A discrete approach to Wirtinger's inequality

Julià Cufí*, Agustí Reventós, Carlos J. Rodríguez

December 20, 2018

Abstract

Considering Wirtinger's inequality for piece-wise equipartite functions we find a discrete version of this classical inequality. The main tool we use is the theorem of classification of isometries. Our approach provides a new elementary proof of Wirtinger's inequality that also allows to study the case of equality. Moreover it leads in a natural way to the Fourier series development of 2π-periodic functions.

1 Introduction

The classical Wirtinger inequality states that for a 2π-periodic $\mathcal{C}^1$ function $f(t)$ with $\int_0^{2\pi} f(t)\,dt = 0$ one has

$$\int_0^{2\pi} f^2(t)\,dt \leq \int_0^{2\pi} f'^2(t)\,dt, \tag{1}$$

with equality if and only if $f(t) = a\sin(t) + b\cos(t)$ for some $a, b \in \mathbb{R}$.

The goal of this note is to give a discrete inequality that will imply the above result, including the case of equality. At the same time our approach leads in a natural way to the Fourier series development of a 2π-periodic function.

Wirtinger did not publish his result, but he communicated it by letter to W. Blascke who included it in [1]. The original proof is based on the theory of Fourier series. Discrete approximations to Wirtinger's inequality have been given by several authors; see for instance [2], [4].

As a motivation for a discrete inequality we consider Wirtinger's inequality for piece-wise equipartite linear functions, that is for continuous functions $f : [0, 2\pi] \longrightarrow \mathbb{R}$, linear on each interval $[\frac{2\pi}{n}(j-1), \frac{2\pi}{n}j]$, $j = 1, \dots, n$ and such that $f(0) = f(2\pi)$. Denoting $f(\frac{2\pi}{n}j)$ by x_j, $j = 1, \dots, n$, and taking $x_0 = x_n$, Wirtinger's inequality for this class of functions is equivalent to the discrete inequality

*The authors were partially supported by grants 2017SGR358, 2017SGR1725 (Generalitat de Catalunya) and MTM2015-66165-P (Ministerio de Economía y Competitividad)

$$\sum_{j=1}^{n} x_j x_{j-1} \leq \frac{3n^2 - 4\pi^2}{3n^2 + 2\pi^2}\,, \tag{2}$$

for $x_j \in \mathbb{R}$, $j = 1, \ldots, n$, $x_0 = x_n$, $\sum_{j=1}^{n} x_j = 0$ and $\sum_{j=1}^{n} x_j^2 = 1$.

Wirtinger's inequality can then be obtained from the above inequality by a limiting process.

We shall obtain (2) as a consequence of the following

Theorem 2.1. *Let $x_1, \ldots, x_n \in \mathbb{R}$, for $n \geq 4$, with $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = 1$. Then*

$$\sum_{i=1}^{n} x_i x_{i-1} \leq \cos(\frac{2\pi}{n}), \tag{3}$$

with $x_0 = x_n$. Equality holds if and only if

$$x_i = a\cos(\frac{2\pi}{n}i) + b\sin(\frac{2\pi}{n}i), \quad i = 1, \ldots, n,$$

for $a, b \in \mathbb{R}$ satisfying $a^2 + b^2 = 2/n$.

This result that can be considered as the *Wirtinger discrete inequality* was obtained by Fan, Taussky and Todd in [2] where it is used to obtain classical Wirtinger inequality (1) but, as the authors say, without the equality clause. Other proofs of Theorem 2.1 have been published later, see for instance [4].

For completeness we provide here a simple different proof of the above result based on the theorem of classification of isometries applied to the cyclic isometry T given by

$$T(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}),$$

since the left hand-side of (3) can be written as $\langle X, T(X)\rangle$, where $X = (x_1, x_2, \ldots, x_n)$.

As we have said our approach, based on inequality (2), leads to inequality (1), and allows to caracterize functions for which equality holds. This characterization is somewhat delicate but the argument used has a surprising consequence: the Fourier series development of a 2π-periodic function.

2 Discrete Wirtinger's inequality

In order to find a discrete version of the Wirtinger inequality we consider this inequality for piece-wise equipartite linear functions.

