes into one singular chain that is the red part $R_{\mathcal{L}^\diamond}$ of the graph

$$(R_{\mathcal{L}^\diamond}) \quad \bullet_{D'} \xrightarrow{s'_0} \bullet_D \xrightarrow{s''_0} \bullet_{D''}$$

and two dead branches $\bullet_{C'} \xrightarrow{s'_\infty} \bullet_{D'}$ and $\bullet_{D''} \xrightarrow{s''_\infty} \bullet_{C''}$ necessarily green. Thus the restriction morphisms $\text{Sym}_{C'}^{\mathcal{L}^\diamond} \rightarrow \text{Sym}_{s'_\infty}^{\mathcal{L}^\diamond}$, $\text{Sym}_{C''}^{\mathcal{L}^\diamond} \rightarrow \text{Sym}_{s''_\infty}^{\mathcal{L}^\diamond}$ are surjective and by Pruning Theorem 2.11, the group $H^1(A_{\mathcal{L}^\diamond}, \text{Sym}^{\mathcal{L}^\diamond})$ is isomorphic to $H^1(R_{\mathcal{L}^\diamond}, \text{Sym}^{\mathcal{L}^\diamond})$. On $R = R_{\mathcal{L}^\diamond}$ the morphism ∂^0 defined in Remark 2.4 decomposes, with additive notations on abelian groups, as

$$\begin{array}{ccccccc} C^0(R, \text{Sym}^{\mathcal{L}^\diamond}) = & C(H_{D'}) & \oplus & C(H_D) & \oplus & C(H_{D''}) \\ \partial^0 \downarrow & \searrow \xi_1 & & \swarrow \xi_2 & \searrow \xi_3 & \swarrow \xi_4 \\ Z^1(R, \text{Sym}^{\mathcal{L}^\diamond}) = & & \text{Sym}_{s'_0}^{\mathcal{L}^\diamond} & \oplus & \text{Sym}_{s''_0}^{\mathcal{L}^\diamond} & \end{array}$$

$$\partial^0(c_1 \oplus c_2 \oplus c_3) = (\xi_1(c_1) + \xi_2(c_2)) \oplus (\xi_3(c_2) + \xi_4(c_3)).$$

On the other hand, $\text{Sing}(\mathcal{L}^\sharp) \cap D'$, resp. $\text{Sing}(\mathcal{L}^\sharp) \cap D''$, is formed by s'_∞ , s'_0 , resp. s''_∞ , s''_0 , and the attaching points s'_i , resp. s''_i , of the strict transforms of the curve $\{y^2 + a_i x^3 = 0\}$, resp. $\{x^2 + b_i y^3 = 0\}$, $i = 1, \dots, p$; and $\text{Sing}(\mathcal{L}^\sharp) \cap D$ is formed by s'_0 , s''_0 and the attaching point s_1 of the strict transform of $\{y - x = 0\}$. The Camacho-Sad indices of \mathcal{L}^\sharp at these points are

$$\text{CS}(D', s'_\infty) = \text{CS}(D'', s''_\infty) = 1/2, \quad \text{CS}(D, s_1) = -\delta, \quad \text{CS}(D', s'_0) = -2\tilde{\alpha},$$

$$\text{CS}(D'', s''_0) = -2\tilde{\beta}, \quad \text{CS}(D', s'_i) = -\frac{\alpha_i}{2}\tilde{\alpha} \quad \text{CS}(D'', s''_i) = -\frac{\beta_i}{2}\tilde{\beta},$$

with $\tilde{\alpha} := (1 + \sum_{i=1}^p \alpha_i)$, $\tilde{\beta} := (1 + \sum_{i=1}^p \beta_i)$. Assuming that $p \geq 3$, we choose α_i , β_i and δ sufficiently generic so that no Camacho-Sad index is a real number, except at the points s'_∞ and s''_∞ . All the singularities of \mathcal{L}^\sharp are linearizable and according to Proposition 5.8 the centralizers $C(H_{D'})$, $C(H_D)$, $C(H_{D''})$ are isomorphic to $\mathbb{C}^* = \mathbb{C}/2\pi i\mathbb{Z}$. Using Remark 4.9 we obtain:

$$\begin{array}{ccccccc} \frac{\mathbb{C}}{2i\pi\mathbb{Z}} & & \oplus & & \frac{\mathbb{C}}{2i\pi\mathbb{Z}} & & \oplus & & \frac{\mathbb{C}}{2i\pi\mathbb{Z}} \\ & \searrow \xi_1 & & \swarrow \xi_2 & \searrow \xi_3 & & \swarrow \xi_4 & \\ & \mathbb{C}/2i\pi(\mathbb{Z} + 2\tilde{\alpha}\mathbb{Z}) & & \oplus & \mathbb{C}/2i\pi(\mathbb{Z} + 2\tilde{\beta}\mathbb{Z}) & \end{array}$$

moreover ξ_2 and ξ_3 are induced by the identity map, but ξ_1 is induced by $z \mapsto \frac{1}{2\tilde{\alpha}}z$ and ξ_4 by $z \mapsto \frac{1}{2\tilde{\beta}}z$. It immediately follows that ∂^0 is surjective and $H^1(R, \text{Sym}^{\mathcal{L}^\diamond}) = 0$. We conclude that

- \mathcal{L}^\diamond is *topologically SL-rigid* i.e. its moduli space is trivial,

$$\text{Mod}([\mathcal{L}^\diamond]) = \{[\mathcal{L}^\diamond]\}.$$

• **Example 1: non-degenerate multicusps**

Let us perform a generic perturbation \mathcal{F}_1 of the previous example, provided by Theorem 6.1, that does not change the Camacho-Sad indices but such that the holonomy groups along D' and D'' are non abelian. According to

Lemma 5.12 their centralizers are finite groups F'_1 , F_1 and F''_1 respectively, however $\text{Sym}_{s'_0}^{\mathcal{F}_1^\circ}$ and $\text{Sym}_{s''_0}^{\mathcal{F}_1^\circ}$ remain isomorphic to $\text{Sym}_{s'_0}^{\mathcal{L}^\circ}$ and $\text{Sym}_{s''_0}^{\mathcal{L}^\circ}$ because the singularities of $\mathcal{F}_1^\#$ at s'_0 and s''_0 are again linearizable.

$$\begin{array}{ccccc}
 F'_1 & & \oplus & & F_1 & & \oplus & & F''_1 \\
 & \searrow \xi_1 & & \swarrow \xi_2 & & \searrow \xi_3 & & \swarrow \xi_4 & \\
 & \mathbb{C}/2i\pi(\mathbb{Z} + 2\tilde{\alpha}\mathbb{Z}) & & \oplus & & \mathbb{C}/2i\pi(\mathbb{Z} + 2\tilde{\beta}\mathbb{Z}) & & &
 \end{array}$$

It follows:

- $\text{Mod}([\mathcal{F}_1^\circ])$ is a finite quotient of a product of two elliptic curves.

• **Example 2: partially degenerate multicusps**

With the induction technique used in the proof of Theorem 6.1, we can perform a perturbation of \mathcal{L} that provides a foliation \mathcal{F}_2 with same Camacho-Sad indices, with non abelian holonomy groups F'_2 and F''_2 along D' and D'' , but such that there is a biholomorphism between neighborhoods of D that conjugates \mathcal{F}_2 with \mathcal{L} . We have

$$\begin{array}{ccccc}
 F'_2 & & \oplus & & \mathbb{C}/2i\pi\mathbb{Z} & & \oplus & & F''_2 \\
 & \searrow \xi_1 & & \swarrow \xi_2 & & \searrow \xi_3 & & \swarrow \xi_4 & \\
 & \mathbb{C}/2i\pi(\mathbb{Z} + \mu'\mathbb{Z}) & & \oplus & & \mathbb{C}/2i\pi(\mathbb{Z} + \mu''\mathbb{Z}) & & &
 \end{array}$$

where again ξ_2 and ξ_3 are induced by the identity map. We easily obtain the exact sequence

$$K \longrightarrow \mathbb{C}/2\pi i(\mathbb{Z} + 2\tilde{\alpha}\mathbb{Z} + 2\tilde{\beta}\mathbb{Z}) \longrightarrow \text{Mod}([\mathcal{F}_2^\circ]) \longrightarrow 0,$$

K being finite. If 1, $2\tilde{\alpha}$, $2\tilde{\beta}$ are \mathbb{Z} -independent then

- $\text{Mod}([\mathcal{F}_2^\circ])$ is not a finite quotient of a product of elliptic curves; in particular, in the statement of Theorem C we cannot replace \mathbb{Z}^p by a finite group in the exact sequence (3).

• **Example 3: infinite type multicusps**

First we choose the coefficients α_i , β_i , δ in the expression of the 1-form ω , so that $\text{CS}(D', s'_0) \in \mathbb{Z}_{<0}$, the other Camacho-Sad indices being in $\mathbb{C} \setminus \mathbb{R}$, except for the points s'_∞ and s''_∞ . At s'_0 the foliation $\mathcal{L}^\#$ possesses now a germ of holomorphic first integral, the local holonomy is a periodic rotation, thus $\text{Sym}_{s'_0}^{\mathcal{L}^\circ}$ is isomorphic to $\text{Diff}(\mathbb{C}, 0)$. Then we perform a perturbation \mathcal{F}_3 of \mathcal{L} changing only the holonomy groups $H_{D'}$ and H_D that become non abelian, without changing $H_{D''}$ neither the local analytic types at any singular point. For such foliation \mathcal{F}_3 the group $\text{Sym}_{s'_0}^{\mathcal{F}_3^\circ}$ is always isomorphic to $\text{Sym}_{s'_0}^{\mathcal{F}_3^\circ} \simeq \text{Diff}(\mathbb{C}, 0)$. The group-graph $\text{Sym}^{\mathcal{F}_3}$ is not abelian and its cohomology is no longer given by the cokernel of a morphism ∂^0 . However

$$\xi_4 : \text{Sym}_{D''}^{\mathcal{F}_3^\circ} \simeq \mathbb{C}/2\pi i\mathbb{Z} \longrightarrow \text{Sym}_{s''_0}^{\mathcal{F}_3^\circ} \simeq \mathbb{C}/2\pi i(\mathbb{Z} + \mu''\mathbb{Z})$$

is always a submersion. Thus performing a new pruning we have an isomor-

phism $H^1(\mathcal{R}_{\mathcal{F}_3^\circ}, \text{Sym}^{\mathcal{F}_3^\circ}) \simeq H^1(\mathcal{R}', \text{Sym}^{\mathcal{F}_3^\circ})$ with $\mathcal{R}' := \bullet_{D'} \xrightarrow{s'_0} \bullet_D$. The centralizer of $H_{D'}$ and H_D being finite by Lemma 5.12, we obtain:

- $\text{Mod}([\mathcal{F}_3^\diamond])$ is a quotient of $\text{Diff}(\mathbb{C}, 0)$ by the action of a finite group and $\mathcal{R}_{\mathcal{F}_3^\diamond}$ is not connected.

