MASS TRANSPORT AND UNIFORM RECTIFIABILITY

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Abstract. In this paper we characterize the so called uniformly rectifiable sets of David and Semmes in terms of the Wasserstein distance W_2 from optimal mass transport. To obtain this result, we first prove a localization theorem for the distance W_2 which asserts that if μ and ν are probability measures in $\mathbb{R}^n$, φ is a radial bump function smooth enough so that $\int \varphi\, d\mu \gtrsim 1$, and μ has a density bounded from above and from below on $\operatorname{supp}(\varphi)$, then $W_2(\varphi\mu, a\varphi\nu) \leq c\, W_2(\mu,\nu)$, where $a = \int \varphi\, d\mu / \int \varphi\, d\nu$.

1. Introduction

In this paper we characterize the so called uniformly rectifiable sets of David and Semmes [DS2] in terms of the Wasserstein distance W_2 from optimal mass transport. To obtain this result, a fundamental tool is a new localization theorem for the distance W_2, which we think that has its own interest.

To state our results in detail we need to introduce some notation and terminology. Recall that, for $1 \leq p < \infty$, the Wasserstein distance W_p between two (Borel) probability measures μ, ν on $\mathbb{R}^d$ is defined by

$$W_p(\mu,\nu) = \inf_{\pi} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^p \, d\pi(x,y) \right)^{1/p},$$

where the infimum is taken over all probability measures π on $\mathbb{R}^n \times \mathbb{R}^n$ whose marginals are μ and ν. That is, $\pi(A \times \mathbb{R}^n) = \mu(A)$ and $\pi(\mathbb{R}^n \times A) = \nu(A)$ for all measurable sets $A \subset \mathbb{R}^n$. The same definition makes sense if instead of probability measures one considers measures μ, ν, π of the same mass. In the particular case $p = 1$, by the Kantorovich duality, the distance W_1 can also be characterized as follows:

$$W_1(\mu,\nu) = \sup\left\{ \left| \int f\, d\mu - \int f\, d\nu \right| : \operatorname{Lip}(f) \leq 1 \right\}. \tag{1.1}$$

The Wasserstein distances play a key role in many problems of optimal mass transport. Further, quite recently they have also been shown to be useful in questions in connection with PDE's (such as the Boltzmann equation) and Riemannian geometry. See [Vi1] and [AGS] for two basic and modern references on Wasserstein distances and mass transport.

Let us turn our attention to uniform rectifiability now. Given $0 < n \leq d$, we say that a Borel measure μ on $\mathbb{R}^d$ is n-dimensional Ahlfors-David regular, or simply

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AD-regular, if there exists some constant c_0 such that $c_0^{-1} r^n \leq \mu(B(x,r)) \leq c_0 r^n$ for all $x \in \operatorname{supp}(\mu)$, $0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu))$. It is not difficult to see that such a measure μ must be of the form $\mu = \rho\, \mathcal{H}^n \lfloor \operatorname{supp}(\mu)$, where ρ is some positive function bounded from above and from below and $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. A Borel set $E \subset \mathbb{R}^d$ is called AD-regular if the measure $\mathcal{H}^n_E$ is AD-regular, where we used the notation $\mathcal{H}^n_E = \mathcal{H}^n \lfloor E$.

The notion of uniform n-rectifiability (or simply, uniform rectifiability) was introduced by David and Semmes in [DS2]. For $n = 1$, an AD-regular 1-dimensional measure is uniformly rectifiable if and only if its support is contained in an AD-regular curve. For an arbitrary integer $n \geq 1$, the notion is more complicated. One of the many equivalent definitions (see Chapter I.1 of [DS2]) is the following: μ is uniformly rectifiable if there exist constants $\theta, M > 0$ so that, for each $x \in \operatorname{supp}(\mu)$ and $R > 0$, there is a Lipschitz mapping g from the n-dimensional ball $B_n(0,r) \subset \mathbb{R}^n$ into $\mathbb{R}^d$ such that g has Lipschitz norm $\leq M$ and

$$\mu\big(B(x,r) \cap g(B_n(0,r))\big) \geq \theta r^n.$$

In the language of [DS2], this means that $\operatorname{supp}(\mu)$ *has big pieces of Lipschitz images of* $\mathbb{R}^n$. A Borel set $E \subset \mathbb{R}^d$ is called uniformly rectifiable if $\mathcal{H}^n_E$ is uniformly rectifiable.

The reason why uniform rectifiability has attracted much attention in the last years is because this is the natural notion in many problems where rectifiability is involved in a quantitative way, such as in problems in connection with singular integral operators. Indeed, as shown in [DS2], it turns out that an n-dimensional AD-regular measure μ is uniformly rectifiable if and only if a sufficiently big class of singular integral operators with an n-dimensional odd Calderón-Zygmund kernel is bounded in $L^2(\mu)$. In the case $n = 1$, it was shown in [MMV] that uniform rectifiability is equivalent the $L^2(\mu)$ boundedness of only one operator, namely the Cauchy transform. The analogous result in higher dimensions involving the n-dimensional Riesz transforms instead of the Cauchy transform is still open and is subject of active investigation. See [MPr] or [MT], for instance.

In this paper we will characterize uniform rectifiability in terms of some scale invariant coefficients α_p. They are defined as follows. Let $\varphi : \mathbb{R}^d \to [0,1]$ be a radial Lipschitz function such that $\chi_{B(0,2)} \leq \varphi \leq \chi_{B(0,3)}$ which also satisfies

(1.2)
$$c_2^{-1} \operatorname{dist}(x, \partial B(0,3))^2 \leq \varphi(x) \leq c_2 \operatorname{dist}(x, \partial B(0,3))^2, \qquad |\nabla \varphi(x)| \leq c_2 \operatorname{dist}(x, \partial B(0,3)),$$

for all $x \in B(0,3)$. Given a ball $B = B(x,r) \subset \mathbb{R}^d$, we denote

$$\varphi_B(y) = \varphi\Big(\frac{y-x}{r}\Big).$$

Given an n-dimensional AD-regular measure μ on $\mathbb{R}^d$ and a ball B with radius $r(B)$ that intersects $\operatorname{supp}(\mu)$, for $1 \leq p < \infty$, we define

$$\alpha_p(B) = \frac{1}{r(B)^{1+\frac{n}{p}}} \inf_L W_p\big(\varphi_B \mu,\, c_{B,L} \varphi_B \mathcal{H}^n_L\big), \tag{1.3}$$

where the infimum is taken over all n-dimensional affine planes that intersect B and the constant $c_{B,L}$ is chosen so that the measures $\varphi_B\mu$ and $c_{B,L}\varphi_B\mathcal{H}^n_L$ have the same mass. That is, $c_{B,L} = \int \varphi_B\,d\mu / \int \varphi_B\,d\mathcal{H}^n_L$. If $B = B(x,r)$, we will also use the notation $\alpha_p(x,r) = \alpha_p(B)$.

Observe that the coefficient $\alpha_p(B)$ measures in an scale invariant way how close is μ to a flat n-dimensional measure in $3B$. Recall that a flat n-dimensional measure is a measure of the form $c\,\mathcal{H}^n_L$, where $c>0$ and L is an affine n-plane. In particular, notice that if μ is flat and $B\cap \operatorname{supp}(\mu)\neq\varnothing$, then $\alpha_p(B)=0$. Notice also that

$$\alpha_p(B)\lesssim \alpha_q(B) \qquad \text{if } p\leq q.$$

Let us remark that, in the case $p=1$, some coefficients quite similar to the α_1's were already introduced in [To2], and they were denoted by α there. The precise relationship between the α_1's and the α's is explained in Lemma 5.1.

The coefficients α_p should also be compared with the coefficients β_p, well known in the area of quantitative rectifiability. Given $1\leq p<\infty$ and a ball $B\subset\mathbb{R}^d$, one sets

$$\beta_p(B) = \inf_L \left\{\frac{1}{r(B)^n}\int_{2B}\left(\frac{\operatorname{dist}(y,L)}{r(B)}\right)^p d\mu(y)\right\}^{1/p},$$

where the infimum is taken over all n-planes in $\mathbb{R}^d$. For $p=\infty$ one has to replace the L^p norm by a supremum:

$$\beta_\infty(B) = \inf_L \left\{\sup_{y\in\operatorname{supp}(\mu)\cap 2B}\frac{\operatorname{dist}(y,L)}{r(B)}\right\},$$

where the infimum is taken over all n-planes L in $\mathbb{R}^d$ again. The coefficients β_p first appeared in [Jo1] and [Jo2], in the case $n=1$, $p=\infty$. In [Jo1] P. Jones showed, among other results, how the β_∞'s can be used to prove the L^2 boundedness of the Cauchy transform on Lipschitz graphs. In [Jo2], he characterized 1-dimensional uniformly rectifiable sets in terms of the β_∞'s. He also obtained other quantitative results on rectifiability without the AD regularity assumption. For other p's and $n\geq 1$, the β_p's were introduced by David and Semmes in their pioneering study of uniform rectifiability in [DS1].

Notice that the β_p's only give information on how close $\operatorname{supp}(\mu)\cap 2B$ is to some n-plane (more precisely, how close is $\operatorname{supp}(\mu)\cap 2B$ to be contained in some n-plane). On the other hand, the coefficients α_p contain more precise information. Indeed, we will see in Lemma 5.2 below that $\beta_p(B)\leq\alpha_p(B)$ for $1<p<\infty$. It is immediate to check that the converse inequality fails, as $\beta_p(B)=0$ does not force μ to be flat in $2B$.

If B, B' are two balls with comparable radii and $B\subset B'$, then it is straightforward to check that $\beta_p(B)\lesssim\beta_p(B')$. However, this property is very far from being clear for the coefficients α_p. So given an AD-regular measure μ, suppose that $3B\subset B'$ and that $\mu(B)\approx\mu(B')\approx r(B)^n\approx r(B')^n$. Is it true that

$$\alpha_p(B)\lesssim\alpha_p(B')\,? \tag{1.4}$$

This question leads naturally to the following. Let μ,ν be probability measures on $\mathbb{R}^d$ and φ a non negative bump function smooth enough. Consider the measures $\varphi\mu$ and $\varphi\nu$. We wish to estimate how different they are in terms of the distance $W_p(\mu,\nu)$. More precisely, let $a = \int \varphi\,d\mu/\int \varphi\,d\nu$ and assume that the mass of the $\varphi\mu$ is quite big, that is, $\int \varphi\,d\mu \geq c_1 \approx 1$. Is it true that

$$W_p(\varphi\mu,\, a\varphi\nu) \leq c\, W_p(\mu,\nu)?,$$

with c depending on c_1. Using Kantorovich's duality, it is easy to check that this holds for $p = 1$, assuming only φ to be Lipschitz (see Subsection 3.1). We will show in this paper that this is also true for the quadratic Wasserstein distance, assuming additional conditions on φ and μ. The precise result, which we will use to prove that (1.4) holds for $p = 2$, is the following.

Theorem 1.1. *Let μ,ν be probability measures on $\mathbb{R}^n$. Let $B \subset \mathbb{R}^n$ be a closed ball with radius $r(B) \geq c_3^{-1}$, and suppose that $\mu(\mathring{B}), \nu(\mathring{B}) > 0$. Assume that μ is absolutely continuous with respect to the Lebesgue measure on B, so that $\mu\lfloor B = f(x)\,dx$, where the density f satisfies $c_4^{-1}\chi_B \leq f(x) \leq c_4$, for a.e. $x \in B$ and for some constant $c_4 > 0$. Let $\varphi : \mathbb{R}^n \to [0,1]$ be a radial Lipschitz function supported on B such that*

$$c_2^{-1}\operatorname{dist}(x,\partial B)^2 \leq \varphi(x) \leq c_2 \operatorname{dist}(x,\partial B)^2 \qquad \text{and} \qquad |\nabla\varphi(x)| \leq c_2 \operatorname{dist}(x,\partial B), \tag{1.5}$$

for all $x \in B$. Then

$$W_2(\varphi\mu,\, a\,\varphi\nu) \leq c\, W_2(\mu,\nu),$$

where $a = \int \varphi\,d\mu/\int \varphi\,d\nu$ and c depends only on c_2,c_3 and c_4.

Notice that the assumptions on μ and φ imply that $r(B) \approx 1$ and $\int \varphi\,d\mu \gtrsim 1$, with constants depending on c_2, c_3, and c_4.

The result also holds if, instead of (1.5), one asks

$$\varphi(x) \approx \operatorname{dist}(x,\partial B)^m \quad \text{and} \qquad |\nabla\varphi(x)| \lesssim \operatorname{dist}(x,\partial B)^{m-1},$$

for $x \in B$ and $m \geq 2$. However, for simplicity we have only considered the case $m = 2$.

The preceding theorem follows from an analogous result where the ball B is replaced by a cube R. We carry out this reduction by a suitable change of coordinates in Subsection 3.2. Roughly speaking, the proof of the corresponding theorem for the cube R consists in estimating $W_2(\varphi\mu, a\,T\#(\varphi\nu))$ in terms of $W_2(\mu,\nu)$, where T is a map such that $T\#\nu = \mu$ which realizes the optimal quadratic transport (the notation $T\#\nu$ stands for the image measure of ν by T, i.e. $T\#\nu(A) = \nu(T^{-1}A)$ for $A \subset \mathbb{R}^n$). This estimate is obtained by a multi-scale analysis, using Haar wavelets. Some of the ideas are partially inspired by [To2].

Roughly speaking, the quadratic decay of φ as one approaches the boundary is used to bound the interchanges of mass between $\varphi\mu$ and $(1-\varphi)\nu$, and between $\varphi\nu$ and $(1-\varphi)\mu$.

With the preceding localization result for W_2 at hand, we will prove the following.

Theorem 1.2. *Let μ be an AD-regular measure on $\mathbb{R}^d$ and $1 \leq p \leq 2$. Then μ is uniformly rectifiable if and only if $\alpha_p(x,t)^2\,d\mu(x)\,\frac{dr}{r}$ is a Carleson measure, that is, if for any ball B centered on $\operatorname{supp}(\mu)$ with radius R,*

$$\int_0^R \int_B \alpha_p(x,r)^2\,d\mu(x)\,\frac{dr}{r} \leq c\,R^n. \tag{1.6}$$

Some remarks are in order. First, let us mention that the same result is already known to hold with β_p instead of α_p (even with in a somewhat wider range of p's). This was shown by David and Semmes in [DS1]. Also, the case α_1 is (essentially) proved in [To2].

Notice that if, for some $p \in [1,2]$, $\alpha_p(x,r)^2\,d\mu(x)\,\frac{dr}{r}$ is a Carleson measure, then so is $\beta_p(x,r)^2\,d\mu(x)\,\frac{dr}{r}$, since $\beta_p(x,r) \leq \alpha_p(x,r)$, and thus by the results of David and Semmes in [DS1] μ is uniformly rectifiable. So the difficult implication in the theorem consists in showing that if μ is uniformly rectifiable, then $\alpha_p(x,r)^2\,d\mu(x)\,\frac{dr}{r}$ is a Carleson measure. As $\alpha_p(x,r) \lesssim \alpha_2(x,r)$ for $p \leq 2$, it suffices to consider the case $p = 2$ for this implication. To this end, the localization Theorem 1.1 will play a key role. First we will prove that $\alpha_2(x,r)^2\,d\mu(x)\,\frac{dr}{r}$ is a Carleson measure in the particular case where μ is comparable to $\mathcal{H}^n$ on an n-dimensional Lipschitz graph, and finally we will prove the result in full generality by means of a geometric corona type decomposition. This technique, which takes its name from the corona theorem of Carleson, has been adapted by David and Semmes to the analysis of uniformly rectifiable sets [DS1, DS2] and has already been shown to be useful in a variety of situations (see also, for example, [Lé], [To1], or [MT]).

The plan of the paper is the following. In Section 2 we introduce some additional notation and terminology. The localization Theorem 1.1 is proved in Sections 3 and 4. In Section 5 we explain the relationship among the coefficients $\alpha, \alpha_p, \beta_p$ and we show that $\alpha_p(B) \lesssim \alpha_p(B')$, under the assumptions just above (1.4). In Section 6 we prove Theorem 1.2 for the particular case of Lipschitz graphs, while the full result is proved in the final Section 7.

2. Preliminaries

As usual, in the paper the letter 'c' stands for an absolute constant which may change its value at different occurrences. On the other hand, constants with subscripts, such as c_1, retain their value at different occurrences. The notation $A \lesssim B$ means that there is a positive absolute constant c such that $A \leq cB$. So $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

Given $x \in \mathbb{R}^n$, $|x|$ stands for its Euclidean norm and $|x|_\infty$ for its ℓ_∞ norm.

By a cube in $\mathbb{R}^n$ we mean a cube with edges parallel to the axes. Most of the cubes in our paper will be dyadic cubes, which are assumed to be half open-closed. The collection of all dyadic cubes is denoted by $\mathcal{D}$. The side length of a cube Q is written as $\ell(Q)$, and its center as z_Q. The lattice of dyadic cubes of side length 2^{-j} is denoted by $\mathcal{D}_j$. On the other hand, if R is a cube, $\mathcal{D}(R)$ stands for the collection of cubes contained in R that are obtained by splitting it dyadically, and $\mathcal{D}_j(R)$ is

the subfamily of those cubes from $\mathcal{D}(R)$ with side length $2^{-j}\ell(R)$. On the other hand, if $Q \in \mathcal{D}$ or $Q \in \mathcal{D}(R)$, then $\mathcal{C}h(Q)$ is the family of dyadic children of Q. One says that Q is the parent of its children. Two cubes $Q, Q' \in \mathcal{D}$ are called brothers if they have the same parent. Also, given $a > 0$ and any cube Q, we denote by $a\,Q$ the cube concentric with Q with side length $a\,\ell(Q)$.

By a measure on $\mathbb{R}^n$, we mean a Radon measure. Its total variation is denoted by $\|\mu\|$. Given a set $A \subset \mathbb{R}^n$, we write $\mu\lfloor A = \chi_A\,\mu$. That is $\mu\lfloor A$ is the restriction of μ to A. The Lebesgue measure is denoted by m or by dx. Given a measure μ and a cube Q, we set $m_Q\mu = \mu(Q)/m(Q)$. That is, $m_Q\mu$ is the mean of μ on Q, with respect to Lebesgue measure. L^p is the usual L^p space with respect to Lebesgue measure, while $L^p(\mu)$ is the one with respect to the measure μ. As usual, we consider the pairing between two functions $\langle f, g\rangle = \int f\,g\,dx$, or between a function and a measure $\langle f, \mu\rangle = \int f\,d\mu$.

Given $A \subset \mathbb{R}^n$, we say that a measure μ on $\mathbb{R}^n$ is doubling on A if there exists a constant c_d such that

$$\mu(B(x,2r)) \leq c_d\,\mu(B(x,r)) \qquad \text{for all } x \in A \cap \operatorname{supp}(\mu),\ 0 < r \leq \operatorname{diam}(A).$$

In the case $A = \mathbb{R}^n$, we just simply say that μ is doubling.

The Hausdorff s-dimensional measure in $\mathbb{R}^n$ is denoted by $\mathcal{H}^s$. Given a set $E \subset \mathbb{R}^n$, we write $\mathcal{H}^s_E = \mathcal{H}^s\lfloor E$.

Concerning mass transport, recall that, by Brenier's theorem (see [Vi1, Chapter 2], for example), if μ and ν are probability measures which are absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^n$, then there exists a unique optimal transference plan (i.e. an optimal measure) π for W_2, and there are maps $S, T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\pi = (Id \times S)\#\mu = (T \times Id)\#\nu$. So $S\#\mu = \nu$ and $T\#\nu = \mu$, and

$$W_2(\mu,\nu)^2 = \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\,d\pi(x,y) = \int_{\mathbb{R}^n} |x - Sx|^2\,d\mu(x) = \int_{\mathbb{R}^n} |Ty - y|^2\,d\nu(y).$$

Moreover, $T \circ S = Id$ μ-a.e. and $S \circ T = Id$ ν-a.e.

Let us remark that the ambient space for the proof of Theorem 1.1 below will be $\mathbb{R}^n$, while the ambient space for the proof of Theorem 1.2 will be $\mathbb{R}^d$, and in this case we will reserve the letter n for the dimension of the measure μ.

3. The localization theorem for Wasserstein distances

3.1. **The case of W_1.** In this subsection we prove a localization result for W_1 under much weaker assumptions on μ and φ than the ones in Theorem 1.1.

Proposition 3.1. *Let μ and ν be probability measures on $\mathbb{R}^n$ and let B be a ball of radius $c_3^{-1} \leq r(B) \leq c_3$. Let $\varphi : \mathbb{R}^n \to [0,\infty)$ be a Lipschitz function supported on B with $\|\nabla\varphi\|_\infty \leq c_6$. Suppose that $\int \varphi\,d\mu \geq c_7$ and $\int \varphi\,d\nu > 0$. Then,*

$$W_1(\varphi\mu,\, a\,\varphi\nu) \leq c\,W_1(\mu,\nu), \tag{3.1}$$

where $a = \int \varphi\,d\mu / \int \varphi\,d\nu$, and c depends on c_3, c_6, c_7.

Proof. We may assume that $W_1(\mu,\nu)$ is small enough. Otherwise, the inequality (3.1) is trivial (for c big enough) because $\varphi\,\mu$ and $\varphi\,\nu$ are both supported on B and $r(B)\leq c_3$. Now, notice that by the Kantorovich duality,

$$\left|\int \varphi\,d\mu - \int \varphi\,d\nu\right| \leq \|\nabla\varphi\|_\infty\, W_1(\mu,\nu). \tag{3.2}$$

Since $a = \int \varphi\,d\mu/\int \varphi\,d\nu$, we have

$$|a-1| = \frac{\left|\int \varphi\,d\mu - \int \varphi\,d\nu\right|}{\int \varphi\,d\nu} \leq \frac{\|\nabla\varphi\|_\infty\, W_1(\mu,\nu)}{\int \varphi\,d\nu}.$$

To estimate $\int \varphi\,d\nu$ from below, we use (3.2) again, and then we obtain

$$\int \varphi\,d\nu \geq \int \varphi\,d\mu - \|\nabla\varphi\|_\infty\, W_1(\mu,\nu).$$

Since $\int \varphi\,d\mu \geq c_7$, if $W_1(\mu,\nu)$ is small enough, we get $\int \varphi\,d\nu \geq c_7/2$. Therefore,

$$|a-1| \leq c\,W_1(\mu,\nu), \tag{3.3}$$

with c depending also on $\|\nabla\varphi\|_\infty$. Let ψ be an arbitrary 1-Lipchitz function such that $\psi(0)=0$. By Kantorovich's duality again,

$$\begin{aligned}\left|\int \psi\,\varphi\,d\mu - \int \psi\,a\,\varphi\,d\nu\right| &\leq \left|\int \psi\,\varphi\,d\mu - \int \psi\,\varphi\,d\nu\right| + |1-a|\int \psi\,\varphi\,d\nu \\ &\leq \|\nabla(\psi\varphi)\|_\infty\, W_1(\mu,\nu) + \int |\psi\,\varphi|\,d\nu\; W_1(\mu,\nu).\end{aligned}$$

By the mean value theorem, it is easy to check that φ and $|\psi|$ are bounded uniformly above on B by some constant depending on c_3 and c_6. Then it follows that $\|\nabla(\psi\varphi)\|_\infty + \int |\psi\,\varphi|\,d\nu \lesssim 1$, and we deduce (3.1). □

Remark 3.2. Notice that (3.3) tells us that

$$|a-1| \leq c\,W_1(\mu,\nu) \leq c\,W_2(\mu,\nu),$$

assuming $W_1(\mu,\nu)$ small enough. Clearly, this also holds under the assumptions of Theorem 1.1, which are more restrictive than the ones in the preceding proposiion.

