# A T(1) theorem for fractional Sobolev spaces on domains

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**Abstract**

In 2012 V. Cruz, J. Mateu and J. Orobitg published a T(1) theorem for the Beurling transform in the complex plane. It asserts that given $0 < s \leqslant 1$, $1 < p < \infty$ with $sp > 2$ and a domain $\Omega \subset \mathbb{C}$ with smooth boundary, the Beurling transform $\mathcal{B}f = -\text{p.v.}\frac{1}{\pi z^2} * f$ is bounded in the Sobolev space $W^{s,p}(\Omega)$ if and only if $\mathcal{B}\chi_\Omega \in W^{s,p}(\Omega)$. They prove this result for a certain family of operators of even kernel in any ambient space $\mathbb{R}^d$ when $sp > d$. In a recent paper, the first author and Tolsa proved that there is a similar situation for Sobolev spaces of smoothness $s \in \mathbb{N}$, and the assumption on the smoothness was relaxed to any Lipschitz domain. They stated that this was also possible for uniform domains.

In this paper we prove that the T(1) theorem remains true for $0 < s < 1$ for any uniform domain and for a larger family of Calderón-Zygmund operators in any ambient space $\mathbb{R}^d$ as long as $sp$ is greater than the dimension of the ambient space. In the process we show the existence of an extension operator for Triebel-Lizorkin spaces in uniform domains $F^s_{p,q}(\Omega)$ and we define some equivalent norms for them which are of some interest by themselves.

## 1 Introduction

The aim of the present article is to find necessary and sufficient conditions on certain singular integral operators to be bounded in fractional Sobolev spaces of a uniform domain $\Omega$ with smoothness $0 < s < 1$. However, the results are valid in $F^s_{p,q}(\Omega)$, that is, the so-called Triebel-Lizorkin spaces, when $s > \max\left\{0, \frac{d}{p} - \frac{d}{q}\right\}$.

Consider $0 < \sigma \leqslant 1$. An operator $T$ defined for $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \backslash \text{supp}(f)$ as

$$Tf(x) = \int_{\mathbb{R}^d} K(x-y) f(y)\, dy,$$

is called an *admissible convolution Calderón-Zygmund operator of order* $\sigma$ if it is bounded in $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$ and its kernel $K$ satisfies the size condition

$$|K(x)| \leqslant \frac{C_K}{|x|^d} \qquad \text{for every } x \neq 0$$

and the Lipschitz smoothness condition

$$|K(x-y) - K(x)| \leqslant \frac{C_K |y|^\sigma}{|x|^{d+\sigma}} \qquad \text{for every } 0 < 2|y| \leqslant |x|$$

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(see Section 2 for more details). In the present article we deal with some properties of the operator $T$ truncated to a domain $\Omega$, defined as $T_\Omega(f) = \chi_\Omega T(\chi_\Omega f)$.

In the complex plane, for instance, the *Beurling transform*, which is defined as the principal value

$$Bf(z) := -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} dm(w),$$

is an admissible convolution Calderón-Zygmund operator of any order with kernel $K(z) = -\frac{1}{\pi z^2}$.

In the article [CMO13], Víctor Cruz, Joan Mateu and Joan Orobitg, seeking for some results on the Sobolev smoothness of quasiconformal mappings proved the next theorem.

**Theorem** (see [CMO13])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1+\varepsilon}$ domain (i.e. a Lipschitz domain with parameterizations of the boundary in $C^{1+\varepsilon}$) for a given $\varepsilon > 0$, and let $1 < p < \infty$ and $0 < s \leqslant 1$ such that $sp > 2$. Then any truncated Calderón-Zygmund operator $T_\Omega$ with smooth, homogeneous and even kernel is bounded in the Sobolev space $W^{s,p}(\Omega)$ if and only if $T(\chi_\Omega) \in W^{s,p}(\Omega)$.*

Later, Xavier Tolsa and the first author of the present paper, studied the case $s \in \mathbb{N}$, finding the following $T(P)$ Theorem.

**Theorem** (see [PT15])**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $T$ a convolution Calderón-Zygmund operator with kernel $K$ satisfying*

$$|\nabla^j K(x)| \leqslant C \frac{1}{|x|^{d+j}} \qquad \text{for all } 0 \leqslant j \leqslant n,\ x \neq 0,$$

*and $p > d$. Then the following statements are equivalent:*

*a) The truncated operator $T_\Omega$ is bounded in $W^{n,p}(\Omega)$.*

*b) For every polynomial $P$ of degree $n-1$, we have that $T_\Omega(P) \in W^{n,p}(\Omega)$.*

The notation is explained in Section 2. Note that the kernels are not assumed to be even, and the conditions on the smoothness of the domain are relaxed. The authors assert that the theorem is valid even for uniform domains.

In the present paper we study again the fractional smoothness, but we deal with the case of uniform domains (see Section 3) for Triebel-Lizorkin spaces $F^s_{p,q}$ with $1 < p, q < \infty$, $\max\left\{0, \frac{d}{p} - \frac{d}{q}\right\} < s < 1$. Let us note here to illustrate that in case $q = 2$ we deal with the Sobolev fractional spaces $W^{s,p}$ and in case $q = p$ then we deal with the Besov spaces $B^s_{p,p}$. To avoid misunderstandings, the reader must be aware that the $B^s_{p,p}$ spaces are called also Sobolev spaces in some books, while the $W^{s,p}$ spaces are sometimes called Bessel potential spaces. See Section 5 for all the definitions of these spaces.

Our main result is the following.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, $T$ an admissible convolution Calderón-Zygmund operator of order $0 < \sigma < 1$. Consider indices $p, q \in (1, \infty)$ and $\frac{d}{p} < s \leqslant \sigma$. Then the truncated operator $T_\Omega$ is bounded in $F^s_{p,q}(\Omega)$ if and only if we have that $T_\Omega(1) \in F^s_{p,q}(\Omega)$.*

To prove this result we will need an equivalent norm for $F^s_{p,q}$. The following result is not present in the literature in its full generality, but it is found for the Sobolev case in [Ste61] and for the general Triebel-Lizorkin case when $s > \frac{d}{\min\{p,q\}}$ in [Tri83, Theorem 2.5.10]. The result as stated below will be a corollary of some results in [Tri06].

**Theorem 1.2** (see Corollary 5.5)**.** *Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then,*

$$F^s_{p,q} = \left\{ f \in L^{\max\{p,q\}} : \|f\|_{L^p} + \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\},$$

*(with the usual modification for $q = \infty$), in the sense of equivalent norms.*

The restriction $s > \frac{d}{p} - \frac{d}{q}$ is sharp, as we will see in Remark 4.2. One can find some equivalent norms for Triebel-Lizorkin spaces in terms of differences using means on balls which avoid this restriction. We refer the reader to [Str67] or [Tri83, Corollary 2.5.11].

Given a domain $\Omega$, we say that $f \in F^s_{p,q}(\Omega)$ if there is a function $h \in F^s_{p,q}(\mathbb{R}^d)$ such that $h|_\Omega = f|_\Omega$. The norm $\|f\|_{F^s_{p,q}(\Omega)}$ will be defined as the infimum of the norms $\|h\|_{F^s_{p,q}(\mathbb{R}^d)}$ for all admissible $h$. Our method is based on an intrinsic characterization of this norm, inspired by the previous theorem. We define

$$\|f\|_{A^s_{p,q}(\Omega)} := \|f\|_{L^p(\Omega)} + \left( \int_\Omega \left( \int_\Omega \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Indeed, this norm will be equivalent to the Triebel-Lizorkin one for uniform domains:

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, $1 < p, q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then $f \in F^s_{p,q}(\Omega)$ if and only if $f \in A^s_{p,q}(\Omega)$ and the norms are equivalent.*

To prove this result we will use Theorem 1.2 and the following extension Theorem:

**Theorem 1.4.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, $1 < p, q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then there exists a bounded operator $\Lambda_0 : A^s_{p,q}(\Omega) \to F^s_{p,q}(\mathbb{R}^d)$ such that $\Lambda_0 f|_\Omega = f$ for every $f \in A^s_{p,q}(\Omega)$.*

However, in the proof of Theorem 1.1 we will make use of a functional which is closely related to $\|\cdot\|_{A^s_{p,q}(\Omega)}$. Call $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. Consider the Carleson boxes (or shadows) $\mathbf{Sh}(x) := \{y \in \Omega : |y - x| \leqslant c_\Omega \delta(x)\}$ with $c_\Omega > 1$ to be fixed (see Section 3). Then we have the following reduction for the Triebel-Lizorkin norm:

**Theorem 1.5** (See Corollary 7.3.)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, $1 < p < q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then $f \in F^s_{p,q}(\Omega)$ if and only if*

$$\|f\|_{L^p(\Omega)} + \left( \int_\Omega \left( \int_{\mathbf{Sh}(x)} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty.$$

*Furthermore, the left-hand side of the inequality above is equivalent to the norm $\|f\|_{F^s_{p,q}(\Omega)}$.*

The situation is even better when $p \geqslant q$:

**Theorem 1.6** (See Corollary 7.3.)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, $1 < q \leqslant p < \infty$, $0 < s < 1$ and $0 < \rho < 1$. Then $f \in F^s_{p,q}(\Omega)$ if and only if*

$$\|f\|_{L^p(\Omega)} + \left( \int_\Omega \left( \int_{B(x, \rho\delta(x))} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty.$$

*Furthermore, the left-hand side of the inequality above is equivalent to the norm $\|f\|_{F^s_{p,q}(\Omega)}$.*

The plan of the paper is the following. In Section 2 we set up the notation. In Section 3 we define uniform domains in the spirit of [Jon81] but from a dyadic point of view and then we prove some basic properties of those domains. The expert reader may skip this part. Section 4 is devoted to proving Lemma 4.4 which is about the change of the domain of integration in the norm $A^s_{p,q}(\Omega)$. This Lemma, together with Theorem 1.3, which is proven in Section 5, leads to Theorem 1.5 (see Corollary 7.3). Section 5 begins with some remarks on Triebel-Lizorkin spaces, followed by the proof of the implicit characterization of Triebel-Lizorkin spaces given in Theorem 1.2, the Extension Theorem 1.4 and, as a corollary, Theorem 1.3. Section 6 is the core of the paper, and it contains the proof of the T(1) Theorem 1.1. The key Lemma 6.6 is a discretization of the transform of a function and it is the cornerstone of the mentioned theorem. Finally, in Section 7 we prove Theorem 1.6 for the sake of completeness of the present article.

# 2 Notation

**On inequalities:** When comparing two quantities $x_1$ and $x_2$ that depend on some parameters $p_1, \dots, p_j$ we will write

$$x_1 \leqslant C_{p_{i_1},\dots,p_{i_j}} x_2$$

if the constant $C_{p_{i_1},\dots,p_{i_j}}$ depends on $p_{i_1}, \dots, p_{i_j}$. We will also write $x_1 \lesssim_{p_{i_1},\dots,p_{i_j}} x_2$ for short, or simply $x_1 \lesssim x_2$ if the dependence is clear from the context or if the constants are universal. We may omit some of these variables for the sake of simplicity. The notation $x_1 \approx_{p_{i_1},\dots,p_{i_j}} x_2$ will mean that $x_1 \lesssim_{p_{i_1},\dots,p_{i_j}} x_2$ and $x_2 \lesssim_{p_{i_1},\dots,p_{i_j}} x_1$.

**On cubes:** Given a cube $Q$, we write $\ell(Q)$ for its side-length. Given two cubes $Q, S$, we define their long distance as $\mathrm{D}(Q,S) = \ell(Q) + \mathrm{dist}(Q,S) + \ell(S)$. Given a real number $\rho$, we define $\rho Q$ as the cube concentric to $Q$, with ratio $\rho$ and faces parallel to the faces of $Q$.

For any cube $Q$ and any function $f$, we call $f_Q = \fint_Q f\,dm$ to the mean of $f$ in $Q$.

**On conjugate indices:** Given $1 \leqslant p \leqslant \infty$ we write $p'$ for its Hölder conjugate, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

# 3 On uniform domains

There is a considerable literature on uniform domains and their properties, we refer the reader e.g. to [GO79] and [Väi88].

**Definition 3.1.** *Given a domain $\Omega$, we say that a collection of open dyadic cubes $\mathcal{W}$ is a* Whitney covering *of $\Omega$ if they are disjoint, the union of the cubes and their boundaries is $\Omega$, there exists a constant $C_{\mathcal{W}}$ such that*

$$C_{\mathcal{W}}\ell(Q) \leqslant \mathrm{dist}(Q, \partial\Omega) \leqslant 4C_{\mathcal{W}}\ell(Q),$$

*two neighbor cubes $Q$ and $R$ (i.e., $\overline{Q} \cap \overline{R} \neq \varnothing$) satisfy $\ell(Q) \leqslant 2\ell(R)$, and the family $\{50Q\}_{Q\in\mathcal{W}}$ has finite superposition. Moreover, we will assume that*

$$S \subset 5Q \implies \ell(S) \geqslant \frac{1}{2}\ell(Q). \tag{3.1}$$

The existence of such a covering is granted for any open set different from $\mathbb{R}^d$ and in particular for any domain as long as $C_{\mathcal{W}}$ is big enough (see [Ste70, Chapter 1] for instance).

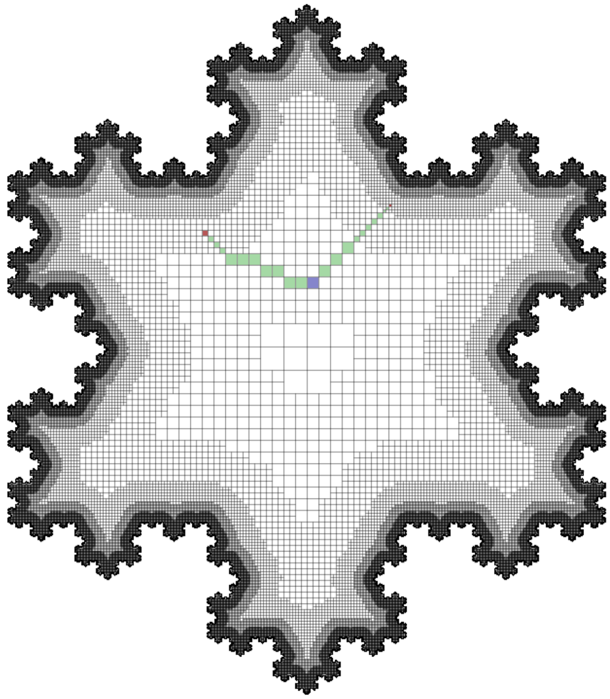


Figure 3.1: A Whitney decomposition of an $(\varepsilon, \infty)$-uniform domain with and an $\epsilon$-admissible chain. The end-point cubes are colored in red and the central one in blue.

**Definition 3.2.** *Let $\Omega$ be a domain and $\mathcal{W}$ a Whitney decomposition of $\Omega$ and $Q, S \in \mathcal{W}$. Given $M$ cubes $Q_1, \ldots, Q_M \in \mathcal{W}$ with $Q_1 = Q$ and $Q_M = S$, the $M$-tuple $(Q_1, \ldots, Q_M)_{j=1}^M \in \mathcal{W}^M$ is a chain connecting $Q$ and $S$ if the cubes $Q_j$ and $Q_{j+1}$ are neighbors for $j < M$. We write $[Q, S] = (Q_1, \ldots, Q_M)_{j=1}^M$ for short.*

*Let $\varepsilon \in \mathbb{R}$. We say that the chain $[Q, S]$ is $\varepsilon$-admissible if*

- *the* length *of the chain is bounded by*

$$\ell([Q, S]) := \sum_{j=1}^{M} \ell(Q_j) \leqslant \frac{1}{\varepsilon} \mathrm{D}(Q, S) \tag{3.2}$$

- *and there exists $j_0 < M$ such that the cubes in the chain satisfy*

$$\ell(Q_j) \geqslant \varepsilon \mathrm{D}(Q_1, Q_j) \text{ for all } j \leqslant j_0 \qquad \text{and} \qquad \ell(Q_j) \geqslant \varepsilon \mathrm{D}(Q_j, Q_M) \text{ for all } j \geqslant j_0. \tag{3.3}$$

*The $j_0$-th cube, which we call* central, *satisfies that $\ell(Q_{j_0}) \gtrsim_d \varepsilon \mathrm{D}(Q, S)$ by (3.3) and the triangle inequality. We will write $Q_S = Q_{j_0}$. Note that this is an abuse of notation because the central cube of $[Q, S]$ may vary for different $\varepsilon$-admissible chains joining $Q$ and $S$.*

*We write (abusing notation again) $[Q, S]$ also for the set $\{Q_j\}_{j=1}^M$. Thus, we will write $P \in [Q, S]$ if $P$ appears in a coordinate of the $M$-tuple $[Q, S]$. For any $P \in [Q, S]$ we call $\mathcal{N}_{[Q,S]}(P)$*

*to the following cube in the chain, that is, for $j < M$ we have that $\mathcal{N}_{[Q,S]}(Q_j) = Q_{j+1}$. We will write $\mathcal{N}(P)$ for short if the chain to which we are referring is clear from the context.*

*Every now and then we will mention subchains. That is, for $1 \leqslant j_1 \leqslant j_2 \leqslant M$, the subchain $[Q_{j_1}, Q_{j_2}]_{[Q,S]} \subset [Q, S]$ is defined as $(Q_{j_1}, Q_{j_1+1}, \dots, Q_{j_2})$. We will write $[Q_{j_1}, Q_{j_2}]$ if there is no risk of confusion.*

**Definition 3.3.** *Let $\varepsilon, \delta \in \mathbb{R}$. We say that a domain $\Omega \subset \mathbb{R}^d$ is an $(\varepsilon, \delta)$-uniform domain if there exists a Whitney covering $\mathcal{W}$ of $\Omega$ such that for any pair of cubes $Q, S \in \mathcal{W}$ with $\mathrm{D}(Q, S) \leqslant \delta$, there exists an $\varepsilon$-admissible chain $[Q, S]$ (see Figure 3.1).*

Next we make some observations on the two subchains $[Q, Q_S]$ and $[Q_S, S]$.

**Remark 3.4.** *Consider a domain $\Omega$ with covering $\mathcal{W}$ and two cubes $Q, S \in \mathcal{W}$ with an $\varepsilon$-admissible chain $[Q, S]$. From Definition 3.2 it follows that*

$$\mathrm{D}(Q, S) \approx_{\varepsilon,d} \ell([Q, S]) \approx_{\varepsilon,d} \ell(Q_S) \approx_{\varepsilon,d} \mathrm{D}(Q, Q_S) \approx_{\varepsilon,d} \mathrm{D}(Q_S, S). \tag{3.4}$$

*If $P \in [Q, Q_S]$, by (3.3) we have that*

$$\mathrm{D}(Q, P) \approx_{d,\varepsilon} \ell(P) \tag{3.5}$$

*On the other hand, by the triangular inequality, (3.2) and (3.3) we have that*

$$\mathrm{D}(P, S) \lesssim_d \ell([P, S]) \leqslant \ell([Q, S]) \leqslant \frac{\mathrm{D}(Q, S)}{\varepsilon} \lesssim_d \frac{\mathrm{D}(Q, P) + \mathrm{D}(P, S)}{\varepsilon} \lesssim_d \frac{\frac{1}{\varepsilon}\ell(P) + \mathrm{D}(P, S)}{\varepsilon},$$

*that is,*

$$\mathrm{D}(P, S) \approx_{\varepsilon,d} \mathrm{D}(Q, S). \tag{3.6}$$

Using (3.6) it is quite easy to see that a domain satisfying this definition satisfies to the one given by Peter Jones in [Jon81] (changing the parameters $\varepsilon$ and $\delta$ if necessary). It is somewhat more involved to prove the converse implication, but it can be done using the ideas of Remark 3.4. In any case it is not transcendent for the present paper to prove this fact, which is left for the reader as an exercise.