• **Example 4: Cremer multicusps**

By gluing techniques and thanks to realization Theorem [28] and Pérez Marco results [26] we can build a foliation \mathcal{F}_4 with same separatrices, thus same resolution, as in previous examples, with non abelian holonomy groups $H_{D'}$, H_D , $H_{D''}$, but whose local holonomies at s'_0 and s''_0 are Cremer with uncountable centralizers. In this case

- $\text{Mod}([\mathcal{F}_4^\diamond])$ is a finite quotient of a product of two uncountable totally discontinuous subgroups of $\mathbb{U}(1) := \{z \in \mathbb{C} \mid |z| = 1\}$.

• **Example 5: non-degenerate foliations with a single separatrix.**

For such a foliation \mathcal{F}_5 , after pruning all dead branches of the dual graph of $\mathcal{E}_{\mathcal{F}_5}$, the obtained graph is the red graph $\mathcal{R}_{\mathcal{F}_5^\diamond}$ which is reduced to a geodesic segment $\bullet_{D_0} \xrightarrow{s_1} \cdots \xrightarrow{s_\ell} \bullet_{D_\ell}$. All Camacho-Sad indices are rational numbers. The singular chains in $\mathcal{R}_{\mathcal{F}_5^\diamond}$ are in two categories: the **normalizable chains** whose edges s correspond to normalizable resonant singularities of \mathcal{F}_5^\sharp and the non-normalizable chains. For the first one the group $\text{Sym}_s^{\mathcal{F}_5^\diamond}$ is isomorphic to \mathbb{C}^* and for non-normalizable chains it is isomorphic with $\mathbb{Z}/m_s\mathbb{Z}$ for a suitable $m_s \in \mathbb{N}$. It follows:

- $\text{Mod}([\mathcal{F}_5^\diamond]) \simeq (\bigoplus_{i=1}^\mu \mathbb{Z}/m_i\mathbb{Z} \oplus \mathbb{C}^{*\nu})/Z$, with Z a finite subgroups, μ , resp ν , the number of non-normalizable, resp. normalizable singular chains; furthermore $\mu + \nu$ is equal to the number of Puiseux pairs of the unique separatrix.

Another specificity of this foliation \mathcal{F}_5 is that the mapping class group of $\mathcal{E}_{\mathcal{F}_5^\diamond}$ is trivial because every singular point of $\mathcal{E}_{\mathcal{F}_5}$ is fixed⁵ by $\text{Mcg}(\mathcal{E}_{\mathcal{F}_5^\diamond})$ and the pure mapping class group of \mathbb{P}^1 punctured at three points is trivial [8, Proposition 2.3]. From (5) we obtain that

$$\text{Mod}([\mathcal{F}_5^\diamond]) \subset [\text{MFol}_{\text{tr}}(\mathcal{E}_{\mathcal{F}_5^\diamond})] \simeq [\text{Fol}_{\text{tr}}(\mathcal{E}_{\mathcal{F}_5^\diamond})]_{\mathbb{C}^0}.$$

• **Example 6: some topologically SL-rigid foliations.**

Whenever for a marked foliation \mathcal{F}^\diamond the red part of any cut-component of $\mathcal{A}_{\mathcal{F}^\diamond}$ is reduced to one vertex, the moduli space $\text{Mod}([\mathcal{F}^\diamond])$ is reduced to one element. In particular this is the case for:

- any non dicritical foliation reduced after only one blow-up, its separatrices being smooth curves mutually transversal, or more generally any topologically quasi-homogeneous germ, see [12],
- absolutely dicritical foliations of Cano-Corral [5],
- dicritical foliations that are non singular after one blow-up, see [1] and [24].

8. EXPONENTIAL AND DISCONNECTED GROUP-GRAPHS

We keep all notations used in Section 5. For technical reasons the last group-graphs that we must consider will be defined uniquely over the red

⁵Each element of the mapping class group of $\mathcal{E}_{\mathcal{F}_5^\diamond}$ preserves dead branches so it must fix every singular point except maybe the attaching points of the two dead branches of the extremity valency 3 divisor. But these two points have different Camacho-Sad indices, as can be easily deduced from [10, p. 164].

graph $R_{\mathcal{F}^\diamond} \subset A_{\mathcal{F}^\diamond}$. Recall that $\mathcal{X}^{\mathcal{F}^\diamond}$, $\mathcal{B}^{\mathcal{F}^\diamond}$ and $\mathcal{T}^{\mathcal{F}^\diamond}$ denote the group-graphs over $A_{\mathcal{F}^\diamond}$ associated to the sheafs $\mathcal{X}^{\mathcal{F}^\sharp}$, $\mathcal{B}^{\mathcal{F}^\sharp}$ and $\mathcal{T}^{\mathcal{F}^\sharp} = \mathcal{B}^{\mathcal{F}^\sharp} / \mathcal{X}^{\mathcal{F}^\sharp}$ of tangent, basic and transverse holomorphic vector fields for \mathcal{F}^\sharp , respectively.

Lemma 8.1. *For $s \in \text{Ed}_{A_{\mathcal{F}^\diamond}}$ the exponential map $\exp : \mathcal{B}_s^{\mathcal{F}^\diamond} \rightarrow \text{Aut}_s^{\mathcal{F}^\diamond}$ induces a well-defined map $\exp_s^{\mathcal{F}^\diamond} : \mathcal{T}_s^{\mathcal{F}^\diamond} \rightarrow \text{Sym}_s^{\mathcal{F}^\diamond}$.*

Proof. We must prove that $\exp(Z + X) \equiv \exp(Z)$ modulo $\text{Fix}_s^{\mathcal{F}^\diamond}$ if $Z \in \mathcal{B}_s^{\mathcal{F}^\diamond}$ and $X \in \mathcal{X}_s^{\mathcal{F}^\diamond}$. For that it suffices to show that for each neighborhood V of $f(s)$ there is another neighborhood U of $f(s)$ such that for each $p \in U$ the curve $\alpha \in [0, 1] \mapsto \exp(Z + \alpha X)(p)$ is contained in a leaf of $\mathcal{F}|_V$. We choose $U \subset V$ such that the map $\phi : U \times \mathbb{D}_2 \times \mathbb{D}_2 \rightarrow V$ given by $\phi(p, t, \alpha) = \exp(t(Z + \alpha X))(p)$ is well-defined. Fix $p \in U$ and take a local holomorphic first integral F defined in a neighborhood W of p . If t is small enough then $\phi(p, t, \alpha) \in W$ and

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial}{\partial \alpha} F(\phi(p, t, \alpha)) \right) &= \frac{\partial}{\partial \alpha} \left(\frac{\partial}{\partial t} F(\phi(p, t, \alpha)) \right) \\ &= \frac{\partial}{\partial \alpha} ((Z + \alpha X)(F) \circ \phi(p, t, \alpha)) \\ &= [X(F) \circ \phi(p, t, \alpha)] \frac{\partial}{\partial \alpha} \phi(p, t, \alpha) = 0 \end{aligned}$$

because X is tangent to \mathcal{F}^\sharp . Since $F(p, 0, \alpha) = p$ does not depend on α we obtain that $\frac{\partial}{\partial \alpha} F(\phi(p, t, \alpha)) = 0$ for t small enough. As ϕ is holomorphic, we conclude that the curve $\alpha \mapsto \phi(p, 1, \alpha)$ is contained in a leaf of $\mathcal{F}|_V$. \square

Remark 8.2. It can be checked that under the identifications $\mathcal{T}_s^{\mathcal{F}^\diamond} \simeq \mathcal{T}_{h_D, s}$ and $C(h_{D, s}) / \langle h_{D, s} \rangle \simeq \text{Sym}_s^{\mathcal{F}^\diamond}$ given by Remark 5.2 and Corollary 4.8, the morphism $\exp_s^{\mathcal{F}^\diamond}$ coincides with the composition of the restriction $\mathcal{T}_{h_D, s} \rightarrow C(h_{D, s})$ of the exponential map on the transverse section Δ_D and the quotient map $C(h_{D, s}) \rightarrow C(h_{D, s}) / \langle h_{D, s} \rangle$. \square

Motivated by the above remark, for $D \in \text{Ve}_{A_{\mathcal{F}^\diamond}}$ we define the map $\exp_D^{\mathcal{F}^\diamond} : \mathcal{T}_D^{\mathcal{F}^\diamond} \rightarrow \text{Sym}_D^{\mathcal{F}^\diamond}$ as the composition $\mathcal{T}_D^{\mathcal{F}^\diamond} \simeq \mathcal{T}_{H_D} \xrightarrow{\exp} C(H_D) \rightarrow \text{Sym}_D^{\mathcal{F}^\diamond}$. From Remark 8.2 it is clear that the following diagram is commutative:

$$(12) \quad \begin{array}{ccc} \mathcal{T}_D^{\mathcal{F}^\diamond} & \xrightarrow{\exp_D^{\mathcal{F}^\diamond}} & \text{Sym}_D^{\mathcal{F}^\diamond} \\ \rho_D^s \downarrow & & \downarrow \rho_D^s \\ \mathcal{T}_s^{\mathcal{F}^\diamond} & \xrightarrow{\exp_s^{\mathcal{F}^\diamond}} & \text{Sym}_s^{\mathcal{F}^\diamond} \end{array}$$

the vertical maps being the restriction maps of the group-graphs $\mathcal{T}^{\mathcal{F}^\diamond}$ and $\text{Sym}^{\mathcal{F}^\diamond}$, written with same notation.