3.2. From a ball B to a cube R in Theorem 1.1.

Definition 3.3. Given a cube R and a function $\varphi : R \to [0,1]$, we write $\varphi \in \mathcal{G}_0(R)$ if φ is Lipschitz, $\varphi(x) \approx 1$ on $\frac{1}{2}R$, and there exists a constant c_2 such that

$$\varphi(x) \leq c_2 \frac{\operatorname{dist}(x,\partial R)^2}{\ell(R)^2} \quad \text{and} \quad |\nabla\varphi(x)| \leq c_2 \frac{\varphi(x)}{\operatorname{dist}(x,\partial R)} \quad \text{for all } x \in R.$$

On the other hand, we write $\varphi \in \mathcal{G}(R)$ if $\varphi : R \to [0,1]$ is Lipschitz and, for all $x \in R$, satisfies

$$c_2^{-1} \frac{\operatorname{dist}(x,\partial R)^2}{\ell(R)^2} \leq \varphi(x) \leq c_2 \frac{\operatorname{dist}(x,\partial R)^2}{\ell(R)^2},$$

and

$$|\nabla\varphi(x)| \leq c_2 \frac{\operatorname{dist}(x,\partial R)}{\ell(R)^2}, \qquad |\nabla_T\varphi(x)| \leq c_2 \frac{\operatorname{dist}(x,\partial R)^2}{\ell(R)^3}.$$

Here, $|\nabla_T\varphi(x)| = \sup_v |v\cdot\nabla\varphi(x)|$, where the supremum is taken over all unit vectors v which are parallel to the face of R which is closest to x. If the face is not unique, then we set $\nabla_T\varphi(x) = 0$.

For example, if φ is a function supported on R such that $\varphi(x) \approx 1$ and is smooth in a neighborhood of $\frac{1}{2}R$, and

$$\varphi(x) = \frac{\operatorname{dist}(x,\partial R)^2}{\ell(R)^2} \qquad \text{for } x \in R \setminus \frac{1}{2}R,$$

then $\varphi \in \mathcal{G}(R)$. To check that $|\nabla_T\varphi(x)| \leq c_2\operatorname{dist}(x,\partial R)^2/\ell(R)^3$, take $x \in R\setminus\frac{1}{2}R$ such that there exists a unique face closest to x, and let t be a vector parallel to this face. Then, $\operatorname{dist}(x+\varepsilon t, \partial R) = \operatorname{dist}(x,\partial R)$ for $|\varepsilon|$ small enough, and thus $\nabla_T\varphi(x) = 0$.

In the next subsections we will prove the following result.

Theorem 3.4. *Let μ,ν be probability measures on $\mathbb{R}^n$. Let $R\subset\mathbb{R}^n$ be the closed cube with side length $\geq c_5$, and suppose that $\mu(\overset{\circ}{R}), \nu(\overset{\circ}{R}) > 0$. Assume that μ absolutely continuous with respect to the Lebesgue measure on R, so that $\mu\llcorner R = f(x)\,dx$, where the density f satisfies $c_4^{-1}\chi_R \leq f(x) \leq c_4\,\chi_R$ for a.e. $x\in R$ and for some constant $c_4 > 0$. If $\varphi\in\mathcal{G}(R)$, then*

$$W_2(\varphi\mu,\, a\,\varphi\nu) \leq c\, W_2(\mu,\nu),$$

where $a>0$ is chosen so that $\int\varphi\,d\mu = a\int\varphi\,d\nu$ and c depends only on c_2, c_4 and c_5.

Let see how Theorem 1.1 follows from the preceding result.

Proof of Theorem 1.1 using Theorem 3.4. Suppose first that $R = [-1,1]^n$ and B is the unit ball, both centered at the origin. Consider the map $F:\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$F(x) = \frac{|x|}{|x|_\infty}\,x.$$

This maps balls centered at the origin of radius r to concentric cubes with side length $2r$. In particular, $F(B) = R$. It is easy to check that F is bilipschitz. Moreover, it can be checked that its Jacobian equals $J(F)(x) = 2|x|^n/|x|_\infty^n \approx 1$.

Now we consider the measures $F\#\mu$ and $F\#\nu$. Notice that

$$dF\#\mu(x) = J(F^{-1})(x)\, f(F^{-1}(x))\,dx.$$

Thus the density of $F\#\mu$ is bounded above and below. On the other hand, in can be checked that $\varphi\circ F^{-1}\in\mathcal{G}(R)$. Indeed, notice that since φ is radial, then $\varphi\circ F^{-1}$ is constant on the boundaries of the cubes centered at the origin.

Then, from Theorem 3.4 we deduce that

$$W_2\big((\varphi\circ F^{-1})F\#\mu,\, a(\varphi\circ F^{-1})F\#\nu\big) \lesssim W_2(F\#\mu, F\#\nu). \tag{3.4}$$

Since F if bilipschitz, $W_2(F\#\mu, F\#\nu) \approx W_2(\mu,\nu)$. Also, since $F\#(\varphi\mu) = (\varphi\circ F^{-1})F\#\mu$ and analogously for ν, we have

$$W_2\big((\varphi\circ F^{-1})F\#\mu, a(\varphi\circ F^{-1})F\#\nu\big) \approx W_2(\varphi\mu, a\varphi\nu),$$

and so the theorem follows from (3.4).

In case B is not the unit ball or R is not the unit cube, we just compose F with an affine map to obtain another $\widetilde{F}$ such that $\widetilde{F}(R) = B$ and does the job. Then, we argue with $\widetilde{F}$ instead of F. □

3.3. **Preliminary lemmas for the proof of Theorem 3.4.** In next lemma we recall the properties of the so called *Whitney decomposition* of a proper open set.

Lemma 3.5. *An open subset $\Omega \subsetneq \mathbb{R}^n$ can be decomposed as follows:*

$$\Omega = \bigcup_{k=1}^{\infty} Q_k,$$

where Q_k are disjoint dyadic cubes (the so called "Whitney cubes") such that for some constants $r > 20$ and $D_0 \geq 1$ the following holds,

(i) $5Q_k \subset \Omega$.
(ii) $rQ_k \cap \Omega^c \neq \varnothing$.
(iii) *For each cube Q_k, there are at most D_0 squares Q_j such that $5Q_k \cap 5Q_j \neq \varnothing$. Moreover, for such squares Q_k, Q_j, we have $\frac{1}{2}\ell(Q_k) \leq \ell(Q_j) \leq 2\,\ell(Q_k)$.*

This is a well known result. See for example [St, pp. 167-169] for the proof.

Roughly speaking, next lemma deals with the existence of a good big subset $G \subset R$ such that the mass on G is transported not too far.

Lemma 3.6. *Let μ, ν be probability measures on $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. Take T such that $T\#\nu = \mu$ and*

$$W_2(\mu,\nu)^2 = \int |Tx - x|^2\, d\nu(x).$$

Let $R \subset \mathbb{R}^n$ be a cube and let $\{Q_i\}_{i\in I}$ be a Whitney decomposition of $\overset{\circ}{R}$ as in Lemma 3.5. For $x \in \overset{\circ}{R}$, let Q_x be the Whitney cube Q_i that contains x. Denote

$$G = \big\{x \in R : |x - Tx| \leq \ell(Q_x)\big\} \cap \big\{x \in T^{-1}(R) : |x - Tx| \leq \ell(Q_{Tx})\big\}. \tag{3.5}$$

Let $\varphi \in \mathcal{G}_0(R)$ and suppose that

$$W_2(\mu,\nu)^2 \leq c_8\,(\varphi\nu)(\mathbb{R}^n)\,\ell(R)^2, \tag{3.6}$$

where c_8 is some positive constant small enough. Then $\int_G \varphi\, d\nu > 0$.

Consider the measure $\widetilde{\nu} = \widetilde{a}\,\chi_G\,\varphi\,\nu$, with $\widetilde{a} = (\varphi\nu)(\mathbb{R}^n)/(\varphi\nu)(G)$ (so that $(\varphi\nu)(\mathbb{R}^n) = \widetilde{\nu}(\mathbb{R}^n)$). Then we have

$$\|\varphi\nu - \widetilde{\nu}\| \leq \frac{c}{\ell(R)^2}\, W_2(\mu,\nu)^2, \tag{3.7}$$

and consequently,

$$W_2(\varphi\nu, \widetilde{\nu}) \lesssim W_2(\mu, \nu). \tag{3.8}$$

Below, to simplify notation we will write $\varphi\nu(E) := (\varphi\nu)(E)$ for any $E \subset \mathbb{R}^n$, and analogously for μ.

Proof. The last estimate follows from (3.7), taking into account that $\operatorname{supp}(\varphi\nu) \cup \operatorname{supp}(\widetilde{\nu}) \subset R$. To prove (3.7), we write

$$\begin{aligned} \|\varphi\nu - \widetilde{\nu}\| &\leq \int |1 - \widetilde{a}|\,\varphi\,d\nu + \widetilde{a} \int |\varphi - \varphi\chi_G|\,d\nu \\ &= |1 - \widetilde{a}|\,\varphi\nu(\mathbb{R}^n) + \widetilde{a} \int_{G^c} \varphi\,d\nu. \end{aligned} \tag{3.9}$$

To estimate $\int_{G^c} \varphi\,d\nu$, denote by G_1 and G_2 the two sets appearing on the right side of (3.5), respectively. Then we set

$$\int_{G^c} \varphi\,d\nu = \int_{G_1^c} \varphi\,d\nu + \int_{G_1 \cap G_2^c} \varphi\,d\nu = I_1 + I_2.$$

We will use the fact that for every $x \in R$,

$$\varphi(x) \lesssim \frac{\ell(Q_x)^2}{\ell(R)^2}.$$

Since $|Tx - x| \geq \ell(Q_x)$ for $x \in R \setminus G_1$, we deduce

$$\varphi(x) \lesssim \frac{|Tx - x|^2}{\ell(R)^2} \qquad \text{for } x \in R \setminus G_1.$$

Therefore,

$$I_1 \lesssim \frac{1}{\ell(R)^2} \int_{\operatorname{supp}(\varphi)} |Tx - x|^2\,d\nu \leq \frac{1}{\ell(R)^2}\, W_2(\mu, \nu)^2. \tag{3.10}$$

Let us turn our attention to the integral

$$I_2 = \int_{R \cap G_1 \cap G_2^c} \varphi\,d\nu.$$

Observe that if $x \in R \cap G_1 \cap G_2^c$, then $|x - Tx| \leq \ell(Q_x)$ and so $Tx \in 3Q_x \subset R$. Thus $3Q_x \cap Q_{Tx} \neq \varnothing$. Thus,

$$\ell(Q_x) \approx \ell(Q_{Tx}) \leq |x - Tx|,$$

using that $Tx \in R$ and $x \notin G_2$ for the last inequality. Therefore,

$$\varphi(x) \lesssim \frac{\ell(Q_x)^2}{\ell(R)^2} \lesssim \frac{|Tx - x|^2}{\ell(R)^2},$$

and then (3.10) also holds with I_2 instead of I_1 (multiplying by a constant if necessary). Thus,

$$\int_{G^c} \varphi\,d\nu \lesssim \frac{1}{\ell(R)^2}\, W_2(\mu, \nu)^2. \tag{3.11}$$

Now we deal with the term $|1-\widetilde{a}|$. Observe that $\widetilde{a} = \int \varphi\,d\nu / \int_G \varphi\,d\nu$, and so

$$\widetilde{a} - 1 = \frac{\int_{G^c} \varphi\,d\nu}{\int_G \varphi\,d\nu}.$$

From the assumption (3.6) and (3.11) we infer that

$$\int_G \varphi\,d\nu \geq \frac{1}{2}\int \varphi\,d\nu = \frac{1}{2}\,\varphi\nu(\mathbb{R}^n) \tag{3.12}$$

if c_8 is chosen small enough, and thus $\widetilde{a} \leq 2$ and

$$|1-\widetilde{a}| \leq \frac{2}{\varphi\nu(\mathbb{R}^n)}\varphi\nu(G^c) \lesssim \frac{1}{\varphi\nu(\mathbb{R}^n)\,\ell(R)^2}\,W_2(\mu,\nu)^2. \tag{3.13}$$

Plugging this estimate and (3.10) into (3.9), the lemma follows. □

Remark 3.7. If $W_2(\mu,\nu)$ is small enough, then from the arguments in Subsection 3.1 it turns out that $\varphi\nu(\mathbb{R}^n) \geq 1/2$, say, and then from (3.13) we deduce

$$|1-\widetilde{a}| \lesssim W_2(\mu,\nu)^2.$$

Lemma 3.8. *Suppose the same notation and assumptions of Lemma 3.6. In particular, assume that*

$$W_2(\mu,\nu)^2 \leq c_8\,(\varphi\mu)(\mathbb{R}^n)\,\ell(R)^2, \tag{3.14}$$

where c_8 is some positive constant small enough. Then $\varphi\big(T\#(\chi_G\,\nu)\big)(\mathbb{R}^n) > 0$.

Consider the measure

$$\widetilde{\mu} = \widetilde{b}\,\varphi\big(T\#(\chi_G\,\nu)\big)$$

with $\widetilde{b}$ chosen so that $(\varphi\mu)(\mathbb{R}^n) = \widetilde{\mu}(\mathbb{R}^n)$. Then we have

$$\|\varphi\mu - \widetilde{\mu}\| \leq \frac{c}{\ell(R)^2}\,W_2(\mu,\nu)^2, \tag{3.15}$$

and consequently,

$$W_2(\varphi\mu,\widetilde{\mu}) \lesssim W_2(\mu,\nu). \tag{3.16}$$

Proof. The arguments are quite similar to the ones of the preceding lemma. However, for completeness, we show the details. The last estimate follows from (3.15), taking into account that $\operatorname{supp}(\varphi\mu)\cup\operatorname{supp}(\widetilde{\mu}) \subset R$.

To prove (3.15), we write

$$\begin{aligned}\|\varphi\mu - \widetilde{\mu}\| &\leq \int |1-\widetilde{b}|\,\varphi\,d\mu + \widetilde{b}\left|\int \big(\varphi\,dT\#\nu - \varphi\,dT\#(\chi_G\,\nu)\big)\right| \\ &= |1-\widetilde{b}|\,\varphi\mu(\mathbb{R}^n) + \widetilde{b}\int_{G^c} \varphi(T(x))\,d\nu(x).\end{aligned} \tag{3.17}$$

To estimate the last integral on the right side, denote by G_1 and G_2 the two sets appearing on the right side of (3.5) respectively. Then we have

$$\int_{G^c} \varphi(T(x))\,d\nu(x) = \int_{G_2^c} \varphi(T(x))\,d\nu(x) + \int_{G_2\cap G_1^c} \varphi(T(x))\,d\nu(x) = I_1 + I_2.$$

First we deal with I_1. Notice that we may assume that for x in the domain of integration of I_1 (and I_2), we have $x \in T^{-1}(R)$. Since $x \notin G_2$, this implies that $|x - Tx| > \ell(Q_{T_x})$, and thus

$$\varphi(Tx) \lesssim \frac{\ell(Q_{Tx})^2}{\ell(R)^2} \leq \frac{|x - Tx|^2}{\ell(R)^2}.$$

Therefore,

$$I_1 \lesssim \frac{1}{\ell(R)^2} \int |Tx - x|^2 \, d\nu(x) = \frac{1}{\ell(R)^2} W_2(\mu, \nu)^2. \tag{3.18}$$

Let us turn our attention to the integral $I_2 = \int_{G_2 \cap G_1^c} \varphi(Tx)\, d\nu(x)$. Observe that if $x \in T^{-1}(R) \cap G_2 \cap G_1^c$, then $|x - Tx| \leq \ell(Q_{Tx})$, and so $x \in 3Q_{Tx} \subset R$. Thus, $Q_x \cap 3Q_{Tx} \neq \varnothing$, and thus $\ell(Q_x) \approx \ell(Q_{T_x})$. So we deduce

$$\varphi(Tx) \lesssim \frac{\ell(Q_{Tx})^2}{\ell(R)^2} \lesssim \frac{\ell(Q_x)^2}{\ell(R)^2} \leq \frac{|x - Tx|^2}{\ell(R)^2},$$

using that $x \in R \setminus G_1$ in the last inequality. Then (3.18) also holds with I_2 instead of I_1. Thus,

$$\int_{G^c} \varphi(Tx)\, d\nu(x) \lesssim \frac{1}{\ell(R)^2} W_2(\mu, \nu)^2. \tag{3.19}$$

Now we deal with the term $|1 - \widetilde{b}|$. Observe that $\widetilde{b} = \int \varphi \circ T \, d\nu / \int_G \varphi \circ T \, d\nu$, and so

$$\widetilde{b} - 1 = \frac{\int_{G^c} \varphi \circ T \, d\nu}{\int_G \varphi \circ T \, d\nu}.$$

From the assumption (3.14) and (3.19) we infer that

$$\int_{G^c} \varphi \circ T \, d\nu \lesssim c_8 \, \varphi\mu(\mathbb{R}^n).$$

Therefore,

$$\int_G \varphi \circ T \, d\nu = \varphi\mu(\mathbb{R}^n) - \int_{G^c} \varphi \circ T \, d\nu \geq \frac{1}{2} \varphi\mu(\mathbb{R}^n), \tag{3.20}$$

choosing c_8 small enough. Thus $\widetilde{b} \leq 2$ and

$$|1 - \widetilde{b}| \leq \frac{2}{\varphi\mu(\mathbb{R}^n)} \int_{G^c} \varphi \circ T \, d\nu \lesssim \frac{1}{\varphi\mu(\mathbb{R}^n)\, \ell(R)^2} W_2(\mu, \nu)^2.$$

Plugging this estimate and (3.19) into (3.17), the lemma follows. □

Remark 3.9. For the record, observe that the inequality (3.19) says that

$$\|\varphi\mu - \widetilde{b}^{-1}\widetilde{\mu}\| \lesssim \frac{1}{\ell(R)^2} W_2(\mu, \nu)^2.$$

3.4. **The key lemma.** We need the following auxiliary result.

Lemma 3.10. *Let R and $\varphi \in \mathcal{G}(R)$ be as in Theorem 3.4. Consider the measure $d\widetilde{\sigma} = \varphi\, dm$ and let $Q \in \mathcal{D}(R)$. Consider a bounded function h supported on Q such that $\|h\|_\infty \leq 1$ and $\int h\, d\widetilde{\sigma} = 0$. Then there exists a map $U : Q \to Q$ such that $U\#(\widetilde{\sigma}\lfloor Q) = (1+h)\,\widetilde{\sigma}\lfloor Q$ which satisfies*

$$|Ux - x| \lesssim \ell(Q)\,\|h\|_\infty \qquad \text{for all } x \in Q.$$

We will prove this lemma by means of the Knothe-Rosenblatt coupling. The arguments involve some calculations that are quite lengthy and rather tedious, and so we will defer the proof to Section 4.

Lemma 3.11. *Let μ, R, and $\varphi \in \mathcal{G}(R)$ be as in Theorem 3.4. Denote $\sigma = \varphi\mu$ and let $Q \in \mathcal{D}(R)$. Consider a bounded function h supported on Q such that $\|h\|_\infty \leq 1$ and $\int h\, d\sigma = 0$. Then*

$$W_2(\sigma\lfloor Q,\, (1+h)\,\sigma\lfloor Q) \lesssim \ell(Q)\,\|h\|_\infty \sigma(Q)^{1/2}.$$

Proof. Recall that the density of μ satisfies $c_4^{-1} \leq f(x) \leq c_4$ for $x \in R$. So if we consider the measure $d\widetilde{\sigma} = \frac{1}{2}\, c_4^{-1} \varphi\, dm$, then we have $2\widetilde{\sigma} \leq \sigma \leq 2c_4^2\,\widetilde{\sigma}$. Notice also that we may assume that $\|h\|_\infty$ is small enough, because otherwise the lemma is trivial. So we suppose that $\|h\|_\infty \leq c_4^{-2}/4$. We write $\sigma = (\sigma - \widetilde{\sigma}) + \widetilde{\sigma}$, and

$$(1+h)\,\sigma = (\sigma - \widetilde{\sigma}) + \widetilde{\sigma} + h\,\frac{d\sigma}{d\widetilde{\sigma}}\,\widetilde{\sigma} = (\sigma - \widetilde{\sigma}) + \Big(1 + 2c_4\, f\, h\Big)\,\widetilde{\sigma}.$$

Notice that the function $\widetilde{h} = 2c_4\, f\, h$ satisfies $\|\widetilde{h}\|_\infty \leq 1/2$ and $\int_Q \widetilde{h}\, d\widetilde{\sigma} = 0$. By Lemma 3.10 applied to $\widetilde{\sigma}$ and $\widetilde{h}$, we deduce that there exists a map $\widetilde{U} : Q \to Q$ such that $\widetilde{U}\#(\widetilde{\sigma}\lfloor Q) = (1+\widetilde{h})\,\widetilde{\sigma}\lfloor Q$ which satisfies

$$|\widetilde{U}x - x| \lesssim \ell(Q)\,\|\widetilde{h}\|_\infty \approx \ell(Q)\,\|h\|_\infty \qquad \text{for all } x \in Q.$$

Since $\sigma - \widetilde{\sigma}$ is a positive measure, we can consider the transference plan

$$d\pi(x,y) = d(\sigma - \widetilde{\sigma})\lfloor Q(x)\,\delta_{x=y} + (Id \times \widetilde{U})\#\widetilde{\sigma}\lfloor Q(x,y).$$

The marginal measures of π are $\sigma\lfloor Q$ and

$$\sigma\lfloor Q - \widetilde{\sigma}\lfloor Q + \widetilde{U}\#\widetilde{\sigma}\lfloor Q = \sigma\lfloor Q + \widetilde{h}\,\widetilde{\sigma}\lfloor Q = (1+h)\,\sigma\lfloor Q.$$

Then,

$$W_2(\sigma\lfloor Q,\, (1+h)\,\sigma\lfloor Q)^2 \leq \int |x-y|^2\, d\pi(x,y) = \int_Q |x - U(x)|^2\, d\widetilde{\sigma} \lesssim \ell(Q)^2\,\|h\|_\infty^2\,\widetilde{\sigma}(Q).$$

Since $\widetilde{\sigma}(Q) \approx \sigma(Q)$, the lemma follows. □

The idea of subtracting a smaller nicer measure $\widetilde{\sigma}$ to σ is inspired by some techniques from Peyre [Pe].

Notice, by the way, that the measures $\widetilde{\sigma}$ and σ in the previous lemmas are doubling, because of the assumptions on μ and φ. This property is used strongly in many of the arguments below.

The following is an easy consequence of Lemma 3.11.

Lemma 3.12. *Let μ, R, and $\varphi \in \mathcal{G}(R)$ be as in Theorem 3.4. Denote $\sigma = \varphi\mu$ and let $c_9 \approx 1$ and $Q \in \mathcal{D}(R)$. Consider a function h supported on Q that is constant on the children of Q and vanishes out of Q, that is,*

$$h = \sum_{P \in \mathcal{C}h(Q)} c_P \chi_P.$$

Assume moreover that $\int h\,d\sigma = 0$ and $\|h\|_\infty \leq 1$. Then,

$$W_2(\sigma\lfloor Q,\, (1+h)\,\sigma\lfloor Q) \lesssim \ell(Q)\,\|h\|_\infty \sigma(Q)^{1/2} \approx \ell(Q)\,\|h\|_{L^2(\sigma)}.$$

The constant depend on c_9.