For $n \in \mathbb{N}$, $n \geq 4$, let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be a continuous function, linear on each interval $[\frac{2\pi}{n}(j-1), \frac{2\pi}{n}j]$, $j = 1, \ldots, n$ and such that $f(0) = f(2\pi)$. Denoting $f(\frac{2\pi}{n}j)$ by x_j, $j = 1, \ldots, n$, and taking $x_0 = x_n$, a computation shows that

$$\int_0^{2\pi} f^2(t)\,dt = \frac{2\pi}{3n}\sum_{j=1}^{n}(2x_j^2 + x_j x_{j-1}) \tag{4}$$

and

$$\int_0^{2\pi} f'^2(t)\,dt = \frac{n}{\pi}\sum_{j=1}^{n}(x_j^2 - x_j x_{j-1}). \tag{5}$$

So the inequality

$$\int_0^{2\pi} f^2(t)\,dt \le \int_0^{2\pi} f'^2(t)\,dt$$

is equivalent to

$$\sum_{j=1}^{n} x_j x_{j-1} \le \frac{3n^2 - 4\pi^2}{3n^2 + 2\pi^2}\sum_{j=1}^{n} x_j^2. \tag{6}$$

Assuming now $\int_0^{2\pi} f(t)\,dt = 0$, that means $\sum_{i=1}^{n} x_j = 0$, it follows that Wirtinger's inequality for piece-wise linear functions is equivalent to (6) with this additional hypothesis or, normalizing,

$$\sum_{j=1}^{n} x_j x_{j-1} \le \frac{3n^2 - 4\pi^2}{3n^2 + 2\pi^2}, \quad \text{with } \sum_{j=1}^{n} x_j = 0, \sum\nolimits_{j=1}^{n} x_j^2 = 1. \tag{7}$$

This is a problem of maximizing a given quadratic form under some restrictions. It can be solved by different methods such as Lagrange multipliers or by the determination of the least characteristic value of a Hermitian matrix, as done in [2]. As said our approach is based on the theorem of classification of isometries.

The canonical expression of the quadratic form.

The left-hand side of (7) leads in a natural way to consider the cyclic isometry

$$T(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}),$$

since

$$\sum_{j=1}^{n} x_j x_{j-1} = \langle X, T(X)\rangle,$$

where $X = (x_1, \ldots, x_n)$ and $\langle\, ,\, \rangle$ is the standard scalar product. Hence, in order to prove (7) we start by analyzing the structure of the isometry T. This will allow us to find the canonical expression of the quadratic form $\langle X, T(X)\rangle$.

The theorem of classification of isometries (see [3]) applied to T asserts that there is an orthonormal basis $(e_1, \ldots, e_n)$ such that, denoting $\alpha_k = \frac{2\pi}{n}k$, one has for n even

$$\begin{aligned}
T(e_1) &= e_1,\\
T(e_2) &= -e_2,\\
T(e_{2k+1}) &= (\cos\alpha_k)\, e_{2k+1} + (\sin\alpha_k)\, e_{2k+2},\\
T(e_{2k+2}) &= (-\sin\alpha_k)\, e_{2k+1} + (\cos\alpha_k)\, e_{2k+2}, \qquad k = 1, \ldots, (n-2)/2,
\end{aligned}$$

and for n odd

$$
\begin{aligned}
T(e_1) &= e_1, \\
T(e_{2k}) &= (\cos\alpha_k)\, e_{2k} + (\sin\alpha_k)\, e_{2k+1}, \\
T(e_{2k+1}) &= (-\sin\alpha_k)\, e_{2k} + (\cos\alpha_k)\, e_{2k+1}, \qquad k = 1, \ldots, (n-1)/2.
\end{aligned}
$$

