QUOTIENT TOPOLOGICAL SPACES UNDER THE ACTION OF THE AFFINE GROUP OF PLANAR QUADRATIC VECTOR FIELDS WITH INVARIANT LINES OF TOTAL MULTIPLICITY FOUR AND WITH ONE REAL AND TWO COMPLEX INFINITE SINGULARITIES

OXANA DIACONESCU[1], DANA SCHLOMIUK[1] AND NICOLAE VULPE[2]

Abstract. In this article we consider the class $\mathbf{QSL}_\mathbf{4}^{\mathbf{u+v^c+w^c},\boldsymbol{\infty}}$ of all real quadratic differential systems $\dfrac{dx}{dt} = p(x,y)$, $\dfrac{dy}{dt} = q(x,y)$ with $\gcd(p,q) = 1$, having invariant lines of total multiplicity four and two complex and one real infinite singularities. We first construct compactified canonical forms for the class $\mathbf{QSL}_\mathbf{4}^{\mathbf{u+v^c+w^c},\boldsymbol{\infty}}$ so as to include limit points in the 12-dimensional parameter space of this class. We next construct the bifurcation diagrams for these compactified canonical forms. These diagrams contain many repetitions of phase portraits and we show that these are due to many symmetries under the group action. To retain the essence of the dynamics we finally construct the quotient spaces under the action of the group $G = Aff(2,\mathbb{R}) \times \mathbb{R}^*$ of affine transformations and time homotheties and we place the phase portraits in these quotient spaces. The final diagrams retain only the necessary information to capture the dynamics under the motion in the parameter space as well as under this group action. We also present here necessary and sufficient for an affine line to be invariant of multiplicity k for a quadratic system.

1. Introduction

We consider here real planar differential systems of the form

$$(1) \qquad (S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y),$$

where $p,\ q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$, their associated vector fields

$$(2) \qquad \tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$$

and differential equations

$$(3) \qquad q(x,y)dx - p(x,y)dy = 0.$$

We call *degree* of a system (1) (or of a vector field (2) or of a differential equation (3)) the integer $n = \deg(S) = \max(\deg p, \deg/, q)$. In particular we call *quadratic* a differential system (1) with $n = 2$.

A system (1) is said to be integrable on an open set U of $\mathbb{R}^2$ if there exists a C^1 function $F(x,y)$ defined on U which is a first integral of the system, i.e. such that $\tilde{D}F(x,y) = 0$ on U and which is nonconstant on any open subset of U. The cases of integrable systems are rare but as Arnold said in [1, p. 405] "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...".

In [9] Darboux gave a method of integration of planar polynomial differential equations in terms of invariant algebraic curves. Roughly speaking, an invariant algebraic curve of system (1) is a curve $f(x,y) = 0$, $f(x,y) \in \mathbb{R}[x,y]$ which is invariant under the flow. For a number of reasons it is not convenient however to consider curves over the reals. Firstly $\mathbb{R}$ is not an algebraically closed field. In particular the curve $x^2 + y^2 + 1 = 0$ is empty over the reals. As all systems (1) over the reals generate systems over $\mathbb{C}$, it is better to talk about

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invariant algebraic curves over $\mathbb{C}$ of differential systems (1) over $\mathbb{C}$. The theory of Darboux was done over $\mathbb{C}$ but it turns out that it can also be applied for real systems. Darboux formulated his theory for differential equations over the complex plane.

Definition 1 (Darboux [9])**.** *An affine algebraic curve $f(x,y)=0$, $f\in\mathbb{C}[x,y]$, $degf\geq 1$ is invariant for a system* (1) *if and only if $f\,|\,\tilde{D}f$ in $\mathbb{C}[x,y]$, i.e. $k=\dfrac{\tilde{D}f}{f}\in\mathbb{C}[x,y]$. In this case k is called the cofactor of f.*

Definition 2 (Darboux [9])**.** *An algebraic solution of an equation* (3) *(respectively of* (1), (2)*) is an invariant algebraic curve $f(x,y)=0$, $f\in\mathbb{C}[x,y]$ $(degf\geq 1)$ with f an irreducible polynomial over $\mathbb{C}$.*

Darboux showed that if an equation (3) (or a system (1) or a vector field (2)) possesses a sufficient number of such invariant algebraic solutions $f_i(x,y)=0$, $f_i\in\mathbb{C}[x,y]$, $i=1,2,\ldots,s$ then the equation has a first integral of the form $F=f_1(x,y)^{\lambda_1}\cdots f_s(x,y)^{\lambda_s}$, $\lambda_i\in\mathbb{C}$.

The simplest class of integrable quadratic systems due to the presence of invariant algebraic curves is the class of integrable quadratic systems due to the presence of invariant lines. The study of this class was initiated in articles [14, 18, 16, 17, 19].

An important ingredient in the classification of quadratic systems possessing invariant lines is the notion of configuration of invariant lines of a polynomial differential system. For this notion we need the concept of multiplicity of an invariant line.

Definition 3. *We say that an invariant straight line $\mathcal{L}(x,y)=ux+vy+w=0$, $(u,v)\neq(0,0)$, $(u,v,w)\in\mathbb{C}^3$ for a quadratic vector field $\tilde{D}$ has multiplicity m if there exists a sequence of real quadratic vector fields $\tilde{D}_k$ converging to $\tilde{D}$, in the topology of their coefficients, such that each $\tilde{D}_k$ has m distinct (complex) invariant straight lines $\mathcal{L}_k^1=0,\ldots,\mathcal{L}_k^m=0$, converging to $\mathcal{L}=0$ as $k\to\infty$, in the topology of the complex projective plane with homogeneous coordinates $[u:v:w]\in\mathbf{P}_2(\mathbb{C})$, and this does not occur for $m+1$.*

Definition 4. *We say that $Z=0$ is an invariant line of multiplicity m for a system (S) of the form (1) if and only if there exists a sequence of systems (S_i) of the form (1) such that (S_i) tend to (S) when $i\to\infty$ and the systems (S_i) have $m-1$ distinct invariant affine lines $\mathcal{L}_i^j=u_i^jx+v_i^jy+w_i^j=0$, $(u_i^j,v_i^j)\neq(0,0)$, $(u_i^j,v_i^j,w_i^j)\in\mathbb{C}^3$ $(j=1,\ldots,m-1)$ such that for every j, $\lim\limits_{i\to\infty}[u_i^j:v_i^j:w_i^j]=[0:0:1]$ and they do not have m invariant such lines L_i^j, $j=1,\ldots,m$ satisfying the above mentioned conditions.*

Definition 5. *We call configuration of invariant lines of a system* (1) *the set of all its (complex) invariant lines (which may have real coefficients), each endowed with its own multiplicity* [14] *and together with all the real singular points of this system located on these lines, each one endowed with its own multiplicity.*

Notation 1. We denote by $\mathbf{QSL_4}$ *the class of all real quadratic differential systems* (1) with p, q relatively prime ($(p,q)=1$), with only a finite number of singularities at infinity, and possessing a configuration of invariant straight lines of total multiplicity $M_{\mathbf{IL}}=4$ including the line at infinity and including possible multiplicities of the lines.

The study of $\mathbf{QSL_4}$ was initiated in [16] where we proved a theorem of classification for this class in terms of the configurations of invariant lines of the systems. This classification, which is taken modulo the action of the group G of real affine transformations and time rescaling, is given in terms of algebraic invariants and comitants and also geometrically, using cycles on the complex projective plane and divisors on the line at infinity.

The following two results were proved in [19].

Theorem 1. *Consider a quadratic system* (1) *in* $\mathbf{QSL_4}$. *Then this system has either a polynomial inverse integrating factor which splits into linear factors over $\mathbb{C}$ or an integrating factor which is Darboux generating in the usual way a Liouvillian first integral. Furthermore the quotient set of* $\mathbf{QSL_4}$ *under the action of the* *group G is formed by: a set of 20 orbits; a set of twenty-three one-parameter families of orbits and a set of ten two-parameter families of orbits. A system of representatives of the quotient space is also given.*

Theorem 2. *i) The total number of topologically distinct phase portraits in the class* $\mathbf{QSL_4}$ *is 69.*

ii) Necessary and sufficient conditions, invariant with respect to the action of the affine group and time rescaling, are given for the realization of each one of the phase portraits corresponding to all possible configurations of invariant lines.

Remark 1. *In classifying planar differential systems the topological equivalence plays an important role. In this work, to reduce the number of phase portraits by half, we say that two planar differential systems* (S_1) *and* (S_2) *are topologically equivalent if and only if there exists an homeomorphism of the plane carrying orbits to orbits and preserving or reversing globally their orientation. In the above theorems also intervenes a finer equivalence relation: two systems are equivalent if they are in the same orbit under the action of the group* G*. The topological equivalence is expected to yield a finite number of over two thousand equivalence classes for the family* **QS** *of planar quadratic systems. On the other hand the set of equivalence classes of this second equivalence relation is infinite. However, this second equivalence relation turns out to have great value for studying the coarser, topological equivalence relation, as it offers the possibility of effective calculations as illustrated in* [19, 17, 18].

We are motivated for studying the systems in the quadratic class **QS** not only because of their usefulness in many applications but also for theoretical reasons. Some hard problems on quadratic systems have been open for over a century. For example the second part of Hilberts 16th problem, which asks for the maximum number of limit cycles which a planar polynomial system of fixed degree n could have, is still open even for quadratic differential systems ($n = 2$). Clearly the topological equivalence relation plays a major role for this problem as the number of limit cycles stays constant within one equivalence class. We expect the number of topologically distinct phase portraits of quadratic differential systems to be finite. However, to exhibit all these distinct phase portraits is a very hard problem.

When studying a specific family of the quadratic class, the finer equivalence relation induced by the action of the group G allows us to choose convenient normal forms depending on fewer than the twelve parameters, the coefficients of a general quadratic system. Clearly the number of limit cycles stays constant within an equivalence class of this finer equivalence relation.

In [2] the authors studied the class $QW2$ of all quadratic differential systems possessing a weak focus of second order. The group action helped in constructing a normal form essentially depending on three parameters for this class. In [2] the authors gave its bifurcation diagram within the 3-dimensional real projective space. Via symmetries we can restrict ourselves to a subspace split by bifurcation surfaces into 373 parts. However we only have 95 topologically distinct phase portraits for $QW2$. In view of these results the following is a legitimate, natural question:

Is it true that for distinct parts of the bifurcation diagram of some family of systems having identical phase portraits, there exist affine transformations and time homotheties carrying one part into the other?

The careful study of this group action on the quadratic class could reduce the number of parts on which the phase portraits need to be obtained. The equivalence relation induced by this group action is also important because of the possibilities it offers for effective calculations using invariant polynomials (see [18, 17, 19]).

Whenever in mathematics we encounter an equivalence relation R on a structured object A it is customary to construct its quotient object A/R, i.e. the set of equivalence classes of A and to inquire about its structure. In particular we have the equivalence relation on **QS** induced by the group G of affine transformations and time homotheties. The quotient object $\mathbf{QS}/G$ is a five-dimensional topological space. In view of the above observations, it is interesting to see what kind of space this is. For example in this work for our canonical form one of the quotient spaces is the real line, while others are disconnected spaces with three components homeomorphic with the interior of a disc completed with one or two open segments of its circumference.

In this article we continue the work begun in [19] by focusing our attention on the class $\mathbf{QSL_4^{u+v^c+w^c,\infty}}$ (i.e. systems in $\mathbf{QSL_4}$ having two complex (non-real) and one real singularities at infinity). Our goals for the study of this class are the following:

- The systems in $\mathbf{QSL_4^{u+v^c+w^c,\infty}}$, modulo the action of the group G, split into three 2-dimensional and two 1-dimensional families of orbits. We want to construct compactified canonical forms and compactified bifurcation diagrams so as to include systems belonging to the border set of $\mathbf{QSL_4^{u+v^c+w^c,\infty}}$ in the 12-parameter space of the quadratic family and also to join whenever possible several of these families into one coherent whole.
- Furthermore we would like to distinguish in these bifurcation diagrams phase portraits which are topologically equivalent but whose algebro-geometric structures of invariant lines are distinct.
- As we shall see in some bifurcation diagrams we consider we have many repetitions of phase portraits. Do we have repetitions of phase portraits which are due to the group action? In other words determine all symmetries under the group action of the systems in the compactified canonical forms.
- Construct quotient spaces with respect to the action of the group G for all the compactified families of systems mentioned above and place the corresponding configurations and phase portraits on these quotient spaces. The resulting diagrams are much simpler and they retain the essence of the dynamics when parameters vary and when we also allow the group to act. Moreover these bifurcation diagrams reveal some interesting phenomena which could not be detected in previous studies. For example, a symmetric system with center and the line at infinity filled up with singularities (*Picture* C_2*.2(b)*) is a limit point in the parameter space of systems with centers of the Lotka-Volterra type (*Picture 4.2(c)*). This fact first observed on Diagram $2(\mathfrak{M})$ was proved in [13, Theorem 3.2] illustrating the action of invariant polynomials in constructing bifurcations in the 12-dimensional space $\mathbb{R}^{12}$.