Given measures σ, τ and a dyadic cube Q, we consider the function defined by

$$\Delta_Q^\sigma \tau(x) = \begin{cases} \dfrac{\tau(P)}{\sigma(P)} - \dfrac{\tau(Q)}{\sigma(Q)} & \text{if } x \in P \in \mathcal{C}h(Q), \\ \\ 0 & \text{if } x \notin Q, \end{cases}$$

assuming that $\sigma(P), \sigma(Q) > 0$ for the cubes P, Q above. If σ coincides with the Lebesgue measure on $\mathbb{R}^n$, then we set $\Delta_Q \tau := \Delta_Q^\sigma \tau$. If $\tau = g(x)\,dx$, then we write $\Delta_Q g := \Delta(g\,dx) = \Delta_Q \tau$. Recall that if $g \in L^2$, then

$$g = \sum_{Q \in \mathcal{D}} \Delta_Q g, \tag{3.21}$$

with the sum converging in L^2. Moreover, the functions $\Delta_Q \tau$, $Q \in \mathcal{D}$, are mutually orthogonal, and thus $\|g\|_2^2 = \sum_{Q \in \mathcal{D}} \|\Delta_Q g\|_2^2$. In fact, (3.21) coincides with the decomposition of g into Haar wavelets: recall that g can be written as

$$g = \sum_{Q \in \mathcal{D}} \sum_{\varepsilon \in \mathcal{E}} \langle g, h_Q^\varepsilon \rangle\, h_Q^\varepsilon,$$

where $\mathcal{E} = \{0,1\}^n \setminus (0, \ldots, 0)$ and each h_Q^ε is a Haar n-dimensional wavelet supported on Q, associated to the index $\varepsilon \in \mathcal{E}$. Then it is easy to check that for every $Q \in \mathcal{D}$,

$$\Delta_Q g = \sum_{\varepsilon \in \mathcal{E}} \langle g, h_Q^\varepsilon \rangle\, h_Q^\varepsilon,$$

which we will also write as $\Delta_Q \tau = \sum_{\varepsilon \in \mathcal{E}} \langle \tau, h_Q^\varepsilon \rangle\, h_Q^\varepsilon$. See [Da, Part I] for more details, for example.

Now we are going to introduce the definition of a *tree* of dyadic cubes. For $R \in \mathcal{D}$, consider a family $\mathcal{S}$ (possibly empty) of disjoint cubes from $\mathcal{D}(R)$ which satisfies the following property:

$$\text{if } Q \in \mathcal{S}, \text{ then every brother of } Q \text{ also belongs to } \mathcal{S}.$$

Denote

$$\mathcal{T} = \big\{Q \in \mathcal{D}(R) : Q \text{ is not contained in any cube from } \mathcal{S}\big\}.$$

Then we say that $\mathcal{T}$ is a tree with *root* R, and the cubes from the family $\mathcal{S}$ are called the *stopping cubes of* $\mathcal{T}$. Notice that in our the definition, the stopping cubes do

not belong to $\mathcal{T}$. Observe also that $\mathcal{T}$ has the property that if $Q \in \mathcal{T}$ and $P \in \mathcal{D}(R)$ is such that $Q \subset P \subset R$, then $P \in \mathcal{T}$ too. This is why $\mathcal{T}$ is called tree.

Using Lemma 3.12 as a kind of building block, we will prove the following.

Lemma 3.13 (Key lemma)**.** *Let μ, R, and $\varphi \in \mathcal{G}(R)$ be as in Theorem 3.4. Denote $\sigma = \varphi\mu$, let $\mathcal{T}$ be a tree with root R and $\mathcal{S} \subset R$ be its family of stopping cubes. Consider a measure τ supported on R such that $\tau(R) = \sigma(R)$ and $\tau(Q) \approx \varphi\mu(Q) = \sigma(Q)$ for all $Q \in \mathcal{T}$. Then, for any $\alpha > 0$, we have*

$$W_2(\sigma,\tau)^2 \lesssim \sum_{Q\in\mathcal{T}} \frac{1}{m_Q\sigma} \|\Delta_Q(\sigma-\tau)\|_{L^2}^2\, \ell(Q)^{2-\alpha}\ell(R)^\alpha + \sum_{Q\in\mathcal{T}} \frac{|m_Q\sigma - m_Q\tau|^2}{(m_Q\sigma)^3}\, \|\Delta_Q\sigma\|_{L^2}^2\, \ell(Q)^{2-\alpha}\ell(R)^\alpha + \sum_{Q\in\mathcal{S}} \ell(Q)^2\,\tau(Q), \tag{3.22}$$

with constants depending on α.

Proof. For simplicity suppose first that $\bigcup_{Q\in\mathcal{S}} Q = R$ and that $\mathcal{T}$ contains a finite number of cubes (and thus all stopping cubes have side length uniformly bounded from below). For $Q \in \mathcal{D}(R)$, consider functions $\Delta_Q^\sigma\tau$ defined above. Then, τ can be written as

$$\tau = \sigma + \left(\sum_{Q\in\mathcal{T}} \Delta_Q^\sigma\tau\right)\sigma + \sum_{Q\in\mathcal{S}}\left(\tau\lfloor Q - \frac{\tau(Q)}{\sigma(Q)}\,\sigma\lfloor Q\right).$$

Observe that the measures $\left(\Delta_Q^\sigma\tau\right)\sigma$, for $Q \in \mathcal{T}$, have zero mean, as well as all the measure inside the last parenthesis above.

By Lemma 3.12, for $Q \in \mathcal{T}$, there exists a map $U_Q : Q \to Q$ be a map such that

$$U_Q\#\left(\frac{\tau(Q)}{\sigma(Q)}\,\sigma\lfloor Q\right) = \left(\Delta_Q^\sigma\tau + \frac{\tau(Q)}{\sigma(Q)}\right)\sigma\lfloor Q,$$

with $h = \Delta_Q^\sigma\tau$ and $c_9 = \tau(Q)/\sigma(Q)$ there (notice that $c_9 \approx 1$ by the assumptions on the cubes $Q \in \mathcal{T}$), so that

$$\int |U_Q x - x|^2\, d\sigma(x) \lesssim \|\Delta_Q^\sigma\tau\|_{L^2(\sigma)}^2\, \ell(Q)^2. \tag{3.23}$$

For $j > 0$, define $U_j(x) = U_Q(x)$ if $x \in Q \in \mathcal{D}_j(R) \cap \mathcal{T}$, and $U_j(x) = x$ if $x \in Q \in \mathcal{D}_j(R) \setminus \mathcal{T}$. Notice that, by definition, we have

$$U_j\left(\sum_{Q\in\mathcal{D}_j(R)\cap\mathcal{T}} \frac{\tau(Q)}{\sigma(Q)}\,\sigma\lfloor Q\right) = \sum_{Q\in\mathcal{D}_j(R)\cap\mathcal{T}}\left(\Delta_Q^\sigma\tau + \frac{\tau(Q)}{\sigma(Q)}\right)\sigma\lfloor Q = \sum_{Q\in\mathcal{D}_j(R)\cap\mathcal{T}}\ \sum_{P\in\mathcal{C}h(Q)} \frac{\tau(P)}{\sigma(P)}\,\sigma\lfloor P.$$

Set also $U_0(x) = x$ and consider the map

$$T = U_m \circ U_{m-1} \circ \cdots \circ U_1,$$

for m big enough so that $\mathcal{T} \subset \bigcup_{j=0}^m D_j(R)$. Observe that

$$T\#\sigma = \sum_{Q\in\mathcal{S}} \frac{\tau(Q)}{\sigma(Q)}\,\sigma\lfloor Q =: \tau_0.$$

The measure τ_0 should be considered as an approximation of τ at the scale of the stopping cubes. Let $V_j = U_j \circ \ldots \circ U_0$, for $0 \leq j \leq m$. We set

$$|x - Tx|^2 \leq \Big(\sum_{j=1}^m |V_{j-1}(x) - V_j(x)|\Big)^2 \lesssim \sum_{j=1}^m 2^{\alpha j} |V_{j-1}(x) - V_j(x)|^2, \tag{3.24}$$

with a constant depending on α. As a consequence, using that $V_j = U_j \circ V_{j-1}$,

$$W_2(\sigma, \tau_0)^2 \leq \int \sum_{j=1}^m 2^{\alpha j} |V_{j-1}x - V_j x|^2 \, d\sigma(x) = \sum_{j=1}^m 2^{\alpha j} \int |x - U_j x|^2 \, dV_{j-1}\#\sigma(x).$$

Recalling the definition of U_j, we get

$$\int |x - U_j x|^2 \, dV_{j-1}\#\sigma(x) = \sum_{Q \in \mathcal{D}_j(R) \cap \mathcal{T}} \int |x - U_Q x|^2 \, dV_{j-1}\#\sigma(x).$$

For the cubes Q in the last sum it turns out that

$$V_{j-1}\#\sigma\lfloor Q = \frac{\tau(Q)}{\sigma(Q)}\, \sigma\lfloor_Q \qquad \text{and} \qquad \tau(Q) \approx \sigma(Q),$$

we obtain

$$W_2(\sigma, \tau_0)^2 \lesssim \sum_{Q \in \mathcal{T}} \frac{\ell(R)^\alpha}{\ell(Q)^\alpha} \int |x - U_Q x|^2 \, d\sigma(x).$$

From (3.23), we infer that

$$W_2(\sigma, \tau_0)^2 \lesssim \sum_{Q \in \mathcal{T}} \frac{\ell(R)^\alpha}{\ell(Q)^\alpha} \, \|\Delta_Q^\sigma \tau\|_{L^2(\sigma)}^2 \, \ell(Q)^2 = \sum_{Q \in \mathcal{T}} \|\Delta_Q^\sigma \tau\|_{L^2(\sigma)}^2 \, \ell(Q)^{2-\alpha} \, \ell(R)^\alpha.$$

Now we wish to write $\Delta_Q^\sigma \tau$ in terms of $\Delta_Q(\sigma - \tau)$. Given $x \in P \in \mathcal{C}h(Q)$, from the definition of $\Delta_Q^\sigma \tau$, we get

$$\begin{aligned}
\Delta_Q^\sigma \tau(x) &= \frac{\tau(P) - \sigma(P)}{\sigma(P)} - \frac{\tau(Q) - \sigma(Q)}{\sigma(Q)} \\
&= \frac{1}{m_P \sigma} \left(\frac{\tau(P) - \sigma(P)}{m(P)} - \frac{\tau(Q) - \sigma(Q)}{m(Q)} \right) + \left(\frac{1}{m_P \sigma} - \frac{1}{m_Q \sigma} \right) \frac{\tau(Q) - \sigma(Q)}{m(Q)} \\
&= \frac{1}{m_P \sigma} \, \Delta_Q(\sigma - \tau)(x) + \frac{1}{m_P \sigma \, m_Q \sigma} \, \Delta_Q \sigma(x) \, \frac{\sigma(Q) - \tau(Q)}{m(Q)}.
\end{aligned}$$

Taking into account that, for $Q \in \mathcal{D}(R)$ and $P \in \mathcal{C}h(Q)$, we have $m_P \sigma \approx m_Q \sigma$ and also $\|\Delta_Q^\sigma \tau\|_{L^2(\sigma)}^2 \approx m_Q \sigma \, \|\Delta_Q^\sigma \tau\|_{L^2}^2$, we get

$$\begin{aligned}
W_2(\sigma, \tau_0)^2 \lesssim \sum_{Q \in \mathcal{T}} \frac{1}{m_Q \sigma} \, \|\Delta_Q(\sigma - \tau)\|_{L^2}^2 \, \ell(Q)^{2-\alpha} \ell(R)^\alpha \\
+ \sum_{Q \in \mathcal{T}} \frac{|m_Q \sigma - m_Q \tau|^2}{(m_Q \sigma)^3} \, \|\Delta_Q \sigma\|_{L^2}^2 \, \ell(Q)^{2-\alpha} \ell(R)^\alpha.
\end{aligned}$$

To complete the proof of the lemma, it is enough to show that

$$W_2(\tau,\tau_0)^2 \lesssim \sum_{Q\in\mathcal{S}} \ell(Q)^2\,\tau(Q). \tag{3.25}$$

To check this, just take

$$\pi = \sum_{Q\in\mathcal{S}} \frac{1}{\tau(Q)}\,(\tau\lfloor Q \otimes \tau_0\lfloor Q).$$

Using that $\tau(Q) = \tau_0(Q)$ for all $Q \in \mathcal{S}$, it is easy to check that π has marginals τ and τ_0. Therefore,

$$\begin{aligned} W_2(\tau,\tau_0)^2 &\leq \int |x-y|^2\,d\pi(x,y) = \sum_{Q\in\mathcal{S}} \frac{1}{\tau(Q)} \int_{Q\times Q} |x-y|^2\,d\tau(x)\,d\tau_0(y) \\ &\leq 2\sum_{Q\in\mathcal{S}} \left(\int_Q |x-z_Q|^2\,d\tau(x) + \int_Q |y-z_Q|^2\,d\tau_0(y)\right) \lesssim \sum_{Q\in\mathcal{S}} \ell(Q)^2\tau(Q), \end{aligned}$$

which gives (3.25). This ends the proof of the lemma under the assumption that $\mathcal{T}$ contains finitely many cubes.

If $\mathcal{T}$ has infinitely many cubes, then we apply the lemma to the tree $\mathcal{T}_m = \mathcal{T} \cap \mathcal{D}_m(R)$ for any fixed $m > 0$, and we let $m \to \infty$. □

Remark 3.14. It is very easy to check that the Lemmas 3.10, 3.11, 3.12, and 3.13 also hold if $\varphi \approx \chi_R$ and φ is Lipschitz on R. In fact, the calculations much easier in this situation (specially the ones in Lemma 3.10). In the particular case where $\sigma = c\chi_R\,\mathcal{L}^n$, where $\mathcal{L}^n$ stands for the Lebesgue measure and the constant c is comparable to 1, Lemma 3.13 becomes

$$W_2(\sigma,\tau)^2 \lesssim \sum_{Q\in\mathcal{T}} \|\Delta_Q\tau\|_{L^2}^2\,\ell(Q)^{2-\alpha}\ell(R)^\alpha + \sum_{Q\in\mathcal{S}} \ell(Q)^2\,\tau(Q), \tag{3.26}$$

still assuming that $\tau(Q) \approx \sigma(Q)$ for all $Q \in \mathcal{T}$.

3.5. **Proof of Theorem 3.4.** We have to prove that $W_2(\varphi\mu, a\varphi\nu) \lesssim W_2(\mu,\nu)$. To this end, we may assume that

$$W_2(\mu,\nu)^2 \leq c_8\,\ell(R)^2\,\|\varphi\mu\|. \tag{3.27}$$

Otherwise, the theorem follows easily, because

$$W_2(\varphi\mu, a\varphi\nu)^2 \lesssim \ell(R)^2\,\|\varphi\mu\| \lesssim W_2(\mu,\nu)^2.$$

Moreover, it also easy to check that ν can be assumed to be absolutely continuous with respect to Lebesgue measure, by a limitting argument.

Let G, T and $\widetilde{\mu}, \widetilde{\nu}$ be as in Lemmas 3.6 and 3.8. We write

$$W_2(\varphi\mu, a\varphi\nu) \leq W_2(\varphi\mu, a\,T\#\widetilde{\nu}) + W_2(a\,T\#\widetilde{\nu}, a\widetilde{\nu}) + W_2(a\widetilde{\nu}, a\varphi\nu).$$

Notice that, by Lemma 3.6, using also that $a \approx 1$, we have

$$W_2(a\varphi\nu, a\widetilde{\nu}) \lesssim W_2(\mu,\nu).$$

Also,

$$W_2(a\,T\#\widetilde{\nu}, a\,\widetilde{\nu})^2 \leq a\int |Tx-x|^2\,d\widetilde{\nu} \leq c\int |Tx-x|^2\,d\nu = c\,W_2(\mu,\nu)^2,$$

where the second inequality follows from the fact that $\widetilde{\nu} = \widetilde{a}\,\chi_G\,\varphi\nu$, with $\widetilde{a} \lesssim 1$. Thus, to prove the theorem it is enough to show that

$$W_2(\varphi\mu, a\,T\#\widetilde{\nu}) \lesssim W_2(\mu,\nu). \tag{3.28}$$

To this end, we will apply Lemma 3.13 to $\sigma = \varphi\mu$ and $\tau = a\,T\#\widetilde{\nu}$ (observe that $\sigma(R) = a\varphi\nu(R) = a\widetilde{\nu}(R) = \tau(R)$). We construct a tree $\mathcal{T}$ whose family $\mathcal{S}$ of stopping cubes is defined as follows:

$$Q \in \mathcal{S} \quad \text{if } Q \text{ is a maximal cube from } \mathcal{D}(R) \text{ such that } \tau(Q) \leq \delta\,\sigma(Q) \tag{3.29}$$

for some small constant $0 < \delta < 1$ to be chosen below. That Q is maximal means that there does not exist another cube $Q' \in \mathcal{D}(R)$ which contains Q satisfying the same estimate. Hence, from the previous definition, we infer that every $Q \in \mathcal{T}$ satisfies $\tau(Q) > \delta\,\sigma(Q)$. To apply Lemma 3.13 to σ, τ and to the tree $\mathcal{T}$ we need to show that there exists some constant c such that $\tau(Q) \leq c\,\sigma(Q)$. This follows quite easily:

$$\tau(Q) = a\,T\#\widetilde{\nu}(Q) = a\,\widetilde{a}\,T\#(\chi_G\,\varphi\,\nu)(Q) = a\,\widetilde{a}\int \chi_Q(Tx)\,\chi_G(x)\,\varphi(x)\,d\nu(x).$$

From Remarks 3.2 and 3.7, taking into account the assumption (3.27) we deduce that $a\,\widetilde{a} \approx 1$. On the other hand, for $x \in G$, we have $|x - Tx| \leq \ell(Q_x)$, where Q_x is the Whitney cube that contains x. This implies that $\operatorname{dist}(x,\partial R) \approx \operatorname{dist}(Tx,\partial R)$ and thus $\varphi(x) \approx \varphi(Tx)$. Therefore,

$$\begin{aligned}\tau(Q) &\approx \int \chi_Q(Tx)\,\chi_G(x)\,\varphi(Tx)\,d\nu(x)\\ &\leq \int \chi_Q(Tx)\,\varphi(Tx)\,d\nu(x) = \int \chi_Q(x)\,\varphi(x)\,dT\#\nu(x) = \sigma(Q),\end{aligned} \tag{3.30}$$

which proves our claim.

Now the key Lemma 3.13 tells us that

$$\begin{aligned}W_2(\sigma,\tau)^2 \lesssim \sum_{Q\in\mathcal{T}} \frac{1}{m_Q\sigma}\left(\|\Delta_Q(\sigma-\tau)\|_{L^2} + \frac{|m_Q\sigma - m_Q\tau|\,\|\Delta_Q\sigma\|_{L^2}}{m_Q\sigma}\right)^2 \ell(Q)\ell(R)\\ + \sum_{Q\in\mathcal{S}} \ell(Q)^2\,\tau(Q),\end{aligned} \tag{3.31}$$

with constants depending on δ. We will estimate each of the terms on the right side of this inequality in separate lemmas. The first one deals with $\sum_{Q\in\mathcal{S}} \ell(Q)^2\,\tau(Q)$.

Lemma 3.15. *If δ is chosen small enough, then*

$$\sum_{Q\in\mathcal{S}} \ell(Q)^2\,\tau(Q) \lesssim W_2(\mu,\nu)^2.$$

Proof. We will show that

$$\tau\Big(\bigcup_{Q\in\mathcal{S}} Q\Big) \lesssim \frac{W_2(\mu,\nu)^2}{\ell(R)^2}. \tag{3.32}$$

Clearly, the lemma follows from this estimate.

For a given $Q \in \mathcal{S}$, notice that the first estimate in (3.30) says that

$$\tau(Q) \approx \int \chi_Q(Tx)\,\chi_G(x)\,\varphi(Tx)\,d\nu(x) =: I(Q),$$

while the last equality in (3.30) states that

$$\sigma(Q) = \int \chi_Q(Tx)\,\varphi(Tx)\,d\nu(x).$$

Therefore, if δ is chosen small enough, then $I(Q) \leq \sigma(Q)/2$, and thus

$$\tau(Q) \approx I(Q) \leq \sigma(Q) - I(Q) = \int \chi_Q(Tx)\,\varphi(Tx)\,\big(1-\chi_G(x)\big)\,d\nu(x).$$

Summing over all $Q \in \mathcal{S}$, we obtain

$$\begin{aligned}\tau\Big(\bigcup_{Q\in\mathcal{S}} Q\Big) &\lesssim \int \chi_R(Tx)\,\varphi(Tx)\,\big(1-\chi_G(x)\big)\,d\nu(x)\\ &= \int_R \varphi(x)\,d(T\#\nu - T\#\chi_G\nu)(x).\end{aligned}$$

Recall now that $T\#\nu = \mu$ and $\widetilde{\mu} = \widetilde{b}\,\varphi\big(T\#(\chi_G\,\nu)\big)$ (see Lemma 3.8). So the preceding inequality says that

$$\tau\Big(\bigcup_{Q\in\mathcal{S}} Q\Big) \lesssim \|\varphi\mu - \widetilde{b}^{-1}\,\widetilde{\mu}\|.$$

Hence, the claim (3.32) follows from this inequality and Remark 3.9. □

To deal with the first sum on the right of (3.31), first we will estimate the L^2 norm of $\Delta_P(\sigma-\tau)$. In the rest of this subsection, we denote by S the map such that $S\#\mu = \nu$ which minimizes $W_2(\mu,\nu)$. Recall that $S\circ T = Id$ ν-a.e. and $T\circ S = Id$ μ-a.e.