In fact, it can be seen by using elementary trigonometric formulas that for n even, this basis is given by

$$
\begin{aligned}
e_1 &= \frac{1}{\sqrt{n}}(1, \ldots, 1), \\
e_2 &= \frac{1}{\sqrt{n}}(1, -1 \ldots, 1, -1), \\
e_{2k+1} &= \sqrt{\frac{2}{n}}\left(1, \cos(\frac{2\pi}{n}k), \cos(\frac{2\pi}{n}2k), \ldots, \cos(\frac{2\pi}{n}(n-1)k)\right), \\
e_{2k+2} &= \sqrt{\frac{2}{n}}\left(0, \sin(\frac{2\pi}{n}k), \sin(\frac{2\pi}{n}2k), \ldots, \sin(\frac{2\pi}{n}(n-1)k)\right),
\end{aligned} \tag{8}
$$

and for n odd by

$$
\begin{aligned}
e_1 &= \frac{1}{\sqrt{n}}(1, \ldots, 1), \\
e_{2k} &= \sqrt{\frac{2}{n}}\left(1, \cos(\frac{2\pi}{n}k), \cos(\frac{2\pi}{n}2k), \ldots, \cos(\frac{2\pi}{n}(n-1)k)\right), \\
e_{2k+1} &= \sqrt{\frac{2}{n}}\left(0, \sin(\frac{2\pi}{n}k), \sin(\frac{2\pi}{n}2k), \ldots, \sin(\frac{2\pi}{n}(n-1)k)\right).
\end{aligned} \tag{9}
$$

Since $\langle e_i, T(e_j)\rangle + \langle e_j, T(e_i)\rangle = 0,$ for $i \neq j$, we get for every vector $X = \sum_{i=1}^n y_i e_i$,

$$
\langle X, T(X)\rangle = \sum_{i,j=1}^{n} y_i y_j \langle e_i, T(e_j)\rangle = \sum_{i=1}^{n} y_i^2 \langle e_i, T(e_i)\rangle.
$$

Hence the canonical expression of the quadratic form $\langle X, T(X)\rangle$ is for even n

$$
\langle X, T(X)\rangle = y_1^2 - y_2^2 + \sum_{k=1}^{(n-2)/2} (y_{2k+1}^2 + y_{2k+2}^2)\cos\alpha_k, \tag{10}
$$

and for odd n

$$
\langle X, T(X)\rangle = y_1^2 + \sum_{k=1}^{(n-1)/2} (y_{2k}^2 + y_{2k+1}^2)\cos\alpha_k. \tag{11}
$$

The discrete inequality

The maximum of the quadratic form $\langle X, T(X)\rangle$ is given by the following

Theorem 2.1 (Discrete Wirtinger's inequality)**.** *Let $x_1, \ldots, x_n \in \mathbb{R}$, for $n \geq 4$, with $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n x_i^2 = 1$. Then*

$$\sum_{i=1}^{n} x_i x_{i-1} \leq \cos(\frac{2\pi}{n}),$$

with $x_0 = x_n$. Equality holds if and only if

$$x_i = a\cos(\frac{2\pi}{n} i) + b\sin(\frac{2\pi}{n} i), \quad i = 1, \ldots, n,$$

for $a, b \in \mathbb{R}$ satisfying $a^2 + b^2 = 2/n$.

Proof. With the previous notation we must prove

$$\langle X, T(X)\rangle \leq \cos(\frac{2\pi}{n}).$$

Since $\langle X, e_1\rangle = 0$, $||X|| = 1$ it is $X = \sum_{j=2}^n y_j e_j$, $\sum_{j=2}^n y_j^2 = 1$ and we get from (10)

$$\langle X, T(X)\rangle \leq \cos(\frac{2\pi}{n}) \sum_{k=1}^{(n-2)/2} (y_{2k+1}^2 + y_{2k+2}^2) \leq \cos(\frac{2\pi}{n})$$

for n even, and from (11)

$$\langle X, T(X)\rangle \leq \cos(\frac{2\pi}{n}) \sum_{k=1}^{(n-1)/2} (y_{2k}^2 + y_{2k+1}^2) = \cos(\frac{2\pi}{n})$$

for n odd. This proves the first part of the Lemma.

Equality holds when $X = y_3 e_3 + y_4 e_4$ for n even and $X = y_2 e_2 + y_3 e_3$ for n odd. Substituting e_2, e_3, e_4 by the expressions in (8) and (9) the Lemma follows. □

As a consequence of this result we obtain inequality (7).

Proposition 2.2. *Let $x_1, \ldots, x_n \in \mathbb{R}$, for $n \geq 4$, with $\sum_{i=1}^n x_i = 0$ and $\sum_{i=1}^n x_i^2 = 1$. Then*

$$\sum_{i=1}^{n} x_i x_{i-1} < \frac{3n^2 - 4\pi^2}{3n^2 + 2\pi^2}, \tag{12}$$

with $x_0 = x_n$.