Now we can define the shadows:

**Definition 3.5.** *Given a cube $P \in \mathcal{W}$ centered at $x_P$ and a real number $\rho$, the $\rho$-shadow of $P$ is the collection of cubes*

$$\mathbf{SH}_\rho(P) = \{Q \in \mathcal{W} : Q \subset B(x_P, \rho\, \ell(P))\}$$

*and its "realization" is the set*

$$\mathbf{Sh}_\rho(P) = \bigcup_{Q \in \mathbf{SH}_\rho(P)} Q$$

*(see Figure 3.2).*

*By the previous remark and the properties of the Whitney covering, we can define $\rho_\varepsilon > 1$ such that the following properties hold:*

- $|\mathrm{diam}(\partial\Omega \cap \overline{\mathbf{Sh}_{\rho_\varepsilon}(P)})| \approx \ell(P)$.
- *For every $\varepsilon$-admissible chain $[Q, S]$, and every $P \in [Q, Q_S]$ we have that $Q \in \mathbf{SH}_{\rho_\varepsilon}(P)$.*
- *Moreover, every cube $P$ belonging to an $\varepsilon$-admissible chain $[Q, S]$ belongs to the shadow $\mathbf{SH}_{\rho_\varepsilon}(Q_S)$.*

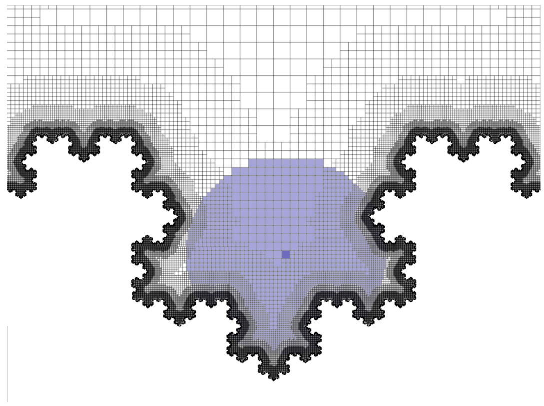


Figure 3.2: The shadow $\mathbf{Sh}_{13}(P)$.

Note that the first property comes straight from the properties of the Whitney covering, while the second is a consequence of (3.5) and the third holds because of the fact that if $P \in [Q,S]$ then $D(P,Q_S) \lesssim_d \ell([Q,S]) \approx \mathrm{D}(Q,S) \approx \ell(Q_S)$ by (3.4).

**Remark 3.6.** *Given an $(\varepsilon,\delta)$-uniform domain $\Omega$ we will write* $\mathbf{Sh}$ *for* $\mathbf{Sh}_{\rho_\varepsilon}$. *We will write also* $\mathbf{SH}$ *for* $\mathbf{SH}_{\rho_\varepsilon}$.

*If $Q \in \mathbf{SH}(P)$ and $\ell(P) \leqslant C$ (with $C$ depending on $\delta$ and the Whitney constants), we can grant that $\mathrm{D}(Q,P) \leqslant \delta$ and, therefore, there exists an $\varepsilon$-admissible chain $[Q,P]$.*

*For $Q \in \mathcal{W}$ and $s > 0$, we have that*

$$\sum_{L: Q \in \mathbf{SH}(L)} \ell(L)^{-s} \lesssim_{s,\mathcal{W}} \ell(Q)^{-s} \tag{3.7}$$

*and, moreover, if $Q \in \mathbf{SH}(P)$ with $\ell(P) \leqslant C$, then*

$$\sum_{L \in [Q,P]} \ell(L)^s \lesssim_{s,\mathcal{W}} \ell(P)^s \qquad \text{and} \qquad \sum_{L \in [Q,P]} \ell(L)^{-s} \lesssim \ell(Q)^{-s}. \tag{3.8}$$

*Proof.* We only need to prove (3.7) and (3.8). Considering the definition of shadow we can deduce that there is a bounded number of cubes with given side-length in the left-hand side of (3.7) and, therefore, the sum is a geometric sum. Again by the definition of shadow we know that the smaller cube in that sum has side-length comparable to $\ell(Q)$.

To prove (3.8), first note that $\ell(Q_P) \approx \mathrm{D}(Q,P) \approx \ell(P)$ by (3.4) and Definition 3.5. For every $L \in [Q,P]$, although it may occur that $L \notin \mathbf{SH}(P)$, we still have that by the triangle inequality $\mathrm{D}(L,P) \lesssim \ell([Q,P]) \approx \mathrm{D}(Q,P)$ and, thus, by the definition of shadow we have that $\mathrm{D}(L,P) \lesssim \ell(P)$, i.e.

$$\mathrm{D}(L,P) \approx \ell(P). \tag{3.9}$$

When $L \in [Q,Q_P]$, (3.5) reads as

$$\ell(L) \approx \mathrm{D}(Q,L),$$

and when $L \in [Q_P,P]$ by (3.5) and (3.9), we have that

$$\ell(L) \approx \mathrm{D}(L,P) \approx \ell(P).$$

In particular, the number of cubes in $[Q_P,P]$ is uniformly bounded. Summing up, for $L \in [Q,P]$ we have that $\ell(Q) \lesssim \ell(L) \lesssim \ell(P)$ and all the cubes of a given sidelength $r$ contained in $[Q,P]$

are situated at a distance from $Q$ bounded by $Cr$. so the number of those cubes is uniformly bounded. Therefore, the left-hand side of both inequalities of (3.8) are geometric sums, bounded by a constant times the bigger term. The constant depends on $s$, but also on the uniformity constants of the domain. □

# 4 Properties of $A^s_{p,q}$ on uniform domains.

Next we introduce a norm which will be the main tool for the proofs in this paper.

**Definition 4.1.** *Consider $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Let $U$ be a open set in $\mathbb{R}^d$. We say that a measurable function $f \in A^s_{p,q}(U)$ if*

- *The function $f \in L^p(U)$.*
- *The seminorm*

$$\|f\|_{\dot{A}^s_{p,q}(U)} := \left( \int_U \left( \int_U \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty. \tag{4.1}$$

*We define the norm*

$$\|f\|_{A^s_{p,q}(U)} := \|f\|_{L^p(U)} + \|f\|_{\dot{A}^s_{p,q}(U)}.$$

**Remark 4.2.** *The condition $\frac{d}{p} - \frac{d}{q} < s$ ensures that the $C^\infty_c$-functions are in the class $A^s_{p,q}(\mathbb{R}^d)$.*

*Proof.* Indeed, given a bump function $\varphi \in C^\infty_c(\mathbb{D})$,

$$\begin{aligned} \|\varphi\|_{A^s_{p,q}(\mathbb{R}^d)} &\geqslant \left( \int_{(2\mathbb{D})^c} \left( \int_{\mathbb{D}} \frac{|\varphi(x) - \varphi(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\ &\approx \left( \int_{(2\mathbb{D})^c} \left( \int_{\mathbb{D}} |\varphi(y)|^q \, dy \right)^{\frac{p}{q}} \frac{1}{|x|^{sp + \frac{dp}{q}}} \, dx \right)^{\frac{1}{p}} \end{aligned}$$

which is finite if and only if $\frac{d}{p} < s + \frac{d}{q}$. The converse implication is an exercise. □

We recall the definition of the non-centered Hardy-Littlewood maximal operator. Given $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define $Mf(x)$ as the supremum of the mean of $f$ in cubes containing $x$, that is,

$$Mf(x) = \sup_{Q: x \in Q} \frac{1}{|Q|} \int_Q f(y) \, dy.$$

It is a well known fact that this operator is bounded in $L^p$ for $1 < p < \infty$. The following lemma is proven in [PT15] and will be used repeatedly along the proofs contained in the present text.

**Lemma 4.3.** *Let $\Omega$ be a bounded uniform domain with an admissible Whitney covering $\mathcal{W}$. Assume that $g \in L^1(\Omega)$ and $r > 0$. For every $\eta > 0$, $Q \in \mathcal{W}$ and $x \in \mathbb{R}^d$, we have*

*1) The non-local inequality for the maximal operator*

$$\int_{|y-x|>r} \frac{g(y)\, dy}{|y-x|^{d+\eta}} \lesssim_d \frac{Mg(x)}{r^\eta} \qquad \text{and} \qquad \sum_{S: \mathrm{D}(Q,S) > r} \frac{\int_S g(y)\, dy}{D(Q,S)^{d+\eta}} \lesssim_d \frac{\inf_{y \in Q} Mg(y)}{r^\eta}. \tag{4.2}$$

*2) The local inequality for the maximal operator*

$$\int_{|y-x|<r} \frac{g(y)\,dy}{|y-x|^{d-\eta}} \lesssim_d r^\eta Mg(x) \qquad \text{and} \qquad \sum_{S:\mathrm{D}(Q,S)<r} \frac{\int_S g(y)\,dy}{D(Q,S)^{d-\eta}} \lesssim_d \inf_{y\in Q} Mg(y)\, r^\eta. \quad (4.3)$$

*3) In particular we have*

$$\sum_{S\in\mathcal{W}} \frac{\ell(S)^d}{\mathrm{D}(Q,S)^{d+\eta}} \lesssim_d \frac{1}{\ell(Q)^\eta} \tag{4.4}$$

*and, by Definition 3.5,*

$$\sum_{S\in\mathbf{SH}(Q)} \int_S g(x)\,dx \lesssim_{d,\varepsilon} \inf_{y\in Q} Mg(y)\,\ell(Q)^d.$$

The first lemma we present an equivalent norm for $A^s_{p,q}(\Omega)$.

**Lemma 4.4.** *Let $\Omega$ be a bounded uniform domain with an admissible Whitney covering $\mathcal{W}$, let $1<p,q<\infty$ and $0<s<1$ wit $s>\frac{d}{p}-\frac{d}{q}$. Then, $f\in A^s_{p,q}(\Omega)$ if and only if*

$$\|f\|_{\tilde{A}^s_{p,q}(\Omega)} = \|f\|_{L^p(\Omega)} + \left(\sum_{Q\in\mathcal{W}} \int_Q \left(\int_{\mathbf{Sh}(Q)} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} < \infty. \tag{4.5}$$

*This quantity defines a norm which is equivalent to $\|f\|^p_{A^s_{p,q}(\Omega)}$ and, moreover, we have that $f\in L^q(\Omega)$.*

*Proof.* Recall that in (4.1) we defined

$$\|f\|_{\dot{A}^s_{p,q}(\Omega)} = \left(\int_\Omega \left(\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}.$$

Trivially

$$\|f\|^p_{\dot{A}^s_{p,q}(\Omega)} \gtrsim \sum_{Q\in\mathcal{W}} \int_Q \left(\int_{\mathbf{Sh}(Q)} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}} dx.$$

For the converse inequality, we start by proving that if (4.5) holds then $f\in L^q(\Omega)$. If we choose $\rho_\varepsilon$ big enough, we can grant that every point in the boundary is in the shadow of a cube of side-length equivalent to $\delta$. Therefore, there is a finite collection of cubes $Q_1,\dots,Q_M$ with $\Omega\subset\bigcup_{j=1}^M \mathbf{Sh}(Q_j)$. Then, the triangular inequality and the Hölder inequality yield

$$\|f\|_{L^q(\Omega)} \leqslant \sum_{j=1}^M \left\|f\chi_{\mathbf{Sh}(Q_j)}\right\|_{L^q(\Omega)} \leqslant \left(\sum_{j=1}^M \left(\int_{\mathbf{Sh}(Q_j)} |f(y)|^q\,dy\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} M^{\frac{1}{p'}}.$$

Therefore, since we assume that (4.5) holds,

$$\begin{aligned}\|f\|_{L^q(\Omega)} &\lesssim \left(\sum_{j=1}^M \fint_{Q_j} \left(\int_{\mathbf{Sh}(Q_j)} |f(x)|^q\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} + \left(\sum_{j=1}^M \fint_{Q_j} \left(\int_{\mathbf{Sh}(Q_j)} |f(x)-f(y)|^q\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \\ &\lesssim \frac{1}{\min_j \ell(Q_j)^{\frac{d}{p}}} \left(\|f\|_{L^p(\Omega)} + \left(\sum_{j=1}^M \int_{Q_j} \left(\int_{\mathbf{Sh}(Q_j)} \mathrm{diam}(\Omega)^{sq+d} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}\right),\end{aligned}$$

and thus

$$\|f\|_{L^q(\Omega)} \lesssim \|f\|_{L^p(\Omega)} + \left( \sum_{Q \in \mathcal{W}} \int_Q \left( \int_{\mathbf{Sh}(Q)} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \tag{4.6}$$

Next, we will use the seminorm in the duality form

$$\|f\|_{\dot{A}^s_{p,q}(\Omega)} = \sup_{\|g\|_{L^{p'}(L^{q'}(\Omega))} \leqslant 1} \int_\Omega \int_\Omega \frac{|f(x)-f(y)|}{|x-y|^{s+\frac{d}{q}}} g(x,y) \, dy \, dx. \tag{4.7}$$

Since the shadow of every cube $Q$ contains $2Q$, we just use Hölder's inequality to find that

$$\sup_{\|g\|_{L^{p'}(L^{q'}(\Omega))} \leqslant 1} \sum_{Q \in \mathcal{W}} \int_Q \int_{2Q} \frac{|f(x)-f(y)|}{|x-y|^{s+\frac{d}{q}}} g(x,y) \, dy \, dx \leqslant \left( \sum_{Q \in \mathcal{W}} \int_Q \left( \int_{2Q} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \tag{4.8}$$

Therefore, we only need to prove that for any function $g > 0$ with $\|g\|_{L^{p'}(L^{q'}(\Omega))} \leqslant 1$, one has that

$$\sum_{Q \in \mathcal{W}} \int_Q \sum_S \int_{S \setminus 2Q} \frac{|f(x)-f(y)|}{|x-y|^{s+\frac{d}{q}}} g(x,y) \, dy \, dx \lesssim \left( \sum_{Q \in \mathcal{W}} \int_Q \left( \int_{\mathbf{Sh}(Q)} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \tag{4.9}$$

If $x \in Q$, $y \in S \setminus 2Q$, then $|x-y| \approx \mathrm{D}(Q,S)$, so we can write

$$\begin{aligned} \sum_{Q \in \mathcal{W}} \int_Q \sum_S \int_{S \setminus 2Q} \frac{|f(x)-f(y)|}{|x-y|^{s+\frac{d}{q}}} g(x,y) \, dy \, dx &\lesssim \sum_Q \sum_{\mathrm{D}(Q,S) \geqslant \delta} \int_Q \int_S \frac{|f(x)-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \, dy \, dx \\ &+ \sum_Q \sum_{\mathrm{D}(Q,S) < \delta} \int_Q \int_S \frac{|f(x)-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \, dy \, dx. \end{aligned} \tag{4.10}$$

For the first term in (4.10),

$$\sum_Q \sum_{\mathrm{D}(Q,S) \geqslant \delta} \int_Q \int_S \frac{|f(x)-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \, dy \, dx \lesssim_\delta \int_\Omega \int_\Omega (|f(x)| + |f(y)|) g(x,y) \, dy \, dx.$$

By Hölder's inequality,

$$\int_\Omega \int_\Omega (|f(x)| + |f(y)|) g(x,y) \, dy \, dx \lesssim \|f\|_{L^p(\Omega)} |\Omega|^{\frac{1}{q}} \|g\|_{L^{p'}(L^{q'}(\Omega))} + |\Omega|^{\frac{1}{p}} \|f\|_{L^q(\Omega)} \|g\|_{L^{p'}(L^{q'}(\Omega))},$$

which, by (4.6), yields

$$\sum_Q \sum_{\mathrm{D}(Q,S) \geqslant \delta} \int_Q \int_S \frac{|f(x)-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \, dy \, dx \lesssim_{|\Omega|,\delta} \|f\|_{\widetilde{A}^s_{p,q}(\Omega)}. \tag{4.11}$$

The second sum in (4.10) deals with the case $\mathrm{D}(Q,S) < \delta$. Since $\Omega$ is a $(\delta,\epsilon)$-uniform domain, for every pair of cubes $Q$ and $S$ in this sum, there exists an amdissible chain $[Q,S]$ joining them.

Thus, writing $f_Q = \fint_Q f\,dm$ for the mean of $f$ in $Q$, the term can be split as follows:

$$\begin{aligned}\sum_{Q,S:\mathrm{D}(Q,S)<\delta} \int_Q \int_S \frac{|f(x)-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\,dy\,dx \leqslant & \sum_{Q,S:\mathrm{D}(Q,S)<\delta} \int_Q \int_S \frac{|f(x)-f_Q|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\,dy\,dx \\ & + \sum_{Q,S:\mathrm{D}(Q,S)<\delta} \int_Q \int_S \frac{|f_Q-f_{Q_S}|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\,dy\,dx \\ & + \sum_{Q,S:\mathrm{D}(Q,S)<\delta} \int_Q \int_S \frac{|f_{Q_S}-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\,dy\,dx \\ & =: ① + ② + ③ \end{aligned} \tag{4.12}$$

The first term can be immediately bounded by Cauchy-Schwarz inequality. Namely, writing $G(x) = \|g(x,\cdot)\|_{L^{q'}(\Omega)}$, by (4.4) we have that

$$\begin{aligned} ① &\leqslant \sum_{Q\in\mathcal{W}} \int_Q |f(x)-f_Q| \left(\sum_{S\in\mathcal{W}} \int_S g(x,y)^{q'}\,dy\right)^{\frac{1}{q'}} \left(\sum_{S\in\mathcal{W}} \frac{\ell(S)^d}{\mathrm{D}(Q,S)^{sq+d}}\right)^{\frac{1}{q}} dx \\ &\leqslant \sum_{Q\in\mathcal{W}} \frac{\int_Q |f(x)-f_Q| G(x)\,dx}{\ell(Q)^s}. \end{aligned}$$

By Jensen's inequality, $|f(x)-f_Q| \leqslant \left(\frac{1}{\ell(Q)^d}\int_Q |f(x)-f(y)|^q\,dy\right)^{\frac{1}{q}}$ and thus, since $\ell(Q) \gtrsim_d |x-y|$ for $x,y\in Q$, we have that

$$① \lesssim \left(\sum_{Q\in\mathcal{W}} \int_Q \left(\int_Q \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \|G\|_{L^{p'}}. \tag{4.13}$$

Since $\|G\|_{L^{p'}} = \|g\|_{L^{p'}(L^{q'})} \leqslant 1$, this finishes this part.

