# Unitary embeddings of finite loop spaces

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Abstract

In this paper we construct faithful representations of saturated fusion systems over discrete $p$-toral groups and use them to find conditions that guarantee the existence of unitary embeddings of $p$-local compact groups. These conditions hold for the Clark-Ewing and Aguadé-Zabrodsky $p$-compact groups as well as some exotic 3-local compact groups. We also show the existence of unitary embeddings of finite loop spaces.

## Introduction

In the theory of compact Lie groups, the existence of a faithful unitary representation for every compact Lie group is a consequence of the Peter-Weyl theorem. This paper is concerned with the analogue of these representations for several objects in the literature which are considered to be homotopical counterparts of compact Lie groups.

W.G. Dwyer and C.W. Wilkerson introduced $p$-compact groups in [DW94]. They are loop spaces which satisfy some finiteness properties at a particular prime $p$. For example, if $G$ is a compact Lie group such that its group of connected components is a finite $p$-group, then its $p$-completion $G^\wedge_p$ in the sense of [BK72] is a $p$-compact group. But there are examples of $p$-compact groups which are not the $p$-completion of any compact Lie group, which are called exotic. Connected $p$-compact groups were classified in [AG09] and [AGMV08], where a bijective correspondence between connected $p$-compact groups and reflection data over the p-adic integers was established.

Many ideas from the theory of compact Lie groups have a homotopical analogue for $p$-compact groups. Faithful unitary representations correspond to homotopy monomorphisms at the prime $p$ from the classifying space of the $p$-compact group into the $p$-completion of the classifying space of a unitary group. By homotopy monomorphism at $p$ we mean a map whose homotopy fiber $F$ satisfies that the pointed mapping space $\mathrm{Map}_*(B\mathbb{Z}/p, F)$ is contractible for all choices of basepoint in $F$. For simplicity, we will call such maps unitary embeddings. The existence of such maps follows from the Peter-Weyl theorem, the classification of connected $p$-compact groups and the works [C00], [C06] for $p > 2$ and [Z05], [Z09] for $p = 2$.

In this article we deal with the same question for the combinatorial structures called $p$-local compact groups, which encode the $p$-local information of some spaces at a prime $p$. They were introduced in [BLO07] to model $p$-completed classifying spaces of compact Lie groups, $p$-compact groups, as well as linear torsion groups, and they have been shown recently to model $p$-completions of classifying spaces of finite loop spaces [BLO13] and other exotic examples [G10].

These structures generalize $p$-local finite groups [BLO03]. In fact, they are given by a fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$ and an associated centric linking system $\mathcal{L}$. More details

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can be found in Section 1 of this article. In Section 2, we construct complex representations of $S$ which are faithful and $\mathcal{F}$-invariant. A representation $\rho$ is fusion-preserving or $\mathcal{F}$-invariant if for any $P \leqslant S$ and any morphism $f$ in $\mathrm{Hom}_{\mathcal{F}}(P,S)$, the representations $\rho_{|P}$ and $\rho_{|f(P)} \circ f$ are isomorphic. The following is Theorem 2.9, the main theorem in this section.

Theorem. *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$. There exists a faithful unitary representation of $S$ which is $\mathcal{F}$-invariant.*

The importance of such representations comes from the maps $\Psi_n$ introduced in Section 3 which take homotopy classes maps $|\mathcal{L}|^\wedge_p \to BU(n)^\wedge_p$ to its restriction to $BS$. This restriction map gives $n$-dimensional complex representations of $S$ which must be $\mathcal{F}$-invariant. We show that if a faithful fusion-preserving representation of $S$ is the image of a map $f$ under $\Psi_n$, then $f$ is a unitary embedding. We also say that $f$ is a unitary embedding of the $p$-local compact group $(S,\mathcal{F},\mathcal{L})$.