Proof. By Theorem 2.1 in order to prove (12) it is enough to show that

$$\cos(\frac{2\pi}{n}) < \frac{3n^2 - 4\pi^2}{3n^2 + 2\pi^2}.$$

Denoting $2\pi/n$ by α the above inequality is equivalent to

$$\cos\alpha < \frac{6 - 2\alpha^2}{6 + \alpha^2},$$

which using that $\cos\alpha < 1 - \alpha^2/2 + \alpha^4/24$ is easily verified. □

We remark that equality in (7) never holds.

Corollary 2.3. *For $n \in \mathbb{N}$, $n \geq 4$, let $f : [0, 2\pi] \longrightarrow \mathbb{R}$ be a continuous function, linear on each interval $[\frac{2\pi}{n}(j-1), \frac{2\pi}{n}j]$, $j = 1, \ldots, n$ and such that $f(0) = f(2\pi)$. Assume that $\int_0^{2\pi} f(t)dt = 0$. Then*

$$\int_0^{2\pi} f^2(t)\,dt \leq \int_0^{2\pi} f'^2(t)\,dt. \tag{13}$$

Proof. As said, inequality (13) with hypothesis $\int_0^{2\pi} f(t)\,dt = 0$ is equivalent to (7). So the Corollary is a direct consequence of Proposition 2.2. □

3 Wirtinger's inequality

Now we can obtain, by a limiting process, the classical Wirtinger's inequality.

Theorem 3.1 (Wirtinger's inequality)**.** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a 2π-periodic $\mathcal{C}^1$ function such that $\int_0^{2\pi} f(t)\,dt = 0$. Then*

$$\int_0^{2\pi} f^2(t)dt \leq \int_0^{2\pi} f'^2(t)\,dt. \tag{14}$$

Equality holds if and only if $f(t) = a\cos(t) + b\sin(t)$ for some $a, b \in \mathbb{R}$.

Proof. For each $n \in \mathbb{N}$, $n \geq 4$, let $\phi_n(t)$ be the function linear on each interval $[\frac{2\pi}{n}(j-1), \frac{2\pi}{n}j]$, with $\phi_n(\frac{2\pi}{n}j) = f(\frac{2\pi}{n}j)$, $j = 1, \ldots, n$.

Set $x_{j,n} = f(\frac{2\pi}{n}j)$, $m_n = \dfrac{1}{n}\sum_{j=1}^n x_j$, and $\tilde{x}_{j,n} = x_{j,n} - m_n$. Let $\tilde{\phi}_n(t)$ be the function linear on each interval $[\frac{2\pi}{n}(j-1), \frac{2\pi}{n}j]$, with $\tilde{\phi}_n(\frac{2\pi}{n}j) = \tilde{x}_{j,n}$, $j = 1, \ldots, n$. Equivalently, $\tilde{\phi}_n(t) = \phi_n(t) - m_n$.

Since $\int_0^{2\pi} \tilde{\phi}(t)\,dt = 0$ it follows, by Corollary 2.3, that

$$\int_0^{2\pi} \tilde{\phi}_n^2(t)\,dt \leq \int_0^{2\pi} \tilde{\phi}_n'^2(t)\,dt.$$

Moreover since f is a $\mathcal{C}^1$ function we have

$$\lim_{n\to\infty} \int_0^{2\pi} \phi_n^2(t)\,dt = \int_0^{2\pi} f^2(t)\,dt$$

and

$$\lim_{n\to\infty} \int_0^{2\pi} \phi_n'^2(t)\,dt = \int_0^{2\pi} f'^2(t)\,dt. \tag{15}$$

Finally the hypothesis $\int_0^{2\pi} f(t)\,dt = 0$ yields $\lim_{n\to\infty} m_n = 0$ and so

$$\lim_{n\to\infty} \int_0^{2\pi} \tilde{\phi}_n^2(t)\,dt = \lim_{n\to\infty} \int_0^{2\pi} \phi_n^2(t)\,dt,$$

and inequality (14) follows.

It remains to analize when equality holds in (14).