For the second one, for all cubes $Q$ and $S$ we consider the subchain $[Q, Q_S) \subset [Q,S]$. Then

$$② \leqslant \sum_{Q,S:\mathrm{D}(Q,S)<\delta} \int_Q \int_S \frac{g(x,y)}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}}\,dy\,dx \sum_{P\in[Q,Q_S)} |f_P - f_{\mathcal{N}(P)}|.$$

Recall that all the cubes $P\in[Q,Q_S]$ contain $Q$ in their shadow and the properties of the Whitney covering grant that $\mathcal{N}(P)\subset 5P$. Moreover, by (3.6) we have that $\mathrm{D}(Q,S)\approx \mathrm{D}(P,S)$. Thus,

$$② \lesssim_d \sum_P \fint_P \fint_{5P} |f(\xi)-f(\zeta)|\,d\zeta\,d\xi \sum_{Q\in\mathbf{SH}(P)} \int_Q \sum_{S\in\mathcal{W}} \int_S \frac{g(x,y)}{\mathrm{D}(P,S)^{s+\frac{d}{q}}}\,dy\,dx$$

and, using Hölder's inequality, and by (4.4), we have that

$$\begin{aligned} ② &\leqslant \sum_P \fint_P \fint_{5P} |f(\xi)-f(\zeta)|\,d\zeta\,d\xi \sum_{Q\in\mathbf{SH}(P)} \int_Q \left(\int_\Omega g(x,y)^{q'}\,dy\right)^{\frac{1}{q'}} \left(\sum_{S\in\mathcal{W}} \frac{\ell(S)^d}{\mathrm{D}(P,S)^{sq+d}}\right)^{\frac{1}{q}} dx \\ &\lesssim_{d,s,q} \sum_P \fint_P \fint_{5P} |f(\xi)-f(\zeta)|\,d\zeta\,d\xi \sum_{Q\in\mathbf{SH}(P)} \int_Q G(x)\,dx \frac{1}{\ell(P)^s}. \end{aligned}$$

By (4.3) we have that $\int_{\mathbf{Sh}(P)} G(x)\,dx \lesssim_{d,\varepsilon} inf_{y\in P} MG(y)\ell(P)^d$, so

$$\textcircled{2} \lesssim_d \sum_P \int_P \int_{5P} |f(\xi) - f(\zeta)|\, d\zeta MG(\xi)\, d\xi \frac{\ell(P)^{d-s}}{\ell(P)^{2d}}$$
$$\lesssim_{d,p} \sum_P \int_P \left( \int_{5P} |f(\xi) - f(\zeta)|^q\, d\zeta \right)^{\frac{1}{q}} \ell(P)^{\frac{d}{q'}} MG(\xi)\, d\xi \frac{1}{\ell(P)^{d+s}}.$$

Note that for $\xi, \zeta \in 5P$, we have that $|\xi - \zeta| \lesssim_d \ell(P)$. Thus, using Hölder's inequality again and the fact that $\|MG\|_{L^{p'}} \lesssim_p \|G\|_{L^{p'}} \leqslant 1$, we bound the second term by

$$\textcircled{2} \lesssim \sum_P \int_P \left( \int_{5P} \frac{|f(\xi) - f(\zeta)|^q}{|\xi - \zeta|^{sq+d}}\, d\zeta \right)^{\frac{1}{q}} MG(\xi)\, d\xi \lesssim \left( \sum_P \int_P \left( \int_{5P} \frac{|f(\xi) - f(\zeta)|^q}{|\xi - \zeta|^{sq+d}}\, d\zeta \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}. \tag{4.14}$$

Now we face the boundedness of

$$\textcircled{3} = \sum_{Q,S:\mathrm{D}(Q,S)<\delta} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\, dy\, dx.$$

Given two cubes $Q$ and $S$ with $\mathrm{D}(Q,S) < \delta$, we have that for every admissible chain $[Q,S]$ the cubes $Q, S \in \mathbf{SH}(Q_S)$ by Definition 3.3 and $\mathrm{D}(Q,S) \approx \ell(Q_S)$ by (3.4). Thus, we can reorder the sum, writing

$$\textcircled{3} \lesssim \sum_R \sum_{Q\in\mathbf{SH}(R)} \sum_{S\in\mathbf{SH}(R)} \int_Q \int_S \frac{|f_R - f(y)|}{\ell(R)^{s+\frac{d}{q}}} g(x,y)\, dy\, dx \tag{4.15}$$
$$\leqslant \sum_R \int_R \sum_{Q\in\mathbf{SH}(R)} \sum_{S\in\mathbf{SH}(R)} \int_Q \int_S \frac{|f(\xi) - f(y)|}{\ell(R)^{s+\left(1+\frac{1}{q}\right)d}} g(x,y)\, dy\, dx\, d\xi.$$

Using Hölder's inequality, Lemma 4.3 and the fact that for $S \in \mathbf{SH}(R)$ one has $\ell(R) \approx \mathrm{D}(S,R)$, we get that

$$\textcircled{3} \leqslant \sum_R \int_R \frac{1}{\ell(R)^{s+\left(1+\frac{1}{q}\right)d}} \sum_{Q\in\mathbf{SH}(R)} \int_Q \sum_{S\in\mathbf{SH}(R)} \left( \int_S |f(\xi) - f(y)|^q\, dy \right)^{\frac{1}{q}} \left( \int_S g(x,y)^{q'}\, dy \right)^{\frac{1}{q'}} dx\, d\xi$$
$$\leqslant \sum_R \int_R \frac{1}{\ell(R)^{s+\left(1+\frac{1}{q}\right)d}} \left( \int_{\mathbf{Sh}(R)} |f(\xi) - f(y)|^q\, dy \right)^{\frac{1}{q}} \sum_{Q\in\mathbf{SH}(R)} \int_Q G(x)\, dx\, d\xi$$
$$\lesssim \sum_R \int_R \left( \int_{\mathbf{Sh}(R)} |f(\xi) - f(y)|^q\, dy \right)^{\frac{1}{q}} \frac{1}{\ell(R)^{s+\left(1+\frac{1}{q}\right)d}} MG(\xi)\ell(R)^d\, d\xi$$

and, using the Hölder inequality again and the boundedness of the maximal operator in $L^{p'}$, we get

$$\textcircled{3} \lesssim \left( \sum_R \int_R \left( \int_{\mathbf{Sh}(R)} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}}\, dy \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}} \|MG\|_{L^{p'}}$$
$$\lesssim \left( \sum_R \int_R \left( \int_{\mathbf{Sh}(R)} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}}\, dy \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}. \tag{4.16}$$

Thus, by (4.12), (4.13), (4.14) and (4.16), we have that

$$\sum_{Q,S:\mathrm{D}(Q,S)<\delta}\int_Q\int_S \frac{|f(x)-f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}}g(x,y)\,dy\,dx \lesssim \left(\sum_R\int_R\left(\int_{\mathbf{Sh}(R)}\frac{|f(\xi)-f(y)|^q}{|\xi-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}d\xi\right)^{\frac{1}{p}}.$$

This fact, together with (4.10) and (4.11) prove (4.9) and thus, using (4.7) and (4.8), we get that

$$\|f\|_{A^s_{p,q}(\Omega)} \lesssim_{d,p,q,s,\varepsilon,\delta,|\Omega|} \|f\|_{\widetilde{A}^s_{p,q}(\Omega)}.$$

□

**Remark 4.5.** *Note that in case $\delta=\infty$, we have that the first term in the right-hand side of (4.10) is zero, and we have proven that in this case the homogeneous seminorms are equivalent, that is,*

$$\sum_{Q\in\mathcal{W}}\int_Q\left(\int_{\mathbf{Sh}(Q)}\frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx \approx \|f\|^p_{\dot{A}^s_{p,q}(\Omega)},$$

*which improves (4.5).*

# 5 Fractional Sobolev spaces

First we recall some results on Triebel-Lizorkin spaces. We refer the reader to [Tri83].

**Definition 5.1.** *Let $\Phi(\mathbb{R}^d)$ be the collection of all the families of smooth functions $\Psi=\{\psi_j\}_{j=0}^{\infty}\subset C_c^\infty(\mathbb{R}^d)$ such that*

$$\begin{cases}\operatorname{supp}\psi_0\subset\mathbb{D}(0,2),\\ \operatorname{supp}\psi_j\subset\mathbb{D}(0,2^{j+1})\setminus\mathbb{D}(0,2^{j-1}) & \text{if } j\geqslant 1,\end{cases}$$

*for every multiindex $\alpha\in\mathbb{N}^d$ there exists a constant $c_\alpha$ such that*

$$\|D^\alpha\psi_j\|_\infty \leqslant \frac{c_\alpha}{2^{j|\alpha|}} \quad \text{for every } j\geqslant 0$$

*and*

$$\sum_{j=0}^{\infty}\psi_j(x)=1 \quad \text{for every } x\in\mathbb{R}^d.$$

We will use the classical notation $\widehat{f}$ for the Fourier transform of a given Schwartz function,

$$\widehat{f}(\xi)=\int_{\mathbb{R}^d}e^{-2\pi i x\cdot\xi}f(x)\,dx,$$

and $\check{f}$ will denote its inverse. It is well known that the Fourier transform can be extended to the whole space of tempered distributions by duality and it induces an isometry in $L^2$ (see for example [Gra08, Chapter 2]).

**Definition 5.2.** *Let $s\in\mathbb{R}$, $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$ and $\Psi\in\Phi(\mathbb{R}^n)$. For any tempered distribution $f\in S'(\mathbb{R}^n)$ we define its non-homogeneous Besov norm*

$$\|f\|^{\Psi}_{B^s_{p,q}}=\left\|\left\{2^{sj}\left\|\left(\psi_j\widehat{f}\right)^{\vee}\right\|_{L^p}\right\}\right\|_{l^q},$$

*and we call $B^s_{p,q} \subset S'$ to the set of tempered distributions such that this norm is finite.*

*Let $s \in \mathbb{R}$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $\Psi \in \Phi(\mathbb{R}^n)$. For any tempered distribution $f \in S'(\mathbb{R}^n)$ we define its non-homogeneous Triebel-Lizorkin norm*

$$\|f\|^{\Psi}_{F^s_{p,q}} = \left\| \left\| \left\{ 2^{sj} \left( \psi_j \hat{f} \right)^{\check{}} \right\} \right\|_{l^q} \right\|_{L^p},$$

*and we call $F^s_{p,q} \subset S'$ to the set of tempered distributions such that this norm is finite.*

These norms are equivalent for different choices of $\Psi$. Usually one works with radial $\psi_j$ and such that $\psi_{j+1}(x) = \psi_j(x/2)$ for $j \geqslant 1$. Of course we will omit $\Psi$ in our notation since it plays no role (see [Tri83, Section 2.3]).

**Remark 5.3.** *For $q = 2$ and $1 < p < \infty$ these spaces are the so-called Bessel-potential spaces $W^{s,p}$. In addition, if $s \in \mathbb{N}$ they coincide with the usual Sobolev spaces of functions in $L^p$ with weak derivatives up to order $s$ in $L^p$, and they coincide with $L^p$ for $s = 0$ ([Tri83, Section 2.5.6]). In the present text, we call Sobolev space to any $W^{s,p}$ with $s > 0$ and $1 < p < \infty$, even if $s$ is not a natural number. Note that complex interpolation between Sobolev spaces is a Sobolev space (see [Tri78, Section 2.4.2, Theorem 1]).*

In some situations, the classical Besov spaces $B^s_{p,p}(U) = A^s_{p,p}(U)$ and the fractional Sobolev spaces $W^{s,p}(U) = A^s_{p,2}(U)$. For instance, when $\Omega$ is a Lipschitz domain then $A^s_{p,2}(\Omega) = W^{s,p}(\Omega)$ (see [Str67]). We will see that this is a property of all uniform domains.

To use the Sobolev embedding for Triebel-Lizorkin spaces, we will use the following proposition.

**Proposition 5.4** (See [Tri83, Section 2.3.2].)**.** *Let $1 \leqslant q \leqslant \infty$ and $1 \leqslant p < \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. Then*

$$F^{s+\varepsilon}_{p,q} \subset W^{s,p}. \tag{5.1}$$

Next we will prove Theorem 1.2. Let us write $\Delta^1_h f(x) := f(x+h) - f(x)$ and, if $M \in \mathbb{N}$ with $M > 1$ we define the $M$-th iterated difference as $\Delta^M_h f(x) := \Delta^1_h(\Delta^{M-1}_h f)(x) = \sum_{j=0}^M \binom{M}{j}(-1)^{M-j} f(x+jh)$. For an index $0 < u \leqslant \infty$ and $t \in \mathbb{R}$, we write

$$d^M_{t,u} f(x) := \left( t^{-d} \int_{|h| \leqslant t} |\Delta^M_h f(x)|^u \, dh \right)^{\frac{1}{u}},$$

with the usual modification for $u = \infty$. In [Tri06, Theorem 1.116] we find the following result.

**Theorem** (See [Tri06].)**.** *Given $1 \leqslant r \leqslant \infty$, $0 < u \leqslant r$, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < s < M$ with $\frac{d}{\min\{p,q\}} - \frac{d}{r} < s$, we have that*

$$F^s_{p,q}(\mathbb{R}^d) = \left\{ f \in L^{\max\{p,r\}} : \|f\|_{L^p} + \left( \int_{\mathbb{R}^d} \left( \int_0^1 \frac{d^M_{t,u} f(x)^q}{t^{sq+1}} \, dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\}$$

*(with the usual modification for $q = \infty$), in the sense of equivalent quasinorms.*

As an immediate consequence of this result, we get the following corollary.

**Corollary 5.5.** *Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $0 < s < M$ with $s > \frac{d}{p} - \frac{d}{q}$. Then*

$$F^s_{p,q}(\mathbb{R}^d) = \left\{ f \in L^{\max\{p,q\}} \text{ s.t. } \|f\|_{A^s_{p,q}(\mathbb{R}^d)} := \|f\|_{L^p} + \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{|\Delta^M_h f(x)|^q}{|h|^{sq+d}} \, dh \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\}$$

*(with the usual modification for $q = \infty$), in the sense of equivalent norms.*

*Proof.* Let $f \in L^{\max\{p,q\}}$ and assume that $1 \leqslant q < \infty$ and $0 < s < M$ with $s > \frac{d}{p} - \frac{d}{q}$. Choosing $q = u = r$ all the conditions in the theorem above are satisfied. Therefore,

$$\|f\|_{F^s_{p,q}(\mathbb{R}^d)} \approx \|f\|_{L^p} + \left(\int_{\mathbb{R}^d}\left(\int_0^1 \frac{d^M_{t,q}f(x)^q}{t^{sq+1}}\,dt\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}. \tag{5.2}$$

Since $d^M_{t,q} = \left(t^{-d}\int_{|h|\leqslant t}|\Delta^M_h f(x)|^q\,dh\right)^{\frac{1}{q}}$, we can change the order of integration to get that

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_0^1 \frac{d^M_{t,q}f(x)^q}{t^{sq+1}}\,dt\right)^{\frac{p}{q}} dx &= \int_{\mathbb{R}^d}\left(\int_{|h|\leqslant 1}\int_{1>t>|h|}\frac{dt}{t^{sq+1+d}}|\Delta^M_h f(x)|^q\,dh\right)^{\frac{p}{q}} dx\\ &= \int_{\mathbb{R}^d}\left(\int_{|h|\leqslant 1}\frac{|\Delta^M_h f(x)|^q}{sq+d}\left(\frac{1}{|h|^{sq+d}}-1\right)dh\right)^{\frac{p}{q}} dx.\end{aligned}$$

This shows that $\|f\|_{F^s_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{A^s_{p,q}(\mathbb{R}^d)}$ and also that

$$\int_{\mathbb{R}^d}\left(\int_{|h|<\frac{1}{2}}\frac{|\Delta^M_h f(x)|^q}{|h|^{sq+d}}\,dh\right)^{\frac{p}{q}} dx \lesssim \int_{\mathbb{R}^d}\left(\int_0^1 \frac{d^M_{t,q}f(x)^q}{t^{sq+1}}\,dt\right)^{\frac{p}{q}} dx \lesssim \|f\|^p_{F^s_{p,q}(\mathbb{R}^d)} \tag{5.3}$$

by (5.2). It remains to see that $\int_{\mathbb{R}^d}\left(\int_{|h|>\frac{1}{2}}\frac{|\Delta^M_h f(x)|^q}{|h|^{sq+d}}\,dh\right)^{\frac{p}{q}} dx \lesssim \|f\|^p_{F^s_{p,q}(\mathbb{R}^d)}$. Using appropriate changes of variables and the triangle inequality, it is enough to check that

$$\textcircled{\small I} := \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{|f(x+h)|^q}{(1+|h|)^{sq+d}}\,dh\right)^{\frac{p}{q}} dx \lesssim \|f\|^p_{F^s_{p,q}(\mathbb{R}^d)}. \tag{5.4}$$

Let us assume that $p \geqslant q$. Then, since the measure $(1+|h|)^{-(sq+d)}\,dh$ is finite, we may apply Jensen's inequality to the inner integral, and then Fubini to obtain

$$\textcircled{\small I} \lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(x+h)|^p}{(1+|h|)^{sp+d}}\,dh\,dx \lesssim \|f\|^p_{L^p},$$

and (5.4) follows.

If, instead, $p < q$, cover $\mathbb{R}^d$ with disjoint cubes $Q_{\vec{j}} = Q_0 + \ell\vec{j}$ for $\vec{j} \in \mathbb{Z}^d$. Fix the side-length $\ell$ of these cubes so that their diameter is 1/3. By the subadditivity of $x \to |x|^{\frac{p}{q}}$, we have that

$$\textcircled{\small I} \lesssim \sum_{\vec{k}}\int_{Q_{\vec{k}}}\sum_{\vec{j}}\left(\int_{Q_{\vec{j}}}\frac{|f(y)|^q}{(1+|x-y|)^{sq+d}}\,dy\right)^{\frac{p}{q}} dx \approx \sum_{\vec{j}}\left(\int_{Q_{\vec{j}}}|f(y)|^q\,dy\right)^{\frac{p}{q}}\sum_{\vec{k}}\frac{1}{(1+|\vec{j}-\vec{k}|)^{sp+\frac{dp}{q}}}.$$

Since $s + \frac{d}{q} > \frac{d}{p}$, the last sum is finite and does not depend on $\vec{j}$. On the other hand, we have that $\sum_{\vec{j}}\int_{Q_{\vec{j}}}\left(\int_{Q_{\vec{j}}}|f(x)|^q\,dy\right)^{\frac{p}{q}} dx \approx \|f\|^p_{L^p}$ because all the cubes have side-length comparable to 1. Let $0 < \sigma < \min\{1, s\}$. By (5.3) we have that

$$\sum_{\vec{j}}\left(\int_{Q_{\vec{j}}}|f(y)|^q\,dy\right)^{\frac{p}{q}} \lesssim \sum_{\vec{j}}\int_{Q_{\vec{j}}}\left(\int_{Q_{\vec{j}}}|f(y)-f(x)|^q\,dy\right)^{\frac{p}{q}} dx + \|f\|^p_{L^p} \lesssim \|f\|^p_{F^\sigma_{p,q}(\mathbb{R}^d)} \lesssim \|f\|^p_{F^s_{p,q}(\mathbb{R}^d)}.$$

Note that in the last step we used (5.1).