The problem is then reduced to studying the obstructions for a faithful representation $\rho$ to be in the image of $\Psi_n$. These obstructions lie in the cohomology of the orbit category of centric radical subgroups with certain functors as coefficients. For our purpose it is enough to know whether $M\rho$ is in the image of $\Psi_{Mn}$ for some $M > 0$. By stabilizing, we find that the obstructions in even dimensions vanish and in odd dimensions we can replace our original functors by the functor which takes a group $P$ to $R(P,\rho) \otimes_{\mathbb{Z}} Z^\wedge_p$, where $R(P,\rho)$ is the Grothendieck ring of subrepresentations of $\rho_{|P}$.

The orbit category of centric radical subgroups has finite length and the obstructions vanish above the length. Thus when the length of this category is smaller than three there are no obstructions, from where we obtain the following theorem, which corresponds to Corollary 3.17 in the text.

Theorem. *Let $(S,\mathcal{F},\mathcal{L})$ be a $p$-local compact group such that $l(\mathcal{O}(\mathcal{F}^{cr})) < 3$. Then there exists a unitary embedding of $(S,\mathcal{F},\mathcal{L})$.*

This is the case for the Clark-Ewing and the Aguadé-Zabrodsky $p$-compact groups, as we show in Section 4. The existence of unitary embeddings for these $p$-compact groups was already shown in [C00] in a different way, but these results were never published. We would also like to add the comment here that the existence of unitary embeddings for $p$-local finite groups was already shown in [CM08].

In Section 5 we show that if $f : X \to Y$ is a finite regular covering of a $p$-good space $Y$ and $X$ has a unitary embedding, then so does $Y^\wedge_p$. This applies to a finite loop space $(X, BX, e)$, since the $p$-completion of the universal cover of $BX$ is the classifying space of a $p$-compact group. Therefore the existence of unitary embeddings of $p$-compact groups implies the existence for $BX^\wedge_p$. In this section we also show the existence of unitary embeddings for the exotic 3-local compact groups constructed in [G10] using the results of Section 3.

The following corollary is a consequence of the classification of $p$-compact groups, the Peter-Weyl theorem, the existence of unitary embeddings of generalized Grassmannians [C06] and of $DI(4)$ [Z05], [Z09], and the results mentioned above.

Corollary. *Let $(S,\mathcal{F},\mathcal{L})$ be a $p$-local compact group which models a finite loop space or a $p$-compact group. Then there exists a unitary embedding of $(S,\mathcal{F},\mathcal{L})$.*

Section 4 also contains a result of independent interest concerning the relationship between the fusion systems of mapping spaces and centralizer fusion systems. In general, if $Q$ is a fully centralized subgroup of $S$, it is not known whether the centralizer $p$-local compact group $(S, C_{\mathcal{F}}(Q), C_{\mathcal{L}}(Q))$ has the homotopy type of $\mathrm{Map}(BQ, |\mathcal{L}|_p^\wedge)_{Bi}$. We have the following partial result, which corresponds to Proposition 4.2.

Proposition. *Let $X$ be a $p$-compact group, $S$ a maximal discrete $p$-toral subgroup and $\mathcal{F}$ the associated fusion system over $S$. Let $E$ be a fully centralized subgroup of $Z(S)$. Then the fusion system $C_{\mathcal{F}}(E)$ coincides with the fusion system of the $p$-compact group $C_X(E)$ over $C_S(E)$.*

Finally, in Section 6 we consider some of the consequences of the existence of a unitary embedding of a $p$-local compact group. We obtain finiteness results for the $p$-local cohomology of $|\mathcal{L}|_p^\wedge$ and a stable elements formula for the Grothendieck ring $\mathbb{K}(|\mathcal{L}|_p^\wedge)$ of vector bundles over $|\mathcal{L}|_p^\wedge$ when the length of the orbit category of centric radical subgroups is small. The stable elements formula comes from the map

$$\Psi : \mathbb{K}(|\mathcal{L}|_p^\wedge) \to \varprojlim_{\mathcal{O}(\mathcal{F}^c)} R(P)$$

induced by the maps $\Psi_n$ introduced in Section 3. We show that if $l(\mathcal{O}(\mathcal{F}^{cr})) < 3$, then $\Psi$ is surjective and if $l(\mathcal{O}(\mathcal{F}^{cr})) < 2$, then $\Psi$ is an isomorphism. In particular, this map is an isomorphism for the Clark-Ewing, the Aguadé-Zabrodsky $p$-compact groups and the exotic 3-local compact groups of [G10].