From now on we will assume that n is an odd integer; the case n even is dealt similarly. Let $H_k = \langle e_{2k}, e_{2k+1} \rangle$ denote the subspace of $\mathbb{R}^n$ generated by e_{2k} and e_{2k+1}, the vectors introduced in Section 2, for $k = 1, \ldots, (n-1)/2$. Let P_k be the orthogonal projection from $\mathbb{R}^n$ on H_k, and let P_0 be the orthogonal projection on $H_0 = \langle e_1 \rangle$.

We need the following

Lemma 3.2. *Let f be a function satisfying the hypotheses of Theorem 3.1 and such that equality holds in (14). For each $n \geq 4$ let X_n be the vector of components $x_{j,n} = f(\frac{2\pi}{n} j)$, $j = 1, \ldots, n$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{(n-1)/2} ||P_k(X_n)||^2 = 0.$$

Proof. By the definition of Riemann's integral we have

$$\lim_{n\to\infty} \frac{2\pi}{n} ||X_n||^2 = \lim_{n\to\infty} \frac{2\pi}{n} \sum_{k=0}^{p} ||P_k(X_n)||^2 = \int_0^{2\pi} f^2(t)\, dt,$$

where $p = (n-1)/2$.

From (5), (11) and (15) it follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{n}{\pi} \left(\sum_{k=0}^{p} ||P_k(X_n)||^2 - \langle T(X_n), X_n \rangle \right) \\ = &\lim_{n\to\infty} \frac{n}{\pi} \sum_{k=1}^{p} (||P_k(X_n)||^2 (1 - \cos(\frac{2\pi}{n} k))) = \int_0^{2\pi} f'^2(t)\, dt. \end{aligned}$$

As a consequence of equality in (14) we get

$$\lim_{n\to\infty} \sum_{k=1}^{p} \left[\frac{n}{\pi} (1 - \cos(\frac{2\pi}{n} k)) - \frac{2\pi}{n} \right] ||P_k(X_n)||^2 = 0.$$

The Lemma follows from the inequality

$$\frac{n}{\pi} (1 - \cos(\frac{2\pi}{n} k)) - \frac{2\pi}{n} \geq \frac{1}{n}$$

which is true for $k \geq 2$ (which implies $n \geq 5$) using that $\cos(x) \leq 1 - x^2/2 + x^4/24$. □

To continue the proof of Theorem 3.1, for each vector $X = (x_1, \ldots, x_n)$ let L_X be the function that is linear on each interval $[\frac{2\pi}{n}(j-1), \frac{2\pi}{n} j]$ with $L_X(\frac{2\pi}{n} j) = x_j$, $j = 1, \ldots, n$, $(x_0 = x_n)$.

When X_n is the vector of components $x_{j,n} = f(\frac{2\pi}{n} j)$, $j = 1, \ldots, n$, L_{X_n} is the function ϕ_n defined at the begining of this proof. So we can assume that $\sum_{j=1}^{n} x_{j,n} = 0$ and we know that $\lim_{n\to\infty} L_{X_n} = f$.

Writing $X_n = y_2 e_2 + y_3 e_3 + \sum_{k=2}^{p} (y_{2k} e_{2k} + y_{2k+1} e_{2k+1})$ we have

$$L_{X_n} = y_2 L_{e_2} + y_3 L_{e_3} + \sum_{k=2}^{p} (y_{2k} L_{e_{2k}} + y_{2k+1} L_{e_{2k+1}}) := \alpha_n + \beta_n.$$

To finish the proof we need to show that

$$\lim_{n\to\infty} \alpha_n = a\cos(t) + b\sin(t), \text{ for some } a, b \in \mathbb{R} \text{ and } \lim_{n\to\infty} \beta_n = 0.$$

Formula (4) can be writen as

$$\int_0^{2\pi} L_X^2\, dt = \frac{4\pi}{3n}||X||^2 + \frac{2\pi}{3n}\langle X, T(X)\rangle$$

which gives, by using the identity of polarization,

$$\langle L_X, L_Y\rangle := \int_0^{2\pi} L_X L_Y\, dt = \frac{4\pi}{3n}\langle X, Y\rangle + \frac{\pi}{3n}\langle T(X), Y\rangle + \frac{\pi}{3n}\langle X, T(Y)\rangle$$

for two vectors X, Y.