In this article we reach the goals stated above. The bifurcation diagrams and quotient spaces we construct here are one and two-dimensional, a long way from the five-dimensional quotient space of the whole quadratic class. However, these diagrams and quotient spaces form stepping stones in that direction. We will use them later on, in the construction of the bifurcation diagrams for the class of quadratic systems with invariant lines of total multiplicity *at least three*, class which contains systems with limit cycles. The quotient space for this class, under the action of the group G, is three-dimensional and will include as subspaces, the quotient spaces we construct here. The class of quadratic systems with invariant lines of total multiplicity *at least three* includes the Lotka-Volterra differential systems, a class of dynamical systems important for many applications. Work by D. Schlomiuk and N. Vulpe on the bifurcation diagram in $\mathbb{R}^{12}$ for the Lotka-Volterra class was done in [20], [21]. The class we study here includes the family of systems with two complex invariant lines intersecting at a real point which is a different family of quadratic systems with invariant lines of total multiplicity at least three and which also contains systems with limit cycles. Work is in progress for completing the study of all quadratic systems with invariant lines of total multiplicity three.

This article is organized as follows:

In Section 2 we give the preliminary definitions and results needed in this article.

In Section 3 we give our main results. In Subsection 3.1 we present necessary and sufficient for an affine line to be invariant of multiplicity k for a quadratic system.

In Subsection 3.2 we construct compactified canonical forms for all families of quadratic systems $(S) \in \mathbf{QS}$ with invariant lines of total multiplicity at least four and having one real and two complex (non-real) singularities at infinity.

In Subsection 3.3 we construct bifurcation diagrams for all the canonical forms obtained in Section 3. We also construct the quotient spaces obtained from these canonical forms when we identify points via the action of the group of G. The information given in these last diagrams is easier to read as many redundancies have been eliminated and they focus on the essence of the dynamics under the group action.

2. Preliminary statements and definitions

Consider real differential systems of the form:

$$(S)\qquad \begin{cases} \dfrac{dx}{dt} &= p_0(a) + p_1(a,x,y) + \ p_2(a,x,y) \equiv p(a,x,y),\\ \dfrac{dy}{dt} &= q_0(a) + q_1(a,x,y) + \ q_2(a,x,y) \equiv q(a,x,y),\end{cases} \tag{4}$$

where $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ and

$$p_0(a) = a_{00},\ p_1(a,x,y) = a_{10}x+a_{01}y,\ p_2(a,x,y) = a_{20}x^2+2a_{11}xy+a_{02}y^2,$$
$$q_0(a) = b_{00},\ \ q_1(a,x,y) = b_{10}x+b_{01}y,\ \ q_2(a,x,y) = b_{20}x^2+2b_{11}xy+b_{02}y^2.$$

Notation 2. $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$. We shall use the notation $\boldsymbol{a} = (\boldsymbol{a}_{00}, \boldsymbol{a}_{10} \ldots, \boldsymbol{b}_{02})$ whenever we need to indicate *a specified point in* $\mathbb{R}^{12}$ *and not the parameter* $a = (a_{00}, a_{10} \ldots, b_{02})$. Each particular system (4) yields an ordered 12-tuple $\boldsymbol{a}$ formed by its coefficients.

Definition 6. *A formal expression of the form* $\Delta = \sum n(w)w$ *where* $w \in \mathbf{P}_2(\mathbb{C})$*,* $n(w)$ *is an integer and only a finite number of the numbers* $n(w)$ *are not zero, is called a zero-cycle of* $\mathbf{P}_2(\mathbb{C})$ *and it is called a divisor of the line* $Z = 0$ *if* w *only belongs to this line. We call degree of the expression* Δ *the integer deg*$(\Delta) = \sum n(w)$*. We call support of* Δ *the set* $\mathrm{Supp}\,(\Delta)$ *of points* w *such that* $n(w) \neq 0$*.*

Notation 3. *Let*

$$P(X,Y,Z) = p_0(a)Z^2 + p_1(a,X,Y)Z + \ p_2(a,X,Y) = 0,$$
$$Q(X,Y,Z) = q_0(a)Z^2 + q_1(a,X,Y)Z + \ q_2(a,X,Y) = 0.$$

Definition 7.

$$\Delta_S(C,Z) = \sum_{w\in\{Z=0\}} I_w(C,Z)w \quad \textit{if} \quad Z \nmid C(X,Y,Z),$$
$$\textit{Supp}\, D_S(C,Z) = \{\omega \in \mathbf{P}_2(\mathbb{C}) | I_w(C,Z) \neq 0\},$$

where $C(X,Y,Z) = YP(X,Y,Z) - XQ(X,Y,Z)$*,* $I_w(F,G)$ *is the intersection number (see,* [10]*) of the curves defined by homogeneous polynomials* F*,* $G \in \mathbb{C}[X,Y,Z]$*, deg*(F)*, deg*$(G) \geq 1$ *and* $\{Z = 0\} = \{[X : Y : 0] \,|\, (X,Y) \in \mathbb{C}^2 \setminus (0,0)\}$*.*

A complex projective line $uX + vY + wZ = 0$ is invariant for the system (S) if either it coincides with $Z = 0$ or it is the projective completion of an invariant affine line $ux + vy + w = 0$.

Notation 4. Let $(S) \in \mathbf{QS}$. Let us denote

$$\mathbf{IL}(S) = \left\{ \ l \ \middle| \ \begin{array}{c} l \textit{ is a line in } \mathbf{P}_2(\mathbb{C}) \textit{ such} \\ \textit{that } l \textit{ is invariant for } (S) \end{array} \right\};$$
$$M(l) = \textit{the multiplicity of the invariant line } l \textit{ of } (S).$$

Remark 2. *We note that the line* $l_\infty : Z = 0$ *is included in* $\mathbf{IL}(S)$ *for any* $(S) \in \mathbf{QS}$ *.*

Let $l_i\ :\ f_i(x,y) = 0$, $i = 1, \ldots, k$, be all the distinct invariant affine lines over $\mathbb{C}$ of a system $(S) \in \mathbf{QS}$. Let $l_i'\ :\ \mathcal{F}_i(X,Y,Z) = 0$ be the complex projective completion of l_i.

It is known that on the set $\mathbf{QS}$ of all quadratic differential systems (4) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [14]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on $\mathbf{QS}$. We can identify the set $\mathbf{QS}$ of systems (4) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS} \longrightarrow \mathbb{R}^{12}$ which associates to each system (4) the 12-tuple $\boldsymbol{a} = (\boldsymbol{a}_{00}, \ldots, \boldsymbol{b}_{02})$ of its coefficients.

Our work here is based on our previous papers [16], [19] where we gave necessary and sufficient conditions, in terms of algebraic invariants, for a quadratic system to have invariant straight lines of total multiplicity four.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [15] (see also [5]). Here we shall only construct the necessary invariant polynomials associated to configurations of invariant lines for the class of quadratic systems with exactly four invariant lines including the line at infinity and including multiplicities.

We recall further below some results in [14], [15], [23] which will be needed.

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ i=0,1,2,\\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ i=1,2. \end{aligned}$$

As it was shown in [22] the polynomials

$$\{\ C_0(a,x,y),\quad C_1(a,x,y),\quad C_2(a,x,y),\quad D_1(a),\quad D_2(a,x,y)\ \} \tag{5}$$

of degree one in the coefficients of systems (4) are GL-comitants of these systems.

Notation 5. *Let $f,\ g \in \mathbb{R}[a,x,y]$ and*

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}. \tag{6}$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [11], [12]*).*

Theorem 3. [23] *Any GL-comitant of systems* (4) *can be constructed from the elements of the set* (5) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f,g)^{(k)}$.*

Notation 6. Consider the polynomial $\Phi_{\alpha,\beta} = \alpha P + \beta Q \in \mathbb{R}[a,X,Y,Z,\alpha,\beta]$ where $P = Z^2 p(X/Z,Y/Z)$, $Q = Z^2 q(X/Z,Y/Z)$, $p,\ q \in \mathbb{R}[a,x,y]$ and $\max(\deg_{(x,y)} p,\ \deg_{(x,y)} q) = 2$. Then

$$\begin{aligned} &\Phi_{\alpha,\beta} = \ c_{11}(\alpha,\beta)X^2 + 2c_{12}(\alpha,\beta)XY + c_{22}(\alpha,\beta)Y^2 + 2c_{13}(\alpha,\beta)XZ +\\ &+ 2c_{23}(\alpha,\beta)YZ \quad + c_{33}(\alpha,\beta)Z^2, \quad \Delta(a,\alpha,\beta) = \ \det ||c_{ij}(\alpha,\beta)||_{i,j\in\{1,2,3\}},\\ &D(a,\alpha,\beta) = 4\Delta(a,-\beta,\alpha), \qquad H(a,\alpha,\beta) = 4\big[\det ||c_{ij}(-\beta,\alpha)||_{i,j\in\{1,2\}}\big]. \end{aligned}$$

Lemma 1. [14] *Consider two parallel invariant affine lines $\mathcal{L}_i(x,y) \equiv ux+vy+w_i = 0$, $\mathcal{L}_i(x,y) \in \mathbb{C}[x,y]$, $(i=1,2)$ of a quadratic system (S) of coefficients $\boldsymbol{a}$. Then $H(\boldsymbol{a},-v,u)=0$, i.e. the T-comitant $H(a,x,y)$ captures the directions of parallel invariant lines of systems* (4).

We construct the following T-comitants:

Notation 7.

$$\begin{aligned} B_3(a,x,y) &= (C_2,D)^{(1)} = Jacob\,(C_2,D)\,,\\ B_2(a,x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,D)^{(3)},\\ B_1(a) &= \operatorname{Res}_x(C_2,D)/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,, \end{aligned} \tag{7}$$

where by $\operatorname{Res}_x(*,*)$ we denote the resultant with respect to x of the corresponding polynomials.

Lemma 2. [14] *A necessary condition for the existence of one (respectively 2; 3) invariant straight line(s) in one (respectively 2; 3 distinct) directions in the affine plane is $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).*

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $p(a, x, y)$ and $q(a, x, y)$. We obtain $\tilde{p}(\tilde{a}(a, x_0, y_0), x', y') = p(a, x' + x_0, y' + y_0)$, $\quad \tilde{q}(\tilde{a}(a, x_0, y_0), x', y') = q(a, x' + x_0, y' + y_0)$. Let us construct the following polynomials

$$\Gamma_i(a, x_0, y_0) \equiv \operatorname{Res}_{x'}\Big(C_i\big(\tilde{a}(a, x_0, y_0), x', y'\big), C_0\big(\tilde{a}(a, x_0, y_0), x', y'\big)\Big)/(y')^{i+1},$$
$$\Gamma_i(a, x_0, y_0) \in \mathbb{R}[a, x_0, y_0], \ (i = 1, 2).$$

Notation 8.

$$\tilde{\mathcal{E}}_i(a, x, y) = \Gamma_i(a, x_0, y_0)|_{\{x_0 = x, \ y_0 = y\}} \in \mathbb{R}[a, x, y] \quad (i = 1, 2). \tag{8}$$

Observation 1. We note that the polynomials $\tilde{\mathcal{E}}_1(a, x, y)$ and $\tilde{\mathcal{E}}_2(a, x, y)$ are affine comitants of systems (4) and are homogeneous polynomials in the coefficients $a_{00}, \ldots, b_{02}$ and non-homogeneous in x, y and $deg_a\tilde{\mathcal{E}}_1 = 3$, $deg_{(x,y)}\tilde{\mathcal{E}}_1 = 5$, $deg_a\tilde{\mathcal{E}}_2 = 4$, $deg_{(x,y)}\tilde{\mathcal{E}}_2 = 6$.

Notation 9. *Let $\mathcal{E}_i(a, X, Y, Z)$ $(i = 1, 2)$ be the homogenization of $\tilde{\mathcal{E}}_i(a, x, y)$, i.e.*

$$\mathcal{E}_1(a, X, Y, Z) = Z^5\tilde{\mathcal{E}}_1(a, X/Z, Y/Z), \qquad \mathcal{E}_2(a, X, Y, Z) = Z^6\tilde{\mathcal{E}}_1(a, X/Z, Y/Z)$$

and $\quad \mathcal{H}(a, X, Y, Z) = \gcd\Big(\mathcal{E}_1(a, X, Y, Z), \ \mathcal{E}_2(a, X, Y, Z)\Big)$ *in* $\mathbb{R}[a, X, Y, Z]$.