The organization of the paper is as follows. Section 1 contains an introduction to fusion systems over discrete $p$-toral groups and $p$-local compact groups. The existence of faithful representations that are fusion-invariant is shown in Section 2. Unitary embeddings of $p$-local compact groups are studied in Section 3. In Section 4 we apply the previous results to the Clark-Ewing and the Aguadé-Zabrodsky $p$-compact groups. Section 5 is concerned with unitary embeddings of finite loop spaces and exotic 3-local compact groups. Finally, in Section 6 we study some of the consequences of the existence of unitary embeddings in $p$-local cohomology and vector bundles.

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## 1. Background on $p$-local compact groups

In this section we recall the definition of a $p$-local compact group in the form given in [BLO07]. Let $\mathbb{Z}/p^\infty \cong \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$ denote the union of the cyclic $p$-groups $\mathbb{Z}/p^n$ under the standard inclusions.

Definition 1.1. A discrete $p$-toral group is a group $P$, with normal subgroup $P_0 \triangleleft P$, such that $P_0$ is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty$, and $P/P_0$ is a finite $p$-group. The subgroup $P_0$ will be called the identity component of $P$, and $P$ will be called connected if $P = P_0$. Set $\pi_0(P) = P/P_0$, the group of components of $P$.

The identity component $P_0$ of a discrete $p$-toral group can be characterized as the characteristic subgroup of all infinitely $p$-divisible elements in $P$, and also as the minimal subgroup of finite index in $P$.

Define $\mathrm{rk}(P) = k$ if $P_0 \cong (\mathbb{Z}/p^\infty)^k$, and set $|P| = (\mathrm{rk}(P), |\pi_0(P)|)$. We regard the order of a discrete $p$-toral group as an element of $\mathbb{N}^2$ with the lexicographical ordering. That is, $|P| \leqslant |P'|$ if and only if $\mathrm{rk}(P) < \mathrm{rk}(P')$, or $\mathrm{rk}(P) = \mathrm{rk}(P')$ and $\pi_0(P) \leqslant \pi_0(P')$. In particular, $P' \leqslant P$ implies $|P| \leqslant |P'|$, with equality only if $P' = P$.

Given two discrete $p$-toral groups $P, Q$, let $\mathrm{Hom}(P,Q)$ denote the set of group homomorphisms from $P$ to $Q$, and let $\mathrm{Inj}(P,Q)$ denote the set of monomorphisms. If $P$ and $Q$ are subgroups of a larger group $S$, then $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Inj}(P,Q)$ denotes the subset of homomorphisms induced by conjugation by elements of $S$, and $\mathrm{Aut}_S(P)$ the group of automorphisms induced by conjugation in $S$

Definition 1.2. A fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$ is a subcategory of the category of groups whose objects are the subgroups subcategory of the category of groups whose objects are the subgroups of $S$, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P, Q \leqslant S$.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

If $\mathcal{F}$ is a fusion system over $S$ and $P, Q \leqslant S$, then we write $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ instead of $\mathrm{Mor}_{\mathcal{F}}(P,Q)$ to emphasize that morphisms in the category $\mathcal{F}$ are all homomorphisms, and $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ for the subset of isomorphisms in $\mathcal{F}$. Note that $\mathrm{Iso}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_{\mathcal{F}}(P,Q)$ if $|P| = |Q|$, and $\mathrm{Iso}_{\mathcal{F}}(P,Q) = \emptyset$ otherwise. We also use the notation $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$ and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. Two subgroups $P, P' \leqslant S$ are called $\mathcal{F}$-conjugate if $\mathrm{Iso}_{\mathcal{F}}(P,P') \neq \emptyset$.

The fusion systems we consider in this article will all satisfy the following additional condition. Here, and throughout the paper, we write $\mathrm{Syl}_p(G)$ for the set of Sylow $p$-subgroups of $G$. Also, for any $P \leqslant G$ and any $g \in N_G(P)$, $c_g \in \mathrm{Aut}(P)$ denotes the automorphism $c_g(x) = gxg^{-1}$.