The case $q = \infty$ follows the same scheme. □

**Definition 5.6.** *Let $X(\mathbb{R}^d)$ be a Banach space of measurable functions in $\mathbb{R}^d$. Let $U \subset \mathbb{R}^d$ be a open set. Then for every measurable function $f : U \to \mathbb{C}$ we define*

$$\|f\|_{X(U)} \approx \inf_{g \in X(\mathbb{R}^d):\, g|_U \equiv f} \|g\|_{X(\mathbb{R}^d)}.$$

Consider a given $(\varepsilon, \delta)$-uniform domain $\Omega$. In [Jon81] Peter Jones defines an extension operator $\Lambda_0 : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ for $1 < p < \infty$, that is, a bounded operator such that $\Lambda_0 f|_\Omega \equiv f|_\Omega$ for every $f \in W^{1,p}(\Omega)$. This extension operator is used to prove that we have the intrinsic characterization of $W^{n,p}(\Omega)$ given by

$$\|f\|_{W^{1,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Next we will see that the same operator is an extension operator for $A^s_{p,q}(\Omega)$ for $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. To define it we need a Whitney covering $\mathcal{W}_1$ of $\Omega$ (see Definition 3.1), a Whitney covering $\mathcal{W}_2$ of $\Omega^c$ and we define $\mathcal{W}_3$ to be the collection of cubes in $\mathcal{W}_2$ with side-lengths small enough, so that for any $Q \in \mathcal{W}_3$ there is a $S \in \mathcal{W}_1$ with $\mathrm{D}(Q, S) \leqslant C\ell(Q)$ and $\ell(Q) = \ell(S)$ (see [Jon81, Lemma 2.4]). We define the symmetrized cube $Q^*$ as one of the cubes satisfying these properties. Note that the number of possible choices for $Q^*$ is uniformly bounded.

**Lemma 5.7.** *[see [Jon81]] For cubes $Q_1, Q_2 \in \mathcal{W}_3$ and $S \in \mathcal{W}_1$ we have that*

- *The symmetrized cubes have finite overlapping: there exists a constant $C$ depending on the parameters $\varepsilon$ and $d$ such that $\#\{Q \in \mathcal{W}_3 : Q^* = S\} \leqslant C$.*
- *The long distance is invariant in the following sense:*

$$\mathrm{D}(Q_1^*, Q_2^*) \approx \mathrm{D}(Q_1, Q_2) \qquad \text{and} \qquad \mathrm{D}(Q_1^*, S) \approx \mathrm{D}(Q_1, S) \tag{5.5}$$

- *In particular, if $Q_1 \cap 2Q_2 \neq \varnothing$ ($Q_1$ and $Q_2$ are neighbors by (3.1)), then $\mathrm{D}(Q_1^*, Q_2^*) \approx \ell(Q_1)$.*

We define the family of bump functions $\{\varphi_Q\}_{Q \in \mathcal{W}_2}$ to be a partition of the unity associated to $\left\{\frac{11}{10}Q\right\}_{Q \in \mathcal{W}_2}$, that is, their sum $\sum \varphi_Q \equiv 1$, we have the pointwise inequalities $0 \leqslant \varphi_Q \leqslant \chi_{\frac{11}{10}Q}$ and $\|\nabla \varphi_Q\|_\infty \lesssim \frac{1}{\ell(Q)}$. We can define the operator

$$\Lambda_0 f(x) = \sum_{Q \in \mathcal{W}_3} \varphi_Q(x) f_{Q^*} \text{ for any } f \in L^1_{loc}(\Omega)$$

(recall that $f_U$ stands for the mean of a function $f$ in a set $U$).

**Lemma 5.8.** *Let $\Omega$ be a bounded uniform domain, let $1 < p, q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then, $\Lambda_0 : A^s_{p,q}(\Omega) \to A^s_{p,q}(\mathbb{R}^d)$ is an extension operator. Furthermore, $\Lambda_0 f \in L^q$ for every $f \in A^s_{p,q}(\Omega)$.*

*Proof.* We have to check that

$$\|\Lambda_0 f\|_{A^s_{p,q}(\mathbb{R}^d)} = \|\Lambda_0 f\|_{L^p} + \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|\Lambda_0 f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} \, dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{q}} \lesssim \|f\|_{A^s_{p,q}(\Omega)}$$

First, note that $\|\Lambda_0 f\|_{L^p} \leqslant \|f\|_{L^p(\Omega)} + \|\Lambda_0 f\|_{L^p(\Omega^c)}$. By Jensen's inequality, we have that

$$\|\Lambda_0 f\|^p_{L^p(\Omega^c)} \lesssim_p \sum_{Q \in \mathcal{W}_3} |f_{Q^*}|^p \|\varphi_Q\|^p_{L^p} \leqslant \sum_{Q \in \mathcal{W}_3} \frac{1}{\ell(Q)^d} \|f\|^p_{L^p(Q^*)} \left(\frac{11}{10}\ell(Q)\right)^d.$$

By the finite overlapping of the symmetrized cubes,

$$\|\Lambda_0 f\|^p_{L^p(\Omega^c)} \lesssim \|f\|^p_{L^p(\Omega)}.$$

The same can be said about $L^q$. It remains to check that

$$\|\Lambda_0 f\|_{\dot{A}^s_{p,q}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\frac{|\Lambda_0 f(x)-\Lambda_0 f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx\right)^{\frac{1}{q}} \lesssim \|f\|_{A^s_{p,q}(\Omega)}.$$

We will argue by duality. We will prove that

$$\textcircled{a}+\textcircled{b}+\textcircled{c} \lesssim \|f\|^p_{A^s_{p,q}(\Omega)},$$

where

$$\textcircled{a} := \int_\Omega\left(\int_{\Omega^c}\frac{|f(x)-\Lambda_0 f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx, \qquad \textcircled{b} := \int_{\Omega^c}\left(\int_{\Omega}\frac{|\Lambda_0 f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx \quad \text{and}$$

$$\textcircled{c} := \int_{\Omega^c}\left(\int_{\Omega^c}\frac{|\Lambda_0 f(x)-\Lambda_0 f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx.$$

Let us begin with

$$\textcircled{a} = \int_\Omega\left(\int_{\Omega^c}\frac{|f(x)-\sum_{S\in\mathcal{W}_3}\varphi_S(y)f_{S*}|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx.$$

Call $\mathcal{W}_4 := \{S\in\mathcal{W}_3 : \text{ all the neighbors of } S \text{ are in } \mathcal{W}_3\}$. Given $y\in\frac{11}{10}S$, where $S\in\mathcal{W}_4$, we have that $\sum_{P\in\mathcal{W}_3}\varphi_P(y)\equiv 1$ and, otherwise $0\leqslant 1-\sum_{P\in\mathcal{W}_3}\varphi_P(y)\leqslant 1$. Thus

$$\begin{aligned}\textcircled{a} \lesssim & \sum_{Q\in\mathcal{W}_1}\int_Q\left(\sum_{S\in\mathcal{W}_3}\frac{|f(x)-f_{S*}|^q}{\mathrm{D}(Q,S)^{sq+d}}\int_{\frac{11}{10}S}\varphi_S(y)\,dy\right)^{\frac{p}{q}}dx \\ & + \sum_{Q\in\mathcal{W}_1}\int_Q\left(\sum_{S\in\mathcal{W}_2\setminus\mathcal{W}_4}\int_{\frac{11}{10}S}\frac{\left|\left(1-\sum_{P\in\mathcal{W}_3}\varphi_P(y)\right)f(x)\right|^q}{\mathrm{D}(Q,S)^{sq+d}}\,dy\right)^{\frac{p}{q}}dx =: \textcircled{a1}+\textcircled{a2}.\end{aligned}$$

In $\textcircled{a1}$ by the choice of the symmetrized cube we have that $\int_{\frac{11}{10}S}\varphi_S(y)\,dy \approx \ell(S^*)^d$. Jensen's inequality implies that $|f(x)-f_{S*}|^q \leqslant \frac{1}{\ell(S^*)^d}\int_{S*}|f(x)-f(\xi)|^q\,d\xi$. By (5.5) and the finite overlapping of the symmetrized cubes, we get that

$$\textcircled{a1} \lesssim \sum_{Q\in\mathcal{W}_1}\int_Q\left(\sum_{S\in\mathcal{W}_3}\int_{S*}\frac{|f(x)-f(\xi)|^q}{\mathrm{D}(Q,S^*)^{sq+d}}\,d\xi\right)^{\frac{p}{q}}dx \lesssim \|f\|^p_{\dot{A}^s_{p,q}(\Omega)}.$$

To bound $\textcircled{a2}$ just note that for $Q\in\mathcal{W}_1$ and $S\in\mathcal{W}_2\setminus\mathcal{W}_4$, we have that $S$ is far from the boundary, say $\ell(S)\geqslant\ell_0$, where $\ell_0$ depends only on $\delta$ and $\varepsilon$. Thus, by (4.2) we have that

$$\textcircled{a2} \lesssim \sum_{Q\in\mathcal{W}_1}\int_Q\left(\sum_{S\in\mathcal{W}_2\setminus\mathcal{W}_4}\int_{\frac{11}{10}S}\frac{|f(x)|^q}{\mathrm{D}(Q,S)^{sq+d}}\,dy\right)^{\frac{p}{q}}dx \lesssim \left(\sum_{S\in\mathcal{W}_2\setminus\mathcal{W}_4}\frac{\ell(S)^d}{\mathrm{D}(\Omega,S)^{sq+d}}\right)^{\frac{p}{q}}\|f\|^p_{L^p}.$$

Recall that Whitney cubes have sidelength equivalent to their distance to $\partial\Omega$. Moreover, the number of cubes of a given side-length bigger than $\ell_0$ is uniformly bounded, so $\sum_{S\in\mathcal{W}_2\setminus\mathcal{W}_4}\frac{\ell(S)^d}{\ell(S)^{sq+d}}$ is a geometric sum, bounded by the maximal element. Therefore,

$$\textcircled{a2}\lesssim\left(\sum_{S\in\mathcal{W}_2\setminus\mathcal{W}_4}\frac{1}{\ell(S)^{sq}}\right)^{\frac{p}{q}}\|f\|_{L^p}^p\leqslant C_{\varepsilon,\delta,\mathrm{diam}(\Omega)}\ell_0^{-sp}\|f\|_{L^p}^p.$$

Next, note that, using the same decomposition as above, we have that

$$\begin{aligned}\textcircled{b}&=\int_{\Omega^c}\left(\int_\Omega\frac{|\sum_{Q\in\mathcal{W}_3}\varphi_Q(x)f_{Q*}-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx\\&\lesssim\sum_{Q\in\mathcal{W}_3}\int_{\frac{11}{10}Q}\varphi_Q(x)\,dx\left(\sum_{S\in\mathcal{W}_1}\int_S\frac{|f_{Q*}-f(y)|^q}{\mathrm{D}(Q,S)^{sq+d}}\,dy\right)^{\frac{p}{q}}\\&\quad+\sum_{Q\in\mathcal{W}_2\setminus\mathcal{W}_4}\int_{\frac{11}{10}Q}(1-\varphi_Q(x))\,dx\left(\sum_{S\in\mathcal{W}_1}\int_S\frac{|f(y)|^q}{\mathrm{D}(Q,S)^{sq+d}}\,dy\right)^{\frac{p}{q}}=:\textcircled{b1}+\textcircled{b2}.\end{aligned}$$

We have that

$$\textcircled{b1}\lesssim\sum_{Q\in\mathcal{W}_3}\ell(Q)^d\left(\sum_{S\in\mathcal{W}_1}\int_S\frac{\left(\frac{1}{\ell(Q)^d}\int_{Q*}|f(\xi)-f(y)|\,d\xi\right)^q}{\mathrm{D}(Q*,S)^{sq+d}}\,dy\right)^{\frac{p}{q}}$$

and, thus, by Minkowsky's integral inequality (see [Ste70, Appendix A1]), we have that

$$\textcircled{b1}\lesssim\sum_{Q\in\mathcal{W}_3}\frac{\ell(Q)^d}{\ell(Q)^{dp}}\left(\int_{Q*}\left(\sum_{S\in\mathcal{W}_1}\int_S\frac{|f(\xi)-f(y)|^q}{|\xi-y|^{sq+d}}\,dy\right)^{\frac{1}{q}}d\xi\right)^p.$$

By Hölder's inequality and the finite overlapping of symmetrized cubes, we get that

$$\textcircled{b1}\lesssim\sum_{Q\in\mathcal{W}_3}\frac{1}{\ell(Q)^{d(p-1)}}\int_{Q*}\left(\int_\Omega\frac{|f(\xi)-f(y)|^q}{|\xi-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}d\xi\,\ell(Q)^{\frac{dp}{p'}}\lesssim\int_\Omega\left(\int_\Omega\frac{|f(\xi)-f(y)|^q}{|\xi-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}d\xi,$$

that is,

$$\textcircled{b1}\lesssim\|f\|^p_{\dot{A}^s_{p,q}(\Omega)}.$$

To bound $\textcircled{b2}$, note that as before,

$$\textcircled{b2}\approx\sum_{Q\in\mathcal{W}_2\setminus\mathcal{W}_4}\ell(Q)^d\left(\sum_{S\in\mathcal{W}_1}\int_S\frac{|f(y)|^q}{\mathrm{D}(Q,\Omega)^{sq+d}}\,dy\right)^{\frac{p}{q}}\lesssim\|f\|^p_{L^q(\Omega)}\sum_{Q\in\mathcal{W}_2\setminus\mathcal{W}_4}\frac{\ell(Q)^d}{\mathrm{dist}(Q,\Omega)^{sp+\frac{dp}{q}}}.$$

Now, since $s>\frac{d}{p}-\frac{d}{q}$ we have that $sp+\frac{dp}{q}>d$. Therefore,

$$\sum_{Q\in\mathcal{W}_2\setminus\mathcal{W}_4}\frac{\ell(Q)^d}{\mathrm{dist}(Q,\Omega)^{sp+\frac{dp}{q}}}\approx\sum_{Q\in\mathcal{W}_2\setminus\mathcal{W}_4}\frac{1}{\ell(Q)^{sp+\frac{dp}{q}-d}}\leqslant C_{\varepsilon,\delta,\mathrm{diam}(\Omega)}\ell_0^{d-sp-\frac{dp}{q}}.$$

On the other hand, $\|f\|_{L^q(\Omega)} \lesssim \|f\|_{A^s_{p,q}(\Omega)}$ by (4.6).

Let us focus on $\textcircled{c}$. We have that

$$\textcircled{c} = \int_{\Omega^c} \left( \int_{\Omega^c} \frac{|\sum_{P\in\mathcal{W}_3} \varphi_P(x) f_{P*} - \sum_{S\in\mathcal{W}_3} \varphi_S(y) f_{S*}|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx.$$

Given $x \in \frac{11}{10}Q$ where $Q \in \mathcal{W}_4$ and $y \in \Omega^c \cap B(x, \frac{\ell_0}{10})$, then neither $x$ nor $y$ are in the support of any bump function of a cube in $\mathcal{W}_2 \backslash \mathcal{W}_3$, so $\sum_{P\in\mathcal{W}_3} \varphi_P(y) \equiv 1$ and $\sum_{P\in\mathcal{W}_3} \varphi_P(x) \equiv 1$. Therefore

$$\sum_{P\in\mathcal{W}_3} \varphi_P(x) f_{P*} - \sum_{S\in\mathcal{W}_3} \varphi_S(y) f_{S*} = \sum_{P\cap 2Q\neq\varnothing} \sum_{S\in\mathcal{W}_3} \varphi_P(x)\varphi_S(y) \left(f_{P*} - f_{S*}\right).$$

If, moreover, $y \in B\left(x, \frac{1}{10}\ell(Q)\right)$, since the points are 'close' to each other, we will use the Hölder regularity of the bump functions, so we write

$$\sum_{P\in\mathcal{W}_3} \varphi_P(x) f_{P*} - \sum_{S\in\mathcal{W}_3} \varphi_S(y) f_{S*} = \sum_{P\in\mathcal{W}_3} \left(\varphi_P(x) - \varphi_P(y)\right) f_{P*}.$$

This decomposition is still valid if $Q \in \mathcal{W}_2 \backslash \mathcal{W}_4$ and $y \in B\left(x, \frac{1}{10}\ell(Q)\right)$, that is, $y \in B\left(x, \frac{\ell_0}{10}\right)$, but we will treat this case apart since we lose the cancellation of the sums of bump functions but we gain a uniform lower bound on the side-lengths of the cubes involved. Finally, we will group the remaining cases, when $x \in \Omega^c$ and $y \notin B(x, \frac{\ell_0}{10})$ in an error term. Considering all these facts we get

$$\begin{aligned}
\textcircled{c} &\lesssim \sum_{Q\in\mathcal{W}_4} \int_Q \left( \int_{\Omega^c \backslash B\left(x, \frac{1}{10}\ell(Q)\right)} \sum_{P\cap 2Q\neq\varnothing} \sum_{S\in\mathcal{W}_3} |\varphi_P(x)\varphi_S(y)| \frac{|f_{P*} - f_{S*}|^q}{\mathrm{D}(P^*, S^*)^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \\
&\quad + \sum_{Q\in\mathcal{W}_4} \int_Q \left( \int_{B\left(x, \frac{1}{10}\ell(Q)\right)} \frac{|\sum_{S\cap 2Q\neq\varnothing} \left(\varphi_S(x) - \varphi_S(y)\right) f_{S*}|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \\
&\quad + \sum_{Q\in\mathcal{W}_2\backslash\mathcal{W}_4} \int_Q \left( \int_{B\left(x, \frac{\ell_0}{10}\right)} \frac{|\sum_{S\in\mathcal{W}_3 : S\cap 2Q\neq\varnothing} \left(\varphi_S(x) - \varphi_S(y)\right) f_{S*}|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \\
&\quad + \int_{\Omega^c} \left( \int_{\Omega^c \backslash B\left(x, \frac{\ell_0}{10}\right)} \frac{|\Lambda_0 f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \\
&=: \textcircled{c1} + \textcircled{c2} + \textcircled{c3} + \textcircled{c4}.
\end{aligned}$$

Using the same arguments as in $\textcircled{a1}$ and $\textcircled{b1}$ we have that

$$\textcircled{c1} \lesssim \|f\|^p_{\dot{A}^s_{p,q}(\Omega)}.$$

Also combining the arguments used to bound $\textcircled{a2}$ and $\textcircled{b2}$ we get

$$\textcircled{c4} \lesssim \left( \|f\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} \right)^p.$$

The novelty comes from the fact that we are integrating in $\Omega^c$ both terms in $\textcircled{c}$, so the variables in the integrals $\textcircled{c2}$ and $\textcircled{c3}$ can get as close as one can imagine. Here we need to use the smoothness of the bump functions, but also the smoothness of $f$ itself. The trick is to use that $\{\varphi_Q\}$ is a partition of the unity with $\varphi_Q$ supported in $\frac{11}{10}Q$, that is, $\sum_{S\in\mathcal{W}_3}\varphi_S(x) = \sum_{S\cap 2Q\neq\varnothing}\varphi_S(x) = 1$ if $x \in \frac{11}{10}Q$ with $Q \in \mathcal{W}_4$. Thus,

$$\textcircled{c2} = \sum_{Q\in\mathcal{W}_4}\int_Q\left(\int_{B\left(x,\frac{1}{10}\ell(Q)\right)}\frac{|\sum_{S\cap 2Q\neq\varnothing}(\varphi_S(x)-\varphi_S(y))\left(f_{S^*}-f_{Q^*}\right)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx,$$

and using the fact that $\|\nabla\varphi_Q\|_\infty \lesssim \frac{1}{\ell(Q)}$ and (4.3), we have that

$$\begin{aligned}\textcircled{c2} &\lesssim_q \sum_{Q\in\mathcal{W}_4}\int_Q\left(\sum_{S\cap 2Q\neq\varnothing}\left|f_{S^*}-f_{Q^*}\right|^q\int_{B\left(x,\frac{1}{10}\ell(Q)\right)}\frac{|x-y|^q}{\ell(Q)^q}\frac{1}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx\\ &\lesssim_s \sum_{Q\in\mathcal{W}_4}\ell(Q)^d\left(\frac{\sum_{S\cap 2Q\neq\varnothing}\left|f_{S^*}-f_{Q^*}\right|^q}{\ell(Q)^{sq}}\right)^{\frac{p}{q}} \approx \sum_{Q\in\mathcal{W}_4}\ell(Q)^d\left(\sum_{S\cap 2Q\neq\varnothing}\frac{\left|f_{S^*}-f_{Q^*}\right|^q}{\mathrm{D}(Q^*,S^*)^{sq}}\right)^{\frac{p}{q}},\end{aligned}$$

which can be bounded as $\textcircled{c1}$.