The geometrical meaning of these affine comitants is given by the following two lemmas:

Lemma 3. [14] *Let $(S) \in$ **QS** and let $\boldsymbol{a} \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. The straight line $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant line for this system if and only if the polynomial $\mathcal{L}(x, y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\boldsymbol{a}, x, y)$ and $\tilde{\mathcal{E}}_2(\boldsymbol{a}, x, y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)\widetilde{W}_i(x, y) \quad (i = 1, 2),$$

where $\widetilde{W}_i(x, y) \in \mathbb{C}[x, y]$.

Lemma 4. [14] *Let $(S) \in$ **QS** and let $\boldsymbol{a} \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. 1) If $\mathcal{L}(x, y) \equiv ux + vy + w = 0$, $u, v, w \in \mathbb{C}$, $(u, v) \neq (0, 0)$ is an invariant straight line of multiplicity k for a quadratic system (4) then $[\mathcal{L}(x, y)]^k \mid \gcd(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2)$ in $\mathbb{C}[x, y]$, i.e. there exist $W_i(\boldsymbol{a}, x, y) \in \mathbb{C}[x, y]$ $(i = 1, 2)$ such that*

$$\tilde{\mathcal{E}}_i(\boldsymbol{a}, x, y) = (ux + vy + w)^k W_i(\boldsymbol{a}, x, y), \quad i = 1, 2. \tag{9}$$

2) If the line $l_\infty : Z = 0$ is of multiplicity $k > 1$ then $Z^{k-1} \mid \gcd(\mathcal{E}_1, \mathcal{E}_2)$.

Let us consider the following GL-comitants of systems (4):

Notation 10.

$$\begin{aligned}
M(a, x, y) &= 2\,\mathrm{Hess}\,\big(C_2(a, x, y)\big), & \eta(a) &= \mathrm{Discriminant}\,\big(C_2(a, x, y)\big),\\
K(a, x, y) &= \mathrm{Jacob}\,\big(p_2(a, x, y), q_2(a, x, y)\big), & \mu(a) &= \mathrm{Discriminant}\,\big(K(a, x, y)\big),\\
N(a, x, y) &= K(a, x, y) + H(a, x, y), & \theta(a) &= \mathrm{Discriminant}\,\big(N(a, x, y)\big).
\end{aligned}$$

The geometrical meaning of these invariant polynomials is revealed by the next 2 lemmas.

Lemma 5. [14] *Let $(S) \in$ **QS** and let $\boldsymbol{a} \in \mathbb{R}^{12}$ be its 12-tuple of coefficients. The common points of $P = 0$ and $Q = 0$ on the line $Z = 0$ are given by the common linear factors over $\mathbb{C}$ of $p_2(x, y)$ and $q_2(x, y)$. Moreover,*

$$deg\,\gcd(p_2(x, y), q_2(x, y)) = \begin{cases} 0 & iff \quad \mu(\boldsymbol{a}) \neq 0;\\ 1 & iff \quad \mu(\boldsymbol{a}) = 0, \ K(\boldsymbol{a}, x, y) \neq 0;\\ 2 & iff \quad K(\boldsymbol{a}, x, y) = 0.\end{cases}$$

Lemma 6. [14] *A necessary condition for the existence of one couple (respectively, two couples) of parallel invariant straight lines of a system (4) corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$ is the condition $\theta(\boldsymbol{a}) = 0$ (respectively, $N(\boldsymbol{a}, x, y) = 0$).*

From [15] it easily follows

Lemma 7. [15] *The type (as defined in [14]) of the divisor $D_S(C,Z)$ on $Z=0$ for systems (1) is determined by the corresponding conditions indicated in Table 1, where we write $\omega_1^c+\omega_2^c+\omega_3$ if two of the points, i.e. ω_1^c,ω_2^c, are complex but not real. Moreover, for each type of the divisor $D_S(C,Z)$ given in Table 1 the quadratic systems (1) can be brought via a linear transformation to one of the following canonical systems $(\mathbf{S}_I)-(\mathbf{S}_V)$ corresponding to their behavior at infinity.*

Table 1

Case	Type of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
1	$\omega_1+\omega_2+\omega_3$	$\eta>0$
2	$\omega_1^c+\omega_2^c+\omega_3$	$\eta<0$
3	$2\omega_1+\omega_2$	$\eta=0,\quad M\neq 0$
4	3ω	$M=0,\quad C_2\neq 0$
5	$D_S(C,Z)$ undefined	$C_2=0$

$$\begin{cases}\dfrac{dx}{dt} &= k+cx+dy+gx^2+(h-b)xy,\\ \dfrac{dy}{dt} &= l+ex+fy+(g-b)xy+hy^2;\end{cases}\qquad (\mathbf{S}_I)$$

$$\begin{cases}\dfrac{dx}{dt} &= k+cx+dy+gx^2+(h+b)xy,\\ \dfrac{dy}{dt} &= l+ex+fy-bx^2+gxy+hy^2;\end{cases}\qquad (\mathbf{S}_{II})$$

$$\begin{cases}\dfrac{dx}{dt} &= k+cx+dy+gx^2+hxy,\\ \dfrac{dy}{dt} &= l+ex+fy+(g-b)xy+hy^2;\end{cases}\qquad (\mathbf{S}_{III})$$

$$\begin{cases}\dfrac{dx}{dt} &= k+cx+dy+gx^2+hxy,\\ \dfrac{dy}{dt} &= l+ex+fy-bx^2+gxy+hy^2,\end{cases}\qquad (\mathbf{S}_{IV})$$

$$\begin{cases}\dfrac{dx}{dt} &= k+cx+dy+gx^2,\\ \dfrac{dy}{dt} &= l+ex+fy+gxy.\end{cases}\qquad (\mathbf{S}_V)$$

In this paper we shall also use the following invariant polynomials, constructed in papers [14]–[19]:

$$
\begin{aligned}
H_1(a) &= -\Big(\big((C_2,C_2)^{(2)},C_2\big)^{(1)},D\Big)^{(3)};\\
H_2(a,x,y) &= (C_1,\ 2H-N)^{(1)} - 2D_1N;\\
H_3(a,x,y) &= (C_2,D)^{(2)};\\
H_4(a) &= \big((C_2,D)^{(2)},(C_2,D_2)^{(1)}\big)^{(2)};\\
H_5(a) &= \big((C_2,C_2)^{(2)},(D,D)^{(2)}\big)^{(2)} + 8\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)};\\
H_6(a,x,y) &= 16N^2(C_2,D)^{(2)} + H_2^2(C_2,C_2)^{(2)};\\
H_7(a) &= (N,C_1)^{(2)};\\
H_8(a) &= 9\big((C_2,D)^{(2)},(D,D_2)^{(1)}\big)^{(2)} + 2\Big[(C_2,D)^{(3)}\Big]^2;\\
H_9(a) &= -\Big(\big((D,D)^{(2)},D,\big)^{(1)}D\Big)^{(3)};\\
H_{10}(a) &= \big((N,D)^{(2)},\ D_2\big)^{(1)};
\end{aligned}
$$

$$
\begin{aligned}
H_{11}(a,x,y) &= 8H\big[(C_2,D)^{(2)} + 8(D,D_2)^{(1)}\big] + 3H_2^2\\
N_1(a,x,y) &= C_1(C_2,C_2)^{(2)} - 2C_2(C_1,C_2)^{(2)},\\
N_2(a,x,y) &= D_1(C_1,C_2)^{(2)} - \big((C_2,C_2)^{(2)},C_0\big)^{(1)},\\
N_3(a,x,y) &= (C_2,C_1)^{(1)},\\
N_4(a,x,y) &= 4\,(C_2,C_0)^{(1)} - 3C_1D_1,\\
N_5(a,x,y) &= \big[(D_2,C_1)^{(1)} + D_1D_2\big]^2 - 4\big(C_2,C_2\big)^{(2)}\big(C_0,D_2\big)^{(1)},\\
N_6(a,x,y) &= 8D + C_2\Big[8(C_0,D_2)^{(1)} - 3(C_1,C_1)^{(2)} + 2D_1^2\Big],\\
\mathcal{G}_1(a) &= ((C_2,\tilde{E})^{(2)},D_2)^{(1)},\\
\mathcal{G}_2(a) &= 8H_8 - 9H_5,\\
\mathcal{G}_3(a) &= (\mu_0-\eta)H_1 - 6\eta(H_4+12H_{10}),
\end{aligned}
$$

where $\mu_0(a) = \mathrm{Res}_{\,x}(p_2,q_2)/y^4$ $(\mu = 4\mu_0)$ and

$$
\begin{aligned}
\tilde{E}(a,x,y) &= \Big[D_1(2\omega_1-\omega_2) - 3(C_1,\omega_1)^{(1)} - D_2(3\omega_3+D_1D_2)\Big]/72,\\
\omega_1(a,x,y) &= (C_2,D_2)^{(1)},\ \omega_2(a,x,y) = (C_2,C_2)^{(2)},\ \omega_3(a,x,y) = (C_1,D_2)^{(1)}.
\end{aligned}
$$

To construct other needed invariant polynomials we use the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[a,x,y]$ (see [4]), where

$$
\begin{aligned}
\mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\
\mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.
\end{aligned}
$$

Using the affine invariant $\mu_0(a)$ we construct the following polynomials:

$$
\begin{aligned}
\mu_i(a,x,y) &= \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i = 1,..,4;\\
\kappa(a) &= (M,K)^{(2)}/4; \quad \kappa_1(a) = (M,C_1)^{(2)};
\end{aligned}
$$

$$\begin{aligned}
L(a,x,y) &= 4K(a,x,y) + 8H(a,x,y) - M(a,x,y);\\
R(a,x,y) &= L(a,x,y) + 8K(a,x,y);\\
K_1(a,x,y) &= p_1(x,y)q_2(x,y) - p_2(x,y)q_1(x,y);\\
K_2(a,x,y) &= 4\,Jacob(J_2,\xi) + 3\,Jacob(C_1,\xi)D_1 - \xi(16J_1 + 3J_3 + 3D_1^2);\\
K_3(a,x,y) &= 2C_2^2(2J_1 - 3J_3) + C_2(3C_0K - 2C_1J_4) + 2K_1(3K_1 - C_1D_2),
\end{aligned}$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and

$$J_1 = Jacob(C_0, D_2),\ J_2 = Jacob(C_0, C_2),\ J_3 = Discrim(C_1),$$
$$J_4 = Jacob(C_1, D_2),\ \xi = M - 2K.$$

We shall also use the following remark:

Remark 3. *Assume $s, \gamma \in \mathbb{R}$, $\gamma > 0$. Then the transformation $x = \gamma^s x_1$, $y = \gamma^s y_1$ and $t = \gamma^{-s} t_1$ does not change the coefficients of the quadratic part of a quadratic system, whereas each coefficient of the linear (respectively constant) part will be multiplied by γ^{-s} (respectively by γ^{-2s}).*

3. Main results

3.1. **Necessary and sufficient conditions for a line to an invariant of multiplicity k of a system in QS.** We use some results, definitions and notations introduced in paper [6].

Definition 8. ([6, Definition 5.1]) *If X is a polynomial vector field on $\mathbb{C}^2$, **the n-th extactic curve** $\mathscr{E}_n(X)$ **of** X is given by the equation*

$$\mathscr{E}_n(X) \equiv \det \begin{pmatrix} v_1 & v_2 & \dots & v_l \\ X(v_1) & X(v_2) & \dots & X(v_l) \\ \vdots & \vdots & \dots & \vdots \\ X^{l-1}(v_1) & X^{l-1}(v_2) & \dots & X^{l-1}(v_l) \end{pmatrix} = 0,$$

where $v_1, v_2, \dots, v_l$ is a basis of $\mathbb{C}_n[x,y]$, the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}_n[x,y]$ of degree at most n, and and we take $l = (k+1)(k+2)/2$, $X^0(v_i) = v_i$ and $X^j(v_1) = X(X^{j-1}(v_1))$.

Proposition 1. ([6, Proposition 5.2]) *Assume that $f(x,y) = 0$ is an invariant algebraic curve of degree n of a polynomial vector field X. Then $f(x,y)$ is a factor of the polynomial $\mathscr{E}_n(X)$.*

Definition 9. ([6, Definition 5.4]) *An invariant algebraic curve $f = 0$ of degree n for a vector field X has **algebraic multiplicity k** if f^k divides $\mathscr{E}_n(X)$ and k is the greatest positive integer satisfying this condition.*

Definition 10. ([6, Definition 6.1]) *An invariant algebraic curve $f = 0$ of degree n of the vector field X has **geometric multiplicity m** if m is the largest integer for which there exists a sequence of vector fields $(X_i)_{i>0}$ of bounded degree, converging to hX, for some polynomial h, not divisible by f , such that each X_r has m distinct invariant algebraic curves, $f_{r,1} = 0, \dots, f_{r,m} = 0$, of degree at most n, which converge to $f = 0$ as r goes to infinity. If we set $h = 1$ in the definition above, then we say that the curve has **strong geometric multiplicity m**.*

Remark 4. *Here by convergence of the vector fields we understand convergence in the topology of their coefficients. By the convergence of a sequence of algebraic algebraic curves $f_{r,1} = 0, \dots, f_{r,m} = 0$ of degree at most n to an algebraic curve $f = 0$ of degree n we understand convergence in the topology of the complex projective space $\mathbf{P}_N(\mathbb{C})$, where $N = (n+1)(n+2)/2$ of all these curve towards $f = 0$.*

Remark 5. *We note that our Definition 3 of multiplicity of an invariant line coincides with the concept of strong geometric multiplicity above in the case of lines.*

According to [6] we have the next result.