Definition 1.3. Let $\mathcal{F}$ be a fusion system over a discrete $p$-toral group $S$.

- A subgroup $P \leqslant S$ is fully centralized in $\mathcal{F}$ if $|C_S(P)| \geqslant |C_S(P')|$ for all $P' \leqslant S$ that are $\mathcal{F}$-conjugate to $P$.
- A subgroup $P \leqslant S$ is fully normalized in $\mathcal{F}$ if $|N_S(P)| \geqslant |N_S(P')|$ for all $P' \leqslant S$ that are $\mathcal{F}$-conjugate to $P$.
- $\mathcal{F}$ is a saturated fusion system if the following three conditions hold:
  (I) For each $P \leqslant S$ which is fully normalized in $\mathcal{F}$, $P$ is fully centralized in $\mathcal{F}$, $\mathrm{Out}_{\mathcal{F}}(P)$ is finite and $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$.
  (II) If $P \leqslant S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that $\phi P$ is fully centralized, and if we set
  $$N_\phi = \{g \in N_S(P) \mid \phi c_g \phi^{-1} \in \mathrm{Aut}_S(\phi P)\}$$
  then there is $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ such that $\bar{\phi}|_P = \phi$.
  (III) If $P_1 \leqslant P_2 \leqslant P_3 \leqslant \dots$ is an increasing sequence of subgroups of $S$, with $P_\infty = \bigcup_{n=1}^{\infty} P_n$, and if $\phi \in \mathrm{Hom}(P_\infty, S)$ is any homomorphism such that $\phi_{|P_n} \in \mathrm{Hom}_{\mathcal{F}}(P_n, S)$ for all $n$, then $\phi \in \mathrm{Hom}_{\mathcal{F}}(P_\infty, S)$.

The motivating example for this definition is the fusion system of a compact Lie group $G$. For any maximal discrete $p$-toral subgroup $S$ of $G$, we let $\mathcal{F}_S(G)$ be the fusion system over $S$ defined by setting $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for all $P, Q \leqslant S$. One can also define fusion systems associated to $p$-compact groups [BLO07].

Lemma 1.4 (Lemma 2.4 in [BLO07]). *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$ with connected component $T = S_0$. Then the following hold for all $P \leqslant T$.*

- *For every $P' \leqslant S$ which is $\mathcal{F}$-conjugate to $P$ and fully centralized in $\mathcal{F}$, $P' \leqslant T$ and there exists some $w \in \mathrm{Aut}_{\mathcal{F}}(T)$ such that $w_{|P} \in \mathrm{Iso}_{\mathcal{F}}(P, P')$.*
- *Every $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, T)$ is the restriction of some $w \in \mathrm{Aut}_{\mathcal{F}}(T)$.*

Definition 1.5. Let $\mathcal{F}$ be any fusion system over a discrete $p$-toral group $S$. A subgroup $P \leqslant S$ is called $\mathcal{F}$-centric if $P$ and all of its $\mathcal{F}$-conjugates contain their $S$-centralizers. A subgroup $Q \leqslant S$ is called $\mathcal{F}$-radical if $\mathrm{Out}_{\mathcal{F}}(Q)$ contains no nontrivial normal $p$-subgroup.

We will denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$ and by $\mathcal{F}^{cr}$ the full subcategory whose objects are the $\mathcal{F}$-centric radical subgroups of $S$.