In particular one gets $\langle L_{e_i}, L_{e_j}\rangle = 0$, $i \neq j$, and hence

$$\langle L_X, L_{e_j}\rangle = y_j \langle L_{e_j}, L_{e_j}\rangle, \quad j = 2, \dots, n,$$

and

$$\langle L_{e_{2k}}, L_{e_{2k}}\rangle = \frac{4\pi}{3n}(1 + \frac{1}{2}\cos\frac{2\pi}{n}k) = \langle L_{e_{2k+1}}, L_{e_{2k+1}}\rangle, \quad k = 1, \dots, \frac{n-1}{2}.$$

Writing $\tilde{e}_2 = \sqrt{\frac{n}{2}}\, e_2$, $\tilde{e}_3 = \sqrt{\frac{n}{2}}\, e_3$ we have

$$\begin{aligned}
\lim_{n\to\infty} \alpha_n &= \lim_{n\to\infty} (y_2 L_{e_2} + y_3 L_{e_3}) \\
&= \lim_{n\to\infty} \frac{\langle L_{X_n}, L_{\tilde{e}_2}\rangle \frac{\sqrt{2}}{\sqrt{n}} L_{\tilde{e}_2} \frac{\sqrt{2}}{\sqrt{n}} + \langle L_{X_n}, L_{\tilde{e}_3}\rangle \frac{\sqrt{2}}{\sqrt{n}} L_{\tilde{e}_3} \frac{\sqrt{2}}{\sqrt{n}}}{\frac{4\pi}{3n}(1 + \frac{1}{2}\cos\frac{2\pi}{n})} \\
&= \frac{1}{\pi}\left(\int_0^{2\pi} f(t)\cos(t)\, dt\right)\cos t + \frac{1}{\pi}\left(\int_0^{2\pi} f(t)\sin(t)\, dt\right)\sin t.
\end{aligned}$$

Thus $\lim_{n\to\infty} \alpha_n = a\cos t + b\sin t$, as wanted, where a, b are the first Fourier coefficients of f.

As concerning $\lim_{n\to\infty} \beta_n$ we have

$$\begin{aligned}
\langle \beta_n, \beta_n\rangle &= \int_0^{2\pi} \beta_n \cdot \beta_n\, dt = \\
&= \sum_{k=2}^{p} (y_{2k}^2 + y_{2k+1}^2)\frac{4\pi}{3n}(1 + \frac{1}{2}\cos\frac{2\pi}{n}k) \leq 2\pi\frac{1}{n}\sum_{k=2}^{p} ||P_k(X_n)||^2,
\end{aligned}$$

and the proof finishes by applying Lema 3.2. □

Remark. Let f be a 2π-periodic $\mathcal{C}^1$ function such that $\int_0^{2\pi} f(t)\, dt = 0$. The same argument used to calculate $\lim_{n\to\infty} \alpha_n$ in the above proof, applied also to β_n shows that f can be written as

$$f(t) = \sum_{j=1}^{\infty} (a_j \cos(jt) + b_j \sin(jt)),$$

with

$$a_j = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(jt) dt, \quad b_j = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(jt) dt.$$

If we drop the assumption $\int_0^{2\pi} f(t)\, dt = 0$ we need to add in the above expression of f the term $\frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt$. So, the discrete approach we have developped here leads, in a natural way, to the well known Fourier series development of a 2π-periodic function.

References

[1] Wilhelm Blaschke. *Kreis und Kugel*. Veit & Co. Leiptzig, 1916. Second edition, Walter de Gruyter & Co. Berlin 1956.

[2] K. Fan, O. Taussky, and J. Todd. Discrete analogs of inequalities of Wirtinger. *Monatsch. Math.*, 59:73–90, 1955.

[3] Agustí Reventós. *Affine Maps, Euclidean Motions and Quadrics*. Springer Undergraduate Mathematics Series, 2011.

[4] O. Shisha. On the discrete version of Wirtinger's inequality. *Amer. Math. Monthly*, 80:755–760, 1973.

Departament de Matemàtiques, Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona, Catalonia

jcufi@mat.uab.cat, agusti@mat.uab.cat, rdrgzcarlos@gmail.com.