Lemma 3.16. *Assume that (3.27) holds and denote $\eta = \varphi T\#(\chi_{G^c}\,\nu)$. For $P \in \mathcal{T}$, we have*

$$\|\Delta_P(\sigma-\tau)\|_{L^2} \lesssim \frac{\eta(P)}{\ell(P)^{n/2}} + \frac{W_2(\mu,\nu)}{\ell(P)^{n/2}\,\ell(R)^{1+n/2}} \int_{P\cap S^{-1}(G)} \varphi\circ S\,d\mu + E(P), \tag{3.33}$$

where $E(P)$ is as follows. If P is contained in some Whitney cube Q_P from the family $\{Q_i\}_{i\in I}$, then

$$E(P) = \frac{\ell(Q_P)}{\ell(R)^2}\,\|Sx - x\|_{L^p(\mu\lfloor P)}.$$

On the other hand, if P is not contained in any Whitney cube from the family $\{Q_i\}_{i\in I}$, then

$$E(P) = \left(\sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)^2}{\ell(R)^4}\,\|Sx-x\|^2_{L^2(\mu\lfloor 3Q_i)}\right)^{1/2}.$$

Proof. Let h_P^ε be the Haar wavelet associated to P and $\varepsilon\in\mathcal{E}$. Then we have

$$\|\Delta_P(\sigma-\tau)\|_{L^2} \leq \sum_{\varepsilon\in\mathcal{E}} \big|\langle \sigma-\tau, h_P^\varepsilon\rangle\big| = \sum_{\varepsilon\in\mathcal{E}} \left|\int h_P^\varepsilon\, d(\sigma-\tau)\right|.$$

Recall that $\sigma=\varphi\mu=\varphi T\#\nu$ and $\tau = a\,T\#\widetilde{\nu} = a\,T\#(\widetilde{a}\,\chi_G\,\varphi\nu)$. So, for each $\varepsilon\in\mathcal{E}$, the last integral equals

(3.34)

$$\begin{aligned}\int h_P^\varepsilon\, d[\varphi\,T\#\nu - a\,T\#(\widetilde{a}\,\chi_G\,\varphi\nu)] &= \int h_P^\varepsilon(Tx)\big[\varphi(Tx) - a\,\widetilde{a}\,\varphi(x)\chi_G(x)\big]\,d\nu(x)\\ &= \int_{G^c} h_P^\varepsilon(Tx)\,\varphi(Tx)\,d\nu(x)\\ &\quad + \int_G h_P^\varepsilon(Tx)\big[\varphi(Tx)-\varphi(x)\big]\,d\nu(x)\\ &\quad + (1-a\widetilde{a})\int_G h_P^\varepsilon(Tx)\,\varphi(x)\,d\nu(x) =: I_1+I_2+I_3.\end{aligned}$$

Concerning I_1, using again that $\|h_p^\varepsilon\|_\infty \lesssim \ell(P)^{-n/2}$ and $\eta=\varphi T\#(\chi_{G^c}\nu)$, we get

$$|I_1| \lesssim \ell(P)^{-n/2}\int_{G^c}\chi_P(Tx)\,\varphi(Tx)\,d\nu(x) = \frac{\eta(P)}{\ell(P)^{n/2}}. \tag{3.35}$$

To deal with I_2, recall that, for $x\in Q_i\cap G$, we have $|x-Tx|\leq \ell(Q_i)$, and thus $Tx\in 3Q_i$ and

$$|\varphi(Tx)-\varphi(x)| \leq \|\nabla\varphi\|_{\infty,3Q_i}\,|Tx-x|.$$

Using that $\|\nabla\varphi\|_{\infty,3Q_i}\lesssim \ell(Q_i)/\ell(R)^2$, we get

$$|I_2| \lesssim \sum_{i\in I}\frac{\ell(Q_i)}{\ell(R)^2}\int_{Q_i\cap G}|h_P^\varepsilon(Tx)|\,|Tx-x|\,d\nu(x). \tag{3.36}$$

Now we distinguish the two cases in the statement of the lemma, according to whether P is contained in some square Q_i, $i\in I$, or not. In the first case we write $P\in\mathcal{T}_a$, and in the second $P\in\mathcal{T}_b$. Suppose first that $P\in\mathcal{T}_a$, and denote by Q_P the square Q_i, $i\in I$, that contains P. Thus, if for $x\in Q_i\cap G$, $h_P^\varepsilon(Tx)\neq 0$, then $Tx\in P\subset Q_P$ and thus $x\in 3Q_P$ (since $|x-Tx|\leq\ell(Q_P)$). As a consequence, $Q_i\cap 3Q_P\neq\varnothing$, and there is bounded number of such cubes Q_i. Moreover, for them we have $\ell(Q_i)\approx\ell(Q_P)$. Thus we obtain

$$|I_2| \lesssim \sum_{i\in I: Q_i\cap 3Q_P\neq\varnothing}\frac{\ell(Q_P)}{\ell(R)^2}\,\|h_P^\varepsilon\circ T\|_{L^2(\nu)}\,\|(Tx-x)\chi_{T^{-1}(P)\cap G}\|_{L^2(\nu)}. \tag{3.37}$$

We have

$$\|h_P^\varepsilon\circ T\|_{L^2(\nu)} = \|h_P^\varepsilon\|_{L^2(\mu)}\lesssim 1,$$

since $\|h_P^\varepsilon\|_{L^2} = 1$ and $\mu = f(x)\,dx$ on R, with $f(x) \le c_4$. Therefore, using also that $\nu = S\#\mu$, we get

$$|I_2| \lesssim \frac{\ell(Q_P)}{\ell(R)^2}\,\|(Tx-x)\chi_{T^{-1}(P)}\|_{L^2(\nu)} = \frac{\ell(Q_P)}{\ell(R)^2}\,\|(x-Sx)\chi_P\|_{L^p(\mu)}.$$

Suppose now that there does not exist any cube Q_i, $i \in I$, which contains P. Notice that if, for some $i \in I$, the integral on the right side of (3.36) does not vanish, then there exists some $x \in Q_i \cap G$ such that $Tx \in P$. From the fact that $x \in G$, we deduce that $|x - Tx| \le \ell(Q_i)$, and thus $Tx \in 3Q_i$. As a consequence, $P \cap 3Q_i \neq \varnothing$. Therefore,

$$\begin{aligned} |I_2| &\lesssim \sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)}{\ell(R)^2}\int_{Q_i\cap G} |h_P^\varepsilon(Tx)|\,|Tx-x|\,d\nu(x) \\ &\lesssim \ell(P)^{-n/2}\sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)}{\ell(R)^2}\int_{Q_i\cap G} |Tx-x|\,d\nu(x), \end{aligned} \tag{3.38}$$

since $\|h_P^\varepsilon\|_\infty \lesssim \ell(P)^{-n/2}$. Using that $\nu = S\#\mu$ and that $TSx = x$ μ-a.e., we have

$$\int_{Q_i\cap G} |Tx-x|\,d\nu(x) = \int \chi_{Q_i\cap G}(Sx)\,|x-Sx|\,d\mu(x).$$

Observe that for μ-a.e x such that $Sx \in G \cap Q_i$,

$$|Sx - x| = |Sx - TSx| \le \ell(Q_{Sx}) = \ell(Q_i).$$

Thus, $x \in 3Q_i$, and so we infer that

$$\int_{Q_i\cap G} |Tx-x|\,d\nu(x) \le \int_{3Q_i} |x-Sx|\,d\mu(x).$$

From (3.38) and the preceding estimate, by Cauchy-Schwartz, we obtain

$$\begin{aligned} |I_2| &\lesssim \ell(P)^{-n/2}\sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)\,\mu(3Q_i)^{1/2}}{\ell(R)^2}\,\|Sx-x\|_{L^2(\mu\lfloor 3Q_i)} \\ &\le \ell(P)^{-n/2}\left(\sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)^2}{\ell(R)^4}\,\|Sx-x\|^2_{L^2(\mu\lfloor 3Q_i)}\right)^{1/2}\left(\sum_{i:3Q_i\cap P\neq\varnothing} \mu(3Q_i)\right)^{1/2}. \end{aligned}$$

Taking into account $\sum_{i:3Q_i\cap P\neq\varnothing}\mu(3Q_i) \lesssim \ell(P)^n$, we deduce

$$|I_2| \lesssim \left(\sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)^2}{\ell(R)^4}\,\|Sx-x\|^2_{L^2(\mu\lfloor 3Q_i)}\right)^{1/2}.$$

Finally we consider the term I_3 in (3.34). Recall that in Remark 3.2 we saw that $|1-a| \lesssim W_2(\mu,\nu) \approx W_2(\mu,\nu)/\ell(R)^{1+n/2}$. On the other hand, from (3.13) we get

$$|1-\widetilde{a}| \le \frac{2}{\varphi\nu(\mathbb{R}^n)\,\ell(R)^2}\,W_2(\mu,\nu)^2 \lesssim \frac{1}{\ell(R)^{n+2}}\,W_2(\mu,\nu)^2 \le \frac{1}{\ell(R)^{1+n/2}}W_2(\mu,\nu),$$

where the last inequality follows from the estimate (3.27). Therefore,

$$|1 - a\widetilde{a}| \leq |1-a| + a\,|1-\widetilde{a}| \lesssim \frac{1}{\ell(R)^{1+n/2}} W_2(\mu,\nu).$$

Then, using that $\nu = S\#\mu$,

$$\begin{aligned}|I_3| &\lesssim \frac{W_2(\mu,\nu)}{\ell(R)^{1+n/2}} \left| \int_G h_P^\varepsilon(Tx)\,\varphi(x)\,d\nu(x)\right| \\ &= \frac{W_2(\mu,\nu)}{\ell(R)^{1+n/2}} \left| \int_{S^{-1}(G)} h_P^\varepsilon(x)\,\varphi(Sx)\,d\mu(x)\right| \leq \frac{W_2(\mu,\nu)}{\ell(P)^{n/2}\,\ell(R)^{1+n/2}} \int_{P\cap S^{-1}(G)} \varphi\circ S\,d\mu.\end{aligned}$$

□

In next lemma we estimate the term $|m_P\sigma - m_P\tau|$.

Lemma 3.17. *Assume that (3.27) holds and let $\eta = \varphi T\#(\chi_{G^c}\nu)$. For $P \in \mathcal{T}$, we have*

$$\frac{|m_P\sigma - m_P\tau|\,\|\Delta_P\sigma\|_{L^2}}{m_P\sigma} \lesssim \frac{\eta(P)}{\ell(P)^{n/2}} + \frac{W_2(\mu,\nu)}{\ell(P)^{n/2}\,\ell(R)^{1+n/2}} \int_{P\cap S^{-1}(G)} \varphi\circ S\,d\mu + E(P),$$

with $E(P)$ defined as in Lemma 3.16.

Proof. From the doubling properties of σ, it follows that $\|\Delta_P\sigma\|_\infty \lesssim m_P\sigma$, and thus

$$\frac{|m_P\sigma - m_P\tau|\,\|\Delta_P\sigma\|_{L^2}}{m_P\sigma} \lesssim |m_P\sigma - m_P\tau|\,\ell(P)^{n/2} = \big|\langle \sigma-\tau,\, w_P\rangle\big|,$$

with $w_P = \chi_P/\ell(P)^{n/2}$.

Now the arguments to finish the proof are very similar to the ones in the preceding lemma. Instead of $\langle \sigma-\tau, h_P^\varepsilon\rangle$, we have to estimate the term $\langle \sigma-\tau, w_P\rangle$. So, all we have to do is to replace h_P^ε by w_P in the preceding proof. Indeed, notice that the cancellation property of h_P^ε was not necessary in the Lemma 3.16. We only used that $\operatorname{supp} h_P^\varepsilon \subset P$, that $\|h_P^\varepsilon\|_\infty \lesssim \ell(P)^{-n/2}$, and that $\|h_P^\varepsilon\|_{L^2} = 1$. These estimates also hold for w_P, and so we deduce that (3.33) also holds replacing its left side by $|m_P\sigma - m_P\tau|\,\|\Delta_P\sigma\|_{L^2}/m_P\sigma$. □

Our next objective consists in estimating the first sum on the right side of (3.31). By Lemmas 3.16 and 3.17, we have

$$\begin{aligned}(3.39)\quad \sum_{P\in\mathcal{T}} \frac{1}{m_P\sigma}&\left(\|\Delta_P(\sigma-\tau)\|_{L^2} + \frac{|m_P\sigma - m_P\tau|\,\|\Delta_P\sigma\|_{L^2}}{m_P\sigma}\right)^2 \ell(P)\ell(R) \\ &\lesssim \sum_{P\in\mathcal{T}} \frac{\ell(P)\ell(R)}{m_P\sigma}\,E(P)^2 + \sum_{P\in\mathcal{T}} \frac{\ell(R)\,\eta(P)^2}{\ell(P)^{n-1}\,m_P\sigma} \\ &\quad + W_2(\mu,\nu)^2 \sum_{P\in\mathcal{T}} \frac{1}{m_P\sigma\,\ell(P)^{n-1}\,\ell(R)^{n+1}} \left(\int_{P\cap S^{-1}(G)} \varphi\circ S\,d\mu\right)^2 \\ &=: S_1 + S_2 + S_3,\end{aligned}$$

where $\eta = \varphi T\#(\chi_{G^c}\nu)$.

In the next lemma we consider the sum S_1:

Lemma 3.18. *We have*

$$S_1 = \sum_{P\in\mathcal{T}} \frac{\ell(P)\ell(R)}{m_P\sigma}\, E(P)^2 \lesssim W_2(\mu,\nu)^2.$$

Proof. We separate the sum S_1 into two sums, according to whether P is contained in some square Q_i, $i \in I$, or not. As above, in the first case we write $P \in \mathcal{T}_a$, and in the second $P \in \mathcal{T}_b$. So we have

$$S_1 = \sum_{P\in\mathcal{T}_a} \frac{\ell(P)\ell(R)}{m_P\sigma}\, E(P)^2 + \sum_{P\in\mathcal{T}_b} \frac{\ell(P)\ell(R)}{m_P\sigma}\, E(P)^2 =: S_1^a + S_1^b.$$

We consider first the case $P \in \mathcal{T}_a$. Then recall that

$$E(P) = \frac{\ell(Q_P)}{\ell(R)^2}\, \|(x - Sx)\chi_P\|_{L^2(\mu)}.$$

Therefore,

$$S_1^a \leq \sum_{P\in\mathcal{T}_a} \frac{\ell(P)\,\ell(Q_P)^2}{m_P\sigma\,\ell(R)^3}\, \|(x - Sx)\chi_P\|^2_{L^2(\mu)}.$$

Since $\varphi \approx \ell(Q_P)^2/\ell(R)^2$ on $P \subset Q_P$, we have $m_P\sigma \approx \ell(Q_P)^2/\ell(R)^2$. Thus,

$$S_1^a \lesssim \sum_{P\in\mathcal{T}_a} \frac{\ell(P)}{\ell(R)}\, \|(x - Sx)\chi_P\|^2_{L^2(\mu)} \lesssim \int_R |Sx - x|^2\, d\mu \leq W_2(\mu,\nu)^2.$$

Let us turn our attention to S_1^b. Recall that if $P \in \mathcal{T}_b$, then

$$E(P) = \left(\sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)^2}{\ell(R)^4}\, \|Sx - x\|^2_{L^2(\mu|3Q_i)} \right)^{1/2},$$

and thus

$$S_1^b = \sum_{P\in\mathcal{T}_b} \frac{\ell(P)\ell(R)}{m_P\sigma} \sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)^2}{\ell(R)^4}\, \|Sx - x\|^2_{L^2(\mu|3Q_i)}.$$

Using that $m_P\sigma \gtrsim \ell(P)^2/\ell(R)^2$, we get

$$S_1^b \lesssim \sum_{P\in\mathcal{T}_b} \sum_{i:3Q_i\cap P\neq\varnothing} \frac{\ell(Q_i)^2}{\ell(P)\,\ell(R)}\, \|Sx - x\|^2_{L^2(\mu|3Q_i)}. \tag{3.40}$$

It is easy to check that for each fixed Q_i,

$$\sum_{P\in\mathcal{T}_b:3Q_i\cap P\neq\varnothing} \frac{1}{\ell(P)} \lesssim \frac{1}{\ell(Q_i)}.$$

Therefore, changing the order of summation in (3.40) we obtain

$$S_1^b \lesssim \sum_{i\in I} \frac{\ell(Q_i)}{\ell(R)}\, \|Sx - x\|^2_{L^2(\mu|3Q_i)} \lesssim \int_R |Sx - x|^2\, d\mu \leq W_2(\mu,\nu)^2.$$

□

Next, we deal with the sum S_2 from (3.39).

Lemma 3.19. *We have*

$$S_2 = \sum_{P\in\mathcal{T}} \frac{\ell(R)\,\eta(P)^2}{\ell(P)^{n-1}\,m_P\sigma} \lesssim W_2(\mu,\nu)^2.$$

Proof. Since $\eta(P) = \varphi T\#(\chi_{G^c}\,\nu)(P) \leq \varphi T\#\nu(P) = \sigma(P)$, we have

$$S_2 \leq \sum_{P\in\mathcal{T}} \ell(R)\,\ell(P)\,\eta(P) \lesssim \ell(R)^2\,\eta(R).$$

By (3.19), we know that

$$\eta(R) = \|\varphi\mu - \varphi T\#(\chi_G\,\nu\| \lesssim \frac{W_2(\mu,\nu)^2}{\ell(R)^2},$$

which proves the lemma. □

Now we consider the last term S_3 from (3.39).

Lemma 3.20. *We have*

$$S_3 = W_2(\mu,\nu)^2 \sum_{P\in\mathcal{T}} \frac{1}{m_P\sigma\,\ell(P)^{n-1}\,\ell(R)^{n+1}} \left(\int_{P\cap S^{-1}(G)} \varphi\circ S\,d\mu\right)^2 \lesssim W_2(\mu,\nu)^2.$$

Proof. Notice that if $x\in P\cap S^{-1}(G)$, then $Sx\in G$, and from the definition of G, it turns out that $|T(Sx)-Sx|\leq \ell(Q_{Sx})$, where Q_{Sx} stands for the Whitney cube that contains Sx. Since $T(Sx) = x$ for μ-a.e. x, then

$$|x - Sx| \leq \ell(Q_{Sx}) \qquad \mu\text{-a.e. } x\in P\cap S^{-1}(G).$$

Then, for these points x, $\varphi(Sx)\approx\varphi(x)$, and thus

$$\int_{P\cap S^{-1}(G)} \varphi\circ S\,d\mu \lesssim \int_{P\cap S^{-1}(G)} \varphi\,d\mu \lesssim \sigma(P).$$

Consequently,

$$S_3 \lesssim W_2(\mu,\nu)^2 \sum_{P\in\mathcal{T}} \frac{\sigma(P)^2}{m_P\sigma\,\ell(P)^{n-1}\,\ell(R)^{n+1}} = W_2(\mu,\nu)^2 \sum_{P\in\mathcal{T}} \frac{\ell(P)\,\sigma(P)}{\ell(R)^{n+1}}.$$

Using that $\sigma(P)\lesssim\ell(P)^n$, we derive

$$S_3 \lesssim W_2(\mu,\nu)^2 \sum_{P\in\mathcal{T}} \frac{\ell(P)^{n+1}}{\ell(R)^{n+1}} \lesssim W_2(\mu,\nu)^2.$$

□

In the preceding lemmas we have shown that $S_1+S_2+S_3\lesssim W_2(\mu,\nu)^2$. So we have shown that the left side of (3.39), which equals the first sum of (3.31), is bounded above by $c\,W_2(\mu,\nu)^2$. This completes the proof of Theorem 3.4. □

4. Proof of Lemma 3.10

Recall that, to conclude the proof of Theorem 1.1, it still remains to prove Lemma 3.10, which we rewrite here for the reader's convenience.

Lemma. *Let R and $\varphi \in \mathcal{G}(R)$ be as in Theorem 3.4. Consider the measure $d\widetilde{\sigma} = \varphi\, dm$ and let $Q \in \mathcal{D}(R)$. Consider a bounded function h supported on Q such that $\|h\|_\infty \leq 1$ and $\int h\, d\widetilde{\sigma} = 0$. Then there exists a map $U : Q \to Q$ such that $U\#(\widetilde{\sigma}\lfloor Q) = (1+h)\,\widetilde{\sigma}\lfloor Q$ which satisfies*

$$|Ux - x| \lesssim \ell(Q)\, \|h\|_\infty \qquad \text{for all } x \in Q. \tag{4.1}$$

Proof. Denote $\sigma_0 = \widetilde{\sigma}\lfloor Q$ and $\rho_0 = \sigma_0 + h\,\sigma_0$. We may assume that $\|h\|_\infty$ is small enough. Otherwise (4.1) holds trivially for the optimal map for quadratic cost, taking into account that σ_0 and ρ_0 are both supported on Q.

We will prove the lemma using the Knothe-Rosenblatt coupling between σ_0 and ρ_0, as explained in [Vi2, p.8]. Let us recall briefly in what consists this coupling. First, one considers the marginals of σ_0 and ρ_0 on the first variable, obtaining $\sigma_1 = \sigma_1(dx_1)$ and $\rho_1 = \rho_1(dx_1)$. Define $y_1 = U_1(x_1)$ by the formula of the increasing rearrangement such that $U_1\#\sigma_1 = \rho_1$. That is, given

$$F_1(t) = \sigma_1(-\infty, t], \qquad G_1(t) = \rho_1(-\infty, t],$$

set $U_1 = G_1^{-1} \circ F_1$.

Take now the marginals with respect to the first two variables, and obtain $d\sigma_2(x_1, x_2)$ and $d\rho_2(x_1, x_2)$. Disintegrate them so that for each x_1 there exist measures $d\sigma_{2;x_1}(x_2)$ and $d\rho_{2;x_1}(x_2)$ such that

$$d\sigma_2(x_1, x_2) = d\sigma_1(x_1)\, d\sigma_{2;x_1}(x_2), \qquad d\rho_2(x_1, x_2) = d\rho_1(x_1)\, d\rho_{2;x_1}(x_2). \tag{4.2}$$

Let us remark that in our precise situation we do not need to apply any delicate disintegration theorem since the densities of $d\sigma_{2;x_1}(x_2)$ and $d\rho_{2;x_1}(x_2)$ can be explicitly calculated by the identities in (4.2). Consider the map $y_2 = U_2(x_1, x_2)$ such that for each x_1, $U_2(x_1, \cdot)\#\sigma_{2;x_1} = \rho_{2;U_1 x_1}$, where U_2 is given by the corresponding increasing rearrangement.

Repeat the construction adding variables one after the other, and obtain also $U_3(x_1, x_2, x_3), \ldots, U_n(x_1, \ldots, x_n)$. Finally, set

$$U(x) = \big(U_1(x_1), U_2(x_1, x_2), \ldots, U_n(x_1, \ldots, x_n)\big).$$

It is easy to check that $U\#\sigma_0 = \rho$.

By translating R, we may assume that $R = [0,1]^n$ (or $(0,1]^n$, to be more precise), and that $Q \cap [0, 1/2]^n \neq \varnothing$, so that either Q is contained in $(0, 1/2]^n$ or $Q = R$. Let a_j, b_j be such that

$$Q = (a_1, b_1] \times \cdots \times (a_n, b_n].$$

By interchanging the coordinates if necessary, we will also suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$. We will show below that for every $x \in Q$, and $1 \leq j \leq n$,

$$|U_j(x_1, \ldots, x_j) - x_j| \lesssim \|h\|_\infty \min\big(x_j - a_j, b_j - x_j\big). \tag{4.3}$$

Clearly, (4.1) follows from this estimate.

To prove (4.3), recall that

$$\varphi(x) \approx \frac{\operatorname{dist}(x,\partial R)^2}{\ell(R)^2} \qquad \text{for all } x \in R.$$

If $Q \cap \partial R = \varnothing$, then $\operatorname{dist}(Q,\partial R) \geq \ell(Q)$, and from the preceding estimate, it turns out that $\varphi(x) \approx m_Q\varphi$ for every $x \in Q$. This fails if $Q \cap \partial R \neq \varnothing$. Indeed, one can easily check that $m_Q\varphi \approx \ell(Q)^2/\ell(R)^2$, while φ vanishes on $\partial Q \cap \partial R$.

We will derive (4.3) by induction on j, first assuming that $\boldsymbol{Q \neq R}$ and thus $Q \subset (0,1/2]^n$. In this case, $\operatorname{dist}(x,\partial R) = \min(x_1,\ldots,x_n)$ for every $x \in Q$, and so

$$\varphi(x) \approx \frac{\min(x_1,\ldots,x_n)^2}{\ell(R)^2} \qquad \text{for } x \in Q. \tag{4.4}$$

The case $\boldsymbol{j=1}$. Recall that $U_1 = G_1^{-1} \circ F_1$, with $F_1(t) = \sigma_1(-\infty,t]$ and $G_1(t) = \rho_1(-\infty,t]$. To estimate σ_1, notice that, using the assumption $a_1 \leq a_2 \leq \cdots \leq a_n$, it turns out that

$$x_1 \geq \min(x_1,\ldots,x_n) \geq \frac{x_1}{2}$$

for $x_2,\ldots,x_n$ in a subset of $[a_2,b_2]\times\cdots\times[a_n,b_n]$ of $(n-1)$-dimensional Lebesgue measure comparable to $\ell(Q)^{n-1}$ (possibly depending on n). From this fact, one infers that

$$d\sigma_1(x_1) \approx \left(\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} \frac{\min(x_1,\ldots,x_n)^2}{\ell(R)^2}\,dx_2\ldots dx_n\right)dx_1 \approx \frac{\ell(Q)^{n-1}\,x_1^2}{\ell(R)^2}\,dx_1.$$

Now we distinguish two cases, according to whether $a_1 = 0$ or not.