Proposition 2. ([6, Main Theorem]) *Algebraic multiplicity and strong geometric multiplicity of an invariant algebraic curve of a polynomial vector field X coincide.*

Using the invariant polynomials $\tilde{\mathcal{E}}_1(a,x,y)$ and $\tilde{\mathcal{E}}_2(a,x,y)$ associated to a quadratic differential system, defined in Section 2 and Lemmas 3 and 4 we arrive at the next result.

Theorem 4. *Consider an affine straight line $\mathcal{L}(x,y)=ux+vy+w=0$, $(u,v)\neq(0,0)$, $(u,v,w)\in\mathbb{C}^3$. This line is an invariant line of multiplicity k for a quadratic vector field $\tilde{D}$ if and only if L^k divides the greatest common divisor of the invariant polynomials $\tilde{\mathcal{E}}_1(a,x,y)$ and $\tilde{\mathcal{E}}_2(a,x,y)$ ($L^k|\gcd\big(\tilde{\mathcal{E}}_1(a,x,y),\,\tilde{\mathcal{E}}_2(a,x,y)\big)$).*

Proof: Consider an affine straight line $\mathcal{L}(x,y)=ux+vy+w=0$, $(u,v)\neq(0,0)$, $(u,v,w)\in\mathbb{C}^3$. According to Lemma 3 this line is an invariant line for a quadratic differential system if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{E}}_1(\boldsymbol{a},x,y)$ and $\tilde{\mathcal{E}}_2(\boldsymbol{a},x,y)$ over $\mathbb{C}$, i.e.

$$\tilde{\mathcal{E}}_i(\boldsymbol{a},x,y)=(ux+vy+w)\widetilde{W}_i(x,y)\quad(i=1,2),$$

where $\widetilde{W}_i(x,y)\in\mathbb{C}[x,y]$.

On the other hand by Definition 9 this invariant line has algebraic multiplicity k, if and only if $\mathcal{L}^k$ divides the first extactic polynomial $\mathscr{E}_1(\tilde{D})$.

For quadratic vector fields X a straightforward calculation gives

$$\mathscr{E}_1(\tilde{D})=P\cdot X(Q)-Q\cdot X(P)=-\tilde{\mathcal{E}}_1(a,x,y).$$

According to Proposition 2 the algebraic multiplicity coincides with the strong geometric multiplicity.

Therefore according to Lemma 4, L^k divides the greatest common divisor of the invariant polynomials $\tilde{\mathcal{E}}_1(a,x,y)$ and $\tilde{\mathcal{E}}_2(a,x,y)$. Since $\mathscr{E}_1(\tilde{D})=-\tilde{\mathcal{E}}_1(a,x,y)$ this completes the proof of the Theorem.

∎

3.2. **Construction of compactified canonical forms.**

Theorem 5. *(i) Any system $(S)\in\mathbf{QSL_4}$ with divisor $D_S(C,Z)=\omega_1^c+\omega_2^c+\omega_3$ can be brought via an affine transformation and time rescaling to one of the following four canonical forms depending on the parameter $[b:g:h]\in\mathbf{P}_2[\mathbb{R}]$ for (10)–(12) and on $[c:d]\in\mathbf{P}_1[\mathbb{R}]$ for (13):*

$$\begin{aligned}\dot{x}&=gx^2+(h+b)xy,\\ \dot{y}&=h[g^2+(h+b)^2]+(g^2+b^2-h^2)x+2ghy-bx^2+gxy+hy^2;\end{aligned}\tag{10}$$

$$\dot{x}=gx^2+(h+b)xy,\quad \dot{y}=-b+g\,x+(h-b)y-bx^2+gxy+hy^2;\tag{11}$$

$$\dot{x}=gx^2+(h+b)xy,\quad \dot{y}=-bx^2+gxy+hy^2;\tag{12}$$

$$\dot{x}=2cx+2dy,\quad \dot{y}=c^2+d^2-x^2-y^2.\tag{13}$$

(ii) All systems $(S)\in\mathbf{QSL_4}$ included in the family (10) (respectively (11); (12); (13)) have the configuration of invariant lines Config. 4.2 (respectively Config. 4.6 or 4.7; Config. 4.8; Config. 4.27) and lie in the affine chart corresponding to $b\neq0$ for (10)–(12) and to $d\neq0$ for (13).

Proof: According to Lemma 7 the systems with this type of divisor can be brought by linear transformations to the canonical form $(\mathbf{S}_{II})$ for which we calculate the main classifying invariant polynomials (see Table 2 [19]):

$$\begin{aligned}\theta&=8b(h+b)[(h-b)^2+g^2],\quad C_2=bx(x^2+y^2),\\ N&=(g^2-2bh+2b^2)x^2+2g(h+b)xy+(h^2-b^2)y^2.\end{aligned}\tag{14}$$

Remark 6. *We observe that the condition* $C_2 = 0$ *is equivalent to* $b = 0$ *and this leads to systems with the line at infinity filled up with singularities. This class was studied in* [18].

Remark 7. *We note that two of the infinite singular points of the systems* $(\mathbf{S}_{II})$ *are not real. Therefore according to* [16] *a system in this class could belong to* $\mathbf{QSL_4}$ *only if for this system the condition* $B_3 = 0$ *holds.*

In what follows we shall assume that for a system $(\mathbf{S}_{II})$ the condition $B_3 = 0$ is fulfilled.

We view $\mathbf{P}_2[\mathbb{R}]$ as a disk with opposite points on the circumference identified. We use the homogeneous coordinates $[b : g : h]$ for $\mathbf{P}_2[\mathbb{R}]$ placing the line $b = 0$ on the circumference of the disk.

3.2.1. **The case** $\theta \neq 0$**.** The condition $\theta \neq 0$ yields $(h + b) \neq 0$ and we may assume $c = d = 0$ in $(\mathbf{S}_{II})$ via the translation $x \to x - d/(h + b)$ and $y \to y + (2dg - c(h + b))/(h + b)^2$. Thus we obtain the systems

$$\dot{x} = k + gx^2 + (h + b)xy, \quad \dot{y} = l + ex + fy - bx^2 + gxy + hy^2, \tag{15}$$

for which we have: Coefficient$[B_3,\ y^4] = -3bk(h + b)^2$. Therefore since $\theta \neq 0$ the condition $B_3 = 0$ implies $k = 0$ and we have

$$B_3 = 3b[ef(h + b) + 2gl(h - b) - f^2g]x^2(x^2 - y^2) + 6b[bf^2 + efg - e^2h + l(h - b)^2 - g^2l]x^3y.$$

So, the condition $B_3 = 0$ yields the following system of equations

$$\begin{aligned} &Eq_1 \equiv ef(h + b) + 2g(h - b)l - f^2g = 0,\\ &Eq_2 \equiv bf^2 + efg - e^2h + \big[(h - b)^2 - g^2\big]l = 0. \end{aligned} \tag{16}$$

Both equations are linear in l. We first note that we cannot have both coefficients of l zero. Indeed, if we suppose $g(h - b) = (h - b)^2 - g^2 = 0$ then this contradicts $\theta \neq 0$.

Hence, at least one of the coefficients in front of l is not zero. We consider two subcases: $g(h-b)\big[(h-b)^2 - g^2\big] \neq 0$ and $g(h - b)\big[(h - b)^2 - g^2\big] = 0$.

3.2.1.1. **The subcase** $g(h - b)\big[(h - b)^2 - g^2\big] \neq 0$**.** In this case we calculate $\text{Res}_l(Eq_1, Eq_2) = (be - eh + fg)[2egh + f(h^2 - b^2 - g^2)]$. We observe that for systems (15) with $k = 0$ we have: $H_7 = 4(h+b)[be - eh + fg]$ and we shall consider two possibilities: $H_7 \neq 0$ and $H_7 = 0$.

3.2.1.1.1. **The possibility** $H_7 \neq 0$**.** Then the equality $\text{Res}_l(Eq_1, Eq_2) = 0$ yields $2egh = f(g^2 + b^2 - h^2)$. Since $\theta \neq 0$ from (14) we have $(gh)^2 + (g^2 + b^2 - h^2)^2 \neq 0$ then without loss of generality we may set: $e = (g^2 + b^2 - h^2)u$ and $f = 2ghu$ where u is a new parameter. Therefore from (16) we obtain

$$g(h - b)\big[l - hu^2(g^2 + (h + b)^2)\big] = 0 = \big[(h - b)^2 - g^2\big]\big[l - hu^2(g^2 + (h + b)^2)\big]$$

and hence, $l = hu^2[g^2 + (h + b)^2]$. In this case $H_7 = 4u(h + b)^2[g^2 + (h - b)^2] \neq 0$ and we may assume $u = 1$ via Remark A ($\gamma = u$, $s = 1$). This leads to the systems:

$$\begin{aligned} &\dot{x} = gx^2 + (h + b)xy,\\ &\dot{y} = h[g^2 + (h + b)^2] + (g^2 + b^2 - h^2)x + 2ghy - bx^2 + gxy + hy^2. \end{aligned} \tag{17}$$

Hence the conditions

$$\eta < 0,\ B_3 = 0,\ \theta \neq 0,\ H_7 \neq 0 \tag{18}$$

necessarily lead us to the canonical form (17), which coincides with (10) and which according to Table 2 ([19]) due to (18) includes the systems with Config. 4.2.

We now observe that the change of parameter $(b, g, h) \mapsto (\lambda b, \lambda g, \lambda h)$ leads to a system (S_λ) equivalent to (17) under the group action. Indeed the change $(x, y, t) \mapsto (\lambda x, \lambda y, \lambda^{-2}t)$ applied to (S_λ) yields a system of the family (17).

Remark 8. *We point out that whenever we have the conditions* (18)*, they lead us to the canonical form* (17)*. However not every system of the form* (17) *satisfies these conditions. Hence if not all three parameters are zero, the parameter space is* $\mathbf{P}_2(\mathbb{R})$ *for the form* (17) *and it contains points in the border of the set of the systems possessing the Config. 4.2.*

3.2.1.1.2. **The possibility** $H_7 = 0$**.** Then $e(h-b) = fg$ and since the condition $\theta \neq 0$ yields $g^2+(h-b)^2 \neq 0$, we may assume $e = gu$ and $f = (h-b)u$, where u is a new parameter. Then from (16) we have $g(h-b)(l+bu^2) = 0 = [(h-b)^2 - g^2](l+bu^2)$. Due to $g^2 + (h-b)^2 \neq 0$ we obtain $l = -bu^2$ and this leads to the systems:

$$\dot{x} = gx^2 + (h+b)xy, \quad \dot{y} = -bu^2 + gu\,x + u(h-b)y - bx^2 + gxy + hy^2. \tag{19}$$

For these systems we have $H_9 = 2304b^4u^8(h+b)^8$.

1) *The case* $H_9 \neq 0$. Then $u \neq 0$ and we may assume $u = 1$ via Remark 3 ($\gamma = u$, $s = 1$). So we get the family of systems

$$\dot{x} = gx^2 + (h+b)xy, \quad \dot{y} = -b + g\,x + (h-b)y - bx^2 + gxy + hy^2 \tag{20}$$

which coincides with (11) and which includes the systems with the Configurations 4.6 and 4.7. Due to the transformation $(x, y, t) \mapsto (x, y, t/\lambda)$ the parameter here could be consider $\Lambda = [b : g : h] \in \mathbf{P}_2(\mathbb{R})$.

2) *The case* $H_9 = 0$. Then $u = 0$ and we obtain the family of systems

$$\dot{x} = gx^2 + (h+b)xy, \quad \dot{y} = -bx^2 + gxy + hy^2, \tag{21}$$

i.e. the family (12), which includes the systems with Configuration 4.8. Due to the transformation $(x, y, t) \mapsto (x, y, t/\lambda))$ the parameter here could be consider $\Lambda = [b : g : h] \in \mathbf{P}_2(\mathbb{R})$.

3.2.1.2. **The subcase** $g(h-b)\big[(h-b)^2 - g^2\big] = 0$**.** Here we have to do a case by case discussion, more precisely, we must consider the cases: $g = 0$, $h = b$ and $g = \pm(h-b)$.