Finally, we bound the error term $\textcircled{c3}$. Here we cannot use the cancellation of the partition of the unity anymore. Instead, we will use the $L^p$ norm of $f$, the Hölder regularity of the bump functions and the fact that all the cubes considered are roughly of the same size:

$$\begin{aligned}\textcircled{c3} &= \sum_{Q\in\mathcal{W}_2\setminus\mathcal{W}_4}\int_Q\left(\int_{B\left(x,\frac{\ell_0}{10}\right)}\frac{|\sum_{S\cap 2Q\neq\varnothing}(\varphi_S(x)-\varphi_S(y))\,f_{S^*}|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx\\ &\lesssim \sum_{\substack{Q\in\mathcal{W}_2\\ \ell_0\leqslant\ell(Q)\leqslant 2\ell_0}}\int_Q\sum_{\substack{S\in\mathcal{W}_3\\ S\cap 2Q\neq\varnothing}}|f_{S^*}|^p\left(\int_{B\left(x,\frac{\ell_0}{10}\right)}\frac{1}{\ell_0^q}\frac{1}{|x-y|^{(s-1)q+d}}\,dy\right)^{\frac{p}{q}}dx\\ &\lesssim_{\varepsilon,\delta,q,p} \sum_{\substack{S\in\mathcal{W}_3\\ \frac{\ell_0}{2}\leqslant\ell(S)\leqslant\ell_0}}\|f\|^p_{L^p(S^*)} \lesssim C\|f\|^p_{L^p(\Omega)}.\end{aligned}$$

The finite number of cubes in the sum allows us to deduce the last inequality.

□

**Corollary 5.9.** *Let $\Omega$ be a bounded uniform domain with an admissible Whitney covering $\mathcal{W}$. Given $1<p<\infty$, $1<q<\infty$ and $0<s<1$ with $s>\frac{d}{p}-\frac{d}{q}$, we have that $A^s_{p,q}(\Omega)=F^s_{p,q}(\Omega)$, and*

$$\|f\|_{F^s_{p,q}(\Omega)} \approx \|f\|_{A^s_{p,q}(\Omega)} \qquad \text{for all } f\in F^s_{p,q}(\Omega).$$

*Proof.* By Corollary 5.5, given $f\in F^s_{p,q}(\Omega)$ we have that

$$\|f\|_{A^s_{p,q}(\Omega)} \leqslant \inf_{g\in L^{\max\{p,q\}}:g|_\Omega\equiv f}\|g\|_{A^s_{p,q}(\mathbb{R}^d)} \approx \inf_{g:g|_\Omega\equiv f}\|g\|_{F^s_{p,q}(\mathbb{R}^d)} = \|f\|_{F^s_{p,q}(\Omega)}.$$

By the Lemma 5.8 we have the converse. Given $f\in A^s_{p,q}(\Omega)$ we have that

$$\|f\|_{F^s_{p,q}(\Omega)} = \inf_{g:g|_\Omega\equiv f}\|g\|_{F^s_{p,q}(\mathbb{R}^d)} \leqslant \|\Lambda_0 f\|_{F^s_{p,q}(\mathbb{R}^d)} \approx \|\Lambda_0 f\|_{A^s_{p,q}(\mathbb{R}^d)} \leqslant C\|f\|_{A^s_{p,q}(\Omega)}.$$

□

# 6 Calderón-Zygmund operators

We will make use of the notion of distributional convolution. Given Schwartz functions $f$ and $g$, the convolution coincide with multiplication at the Fourier side, that is, $f * g(x) = (\widehat{f} \cdot \widehat{g})\check{}$. Given a tempered distribution $W$, a function $f \in \mathcal{S}$ and $x \in \mathbb{R}^d$, the tempered distribution $W * f$ is defined as

$$\langle W * f, g \rangle = \langle \widehat{W}, \widehat{f} \cdot \check{g} \rangle = \langle W, f_- * g \rangle \qquad \text{for every } g \in \mathcal{S},$$

where $f_-(x) = f(-x)$. Note that $f_- * g(x) = \int f(-y)g(x-y)\,dy$, so in case $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$ then $f_- * g \equiv 0$ in a neghborhood of 0. The distribution $W * f$ coincides with the $C^\infty$ function $W * f(x) = \langle W, \tau_x f_- \rangle$, where $\tau_x f_-(y) = f_-(y - x)$ (see [SW71, Chapter I, Theorem 3.13]).

**Definition 6.1.** *We say that a measurable function $K \in L^1_{loc}(\mathbb{R}^d \backslash \{0\})$ is an* admissible convolution Calderón-Zygmund kernel of order $\sigma \leqslant 1$ *if*

$$|K(x)| \leqslant \frac{C_K}{|x|^d} \quad \text{for } x \neq 0, \tag{6.1}$$

$$|K(x-y) - K(x)| \leqslant \frac{C_K |y|^\sigma}{|x|^{d+\sigma}} \quad \text{for } 0 < 2|y| \leqslant |x|, \tag{6.2}$$

*for a positive constant $C_K$ and that kernel can be extended to a convolution with a tempered distribution $W_K$ in $\mathbb{R}^d$ in the sense that for every Schwartz functions $f, g \in \mathcal{S}$ with* $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$, *one has*

$$\langle W_K * f, g \rangle = \int_{\mathbb{R}^d \backslash \{0\}} K(x)\,(f_- * g)\,(x)\,dx. \tag{6.3}$$

There are some cancellation conditions that one can impose to a kernel satisfying the size condition (6.1) to grant that it can be extended to a convolution with a tempered distribution. For instance, if $K$ satisfies (6.1) and $W_K$ is a principal value operator in the sense that

$$\langle W_K, \varphi \rangle = \lim_{j \to \infty} \int_{|x| \geqslant \delta_j} K(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in \mathcal{S}, \tag{6.4}$$

for a certain sequence $\delta_j \searrow 0$, then $W_K$ satisfies (6.3) (see [Gra08, Section 4.3.2]).

**Definition 6.2.** *Let $1 < p, q < \infty$. We say that an operator $T : \mathcal{S} \to \mathcal{S}'$ is a* $p, q$-admissible convolution Calderón-Zygmund operator of order $\sigma \in (0, 1]$ *with kernel $K$ if*

1. *$K$ is an admissible convolution Calderón-Zygmund kernel of order $\sigma$ which can be extended to a convolution with a tempered distribution $W_K$,*

2. *$T$ satisfies that $Tf = W_K * f$ for all $f \in \mathcal{S}$ and*

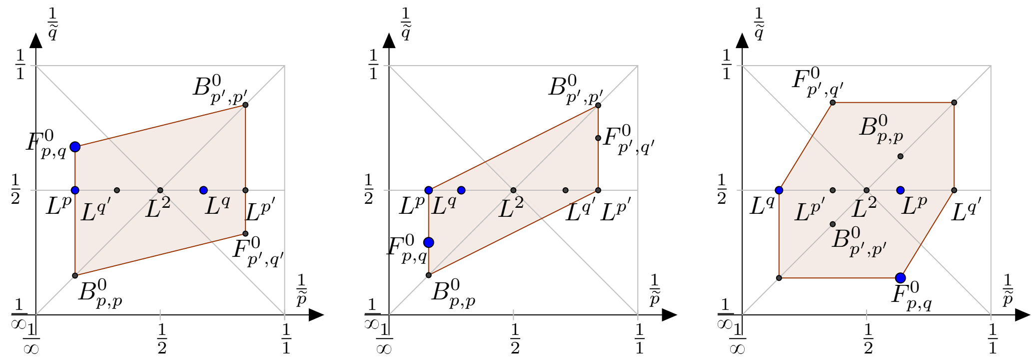
3. *$T$ extends to an operator bounded in $F^0_{p,q}$, in $L^p$ and in $L^q$.*

**Remark 6.3.** *The Fourier transform of a $p, q$-admissible convolution Calderón-Zygmund operator $T$ is a Fourier multiplier for $F^0_{p,q}$, $L^p$ and $L^q$, following the notation in [Tri83, Section 2.6]. This Section also contains some results on Fourier multipliers that we sum up now. Being a Fourier multiplier for $F^0_{p,q}$ implies being a Fourier multiplier also for $F^s_{p,q}$ for every $s$, also for $F^0_{p,p}$ and for $F^0_{p',q'}$, and the property is stable under interpolation (i.e., the set of indices $\frac{1}{p}$, $\frac{1}{q}$ such that a $T$ is bounded in $F^0_{p,q}$ is a convex set, see Figure 6.1). In particular it is a Fourier multiplier for $L^2$ and this implies that $\widehat{W_K} \in L^\infty$ (see [SW71, Chapter I, Theorem 3.18]).*

*Therefore, condition 3) in Definition 6.2, can be reduced to asking*

- *that $T$ extends to an operator bounded in $F^0_{p,q}$ when $\infty > p \geqslant q' \geqslant 2$ or $\infty > p' \geqslant q \geqslant 2$,*
- *that $T$ extends to an operator bounded in $L^p$ when $\infty > p \geqslant q \geqslant 2$ or $\infty > p' \geqslant q' \geqslant 2$ and*
- *that $T$ extends to an operator bounded in $F^0_{p,q}$ and in $L^q$ when $\infty > q \geqslant p \geqslant q'$ or $\infty > q' \geqslant p \geqslant q$.*

Figure 6.1: Indices $\frac{1}{\tilde{p}}$, $\frac{1}{\tilde{q}}$ such that a $p,q$-admissible operator is bounded in $F^s_{\tilde{p},\tilde{q}}$.




(a) The case $\infty > p \geqslant q' \geqslant 2$. (b) The case $\infty > p \geqslant q \geqslant 2$. (c) The case $\infty > q \geqslant p \geqslant q'$.

**Remark 6.4.** *If $Tf = W_K * f$ for an admissible convolution Calderón-Zygmund kernel $K$ of order $\sigma$ which can be extended to the convolution with tempered distribution $W_K$ satisfying (6.4), then $\widehat{W_K} \in L^\infty$ implies that $T$ extends to an operator bounded in $L^p$ for every $1 < p < \infty$ (see [Gra08, Theorem 4.3.3]).*

It is a well-known fact that the Schwartz class is dense in $F^s_{p,q}$ for $p < \infty$, $1 \leqslant q \leqslant \infty$. Thus, if $f \in L^p$ and $x \notin \operatorname{supp}(f)$, then

$$Tf(x) = \int K(x-y)f(y)dy. \tag{6.5}$$

To prove Theorem 1.1 we need the following lemma which says that it is equivalent to bound the transform of a function and its approximation by constants on Whitney cubes.

To do so, we define the fractional derivative,

**Definition 6.5.** *Given a uniform domain $\Omega$ and $f \in L^p(\Omega)$ for certain values $0 < s < 1$ and $1 < q < \infty$, the $s$-th fractional gradient of index $q$ of $f$ in a point $x \in Q \in \mathcal{W}$ is*

$$\nabla^s_q f(x) := \left( \int_{\mathbf{Sh}(Q)} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{1}{q}}.$$

*Then, by Corollary 5.9 and Lemma 4.4, for $1 < p < \infty$ with $\frac{d}{p} - \frac{d}{q} < s$, we have that*

$$\|f\|_{F^s_{p,q}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left\|\nabla^s_q f\right\|_{L^p(\Omega)}. \tag{6.6}$$

**Key Lemma 6.6.** *Let $\Omega$ be a uniform domain with Whitney covering $\mathcal{W}$, let $T$ a $p,q$-admissible convolution Calderón-Zygmund operator of order $0 < \sigma < 1$, $1 < p < \infty$, $1 < q < \infty$ and $0 < s \leqslant \sigma$ with $s > \frac{d}{p} - \frac{d}{q}$. Then*

$$\sum_{Q \in \mathcal{W}} \int_Q |\nabla^s_q T_\Omega (f - f_Q)(x)|^p \, dx \leqslant C \|f\|^p_{F^s_{p,q}(\Omega)}. \tag{6.7}$$

*Proof.* Again we use duality. That is, to prove (6.7) it suffices to prove that given a function $g \in L^{p'}(L^{q'}(\Omega))$ with $\|g\|_{L^{p'}(L^{q'}(\Omega))} = 1$, we have that

$$\sum_Q \int_Q \int_{\mathbf{Sh}(Q)} \frac{|T_\Omega (f - f_Q)(x) - T_\Omega (f - f_Q)(y)|}{|x - y|^{s + \frac{d}{q}}} g(x, y)\, dy\, dx \lesssim \|f\|_{F^s_{p,q}(\Omega)}.$$

Given a cube $Q \in \mathcal{W}$, we can define a bump funcion $\varphi_Q$ such that $\chi_{6Q} \leqslant \varphi_Q \leqslant \chi_{7Q}$ and $\|\nabla \varphi_Q\|_{L^\infty} \leqslant C\ell(Q)^{-1}$. Given a cube $S \subset 5Q$ we define $\varphi_{QS} := \varphi_Q$. Otherwise, take $\varphi_{QS} := \varphi_S$. Note that in both situations, by (3.1) we have that $\mathrm{supp}\varphi_{QS} \subset 23S$. Then, we can express the difference between $T_\Omega(f - f_Q)$ evaluated at $x \in Q$ and in $y \in S$ as

$$\begin{aligned} T_\Omega(f - f_Q)(x) - T_\Omega(f - f_Q)(y) &= T_\Omega [(f - f_Q)\varphi_Q](x) - T_\Omega [(f - f_Q)\varphi_{QS}](y) \\ &\quad + T_\Omega [(f - f_Q)(1 - \varphi_Q)](x) - T_\Omega [(f - f_Q)(1 - \varphi_{QS})](y). \end{aligned} \tag{6.8}$$

Note that the first two terms in the right-hand side of (6.8) are 'local' terms in the sense that the functions to which we apply the operator $T_\Omega$ are supported in a small neighborhood of the point of evaluation (and are globally $F^s_{p,q}$) and the other two terms are 'non-local'. What we will prove is that the non-local part

$$\boxed{1} := \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T_\Omega [(f - f_Q)(1 - \varphi_Q)](x) - T_\Omega [(f - f_Q)(1 - \varphi_{QS})](y)|}{|x - y|^{s + \frac{d}{q}}} g(x, y)\, dy\, dx$$

and the local part

$$\boxed{2} := \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T_\Omega [(f - f_Q)\varphi_Q](x) - T_\Omega [(f - f_Q)\varphi_{QS}](y)|}{|x - y|^{s + \frac{d}{q}}} g(x, y)\, dy\, dx$$

are both bounded as

$$\boxed{1} + \boxed{2} \leqslant C\|f\|_{F^s_{p,q}(\Omega)}. \tag{6.9}$$

We begin by the non-local part. Consider $x \in Q \in \mathcal{W}$. By (6.5), since $x$ is not in the support of $(f - f_Q)(1 - \varphi_Q)$, we have that

$$T_\Omega [(f - f_Q)(1 - \varphi_Q)](x) = \int_\Omega K(x - z)(f(z) - f_Q)(1 - \varphi_Q(z))\, dm(z)$$

and by the same token for $y \in S \in \mathbf{SH}(Q)$

$$T_\Omega [(f - f_Q)(1 - \varphi_{QS})](y) = \int_\Omega K(y - z)(f(z) - f_Q)(1 - \varphi_{QS}(z))\, dm(z).$$

To shorten the notation, we will write

$$\lambda_{QS}(z_1, z_2) = K(z_1 - z_2)(f(z_2) - f_Q)(1 - \varphi_{QS}(z_2)),$$

for $z_1 \neq z_2$. Then we have that

$$\Big|T_\Omega [(f - f_Q)(1 - \varphi_Q)](x) - T_\Omega [(f - f_Q)(1 - \varphi_{QS})](y)\Big| = \left|\int_\Omega (\lambda_{QQ}(x, z) - \lambda_{QS}(y, z))\, dm(z)\right|,$$

that is,

$$\boxed{1} = \sum_Q \int_Q \sum_{S \subset \mathbf{Sh}(Q)} \int_S \frac{\left|\int_\Omega (\lambda_{QQ}(x, z) - \lambda_{QS}(y, z))\, dz\right|}{|x - y|^{s + \frac{d}{q}}} g(x, y)\, dy\, dx.$$

For $\rho_2$ big enough, $\mathbf{Sh}^2(Q) := \mathbf{Sh}_{\rho_2}(Q) \supset \bigcup_{S\in\mathbf{SH}(Q)} \mathbf{Sh}(S)$ (call $\mathbf{SH}^2(Q) := \mathbf{SH}_{\rho_2}(Q)$), we can decompose

$$\boxed{\boxed{1}} \leqslant \sum_Q \int_Q \sum_{S\subset \mathbf{Sh}(Q)\setminus 2Q} \int_S \frac{\int_{\mathbf{Sh}^2(Q)} |\lambda_{QQ}(x,z)-\lambda_{QS}(y,z)|\,dz}{|x-y|^{s+\frac{d}{q}}} g(x,y)\,dy\,dx \tag{6.10}$$
$$+ \sum_Q \int_Q \sum_{S\subset \mathbf{Sh}(Q)\setminus 2Q} \int_S \frac{\int_{\Omega\setminus\mathbf{Sh}^2(Q)} |\lambda_{QQ}(x,z)-\lambda_{QS}(y,z)|\,dz}{|x-y|^{s+\frac{d}{q}}} g(x,y)\,dy\,dx$$
$$+ \sum_Q \int_Q \int_{5Q} \frac{\int_{\Omega} |\lambda_{QQ}(x,z)-\lambda_{QQ}(y,z)|\,dz}{|x-y|^{s+\frac{d}{q}}} g(x,y)\,dy\,dx =: \boxed{\boxed{A}} + \boxed{\boxed{B}} + \boxed{\boxed{C}}.$$

In the first term in the right-hand side of (6.10) the variable $z$ is 'close' to either $x$ or $y$, so smoothness does not help. Thus, we will take absolute values, giving rise to two terms separating $\lambda_{QQ}$ and $\lambda_{QS}$. That is, we use that

$$\boxed{\boxed{A}} \leqslant \sum_Q \int_Q \sum_{S\subset \mathbf{Sh}(Q)\setminus 2Q} \int_S \frac{\int_{\mathbf{Sh}^2(Q)} \left(|\lambda_{QQ}(x,z)|+|\lambda_{QS}(y,z)|\right)\,dz}{|x-y|^{s+\frac{d}{q}}} g(x,y)\,dy\,dx.$$

Using (6.1),

$$|\lambda_{QQ}(x,z)| \leqslant C_K \frac{|f(z)-f_Q|}{|x-z|^d} |1-\varphi_Q(z)|$$

and

$$|\lambda_{QS}(y,z)| \leqslant C_K \frac{|f(z)-f_Q|}{|y-z|^d} |1-\varphi_{QS}(z)|.$$

Summing up,

$$\boxed{\boxed{A}} \lesssim_{C_K} \sum_Q \int_Q \int_{\mathbf{Sh}(Q)\setminus 2Q} \int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f_Q|\,|1-\varphi_Q(z)|\,dz}{|x-y|^{s+\frac{d}{q}}|x-z|^d} g(x,y)\,dy\,dx \tag{6.11}$$
$$+ \sum_Q \int_Q \int_{\mathbf{Sh}(Q)\setminus 2Q} \int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f_Q|\,|1-\varphi_{QS}(z)|\,dz}{|x-y|^{s+\frac{d}{q}}|y-z|^d} g(x,y)\,dy\,dx =: \boxed{\boxed{1.1}} + \boxed{\boxed{1.2}},$$

with constants depending linearly on the Calderón-Zygmund constant $C_K$ in (6.1).