Although the exponential map

$$\exp : \mathbb{C}\{z\}z\partial_z \rightarrow \text{Diff}(\Delta_D, f(o_D)) \simeq \text{Diff}(\mathbb{C}, 0),$$

$z : (\Delta_D, f(o_D)) \rightarrow (\mathbb{C}, 0)$ being a germ of coordinate, is not a morphism of groups, its restriction to a subspace of complex dimension ≤ 1 is. On

the other hand it is well-known that $\dim_{\mathbb{C}} \mathcal{T}_s^{\mathcal{F}^\diamond} \leq 1$ if $\mathcal{J}_s^{\mathcal{F}^\diamond} = \mathbb{C}$. Since $\mathcal{T}_D^{\mathcal{F}^\diamond} \subset \mathcal{T}_s^{\mathcal{F}^\diamond}$ for $s \in D$ we deduce that $\exp_D^{\mathcal{F}^\diamond}$ and $\exp_s^{\mathcal{F}^\diamond}$ define a morphism

$$\exp^{\mathcal{F}^\diamond} : \mathcal{T}^{\mathcal{F}^\diamond} \rightarrow \text{Sym}^{\mathcal{F}^\diamond}$$

of abelian group-graphs over $\mathbf{R}_{\mathcal{F}^\diamond}$.

Definition 8.3. *The group-graph over $\mathbf{R}_{\mathcal{F}^\diamond}$ image of $\exp^{\mathcal{F}^\diamond}$ is called the **exponential group-graph** of \mathcal{F}^\diamond . We denote it by $\text{Exp}^{\mathcal{F}^\diamond}$.*

At this point it is clear that the subset of $\mathbf{R}_{\mathcal{F}^\diamond}$ consisting of all $a \in \text{Ve}_{\mathbf{A}_{\mathcal{F}^\diamond}} \cup \text{Ed}_{\mathbf{A}_{\mathcal{F}^\diamond}}$ such that $\text{Exp}_a^{\mathcal{F}^\diamond} = 0$ is just the subgraph $\mathbf{R}_{\mathcal{F}^\diamond}^0$ of $\mathbf{R}_{\mathcal{F}^\diamond}$ previously defined in Section 5 and characterized by the second equivalence in Theorem 5.9. Let us denote by $\mathbf{R}_{\mathcal{F}^\diamond}^1$ the **completion** of $\mathbf{R}_{\mathcal{F}^\diamond} \setminus \mathbf{R}_{\mathcal{F}^\diamond}^0$, i.e. the minimal subgraph of $\mathbf{R}_{\mathcal{F}^\diamond}$ containing $\mathbf{R}_{\mathcal{F}^\diamond} \setminus \mathbf{R}_{\mathcal{F}^\diamond}^0$.

Lemma 8.4. *If $(D, s) \in I_{\mathbf{R}_{\mathcal{F}^\diamond}}$ and $\mathcal{T}_D^{\mathcal{F}^\diamond} \neq 0$, then the restriction map $\rho' : \mathcal{T}_D^{\mathcal{F}^\diamond} \rightarrow \mathcal{T}_s^{\mathcal{F}^\diamond}$ is an isomorphism and all the red singular points in D share the same character linearizable or resonant; we will say that D is resonant or linearizable according to the case. Furthermore, the isomorphism class of the group $\text{Exp}_D^{\mathcal{F}^\diamond}$ is given by the following table*

	$\text{Exp}_D^{\mathcal{F}^\diamond}$	$\text{val}_\Sigma(D) \leq 2$	$\text{val}_\Sigma(D) \geq 3$
(13)	D resonant	\mathbb{C}/\mathbb{Z}	\mathbb{C}
	D linearizable	$\mathbb{C}^*/\alpha^{\mathbb{Z}}$	\mathbb{C}^*

the restriction morphism $\rho_D^s : \text{Exp}_D^{\mathcal{F}^\diamond} \rightarrow \text{Exp}_s^{\mathcal{F}^\diamond}$ is surjective and

$$\ker \rho_D^s \simeq \begin{cases} \mathbb{Z} & \text{if } \text{val}_\Sigma(D) \geq 3, \\ 0 & \text{if } \text{val}_\Sigma(D) \leq 2. \end{cases}$$

Proof. The homogeneity of singular types in D is given by Proposition 5.13. Notice that if a basic vector field for \mathcal{F}^\sharp defined on a connected open set $U \subset M_{\mathcal{F}}$ is tangent to the foliation in a neighborhood of a point of U then it is tangent to the foliation on the whole U . Using this fact it is easy to see that if W is a connected subset of a \mathcal{F}^\sharp -invariant component of $\mathcal{E}_{\mathcal{F}}$, the stalk maps $\mathcal{T}^{\mathcal{F}^\sharp}(W) \rightarrow \mathcal{T}_m^{\mathcal{F}^\sharp}$, $m \in W$ are injective. Taking $W = D$ we deduce that the restriction map $\rho_D^s : \mathcal{T}_D^{\mathcal{F}^\diamond} \rightarrow \mathcal{T}_s^{\mathcal{F}^\diamond}$ is injective. Since $\dim \mathcal{T}_D^{\mathcal{F}^\diamond} = \dim \mathcal{T}_s^{\mathcal{F}^\diamond} = 1$, ρ_D^s is an isomorphism. In fact, $\mathcal{T}_s^{\mathcal{F}^\diamond} \simeq \mathcal{T}_D^{\mathcal{F}^\diamond}$ can be identified to a line $\mathbb{C}X$ in the space of germs of vector fields on $(\Delta_D, f(o_D))$. Then Table (13) follows from Proposition 5.8.

On the other hand $\exp_s^{\mathcal{F}^\diamond} : \mathcal{T}_s^{\mathcal{F}^\diamond} \rightarrow \text{Exp}_s^{\mathcal{F}^\diamond}$ being surjective by definition and the restriction map ρ_D^s being an isomorphism, it follows that $\rho_D^s \circ \exp_D^{\mathcal{F}^\diamond} = \exp_s^{\mathcal{F}^\diamond} \circ \rho_D^s$ is surjective. Therefore ρ_D^s is also surjective. We conclude thanks to Remark 8.2 and the commutativity of the diagram (12). \square

Thanks to Theorem 5.9, $H^1(\mathbf{R}_{\mathcal{F}^\diamond}, \text{Sym}^{\mathcal{F}^\diamond})$ is an abelian group and the natural inclusion $\text{Exp}^{\mathcal{F}^\diamond} \rightarrow \text{Sym}^{\mathcal{F}^\diamond}$ is an injective morphism of abelian group-graphs over $\mathbf{R}_{\mathcal{F}^\diamond}$.

Definition 8.5. *The quotient group-graph over $\mathbf{R}_{\mathcal{F}^\diamond}$*

$$\text{Dis}^{\mathcal{F}^\diamond} := \text{Sym}^{\mathcal{F}^\diamond} / \text{Exp}^{\mathcal{F}^\diamond}$$

*given by $\text{Dis}_a^{\mathcal{F}^\diamond} := \text{Sym}_a^{\mathcal{F}^\diamond} / \text{Exp}_a^{\mathcal{F}^\diamond}$ for every $a \in \text{Ve}_{\mathbf{R}_{\mathcal{F}^\diamond}} \cup \text{Ed}_{\mathbf{R}_{\mathcal{F}^\diamond}}$, is called the **disconnected group-graph** of \mathcal{F}^\diamond .*

Obviously, we have over $\mathbf{R}_{\mathcal{F}^\diamond}$ the short exact sequence of abelian group-graphs:

$$(14) \quad 0 \rightarrow \text{Exp}^{\mathcal{F}^\diamond} \rightarrow \text{Sym}^{\mathcal{F}^\diamond} \rightarrow \text{Dis}^{\mathcal{F}^\diamond} \rightarrow 0.$$

The name “disconnected” is explained by the following proposition.

Proposition 8.6. *For $s \in \text{Ed}_{\mathbf{R}_{\mathcal{F}^\diamond}}$ and $D \in \text{Ve}_{\mathbf{R}_{\mathcal{F}^\diamond}}$ we have that*

- (1) *the abelian group $\text{Dis}_s^{\mathcal{F}^\diamond} \simeq C(h_{D,s})/\langle h_{D,s}, \exp(\mathcal{T}_{h_{D,s}}) \rangle$ is:*
 - (a) *trivial if $f(s)$ is a linearizable (not-periodic) singularity*
 - (b) *a finite abelian group if $f(s)$ is a resonant (non-periodic) singularity, cyclic in the normalizable case;*
 - (c) *a cyclic quotient of a totally disconnected subgroup of $\mathbb{U}(1)$ if $f(s)$ is a non-resonant and non-linearizable singularity;*
- (2) *the abelian group $\text{Dis}_D^{\mathcal{F}^\diamond}$ is*
 - (a) *infinite and of finite type if H_D is abelian and all the red singularities on $f(D)$ are resonant non-normalizable;*
 - (b) *a cyclic quotient of a totally disconnected subgroup of $\mathbb{U}(1)$ if H_D is abelian and all the red singularities on $f(D)$ are non-resonant non-linearizable;*
 - (c) *finite in all the remaining cases;*
- (3) *$\text{Dis}_s^{\mathcal{F}^\diamond}$ and $\text{Dis}_D^{\mathcal{F}^\diamond}$ are finite if $s \in \text{Ed}_{\mathbf{R}_{\mathcal{F}^\diamond}^1}$ and $D \in \text{Ve}_{\mathbf{R}_{\mathcal{F}^\diamond}^1}$.*

Proof. Assertions (1) result directly from Proposition 5.8.

To obtain Assertions (2) we can suppose that the singular valency of D is at least three, otherwise $\text{Dis}_D^{\mathcal{F}^\diamond} = \text{Dis}_s^{\mathcal{F}^\diamond}$ for $s \in D \cap \Sigma$ and Assertions (2) result again directly from Proposition 5.8. Now let us suppose also that $\text{Dis}_D^{\mathcal{F}^\diamond}$ -thus also $C(H_D)$ - is infinite. Because D is red, H_D is infinite and it follows from Lemmas 5.11 and 5.12 that the set H' of all non periodic elements of H_D is non empty and for every $h \in H'$ we have $C(H_D) = C(h)$, therefore $\text{Dis}_D^{\mathcal{F}^\diamond} = C(h)/\exp(\mathcal{T}_h)$. By Proposition 5.8 the only case where this group is infinite is when h is resonant non normalizable or non resonant non linearizable. To see that these two possibilities correspond to the cases (2a) and (2b) above, it is enough to notice that the local holonomies $h_{D,s}$, $s \in D \cap \Sigma$, that generate H_D , cannot be all periodic (otherwise by abelianity H_D would be finite), and to use Proposition 5.13.