Considering the equations (16) it is not too hard to show, that in each one of the cases mentioned above we also arrive at one of the systems (17) or (19).

3.2.2. **The case** $\theta = 0$**.** According to (14) we have $b(h+b)[(h-b)^2 + g^2] = 0$.

3.2.2.1. **The subcase** $N \neq 0$**.** Then assuming $b \neq 0$ (see Remark 6) by (14) the condition $\theta = 0$ yields $h = -b$ and in addition we may assume $f = 0$ due to the translation: $x \to x$ and $y \to y + f/(2b)$. Hence, we obtain the systems

$$\dot{x} = k + cx + dy + gx^2, \quad \dot{y} = l + ex - bx^2 + gxy - by^2, \tag{22}$$

for which by Remark 7 the condition $B_3 = 0$ must be satisfied. Calculations yield: $H_7 = 4d(g^2 + 4b^2)$, Coefficient$[B_3, y^4] = -3\,bd^2g$. So the condition $B_3 = 0$ implies $dg = 0$. According to [16] for systems (22) to be in the class $\mathbf{QSL_4}$ the condition $d \neq 0$ (i.e. $H_7 \neq 0$) must be fulfilled. Then $g = 0$ and we may assume $e = 0$ via the translation: $x \to x + e/(2b)$, $y \to y$. After that for systems (22) calculations yield: $B_3 = 12b^3kx^2(x^2 - y^2) - 6(c^2 - 4bl + d^2)x^3y$. Therefore the condition $B_3 = 0$ yields $k = 0$ and $4bl = c^2 + d^2$. As $b \neq 0$ due to time rescalling we may set $b = 1$ and replacing c with $2c$ and d with $2d$ we get the systems:

$$\dot{x} = 2cx + 2dy, \quad \dot{y} = c^2 + d^2 - x^2 - y^2, \tag{23}$$

i.e. the family (13), which includes the systems with Configuration 4.27. Due to the transformation $(x, y, t) \mapsto (\lambda x, \lambda y, \lambda^{-1}t)$ the parameter here is $\Lambda = [c : d] \in \mathbf{P}_1(\mathbb{R})$.

3.2.2.2. **The subcase** $N = 0$. According to [16] in this case systems $(\mathbf{S}_{II})$ cannot belong to the class $\mathbf{QSL_4}$.

Since all the cases were analysed Theorem 5 is proved. ∎

3.3. **Construction of bifurcation diagrams.** There are the systems for which we have $\eta < 0$ (see Table 1, page 8) and each such system has the configuration of invariant lines corresponding to one of the configurations *Config. 4.j, j=2,6,7,8,27* (see [16]). Pictures for corresponding configurations together with invariant conditions are listed in DIAGRAM 1 where we also have the bifurcation diagram in $\mathbb{R}^{12}$ for this family in terms of invariant polynomials.

We shall examine these configurations one by one.

3.3.1. **Configuration 4.2.**

Theorem 6. *(i) The family* (10) *contains as a subfamily representatives of all the systems with configuration Config. 4.2 and has the projective plane as a parameter space. The bifurcation diagram of the family* (10) *is indicated in* DIAGRAM 2 *which contains 10 topologically distinct phase portraits. These are stratified as folllows: 4 of them have systems with configuration Config. 4.2 and Pictures 4.2(i)* $(i \in \{a, b, c, d\}$. *The remaining portraits correspond to systems which are*

- *degenerate* (D_4, D_8, D_9, D_{10})
- *with the line at infinity filled up with singularities (Picture* $C_2.2(a)$ *and Picture* $C_2.2(b)$*)*
- *with invariant lines of total multiplicity five (Picture 5.10, which is topologically equivalent to Picture* $D.10$*)*
- *with 6 invariant lines (Picture 6.4, which is topologically equivalent to Picture 4.2(d))*

All systems have at least 3 distinct affine invariant lines (one real line and two complex)

(ii) The family (10) *yields a quotient topological space homeomorphic to a closed disk modulo the group action. This orbit space carries the bifurcation diagram indicated in* DIAGRAM 2*(*$\mathfrak{M}$*) where we give all phase portraits of systems in the family as well as their corresponding configurations of invariant lines.*

(iii) In this diagram the subspace of orbits of systems with configuration Config. 4.2 is disconnected with three connected components, two of which homeomorphic to the interior of a disc completed with an open arc of its circumference and one of them is homeomorphic to the interior of a disc completed with two open arcs with a common limiting point on its circumference. Two of these three components have systems with the same phase portrait Picture 4.2 (a) with a focus and a saddle and the third component with Picture 4.2(b) has a focus and a center. On the first two arcs we have a center and a saddle and on the arcs for the third component we have two centers.

Proof: According to Theorem 5 all the systems having the Configuration 4.2 are included in the family:

$$\begin{aligned} \dot{x} &= gx^2 + (h+b)xy, \\ \dot{y} &= h[g^2 + (h+b)^2] + (g^2 + b^2 - h^2)x + 2ghy - bx^2 + gxy + hy^2, \end{aligned} \tag{24}$$

where $[b : g : h] \in \mathbf{P}_2(\mathbb{R})$.

We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line $b = 0$.

3.3.1.1. **The case** $b \neq 0$**.** Then we may consider $b = 1$ and following the Table 2 of [19] we compute invariant polynomials necessary for each one of the Pictures 4.2(u), $u \in \{a, b, c, d\}$:

$$\begin{aligned} &\theta = 8(h+1)[g^2 + (h-1)^2],\ \mu_0 = -h[g^2 + (h+1)^2],\ B_3 = 0, \\ &\mathcal{G}_1 = 2g(h+1)[g^2 + (3h+1)^2],\ H_7 = 4(h+1)^2[g^2 + (h-1)^2], \\ &N = (g^2 - 2h + 2)x^2 + 2g(h+1)xy + (h^2 - 1)y^2. \end{aligned} \tag{25}$$

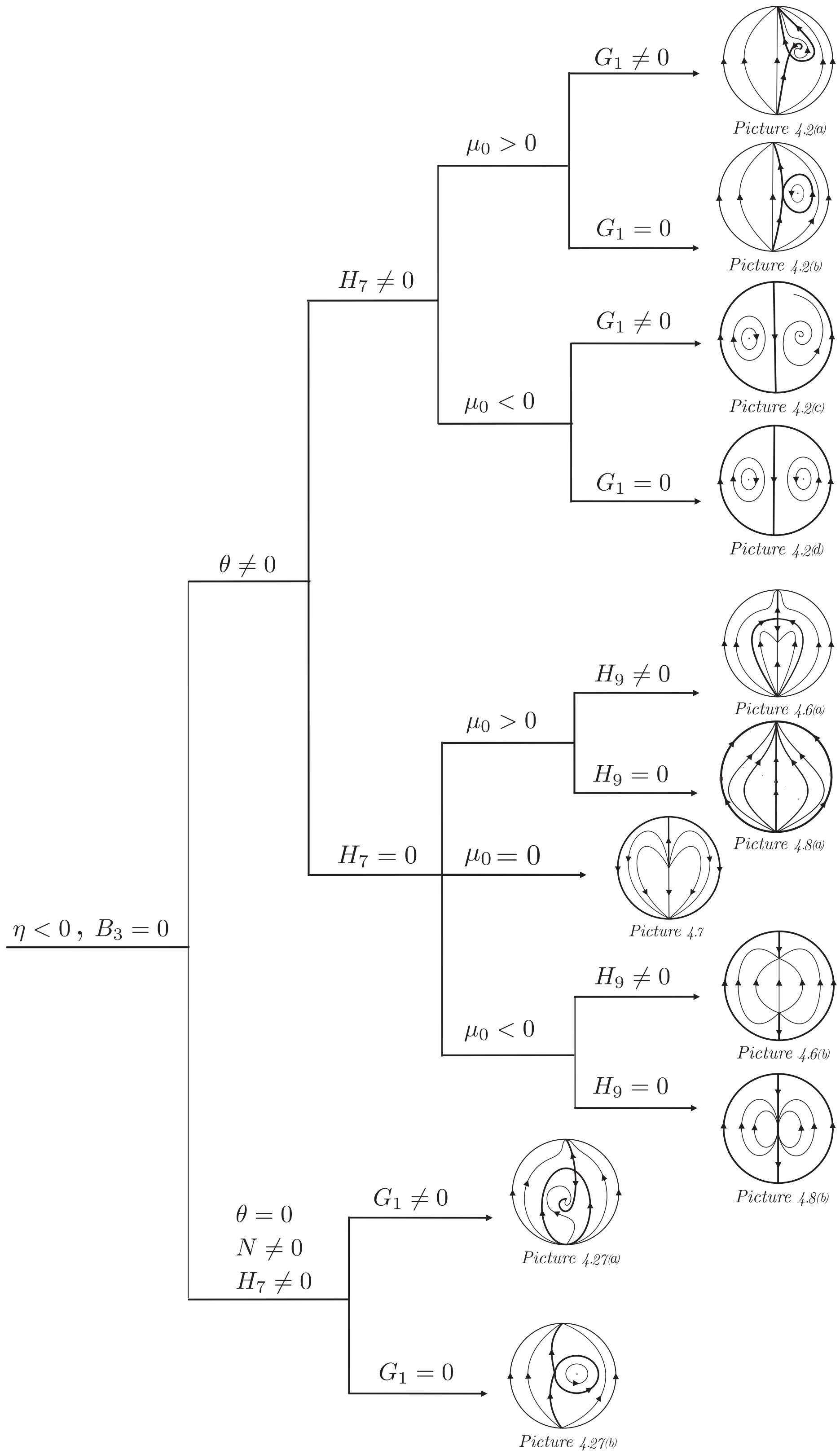

DIAGRAM 1. **Phase portraits: the case $\boldsymbol{\eta < 0}$**

3.3.1.1.1. **The subcase $\mu_0 \neq 0$.** We consider two possibilities: $\theta \neq 0$ and $\theta = 0$.

1) *The possibility* $\theta \neq 0$. This implies $H_7 \neq 0$ and according to [19] the phase portrait of a system (24) with $b = 1$ corresponds to *Picture 4.2(a)* (respectively *Picture 4.2(b)*; *Picture 4.2(c)*; *Picture 4.2(d)*) if $\mu_0 > 0$, $G_1 \neq 0$ (respectively $\mu_0 > 0$, $G_1 = 0$; $\mu_0 < 0$, $G_1 \neq 0$; $\mu_0 < 0$, $G_1 = 0$). So, if $\theta\mu_0 \neq 0$, i.e. $h(h+1)[g^2 + (h-1)^2][g^2 + (h+1)^2] \neq 0$ then we end up with the phase portraits above indicated placed inside the circle of DIAGRAM 2.