We begin by the shorter part, that is

$$\boxed{\boxed{1.1}} = \sum_Q \int_Q \int_{\mathbf{Sh}(Q)\setminus 2Q} \int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f_Q|\,|1-\varphi_Q(z)|\,dz}{|x-y|^{s+\frac{d}{q}}|x-z|^d} g(x,y)\,dy\,dx.$$

Using the fact that $1-\varphi_Q(z)=0$ when $z$ is close to the cube $Q$, we can say that

$$\boxed{\boxed{1.1}} \lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}+d}} \int_{\mathbf{Sh}^2(Q)\setminus 6Q} |f(z)-f_Q| \int_Q \int_{\mathbf{Sh}(Q)\setminus 2Q} g(x,y)\,dy\,dx\,dz.$$

Now, by the Hölder inequality we have that

$$\int_{\mathbf{Sh}(Q)\setminus 2Q} g(x,y)\,dy \leqslant \left(\int_{\mathbf{Sh}(Q)\setminus 2Q} g(x,y)^{q'}\,dy\right)^{\frac{1}{q'}} |\mathbf{Sh}(Q)|^{\frac{1}{q}} \lesssim_{\rho_\varepsilon,d} G(x)\ell(Q)^{\frac{d}{q}}$$

where $G(x) = \|g(x,\cdot)\|_{L^{q'}}$. Thus,

$$\boxed{\boxed{1.1}} \lesssim \sum_Q \int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f_Q|}{\ell(Q)^{s+d}} \int_Q G(x)\,dx\,dz \lesssim \sum_Q \int_Q \int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f(\xi)|}{\ell(Q)^{s+d}} MG(\xi)\,dz\,d\xi.$$

Finally, by Jensen's inequality and the boundedness of the maximal operator in $L^{p'}$ we have that

$$\begin{aligned}\sum_Q \int_Q \int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f(\xi)|}{\ell(Q)^{s+d}} MG(\xi)\,dz\,d\xi &\lesssim \sum_Q \int_Q \left(\int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f(\xi)|^q}{\ell(Q)^{sq+d}}\,dz\right)^{\frac{1}{q}} MG(\xi)\,d\xi \qquad (6.12)\\ &\lesssim \left(\int_\Omega \left(\int_\Omega \frac{|f(z)-f(\xi)|^q}{|z-\xi|^{sq+d}}\,dz\right)^{\frac{p}{q}} d\xi\right)^{\frac{1}{p}} \|MG\|_{L^{p'}},\end{aligned}$$

that is,

$$\boxed{\boxed{1.1}} \lesssim \|f\|_{F^s_{p,q}(\Omega)}. \qquad (6.13)$$

The second term in (6.11) is the most delicate one. Take cubes $Q$, $S$ and $P$ and points $y \in S$ and $z \in P$ with $1-\varphi_{QS}(z) \neq 0$. Then $|z-y| \approx \mathrm{D}(S,P)$ and, therefore, we can write

$$\begin{aligned}\boxed{\boxed{1.2}} &= \sum_Q \int_Q \int_{\mathbf{Sh}(Q)\setminus 2Q} \int_{\mathbf{Sh}^2(Q)} \frac{|f(z)-f_Q|\,|1-\varphi_{QS}(z)|\,dz}{|x-y|^{s+\frac{d}{q}}|y-z|^d} g(x,y)\,dy\,dx \qquad (6.14)\\ &\lesssim_d \sum_Q \int_Q \sum_{S\in\mathbf{SH}(Q)} \int_S \sum_{P\in\mathbf{SH}^2(Q):\mathrm{D}(P,S)\geqslant\delta} \frac{1}{\delta^d} \int_P \frac{|f(z)-f_Q|\;dz}{\ell(Q)^{s+\frac{d}{q}}} g(x,y)\,dy\,dx\\ &\quad+ \sum_Q \int_Q \sum_{S\in\mathbf{SH}(Q)} \int_S \sum_{P\in\mathbf{SH}^2(Q):\mathrm{D}(P,S)<\delta} \int_P \frac{|f(z)-f_Q|\;dz}{\ell(Q)^{s+\frac{d}{q}}\mathrm{D}(S,P)^d} g(x,y)\,dy\,dx =: \boxed{\boxed{1.2.1}} + \boxed{\boxed{1.2.2}}\end{aligned}$$

To bound $\boxed{\boxed{1.2.1}}$, note that given $Q$, $S \in \mathbf{SH}(Q)$ and $P \in \mathbf{SH}^2(Q)$ with $\mathrm{D}(P,S) \geqslant \delta$, then also $\ell(Q) \geqslant C_\delta$. Moreover, $\int_{\mathbf{Sh}(Q)} g(x,y)\,dy \lesssim_{\varepsilon,d,q} G(x)\ell(Q)^{\frac{d}{q}}$ by the Hölder inequality, so

$$\begin{aligned}\boxed{\boxed{1.2.1}} &\lesssim \sum_Q \int_Q \sum_{P\in\mathbf{SH}^2(Q)} \int_P |f(z)-f_Q|\,G(x)\,dz\,dx \leqslant \sum_Q \int_Q \int_{\mathbf{Sh}^2(Q)} |f(z)-f(\xi)|\,MG(\xi)\,dz\,d\xi\\ &\lesssim \|f\|_{L^p(\Omega)} \lesssim \|f\|_{F^s_{p,q}(\Omega)}. \qquad (6.15)\end{aligned}$$

To bound $\boxed{\boxed{1.2.2}}$, we change the focus on the sum. Consider an admissible chain connecting two given cubes $S$ and $P$ both in $\mathbf{SH}^2(Q)$. Then $\mathrm{D}(S,P) \approx \ell(S_P)$. Of course, since $S$ and $P$ are in $\mathbf{SH}^2(Q)$ we have that

$$\mathrm{D}(Q,S_P) \lesssim \mathrm{D}(Q,S) + \mathrm{D}(S,S_P) \approx \mathrm{D}(Q,S) + \mathrm{D}(S,P) \lesssim 2\mathrm{D}(Q,S) + \mathrm{D}(Q,P) \lesssim \ell(Q)$$

and, therefore, it is contained in some $\mathbf{SH}_{\rho_3}(Q)$ for a certain constant $\rho_3$ depending on $d$ and $\varepsilon$. For short, we write $L := S_P \in \mathbf{SH}^3(Q)$ and $\mathbf{Sh}^3(Q) := \mathbf{Sh}_{\rho_3}(Q)$. Then

$$\begin{aligned}\boxed{\boxed{1.2.2}} &\lesssim \sum_Q \int_Q \sum_{L\in\mathbf{SH}^3(Q)} \sum_{S\in\mathbf{SH}(L)} \int_S \sum_{P\in\mathbf{SH}(L)} \int_P \frac{|f(z)-f_Q|\;dz}{\ell(Q)^{s+\frac{d}{q}}\ell(L)^d} g(x,y)\,dy\,dx\\ &\leqslant \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \sum_{L\in\mathbf{SH}^3(Q)} \int_{\mathbf{Sh}(L)} |f(z)-f_Q|\;dz \int_Q \frac{1}{\ell(L)^d} \int_{\mathbf{Sh}(L)} g(x,y)\,dy\,dx. \qquad (6.16)\end{aligned}$$

If we write $g_x(y) = g(x,y)$, we have that for any cube $L$ the integral

$$\int_{\mathbf{Sh}(L)} g(x,y)\,dy \leqslant \ell(L)^d \inf_L Mg_x.$$

By the same token, for $\rho_4$ big enough, we have that $\mathbf{Sh}(L) \subset \mathbf{Sh}_{\rho_4}(Q) =: \mathbf{Sh}^4(Q)$ and therefore

$$\int_{\mathbf{Sh}(L)} |f(z) - f_Q|\,dz = \int_{\mathbf{Sh}(L)} |f(z) - f_Q|\,\chi_{\mathbf{Sh}^4(Q)}(z)\,dz \leqslant \int_L M[(f - f_Q)\chi_{\mathbf{Sh}^4(Q)}](\xi)\,d\xi.$$

Thus, by (6.16) we have that

$$\boxed{\boxed{1.2.2}} \lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \sum_{L\in\mathbf{SH}^3(Q)} \int_Q \int_L M[(f - f_Q)\chi_{\mathbf{Sh}^4(Q)}](\xi) Mg_x(\xi)\,d\xi\,dx$$
$$= \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \int_{\mathbf{Sh}^3(Q)} M[(f - f_Q)\chi_{\mathbf{Sh}^4(Q)}](\xi) Mg_x(\xi)\,d\xi\,dx$$

and, by Hölder's inequality and the boundedness of the maximal operator in $L^q$ and $L^{q'}$, we have that

$$\boxed{\boxed{1.2.2}} \lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \left(\int_{\mathbf{Sh}^3(Q)} M[(f - f_Q)\chi_{\mathbf{Sh}^4(Q)}](\xi)^q\,d\xi\right)^{\frac{1}{q}} \left(\int_{\mathbf{Sh}^3(Q)} Mg_x(\xi)^{q'}\,d\xi\right)^{\frac{1}{q'}} dx$$
$$\lesssim_q \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \left(\int_{\mathbf{Sh}^4(Q)} |f(\xi) - f_Q|^q\,d\xi\right)^{\frac{1}{q}} \left(\int_\Omega g(x,\xi)^{q'}\,d\xi\right)^{\frac{1}{q'}} dx.$$

Again, we write $G(x) = \|g(x,\cdot)\|_{L^{q'}}$ and by Minkowski's integral inequality (see [Ste70, Appendix A1]) we get that

$$\boxed{\boxed{1.2.2}} \lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}+d}} \left(\int_{\mathbf{Sh}^4(Q)} \left(\int_Q |f(\xi) - f(\zeta)|\,d\zeta\right)^q d\xi\right)^{\frac{1}{q}} \int_Q G(x)\,dx$$
$$\lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \left(\int_{\mathbf{Sh}^4(Q)} |f(\xi) - f(\zeta)|^q\,d\xi\right)^{\frac{1}{q}} MG(\zeta)\,d\zeta.$$

Thus,

$$\boxed{\boxed{1.2.2}} \lesssim \left(\int_\Omega \left(\int_\Omega \frac{|f(\xi) - f(\zeta)|^q}{|\xi - \zeta|^{sq+d}}\,d\xi\right)^{\frac{p}{q}} d\zeta\right)^{\frac{1}{p}} \|MG\|_{L^{p'}} \lesssim \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.17}$$

Back to (6.10), it remains to bound $\boxed{\boxed{B}}$ and $\boxed{\boxed{C}}$. For the first one,

$$\boxed{\boxed{B}} = \sum_Q \int_Q \sum_{S\subset\mathbf{Sh}(Q)\setminus 2Q} \int_S \frac{\int_{\Omega\setminus\mathbf{Sh}^2(Q)} |\lambda_{QQ}(x,z) - \lambda_{QS}(y,z)|\,dz}{|x-y|^{s+\frac{d}{q}}} g(x,y)\,dy\,dx,$$

just note that if $x \in Q$, $y \in S \subset \mathbf{Sh}(Q)$ and $z \notin \mathbf{Sh}^2(Q)$ we have that $\varphi_{QQ}(z) = \varphi_{QS}(z) = 0$ and, if $\rho_2$ is big enough, $|x - z| > 2|x - y|$. Thus, we can use the smoothness assumption, that is, $|\lambda_{QQ}(x,z) - \lambda_{QS}(y,z)| \leqslant |K(x - z) - K(y - z)|\,|f(z) - f_Q| \leqslant C_K \frac{|f(z)-f_Q||x-y|^\sigma}{|x-z|^{d+\sigma}}$ by (6.2).

In the last term in (6.10),

$$\boxed{\boxed{\mathrm{C}}} = \sum_Q \int_Q \int_{5Q} \frac{\int_\Omega |\lambda_{QQ}(x,z) - \lambda_{QQ}(y,z)|\, dz}{|x-y|^{s+\frac{d}{q}}} g(x,y)\, dy\, dx$$

we are integrating in the region where $x \in Q$, $y \in 5Q$ and $z \notin 6Q$ because otherwise $1 - \varphi_Q(z)$ would vanish. Also $|x-z| > C_d|x-y|$ and $|x-z| \approx |y-z|$. Thus, we have again that $|\lambda_{QQ}(x,z) - \lambda_{QQ}(y,z)| \leqslant |K(x-z) - K(y-z)|\,|f(z) - f_Q| \lesssim C_K \frac{|f(z)-f_Q||x-y|^\sigma}{|x-z|^{d+\sigma}}$ by (6.2) and (6.1) (one may use the last one when $2|x-y| \geqslant |x-z| > C_d|x-y|$, that is $|x-y| \approx |x-z| \approx |y-z|$).

Summing up,

$$\boxed{\boxed{\mathrm{B}}} + \boxed{\boxed{\mathrm{C}}} \lesssim_{C_K} \sum_Q \int_Q \int_{\mathbf{Sh}(Q)} \int_{\Omega \setminus 6Q} \frac{|f(z) - f_Q||x-y|^\sigma\, dz}{|x-y|^{s+\frac{d}{q}}|x-z|^{d+\sigma}} g(x,y)\, dy\, dx =: \boxed{\boxed{1.3}}. \tag{6.18}$$

with constants depending linearly on the Calderón-Zygmund constant $C_K$. Reordering,

$$\boxed{\boxed{1.3}} = \sum_Q \int_Q \int_{\Omega \setminus 6Q} \frac{|f(z) - f_Q|\, dz}{|x-z|^{d+\sigma}} \int_{\mathbf{Sh}(Q)} \frac{g(x,y)\, dy}{|x-y|^{s-\sigma+\frac{d}{q}}}\, dx.$$

The last integral above is easy to bound by the same techniques as before: Given $x \in Q \in \mathcal{W}$, since $s - \sigma + \frac{d}{q} < d$, by (4.3), Hölder's Inequality and the boundedness of the maximal operator in $L^{q'}$ we have that

$$\begin{aligned}\int_{\mathbf{Sh}(Q)} \frac{g(x,y)\, dy}{|x-y|^{s-\sigma+\frac{d}{q}}} &\lesssim \ell(Q)^{\sigma-s-\frac{d}{q}+d} \inf_Q Mg_x \leqslant \ell(Q)^{\sigma-s-\frac{d}{q}} \int_Q Mg_x \\ &\leqslant \ell(Q)^{\sigma-s}\|Mg_x\|_{L^{q'}} \lesssim_q \ell(Q)^{\sigma-s}G(x).\end{aligned}$$

Thus, since $s \leqslant \sigma$, we

$$\begin{aligned}\boxed{\boxed{1.3}} &\lesssim \sum_Q \ell(Q)^{\sigma-s} \int_Q \sum_P \int_P \frac{|f(z) - f_Q|\, dz}{\mathrm{D}(P,Q)^{d+\sigma}} G(x)\, dx \leqslant \sum_Q \int_Q \sum_{P:\mathrm{D}(P,Q)\geqslant\delta} \int_P \frac{|f(z) - f_Q|\, dz}{\mathrm{D}(P,Q)^{d+s}} G(x)\, dx \\ &+ \sum_Q \int_Q \sum_{P:\mathrm{D}(P,Q)<\delta} \int_P \frac{|f(z) - f_Q|\, dz}{\mathrm{D}(P,Q)^{d+s}} G(x)\, dx =: \boxed{\boxed{1.3.1}} + \boxed{\boxed{\mathrm{D}}}\end{aligned} \tag{6.19}$$

Using the same techniques as in $\boxed{\boxed{1.2.1}}$ we can see that

$$\boxed{\boxed{1.3.1}} \lesssim \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.20}$$

For $\boxed{\boxed{\mathrm{D}}}$ using admissible chains and writing $[P, P_Q) = [P, P_Q]\backslash\{P_Q\}$, we get

$$\begin{aligned}\boxed{\boxed{\mathrm{D}}} &\lesssim \sum_Q \int_Q \sum_P \int_P \frac{|f(z) - f_P|\, dz}{\mathrm{D}(P,Q)^{d+s}} G(x)\, dx \\ &+ \sum_Q \int_Q \sum_P \sum_{L\in[P,P_Q)} \frac{|f_L - f_{\mathcal{N}(L)}|\ell(P)^d}{\mathrm{D}(P,Q)^{d+s}} G(x)\, dx \\ &+ \sum_Q \int_Q \sum_P \sum_{L\in[P_Q,Q)} \frac{|f_L - f_{\mathcal{N}(L)}|\ell(P)^d}{\mathrm{D}(P,Q)^{d+s}} G(x)\, dx =: \boxed{\boxed{1.3.2}} + \boxed{\boxed{1.3.3}} + \boxed{\boxed{1.3.4}}.\end{aligned} \tag{6.21}$$

The first term in (6.21) can be bounded by reordering and using (4.2). Indeed, we have that

$$\boxed{1.3.2} \leqslant \sum_P \int_P \int_P \frac{|f(z) - f(\xi)|\, d\xi\, dz}{\ell(P)^d} \sum_Q \int_Q \frac{G(x)}{\mathrm{D}(P,Q)^{d+s}}\, dx \lesssim \sum_P \int_P \int_P \frac{|f(z) - f(\xi)|\, d\xi MG(z)\, dz}{\ell(P)^{d+s}},$$

that is, by (6.12) we have that

$$\boxed{1.3.2} \lesssim \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.22}$$

For the second term in (6.21) note that given cubes $L \in [P, P_Q]$ we have that $\mathrm{D}(P,Q) \approx \mathrm{D}(L,Q)$ by (3.6) and $P \in \mathbf{Sh}(L)$ by Definition 3.5. Therefore, by (4.2) we have that

$$\begin{aligned}\boxed{1.3.3} &\lesssim \sum_L \frac{1}{\ell(L)^{2d}} \int_L \int_{5L} |f(\xi) - f(\zeta)|\, d\zeta\, d\xi \sum_Q \frac{1}{\mathrm{D}(L,Q)^{d+s}} \int_Q G(x)\, dx \sum_{P \in \mathbf{SH}(L)} \ell(P)^d \\ &\leqslant \sum_L \frac{1}{\ell(L)^{2d}} \int_L \int_{5L} |f(\xi) - f(\zeta)| \frac{MG(\zeta)}{\ell(L)^s}\, d\zeta\, d\xi \ell(L)^d = \sum_L \int_L \int_{5L} \frac{|f(\xi) - f(\zeta)| MG(\zeta)}{\ell(L)^{d+s}}\, d\zeta\, d\xi,\end{aligned}$$

and, again by (6.12), we have that

$$\boxed{1.3.3} \lesssim \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.23}$$

Finally, the last term of (6.21) can be bounded analogously: Given cubes $L \in [P_Q, Q]$ we have that $\mathrm{D}(Q,P) \approx \mathrm{D}(L,P)$ by (3.6), and

$$\begin{aligned}\boxed{1.3.4} &\lesssim \sum_L \frac{1}{\ell(L)^{2d}} \int_L \int_{5L} |f(\xi) - f(\zeta)|\, d\zeta\, d\xi \sum_{Q \in \mathbf{SH}(L)} \int_Q G(x)\, dx \sum_P \frac{\ell(P)^d}{\mathrm{D}(P,L)^{d+s}} \\ &\leqslant \sum_L \frac{1}{\ell(L)^{2d}} \int_L \int_{5L} |f(\xi) - f(\zeta)| MG(\zeta)\, d\zeta\, d\xi \ell(L)^{d-s} = \sum_L \int_L \int_{5L} \frac{|f(\xi) - f(\zeta)| MG(\zeta)}{\ell(L)^{d+s}}\, d\zeta\, d\xi,\end{aligned}$$

and

$$\boxed{1.3.4} \lesssim \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.24}$$

Now, putting together (6.10), (6.11), (6.14), (6.18), (6.19) and (6.21) we have that

$$\boxed{1} \lesssim_{C_K} \boxed{1.1} + \boxed{1.2.1} + \boxed{1.2.2} + \boxed{1.3.1} + \boxed{1.3.2} + \boxed{1.3.3} + \boxed{1.3.4},$$

and by (6.13), (6.15), (6.17), (6.20), (6.22), (6.23) and (6.24) we have that

$$\boxed{1} \lesssim C_K \|f\|_{F^s_{p,q}(\Omega)} \tag{6.25}$$

with constants depending on $\delta$, $\varepsilon$, $|\Omega|$, $p$, $q$, $s$ and $d$.