Assertion (3) follows immediately from Assertions (1) and (2) except for $\text{Dis}_D^{\mathcal{F}^\diamond}$ when D is a common vertex of $\mathbf{R}_{\mathcal{F}^\diamond}^1$ and $\mathbf{R}_{\mathcal{F}^\diamond}^0$. In this case although $\text{Exp}_D^{\mathcal{F}^\diamond} = 0$, at the meeting points s of D with components D' of $\mathbf{R}_{\mathcal{F}^\diamond}^1$ we have $\text{Exp}_s^{\mathcal{F}^\diamond} \neq 0$ because $\text{Exp}_{D'}^{\mathcal{F}^\diamond} \neq 0$. Therefore D does not correspond to case (2a) nor case (2b) and $\text{Dis}_D^{\mathcal{F}^\diamond}$ is finite according to (2c). \square

9. PROOF OF THEOREM C

In order to simplify the notations in the proofs below, we will write again \mathbf{A} , \mathbf{R} , \mathbf{R}^0 , \mathbf{R}^1 , Aut , Sym , Exp , Dis and τ , instead of $\mathbf{A}_{\mathcal{F}^\diamond}$, $\mathbf{R}_{\mathcal{F}^\diamond}$, $\mathbf{R}_{\mathcal{F}^\diamond}^0$, $\mathbf{R}_{\mathcal{F}^\diamond}^1$, $\text{Aut}^{\mathcal{F}^\diamond}$, $\text{Sym}^{\mathcal{F}^\diamond}$, $\text{Exp}^{\mathcal{F}^\diamond}$, $\text{Dis}^{\mathcal{F}^\diamond}$ and $\tau_{\mathcal{F}^\diamond}$.

We have already shown that there are “natural” bijections:

$$(15) \quad \text{Mod}([\mathcal{F}^\diamond]) \stackrel{(6)}{\simeq} H^1(\mathbf{A}, \text{Aut}) \stackrel{(9)}{\simeq} H^1(\mathbf{A}, \text{Sym}) \stackrel{(10)}{\simeq} H^1(\mathbf{R}, \text{Sym}),$$

the bijection (10) being only valid when \mathcal{F} has finite type. Moreover Sym is an abelian group-graph over \mathbf{R} and consequently $H^1(\mathbf{R}, \text{Sym})$ is an abelian group, cf. Theorem 5.9. Recall that \mathbf{R}^0 is the subgraph of \mathbf{R} constituted by all the vertices and edges b satisfying $\text{Exp}_b^{\mathcal{F}} = 0$ and \mathbf{R}^1 is the completion of $\mathbf{R} \setminus \mathbf{R}^0$. The rest of the proof is divided in several steps:

- (i) The abelian group $H^1(\mathbf{R}, \text{Sym})$ fits into an exact sequence

$$0 \rightarrow F \rightarrow H^1(\mathbf{R}, \text{Exp}) \rightarrow H^1(\mathbf{R}, \text{Sym}) \rightarrow \mathbf{D} \rightarrow 0,$$

where F is a finite abelian group and \mathbf{D} is a totally disconnected topological abelian group.

- (ii) We have group isomorphisms

$$H^1(\mathbf{R}, \text{Exp}) \simeq H^1(\mathbf{R}^1, \text{Exp}) \simeq \bigoplus_{\alpha} H^1(\mathbf{Z}^{\alpha}, \text{Exp}),$$

where $\mathbf{R}^1 =: \bigcup_{\alpha \in \pi_0(\mathbf{R} \setminus \mathbf{R}^0)} \mathbf{Z}^{\alpha}$ where each **zone** \mathbf{Z}^{α} is the completion of a connected component of $\mathbf{R} \setminus \mathbf{R}^0 = \mathbf{R}^1 \setminus \mathbf{R}^0$.

- (iii) To simplify the computation of the cohomology groups $H^1(\mathbf{Z}^{\alpha}, \text{Exp})$ we modify each zone (not reduced to a single vertex) without changing the number of its extremities neither its cohomology, by adding a vertex and an edge, for each of its extremities. The modified zones fulfill the following property:

(*) *each extremity of \mathbf{Z}^{α} is joined by its edge to a vertex of valency 2 in \mathbf{Z}^{α} .*

- (iv) We decompose each modified zone \mathbf{Z} as $\mathbf{Z} = \mathbf{Z}_0 \cup \mathbf{Z}_1$ where \mathbf{Z}_0 is either empty or a disjoint union of $n + 1 \geq 1$ segments $\bullet_{D'_i} \text{---} \bullet_{D_i}$ with $D'_i \in \mathbf{R}^0$, $D_i \in \mathbf{R}^1$ and $\mathbf{Z}_0 \cap \mathbf{Z}_1 = \{D_0, \dots, D_n\}$. We prove that $H^1(\mathbf{Z}, \text{Exp})$ is trivial if \mathbf{Z}_0 is empty, and it is a finite type quotient of $\bigoplus_{i=1}^n \text{Exp}_{D_i}$ if $\mathbf{Z}_0 \neq \emptyset$.
- (v) Since Exp_{D_i} is isomorphic to \mathbb{C} or $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ or $\mathbb{C}^*/\alpha^{\mathbb{Z}}$ by Lemma 8.4, we can construct a morphism $\Lambda : \mathbb{C}^{\tau} \rightarrow \text{Mod}([\mathcal{F}^{\diamond}])$ with totally disconnected cokernel and finite type kernel.
- (vi) We will specify the notion of semi-local-equisingularity, denoted by SL-equisingularity. This notion was introduced in [21] for germs of deformations and in this paper we adapt it to the context of a global parameter space.
- (vii) We construct SL-equisingular families of foliations $\mathcal{F}_{t,i}^U$ satisfying Theorem C.

Step (i). Consider the long exact sequences

$$(16) \quad \dots \rightarrow H^0(\mathbf{R}, \text{Dis}) \rightarrow H^1(\mathbf{R}, \text{Exp}) \xrightarrow{\chi} H^1(\mathbf{R}, \text{Sym}) \rightarrow H^1(\mathbf{R}, \text{Dis}) \rightarrow 0$$

and

$$\dots \rightarrow H^0(\mathbf{R}_1, \text{Dis}) \rightarrow H^1(\mathbf{R}_1, \text{Exp}) \xrightarrow{\chi^1} H^1(\mathbf{R}_1, \text{Sym}) \rightarrow H^1(\mathbf{R}_1, \text{Dis}) \rightarrow 0$$

associated by Lemma 2.6 to the short exact sequence (14) of abelian group-graphs. By the first part of Proposition 8.6, $Z^1(\mathbf{R}, \text{Dis})$ is a finite product of totally disconnected subgroups of $\mathbb{U}(1)$ and $H^1(\mathbf{R}, \text{Dis})$ is thus a totally disconnected abelian topological group. Moreover when all the singularities of the foliation are resonant or linearizable, the case 1c) is excluded and $Z^1(\mathbf{R}, \text{Dis})$ is finite. In order to conclude this step it only remains to prove that $\ker \chi$ is finite. Let us notice that $H^0(\mathbf{R}_0 \cap \mathbf{R}_1, \text{Exp}) = 0$, $H^1(\mathbf{R}_0, \text{Exp}) = 0$ and $H^1(\mathbf{R}_0 \cap \mathbf{R}_1, \text{Exp}) = 0$. By applying Mayer-Vietoris Lemma 2.7 to the union $\mathbf{R} = \mathbf{R}^0 \cup \mathbf{R}^1$ we obtain the following commutative diagram with exact rows

(17)

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathbf{R}, \text{Exp}) & \longrightarrow & 0 \oplus H^1(\mathbf{R}_1, \text{Exp}) & \longrightarrow & 0 \\ & & \downarrow \chi & & \downarrow 0 \oplus \chi^1 & & \\ \cdots & \longrightarrow & H^1(\mathbf{R}, \text{Sym}) & \longrightarrow & H^1(\mathbf{R}_0, \text{Sym}) \oplus H^1(\mathbf{R}_1, \text{Sym}) & \longrightarrow & \cdots \end{array}$$

Thus $\ker \chi$ is isomorphic to a subgroup of $\ker \chi^1$ and it is sufficient to prove that $H^0(\mathbf{R}_1, \text{Dis}^{\mathcal{F}})$ is finite. But using the second part of Proposition 8.6 we obtain that $C^0(\mathbf{R}_1, \text{Dis}^{\mathcal{F}})$ is finite.

Step (ii). Diagram (17), coming from the Mayer-Vietoris sequence, gives us the isomorphism

$$(18) \quad H^1(\mathbf{R}, \text{Exp}) \simeq H^1(\mathbf{R}^1, \text{Exp}).$$

Clearly \mathbf{R}^1 is a finite union of zones Z^α . Moreover, $Z^\alpha \cap Z^\beta$ is either empty or a single vertex of \mathbf{R}^0 . Therefore

$$H^0(Z^\alpha \cap Z^\beta, \text{Exp}) = 0 \quad \text{and} \quad H^1(Z^\alpha \cap Z^\beta, \text{Exp}) = 0.$$

By applying recursively Mayer-Vietoris Lemma 2.7 we deduce that

$$(19) \quad H^i(\mathbf{R}^1, \text{Exp}) \simeq \bigoplus_{\alpha \in \pi_0(\mathbf{R} \setminus \mathbf{R}^0)} H^i(Z^\alpha, \text{Exp}), \quad i = 0, 1.$$

Step (iii). If a zone Z^α is reduced to a single vertex then $H^1(Z^\alpha, \text{Exp})$ is clearly trivial. If this is not the case, we modify Z^α in the following way: if v' is an extremity of Z^α and $v'' \in \text{Ve}_{Z^\alpha}$ is the unique vertex joined to v' by an edge e' , we replace the segment $\bullet_{v''} \xrightarrow{e'} \bullet_{v'}$ by $\bullet_{v''} \xrightarrow{e''} \bullet_v \xrightarrow{e} \bullet_{v'}$. We also extend the group-graph Exp to the new edges and vertices by defining

$$(20) \quad \text{Exp}_{e''} := \text{Exp}_e := \text{Exp}_v := \text{Exp}_{e'}, \quad \rho_v^{e''} := \rho_v^e := \text{id}_{\text{Exp}_{e'}},$$

$$\rho_{v''}^{e''} = \rho_{v''}^{e'}, \quad \rho_{v'}^e = \rho_{v'}^{e'}.$$

We call this operation the **blow-up of the edge e'** . By performing these blow-ups for each extremity of Z^α we get a new graph \tilde{Z}^α called a **modified zone**. Clearly \tilde{Z}^α fulfills property (*) of (iii).