Theorem 1.6 (Alperin's fusion theorem). *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$. Then for each $\phi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$, there exist sequences of subgroups of $S$*

$$P = P_0, P_1, \ldots, P_k = P' \text{ and } Q_1, Q_2, \ldots Q_k$$

*and elements $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that*

*(a)* *$Q_i$ is fully normalized in $\mathcal{F}$, $\mathcal{F}$-radical, and $\mathcal{F}$-centric for each $i$ ;*

*(b)* *$P_{i-1}$, $P_i \leqslant Q_i$ and $\phi_i(P_{i-1}) = P_i$ for each $i$ ; and*

*(c)* *$\phi = \phi_k \circ \phi_{k-1} \circ \ldots \circ \phi_1$.*

*Proof.* See [BLO07], Theorem 3.6. □

Definition 1.7. Let $\mathcal{F}$ be a fusion system over the $p$-group $S$. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$, together with a functor

$$\pi : \mathcal{L} \to \mathcal{F}^c$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leqslant S$, which satisfy the following conditions:

(A) $\pi$ is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leqslant \mathrm{Aut}_{\mathcal{L}}(P)$), and $\pi$ induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q)$$

(B) For each $\mathcal{F}$-centric subgroup $P \leqslant S$ and each $g \in P$, $\pi$ sends the element $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(f)(g))"] \\
P \arrow[r, "f"] & Q
\end{tikzcd}$$

Definition 1.8. A $p$-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over the discrete $p$-toral group $S$ and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space of the $p$-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|^{\wedge}_p$.

The orbit category $\mathcal{O}(\mathcal{F})$ is the category whose objects are the subgroups of $S$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_{\mathcal{F}}(P,Q)/\,\mathrm{Inn}(Q).$$

For any full subcategory $\mathcal{F}_0$ of $\mathcal{F}$ we also consider $\mathcal{O}(\mathcal{F}_0)$, the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are those of $\mathcal{F}_0$.

Proposition 1.9 (Proposition 4.6 in [BLO07]). *Fix a saturated fusion system $\mathcal{F}$ over a discrete $p$-toral group $S$, and let $\mathcal{F}_0 \subseteq \mathcal{F}^c$ be any full subcategory. For any linking system $\mathcal{L}_0$ associated to $\mathcal{F}_0$, the left homotopy Kan extension $\widetilde{B} : \mathcal{O}(\mathcal{F}_0) \to \mathrm{Top}$ of the constant functor $\mathcal{L}_0 \xrightarrow{*} \mathrm{Top}$ along the projection $\tilde{\pi}_0 : \mathcal{L}_0 \to \mathcal{O}(\mathcal{F}_0)$ is a rigidification of $B$, and there is a homotopy equivalence:*

$$|\mathcal{L}_0| \simeq \operatorname*{hocolim}_{\overrightarrow{\mathcal{O}(\mathcal{F}_0)}} \widetilde{B}$$

It is possible to construct $p$-local compact groups associated to compact Lie groups, $p$-compact groups and some locally finite discrete groups in such a way that their classifying spaces are homotopy equivalent to the $p$-completion of the classifying spaces of the given groups [BLO07].

## 2. Fusion-preserving representations

The main result in this section is Theorem 2.9, which shows the existence of a faithful unitary representation of any saturated fusion system $\mathcal{F}$. All representations in this section are complex representations.

Definition 2.1. Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$. A representation $\rho : S \to GL(V)$ is fusion-preserving or $\mathcal{F}$-invariant if for any $P \leqslant S$ and any $f \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, we have an isomorphism $\rho_{|P} \cong \rho_{|f(P)} \circ f$. We will also refer to $\rho$ or $V$ as a representation of $(S,\mathcal{F})$.

Definition 2.2. We say that a class-invariant map $\chi : S \to \mathbb{C}$ is fusion-preserving or $\mathcal{F}$-invariant if for any $P \leqslant S$ and any $f \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, we have $\chi_{\rho_{|P}} = \chi_{\rho_{|f(P)\circ f}}$.

Lemma 2.3. *Let $V$ be a unitary representation of $S$. Then $V$ is fusion-preserving if and only if its character $\chi_V$ is fusion-preserving.*

*Proof.* This follows from Corollary 2.3.21 of [Z05], which shows that two unitary representations of a locally finite group are isomorphic if and only if their characters are equal. $\square$

Let us consider a discrete $p$-toral group $S$ of rank $n$ with $T = S_0$ and $G = \pi_0(S)$. Given a representation $\rho : T \to GL(V)$, let $\mathrm{ind}_T^S(V)$ be the finite-dimensional vector space $\mathbb{C}G \otimes V$. This vector space can be identified with $\mathbb{C}S \otimes_T V$, where $[s,v] = [st^{-1}, \rho(t)(v)]$ for $t \in T$. Then $S$ acts on $\mathrm{ind}_T^S(V)$ by $s'[s,v] = [s's,v]$. This action is well defined and linear, so it determines a finite-dimensional representation $\mathrm{ind}_T^S(\rho) : S \to GL(\mathbb{C}G \otimes V)$.