Suppose first that $\boldsymbol{a_1 = 0}$. Then,

$$F_1(t) = \sigma_1(-\infty,t] \approx \int_0^t \frac{\ell(Q)^{n-1}\,x_1^2}{\ell(R)^2}\,dx_1 \approx \frac{\ell(Q)^{n-1}}{\ell(R)^2}\,t^3 = A_0\,t^3, \quad \text{for } t \in [a_1,b_1], \tag{4.5}$$

where, to simplify notation, we set $A_0 = \frac{\ell(Q)^{n-1}}{\ell(R)^2}$. Assuming $\|h\|_\infty \leq 1/2$, the density functions of σ_0 and ρ_0 are comparable, and thus we also have

$$d\rho_1(x_1) \approx \chi_{[a_1,b_1]}(x_1)\,A_0\,x_1^2\,dx_1, \qquad G_1(t) = \rho_1(-\infty,t] \approx A_0\,t^3 \quad \text{for } t \in [a_1,b_1],$$

and therefore

$$G_1^{-1}(s) \approx A_0^{-1/3}\,s^{1/3},$$

and the derivative of G_1^{-1} satisfies

$$(G_1^{-1})'(s) = \frac{1}{G_1'(G_1^{-1}(s))} \approx \frac{1}{A_0\,G_1^{-1}(s)^2} \approx A_0^{-1/3}\,s^{-2/3}.$$

Then we deduce

$$|U_1(x_1) - x_1| = |G_1^{-1}(F_1(x_1)) - G_1^{-1}(G_1(x_1))| = \left| \int_{G_1(x_1)}^{F_1(x_1)} (G_1^{-1})'(s)\, ds \right|$$

$$\approx A_0^{-1/3} \left| \int_{G_1(x_1)}^{F_1(x_1)} s^{-2/3}\, ds \right| \approx A_0^{-1/3} |F_1(x_1)^{1/3} - G_1(x_1)^{1/3}|. \tag{4.6}$$

Since $|c^{1/3} - 1| \approx |c - 1|$ for c close to 1, assuming $\|h\|_\infty$ small enough we get

$$|U_1(x_1) - x_1| \approx A_0^{-1/3}\, F_1(x_1)^{1/3} \left| \frac{G_1(x_1)}{F_1(x_1)} - 1 \right| = A_0^{-1/3}\, F_1(x_1)^{-2/3}\, |F_1(x_1) - G_1(x_1)|. \tag{4.7}$$

Using that $|F_1(x_1) - G_1(x_1)| \leq F_1(x_1)\, \|h\|_\infty$ and $F_1(x_1) \approx A_0\, x_1^3$, we obtain

$$|U_1(x_1) - x_1| \lesssim A_0^{-1/3}\, F_1(x_1)^{-2/3} F_1(x_1)\, \|h\|_\infty \approx x_1\, \|h\|_\infty,$$

which proves (4.3) in this case.

Suppose now that $\boldsymbol{a_1 \neq 0}$. Then we have $x_1 \approx a_1$ for all $x \in Q$, and we obtain

$$F_1(t) = \sigma_1(-\infty, t] \approx \int_{a_1}^{t} \frac{\ell(Q)^{n-1}\, a_1^2}{\ell(R)^2}\, dx_1 = (t - a_1) \frac{\ell(Q)^{n-1}\, a_1^2}{\ell(R)^2} \quad \text{for } t \in [a_1, b_1]. \tag{4.8}$$

To simplify notation, we set $A_1 = \frac{\ell(Q)^{n-1}\, a_1^2}{\ell(R)^2}$, so that

$$F_1(t) \approx A_1\, (t - a_1) \quad \text{for } t \in [a_1, b_1].$$

By the comparability of the measures σ_0 and ρ_0, we also get

$$d\rho_1(x_1) \approx \chi_{[a_1, b_1]}(x_1)\, A_1\, dx_1, \qquad G_1(t) = \rho_1(-\infty, t] \approx A_1\, (t - a_1) \quad \text{for } t \in [a_1, b_1],$$

and thus we derive

$$G_1^{-1}(s) - a_1 \approx \frac{s}{A_1}$$

and

$$(G_1^{-1})'(s) = \frac{1}{G_1'(G_1^{-1}(s))} \approx \frac{1}{A_1}.$$

Arguing as in (4.6), we deduce

$$|U_1(x_1) - x_1| = \left| \int_{G_1(x_1)}^{F_1(x_1)} (G_1^{-1})'(s)\, ds \right| \approx \frac{1}{A_1}\, |F_1(x_1) - G_1(x_1)|.$$

Using that $|F_1(x_1) - G_1(x_1)| \leq F_1(x_1)\, \|h\|_\infty$ and $F_1(x_1) \approx A_1\, (x_1 - a_1)$, we obtain

$$|U_1(x_1) - x_1| \lesssim \frac{1}{A_1} F_1(x_1)\, \|h\|_\infty \approx |x_1 - a_1|\, \|h\|_\infty.$$

Thus (4.3) also holds in this case.

The case $\boldsymbol{j > 1}$. Suppose that $U_1(x_1), \dots, U_{j-1}(x_1, \dots, x_{j-1})$ satisfy (4.3). Recall that

$$U_j(x_1, \dots, x_{j-1}, \cdot)\#\sigma_{j;x_1,\dots,x_{j-1}} = \rho_{j;y_1,\dots,y_{j-1}},$$

where $\sigma_{j;x_1,\ldots,x_{j-1}}$ is defined by

$$d\sigma_j(x_1,\ldots,x_j) = d\sigma_{j-1}(x_1,\ldots,x_{j-1})\,d\sigma_{j;x_1,\ldots,x_{j-1}}(x_j),$$

and analogously for ρ_0. Recall also that $y_1 = U_1(x_1)$, $y_2 = U_2(x_1,x_2),\ldots$ and, further, σ_{j-1} and σ_j stand for the marginals of σ_0 with respect to the first $j-1$ and j variables, respectively. Thus,

$$d\sigma_{j-1}(x_1,\ldots,x_{j-1}) \approx \left(\int_{a_j}^{b_j}\cdots\int_{a_n}^{b_n}\frac{\min(x_1,\ldots,x_n)^2}{\ell(R)^2}\,dx_j\ldots dx_n\right)dx_1\ldots dx_{j-1},$$

and similarly for σ_j.

To estimate the integral inside the parentheses we argue as in the case $j=1$: since $a_1\leq\ldots\leq a_n$, there exists a subset of $[a_j,b_j]\times\cdots\times[a_n,b_n]$ of measure comparable to $\ell(Q)^{n-j+1}$ such that

$$\min(x_1,\ldots,x_{j-1}) \geq \min(x_1,\ldots,x_n) \geq \frac{1}{2}\,\min(x_1,\ldots,x_{j-1}).$$

Consequently,

$$d\sigma_{j-1}(x_1,\ldots,x_{j-1}) \approx \frac{\ell(Q)^{n-j+1}}{\ell(R)^2}\min(x_1,\ldots,x_{j-1})^2\,dx_1\ldots dx_{j-1}. \tag{4.9}$$

Analogously, we have

$$d\sigma_j(x_1,\ldots,x_j) \approx \frac{\ell(Q)^{n-j}}{\ell(R)^2}\min(x_1,\ldots,x_j)^2\,dx_1\ldots dx_j, \tag{4.10}$$

and thus

$$d\sigma_{j;x_1,\ldots,x_{j-1}}(x_j) \approx \frac{\min(x_1,\ldots,x_j)^2}{\ell(Q)\,\min(x_1,\ldots,x_{j-1})^2}\,dx_j.$$

Since the densities of ρ_0 and σ_0 are comparable, a similar estimate holds for $\rho_{j;y_1,\ldots,y_{j-1}}$:

$$d\rho_{j;y_1,\ldots,y_{j-1}}(x_j) \approx \frac{\min(y_1,\ldots,y_{j-1},x_j)^2}{\ell(Q)\,\min(y_1,\ldots,y_{j-1})^2}\,dx_j.$$

Moreover, since $|y_k - x_k| \lesssim \|h\|_\infty x_k$ for $1\leq k\leq j-1$, by (4.3) and the induction hypothesis, we deduce that $y_k\approx x_k$ for these k's, assuming $\|h\|_\infty$ small enough, and then we get

$$d\sigma_{j;x_1,\ldots,x_{j-1}}(x_j) \approx d\rho_{j;y_1,\ldots,y_{j-1}}(x_j).$$

For fixed points $x_1,\ldots,x_{j-1}$, $y_1,\ldots,y_{j-1}$, denote

$$F_j(t) = \sigma_{j;x_1,\ldots,x_{j-1}}(-\infty,t], \qquad G_j(t) = \rho_{j;y_1,\ldots,y_{j-1}}(-\infty,t],$$

so that $U_j(x_1,\ldots,x_{j-1},x_j) = G_j^{-1}\circ F_j(x_j)$. Then, as in the case $j=1$, we have

$$|U_j(x_1,\ldots,x_{j-1},x_j) - x_j| = |G_j^{-1}\circ F_j(x_j) - G_j^{-1}\circ G_j(x_j)| = \left|\int_{G_j(x_j)}^{F_j(x_j)}(G_j^{-1})'(s)\,ds\right|. \tag{4.11}$$

We need now to estimate $G_j(x_j)$, $F_j(x_j)$, and $(G_j^{-1})'(s)$. To this end, notice that

$$d\sigma_{j;x_1,\dots,x_{j-1}}(x_j) \approx \begin{cases} \dfrac{x_j^2}{\ell(Q)\,\min(x_1,\dots,x_{j-1})^2}\,dx_j & \text{if } x_j \le \min(x_1,\dots,x_{j-1}), \\ \dfrac{1}{\ell(Q)}\,dx_j & \text{if } x_j > \min(x_1,\dots,x_{j-1}). \end{cases} \tag{4.12}$$

Below we will show that

$$|d\sigma_{j;x_1,\dots,x_{j-1}}(t) - d\rho_{j;y_1,\dots,y_{j-1}}(t)| \lesssim \|h\|_\infty\, d\sigma_{j;x_1,\dots,x_{j-1}}(t) \tag{4.13}$$

(this notation means that the difference of the densities of $\sigma_{j;x_1,\dots,x_{j-1}}$ and $\rho_{j;y_1,\dots,y_{j-1}}$ is bounded above by some constant times $\|h\|_\infty$ times the density $\sigma_{j;x_1,\dots,x_{j-1}}$). We defer the proof of this estimate to Lemma 4.1. Integrating on t, we derive

$$|F_j(t) - G_j(t)| \lesssim F_j(t)\,\|h\|_\infty. \tag{4.14}$$

Suppose first that $\boldsymbol{a_j = 0}$. For $t \le \min(x_1,\dots,x_{j-1})$, we have

$$F_j(t) \approx \int_0^t \frac{x_j^2}{\ell(Q)\,\min(x_1,\dots,x_{j-1})^2}\,dx_j \approx \frac{t^3}{\ell(Q)\,\min(x_1,\dots,x_{j-1})^2} =: B_0\,t^3,$$

while for $t > \min(x_1,\dots,x_{j-1})$,

$$F_j(t) \approx \int_0^{\min(x_1,\dots,x_{j-1})} \frac{x_j^2}{\ell(Q)\,\min(x_1,\dots,x_{j-1})^2}\,dx_j + \int_{\min(x_1,\dots,x_{j-1})}^t \frac{1}{\ell(Q)}\,dx_j \approx \frac{t}{\ell(Q)}.$$

For $G_j(t)$ we have analogous estimates, because $G_j(t) \approx F_j(t)$.

Observe that, for $t \le \min(x_1,\dots,x_{j-1})$, this is the same estimate as the one in (4.5), replacing A_0 by B_0. Thus, for $0 \le s \le \min(x_1,\dots,x_{j-1})/\ell(Q)$ we deduce

$$G_j^{-1}(s) \approx B_0^{-1/3}\,s^{1/3}, \qquad (G_j^{-1})'(s) = \frac{1}{G_j'(G_j^{-1}(s))} \approx B_0^{-1/3}\,s^{-2/3}.$$

On the other hand, for $s > \min(x_1,\dots,x_{j-1})/\ell(Q)$,

$$G_j^{-1}(s) \approx \ell(Q)\,s, \qquad (G_j^{-1})'(s) = \frac{1}{G_j'(G_j^{-1}(s))} \approx \ell(Q).$$

For $x_j \le \min(x_1,\dots,x_{j-1})$, by (4.11) and arguing as in (4.7), we will obtain

$$|U_j(x_1,\dots,x_{j-1},x_j) - x_j| \lesssim B_0^{-1/3}\,F_j(x_j)^{-2/3}\,|F_j(x_j) - G_j(x_j)|.$$

Hence, from (4.14) and the fact that $F_j(x_j) \approx B_0\,x_j^3$, we get

$$|U_j(x_1,\dots,x_{j-1},x_j) - x_j| \lesssim B_0^{-1/3}\,F_j(x_j)^{-2/3} F_j(x_j)\,\|h\|_\infty \approx x_j\,\|h\|_\infty.$$

For $x_j > \min(x_1,\dots,x_{j-1})$, $F_j(x_j) \approx G_j(x_j) \gtrsim \min(x_1,\dots,x_{j-1})/\ell(Q)$, and then, for values of s between $F_j(x_j)$ and $G_j(x_j)$, we have $(G_j^{-1})'(s) \approx \ell(Q)$, and so, by

(4.11),

$$|U_j(x_1,\ldots,x_{j-1},x_j) - x_j| = \left|\int_{G_j(x_j)}^{F_j(x_j)} (G_j^{-1})'(s)\,ds\right| \approx \ell(Q)\,|F_j(x_j) - G_j(x_j)|$$

By (4.14) and the fact that $F_j(x_j) \approx x_j/\ell(Q)$,

$$|U_j(x_1,\ldots,x_{j-1},x_j) - x_j| \lesssim \ell(Q)\,F_j(x_j)\,\|h\|_\infty \approx x_j\,\|h\|_\infty.$$

Therefore, (4.3) holds if $a_j = 0$.

Consider now the case $\boldsymbol{a_j \neq 0}$, and so $Q \subset [0,1/2]^n$. In this case, for all $x \in Q$ we have $x_j \approx a_j$. Then, since $a_j \geq a_{j-1} \geq \cdots \geq a_1$, it easily follows that $x_j \gtrsim x_k$ for $1 \leq k \leq j-1$. So from (4.12), we infer that

$$d\sigma_{j;x_1,\ldots,x_{j-1}}(x_j) \approx \frac{1}{\ell(Q)}\,dx_j.$$

In this case, we get

$$F_j(t) = \sigma_{j;x_1,\ldots,x_{j-1}}(-\infty,t] \approx \frac{1}{\ell(Q)}\,(t-a_j) \quad \text{for } t \in [a_j,b_j].$$

This estimate is analogous to the one in (4.8), replacing A_1 by $B_1 = 1/\ell(Q)$ and a_1 by a_j. By similar arguments, we get again

$$|U_j(x_1,\ldots,x_{j-1},x_j) - x_j| \lesssim x_j\,\|h\|_\infty.$$

This finishes the proof of the lemma for $Q \neq R$.

For $\boldsymbol{Q = R}$, the proof of (4.3) for $x \in [0,1/2]^n$ is analogous to the one above for $Q \neq R$ in the case $a_j = 0$ (for every j). The details are left for the reader. The estimate (4.3) for $x \in R \setminus [0,1/2]^n$ also holds, because of the symmetry of the assumptions on R, σ_0, and ρ_0. Indeed, notice that we also have $U_1 = \widetilde{G}_1^{-1} \circ \widetilde{F}_1$, with $\widetilde{F}_1(t) = \sigma_1[t,+\infty)$, $\widetilde{G}_1(t) = \rho_1[t,+\infty)$, and analogously for the other indices $j > 1$. □

To complete the proof of the preceding lemma, it remains to prove the estimate (4.13). This is what we do below.

Lemma 4.1. *Under the notation and assumptions of Lemma 3.10, if (4.3) holds for $1 \leq k \leq j-1$, then*

$$|d\sigma_{j;x_1,\ldots,x_{j-1}}(x_j) - d\rho_{j;y_1,\ldots,y_{j-1}}(x_j)| \lesssim \|h\|_\infty\,d\sigma_{j;x_1,\ldots,x_{j-1}}(x_j).$$

Proof. We assume that $Q \neq R$ and $Q \subset [0,1/2]^n$. Let s_j, r_j be such that

$$d\sigma_j(x_1,\ldots,x_j) = s_j(x_1,\ldots,x_j)\,dx_1\ldots dx_j,$$

and

$$d\rho_j(x_1,\ldots,x_j) = r_j(x_1,\ldots,x_j)\,dx_1\ldots dx_j.$$

So we have

$$d\sigma_{j;x_1,\ldots,x_{j-1}}(x_j) - d\rho_{j;y_1,\ldots,y_{j-1}}(x_j) = \left(\frac{s_j(x_1,\ldots,x_j)}{s_{j-1}(x_1,\ldots,x_{j-1})} - \frac{r_j(y_1,\ldots,y_{j-1},x_j)}{r_{j-1}(y_1,\ldots,y_{j-1})}\right)dx_j$$

We split the term inside the parentheses on the right side as follows:

(4.15)
$$\left(\frac{s_j(x_1,\dots,x_j)}{s_{j-1}(x_1,\dots,x_{j-1})}-\frac{s_j(y_1,\dots,y_{j-1},x_j)}{s_{j-1}(y_1,\dots,y_{j-1})}\right)+\left(\frac{s_j(y_1,\dots,y_{j-1},x_j)}{s_{j-1}(y_1,\dots,y_{j-1})}-\frac{r_j(y_1,\dots,y_{j-1},x_j)}{r_{j-1}(y_1,\dots,y_{j-1})}\right) = A+B.$$

Denote by s and r the densities of σ_0 and ρ_0 on Q. Recall that $s(x)=\varphi(x)$, where φ satisfies the assumptions of Theorem 3.4. In particular, recall that

$$|\nabla s(x)| \lesssim \frac{\operatorname{dist}(x,\partial R)}{\ell(R)^2} \qquad \text{and} \qquad |\nabla_T s(x)| \lesssim \frac{\operatorname{dist}(x,\partial R)^2}{\ell(R)^3}. \tag{4.16}$$

We write A as follows:

$$\begin{aligned} A &= \frac{s_j(x_1,\dots,x_j)-s_j(y_1,\dots,y_{j-1},x_j)}{s_{j-1}(x_1,\dots,x_{j-1})} \\ &\quad + s_j(y_1,\dots,y_{j-1},x_j)\left(\frac{1}{s_{j-1}(x_1,\dots,x_{j-1})}-\frac{1}{s_{j-1}(y_1,\dots,y_{j-1})}\right) = A_1+A_2. \end{aligned} \tag{4.17}$$

First we consider the term A_2. To this end, notice that

$$\begin{aligned} &s_{j-1}(x_1,\dots,x_{j-1})-s_{j-1}(y_1,\dots,y_{j-1}) \\ &\quad = \int_{a_j}^{b_j}\cdots\int_{a_n}^{b_n}\big(s(x_1,\dots,x_{j-1},z_j,\dots,z_n)-s(y_1,\dots,y_{j-1},z_j,\dots,z_n)\big)\,dz_j\dots dz_n. \end{aligned} \tag{4.18}$$

To estimate the difference inside the integral, we distinguish two cases. Assume first that

$$\min(z_j,\dots,z_n) < 2\,\min(x_1,\dots,x_{j-1}).$$

Recall that, for $u\in Q$, we have $\operatorname{dist}(u,\partial R)=\min(u_1,\dots,u_n)$ and denote

$$Q_{z_j,\dots,z_n} = \{u\in Q : u_k = z_k \text{ for } k\geq j\}.$$

From (4.16), taking into account that $|x_k-y_k|\lesssim x_k\,\|h\|_\infty \leq \ell(Q)\,\|h\|_\infty$ for $1\leq k\leq j-1$, we deduce that

$$\begin{aligned} \big|s(x_1,\dots,x_{j-1},z_j,\dots,z_n)-s(y_1,\dots,y_{j-1},z_j,\dots,z_n)\big| &\lesssim \|\nabla s\|_{\infty,Q_{z_j,\dots,z_n}}\sum_{k=1}^{j-1}|x_k-y_k| \\ &\lesssim \frac{\min(z_j,\dots,z_n)}{\ell(R)^2}\,\ell(Q)\,\|h\|_\infty \lesssim \frac{\min(x_1,\dots,x_{j-1})}{\ell(R)^2}\,\ell(Q)\,\|h\|_\infty. \end{aligned} \tag{4.19}$$

Suppose now that

$$\min(z_j,\dots,z_n) \geq 2\,\min(x_1,\dots,x_{j-1}).$$

Since $|y_k-x_k|\lesssim x_k\|h\|_\infty$ for $0\leq k\leq j-1$, for $\|h\|_\infty$ small enough we have $y_k\leq 2x_k$, and thus

$$\min(z_j,\dots,z_n) > \min(y_1,\dots,y_{j-1}).$$

Assume that $\min(x_1,\ldots,x_{j-1}) \leq \min(y_1,\ldots,y_{j-1})$, and let i be such that $x_i = \min(x_1,\ldots,x_{j-1})$. Then we have

$$\operatorname{dist}\big((x_1,..,x_{j-1},z_j,..,z_n),\,\partial R\big) = \operatorname{dist}\big((y_1,..,x_i,..,y_{j-1},z_j,..,z_n),\,\partial R\big) = x_i.$$

where $(y_1,..,x_i,..,y_{j-1},z_j,..,z_n)$ is obtained by replacing y_i by x_i in $(y_1,..,y_{j-1},z_j,..,z_n)$. By (4.16), denoting

$$Q_{x_i,z_j,\ldots,z_n} = \{u \in Q : u_i = x_i,\ u_k = z_k \text{ for } k \geq j\}$$

and

$$L = L_{y_1,..,y_{i-1},y_i,..,y_{j-1},z_j,..,z_n} = \big\{(y_1,..,u_i,..,y_{j-1},z_j,..,z_n) : x_i \leq u_i \leq y_i\big\},$$

we get

$$\begin{aligned}
&\big|s(x_1,..,x_{j-1},z_j,..,z_n) - s(y_1,..,y_{j-1},z_j,..,z_n)\big| \\
&\qquad\leq \big|s(x_1,..,x_{j-1},z_j,..,z_n) - s(y_1,..,x_i,..,y_{j-1},z_j,..,z_n)\big| \\
&\qquad\quad + \big|s(y_1,..,x_i,..,y_{j-1},z_j,..,z_n) - s(y_1,..,y_{j-1},z_j,..,z_n)\big| \\
&\qquad\lesssim \|\nabla_T s\|_{\infty,Q_{x_i,z_j,\ldots,z_n}} \sum_{\substack{1\leq k\leq j-1\\ k\neq i}} |x_k - y_k| + \|\nabla s\|_{\infty,L}\,|x_i - y_i| \\
&\qquad\lesssim \frac{x_i^2}{\ell(R)^3} \sum_{\substack{1\leq k\leq j-1\\ k\neq i}} |x_k - y_k| + \frac{x_i}{\ell(R)^2}\,|x_i - y_i|.
\end{aligned}$$

Since $|x_k - y_k| \lesssim x_k\,\|h\|_\infty \leq \ell(Q)\,\|h\|_\infty$, we get

(4.20)

$$\begin{aligned}
\big|s(x_1,..,x_{j-1},z_j,..,z_n) - s(y_1,..,y_{j-1},z_j,..,z_n)\big| &\lesssim \frac{x_i^2}{\ell(R)^3}\,\ell(Q)\,\|h\|_\infty + \frac{x_i}{\ell(R)^2}\,x_i\,\|h\|_\infty \\
&\lesssim \frac{x_i^2}{\ell(R)^2}\,\|h\|_\infty = \frac{\min(x_1,\ldots,x_{j-1})^2}{\ell(R)^2}\,\|h\|_\infty.
\end{aligned}$$

If $\min(x_1,\ldots,x_{j-1}) > \min(y_1,\ldots,y_{j-1}) = y_i$, we get the same estimate just interchanging the roles of x_k and y_k, $1 \leq k \leq j-1$.