By doing this process on each zone, we get a **modified graph $\tilde{\mathbf{R}}$** endowed with a group-graph still denoted by Exp . We define now a contraction map:

$$Z^1(\tilde{\mathbf{R}}, \text{Exp}) \xrightarrow{b} Z^1(\mathbf{R}, \text{Exp}), \quad c = (\phi_{ve}) \mapsto c^b = (\phi_{ve}^b)$$

where $\phi_{ve}^b = \phi_{ve}$ if e is not produced by a blow-up, and $\phi_{v'e'} := \phi_{v''e''}\phi_{ve} =: \phi_{v'e'}^{-1}$ if $\bullet_{v''} \xrightarrow{e''} \bullet_v \xrightarrow{e} \bullet_{v'}$ is given by the blow-up of $\bullet_{v''} \xrightarrow{e'} \bullet_{v'}$. It is easy to see that this map induces group isomorphisms

$$(21) \quad H^1(\tilde{Z}^\alpha, \text{Exp}) \simeq H^1(Z^\alpha, \text{Exp}), \quad H^1(\tilde{R}, \text{Exp}) \simeq H^1(R, \text{Exp}).$$

Step (iv). Fix $Z = \tilde{Z}^\alpha$ a modified zone of R^1 . Let Z_1 be the maximal subgraph of Z with all vertices and edges b satisfying $\text{Exp}_b \neq 0$. Denote by Z_0 the completion of $Z \setminus Z_1$. Clearly $Z = Z_0 \cup Z_1$ and Z_0 is either empty or a disjoint union of $n+1 \geq 1$ segments $\bullet_{D'} \text{---} \bullet_D$ with $\text{Exp}_{D'} = 0$, $\text{Exp}_D \neq 0$, D' being an extremity of Z and $\text{val}_Z(D) = 2$. Notice that $H^1(Z_1, \text{Exp}) = 0$. Indeed the restriction morphisms of the group-graph $\text{Exp}^\mathcal{F}$ over Z_1 are surjective by Lemma 8.4. We apply recursively Pruning Theorem 2.11 and we conclude by Remark 2.12 that $H^1(Z, \text{Exp}) = 0$ if Z_0 is empty.

Now suppose that $Z_0 \neq \emptyset$. We will apply Mayer-Vietoris Lemma 2.7 to $Z = Z_0 \cup Z_1$. Using again Lemma 8.4 we see that $H^1(Z_0, \text{Exp}) = 0$ and $H^0(Z_0, \text{Exp}^\mathcal{F}) = 0$ by construction of the modified zones. We obtain the exact sequence

$$(22) \quad H^0(Z_1, \text{Exp}^\mathcal{F}) \xrightarrow{\sigma_\alpha} H^0(Z_0 \cap Z_1, \text{Exp}) \xrightarrow{\delta_\alpha} H^1(Z, \text{Exp}^\mathcal{F}) \rightarrow 0,$$

σ_α being the restriction map and δ_α the connecting map.

In the sequel we will choose one vertex D_0 in $Z_0 \cap Z_1 = \{D_0, \dots, D_n\}$ and we will call the remaining vertices D_1, \dots, D_n the **active vertices of the zone Z** .

Lemma 9.1. *The projection $\pi_\alpha : H^0(Z_1, \text{Exp}) \rightarrow H^0(D_0, \text{Exp}) \simeq \text{Exp}_{D_0}$ is surjective and its kernel is an abelian group of finite type.*

Proof. From Lemma 8.4 for each $(s, D) \in I_{Z_1}$ the restriction map ρ_D^s is surjective with kernel 0 or \mathbb{Z} . Let s_i , $i = 1, \dots, \ell$, be the edges such that $D_0 \in \partial s_i$. For each $y_0 \in \text{Exp}_{D_0}$ there are $y_i \in \text{Exp}_{s_i}$, $i = 1, \dots, \ell$, such that $\rho_{D_i}^{s_0}(y_i) = \rho_{D_0}^{s_0}(y_0)$, see Figure 3. Moreover, the different choices of y_i are parametrized by \mathbb{Z}^k with $k \leq \ell$. By induction we easily deduce that π_α is surjective and its kernel is of finite type. \square

Denote by $\rho : H^0(Z_0 \cap Z_1, \text{Exp}) \rightarrow H^0(D_0, \text{Exp}) \simeq \text{Exp}_{D_0}$ the projection map. Then we have the following morphism of exact sequences

$$\begin{array}{ccccccc} H^0(Z_1, \text{Exp}) & \xrightarrow{\sigma_\alpha} & H^0(Z_0 \cap Z_1, \text{Exp}) & \xrightarrow{\delta_\alpha} & H^1(Z, \text{Exp}) & \longrightarrow & 0 \\ \pi_\alpha \downarrow & & \downarrow \rho & & \downarrow & & \\ 0 & \longrightarrow & \text{Exp}_{D_0} & \xrightarrow{\text{id}} & \text{Exp}_{D_0} & \longrightarrow & 0 \end{array}$$

Since $Z_0 \cap Z_1$ only contains the vertices D_i we have that $H^0(Z_0 \cap Z_1, \text{Exp}) = \bigoplus_{i=0}^n \text{Exp}_{D_i}$ and $\ker \rho$ is just $\bigoplus_{i=1}^n \text{Exp}_{D_i}$. Because π_α is surjective, by

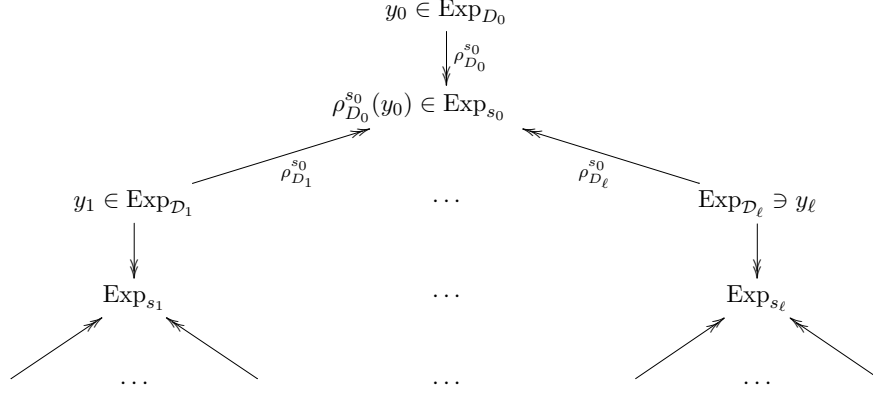


FIGURE 3. Schematic diagram for the group-graph Exp restricted to Z_1 .

applying Snake Lemma we obtain the following exact sequence:

$$(23) \quad \ker(\pi_\alpha) \rightarrow \bigoplus_{i=1}^n \text{Exp}_{D_i} \xrightarrow{\delta_\alpha} H^1(Z, \text{Exp}) \rightarrow \text{coker}(\pi_\alpha).$$

Since π_α is surjective and $\ker(\pi_\alpha)$ is of finite type, thanks to Lemma 9.1 we get that $H^1(Z, \text{Exp})$ is a quotient of \mathbb{C}^n by a finite type subgroup.

Step (v). First we notice that the number of active vertices of a modified zone $Z = \tilde{Z}^\alpha$ is equal to the rank of the homology groups $H_1(Z/Z^0, \mathbb{Z}) \simeq H_1(Z^\alpha/(Z^\alpha \cap R^0), \mathbb{Z})$ of the corresponding quotient graphs. We easily deduce that the number of all active vertices a_r for all the modified zones is equal to the rank $\tau := \text{rank } H_1(R/R_0, \mathbb{Z})$ introduced in Definition 5.7.

Each active vertex a_r , $r = 1, \dots, \tau$, belonging to some modified zone, is produced by the blow-up of an edge $\bullet_{v_r''} \xrightarrow{s_r} \bullet_{v_r'}, v_r'$ being an extremity of the zone. By construction, $\text{Exp}_{a_r} = \text{Exp}_{s_r}$, cf. (20). With this identification and thanks to the isomorphisms (18), (19) and (21) it can be easily checked, using the proof of Lemma 2.7, that the map $\delta = \oplus_\alpha \delta_\alpha$ given by the connecting maps δ_α of the Mayer-Vietoris exact sequences (22) is the surjective morphism

$$\delta : \bigoplus_{r=1}^{\tau} \text{Exp}_{a_r} = \bigoplus_{r=1}^{\tau} \text{Exp}_{s_r} \ni (\varphi_r) \mapsto [(\phi_{ve})] \in H^1(R, \text{Exp})$$

with $\phi_{v_r'' s_r} := \varphi_r$, $\phi_{v_r' s_r} := \varphi_r^{-1}$ and ϕ_{ve} is trivial otherwise.