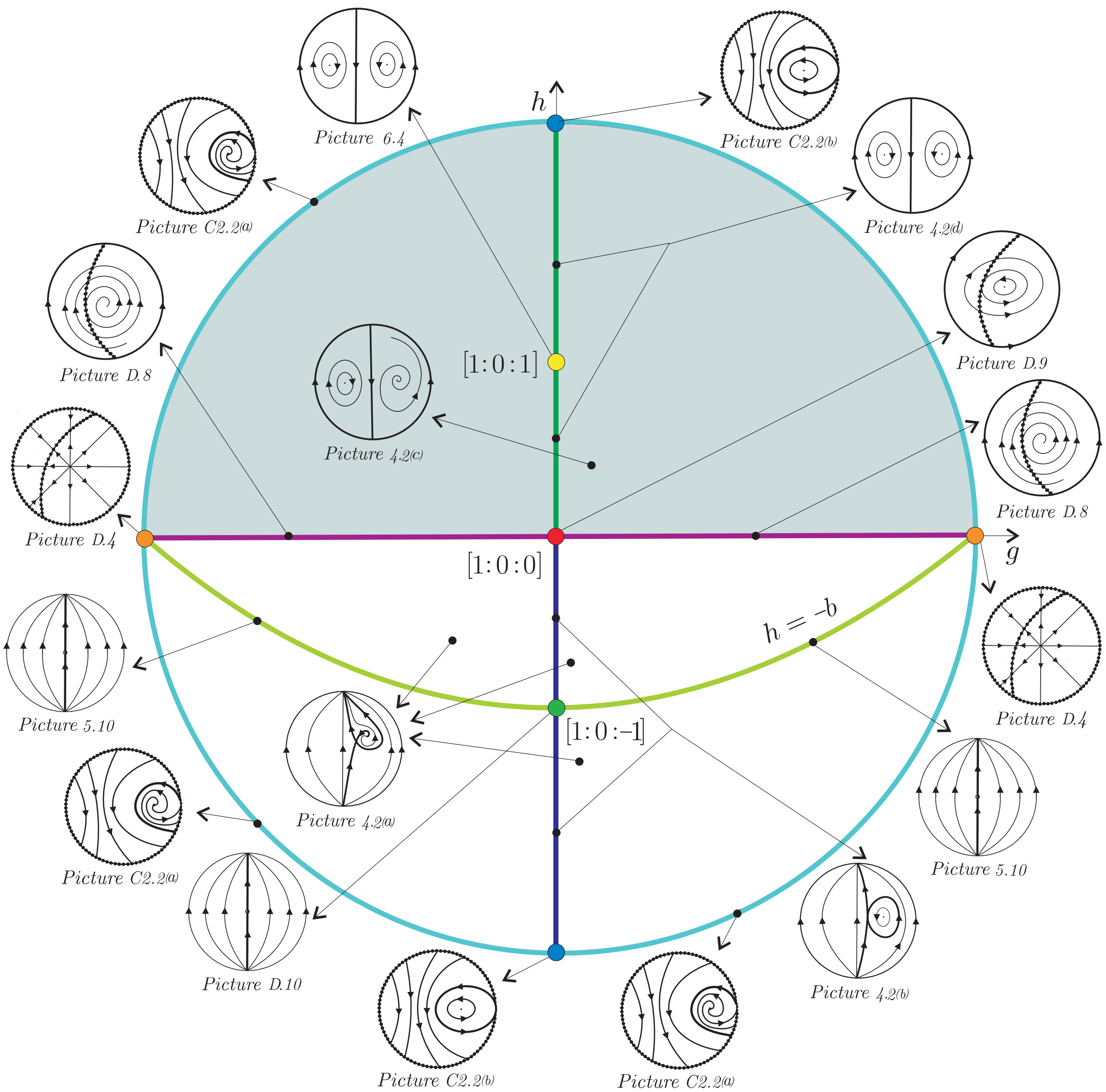

DIAGRAM 2. **Bifurcation diagram associated to *Config. 4.2***

2) *The possibility* $\theta = 0$. Then $(h+1)[g^2 + (h-1)^2] = 0$ and we consider two cases: $N \neq 0$ and $N = 0$.

a) *Assume first* $N \neq 0$. Then considering (25) we get $h = -1$ and we have:

$$B_3 = \theta = H_1 = 0, \ \mu_0 = g^2 \neq 0, \ N = (g^2 + 4)x^2 \neq 0.$$

So, according to [17] the phase portrait of systems (24) in this case corresponds to *Picture 5.10*.

b) *Suppose now* $N = 0$. From (25) we obtain $h = 1$, $g = 0$ and then we have $B_3 = N = 0$, $H_1 = 36864 > 0$. Hence, by [15] we get *Picture 6.4*.

3.3.1.1.2. **The subcase $\mu_0 = 0$.** Then $h[g^2 + (h+1)^2] = 0$.

1) If $h = 0$ then we get the family of degenerate systems

$$\dot{x} = gx^2 + xy, \quad \dot{y} = (g^2+1)x - x^2 + gxy. \tag{26}$$

We observe, that the associated linear systems possess a focus if $g \neq 0$ and for $g = 0$ the system (26) possesses a center. Considering the affine line $x = 0$ filled up with singularities, we get *Picture D.8* if $g \neq 0$ and *Picture D.9* if $g = 0$.

2) Assuming $g = h+1 = 0$ we get the degenerate system $\dot{x} = 0$, $\dot{y} = -(x^2+y^2)$ the phase portrait of which is given by *Picture D.10.*

3.3.1.2. **The case $b = 0$.** Then from (24) we get the following family of systems:

$$\dot{x} = gx^2 + hxy, \quad \dot{y} = h[g^2+h^2] + (g^2-h^2)x + 2ghy + gxy + hy^2, \tag{27}$$

possessing the infinite line filled up with singularities. For these systems according to Table 1 of [18] we calculate:

$$C_2 = 0,\ H_{10} = 36h^4(g^2+h^2)^2,\ H_9 = 2304h^{12}(g^2+h^2)^4,\ N_7 = 16gh^3(g^2+9h^2).$$

Hence, if $H_{10} \neq 0$ then $h \neq 0$ and this implies $H_9 > 0$. According to [18] in this case the phase portrait of systems (27) corresponds to *Picture* $C_2.2(a)$ if $N_7 \neq 0$ (i.e. $g \neq 0$) and to *Picture* $C_2.2(b)$ if $N_7 = 0$ (i.e. $g = 0$).

For $H_{10} = 0$ we get $h = 0$ and this leads to degenerate systems $\dot{x} = gx^2$, $\dot{y} = gx(g+y)$, where $g \neq 0$ due to the condition $g^2 + h^2 + b^2 \neq 0$ for the family of systems (24). Clearly in this case we get *Picture D.4.*

We now see that the portraits on the left-hand side ($g < 0$) of Diagram 2 coincide with those on the right-hand side ($g > 0$) and we wonder if they could be identified via the group action. Indeed, this is the case as we see by using the transformation $(x, y, t) \mapsto (x, -y, -t)$ inducing the map $(b, g, h) \mapsto (b, -g, h)$. Therefore we can limit ourselves to the case $g \geq 0$ and thus we can discard the left-hand side of this diagram. We check if under the group action we can still identify points. It can be easily verified that two systems corresponding to two distinct points inside the half disk cannot lie on the same orbit. Secondly we limit ourselves to the line $b = 0$. In the affine chart corresponding to $g = 1$ using the coordinates (b, h), the line $b = 0$ becomes the h-axis. In the resulting equations, via the transformation $(x, y, t) \mapsto (-x, y, -t)$ we can change h to $-h$ obtaining a system in the same orbit. So we can identify the points with $h \geq 0$ with those with $h \leq 0$. On the other hand one can easily prove that two systems with $b = 0$ and $|h_1| \neq |h_2|$ cannot lie on the same orbit. Projecting the cone obtained by identifying the points with $h > 0$ with those with $h < 0$ on the disk with circumference the line $g = 0$, and placing on this picture the portraits previously obtained for the half disk of Diagram 2 we obtain Diagram $2(\mathfrak{M})$. ∎

3.3.2. **Configurations 4.6 and 4.7.**

Theorem 7. *(i) The family* (11) *contains as a subfamily representatives of all the systems with configurations Config. 4.6 and Config. 4.7 and has the projective plane as a parameter space. The bifurcation diagram of the family* (11) *is indicated in* Diagram 3 *which contains 5 topologically distinct phase portraits. These are stratified as folllows: 3 of them have systems with configuration Config. 4.6 with Pictures 4.6(i)* $(i \in \{a, b, \}$ *and Config. 4.7 with Picture 4.7. The remaining portraits correspond to systems which are*

- *degenerate* (D_4, D_{10})
- *with invariant lines of total multiplicity five (Picture 5.10, which is topologically equivalent to* D_{10} *)*
- *with 6 invariant lines (Picture 6.3, which is topologically equivalent to the Picture 4.6(b))*

(ii) The family (11) *yields a quotient topological space homeomorphic to a closed disk modulo the group action. This orbit space carries the bifurcation diagram indicated in* Diagram $3(\mathfrak{M})$ *where we give all phase portraits of systems in the family as well as their corresponding configurations of invariant lines.*

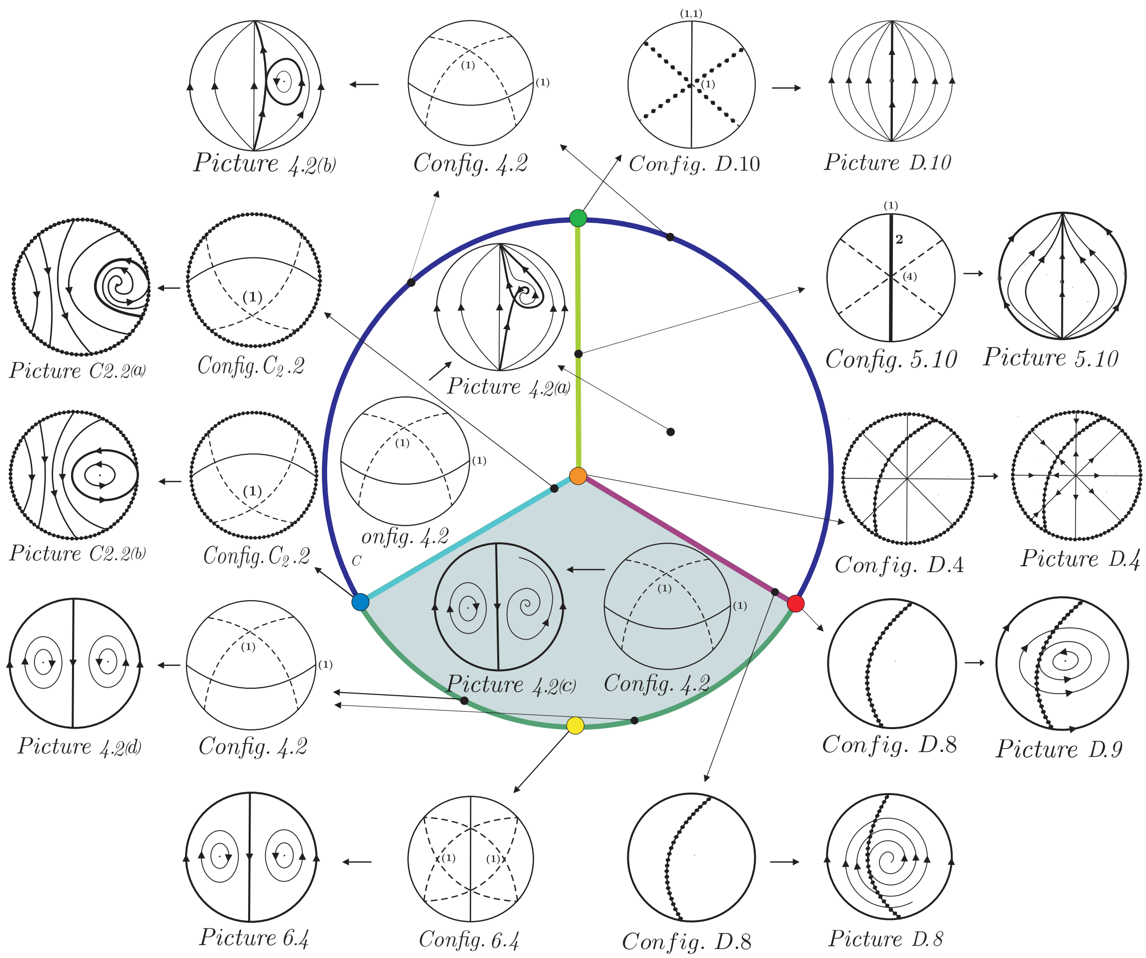

DIAGRAM 2($\mathfrak{M}$) **Bifurcation diagram on the quotient space**

(iii) In this diagram the subspace of orbits of systems with configurations Config. 4.6 and Config. 4.7 is disconnected with three connected components, each of which is homeomorphic to the interior of a disk completed with an open arc of its circumference. Two of these components have the same phase portrait Picture 4.6(a) with a node and a saddle and the third one with phase portrait Picture 4.6(b) with two nodes.

Proof: According to Theorem 5 all the systems having the Configurations 4.6 and 4.7 are included in the family:

$$\dot{x} = gx^2 + (h+b)xy, \quad \dot{y} = -b + g\,x + (h-b)y - bx^2 + gxy + hy^2, \tag{28}$$

where $[b : g : h] \in \mathbf{P}_2(\mathbb{R})$. We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line $b = 0$.

3.3.2.1. **The case** $b = 1$**.** Then for systems (28) calculations yield:

$$\begin{aligned} &\theta = 8(h+1)[g^2 + (h-1)^2],\ \mu_0 = -h[g^2 + (h+1)^2],\ B_3 = H_7 = 0,\\ &N = (g^2 - 2h + 2)x^2 + 2g(h+1)xy + (h^2 - 1)y^2,\ H_9 = 2304(h+1)^8. \end{aligned} \tag{29}$$

3.3.2.1.1. **The subcase** $\mu_0 \neq 0$**,** i.e. $h[g^2 + (h+1)^2] \neq 0$. We shall consider two possibilities: $\theta \neq 0$ and $\theta = 0$.

1) Assume $\theta \neq 0$. Then we have $h + 1 \neq 0$ and this implies $H_9 \neq 0$. According to [19] the phase portrait of a system (28) with $b = 1$ corresponds to *Picture 4.6(a)* if $\mu_0 > 0$ (i.e. $h < 0$) and to *Picture 4.6(b)* if

$\mu_0 < 0$ (i.e. $h > 0$). So, if $g(h+1)[g^2+(h-1)^2][g^2+(h+1)^2] \neq 0$ then we arrive to the situation given by Diagram 3.