If we plug the estimates (4.19) and (4.20) into the identity (4.18), we obtain

$$\begin{aligned}
\big|s_{j-1}(x_1,\ldots,x_{j-1}) - s_{j-1}(y_1,\ldots,y_{j-1})\big| &\leq \ell(Q)\,\|h\|_\infty \int_E \frac{\min(x_1,\ldots,x_{j-1})}{\ell(R)^2}\,dz_j\ldots dz_n \\
&\quad + \|h\|_\infty \int_F \frac{\min(x_1,\ldots,x_{j-1})^2}{\ell(R)^2}\,dz_j\ldots dz_n,
\end{aligned}$$

where E denotes the subset of those points $(z_j,..,z_n) \in [a_j,b_j]\times\cdots\times[a_n,b_n]$ such that $\min(z_j,\ldots,z_n) < 2\,\min(x_1,\ldots,x_{j-1})$, and $F = [a_j,b_j]\times\cdots\times[a_n,b_n]\setminus E$. It is easy to check that the $(n-j+1)$-dimensional Lebesgue measure of E is smaller than

$c\min(x_1,\ldots,x_{j-1})\,\ell(Q)^{n-j}$. Then we deduce

$$\big|s_{j-1}(x_1,\ldots,x_{j-1}) - s_{j-1}(y_1,\ldots,y_{j-1})\big| \lesssim \frac{\min(x_1,\ldots,x_{j-1})^2}{\ell(R)^2}\,\ell(Q)^{n-j+1}\,\|h\|_\infty. \tag{4.21}$$

Recalling the definition of A_2 in (4.17), we derive

$$|A_2| \lesssim s_j(y_1,\ldots,y_{j-1},x_j)\,\frac{\ell(Q)^{n-j+1}\,\min(x_1,\ldots,x_{j-1})^2}{\ell(R)^2\,s_{j-1}(x_1,\ldots,x_{j-1})\,s_{j-1}(y_1,\ldots,y_{j-1})}\,\|h\|_\infty.$$

By (4.9) we know that

$$s_{j-1}(x_1,\ldots,x_{j-1}) \approx \frac{\ell(Q)^{n-j+1}}{\ell(R)^2}\,\min(x_1,\ldots,x_{j-1})^2.$$

Using also that

$$s_j(y_1,\ldots,y_{j-1},x_j) \approx s_j(x_1,\ldots,x_{j-1},x_j), \qquad s_{j-1}(y_1,\ldots,y_{j-1}) \approx s_{j-1}(x_1,\ldots,x_{j-1}),$$

by (4.10) and (4.9), we deduce that

$$|A_2| \lesssim \frac{s_j(x_1,\ldots,x_{j-1},x_j)}{s_{j-1}(x_1,\ldots,x_{j-1})}\,\|h\|_\infty. \tag{4.22}$$

To deal with the term A_1 in (4.17), first we have to estimate the difference

$$s_j(x_1,\ldots,x_j) - s_j(y_1,\ldots,y_{j-1},x_j). \tag{4.23}$$

The calculations are analogous to the ones above for

$$s_{j-1}(x_1,\ldots,x_{j-1}) - s_{j-1}(y_1,\ldots,y_{j-1}).$$

Indeed, recall that for the latter difference we used the estimate $|x_k - y_k| \lesssim x_k\|h\|_\infty$, for $1 \le k \le j-1$. The same inequalities are the ones required to estimate (4.23), taking into account the j-th coordinate is the same in the two terms involving $s_j(\ldots)$. Then, as in (4.21), we get

$$\big|s_j(x_1,\ldots,x_j) - s_j(y_1,\ldots,y_{j-1},x_j)\big| \lesssim \frac{\min(x_1,\ldots,x_j)^2}{\ell(R)^2}\,\ell(Q)^{n-j}\,\|h\|_\infty.$$

By the definition of A_1 in (4.17), we deduce

$$|A_1| \lesssim \frac{\ell(Q)^{n-j}\,\min(x_1,\ldots,x_j)^2}{\ell(R)^2\,s_{j-1}(x_1,\ldots,x_{j-1})}\,\|h\|_\infty.$$

Recalling that, by (4.9),

$$s_j(x_1,\ldots,x_j) \approx \frac{\ell(Q)^{n-j}}{\ell(R)^2}\,\min(x_1,\ldots,x_j)^2,$$

we obtain

$$|A_1| \lesssim \frac{s_j(x_1,\ldots,x_{j-1},x_j)}{s_{j-1}(x_1,\ldots,x_{j-1})}\,\|h\|_\infty. \tag{4.24}$$

Now it remains to estimate the term B in (4.15):

$$\begin{aligned} B &= \frac{s_j(y_1,\ldots,y_{j-1},x_j)}{s_{j-1}(y_1,\ldots,y_{j-1})} - \frac{r_j(y_1,\ldots,y_{j-1},x_j)}{r_{j-1}(y_1,\ldots,y_{j-1})} \\ &= \frac{s_j(y_1,\ldots,y_{j-1},x_j) - r_j(y_1,\ldots,y_{j-1},x_j)}{s_{j-1}(y_1,\ldots,y_{j-1})} \\ &\quad + r_j(y_1,\ldots,y_{j-1},x_j)\left(\frac{1}{s_{j-1}(y_1,\ldots,y_{j-1})} - \frac{1}{r_{j-1}(y_1,\ldots,y_{j-1})}\right) = B_1 + B_2. \end{aligned}$$

Taking into account that $|s(x) - r(x)| \leq s(x)\,\|h\|_\infty$ for all $x \in Q$, one easily gets

$$|s_j(y_1,\ldots,y_{j-1},x_j) - r_j(y_1,\ldots,y_{j-1},x_j)| \lesssim s_j(y_1,\ldots,y_{j-1},x_j)\,\|h\|_\infty$$

and

$$|s_{j-1}(y_1,\ldots,y_{j-1}) - r_{j-1}(y_1,\ldots,y_{j-1})| \lesssim s_{j-1}(y_1,\ldots,y_{j-1})\,\|h\|_\infty.$$

The first estimate readily implies that

$$|B_1| \lesssim \frac{s_j(y_1,\ldots,y_{j-1},x_j)}{s_{j-1}(y_1,\ldots,y_{j-1})}\,\|h\|_\infty \approx \frac{s_j(x_1,\ldots,x_j)}{s_{j-1}(x_1,\ldots,x_{j-1})}\,\|h\|_\infty.$$

The calculations for B_2 are also straightforward:

$$|B_2| \lesssim r_j(y_1,\ldots,y_{j-1},x_j)\,\frac{s_{j-1}(y_1,\ldots,y_{j-1})\,\|h\|_\infty}{s_{j-1}(y_1,\ldots,y_{j-1})\,r_{j-1}(y_1,\ldots,y_{j-1})} \lesssim \frac{s_j(x_1,\ldots,x_j)}{s_{j-1}(x_1,\ldots,x_{j-1})}\,\|h\|_\infty,$$

where we used that $r_j(y_1,\ldots,y_{j-1},x_j) \approx s_j(x_1,\ldots,x_j)$ and the analogous estimate for $r_{j-1}(y_1,\ldots,y_{j-1})$.

The lemma follows by gathering the inequalities obtained for A_1, A_2, B_1, and B_2. □

Notice that the preceding lemma is the only place of the paper where we used the smallness of the tangential derivatives of φ near the boundary of R.

5. Relationship between the coefficients α, α_1, α_2, and β_2

Let us recall the definition of the coefficients α from [To2]. Given a closed ball $B \subset \mathbb{R}^d$ and two finite Borel measures σ, ν on $\mathbb{R}^d$, we set

$$\operatorname{dist}_B(\sigma,\nu) := \sup\left\{\left|\int f\,d\sigma - \int f\,d\nu\right| : \operatorname{Lip}(f) \leq 1,\ \operatorname{supp}(f) \subset B\right\},$$

where $\operatorname{Lip}(f)$ stands for the Lipschitz constant of f. Notice the similarities between the distances W_1 and d_B. Given an AD regular measure μ on $\mathbb{R}^d$ and a ball B that intersects $\operatorname{supp}(\mu)$, we set

$$\alpha(B) = \frac{1}{r(B)^{n+1}} \inf_{a\geq 0, L} \operatorname{dist}_{3B}(\mu,\, a\mathcal{H}^n_{|L}),$$

where the infimum is taken over all the constants $a \geq 0$ and all the n-planes L.

Lemma 5.1. *Let μ be n-dimensional AD regular and let B be a closed ball that intersects* $\operatorname{supp}(\mu)$. *Then,*

$$\alpha(B) \le \alpha_1(2B)$$

and

$$\alpha_1(B) \le c\,\alpha(B),$$

where $c = c_{10}(\|\nabla\varphi\|_\infty + 1)$, with c_{10} being an absolute constant.

Proof. To prove the first inequality, take an n-plane L that intersects $2B$ and let $c_{2B,L}$ be as in (1.3), with $2B$ instead of B. Consider a Lipschitz function f supported in $3B$ with $\|\nabla f\|_\infty \le 1$. Since φ_{2B} equals 1 in the $4B \supset \operatorname{supp} f$, we have

$$\int f\,d\mu - \int f\,c_{2B,L}\,d\mathcal{H}^n_L = \int f\,\varphi_{2B}\,d\mu - \int f\,\varphi_{2B}\,c_{2B,L}\,d\mathcal{H}^n_L.$$

Taking supremums over such functions f, we get

$$\operatorname{dist}_{3B}(\mu,\, c_{2B,L}\mathcal{H}^n_L) \le W_1(\varphi_{2B}\,\mu,\, c_{2B,L}\varphi_{2B}\,\mathcal{H}^n_L).$$

Taking the infimum over all n planes L intersecting $2B$ and dividing by $r(2B)^{n+1}$, we deduce $\alpha(B) \le \alpha_1(2B)$.

Let us turn our attention to the second inequality in the lemma. let a and L be the constant and the n-plane that minimize $\alpha(B)$. To estimate $W_1\big(\varphi_B\mu,\, c_{B,L}\varphi_B\mathcal{H}^n_L\big)$ we may assume that the Lipschitz functions f in the supremum that defines the distance W_1 vanish in the center of B, since the difference of integrals in (1.1) vanishes on constant functions. So consider an arbitrary Lipschitz function with $\operatorname{Lip} f \le 1$ which vanishes in the center of B. Then

$$\left|\int f\,\varphi_B\,d\mu - \int f\,\varphi_B\,c_{B,L}\,d\mathcal{H}^n_L\right| \le \left|\int f\,\varphi_B\,d\mu - \int f\,\varphi_B\,a\,d\mathcal{H}^n_L\right| + |a - c_{B,L}|\left|\int f\,\varphi_B\,d\mathcal{H}^n_L\right|. \tag{5.1}$$

Concerning the first sum on the right side, since $f\varphi_B$ is supported on $3B$,

$$\begin{aligned}\left|\int f\,\varphi_B\,d\mu - \int f\,\varphi_B\,a\,d\mathcal{H}^n_L\right| &\le \|\nabla(f\varphi_B)\|_\infty \operatorname{dist}_{3B}(\mu,\, a\mathcal{H}^n_L)\\ &= \|\nabla(f\varphi_B)\|_\infty\,\alpha(B)\,r(B)^{1+n}.\end{aligned}$$

To estimate $\|\nabla(f\varphi_B)\|_\infty$, notice that, by the mean value theorem, $\|f\|_\infty \le r(B)$, since f vanishes in the center of B and $\|\nabla f\|_\infty \le 1$, and thus,

$$\|\nabla(f\varphi_B)\|_\infty \le \|\nabla f\|_\infty\|\varphi_B\|_\infty + \|\nabla\varphi_B\|_\infty\|f\|_\infty \le 1 + \frac{1}{r(B)}\,\|\nabla\varphi\|_\infty r(B) = 1 + \|\nabla\varphi\|_\infty.$$

Therefore,

$$\left|\int f\,\varphi_B\,d\mu - \int f\,\varphi_B\,a\,d\mathcal{H}^n_L\right| \le \big(1 + \|\nabla\varphi\|_\infty\big)\,\alpha(B)\,r(B)^{1+n}.$$

Finally we deal with the last summand on the right side of (5.1):

$$|a - c_{B,L}|\left|\int f\,\varphi_B\,d\mathcal{H}^n_L\right| \leq |a - c_{B,L}|\,\|f\|_\infty \mathcal{H}^n(3B\cap L) \leq c\,|a - c_{B,L}|\,r(B)^{1+n}.$$

At last, we estimate $|a - c_{B,L}|$. To this end, we set

$$\left|\int \varphi_B\,a\,d\mathcal{H}^n_L - \int \varphi_B\,d\mu\right| \leq \alpha(B)\,\|\nabla\varphi_B\|_\infty\,r(B)^{n+1} = \alpha(B)\,\|\nabla\varphi\|_\infty\,r(B)^n.$$

If we divide this inequality by $\int \varphi_B\,d\mathcal{H}^n_L$, we get

$$|a - c_{B,L}| \leq \frac{\alpha(B)\,\|\nabla\varphi\|_\infty\,r(B)^n}{\int \varphi_B\,d\mathcal{H}^n_L}.$$

Notice that $\int \varphi_B\,d\mathcal{H}^n_L \geq \mathcal{H}^n(2B\cap L) \geq c^{-1}r(B)^n$, since $L\cap B\neq\varnothing$. Thus, $|a - c_{B,L}| \leq c\alpha(B)\,\|\nabla\varphi\|_\infty$, and so the last summand on the right side of (5.1) is bounded above by $c\,\alpha(B)\,\|\nabla\varphi\|_\infty\,r(B)^{1+n}$. Together with the estimate obtained for the first summand, this yields

$$\left|\int f\,\varphi_B\,d\mu - \int f\,\varphi_B\,c_{B,L}\,d\mathcal{H}^n_L\right| \leq c\,\alpha(B)\,(1+\|\nabla\varphi\|_\infty)\,r(B)^{1+n}.$$

Taking the supremum over all functions f as above, we deduce that

$$W_1\big(\varphi_B\mu,\, c_{B,L}\varphi_B\mathcal{H}^n_L\big) \leq c\,(1+\|\nabla\varphi\|_\infty)\,\alpha(B)\,r(B)^{1+n},$$

and thus $\alpha_1(B) \leq c\,(1+\|\nabla\varphi\|_\infty)\,\alpha(B)$. □

Recall that, for a ball B that intersects $\operatorname{supp}(\mu)$, we have

$$\beta_2(B) = \left(\inf_L \frac{1}{r(B)^n}\int_{2B}\left(\frac{\operatorname{dist}(x,L)}{r(B)}\right)^2 d\mu(x)\right)^{1/2},$$

where the infimum is taken over all n-planes in $\mathbb{R}^d$.

Lemma 5.2. *Let B be a ball that intersects $\operatorname{supp}(\mu)$ and fix an n-plane L that minimizes $\alpha_2(B)$. Denote by Π_L the orthogonal projection onto L. Then we have*

$$W_2\big(\varphi_B\,\mu,\,\Pi_L\#(\varphi_B\,\mu)\big) \leq W_2\big(\varphi_B\,\mu,\,c_{B,L}\,\varphi_B\,\mathcal{H}^n_L\big).$$

Moreover, $\beta_2(B) \leq \alpha_2(B)$.

Proof. Notice that, by the definition of W_2,

(5.2)
$$W_2\big(\varphi_B\,\mu,\,\Pi_L\#(\varphi_B\,\mu)\big)^2 \leq \int |\Pi_L(x) - x|^2\,\varphi_B(x)\,d\mu(x) = \int \operatorname{dist}(x,L)^2\,\varphi_B(x)\,d\mu(x).$$

Consider now an optimal transference plan π between $\varphi_B\,\mu$ and $c_{B,L}\,\varphi_B\,\mathcal{H}^n_L$, that is, π is a measure on $\mathbb{R}^d\times\mathbb{R}^d$ with two marginals given by the preceding measures such that

$$\int_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\,d\pi(x,y) = W_2(\varphi_B\,\mu,\,c_{B,L}\,\varphi_B\,\mathcal{H}^n_L)^2. \tag{5.3}$$

It turns out that such a measure π must be supported on

$$\operatorname{supp}(\varphi_B\,\mu)\times\operatorname{supp}(\varphi_B\,\mathcal{H}^n_L)\subset\mathbb{R}^d\times L.$$

Therefore,

$$\int_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\,d\pi(x,y)=\int_{(x,y)\in\mathbb{R}^d\times L}|x-y|^2\,d\pi(x,y).$$

For (x,y) in the domain of integration of the last integral we have $|x-y|\geq\operatorname{dist}(x,L)$, and thus

$$\begin{aligned}\int_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}|x-y|^2\,d\pi(x,y)&\geq\int_{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d}\operatorname{dist}(x,L)^2\,d\pi(x,y)\\&=\int\operatorname{dist}(x,L)^2\,\varphi_B(x)\,d\mu(x),\end{aligned}\tag{5.4}$$

which proves the first claim in the lemma, by (5.2) and (5.3).

To prove the second assertion, just notice that

$$\int\operatorname{dist}(x,L)^2\,\varphi_B(x)\,d\mu(x)\geq\int_{2B}\operatorname{dist}(x,L)^2\,d\mu(x)\geq\beta_2(B)^2\,r(B)^{n+2}.$$

Together with (5.3) and (5.4), this implies that $\beta_2(B)\leq\alpha_2(B)$. □

Of course, by analogous arguments, we get $\beta_p(B)\leq\alpha_p(B)$ for all $1<p<\infty$.

Our next objective consists in showing that if μ is AD-regular and B, B' are two balls in $\mathbb{R}^d$, centered in $\operatorname{supp}(\mu)$, such that $3B\subset B'$, which have comparable radii, then $\alpha_2(B)\lesssim\alpha_2(B')$. First we will prove a slightly more general result, where μ is not assumed to be AD-regular.

Lemma 5.3. *Let μ be a measure supported on a ball $B'\subset\mathbb{R}^d$. Let $B\subset\mathbb{R}^d$ be another ball such that $3B\subset B'$, with $r(B)\approx r(B')$ and $\mu(B)\approx\mu(B')\approx r(B)^n$. Let L be an n-plane which intersects B and let $f:L\to[0,1]$ be a function which equals 1 identically on $3B$ and vanishes in $L\setminus B'$, so that $\int f\,d\mathcal{H}^n_L=\mu(B')$. Then, $W_2(\varphi_B\mu,\,c\varphi_B\mathcal{H}^n_L)\lesssim W_2(\mu,\,f\mathcal{H}^n_L)$, for the appropriate constant c.*

Proof. Denote $B=B(z_B,r)$, and let $P^\perp$ the orthogonal projection onto L. Also, for $x\in B'$ such that $|x-z_B|>\operatorname{dist}(z_B,L)$ and $P^\perp(x)\neq P^\perp(z_B)$ we define the angular projection $P^a(z)$ onto L with center in z_B as follows:

$$P^a(x)=P^\perp(z_B)+\frac{\sqrt{|x-z_B|^2-\operatorname{dist}(z_B,L)^2}}{|P^\perp(x)-P^\perp(z_B)|}\big(P^\perp(x)-P^\perp(z_B)\big).$$

Notice $P^a(x)\in L$ and

$$|P^a(x)-z_B|=|x-z_B|.\tag{5.5}$$

If $|x-z_B|<\operatorname{dist}(z_B,L)$ but $P^\perp(x)=P^\perp(z_B)$, we let $P^a(x)$ be an arbitrary point from L satisfying (5.5). Now we consider a new map $P:B'\to L$ defined as follows:

$$P(x)=\begin{cases}P^\perp(x) & \text{if } x\in\frac{3}{2}B,\\[2ex] P^a(x) & \text{if } x\in B'\setminus\frac{3}{2}B.\end{cases}$$

Observe that $P_{|L} = Id_{|L}$.

We claim that for any $x \in B'$, $\varphi_B(P(x)) = \varphi_B(x)$. For $x \in B' \setminus \frac{3}{2}B$ this follows from (5.5) and the fact that φ_B is radial (with respect to the center z_B). For $x \in \frac{3}{2}B$, we have $|P(x) - P(z_B)| = |P^\perp(x) - P^\perp(z_B)| \leq |x - z_B| \leq \frac{3}{2}r$ and since $|P^\perp(z_B) - z_B| = \operatorname{dist}(z_B, L) \leq r$, we deduce

$$|P(x) - z_B|^2 = |P^\perp(x) - P^\perp(z_B)|^2 + |P^\perp(z_B) - z_B|^2 \leq \frac{13}{4}\, r^2 < (2r)^2,$$

and thus $P(x) \in 2B$ and $\varphi_B(x) = \varphi_B(P(x)) = 1$.