Now for each $r = 1, \dots, \tau$ we choose a local holomorphic basic vector field X_r transverse to the foliation \mathcal{F}^\sharp and defined on a neighborhood of $f(s_r)$ in the ambient space of \mathcal{F}^\sharp . We define the group morphism $\Lambda : \mathbb{C}^\tau \rightarrow \text{Mod}([\mathcal{F}^\diamond])$ of the theorem C, as the composition

$$\mathbb{C}^\tau \xrightarrow{\xi} \bigoplus_{r=1}^{\tau} \text{Exp}_{s_r} \xrightarrow{\delta} H^1(R, \text{Exp}) \xrightarrow{\chi} H^1(R, \text{Sym}) \stackrel{(15)}{\simeq} \text{Mod}([\mathcal{F}^\diamond]),$$

where $\xi(t) = (\text{expt}_1 X_1, \dots, \text{expt}_r X_r)$, $\text{expt}_r X_r$ denotes the class of $\exp t_r X_r$ in $\text{Aut}_{s_r}/\text{Fix}_{s_r} = \text{Sym}_{s_r}$ and χ is induced by the natural inclusion of group-graphs $\text{Exp} \hookrightarrow \text{Sym}$, see (14). The last bijection (15) induces an abelian group structure on $\text{Mod}([\mathcal{F}^\diamond])$. Moreover, if we define $\mathbf{D} := H^1(\mathbb{R}, \text{Dis}^{\mathcal{F}})$ and $\Gamma : \text{Mod}([\mathcal{F}^\diamond]) \rightarrow \mathbf{D}$ as the composition of the isomorphism (15) and the last arrow in the sequence (16), then the sequence

$$\mathbb{C}^\tau \xrightarrow{\Lambda} \text{Mod}([\mathcal{F}^\diamond]) \xrightarrow{\Gamma} \mathbf{D} \rightarrow 0$$

is exact. It remains to check that $\ker \Lambda = \ker(\chi \circ \delta \circ \xi)$ is of finite type. Since $\ker \chi$ is finite by step (i) and $\ker \delta$ is of finite type thanks to Lemma 9.1 and (23), it suffices to see that $\ker \xi$ is also of finite type. In fact, we will conclude by proving that the kernel of each group morphism $\xi_r : \mathbb{C} \rightarrow \text{Sym}_{s_r}$, $t \mapsto \text{expt}_r X_r$ is of finite type. Since Remark 8.2 allows us to work on a transversal, we can use Proposition 5.8 to describe $\ker(\xi_r)$ as the kernel of the morphism $\mathbb{C} \rightarrow C(h_r)/\langle h_r \rangle$ given by $t \mapsto [\exp t X_r]$, which is of finite type thanks to Lemma 8.4.

Step (vi). Let P be a holomorphic connected manifold and t_0 a point of P . A **deformation of \mathcal{F}** with parameter space the manifold P pointed at t_0 , is a germ along all $\{0\} \times P$ of a 1-dimensional holomorphic singular foliation \mathcal{F}_P defined on a open neighborhood of $\{0\} \times P$ in $\mathbb{C}^2 \times P$, which is locally tangent to the fibers of the projection $\pi_P : \mathbb{C}^2 \times P \rightarrow P$ and such that \mathcal{F} is equal to the restriction of \mathcal{F}_P to $\mathbb{C}^2 \times \{t_0\}$, with the identification $\mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2 \times \{t_0\}$, $(x, y) \mapsto (x, y, t_0)$.

We say that \mathcal{F}_P is **equireducible** if there exists a map $E_{\mathcal{F}_P} : \mathcal{M} \rightarrow \mathbb{C}^2 \times P$ obtained by composition of blow-up maps $E_j : \mathcal{M}_j \rightarrow \mathcal{M}_{j-1}$ fulfilling:

- (1) each center of blow-up $C_j \subset \mathcal{M}_j$ of E_j is biholomorphic to P by the map $\pi_j := \pi_P \circ E_1 \circ E_1 \circ \dots \circ E_{j-1}$,
- (2) the singular locus of the foliation $E_{\mathcal{F}_P}^* \mathcal{F}_P$ is smooth, contained in the exceptional divisor $\mathcal{E}_{\mathcal{F}_P} := E_{\mathcal{F}_P}^{-1}(\{0\} \times P)$ and the restriction of $\pi_P \circ E_{\mathcal{F}_P}$ to each of its connected component is a biholomorphism onto P ,
- (3) the restriction of $E_{\mathcal{F}_P}$ to $\mathcal{M}_t := (\pi_P \circ E_{\mathcal{F}_P})^{-1}(t)$ is exactly the minimal reduction map of the foliation \mathcal{F}_t on $\mathbb{C}^2 \times \{t\}$ induced by \mathcal{F}_P ;

Notice that $\mathcal{E}_{\mathcal{F}_P}$ is a topological product over P , i.e. there is a homeomorphism $\Phi_P : \mathcal{M}_{t_0} \times P \xrightarrow{\sim} \mathcal{M}$ such that $\pi_P \circ \Phi_P$ is the second projection map. By identifying $\mathbb{C}^2 \times \{t_0\}$ with \mathbb{C}^2 , each marking $f : \mathcal{E} \rightarrow \mathcal{E}_{\mathcal{F}_{t_0}}$ of \mathcal{F}_{t_0} by \mathcal{E}^\diamond extends via Φ_P to markings $f_t : \mathcal{E} \rightarrow \mathcal{E}_{\mathcal{F}_t} \subset \mathcal{M}_t$ of \mathcal{F}_t , $t \in P$, defining in this way a map

$$P \rightarrow \text{Mod}([\mathcal{F}^\diamond]), \quad t \mapsto [\mathcal{F}_t, f_t].$$

On the other hand, given a point $t' \in P$, for each base point o_D in a component D of \mathcal{E} introduced at the beginning of Section 4, let us choose a $(1 + \dim P)$ -dimensional submanifold $\Delta_{P,D}$ of \mathcal{M} , transverse to $f_{t'}(D)$ at the point $f_{t'}(o_D)$. The representation of \mathcal{F}_P -holonomy of the leaf $f_{t'}(D \setminus \Sigma)$ defines a representation $\mathcal{H}_{P,D}^{t'}$ of the fundamental group $\pi_1(D \setminus \Sigma, o_D)$ in the group $\text{Diff}(\Delta_{P,D}, f_{t'}(o_D))$ of germs of holomorphic automorphisms of $(\Delta_{P,D}, f_{t'}(o_D))$.

Definition 9.2. We say that \mathcal{F}_P is **SL-equisingular at a point t' of P** if

- (1) \mathcal{F}_P is equireducible,
- (2) for $t \in P$ sufficiently close to t' and for each $(s, D) \in \text{Ed}_{\mathcal{A}_{\mathcal{F}^\circ}} \times \text{Ve}_{\mathcal{A}_{\mathcal{F}^\circ}}$, $D \in \partial s$, the Camacho-Sad indices $CS(E_t^* \mathcal{F}_t, f_t(D), f_t(s))$ does not depend on t ;
- (3) there is a germ of biholomorphism $\psi : (\Delta_{P,D}, f_{t'}(o_D)) \xrightarrow{\sim} (\mathbb{C} \times P, (0, t'))$ such that
 - (a) the composition of ψ by the second projection $\mathbb{C} \times P \rightarrow P$ is equal to $\pi_P \circ E_{\mathcal{F}_P}$ restricted to $\Delta_{P,D}$;
 - (b) for all $\gamma \in \pi_1(D \setminus \Sigma, o_D)$ the biholomorphism $(z, t) \mapsto \psi \circ \mathcal{H}_{P,D}'(\gamma) \circ \psi^{-1}(z, t)$ does not depend on t .

We say that \mathcal{F}_P is **SL-equisingular** if it is SL-equisingular at each point of P .

Step (vii). Consider the elements $[\mathcal{F}_i, f_i]$ of $\text{Mod}([\mathcal{F}^\circ])$, $i \in \mathbf{D}$, given in the statement of Theorem C. By Isomorphism (15) they are represented by cocycles $\mathbf{c}^i = (c_{D,s}^i) \in Z^1(\mathbf{R}, \text{Sym})$. Now we fix an orientation \prec of \mathbf{A} and as in the proof of Theorem 4.14, we lift this cocycle to a cocycle $(\varphi_{D,s}^i) \in Z^1(\mathbf{R}, \text{Aut})$ and we continue to denote by s_1, \dots, s_τ the edges that produce the complete system a_1, \dots, a_τ of active vertices used in step (v). We define then $(\varphi_{D,s}^{i,t}) \in Z^1(\mathbf{A}, \text{Aut})$ by setting

$$\varphi_{D,s}^{i,t} = \begin{cases} \text{id} & \text{if } s \in \text{Ed}_{\mathbf{A}} \setminus \text{Ed}_{\mathbf{R}}, \\ \varphi_{D,s}^i & \text{if } s \in \text{Ed}_{\mathbf{R}} \setminus \{s_1, \dots, s_\tau\}, \\ \varphi_{D,s}^i \circ \exp t_j X_j & \text{if } s = s_j \in \{s_1, \dots, s_\tau\}, \end{cases}$$

$$\varphi_{D',s}^{i,t} = (\varphi_{D,s}^{i,t})^{-1},$$

for $s \in \text{Ed}_{\mathbf{A}}$ with $\partial s = \{D, D'\}$ and $D \prec D'$. Using these cocycles, for each $i \in \mathbf{D}$ we glue suitable neighborhoods W_D of $D \times \mathbb{C}^\tau$ inside $M_{\mathcal{F}} \times \mathbb{C}^\tau$. We obtain a manifold \mathcal{M}_i endowed with a submersion map onto \mathbb{C}^τ , a flat divisor over \mathbb{C}^τ and a foliation by curves tangent to the fibers of the submersion and to the divisor. By the same arguments used in Theorem 3.5 we obtain an open neighborhood of $\{0\} \times \mathbb{C}^\tau$ in $\mathbb{C}^2 \times \mathbb{C}^\tau$ and on this neighborhood an holomorphic vector field defining a one-dimensional equireducible foliation tangent to the fibers of the projection onto \mathbb{C}^τ , whose singular locus is $\{0\} \times \mathbb{C}^\tau$. By construction, after equireduction the exceptional divisor, as intrinsic analytic space, is holomorphically trivial over \mathbb{C}^τ and along each of its irreducible components the reduced foliation is holomorphically trivial. Hence we have obtained a SL-equisingular deformation $\mathcal{F}_{i,t}^U$ of \mathcal{F}_i , see [20], and biholomorphisms $h_{i,t} : \mathcal{E}_{\mathcal{F}_i} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}_{i,t}^U}$, $i \in \mathbf{D}$. We define the markings $f_{i,t}^U : \mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}_{i,t}^U}$ by $f_{i,t}^U := h_{i,t} \circ f_i$.