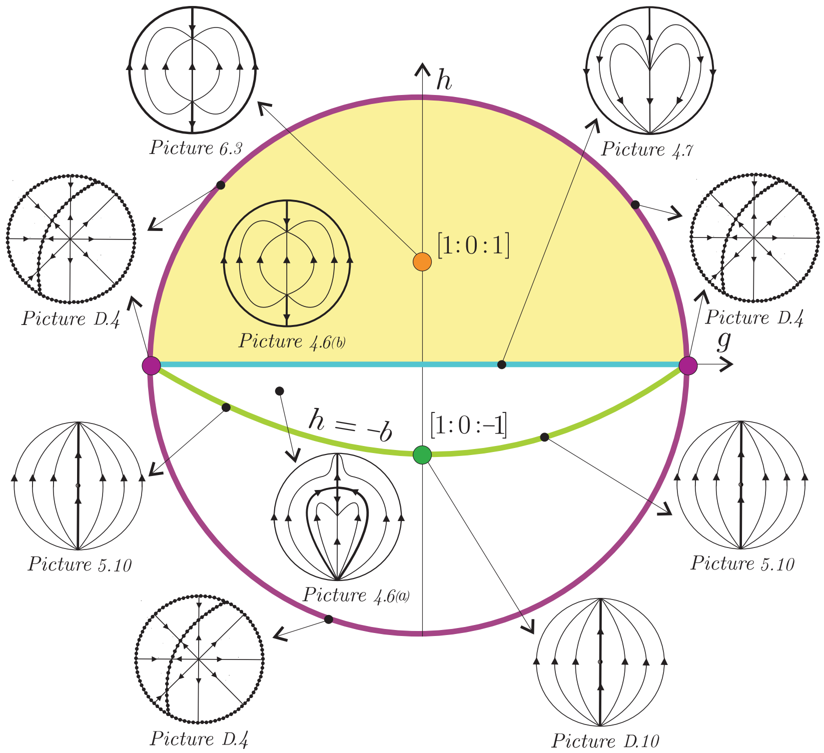

Diagram 3. **Bifurcation diagram associated to *Configs. 4.6* and *4.7***

2) Suppose now $\theta = 0$. Then $(h+1)[g^2+(h-1)^2] = 0$ and we shall consider two subcases: $H_9 \neq 0$ and $H_9 = 0$.

a) If $H_9 \neq 0$ then from (29) we have $h+1 \neq 0$ and hence we get $g = 0$ and $h = 1$. So, from (28) we get a single system, for which calculation yields: $B_3 = N = 0$ and $H_1 = -9216 < 0$. According to [17] the phase portrait of this system corresponds to *Picture 6.3*.

b) Assume now $H_9 = 0$, i.e. $h = -1$. Then we get the family of systems

$$\dot{x} = gx^2, \quad \dot{y} = -1 + g\,x - 2y - x^2 + gxy - y^2, \tag{30}$$

for which we have $\theta = B_3 = H_1 = 0$, $\mu_0 = g^2$, $N = (g^2+4)x^2$. Since $\mu_0 \neq 0$ according to [17] the phase portrait of systems (30) corresponds to *Picture 5.10*.

3.3.2.1.2. **The subcase $\mu_0 = 0$.** In this case we have $h[g^2+(h+1)^2] = 0$.

1) If $h = 0$ then we get the family of systems

$$\dot{x} = gx^2 + xy, \quad \dot{y} = -1 + g\,x - y - x^2 + gxy, \tag{31}$$

for which we have $\theta = 8(g^2+1)$, $B_3 = H_7 = \mu_0 = 0$. So, by [19] the phase portrait of these systems corresponds to *Picture 4.7*.

2) Assuming $g = h+1 = 0$ we get the degenerate system $\dot{x} = 0$, $\dot{y} = -x^2-(y+1)^2$ the phase portrait of which is given by *Picture D.10*.

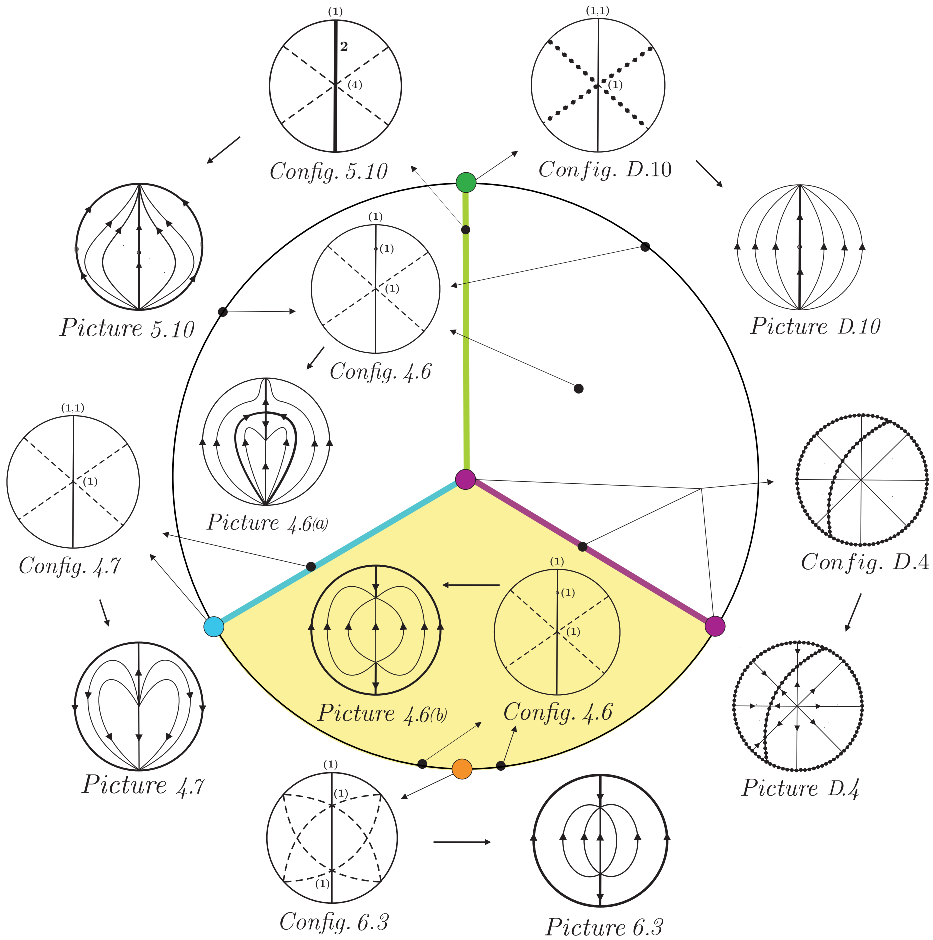

DIAGRAM 3($\mathfrak{M}$) **Bifurcation diagram on the quotient space**

3.3.2.2. **The case $b = 0$.** Then from (28) we get the following family of degenerate systems:

$$\dot{x} = x(gx + hy), \quad \dot{y} = (y + 1)(gx + hy), \tag{32}$$

possessing the affine singular line $gx + hy = 0$ (as for for systems (28) we have $b^2 + g^2 + h^2 \neq 0$). Then we get the phase portrait *Picture D.4*.

We observe that the transformation $(x, y) \mapsto (-x, y)$ change the sign of the parameter g in systems (28). Then the portraits on the left-hand side ($g < 0$) of DIAGRAM 3 coincide with those on the right-hand side ($g > 0$) and they are identified under the group action. Therefore we can limit ourselves to the case $g \geq 0$ and thus we can discard the left-hand side of this diagram.

Firstly we observe that under the group action we cannot identify systems corresponding to points inside the half disk as it can easily be seen. However on the line $b = 0$ in the affine chart corresponding to $g = 1$ using the coordinates (b, h), the line $b = 0$ becomes the h-axis. In the resulting equations, via the transformation $(x, y, t) \mapsto (-x, y, -t)$ we can change h to $-h$ obtaining a system in the same orbit. Moreover we can actually identify via the group action any two systems corresponding to two points $0 \neq |h_1| \neq |h_2| \neq 0$ on the line $b = 0$.

So we first identify the points with $h \geq 0$ with those with $h \leq 0$. Projecting the cone thus obtained on the disk with circumference the line $g = 0$ and placing on this picture the portraits previously obtained for the half disk of DIAGRAM 3 we obtain DIAGRAM 3 ($\mathfrak{M}$). In view of the previous arguments we can identify all the points from the open segment corresponding to *Picture D.4* with the point on the disk with phase portrait *Picture D.4*. ∎

Remark 9. *We observe that the moduli space thus obtained is not Hausdorff. Indeed, the two points obtained via the group action from the segment mentioned above cannot be separated in the topology of this moduli space.*

3.3.3. **Configuration 4.8.**

Theorem 8. *(i) The family* (12) *contains as a subfamily representatives of all the systems with configuration Config. 4.8 and has the projective plane as a parameter space. The bifurcation diagram of the family* (12) *is indicated in* DIAGRAM 4 *which contains 5 topologically distinct phase portraits. These are stratified as follows: 2 of them have systems with configuration Config. 4.8 and phase portraits Pictures 4.8(i)* ($i \in \{a, b, \}$. *The remaining portraits correspond to systems which are*

- *degenerate* (D_7, D_{10} *(equivalent to Picture 4.8(a)),* D_{11}, D_{12}*)*
- *with invariant lines of total multiplicity five (Picture 5.10, topologically equivalent to pictures* D_{10} *and* 4.8(a))
- *with 6 invariant lines (Picture 6.6 which is topologically equivalent to Pictture 4.8(b))*

(ii) The family (13) *yields a quotient topological space homeomorphic to a closed disk modulo the group action. This orbit space carries the bifurcation diagram indicated in* DIAGRAM 4($\mathfrak{M}$) *where we give all phase portraits of systems in the family as well as their corresponding configurations of invariant lines.*

(iii) In this diagram the subspace of orbits of systems with configuration Config. 4.8 is disconnected with three connected components, each of which is homeomorphic to the interior of a disc completed with an open arc of its circumference. Two of these components have systems phase portraits Picture 4.8(a) and the third one has Picture 4.8(b).

Proof: According to Theorem 5 all the systems having the Configuration 4.8 are included in the family:

$$\dot{x} = gx^2 + (h+b)xy, \quad \dot{y} = -bx^2 + gxy + hy^2, \tag{33}$$

where $[b : g : h] \in \mathbf{P}_2(\mathbb{R})$. We construct now the bifurcation diagram for this canonical form. This diagram will be drawn on the projective plane viewed on the disk with opposite points on the circumference identified. We place on the circumference the line $b = 0$.

3.3.3.1. **The case** $b = 1$**.** Then for systems (33) calculations yield:

$$\begin{aligned} &\theta = 8(h+1)[g^2 + (h-1)^2],\ \mu_0 = -h[g^2 + (h+1)^2],\ B_3 = H_7 = 0, \\ &N = (g^2 - 2h + 2)x^2 + 2g(h+1)xy + (h^2 - 1)y^2,\ H_9 = 0. \end{aligned} \tag{34}$$

3.3.3.1.1. **The subcase** $\mu_0 \neq 0$**,** i.e. $h[g^2 + (h+1)^2] \neq 0$ and we shall consider two cases: $\theta \neq 0$ and $\theta = 0$.

1) Assume first $\theta \neq 0$. Then we have $h + 1 \neq 0$ and according to [19] the phase portrait of a system (33) with $b = 1$ corresponds to *Picture 4.8(a)* if $\mu_0 > 0$ (i.e. $h < 0$) and to *Picture 4.8(b)* if $\mu_0 < 0$ (i.e. $h > 0$). So, if $g(h+1)[g^2 + (h-1)^2][g^2 + (h+1)^2] \neq 0$ then we arrive to the situation given by DIAGRAM 4.

2) Admit now $\theta = 0$. Then $(h+1)[g^2 + (h-1)^2] = 0$.