Now we bound the local part in (6.9), that is, we want to prove that

$$\boxed{2} = \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T_\Omega [(f - f_Q) \varphi_Q](x) - T_\Omega [(f - f_Q) \varphi_{QS}](y)|}{|x - y|^{s + \frac{d}{q}}} g(x,y)\, dy\, dx \lesssim \|f\|_{F^s_{p,q}(\Omega)}.$$

Note that for $x \in Q$ and $y \cap S$, if $y \in 3Q$ then $\varphi_{QS} = \varphi_Q$ and, otherwise $|x - y| \approx \ell(Q)$. Thus,

$$\begin{aligned}\boxed{2} &\leqslant \sum_Q \int_Q \int_{3Q} \frac{|T_\Omega [(f - f_Q) \varphi_Q](x) - T_\Omega [(f - f_Q) \varphi_{QS}](y)|}{|x - y|^{s + \frac{d}{q}}} g(x,y)\, dy\, dx \\ &+ \sum_Q \int_Q \int_{\mathbf{Sh}(Q) \setminus 3Q} \frac{|T_\Omega [(f - f_Q) \varphi_Q](x)|}{\ell(Q)^{s + \frac{d}{q}}} g(x,y)\, dy\, dx \\ &+ \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T_\Omega [(f - f_Q) \varphi_{QS}](y)|}{\ell(Q)^{s + \frac{d}{q}}} g(x,y)\, dy\, dx =: \boxed{2.1} + \boxed{2.2} + \boxed{2.3}.\end{aligned} \tag{6.26}$$

Of course, by Hölder's inequality we have that

$$\boxed{\boxed{2.1}}^p \leqslant \sum_Q \int_Q \left( \int_{3Q} \frac{|T_\Omega \left[ (f - f_Q) \varphi_Q \right](x) - T_\Omega \left[ (f - f_Q) \varphi_{QS} \right](y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \|g\|^p_{L^{p'}(L^{q'}(\Omega))}.$$

If $x \in \Omega$, then $T_\Omega \left[ (f - f_Q) \varphi_Q \right](x) = T \left[ (f - f_Q) \varphi_Q \right](x)$. Thus,

$$\boxed{\boxed{2.1}}^p \lesssim \sum_Q \|T \left[ (f - f_Q) \varphi_Q \right]\|^p_{F^s_{p,q}(\Omega)} \leqslant \sum_Q \|T \left[ (f - f_Q) \varphi_Q \right]\|^p_{F^s_{p,q}(\mathbb{R}^d)}$$

Now, the operator $T$ is bounded in $F^s_{p,q}$ by assumption (see Definition 6.2 and Remark 6.3). Using Corollary 5.5, it follows that

$$\boxed{\boxed{2.1}}^p \lesssim \sum_Q \|(f - f_Q) \varphi_Q\|^p_{F^s_{p,q}(\mathbb{R}^d)} \lesssim \sum_Q \|(f - f_Q) \varphi_Q\|^p_{L^p(\mathbb{R}^d)} + \sum_Q \|(f - f_Q) \varphi_Q\|^p_{\dot{A}^s_{p,q}(\mathbb{R}^d)}.$$

Since $\varphi_Q \leqslant \chi_{7Q}$, the first term is bounded by $\|f\|_{L^p}$ by the finite overlapping of the Whitney cubes and the Jensen inequality, and the second is

$$\sum_Q \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{|(f(x) - f_Q) \varphi_Q(x) - (f(y) - f_Q) \varphi_Q(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx,$$

where the integrand vanishes if both $x, y \notin 8Q$. Therefore, we can write

$$\begin{aligned} \boxed{\boxed{2.1}}^p \lesssim \|f\|_{L^p} &+ \sum_Q \int_{8Q} \left( \int_{8Q} \frac{|(f(x) - f_Q) \varphi_Q(x) - (f(y) - f_Q) \varphi_Q(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \\ &+ \sum_Q \int_{\mathbb{R}^d \setminus 8Q} \left( \int_{7Q} \frac{|(f(y) - f_Q) \varphi_Q(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \\ &+ \sum_Q \int_{7Q} \left( \int_{\mathbb{R}^d \setminus 8Q} \frac{|(f(x) - f_Q) \varphi_Q(x)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx =: \|f\|_{L^p} + \boxed{\boxed{2.1.1}} + \boxed{\boxed{2.1.2}} + \boxed{\boxed{2.1.3}}, \end{aligned} \tag{6.27}$$

where the constant depends linearly on the operator norm $\|T\|^p_{F^s_{p,q} \to F^s_{p,q}}$.

Adding and substracting $(f(x) - f_Q) \varphi_Q(y)$ in the numerator of the integral in $\boxed{\boxed{2.1.1}}$ we get that

$$\begin{aligned} \boxed{\boxed{2.1.1}} \lesssim &\sum_Q \int_{8Q} \left( \int_{8Q} \frac{|f(x) - f_Q|^q \, |\varphi_Q(x) - \varphi_Q(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \\ &+ \sum_Q \int_{8Q} \left( \int_{8Q} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} dx \end{aligned}$$

The second term is bounded by $\|f\|^p_{F^s_{p,q}(\Omega)}$, so

$$\boxed{\boxed{2.1.1}} \lesssim \sum_Q \int_{8Q} \left( \int_{8Q} \frac{\|\nabla \varphi_Q\|^q_{L^\infty} |x-y|^q}{|x-y|^{sq+d}} \, dy \right)^{\frac{p}{q}} |f(x) - f_Q|^p \, dx + \|f\|^p_{F^s_{p,q}(\Omega)}.$$

Using $\|\nabla\varphi_Q\|_{L^\infty} \lesssim \frac{1}{\ell(Q)}$ and the local inequality for the maximal operator (4.3) we get that

$$\boxed{2.1.1} \lesssim \sum_Q \int_{8Q} \ell(Q)^{(1-s)p} \frac{|f(x)-f_Q|^p}{\ell(Q)^p}\,dx + \|f\|^p_{F^s_{p,q}(\Omega)} \tag{6.28}$$
$$\lesssim \sum_Q \int_{8Q} \left(\frac{\int_Q |f(x)-f(\xi)|\,d\xi}{\ell(Q)^{s+d}}\right)^p dx + \|f\|^p_{F^s_{p,q}(\Omega)}.$$

By Jensen's inequality $\frac{1}{\ell(Q)^d}\int_Q |f(x)-f(\xi)|\,d\xi \lesssim \left(\int_Q \frac{1}{\ell(Q)^d}|f(x)-f(\xi)|^q\,d\xi\right)^{\frac{1}{q}}$ and, therefore,

$$\boxed{2.1.1} \lesssim \|f\|^p_{F^s_{p,q}(\Omega)}. \tag{6.29}$$

Now we undertake the task of bounding $\boxed{2.1.2}$ in (6.27). Writing $x_Q$ for the center of a given cube $Q$, we have that

$$\boxed{2.1.2} \lesssim \sum_Q \int_{\mathbb{R}^d\setminus 8Q} \frac{dx}{|x-x_Q|^{sp+\frac{dp}{q}}}\left(\int_{7Q}|f(y)-f_Q|^q\,dy\right)^{\frac{p}{q}}$$

Since $s > \frac{d}{p}-\frac{d}{q}$ we have that $sp+\frac{dp}{q} > d$. Thus

$$\boxed{2.1.2} \lesssim \sum_Q \frac{1}{\ell(Q)^{sp+\frac{dp}{q}-d}}\left(\int_{7Q}|f(y)-f_Q|^q\,dy\right)^{\frac{p}{q}} \leqslant \sum_Q \frac{\left(\int_{7Q}\left(\int_Q|f(y)-f(\xi)|\,d\xi\right)^q dy\right)^{\frac{p}{q}}}{\ell(Q)^{sp+\frac{dp}{q}-d+dp}}.$$

By Minkowski's inequality we have that

$$\boxed{2.1.2} \lesssim \sum_Q \frac{\left(\int_Q\left(\int_{7Q}|f(y)-f(\xi)|^q\,dy\right)^{\frac{1}{q}}d\xi\right)^p}{\ell(Q)^{sp+\frac{dp}{q}+d(p-1)}},$$

and by Hölder's inequality, using that $p-1=\frac{p}{p'}$ we get that

$$\boxed{2.1.2} \lesssim \sum_Q \frac{\int_Q\left(\int_{7Q}|f(y)-f(\xi)|^q\,dy\right)^{\frac{p}{q}}d\xi\,\ell(Q)^{\frac{dp}{p'}}}{\ell(Q)^{sp+\frac{dp}{q}+\frac{dp}{p'}}} \lesssim \sum_Q \int_Q \left(\int_{7Q}\frac{|f(y)-f(\xi)|^q\,dy}{|y-\xi|^{sq+d}}\right)^{\frac{p}{q}} d\xi$$

and

$$\boxed{2.1.2} \lesssim \|f\|^p_{F^s_{p,q}(\Omega)}. \tag{6.30}$$

The last term in (6.27) is somewhat easier. Note that by (4.2)

$$\boxed{2.1.3} \leqslant \sum_Q \int_{7Q}|f(x)-f_Q|^p\left(\int_{\mathbb{R}^d\setminus 8Q}\frac{1}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}}dx \leqslant \sum_Q\int_{7Q}\frac{|f(x)-f_Q|^p}{\ell(Q)^{sp}}\,dx$$

and, since this quantity is bounded by the right-hand side of (6.28), we have that

$$\boxed{2.1.3} \lesssim \|f\|^p_{F^s_{p,q}(\Omega)}. \tag{6.31}$$

Summing up, by (6.27), (6.29), (6.30) and (6.31) we have that

$$\boxed{2.1} \lesssim \|T\|_{F^s_{p,q}\to F^s_{p,q}} \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.32}$$

Back to (6.26), it remains to bound $\boxed{2.2}$ and $\boxed{2.3}$. Recall that

$$\boxed{2.2} = \sum_Q \int_Q \frac{|T_\Omega\left[(f-f_Q)\,\varphi_Q\right](x)|}{\ell(Q)^{s+\frac{d}{q}}} \int_{\mathbf{Sh}(Q)\setminus 3Q} g(x,y)\,dy\,dx.$$

Writing $G(x) = \|g(x,\cdot)\|_{L^{q'}(\Omega)}$ and using Hölder's inequality twice we get that

$$\begin{aligned}\boxed{2.2} &\lesssim \sum_Q \int_Q \frac{|T_\Omega\left[(f-f_Q)\,\varphi_Q\right](x)|\,G(x)\ell(Q)^{\frac{d}{q}}}{\ell(Q)^{s+\frac{d}{q}}}\,dx \\ &\lesssim \left(\sum_Q \int_Q \frac{|T_\Omega\left[(f-f_Q)\,\varphi_Q\right](x)|^p}{\ell(Q)^{sp}}\,dx\right)^{\frac{1}{p}} \|G\|_{L^{p'}(\Omega)}.\end{aligned}$$

Of course, $\|G\|_{L^{p'}(\Omega)} \leqslant 1$. Now, by Definition 6.2 we can use the boundedness of $T_\Omega$ in $L^p$ to find that

$$\boxed{2.2} \lesssim \|T\|_{L^p\to L^p} \left(\sum_Q \frac{\|(f-f_Q)\,\varphi_Q\|^p_{L^p(\Omega)}}{\ell(Q)^{sp}}\right)^{\frac{1}{p}} \leqslant \left(\sum_Q \frac{\|f-f_Q\|^p_{L^p(7Q)}}{\ell(Q)^{sp}}\right)^{\frac{1}{p}},$$

and we can argue again as in (6.28) to prove that

$$\boxed{2.2} \lesssim \|T\|_{L^p\to L^p} \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.33}$$

Finally, for the last term in (6.26), that is, for

$$\boxed{2.3} = \sum_Q \int_Q \sum_{S\in\mathbf{SH}(Q)} \int_S \frac{|T_\Omega\left[(f-f_Q)\,\varphi_{QS}\right](y)|}{\ell(Q)^{s+\frac{d}{q}}} g(x,y)\,dy\,dx,$$

by Hölder's inequality we have that

$$\boxed{2.3} \leqslant \sum_Q \int_Q \left(\sum_{S\in\mathbf{SH}(Q)} \int_S \frac{|T_\Omega\left[(f-f_Q)\,\varphi_{QS}\right](y)|^q}{\ell(Q)^{sq+d}}\,dy\right)^{\frac{1}{q}} G(x)\,dx.$$

The boundedness of $T$ in $L^q$ leads to

$$\boxed{2.3} \lesssim \|T\|_{L^q\to L^q} \sum_Q \left(\sum_{S\in\mathbf{SH}(Q)} \int_{\mathrm{supp}(\varphi_{QS})} \frac{|(f(y)-f_Q)\,\varphi_{QS}(y)|^q}{\ell(Q)^{sq+d}}\,dy\right)^{\frac{1}{q}} \ell(Q)^d \inf_Q MG.$$

Given a cube $Q$, the finite overlapping of the family $\{50S\}_{S\in\mathcal{W}}$ (see Definition 3.1) implies the finite overlapping of the supports of the family $\{\varphi_{QS}\}$ (recall that $\mathrm{supp}(\varphi_{QS}) \subset 23S$), so there is a certain ratio $\rho_5$ such that naming $\mathbf{Sh}_5(Q) := \mathbf{Sh}_{\rho_5}(Q)$ we have that

$$\boxed{2.3} \lesssim \sum_Q \left(\int_{\mathbf{Sh}_5(Q)} \frac{|f(y)-f_Q|^q}{\ell(Q)^{sq+d-dq}}\,dy\right)^{\frac{1}{q}} \inf_Q MG = \sum_Q \frac{\left(\int_{\mathbf{Sh}_5(Q)} \left(\int_Q |f(y)-f(\xi)|\,d\xi\right)^q dy\right)^{\frac{1}{q}}}{\ell(Q)^{s+\frac{d}{q}-d+d}} \inf_Q MG.$$

Finally, using Minkowski's inequality and Hölder's inequality we get that

$$\boxed{\boxed{2.3}} \lesssim \sum_Q \frac{\int_Q \left(\int_{\mathbf{Sh}_5(Q)} |f(y)-f(\xi)|^q \, dy\right)^{\frac{1}{q}} MG(\xi)\, d\xi}{\ell(Q)^{s+\frac{d}{q}}} \lesssim \left(\sum_Q \int_Q \left(\int_\Omega \frac{|f(y)-f(\xi)|^q}{|x-y|^{sq+d}}\, dy\right)^{\frac{p}{q}} d\xi\right)^{\frac{1}{p}},$$

that is,

$$\boxed{\boxed{2.3}} \lesssim \|T\|_{L^q \to L^q} \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.34}$$

Now, by (6.26), (6.32), (6.33) and (6.34) we have that

$$\boxed{\boxed{2}} \lesssim \left(\|T\|_{F^s_{p,q}\to F^s_{p,q}} + \|T\|_{L^p\to L^p} + \|T\|_{L^q\to L^q}\right) \|f\|_{F^s_{p,q}(\Omega)}. \tag{6.35}$$

□

**Corollary 6.7.** *Let $\Omega$ be a uniform domain with Whitney covering $\mathcal{W}$, let $T$ a $p,q$-admissible convolution Calderón-Zygmund operator of order $0<\sigma<1$, $1<p<\infty$, $1<q<\infty$ and $0<s\leqslant\sigma$ with $s>\frac{d}{p}-\frac{d}{q}$. The following statements are equivalent:*

*i) For every $f \in F^s_{p,q}(\Omega)$ one has*

$$\|T_\Omega f\|_{F^s_{p,q}(\Omega)} \lesssim \|f\|_{F^s_{p,q}(\Omega)}.$$

*ii) For every $f \in F^s_{p,q}(\Omega)$ one has*

$$\sum_{Q\in\mathcal{W}} |f_Q|^p \left\|\nabla^s_q T\chi_\Omega\right\|^p_{L^p(Q)} \lesssim \|f\|^p_{F^s_{p,q}(\Omega)}.$$

*Proof.* By Definition 6.5, we have that

$$\begin{aligned}\|T_\Omega f\|^p_{F^s_{p,q}(\Omega)} &\approx \|T_\Omega f\|^p_{L^p(\Omega)} + \sum_{Q\in\mathcal{W}} \int_Q |\nabla^s_q T_\Omega f(x)|^p\, dx \\ &= \|T_\Omega f\|^p_{L^p(\Omega)} + \sum_{Q\in\mathcal{W}} \int_Q |\nabla^s_q T_\Omega (f - f_Q + f_Q)(x)|^p\, dx \end{aligned} \tag{6.36}$$

Since $T_\Omega$ is bounded in $L^p(\Omega)$ we have that $\|T_\Omega f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$.

Inequality (6.7) proves that

$$\sum_{Q\in\mathcal{W}} \int_Q |\nabla^s_q T_\Omega f(x)|^p\, dx \lesssim \|f\|^p_{F^s_{p,q}(\Omega)} \iff \sum_{Q\in\mathcal{W}} \int_Q |f_Q|^p |\nabla^s_q T_\Omega 1(x)|^p\, dx \lesssim \|f\|^p_{F^s_{p,q}(\Omega)}.$$

□

**Theorem 6.8.** *Let $\Omega$ be a bounded uniform domain with Whitney covering $\mathcal{W}$, $T$ a $p,q$-admissible convolution Calderón-Zygmund operator of order $0<\sigma<1$, $1<p<\infty$, $1<q<\infty$ and $\frac{d}{p}<s\leqslant\sigma$. Then*

$$\|T_\Omega 1\|_{F^s_{p,q}(\Omega)} < \infty \qquad \iff \qquad T_\Omega \text{ is bounded in } F^s_{p,q}(\Omega).$$

*Proof.* Note that since $s > \frac{d}{p} > \frac{d}{p} - \frac{d}{q}$, we can use (6.6) from Definition 6.5 and, by the Corollary 6.7 of the Key Lemma we have that

$$\sum_{Q\in\mathcal{W}} \int_Q |\nabla^s_q T_\Omega(f - f_Q)(x)|^p \, dx \leqslant C\|f\|^p_{F^s_{p,q}(\Omega)},$$

where $C$ depends only on $d$, $s$, $p$, $T$ and the uniform character of $\Omega$.