On the other hand, it is also easy to check that $|x - P(x)| \lesssim \operatorname{dist}(x, L)$ for $x \in B'$, and thus

$$\begin{aligned} W_2(P\#(\varphi_B\mu), \varphi_B\mu)^2 &\leq \int |Px - x|^2\, d\mu(x) \\ &\lesssim \int d(x, L)^2\, d\mu(x) \lesssim W_2(\mu, f\mathcal{H}^n_L)^2, \end{aligned} \tag{5.6}$$

where the last inequality is proved arguing as in Lemma 5.2. The same arguments yield

$$W_2(P\#\mu, \mu) \lesssim W_2(\mu, f\mathcal{H}^n_L). \tag{5.7}$$

Notice now that $P\#(\varphi_B\mu) = \varphi_B(P\#\mu)$. Indeed, using that $\varphi_B(P(x)) = \varphi_B(x)$ for all $x \in \operatorname{supp}(\mu)$, for any subset $A \subset L$ we have

$$\begin{aligned} \int \chi_A dP\#(\varphi_B\mu) &= \int \chi_A(P(x))\, \varphi_B(x)\, d\mu(x) \\ &= \int \chi_A(P(x))\varphi_B(P(x))\, d\mu(x) = \int \chi_A\varphi_B\, dP\#\mu. \end{aligned}$$

Using that $f\varphi_B = \varphi_B$, by Theorem 1.1 and (5.7), for an appropriate constant c we get

$$\begin{aligned} W_2(P\#(\varphi_B\mu), c\varphi_B\mathcal{H}^n_L) = W_2\big(\varphi_B(P\#\mu), c\varphi_B\, f\mathcal{H}^n_L\big) &\lesssim W_2(P\#\mu, f\mathcal{H}^n_L) \\ &\leq W_2(P\#\mu, \mu) + W_2(\mu, f\mathcal{H}^n_L) \lesssim W_2(\mu, f\mathcal{H}^n_L). \end{aligned} \tag{5.8}$$

Then, from (5.6) and (5.8),

$$W_2(\varphi_B\mu, c\varphi_B\mathcal{H}^n_L) \leq W_2(\varphi_B\mu, P\#(\varphi_B\mu)) + W_2(P\#(\varphi_B\mu), c\varphi_B\mathcal{H}^n_L) \lesssim W_2(\mu, f\mathcal{H}^n_L).$$

□

Lemma 5.4. *Let μ be an n-dimensional AD-regular measure on $\mathbb{R}^d$. Let $B, B' \subset \mathbb{R}^d$ be balls such that $3B \subset B'$, $\mu(B) \approx r(B)^n$, and $r(B) \approx r(B')$. Then,*

$$\alpha_2(B) \lesssim \alpha_2(B').$$

Proof. We apply the preceding lemma with $\varphi_{B'}\mu$ instead of μ there, L equal to an n-plane that minimizes $\alpha_2(B')$, and $f = \varphi_{B'}$. To prove the lemma we may assume that $\alpha_2(B') \leq \delta$, with $\delta > 0$ small enough. This implies that $\beta_2(B') \leq \delta$. Then, using that $\mu(B) \approx r(B)^n$, it is easy to check that if δ is small enough, L intersects B, and thus the assumptions in the preceding lemma are satisfied. □

6. The coefficients α_2 on Lipschitz graphs

In this section we will prove the particular case of Theorem 1.2 for Lipschitz graphs. Given a Lipschitz function $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$, we set $\Gamma = \{(x, A(x)) : x \in \mathbb{R}^n\}$. Then we take $\mu = g\,\mathcal{H}^n_\Gamma$, where $g : \Gamma \to (0, +\infty)$ satisfies $g(x) \approx 1$ for all $x \in \Gamma$. It is clear that μ is n-dimensional AD-regular. We consider the following special "v-cubes" associated to μ: we say that $Q \subset \mathbb{R}^d$ is a v-cube if it is of the form $Q = Q_0 \times \mathbb{R}^{d-n}$, where $Q_0 \subset \mathbb{R}^n$ is an n-dimensional cube. We denote $\ell(Q) := \ell(Q_0)$. We say that Q is a dyadic v-cube if Q_0 is a dyadic cube. The collection of dyadic v-cubes Q with $\ell(Q) = 2^{-j}$ is denoted by $\mathcal{D}_{\mathrm{v,j}}$. Also, we set $\mathcal{D}_{\mathrm{v}} = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_{\mathrm{v,j}}$ and $\mathcal{D}^k_{\mathrm{v}} = \bigcup_{j\geq k} \mathcal{D}_{\mathrm{v,j}}$, Given a v-cube Q, we let $\alpha_p(Q) = \alpha_p(B_Q)$, where B_Q is a smallest closed ball centered at some point from $Q \cap \Gamma$ that contains $Q \cap \Gamma$.

Now, for technical reasons, we need to introduce other scale invariant coefficient of "α and β type". Given Γ and μ as above and a v-cube $Q \subset \mathbb{R}^d$, we denote

$$\widetilde{\alpha}_2(Q) = \frac{1}{\ell(Q)^{1+\frac{n}{2}}} \inf_L W_2\big(\chi_Q\,\mu,\, c_{Q,L}\chi_Q \mathcal{H}^n_L\big),$$

where the infimum is taken over all n-dimensional planes that intersect $\overset{\circ}{Q}$ and $c_{Q,L} = \mu(Q)/\mathcal{H}^n(L \cap Q)$. We also set

$$\widetilde{\beta}_2(Q) = \left(\inf_L \frac{1}{\ell(Q)^n} \int_Q \left(\frac{\operatorname{dist}(x, L)}{\ell(Q)}\right)^2 d\mu(x)\right)^{1/2},$$

where the infimum is taken over all n-planes in $\mathbb{R}^d$. Notice that the integral is over Q instead of $2Q$.

Lemma 6.1. *As above, let $\Gamma \subset \mathbb{R}^d$ be an n-dimensional Lipschitz graph and $\mu = \rho\mathcal{H}^n_\Gamma$, with $\rho(x) \approx 1$ for all $x \in \Gamma$. Let $B, Q \subset \mathbb{R}^d$ be a ball and a v-cube such that $3B \subset Q$ and $\frac{1}{2}B \cap \operatorname{supp}(\mu) \neq \varnothing$. Suppose also that $r(B) \approx \ell(Q)$. Then,*

$$\alpha_2(B) \lesssim \widetilde{\alpha}_2(Q).$$

Proof. The arguments are the same as in Lemma 5.4. Indeed, notice that in its proof the smoothness of $\varphi_{B'}$ was not used. It was only necessary that $\varphi_{B'}(x) = 1$ for $x \in \operatorname{supp}(\varphi_B)$, which also holds with χ_Q instead of $\varphi_{B'}$. □

We identify $\mathbb{R}^n$ with the subspace of $\mathbb{R}^d$ formed by those points whose last $d-n$ coordinates are zero. Let $P_{\mathbb{R}^n}$ and P_Γ projections from $\mathbb{R}^d$ onto $\mathbb{R}^n$ and Γ, respectively, both orthogonal to $\mathbb{R}^n$. Observe that $P_{\mathbb{R}^n|\Gamma} : \Gamma \to \mathbb{R}^n$ is a bilipschitz mapping. We denote $\sigma_\Gamma = P_\Gamma \# \mathcal{L}^n_{\mathbb{R}^n}$, where $\mathcal{L}^n_{\mathbb{R}^n}$ stands for the Lebesgue measure on $\mathbb{R}^n$. Clearly, $\mathcal{H}^n_\Gamma$ and σ_Γ are comparable. However, for the arguments below σ_Γ will be more convenient that $\mathcal{H}^n_\Gamma$.

Lemma 6.2. *As above, let $\Gamma \subset \mathbb{R}^d$ be an n-dimensional Lipschitz graph and $\mu = g\sigma_\Gamma$, with $g(x) \lesssim 1$ for all $x \in \Gamma$. Consider a v-cube $Q \subset \mathbb{R}^d$ and a tree $\mathcal{T}$ with root Q. Denote by $\mathcal{S}(\mathcal{T}) \subset Q$ its family of stopping v-cubes and suppose that $\mu(P) \approx \ell(P)^n$*

for every $P \in \mathcal{T}$. Then we have

$$\widetilde{\alpha}_2(Q)^2 \lesssim \widetilde{\beta}_2(Q)^2 + \sum_{P\in\mathcal{T}} \|\Delta_P^{\sigma_\Gamma} g\|^2_{L^2(\sigma_\Gamma)} \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{P\in\mathcal{S}(\mathcal{T})} \frac{\ell(P)^2}{\ell(Q)^{n+2}}\,\mu(P).$$

Proof. Let L be the n plane that minimizes $\widetilde{\alpha}_2(Q)$, and assume that it is not orthogonal to $\mathbb{R}^n$ (otherwise, just rotate it slightly). Let P_L be the projection from $\mathbb{R}^d$ onto L which is orthogonal to $\mathbb{R}^n$, and $P_{\mathbb{R}^n}, P_\Gamma$ as above. Consider the flat measure $\sigma_L = P_L\#\sigma_\Gamma = P_L\#\mathcal{L}^n_{\mathbb{R}^n}$. We have,

$$W_2(\chi_Q\mu,\, a\,\chi_Q\sigma_L) \leq W_2(\chi_Q\mu,\, \chi_Q\, P_L\#\mu) + W_2(\chi_Q\, P_L\#\mu,\, a\,\chi_Q\,\sigma_L), \tag{6.1}$$

where

$$a = \frac{\mu(Q)}{\sigma_L(Q)} = \frac{\mu(Q)}{\mathcal{L}^n_{\mathbb{R}^n}(Q)} = m_Q g_0, \qquad g_0 = g\circ P_\Gamma.$$

The first summand on the right side of (6.1) is easily estimated in terms of $\widetilde{\beta}_2(Q)$:

$$W_2(\chi_Q\mu,\, \chi_Q\, P_L\#\mu)^2 \leq \int_Q |x - P_L x|^2\, d\mu(x) \lesssim \widetilde{\beta}_2(Q)^2\,\ell(Q)^{n+2}.$$

Concerning the last term in (6.1), using that $P_{\mathbb{R}^n|L\cap Q} : L\cap Q \to \mathbb{R}^n\cap Q$ is bilipschitz, we have

$$\begin{aligned} W_2(\chi_Q\, P_L\#\mu,\, a\,\chi_Q\,\sigma_L) &\approx W_2(\chi_Q\, P_{\mathbb{R}^n}\#(P_L\#\mu),\, a\,\chi_Q\, P_{\mathbb{R}^n}\#\sigma_L) \\ &= W_2(\chi_Q\, g_0\mathcal{L}^n_{\mathbb{R}^n},\, a\,\chi_Q\,\mathcal{L}^n_{\mathbb{R}^n}). \end{aligned}$$

Now recall that, by (3.26) in Remark 3.14, we have

$$W_2(\chi_Q\, g_0\mathcal{L}^n_{\mathbb{R}^n},\, a\,\chi_Q\,\mathcal{L}^n_{\mathbb{R}^n})^2 \lesssim \sum_{P\in\mathcal{T}_{\mathbb{R}^n}} \|\Delta_P g_0\|^2_{L^2(\mathbb{R}^n)}\,\ell(P)\ell(Q) + \sum_{P\in\mathcal{S}(\mathcal{T})} \ell(P)^2\,\mu(P),$$

where $\mathcal{T}_{\mathbb{R}^n} \subset \mathcal{D}(\mathbb{R}^n)$ is the tree formed by the cubes $P_{\mathbb{R}^n}(\widehat{P})$, $\widehat{P}\in\mathcal{T}$. From (6.1) and the preceding estimates, we infer that

$$\begin{aligned} &W_2(\chi_Q\mu,\, a\,\chi_Q\sigma_L)^2 \\ &\qquad \lesssim \widetilde{\beta}_2(Q)^2\,\ell(Q)^{n+2} + \sum_{P\in\mathcal{T}_{\mathbb{R}^n}} \|\Delta_P g_0\|^2_{L^2(\mathbb{R}^n)}\,\ell(P)\ell(Q) + \sum_{P\in\mathcal{S}(\mathcal{T})} \ell(P)^2\,\mu(P). \end{aligned}$$

To conclude the proof of the lemma, just notice that, from the definition of g, g_0 and σ_Γ, it follows that for each cube $P\subset\mathbb{R}^n$ and the corresponding v-cube $\widehat{P} = P\times\mathbb{R}^{d-n}$,

$$\|\Delta_P g_0\|_{L^2(\mathbb{R}^n)} = \|\Delta_{\widehat{P}}^{\sigma_\Gamma} g\|_{L^2(\sigma_\Gamma)}.$$

□

Lemma 6.3. *As above, let $\Gamma\subset\mathbb{R}^d$ be an n-dimensional Lipschitz graph and $\mu = g\sigma_\Gamma$, with $g(x)\lesssim 1$ for all $x\in\Gamma$. Consider a tree $\mathcal{T}$ of v-cubes such that every $Q\in\mathcal{T}$ satisfies the following property:*

If P is a v-cube (non necessarily dyadic) such that $P\cap Q\neq\varnothing$ and $\ell(P) = \ell(Q)$, then $\mu(P)\approx\ell(P)^n$. (6.2)

Then we have

$$\sum_{Q\in\mathcal{T}:Q\subset R} \alpha_2(Q)^2\mu(Q) \lesssim \mu(R) \qquad \text{for all } R\in\mathcal{T}. \tag{6.3}$$

Proof. We will use a well known trick which goes back to Okikiolu [Ok], as far as we know. Given a fixed $k\in\mathbb{Z}$ and $j\geq k$, for $e\in\{0,1\}^n$ consider the translated grid $\frac{2^{-k}}{3}\,e+\mathcal{D}_{\mathbb{R}^n,j}$ and denote

$$\widetilde{\mathcal{D}}_{\mathbb{R}^n,j} = \bigcup_{e\in\{0,1\}^n}\left(\frac{2^{-k}}{3}\,e+\mathcal{D}_{\mathbb{R}^n,j}\right), \qquad \widetilde{\mathcal{D}}^k_{\mathbb{R}^n} = \bigcup_{j\geq k}\widetilde{\mathcal{D}}_{\mathbb{R}^n,j}.$$

It turns out that for every $j\geq k$ and $x\in\mathbb{R}^n$,

$$\exists\, Q\in\widetilde{\mathcal{D}}_{\mathbb{R}^n,j} \quad \text{such that } x\in\frac{2}{3}Q. \tag{6.4}$$

See [Lr] for a further generalization and a very transparent proof.

To prove (6.3), we may assume that the root of $\mathcal{T}$ is R. Let $k\in\mathbb{Z}$ be such that $R\in\mathcal{D}_{\mathrm{v},k}$, that is, $\ell(R)=2^{-k}$. Consider the "extended lattice" $\widetilde{\mathcal{D}}^k_{\mathrm{v}}$ associated to $\widetilde{\mathcal{D}}^k_{\mathbb{R}^n}$. If n_0 is chosen big enough (depending on d, n, $\|\nabla A\|_\infty$, and various absolute constants), from (6.4) we infer that for any $Q\in\mathcal{D}_{\mathrm{v}}(R)$ with $\ell(Q)\leq 2^{-n_0}\ell(R)$ there exists another v-cube $\widetilde{Q}\in\widetilde{\mathcal{D}}^k_{\mathrm{v}}$ such that $3B_Q\subset\widetilde{Q}$ and $\ell(\widetilde{Q})=2^{n_0}\ell(Q)$ (recall that B_Q is the smallest closed ball that contains $Q\cap\Gamma$).

For $e\in\{0,1\}^n$, denote by $\mathcal{T}_e$ the collection of v-cubes $P\in\frac{2^{-k}}{3}\,e+\mathcal{D}^k_{\mathrm{v}}$ for which there exists some $Q\in\mathcal{T}$ such that $Q\cap P\neq\varnothing$ and $\ell(Q)=\ell(P)$. It is immediate to check that although, in general, $\mathcal{T}_e$ is not a tree, it is made of a finite collection of trees whose roots are the v-cubes from $\frac{2^{-k}}{3}\,e+\mathcal{D}_{\mathrm{v},k}$ that intersect R.

Observe that every $Q\in\mathcal{T}_e$ satisfies $\mu(Q)\approx\ell(Q)^n$, by the condition (6.2). Then, by Lemma 6.2, we obtain

$$\begin{aligned}\sum_{Q\in\mathcal{T}_e}\widetilde{\alpha}_2(Q)^2\mu(Q) &\lesssim \sum_{Q\in\mathcal{T}_e}\widetilde{\beta}_2(Q)^2\,\mu(Q) + \sum_{Q\in\mathcal{T}_e}\sum_{P\in\mathcal{T}_e:P\subset Q}\|\Delta_P^{\sigma_\Gamma}g\|^2_{L^2(\sigma_\Gamma)}\frac{\ell(P)}{\ell(Q)}\\ &\quad+\sum_{Q\in\mathcal{T}_e}\sum_{P\in\mathcal{S}(\mathcal{T}_e):P\subset Q}\frac{\ell(P)^2}{\ell(Q)^2}\,\mu(P).\end{aligned} \tag{6.5}$$

The first sum on the right side is bounded by $c\mu(R)$, since $\widetilde{\beta}_2(Q)\leq\beta_2(Q)$ and the $\beta_2(Q)$ coefficients satisfy a Carleson packing condition on Lipschitz graphs (see [DS1]). Concerning the second one, interchanging the sums, it equals

$$\sum_{P\in\mathcal{T}_e}\|\Delta_P^{\sigma_\Gamma}g\|^2_{L^2(\sigma_\Gamma)}\sum_{Q\in\mathcal{T}_e:Q\supset P}\frac{\ell(P)}{\ell(Q)}\lesssim\sum_{P\in\mathcal{T}_e}\|\Delta_P^{\sigma_\Gamma}g\|^2_{L^2(\sigma_\Gamma)}\lesssim\mu(R),$$

since g is a bounded function. Finally, interchanging the order of summation again, the last term in (6.5) equals

$$\sum_{P\in\mathcal{S}(\mathcal{T}_e)}\mu(P)\sum_{Q\in\mathcal{T}_e:Q\supset P}\frac{\ell(P)^2}{\ell(Q)^2}\lesssim\sum_{P\in\mathcal{S}(\mathcal{T}_e)}\mu(P)\lesssim\mu(R).$$

Therefore, $\sum_{Q\in\mathcal{T}_e}\widetilde{\alpha}_2(Q)^2\mu(Q)\lesssim\mu(R)$.

By the discussion just below (6.4), for every $Q\in\mathcal{T}$ with $\ell(Q)\leq 2^{-n_0}\ell(R)$, there exists some $e\in\{0,1\}^n$ and some $Q'\in\mathcal{T}_e$ such that $3B_Q\subset Q'$ and thus $\alpha_2(Q)\lesssim\widetilde{\alpha}_2(Q')$, by Lemma 6.1. Then we get

$$\sum_{Q\in\mathcal{T}:\ell(Q)\leq 2^{-n_0}\ell(R)}\alpha_2(Q)^2\mu(Q)\lesssim\sum_{Q\in\mathcal{T}_e}\widetilde{\alpha}_2(Q)^2\mu(Q)\lesssim\mu(R).$$

On the other hand, since there is a bounded number of v-cubes $Q\in\mathcal{T}$ such that $\ell(Q)>2^{-n_0}\ell(R)$, also

$$\sum_{Q\in\mathcal{T}:\ell(Q)>2^{-n_0}\ell(R)}\alpha_2(Q)^2\mu(Q)\lesssim\mu(R).$$

□

A direct consequence of the preceding lemma is the following.

Theorem 6.4. *Let $\Gamma\subset\mathbb{R}^d$ be an n-dimensional Lipschitz graph in $\mathbb{R}^d$ and $\mu=g\sigma_\Gamma$, if $g:\Gamma\to(0,+\infty)$ satisfies $g(x)\approx 1$ for all $x\in\Gamma$, then $\alpha_2(x,t)^2\,d\mu(x)\,\dfrac{dr}{r}$ is a Carleson measure, that is, if for any ball B with radius R,*

$$\int_0^R\int_B\alpha_2(x,t)^2\,d\mu(x)\,\frac{dr}{r}\leq c\,R^n.$$

Proof. It follows by standard methods from the previous lemma, taking into account Lemma 5.4. □

Notice that the assumptions in the Lemma 6.3 were somewhat more general than the ones needed for the last theorem. The broader generality of Lemma 6.3 will be needed below, to deal with the case of uniformly rectifiable sets.

7. The coefficients α_2 on uniformly rectifiable sets

Throughout all this section we will assume that μ is an n-dimensional AD-regular measure on $\mathbb{R}^d$. As in the preceding section, the coefficients α_p and β_p are defined with respect to μ. If they are taken with respect to a different measure σ, they are denoted by $\alpha_{p,\sigma}$ or $\beta_{p,\sigma}$.

7.1. μ-cubes and the corona decomposition.

For the study of the uniformly rectifiable sets we will use the "dyadic cubes" built by David in [Da, Appendix 1] (see also [Ch] for an alternative construction). These dyadic cubes are not true cubes, but they play this role with respect to μ, in a sense. To distinguish them from the usual cubes, we will call them "μ-cubes".

Let us explain which are the precise results and properties about the lattice of dyadic μ-cubes. For each $j\in\mathbb{Z}$, there exists a family $\mathcal{D}_j^\mu$ of Borel subsets of $\operatorname{supp}(\mu)$ (the dyadic μ-cubes of the j-th generation) such that:

(i) each $\mathcal{D}_j^\mu$ is a partition of $\operatorname{supp}(\mu)$, i.e. $\operatorname{supp}(\mu)=\bigcup_{Q\in\mathcal{D}_j^\mu}Q$ and $Q\cap Q'=\varnothing$ whenever $Q,Q'\in\mathcal{D}_j^\mu$ and $Q\neq Q'$;

(ii) if $Q \in \mathcal{D}_j^\mu$ and $Q' \in \mathcal{D}_k^\mu$ with $k \le j$, then either $Q \subset Q'$ or $Q \cap Q' = \varnothing$;
(iii) for all $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_j^\mu$, we have $2^{-j} \lesssim \operatorname{diam}(Q) \le 2^{-j}$ and $\mu(Q) \approx 2^{-jn}$;
(iv) if $Q \in \mathcal{D}_j^\mu$, there is a point $z_Q \in Q$ (the center of Q) such that $\operatorname{dist}(z_Q, \operatorname{supp}(\mu) \setminus Q) \gtrsim 2^{-j}$.

We denote $\mathcal{D}^\mu = \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j^\mu$. Given $Q \in \mathcal{D}_j^\mu$, the unique μ-cube $Q' \in \mathcal{D}_{j-1}^\mu$ which contains Q is called the parent of Q. We say that Q is a sibling or son of Q'.

For $Q \in \mathcal{D}_j^\mu$, we define the side length of Q as $\ell(Q) = 2^{-j}$. Notice that $\ell(Q) \lesssim \operatorname{diam}(Q) \le \ell(Q)$. Actually it may happen that a cube Q belongs to $\mathcal{D}_j^\mu \cap \mathcal{D}_k^\mu$ with $j \ne k$, because there may exist cubes with only one sibling. In this case, $\ell(Q)$ is not well defined. However this problem can be solved in many ways. For example, the reader may think that a cube is not only a subset of $\operatorname{supp}(\mu)$, but a couple (Q, j), where Q is a subset of $\operatorname{supp}(\mu)$ and $j \in \mathbb{Z}$ is such that $Q \in \mathcal{D}_j^\mu$.

Given $\lambda > 1$, we set

$$\lambda Q := \big\{x \in \operatorname{supp}(\mu) : \operatorname{dist}(x, Q) \le (\lambda - 1)\ell(Q)\big\}.$$

Observe that $\operatorname{diam}(\lambda Q) \le \operatorname{diam}(Q) + 2(\lambda-1)\ell(Q) \le (2\lambda - 1)\ell(Q)$. For $R \in \mathcal{D}^\mu$, we denote $\mathcal{D}^\mu(R) = \{Q \in \mathcal{D}^\mu : Q \subset R\}$.

Given a μ-cube Q (or an arbitrary subset of $\operatorname{supp}(\mu)$), we denote by B_Q a smallest ball centered at some point from Q which contains Q. Then we define $\alpha_p(Q) := \alpha_p(B_Q)$ and $\beta_p(Q) := \beta_p(B_Q)$. If B_Q is not unique it does not matter which one we choose. The "bilateral β coefficient" of Q is:

$$b\beta_\infty(Q) = \inf_L \left[\sup_{x \in \operatorname{supp}(\mu) \cap 2B_Q} \frac{\operatorname{dist}(x, L)}{\ell(Q)} + \sup_{x \in L \cap 2B_Q} \frac{\operatorname{dist}(x, \operatorname{supp}(\mu))}{\ell(Q)} \right].$$

Now we wish to recall the notion of corona decomposition from David and Semmes [DS1], [DS2] (adapted to our specific situation). It involves the notion of a tree of dyadic μ-cubes, which is analogous to the one of a tree of dyadic cubes that was introduced just before Lemma 3.13.