***a*)** If $h = -1$ then we get the family of systems

$$\dot{x} = gx^2, \quad \dot{y} = -x^2 + gxy - y^2, \tag{35}$$

for which we have $\theta = B_3 = H_1 = 0,\ \mu_0 = g^2,\ N = (g^2 + 4)x^2$. Since $\mu_0 \neq 0$ according to [17] the phase portrait of systems (30) corresponds to *Picture 5.10.*

***b*)** Assume now $g = 0$ and $h = 1$. So, from (33) we get a single system, for which calculation yields: $B_3 = N = H_1 = 0$. So, according to [17] the phase portrait of this system corresponds to *Picture 6.6.*

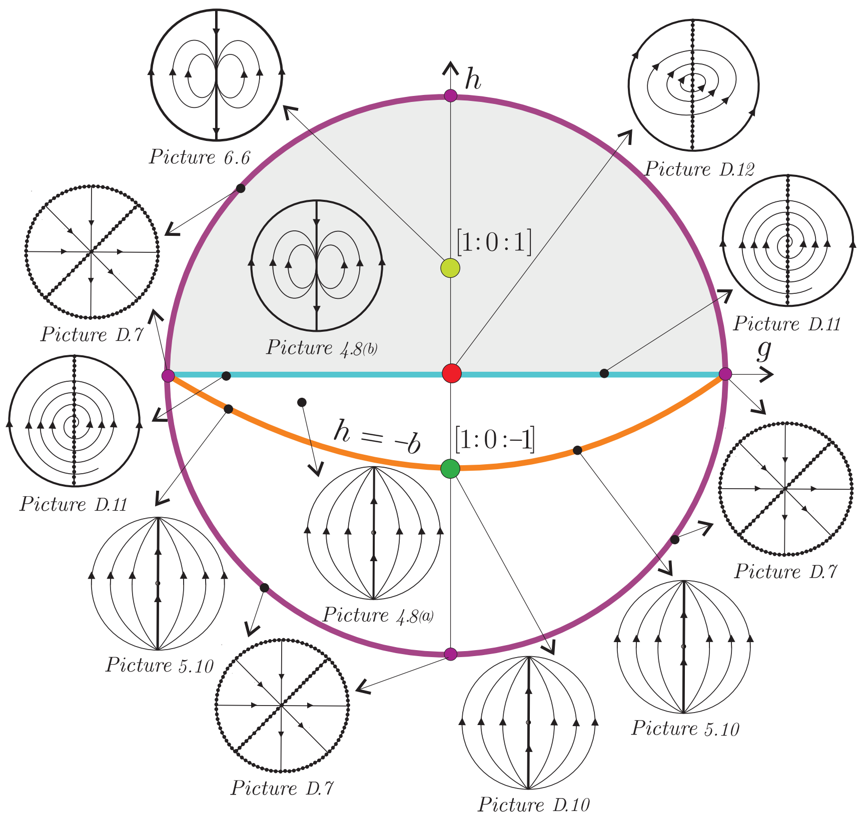

Diagram 4. **Bifurcation diagram associated to *Config. 4.8***

3.3.3.1.2. **The subcase $\mu_0 = 0$.** In this case we have $h[g^2 + (h+1)^2] = 0$.

1) If $h = 0$ then we get the family of degenerate systems

$$\dot{x} = x(gx + y), \quad \dot{y} = x(-x + gy). \tag{36}$$

We observe that for the respective linear systems the point $(0,0)$ is a focus if $g \neq 0$ (*Picture D.11*) and it is a center if $g = 0$ (*Picture D.12*).

2) Assuming $g = h+1 = 0$ we get the degenerate system $\dot{x} = 0$, $\dot{y} = -(x^2 + y^2)$ the phase portrait of which is given by *Picture D.10*.

3.3.3.2. **The case $b = 0$.** Then from (33) we get the following family of degenerate systems:

$$\dot{x} = x(gx + hy), \quad \dot{y} = y(gx + hy), \tag{37}$$

possessing the affine singular line $gx + hy = 0$ (as for systems (33) we have $b^2 + g^2 + h^2 \neq 0$). Then we get the phase portrait *Picture D.7*.

We now see that the portraits on the left-hand side ($g < 0$) of Diagram 4 coincide with those on the right-hand side ($g > 0$) and we wonder if they could be identified via the group action. Indeed, this is the case as we see by using the transformation $(x, y, t) \mapsto (-x, y, t)$. Therefore we can limit ourselves to the case $g \geq 0$ and thus we can discard the left-hand side of this diagram.

Firstly we observe that under the group action we cannot identify systems corresponding to points inside the half disk as it can easily be shown. However on the line $b = 0$ in the affine chart corresponding to $g = 1$ using the coordinates (b, h), the line $b = 0$ becomes the h-axis. In the resulting equations, via the transformation $(x, y, t) \mapsto (x, -y, t)$ we can change h to $-h$ obtaining a system in the same orbit. Moreover we can actually identify via the group action any two systems corresponding to two points $|h_1| \neq |h_2|$ on the line $b = 0$.

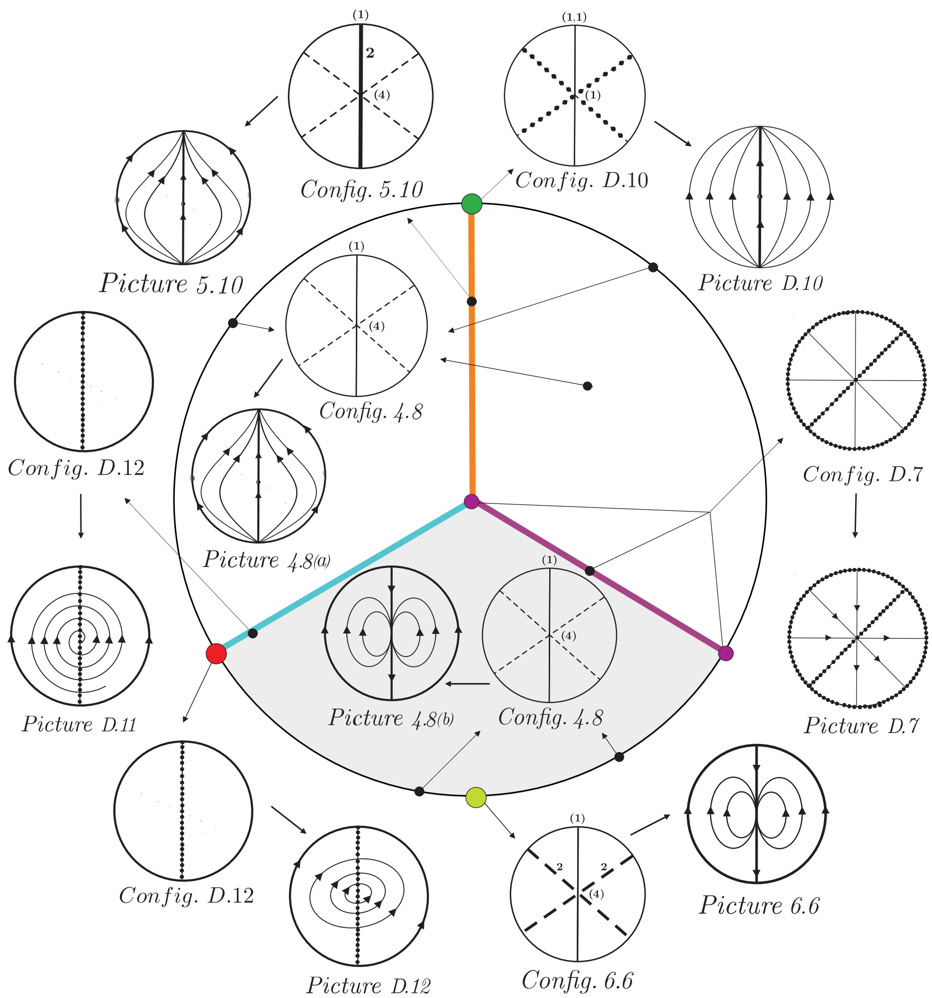

DIAGRAM 4($\mathfrak{M}$) **Bifurcation diagram on the quotient space**

So we first identify the points with $h \geq 0$ with those with $h \leq 0$. Projecting the cone thus obtained on the disk with circumference the line $g = 0$ and placing on this picture the portraits previously obtained for the half disk of DIAGRAM 4 we obtain DIAGRAM 4($\mathfrak{M}$). In view of the previous arguments we can identify all the points from the segment corresponding to *Picture D.7* with the point on the disk with phase portrait *Picture D.7*. Because all the cases were considered Theorem 8 is proved.

3.3.4. **Configuration 4.27.**

Theorem 9. *(i) The family* (13) *contains as a subfamily representatives of all the systems with configuration Config. 4.27 and has the projective line as a parameter space. The bifurcation diagram of the family* (13) *is indicated in* DIAGRAM 5 *which contains 3 topologically distinct phase portraits. These are stratified as folllows: 2 of them have systems with configuration Config. 4.27 with phase portraits Pictures 4.27(i)* $(i \in \{a, b, \}$. *The remaining portrait corresponds to systems with invariant lines of total multiplicity five (Picture 5.9)*

(ii) The family (13) *yields a quotient topological space homeomorphic to* S^1 *modulo the group action. This orbit space carries the bifurcation diagram indicated in* DIAGRAM 5($\mathfrak{M}$) *where we give all phase portraits of systems in the family as well as their corresponding configurations of invariant lines.*

(iii) In this diagram the subspace of orbits of systems with configuration 4.27 is the complement of a point in S^1 *which have phase portraits with a focus and a saddle and a phase portrait with saddle and a center.*

Proof: According to Theorem 5 all the systems having the Configuration 4.27 are included in the family:

$$\dot{x} = 2cx + 2dy, \quad \dot{y} = c^2 + d^2 - x^2 - y^2 \tag{38}$$

with $[c:d] \in \mathbf{P}_1(\mathbb{R})$, for which we calculate:

$$B_3 = \theta = 0,\ N = x^2,\ H_7 = 32d,\ \mathcal{G}_1 = 16c.$$

1) If $H_7 \neq 0$ then according to [19] the phase portrait corresponds to *Picture 4.27(a)* if $\mathcal{G}_1 \neq 0$ (i.e. $c \neq 0$) and to *Picture 4.27(b)* if $\mathcal{G}_1 = 0$ (i.e. $c = 0$).

2) Assume now $H_7 = 0$, i.e. $d = 0$. In this case for systems (38) we have $\mu_0 = 0$, $H_6 = 0$ and according to [17] the phase portrait of these systems with $d = 0$ and $c \neq 0$ corresponds to *Picture 5.9*.

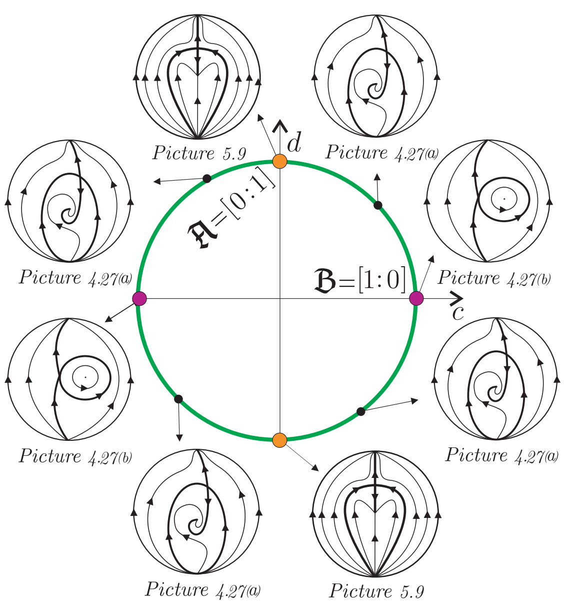

Diagram 5 **Diagram associated to *Config. 4.27***

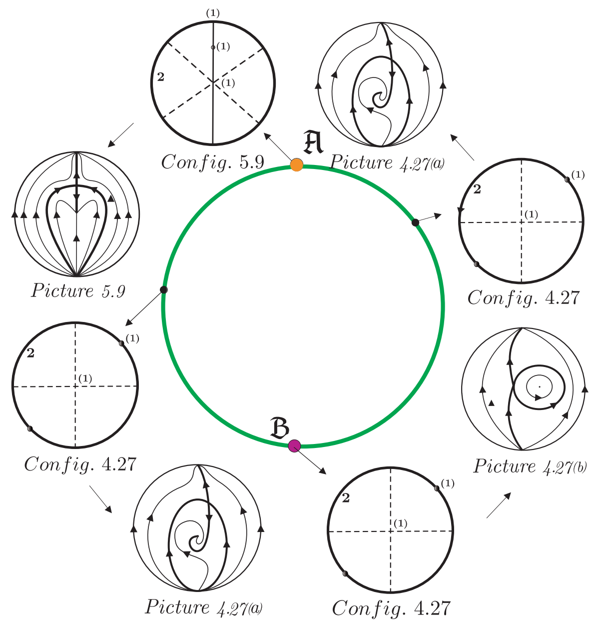

Diagram 5($\mathfrak{M}$) **Bifurcation in quotient space**

We consider the projective line as a circle with opposite points on diameters identified. Then the bifurcation diagram is indicated in Diagram 5. We observe that under the action of the affine group and time rescaling the systems corresponding to the points points $[c_1 : d_1]$ and $[c_2 : d_2]$ could not be identified, unless these are opposite points on the circle.

Discarding the left hand-side of this diagram and identifying the north and south poles we obtain the Diagram 5($\mathfrak{M}$). ∎

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[1] Département de Mathématiques et de Statistiques, Université de Montréal

E-mail address: ksiushaa@gmail.com ; dasch@dms.umontreal.ca

[3] Institute of Mathematics and Computer Science, Academy of Science of Moldova, 5 Academiei str, Chişinău, MD-2028, Moldova

E-mail address: nvulpe@gmail.com