Given $f \in F^s_{p,q}(\Omega)$, by (6.6) we have that $T_\Omega$ is bounded if and only if

$$\sum_{Q\in\mathcal{W}} |f_Q|^p \left\|\nabla^s_q T\chi_\Omega\right\|^p_{L^p(Q)} \lesssim \|f\|^p_{F^s_{p,q}(\Omega)}.$$

Since $sp > d$, by (5.1) and the Sobolev Embedding Theorem we have the continuous embedding $F^s_{p,q}(\Omega) \subset L^\infty$. Therefore, given a cube $Q$ we have that $|f_Q| \leqslant \|f\|_{L^\infty} \leqslant \|f\|_{F^s_{p,q}(\Omega)}$ and, by (6.36) we get

$$\|T_\Omega f\|_{F^s_{p,q}(\Omega)} \lesssim \left(1 + \|T_\Omega 1\|_{F^s_{p,q}(\Omega)}\right) \|f\|_{F^s_{p,q}(\Omega)} \tag{6.37}$$

with $C$ depending only on $d$, $p$, $q$, $s$, $\varepsilon$, $\delta$, $\operatorname{diam}(\Omega)$ and $T$. □

To end this section, we make some observations.

**Remark 6.9.** *Note that putting together (6.25), (6.35) and (6.36), the precise dependence on $T$ in equation (6.37) can be shown to be*

$$\|T_\Omega f\|_{F^s_{p,q}(\Omega)} \lesssim \left(C_K + \|T\|_{F^s_{p,q}\to F^s_{p,q}} + \|T\|_{L^p\to L^p} + \|T\|_{L^q\to L^q} + \|T_\Omega 1\|_{\dot F^s_{p,q}(\Omega)}\right) \|f\|_{F^s_{p,q}(\Omega)}.$$

**Remark 6.10.** *The Key Lemma is valid in a wider range of indices than Theorem 6.8 because in the second case we have the restriction of the Sobolev embedding. In the cases where the Key Lemma can be applied but not the theorem above, that is, when*

$$\max\left\{0, \frac{d}{p} - \frac{d}{q}\right\} < s \leqslant \min\left\{\sigma, \frac{d}{p}\right\},$$

*there is room to do some steps forward.*

*In [PT15, Theorems 1.2 and 1.3], the authors consider the measures $\mu_P(x) = |\nabla^s T_\Omega P(x)|^p \, dx$ for polinomials $P$ of degree smaller than the smoothness $s \in \mathbb{N}$ (here the $s$-th gradient has its usual meaning). They conclude that if $\mu_P$ is a $p$-Carleson measure for every such $P$, that is, if*

$$\int_{\widetilde{\mathbf{Sh}}(a)} \operatorname{dist}(x, \partial\Omega)^{(d-p)(1-p')} (\mu_P(\mathbf{Sh}(x) \cap \mathbf{Sh}(a)))^{p'} \frac{dx}{\operatorname{dist}(x, \partial\Omega)^d} \leqslant C\mu_P(\mathbf{Sh}(a)),$$

*then $T_\Omega$ is bounded in $W^{s,p}(\Omega)$, and, in case $s = 1$, the condition is necessary and sufficient.*

*The authors of the present article expect that some similar result can be found in the case $\max\left\{0, \frac{d}{p} - \frac{d}{q}\right\} < s \leqslant \min\left\{\sigma, \frac{d}{p}\right\}$.*

*Furthermore, the restriction $\frac{d}{p} - \frac{d}{q} < s$ comes from the intrinsic characterization that we use for the present article, which we think is the easier to handle in our proofs. However, there are other characterizations (see [Str67] or [Tri06, Section 1.11.9]) which cover all the range of indices. There is hope that this characterizations may be used to obtain a result analogous to the Key Lemma 6.6 for a wider range.*

**Remark 6.11.** *For $1 < p, q < \infty$ and $0 < s < \frac{1}{p}$, we have that the multiplication by the characteristic functions of a half plane is bounded in $F^s_{p,q}(\mathbb{R}^d)$. This implies that for domains $\Omega$ whose boundary consists on a finite number of polygonal boundaries, the pointwise multiplication with $\chi_\Omega$ is also bounded and, using characterizations by differences, this property can be seen to be stable under bi-Lipschitz changes of coordinates. Summing up, we have that given any Lipschitz domain $\Omega$ and any function $f \in F^s_{p,q}(\mathbb{R}^d)$, we have that*

$$\|\chi_\Omega f\|_{F^s_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{F^s_{p,q}(\mathbb{R}^d)}.$$

*Therefore, if $s > \frac{d}{p} - \frac{d}{q}$ and $T$ is an operator bounded in $F^s_{p,q}$, using the extension $\Lambda_0 : F^s_{p,q}(\Omega) \to F^s_{p,q}(\mathbb{R}^d)$ (see Corollary 5.9), for every $f \in F^s_{p,q}(\Omega)$ we have that*

$$\begin{aligned}\|T_\Omega f\|_{F^s_{p,q}(\Omega)} &= \|T(\chi_\Omega \Lambda_0 f)\|_{F^s_{p,q}(\Omega)} \leqslant \|T(\chi_\Omega \Lambda_0 f)\|_{F^s_{p,q}} \leqslant \|T\|_{F^s_{p,q}\to F^s_{p,q}} \|\chi_\Omega \Lambda_0 f\|_{F^s_{p,q}} \lesssim \|\Lambda_0 f\|_{F^s_{p,q}} \\ &\lesssim \|f\|_{F^s_{p,q}(\Omega)}.\end{aligned}$$

*In particular, given a $p, q$-admissible convolution Calderón-Zygmund operator $T$ and a Lipschitz domain $\Omega$ we have that $T_\Omega$ is bounded in $F^s_{p,q}(\Omega)$ for any $0 < s < \frac{1}{p}$.*

# 7 Refinement of Lemma 4.4 for $p \geqslant q$

In some situations we can improve Lemma 4.4.

**Lemma 7.1.** *Let $\Omega$ be a bounded uniform domain with an admissible Whitney covering $\mathcal{W}$, let $1 < q \leqslant p < \infty$ and $\max\left\{\frac{d}{p} - \frac{d}{q}, 0\right\} < s < 1$. Then, $f \in A^s_{p,q}(\Omega)$ if and only if*

$$\|f\|_{L^p(\Omega)} + \left(\sum_{Q\in\mathcal{W}} \int_Q \left(\int_{5Q} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}}\, dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} < \infty.$$

*Furthermore, this quantity defines a norm which is equivalent to $\|f\|_{A^s_{p,q}(\Omega)}$.*

*Proof.* Arguing as before by duality, we consider a function $g > 0$ with $\|g\|_{L^{p'}(L^{q'}(\Omega))} \leqslant 1$. Consider a constant $C_\delta < \delta$ to be fixed. Combining (4.13) and (4.14) we know that

$$\sum_{\substack{Q,S: S\cap 2Q=\varnothing \\ \mathrm{D}(Q,S)<C_\delta}} \int_Q \int_S \frac{|f(x) - f_{Q_S}|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\, dy\, dx \lesssim \left(\sum_{Q\in\mathcal{W}} \int_Q \left(\int_{5Q} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}}\, dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}}$$

and, thus, we have

$$\sum_{\substack{Q,S: S\cap 2Q=\varnothing \\ \mathrm{D}(Q,S)<C_\delta}} \int_Q \int_S \frac{|f(x) - f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\, dy\, dx \approx \left(\sum_{Q\in\mathcal{W}} \int_Q \left(\int_{5Q} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}}\, dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} + \textcircled{3}.$$

where

$$\begin{aligned}\textcircled{3} &:= \sum_{Q,S:\mathrm{D}(Q,S)<C_\delta} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y)\, dy\, dx \\ &\lesssim \sum_{R:\ell(R)\lesssim C_{\delta,\varepsilon}} \sum_{Q,S\in\mathbf{SH}(R)} \int_Q \int_S \frac{|f_R - f(y)|}{\ell(R)^{s+\frac{d}{q}}} g(x,y)\, dy\, dx\end{aligned}$$

by (4.15).

Using Hölder's inequality and Lemma 4.3 we get that

$$
\begin{aligned}
\textcircled{3} &\lesssim \sum_{R:\ell(R)\lesssim C_{\delta,\varepsilon}} \frac{1}{\ell(R)^{s+\frac{d}{q}}} \left( \sum_{S\in\mathbf{SH}(R)} \int_S |f_R - f(y)|^q \, dy \right)^{\frac{1}{q}} \sum_{Q\in\mathbf{SH}(R)} \int_Q G(x)\, dx \\
&\lesssim \sum_{R:\ell(R)\lesssim C_{\delta,\varepsilon}} \left( \sum_{S\in\mathbf{SH}(R)} \int_S |f_R - f(y)|^q \, dy \right)^{\frac{1}{q}} \frac{\int_R MG(\xi)\, d\xi}{\ell(R)^{s+\frac{d}{q}}}
\end{aligned}
$$

and, using the Hölder inequality again, we get

$$
\textcircled{3} \lesssim \left( \sum_{R:\ell(R)\lesssim C_{\delta,\varepsilon}} \left( \sum_{S\in\mathbf{SH}(R)} \int_S |f_R - f(y)|^q \, dy \right)^{\frac{p}{q}} \frac{\ell(R)^d}{\ell(R)^{sp+\frac{dp}{q}}} \right)^{\frac{1}{p}} \|MG\|_{L^{p'}}.
$$

Now, given $R$ and $S \in \mathbf{SH}(R)$, there exists a chain $[S, R]$ as in Remark 3.6 (taking $C_\delta$ small enough). Using the boundedness of the maximal operator in $L^{p'}$ we can decompose the previous expression as

$$
\begin{aligned}
\textcircled{3}^p &\lesssim \sum_{R:\ell(R)\lesssim C_{\delta,\varepsilon}} \left( \sum_{S\in\mathbf{SH}(R)} \left| \sum_{P\in[S,R)} \left(f_P - f_{\mathcal{N}(P)}\right) \frac{\ell(P)^{\frac{s}{q}}}{\ell(P)^{\frac{s}{q}}} \right|^q \ell(S)^d \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}} \qquad (7.1) \\
&\quad + \sum_R \left( \sum_{S\in\mathbf{SH}(R)} \int_S |f_S - f(y)|^q \, dy \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}} =: \textcircled{3.1} + \textcircled{3.2},
\end{aligned}
$$

where we wrote $[S, R) = [S, R]\backslash\{R\}$.

Using Hölder's inequality

$$
\textcircled{3.1} \lesssim \sum_{R:\ell(R)\lesssim C_\delta} \left( \sum_{S\in\mathbf{SH}(R)} \sum_{P\in[S,R)} \frac{|f_P - f_{\mathcal{N}(P)}|^q}{\ell(P)^s} \left( \sum_{P\in[S,R)} \ell(P)^{\frac{sq'}{q}} \right)^{\frac{q}{q'}} \ell(S)^d \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}}.
$$

But for $S \in \mathbf{SH}(R)$ by Remark 3.6 we have that $\sum_{P\in[S,R)} \ell(P)^{\frac{sq'}{q}} \lesssim \ell(R)^{\frac{sq'}{q}}$. We also know that $\sum_{S\in\mathbf{SH}(P)} \ell(S)^d \lesssim \ell(P)^d$, so writing $U_P$ for the union of the neighbors of $P$, we get

$$
\textcircled{3.1} \lesssim \sum_R \left( \sum_{P\in\mathbf{SH}(R)} \frac{\left( ⨍_{U_P} |f(\xi) - f_P|\, d\xi \right)^q \ell(P)^d}{\ell(P)^s} \right)^{\frac{p}{q}} \ell(R)^{d+\frac{sp}{q}-sp-\frac{dp}{q}}.
$$

Recall that $p \geqslant q$ and, therefore, by Hölder's inequality and (3.7) we have that

$$
\begin{aligned}
\textcircled{3.1} &\lesssim \sum_R \sum_{P\in\mathbf{SH}(R)} \frac{\left( ⨍_{U_P} |f(\xi) - f_P|\, d\xi \right)^p \ell(P)^d}{\ell(P)^{\frac{sp}{q}}} \left( \sum_{P\in\mathbf{SH}(R)} \ell(P)^d \right)^{\left(1-\frac{q}{p}\right)\frac{p}{q}} \ell(R)^{d-\frac{sp}{q'}-\frac{dp}{q}} \\
&= \sum_P \frac{\left( ⨍_{U_P} |f(\xi) - f_P|\, d\xi \right)^p \ell(P)^d}{\ell(P)^{\frac{sp}{q}}} \sum_{R:P\in\mathbf{SH}(R)} \ell(R)^{-\frac{sp}{q'}} \approx \sum_P \frac{\left( ⨍_{U_P} |f(\xi) - f_P|\, d\xi \right)^p \ell(P)^d}{\ell(P)^{sp}}
\end{aligned}
$$

Using Jensen's inequality we get

$$\textcircled{3.1} \lesssim \sum_P \int_{U_P} \frac{|f(\xi) - f_P|^p}{\ell(P)^{sp}} \, d\xi, \tag{7.2}$$

and Jensen's inequality again leads to

$$\textcircled{3.1} \lesssim \sum_P \int_{U_P} \left( \frac{\int_P |f(\xi) - f(\zeta)|^q \, d\zeta}{\ell(P)^d} \right)^{\frac{p}{q}} \frac{1}{\ell(P)^{sp}} \, d\xi \lesssim \sum_P \int_P \left( \frac{\int_{5P} |f(\xi) - f(\zeta)|^q \, d\zeta}{|\xi - \zeta|^{sq+d}} \right)^{\frac{p}{q}} d\xi. \tag{7.3}$$

To bound $\textcircled{3.2}$ we follow the same scheme. Since $p \geqslant q$ we have that

$$\begin{aligned} \textcircled{3.2} &= \sum_R \left( \sum_{S \in \mathbf{SH}(R)} \int_S |f_S - f(y)|^q \, dy \frac{\ell(S)^{d\left(1-\frac{q}{p}\right)}}{\ell(S)^{d\left(1-\frac{q}{p}\right)}} \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}} \\ &\leqslant \sum_R \left( \sum_{S \in \mathbf{SH}(R)} \frac{\left(\int_S |f_S - f(y)|^q \, dy\right)^{\frac{p}{q}}}{\ell(S)^{d\left(\frac{p}{q}-1\right)}} \right)^{\frac{q}{p}\cdot\frac{p}{q}} \left( \sum_{S \in \mathbf{SH}(R)} \ell(S)^d \right)^{\left(1-\frac{q}{p}\right)\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}}, \end{aligned}$$

and, since $\sum_{S \in \mathbf{SH}(R)} \ell(S)^d \approx \ell(R)^d$, reordering and using (3.7) we get that

$$\textcircled{3.2} \lesssim \sum_S \frac{\left(\int_S |f_S - f(y)|^q \, dy\right)^{\frac{p}{q}}}{\ell(S)^{d\left(\frac{p}{q}-1\right)}} \sum_{R: S \in \mathbf{SH}(R)} \ell(R)^{-sp} \lesssim \sum_S \left( \frac{\int_S |f_S - f(y)|^q \, dy}{\ell(S)^d} \right)^{\frac{p}{q}} \frac{\ell(S)^d}{\ell(S)^{sp}}.$$

Thus, by Jensen's inequality,

$$\textcircled{3.2} \lesssim \sum_S \frac{\int_S |f_S - f(y)|^p \, dy}{\ell(S)^d} \frac{\ell(S)^d}{\ell(S)^{sp}}$$

and, arguing as in (7.2), we get that

$$\textcircled{3.2} \lesssim \sum_S \int_S \left( \frac{\int_S |f(y) - f(\zeta)|^q \, d\zeta}{|y - \zeta|^{sq+d}} \right)^{\frac{p}{q}} dy. \tag{7.4}$$

Thus, by (4.12), (4.13), (4.14), (7.1), (7.3) and (7.4), we have that

$$\sum_{Q,S: \mathrm{D}(Q,S) < C_\delta} \int_Q \int_S \frac{|f(x) - f(y)|}{\mathrm{D}(Q,S)^{s+\frac{d}{q}}} g(x,y) \, dy \, dx \lesssim \left( \sum_S \int_S \left( \int_{5S} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} \, dy \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}$$

with constants depending on $d$, $p$ and $\varepsilon$. This fact, together with (4.7), (4.8), (4.9), (4.10) and (4.11) finishes the proof of Lemma 7.1. □

**Remark 7.2.** *An analogous result to Lemma 7.1 for Besov spaces $B^s_{p,p}$ can be found in [Dyd06, Proposition 5] where it is stated in the case of Lipschitz domains.*

**Corollary 7.3.** *Let $\Omega$ be a uniform domain. Let $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ for every $x \in \mathbb{C}$.*

*Given $1 < p < q < \infty$, and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$, we have that $A^s_{p,q}(\Omega) = F^s_{p,q}(\Omega)$ and, moreover, for $\rho_1 > 1$ big enough, we have that*

$$\|f\|_{F^s_{p,q}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_\Omega \left(\int_{B_{\rho_1 \delta(x)}(x)\cap\Omega} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \qquad \text{for all } f \in F^s_{p,q}(\Omega).$$

*Given $1 < q \leqslant p < \infty$, and $0 < s < 1$, we have that $A^s_{p,q}(\Omega) = F^s_{p,q}(\Omega)$ and, moreover, for $0 < \rho_0 < 1$ we have that*

$$\|f\|_{F^s_{p,q}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_\Omega \left(\int_{B_{\rho_0 \delta(x)}(x)} \frac{|f(x)-f(y)|^q}{|x-y|^{sq+d}}\,dy\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \qquad \text{for all } f \in F^s_{p,q}(\Omega).$$

*Proof.* This comes straight forward from Corollary 5.9, Lemma 4.4 and Lemma 7.1, taking smaller cubes in the Whitney covering if necessary when $\rho_0 << 1$. □

**Remark 7.4.** *In particular, for every $1 < p < \infty$ and $0 < s < 1$ we have that $A^s_{p,p}(\Omega) = B^s_{p,p}(\Omega)$, with*

$$\|f\|_{B^s_{p,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_\Omega \int_{B_{\rho_0 \delta(x)}(x)} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+d}}\,dy\,dx\right)^{\frac{1}{p}} \qquad \text{for all } f \in B^s_{p,p}(\Omega).$$

*If in addition $s > \frac{d}{p} - \frac{d}{2}$, then $A^s_{p,2}(\Omega) = W^{s,p}(\Omega)$. If $p \geqslant 2$ we have that*

$$\|f\|_{W^{s,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_\Omega \left(\int_{B_{\rho_0 \delta(x)}(x)} \frac{|f(x)-f(y)|^2}{|x-y|^{2s+d}}\,dy\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \qquad \text{for all } f \in W^{s,p}(\Omega),$$

*and, if $1 < p < 2$, we have that*

$$\|f\|_{W^{s,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_\Omega \left(\int_{B_{\rho_1 \delta(x)}(x)\cap\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{2s+d}}\,dy\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \qquad \text{for all } f \in W^{s,p}(\Omega).$$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