Notice that by the construction of Λ in step (iv), $\alpha(t)$ is represented in $H^1(\mathbf{A}, \text{Aut})$ by the following 1-cocycle with support in \mathbf{R}^1 :

$$a_{D,s}^t := \begin{cases} \text{id} & \text{if } s \in \text{Ed}_{\mathbf{A}} \setminus \text{Ed}_{\mathbf{R}}, \\ \text{id} & \text{if } s \in \text{Ed}_{\mathbf{R}} \setminus \{s_1, \dots, s_\tau\}, \\ \exp t_j X_j & \text{if } s = s_j. \end{cases}$$

Thanks to Theorems 5.6 and 5.9 we have in $H^1(\mathbf{R}, \text{Sym})$ the equality

$$\left[(\bullet \varphi_{D,s}^{i,t}) \right] = [\mathbf{c}^i] \cdot \left[(\bullet a_{D,s}^t) \right],$$

where $\bullet \varphi_{D,s}^{i,t}$ and $\bullet a_{D,s}^t$ denote the classes of $\varphi_{D,s}^{i,t}$ and $a_{D,s}^t$ in Sym_s . The abelian group structure on $\text{Mod}([\mathcal{F}^\diamond])$ being induced by the one on $H^1(\mathbf{R}, \text{Sym}^{\mathcal{F}})$ by (15), the previous equality proves that in $\text{Mod}([\mathcal{F}^\diamond])$ we have

$$[\mathcal{F}_{i,t}^U, f_{i,t}^U] = \alpha(t) \cdot [\mathcal{F}_i, f_i], \quad i \in \mathbf{D}.$$

10. APPENDIX

Let us denote by $B_r \subset \mathbb{C}^2$ the ball $\{|x|^2 + |y|^2 \leq r\}$ and for a curve $S \ni 0$ in \mathbb{C}^2 let us call **Milnor ball** any ball $B = B_R$ such that $S \cap B \setminus \{0\}$ is regular and meets transversely each sphere ∂B_r , $0 < r \leq R$. We fix a germ \mathcal{F} at $0 \in \mathbb{C}^2$ of a singular holomorphic foliation.

Definition 10.1. *A germ of an invariant curve S at $0 \in \mathbb{C}^2$ will be called \mathcal{F} -appropriate if S is invariant by \mathcal{F} , contains all the **isolated separatrices**⁶ and its strict transform by the reduction of \mathcal{F} meets any dicritical component D of $\mathcal{E}_{\mathcal{F}}$ -valency one, i.e. $\text{card}(D \cap \text{Sing}(\mathcal{E}_{\mathcal{F}})) = 1$.*

The following incompressibility property is proven under some additional assumptions in [13], [16] and an optimal version is obtained by L. Teyssier in [32]:

Theorem 10.2. *If \mathcal{F} is a generalized curve and S is an \mathcal{F} -appropriate curve in a Milnor ball B . Then there exists a fundamental system $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ of open neighborhoods of S in B such that for each $n \in \mathbb{N}$*

- (1) *the inclusion map $U_n \hookrightarrow B$ induces an isomorphism between the fundamental groups of $U_n \setminus S$ and $B \setminus S$;*
- (2) *for each leaf L of the foliation $\mathcal{F}|_{(U_n \setminus S)}$ the inclusion map $L \hookrightarrow U_n \setminus S$ induces an injective morphism $\pi_1(L, \cdot) \hookrightarrow \pi_1(U_n \setminus S, \cdot)$;*
- (3) *there is a finite union of curves on $U_n \setminus S$ whose preimage Ω in the universal covering \tilde{U}_n^* of $U_n \setminus S$ is a disjoint union of embedded conformal discs Ω_α such that each leaf L of the foliation $\tilde{\mathcal{F}}_n$ induced by \mathcal{F} on \tilde{U}_n^* meets Ω and $\text{card}(L \cap \Omega_\alpha) \leq 1$ for any α .*

Remark 10.3. Two direct consequences of this result are the simple connectedness of the leaves of the foliation $\tilde{\mathcal{F}}_n$ and a structure of (non Hausdorff) Riemann surface of its the leaf space $\tilde{\mathcal{Q}}_{U_n}^{\mathcal{F}}$, whose atlas is given by the transversals Ω_α . \square

Now, in the sequel we consider the following situation: \mathcal{F} and \mathcal{G} are two topologically equivalent germs of foliations at $0 \in \mathbb{C}^2$ and $\psi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ is a germ of homeomorphism that conjugates them, $\psi^* \mathcal{G} = \mathcal{F}$.

Previous Theorem 10.2 will allow us to extend Theorem 1.6 of [15] with weaker assumptions.

⁶i.e. their strict transforms meet invariant components of the exceptional divisor.

Theorem 10.4. *If \mathcal{F} is a generalized curve fulfilling Conditions (TC) and (TR) stated in the introduction, then there exists a germ of a homeomorphism $\phi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that:*

- (1) *the lifting $E_{\mathcal{G}} \circ \phi \circ E_{\mathcal{F}}^{-1}$ of ϕ through the reduction maps of \mathcal{F} and \mathcal{G} extends to the exceptional divisor as a germ of homeomorphism $\Phi : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \rightarrow (M_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}})$ along the exceptional divisors;*
- (2) *Φ is holomorphic at each non-nodal singular point of \mathcal{F}^{\sharp} ;*
- (3) *Φ is transversely holomorphic at each point of the exceptional divisor which is regular for \mathcal{F}^{\sharp} and not contained in a dicritical component.*

Beside the possible existence of dicritical components, the new difficulty of this theorem lies in the fact that ψ may not be transversely holomorphic on a whole neighborhood of 0. Indeed, let us denote by $\mathcal{E}_{\mathcal{F}}^{\text{cut}}$ the disjoint union of the cut-components of $\mathcal{E}_{\mathcal{F}}$. Conditions (TC) and (TR) of the introduction do not exclude the existence of exceptional cut-components of $\mathcal{E}_{\mathcal{F}}$, i.e. irreducible components containing at most two singular points of \mathcal{F}^{\sharp} . Around such cut-components, in a meaning which will be specified later, the conjugation ψ may not be transversely holomorphic.

Because \mathcal{F} , and therefore \mathcal{G} are generalized curves [2] ψ defines one-to-one correspondences

$$(24) \quad D \mapsto D' \quad \text{and} \quad s \mapsto s'$$

between the irreducible components of the exceptional divisors $\mathcal{E}_{\mathcal{F}}$ and $\mathcal{E}_{\mathcal{G}}$ and between the points of $\text{Sing}(\mathcal{E}_{\mathcal{F}}) \cup \text{Sing}(\mathcal{F}^{\sharp})$ and $\text{Sing}(\mathcal{E}_{\mathcal{G}}) \cup \text{Sing}(\mathcal{G}^{\sharp})$. Moreover we have the equalities of intersection numbers:

$$(25) \quad (D'_1, D'_2) = (D_1, D_2) \quad \text{for} \quad D_1 \mapsto D'_1, \quad D_2 \mapsto D'_2.$$

Indeed the reduction map of a foliation is also equal to the reduction map of the curve formed by all its isolated separatrices and two dicritical separatrices for each dicritical component of the exceptional divisor. Thus equality (25) follows from classical topological properties of germs of curves.

Let us point out that property (2) in Theorem 10.4 implies equality of Camacho-Sad indices of these foliations. These equalities will be strongly used in the proof of the above theorem and in fact we need to prove them first.

Lemma 10.5. *Under the assumptions of Theorem 10.4, $s \in D \subset \mathcal{E}_{\mathcal{F}}$ if $s' \in D' \subset \mathcal{E}_{\mathcal{G}}$ correspond by (24), then*

$$(26) \quad \text{CS}(\mathcal{F}^{\sharp}, D, s) = \text{CS}(\mathcal{G}^{\sharp}, D', s').$$

Proof. It is enough to prove these equalities when s is an intersection point of the strict transform of a separatrix S of \mathcal{F} with an irreducible component D of $\mathcal{E}_{\mathcal{F}}$. Indeed according to an extension of [23] given in [16, Lemma 1.9] or in [3, Theorem 8], there is such a point on any cut-component \mathcal{C} of $\mathcal{E}_{\mathcal{F}}$. Thus the induction given in [15, § 7.3] will remain valid and equalities (26) will be satisfied at every singular point of the foliation.

We distinguish three possibilities.

- a) $\lambda := \text{CS}(\mathcal{F}^\sharp, D, s)$ is an irrational real number. If λ is positive, s is a nodal singular point, and (26) was obtained by R. Rosas in [30, Proposition 13]. Another proof is given in [16, Theorem 1.12] that remains valid for $\lambda < 0$.
- b) \mathcal{C} is not exceptional and s is not a nodal singular point. Then by using (TR) and thanks to an extended version [15] of the rigidity theorem in [29], ψ is transversely holomorphic on the image by $E_{\mathcal{F}}$ of a neighborhood of \mathcal{C} and specifically at the points of the separatrix S . In this case the proof of (26) given in [15, Chapter 2] remains valid.
- c) \mathcal{C} is exceptional and s is not a nodal point. Then \mathcal{C} is a "chain" of components of $\mathcal{E}_{\mathcal{F}}$, $\mathcal{C} = D_1 \cup \dots \cup D_\ell$, $\ell \geq 1$, D_i meeting D_{i+1} in one point s_i and $D_i \cap D_j = \emptyset$ if $|i - j| \neq 1$. Perhaps \mathcal{C} meets several dicritical components of $\mathcal{E}_{\mathcal{F}}$, but we only have two possibilities fulfilling Assumption (TC):
 - (i) $s \in D_1$ and the other singular points of \mathcal{F}^\sharp belonging to \mathcal{C} are $s_1, \dots, s_{\ell-1}$;
 - (ii) $s \in D_1$, D_ℓ contains a nodal singular point $s_\ell \neq s_{\ell-1}$, the other singular points of \mathcal{F}^\sharp belonging to \mathcal{C} being $s_1, \dots, s_{\ell-1}$;
 In case (ci) using the classical index formula, we see that $\text{CS}(\mathcal{F}^\sharp, D_1, m_1)$ is given by a continuous fraction whose coefficients are the self-intersections (D_i, D_i) , $i = 1, \dots, \ell$; thus (26) follows from (25). In the same way we obtain in case (cii) that $\text{CS}(\mathcal{F}^\sharp, D_1, s)$ is an irrational (negative) real number, but this case was already examined above.