Definition 7.1. Fix some constants $\lambda > 2$ and $\eta, \varepsilon > 0$. A corona decomposition of μ (with parameters λ, η, θ) is a partition of $\mathcal{D}^\mu$ into a family of trees $\{\mathcal{T}_i\}_{i\in I}$ of dyadic μ-cubes and a collection $\mathcal{B}$ of *bad* μ-cubes (that is, $\mathcal{D}^\mu = \mathcal{B} \cup \bigcup_{i\in I} \mathcal{T}_i$, with $\mathcal{B} \cap \mathcal{T}_i = \varnothing$ for all $i \in I$, and $\mathcal{T}_i \cap \mathcal{T}_j = \varnothing$ for all $i \ne j$) which satisfies the following properties:

- The family $\mathcal{B}$ and the collection $\mathcal{R}$ of all roots of the trees $\{\mathcal{T}_i\}_{i\in I}$ satisfy a Carleson packing condition. That is, there exists $c > 0$ such that for every $R \in \mathcal{D}^\mu$,
$$\sum_{Q \in \mathcal{B} \cup \mathcal{R} : Q \subset R} \mu(Q) \le c\,\mu(R).$$
- Each $Q \in \bigcup_{i\in I} \mathcal{T}_i$ satisfies $b\beta_\infty(\lambda Q) \le \varepsilon$.
- For each tree $\mathcal{T}_i$ there exists a (possibly rotated) n-dimensional Lipschitz graph Γ_i with Lipschitz constant $\le \eta$ such that $\operatorname{dist}(x, \Gamma_i) \le \varepsilon\,\ell(Q)$ whenever $x \in \lambda Q$ and $Q \in \mathcal{T}_i$.

It is shown in [DS1] (see also [DS2]) that if μ is uniformly rectifiable then it admits a corona decomposition for all parameters $\lambda > 2$ and $\eta, \varepsilon > 0$. Conversely, the existence of a corona decomposition for a single set of parameters $\lambda > 2$ and $\eta, \varepsilon > 0$ implies that μ is uniformly rectifiable.

For a given tree $\mathcal{T}$ from the corona decomposition of a uniformly rectifiable measure μ, we denote by $R_{\mathcal{T}}$ its root and by $\Gamma_{\mathcal{T}}$ its associated Lipschitz graph.

Let us remark that, in general, the stopping μ-cubes $\mathcal{S}(\mathcal{T})$ may have very different side lengths. This may cause some troubles in some of the arguments below. As in [DS1], this problem is easily solved by defining the following function associated to $\mathcal{T}$. Given $x \in \operatorname{supp}(\mu)$, one sets

$$d_{\mathcal{T}}(x) = \inf_{Q \in \mathcal{T}} \big[\ell(Q) + \operatorname{dist}(x, Q)\big].$$

Observe that $d_{\mathcal{T}}$ is Lipschitz with constant 1. We denote

$$G(\mathcal{T}) = \big\{x \in \lambda R_{\mathcal{T}} : d_{\mathcal{T}}(x) = 0\big\}.$$

It is easy to check that $G(\mathcal{T}) \subset \Gamma_{\mathcal{T}} \cap \overline{R_{\mathcal{T}}}$. For each $x \in \operatorname{supp}(\mu)$ such that $d_{\mathcal{T}}(x) > 0$, let Q_x be a dyadic μ-cube containing x such that

$$\frac{d_{\mathcal{T}}(x)}{20A} < \ell(Q_x) \leq \frac{d_{\mathcal{T}}(x)}{10A}, \tag{7.1}$$

where $A \gg 1$ is some constant to be fixed below. Then, $\operatorname{Reg}(\mathcal{T})$ is a maximal (and thus disjoint) subfamily of $\{Q_x\}_{x \in AR_{\mathcal{T}}}$ (recall that $R_{\mathcal{T}}$ is the root of $\mathcal{T}$). If $d_{\mathcal{T}}(x) = 0$, then Q_x can be identified with the point x.

Lemma 7.2. *Let μ be uniformly rectifiable and $\mathcal{T}$ a tree of its corona decomposition and denote by $R_{\mathcal{T}}$ its root. If we choose $1 \ll A^2 \ll \lambda$ big enough, then the family of μ-cubes $\operatorname{Reg}(\mathcal{T})$ satisfies:*

(a) *$AR_{\mathcal{T}} \subset G(\mathcal{T}) \cup \bigcup_{Q \in \operatorname{Reg}(\mathcal{T})} Q$.*
(b) *If $Q \in \mathcal{T}$, $P \in \operatorname{Reg}(\mathcal{T})$, and $P \cap Q \neq \varnothing$, then $\ell(P) < \ell(Q)/2$, and so $P \subset Q$.*
(c) *If $P, Q \in \operatorname{Reg}(\mathcal{T})$ and $AP \cap AQ \neq \varnothing$, then $\ell(Q)/2 \leq \ell(P) \leq 2\ell(Q)$.*
(d) *If $Q \in \mathcal{D}^\mu$ is contained in $AR_{\mathcal{T}}$ and Q contains some μ-cube from $\operatorname{Reg}(\mathcal{T})$, then $b\beta_\infty(Q) \leq c(A, \lambda)\,\varepsilon$ and $\operatorname{dist}(x, \Gamma_{\mathcal{T}}) \leq c(A, \lambda)\varepsilon \ell(Q)$ for all $x \in Q$.*

Because of (b) and (c), in a sense, the family $\operatorname{Reg}(\mathcal{T})$, can be considered as a regularized version of the μ-cubes from $\mathcal{S}(\mathcal{T})$. This is why we use the notation $\operatorname{Reg}(\Gamma_{\mathcal{T}})$.

Proof. (a) This is a straightforward consequence of the definition of $\operatorname{Reg}(\mathcal{T})$.

(b) By the construction of $\operatorname{Reg}(\mathcal{T})$, there exists some $x \in P$ such that $\ell(P) \leq d_{\mathcal{T}}(x)/10A$. Since $Q \cap P \neq \varnothing$, we have

$$d_{\mathcal{T}}(x) \leq \ell(Q) + \operatorname{dist}(x, Q) \leq \ell(Q) + \ell(P).$$

Thus,

$$\ell(P) \leq \frac{d_{\mathcal{T}}(x)}{10A} \leq \frac{\ell(Q) + \ell(P)}{10A},$$

and then it follows that $\ell(P) \leq \ell(Q)/(10A-1) < \ell(Q)/2$.

(c) Consider $P, Q \in \mathrm{Reg}(\mathcal{T})$ such that $AP \cap AQ \neq \varnothing$. By construction, there exists some $x \in P$ such that $\ell(P) \geq d_{\mathcal{T}}(x)/20A$ and some μ-cube $P_0 \in \mathcal{T}$ such that

$$\mathrm{dist}(x, P_0) + \ell(P_0) \leq 1.1 d_{\mathcal{T}}(x) \leq 22A\ell(P).$$

Thus, for any $y \in Q$,

$$\begin{aligned} \mathrm{dist}(y, P_0) + \ell(P_0) &\leq \mathrm{diam}(AQ) + \mathrm{diam}(AP) + \mathrm{dist}(x, P_0) + \ell(P_0) \\ &\leq (2A-1)\,\ell(Q) + (2A-1)\,\ell(P) + 22A\ell(P). \end{aligned}$$

So $d_{\mathcal{T}}(y) \leq (2A-1)\,\ell(Q) + (24A-1)\,\ell(P)$ for all $y \in Q$. Therefore,

$$\ell(Q) \leq \frac{1}{10A}\Big(2A\,\ell(Q) + 24A\,\ell(P)\Big) = 0.2\,\ell(Q) + 2.4\,\ell(P),$$

which yields $\ell(Q) \leq 3\,\ell(P)$. This implies that $\ell(Q) \leq 2\ell(P)$ as $\ell(P)$ and $\ell(Q)$ are dyadic numbers.

The inequality $\ell(P) \leq 2\ell(Q)$ is proved in an analogous way.

(d) Take now $Q \in \mathcal{D}^\mu$ with $Q \subset AR_{\mathcal{T}}$ which contains some μ-cube from $\mathrm{Reg}(\mathcal{T})$. By (7.1), this implies that there exists some $x \in Q$ such that $d_{\mathcal{T}}(x) \leq 20A\,\ell(Q)$, and thus there exists some $P_0 \in \mathcal{T}$ such that

$$\mathrm{dist}(x, P_0) + \ell(P_0) \leq 1.1\,d_{\mathcal{T}}(x) \leq 22A\,\ell(Q). \tag{7.2}$$

In particular, $\ell(P_0) \leq 22A\,\ell(Q)$. Suppose for simplicity that A is chosen so that $22A$ is a dyadic number. Consider now $Q_0 \in \mathcal{D}^\mu$ which contains P_0 with $\ell(Q_0) = 22A\,\ell(Q)$. Observe that, by (7.2),

$$\mathrm{dist}(Q, Q_0) \leq \mathrm{dist}(Q, P_0) \leq 22A\,\ell(Q) = \ell(Q_0),$$

and thus $Q \subset 3Q_0$.

So we have shown that there exists some μ-cube Q_0 with $3Q_0 \supset Q$ and $Q_0 \supset P_0 \in \mathcal{S}(\mathcal{T})$, and moreover $\ell(Q_0) = 22A\,\ell(Q)$, with $Q \subset AR$. If the parameter λ is taken big enough in the corona decomposition ($\lambda \gg A^2$, say), then it is easy to check that $3Q_0 \subset \lambda R$, for some $R \in \mathcal{T}$, with $\ell(R) \approx \ell(Q_0)$ (with some constant depending on A and λ). As a consequence, $b\beta_\infty(Q) \lesssim b\beta_\infty(Q_0) \lesssim b\beta_\infty(R) \lesssim \varepsilon$, with all the constants here depending on A and λ. Analogously, we infer that $\mathrm{dist}(x, \Gamma_{\mathcal{T}}) \leq c(A,\lambda)\varepsilon\ell(Q)$ for all $x \in Q$. □

Observe that, if $Q \in \mathcal{T}$, then $Q = \big(G(\mathcal{T}) \cap Q\big) \cup \bigcup_{P \in \mathrm{Reg}(T): P \subset Q} P$. Further, the union is disjoint since, from the property (b) above, it turns out that

$$G(\mathcal{T}) \cap R_{\mathcal{T}} \cap \bigcup_{P \in \mathrm{Reg}(\mathcal{T})} P = \varnothing. \tag{7.3}$$

7.2. **The proof of Theorem 1.2.** If $\alpha_p(x,t)^2\,d\mu(x)\,\frac{dr}{r}$ is a Carleson measure for some $p\in[1,2]$, then $\alpha(x,t)^2\,d\mu(x)\,\frac{dr}{r}$ is also a Carleson measure, by Lemma 5.1. By the results in [To2], this implies that μ is uniformly rectifiable. Therefore, to prove Theorem 1.2, it is enough to show that $\alpha_2(x,t)^2\,d\mu(x)\,\frac{dr}{r}$ is a Carleson measure, taking into account that $\alpha_p(B)\lesssim\alpha_2(B)$ for $p\leq 2$. By standard arguments, using Lemma 5.4, this is equivalent to showing that

$$\sum_{Q\in\mathcal{D}^\mu:Q\subset R}\alpha_2(Q)^2\mu(Q)\lesssim\mu(R)\qquad\text{for all } R\in\mathcal{D}^\mu,. \tag{7.4}$$

Our main tool to prove this inequality will be the corona decomposition of μ described above.

We will show that (7.4) holds for uniformly rectifiable sets following arguments analogous to the ones of [DS1, Chapter 15], where it is shown that the existence of a corona decomposition implies that $\beta_2(x,t)^2\,d\mu(x)\,\frac{dr}{r}$ is a Carleson measure. To this end, arguing as in [DS1, Chapter 15], it turns out that it is enough to show that for every tree $\mathcal{T}$ from the corona decomposition of μ,

$$\sum_{Q\in\mathcal{T}:Q\subset R}\alpha_2(Q)^2\mu(Q)\lesssim\mu(R)\qquad\text{for all } R\in\mathcal{T}. \tag{7.5}$$

The rest of this subsection is devoted to prove this inequality.

Lemma 7.3. *Let μ be an AD-regular measure which is uniformly rectifiable, and $\mathcal{T}$ be a tree from its corona decomposition, as described above. For each $Q\in\mathrm{Reg}(\mathcal{T})$ there exists a function g^Q supported on $2B_Q\cap\Gamma_{\mathcal{T}}$ such that*

$$\int_{\Gamma_{\mathcal{T}}} g^Q d\mathcal{H}^n=\mu(Q)\qquad\text{and}\qquad\sum_{Q\in\mathrm{Reg}(\mathcal{T})}g^Q\lesssim 1. \tag{7.6}$$

Proof. Recall that, for $Q\in\mathrm{Reg}(\mathcal{T})$ and $x\in Q$, $\mathrm{dist}(x,\Gamma_{\mathcal{T}})\leq c(A,\lambda)\varepsilon\ell(Q)$. Thus, if ε is chosen small enough in Definition 7.1 (for given A and λ), $\mathrm{dist}(x,\Gamma_{\mathcal{T}})\leq r(B_Q)/10$ for all $x\in Q$. Therefore, $\frac{3}{2}B_Q\cap\Gamma_{\mathcal{T}}\neq\varnothing$ and so $\mathcal{H}^n(2B_Q\cap\Gamma_{\mathcal{T}})\approx\ell(Q)^n$. We define

$$g^Q=\frac{\mu(Q)}{\mathcal{H}^n(2B_Q\cap\Gamma_{\mathcal{T}})}\,\chi_{2B_Q\cap\Gamma_{\mathcal{T}}}.$$

Since $\mu(Q)\approx\mathcal{H}^n(2B_Q\cap\Gamma_{\mathcal{T}})$, we deduce that $\|g^Q\|_{L^\infty(\Gamma_{\mathcal{T}})}\lesssim 1$. From the property (b) in Lemma 7.2, it easily follows that $\sum_{Q\in\mathrm{Reg}(\mathcal{T})}\chi_{2B_Q}\lesssim 1$ if A is big enough, and thus (7.6) follows. □

For $Q\in\mathrm{Reg}(\mathcal{T})$, let g^Q be the function associated to Q, given by the preceding lemma. Recall that in $AR_{\mathcal{T}}\setminus\bigcup_{Q\in\mathrm{Reg}(\mathcal{T})}Q\subset G(\mathcal{T})$, μ is absolutely continuous with respect to $\mathcal{H}^n_\Gamma$. In fact,

$$\mu\lfloor\Big(AR_{\mathcal{T}}\setminus\bigcup_{Q\in\mathrm{Reg}(\mathcal{T})}Q\Big)=g_0\,\mathcal{H}^n_\Gamma,$$

with $\|g_0\|_{L^\infty(\mathcal{H}^n_\Gamma)} \lesssim 1$. Consider the measure

$$\sigma = g_0 + \sum_{Q\in \mathrm{Reg}(\mathcal{T})} g^Q.$$

To prove (7.9), we will use that, roughly speaking, σ approximates μ on $\mathcal{T}$, and we will apply the results obtained in the previous section for Lipschitz graphs to the measure σ.

Lemma 7.4. *Let $\mathcal{T}$ and σ be as above. Then, every $Q \in \mathcal{T}$ satisfies $\sigma(2B_Q) \approx \ell(Q)^n$. Also, if L is an n-plane that minimizes $\alpha_{2,\sigma}(10B_Q)$,*

$$W_2(\varphi_Q\mu,\, c\,\varphi_Q\mathcal{H}^n_L)^2 \lesssim \alpha_{2,\sigma}(10B_Q)^2\,\ell(Q)^n + \sum_{P\in\mathrm{Reg}(\mathcal{T}):P\subset 50Q} \mu(P)\,\ell(P)^2.$$

Proof. The estimate $\sigma(2B_Q) \lesssim \ell(Q)^n$ for $Q \in \mathcal{T}$ follows easily from (7.6) in the preceding lemma and the definition of σ. Let us prove the converse inequality. Notice first that, for $Q \in \mathrm{Reg}(\mathcal{T})$, we have $\sigma(2B_Q) \approx \ell(Q)^n$, also by the definition of σ and the preceding lemma. Consider now an arbitrary μ-cube $Q \in \mathcal{T}$. If $\mu\big(G(\mathcal{T}) \cap Q\big) \geq \frac{1}{2}\,\mu(Q)$, then we also have $\sigma(2B_Q) \approx \ell(Q)^n$, because $\sigma\lfloor Q \geq g_0\,\mathcal{H}^n_{\Gamma\cap Q} = \mu\lfloor\Gamma_{\mathcal{T}} \cap Q$, by (7.3) and the definition of σ. On the other hand, if $\mu\big(G(\mathcal{T}) \cap Q\big) < \frac{1}{2}\,\mu(Q)$, then

$$\sigma\Big(\bigcup_{P\in\mathrm{Reg}(\mathcal{T}):P\subset Q} 2B_P\Big) \geq \mu\Big(\bigcup_{P\in\mathrm{Reg}(\mathcal{T}):P\subset Q} P\Big) \geq \frac{1}{2}\,\mu(Q). \tag{7.7}$$

We may assume that, for all $P \in \mathrm{Reg}(\mathcal{T})$ contained in Q, $\ell(P) \leq c_{11}\ell(Q)$ with $0 < c_{11} < 1$ small enough. Otherwise, if for some $P \in \mathrm{Reg}(\mathcal{T})$ contained in Q this fails, we get $\sigma(Q) \geq \sigma(P) \approx \ell(P)^n \approx \ell(Q)^n$. The assumption that $\ell(P) \leq c_{11}\ell(Q)$, with c_{11} small enough for all $P \in \mathrm{Reg}(\mathcal{T})$ contained in Q, ensures that for all these P's, $2B_P \subset 2B_Q$. Then, from (7.7) we deduce that $\sigma(2B_Q) \gtrsim \ell(Q)^n$.

Let us turn our attention to the second statement in the lemma. Denote by I_Q the family of those μ-cubes $P \in \mathrm{Reg}(\mathcal{T})$ such that $2B_P \cap 30B_Q \neq \varnothing$. It is easy to check that such μ-cubes P are contained in $50Q$. Let

$$\sigma_Q = g_0\chi_{30B_Q}\,\mathcal{H}^n_\Gamma + \sum_{P\in I_Q} g^P\,\mathcal{H}^n_\Gamma.$$

Since $\sigma\lfloor 30B_Q = \sigma_Q\lfloor 30B_Q$, we have

$$W_2(\varphi_{10B_Q}\sigma_Q,\, c'\varphi_{10B_Q}\mathcal{H}^n_L)^2 = W_2(\varphi_{10B_Q}\sigma,\, c'\varphi_{10B_Q}\mathcal{H}^n_L)^2 \lesssim \alpha_{2,\sigma}(10B_Q)^2\ell(Q)^{n+2}, \tag{7.8}$$

for the appropriate constant c'. Consider now the measure

$$\mu_{\varphi,Q} = g_0\varphi_{10B_Q}\,\mathcal{H}^n_\Gamma + \sum_{P\in I_Q} a_P\,\mu\lfloor P,$$

where $a_P = \int \varphi_{10B_Q}\,g^P\,d\mathcal{H}^n_\Gamma/\mu(P)$. Notice that

$$\varphi_{10B_Q}\,\sigma_Q = \sum_{P\in I_Q} \varphi_{10B_Q}\,\big(g^P + g_0\chi_P\big)\,\mathcal{H}^n_\Gamma =: \sum_{P\in I_Q} \widetilde{\sigma}_P,$$

with $\|\widetilde{\sigma}_P\| = \mu_{\varphi,Q}(P)$ for each $P \in I_Q$, by construction. By taking the following transference plan between $\mu_{\varphi,Q}$ and $\varphi_{10B_Q}\sigma$:

$$\pi = \sum_{P\in I_Q} \frac{1}{\mu_{\varphi,Q}(P)}\,(\mu_{\varphi,Q}\lfloor P \otimes \widetilde{\sigma}_P),$$

one easily deduces that $W_2(\mu_{\varphi,Q}, \varphi_{10B_Q}\sigma)^2 \lesssim \sum_{P\in I_Q} \mu(P)\,\ell(P)^2$. Therefore, by (7.8) and the triangle inequality,

$$W_2(\mu_{\varphi,Q},\, c'\,\varphi_{10B_Q}\mathcal{H}^n_L)^2 \lesssim \alpha_{2,\sigma}(10B_Q)^2\ell(Q)^{n+2} + \sum_{P\in I_Q} \mu(P)\,\ell(P)^2.$$

Notice also that $\mu_{\varphi,Q}\lfloor 3B_Q = \mu\lfloor 3B_Q$. Thus, by Lemma 5.3, we obtain

$$W_2(\varphi_{B_Q}\mu,\, c\varphi_{B_Q}\mathcal{H}^n_L)^2 \lesssim W_2(\mu_{\varphi,Q},\, c'\,\varphi_{10B_Q}\mathcal{H}^n_L)^2,$$

and then the lemma follows. □

Now we are ready to prove (7.5) and conclude the proof of Theorem 1.2.

Lemma 7.5. *Let μ be an AD-regular measure which is uniformly rectifiable, and $\mathcal{T}$ be a tree of μ-cubes from its corona decomposition. Then,*

$$\sum_{Q\in\mathcal{T}:Q\subset R} \alpha_2(Q)^2\mu(Q) \lesssim \mu(R) \qquad \text{for every } R \in \mathcal{T}, \tag{7.9}$$

Proof. Consider the measure σ defined above and take $Q \in \mathcal{T}$. By the preceding lemma, if L is an n-plane that minimizes $\alpha_{2,\sigma}(Q)$, we have

$$W_2(\varphi_Q\mu,\, c_{Q,L}\varphi_Q\mathcal{H}^n_L)^2 \lesssim \alpha_{2,\sigma}(10B_Q)^2\ell(Q)^{n+2} + \sum_{P\in\mathrm{Reg}(\mathcal{T}):P\subset 50Q} \mu(P)\,\ell(P)^2.$$

Therefore,

$$\sum_{Q\in\mathcal{T}:Q\subset R} \alpha_2(Q)^2\mu(Q) \lesssim \sum_{Q\in\mathcal{T}:Q\subset R}\left[\alpha_{2,\sigma}(10B_Q)^2\ell(Q)^n + \frac{1}{\ell(Q)^2}\sum_{P\in\mathrm{Reg}(\mathcal{T}):P\subset 50Q}\mu(P)\,\ell(P)^2\right].$$

We claim now that

$$\sum_{Q\in\mathcal{T}:Q\subset R} \alpha_{2,\sigma}(10B_Q)^2\ell(Q)^n \lesssim \ell(R)^n.$$

This follows easily from Lemma 6.3, taking into account that $b\beta(\lambda Q) \le \varepsilon$ for every $Q \in \mathcal{T}$. Indeed, if ε is small enough it is immediate to check that the latter condition implies that the assumption (6.2) is satisfied (taking also A and λ big enough). On the other hand,

$$\begin{aligned}\sum_{Q\in\mathcal{T}:Q\subset R}\frac{1}{\ell(Q)^2}\sum_{P\in\mathrm{Reg}(\mathcal{T}):P\subset 50Q}\mu(P)\,\ell(P)^2 &\le \sum_{P\in\mathrm{Reg}(\mathcal{T}):P\subset 50R}\mu(P)\sum_{\substack{Q\in\mathcal{T}:Q\subset R\\ 50Q\supset P}}\frac{\ell(P)^2}{\ell(Q)^2}\\ &\lesssim \sum_{P\in\mathrm{Reg}(\mathcal{T}):P\subset 50R}\mu(P) \lesssim \mu(R),\end{aligned}$$

and thus the lemma follows. □

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