□

Because the dicriticality of an irreducible component D can be characterized by the vanishing of the Camacho-Sad indices along all the adjacent components at their intersection points with D , we have:

Corollary 10.6. *By Correspondences (24) the image of a dicritical component, a exceptional cut-component, a non exceptional cut-component of \mathcal{F} is respectively a dicritical component, an exceptional cut-component, a non exceptional cut-component of \mathcal{G} .*

Proof of Theorem 10.4. This result extends Theorem 5.0.2 of [15] and we will sketch an inductive proof similar to that described in Chapter 8 of [15]. We proceed in four steps: first we extend to our new context the notion of monodromy; then we construct a conjugation Φ_1 between \mathcal{F}^\sharp and \mathcal{G}^\sharp on a neighborhood of the union of all non exceptional cut-components of $\mathcal{E}_{\mathcal{F}}$; in a third step we define a conjugation Φ_2 along the exceptional cut-component except at the nodal singularities; finally in the fourth and last step we extend and glue Φ_1 and Φ_2 at the nodal singularities and along the dicritical components.

- **Step 1.** Let us fix a \mathcal{F} -appropriate curve S and Milnor balls B and B' for S and $S' := \psi(S)$. Let \tilde{B}^* and \tilde{B}'^* be universal coverings of $B \setminus S$ and $B' \setminus S'$ respectively. We suppose that $\psi(B) \subset B'$, we choose a lifting $\tilde{\psi} : \tilde{B}^* \rightarrow \tilde{B}'^*$ of ψ and we denote by $\tilde{\psi}_* : \Gamma \xrightarrow{\sim} \Gamma'$ the isomorphisms induced by $\tilde{\psi}$ between the deck transformation groups of these coverings.

As in Chapter 3 of [15] we call **monodromy of \mathcal{F}** the natural group morphism

$$\mathfrak{M}_S^{\mathcal{F}} : \Gamma \rightarrow \text{Aut}_{\overleftarrow{\text{An}}}(\tilde{\mathcal{Q}}_{\infty}^{\mathcal{F}}) \subset \text{Aut}_{\overleftarrow{\text{Top}}}(\tilde{\mathcal{Q}}_{\infty}^{\mathcal{F}})$$

where with notations of Remark 10.3, $\tilde{\mathcal{Q}}_{\infty}^{\mathcal{F}}$ is the inverse system $(\tilde{\mathcal{Q}}_{U_n}^{\mathcal{F}})_{n \in \mathbb{N}}$, $\overleftarrow{\text{An}}$ is the category of pro-objects associated to the category of analytic spaces and $\overleftarrow{\text{Top}}$ is the category of pro-objects associated to the category of topological spaces and continuous functions. The monodromy

$$\mathfrak{M}_{S'}^{\mathcal{G}} : \Gamma' \rightarrow \text{Aut}_{\overleftarrow{\text{An}}}(\tilde{\mathcal{Q}}_{\infty}^{\mathcal{G}}) \subset \text{Aut}_{\overleftarrow{\text{Top}}}(\tilde{\mathcal{Q}}_{\infty}^{\mathcal{G}})$$

of \mathcal{G} is defined in the same way, after the choice of $S' := \psi(S)$ as \mathcal{G} -appropriate curve. The conjugation $\tilde{\psi}$ induces an automorphism $h_{\tilde{\psi}} : \tilde{\mathcal{Q}}_{\infty}^{\mathcal{F}} \xrightarrow{\sim} \tilde{\mathcal{Q}}_{\infty}^{\mathcal{G}}$ in the category $\overleftarrow{\text{Top}}$; however, because here Condition (G) of [15, page 406] may not be satisfied, h may not be \mathcal{N} -analytic in the sense of [15, Definition 3.4.2]. Thus we extend the notion of conjugation of [15, page 416] by calling **topological conjugation between the monodromies $\mathfrak{M}_S^{\mathcal{F}}$ and $\mathfrak{M}_{S'}^{\mathcal{G}}$** any pair (\mathfrak{g}, h) where $\mathfrak{g} : \Gamma \xrightarrow{\sim} \Gamma'$ is a group isomorphism and $h : \tilde{\mathcal{Q}}_{\infty}^{\mathcal{F}} \xrightarrow{\sim} \tilde{\mathcal{Q}}_{\infty}^{\mathcal{G}}$ is an isomorphism in the category $\overleftarrow{\text{Top}}$ such that $h_* \circ \mathfrak{M}_S^{\mathcal{F}} = \mathfrak{M}_{S'}^{\mathcal{G}} \circ \mathfrak{g}$, with $h_* : \text{Aut}_{\overleftarrow{\text{Top}}}(\tilde{\mathcal{Q}}_{\infty}^{\mathcal{F}}) \rightarrow \text{Aut}_{\overleftarrow{\text{Top}}}(\tilde{\mathcal{Q}}_{\infty}^{\mathcal{G}})$, $\varphi \mapsto h \circ \varphi \circ h^{-1}$. The notions of **geometric conjugation** and **realization of geometric conjugation** given in [15, Definitions 3.3.3 and 3.6.1] which are already defined in the topological category, remain unchanged for topological conjugations. Now we highlight the fact that

- A. the pair $(\tilde{\psi}_*, h_{\tilde{\psi}})$ is a geometric topological conjugation between $\mathfrak{M}_S^{\mathcal{F}}$ and $\mathfrak{M}_{S'}^{\mathcal{G}}$, that is realized on germs (Δ, S) and $(\psi(\Delta), S')$ for any subset Δ of B meeting S ;
- B. Theorem 4.3.1 of [15] which gives a relation between conjugations of monodromies and conjugation of holonomies, remains valid for topological conjugations;
- C. Key Lemma 8.3.2 of [15] of extension of realizations, is also valid when the conjugation of holonomies (\mathfrak{g}, h) is topological, the realization $\phi_S : (T, c) \rightarrow (T', c')$ remaining biholomorphic.

Assertion A that extends Assertion (2) of Invariance Theorem 5.0.1 of [15], is immediate; the other two follow directly from the proofs in [15].

- **Step 2.** We will perform an induction process as in [15, Chapter 8].

a) First we define **elementary pieces** as in [15, §8.2], however the unions defining the real hypersurfaces \mathcal{H} and \mathcal{H}' are now indexed by the set of all singular points of the exceptional divisor and of the foliation. In this way we have three new types of elementary pieces:

- (i) K_D with D an invariant component of the exceptional divisor meeting a dicritical component,
- (ii) K_s with s an intersection point of a dicritical and a non dicritical component of the exceptional divisor,
- (iii) K_D , with D a dicritical component.

b) To start the induction, we proceed as in [15, §8.4] but with an \mathcal{F} -**suitable collection of transversals** $(\Delta_C)_{C \in \mathfrak{C}}$ in the meaning that it is obtained in the following way: \mathfrak{C} is the set of non exceptional cut-components of $\mathcal{E}_{\mathcal{F}}$; for each $C \in \mathfrak{C}$ we choose one separatrix S_C whose strict transform meets C and an embedded conformal disc Δ_C transversal to Δ_C at a regular point; these discs are small enough so their pairwise intersections are empty and they are transversal to the foliation. Up to suitable foliated isotopies we suppose that $\psi(\Delta_C)$ are also conformal embedded discs. Thus thanks to Corollary 10.6 the collection given by $\Delta'_{C'} := \psi(\Delta_C)$, $C \in \mathfrak{C}$, where C' is the cut-component corresponding to C by (24), forms a \mathcal{G} -suitable collection of transversals. We begin the induction with the realization $(\psi|_{\Delta}, \tilde{\psi}|_{\tilde{\Delta}}, h_{\tilde{\psi}})$ of the geometric conjugation $(\psi_*, h_{\tilde{\psi}})$ on $\Delta := \cup_{C \in \mathfrak{C}} \Delta_C$ and $\Delta' := \cup_{C' \in \mathfrak{C}'} \Delta'_{C'}$. Because the cut-components are non exceptional and thanks to the extended version of the main theorem of [29], Rigidity condition (TR) of the introduction implies the analyticity of the restrictions $\psi|_{\Delta_C}$. Then we finish the base case of the induction by the construction of a geometric representation of the conjugation $(\psi_*, h_{\tilde{\psi}})$ satisfying condition (23) of [15, Extension Lemma 8.3.2], as in [15, §8.4].

c) The process of the induction described in [15, §8.1] and started in this way, stops when it requires to make an extension to an elementary piece of type (ii) or to elementary piece containing a nodal singularity belonging to $\text{Sing}(\mathcal{E}_{\mathcal{F}})$. In this way we obtain the announced conjugation Φ_1 .

- **Step 3.** Let \mathcal{C} be an exceptional cut-component of $\mathcal{E}_{\mathcal{F}}$ and let us keep the notations introduced in the proof of Lemma 26: $\mathcal{C} = D_1 \cup \dots \cup D_{\ell}$ has two possible configurations (ci) and (cii).

In the first case (ci), D_{ℓ} contains only one singular point of the foliation, hence there is a holomorphic first integral defined on a neighborhood of \mathcal{C} and specifically the foliation is linearizable at each singular point. Thus by equality of Camacho-Sad indices given by Lemma 10.5 the considered foliations are locally holomorphically conjugated at the singular points corresponding by (24). The equalities (25) of self-intersections of the components of \mathcal{C} and \mathcal{C}' allow us to glue these conjugacies and to obtain a homeomorphism defined on a neighborhood of \mathcal{C} . We leave to the reader the details of this construction.

The situation in case (cii) is similar: we have again equality of Camacho-Sad indices and therefore local conjugacies, and then equality of self-intersections allowing to glue and to obtain a global \mathcal{C}^0 conjugation.

- **Step 4.** On the elementary pieces K_s corresponding to a nodal singular point s , we perform the gluing of the homeomorphisms already constructed by the process described in [15, §8.5]. It remains to extend the obtained homeomorphisms to the dicritical components. Notice that in all the above constructions the homeomorphisms can be built by respecting the dicritical components meeting their definition domains. Finally we arrive at the following situation described in [16, page 147]: if we identify tubular neighborhoods of corresponding dicritical components $D \subset \mathcal{E}_{\mathcal{F}}$ and $D' \subset \mathcal{E}_{\mathcal{G}}$ with

a same⁷ tubular neighborhood of the zero section of the normal bundle of D , the corresponding foliations being identified with the natural normal fibration, we have to extend to the whole D a continuous map g from a union \mathcal{K} of disjoint closed discs to the group $\text{Aut}_0(\mathbb{C}, 0)$ of germs of homeomorphisms of $(\mathbb{C}, 0)$. This can be easily made by extending g to a union of bigger discs \mathcal{K}' being a constant automorphism on $\partial\mathcal{K}'$. This ends the proof of Theorem 10.4. \square

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⁷It is possible because D and D' have same negative self-intersection.

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