Proposition 2.4. *Let $\mathcal{F}$ be a saturated fusion system over $S$ and $\rho : T \to U(m)$ a representation that is invariant under $\mathrm{Aut}_{\mathcal{F}}(T)$ and such that $\chi_\rho(x) = 0$ if $x \in P \cap T - P_0$ for some $P \in \mathcal{F}^{cr}$. Then $\mathrm{ind}_T^S(\rho) : S \to U(m|S/T|)$ is a fusion-preserving representation of $S$. Moreover, if $\rho$ is injective, so is $\mathrm{ind}_T^S(\rho)$.*

*Proof.* Let $\widetilde{\rho} = \mathrm{ind}_T^S(\rho)$. We will show that $\widetilde{\rho}$ is $\mathcal{F}$-invariant. Let $P$ be a subgroup of $S$ in $\mathcal{F}^{cr}$ and $\phi : P \to S$ in $\mathcal{F}$. Let $X$ be a set of coset representatives for $S/T$.

$$\chi_{\widetilde{\rho}}(p) = \sum_{s \in X} \chi_\rho(s^{-1}ps) = \begin{cases} |G|\chi_\rho(p) & \text{if } p \in T \cap P \\ 0 & \text{otherwise,} \end{cases}$$

since $c_s$ belongs to $\mathrm{Aut}_S(T) \subseteq \mathrm{Aut}_{\mathcal{F}}(T)$ for all $s \in S$ and we are assuming that $\rho$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. By the hypotheses, we actually have:

$$\chi_{\widetilde{\rho}}(p) = \begin{cases} |G|\chi_\rho(p) & \text{if } p \in P_0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\phi(P)$ also belongs to $\mathcal{F}^{cr}$, we have similarly:

$$\chi_{\widetilde{\rho}\circ\phi}(p) = \begin{cases} |G|\chi_\rho(\phi(p)) & \text{if } \phi(p) \in \phi(P)_0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $P_0$ is the characteristic subgroup of infinitely $p$-divisible elements of $P$. Thus, $p \in P_0$ if and only if $\phi(p) \in \phi(P)_0$. In fact, $\phi$ restricts to a homomorphism $\widetilde{\phi} : P_0 \to \phi(P)_0$ in $\mathcal{F}$. By Lemma 1.4, $\widetilde{\phi}$ extends to a map $\psi \in \mathrm{Aut}_{\mathcal{F}}(T)$. Therefore, for $p \in P_0$:

$$\chi_\rho(\phi(p)) = \chi_\rho(\widetilde{\phi}(p)) = \chi_\rho(\psi(p)) = \chi_\rho(p)$$

Thus $\chi_{\widetilde{\rho}_{|P}} = \chi_{\widetilde{\rho}_{|\phi(P)}\circ\phi}$.

Now, let $\psi : Q \to S$ in $\mathcal{F}$. By Alperin's fusion theorem 1.6, $\psi$ can be factored into automorphisms of $\mathcal{F}$-centric radical subgroups of $S$ via certain inclusions. That is, $\psi = \phi_m \circ \phi_{m-1} \circ \ldots \circ \phi_1$ where $\phi_j \in \mathrm{Aut}_{\mathcal{F}}(P_j)$ and the groups $P_j$ are $\mathcal{F}$-centric radical. And so the previous paragraph shows that $\chi_{\widetilde{\rho}_{|Q}} = \chi_{\widetilde{\rho}_{|\psi(Q)}\circ\psi}$, that is, $\widetilde{\rho}$ is fusion-invariant. The last statement is a general property of induced representations. □

The rest of this section shows the existence of representations of $T$ with the characteristics needed to apply Proposition 2.4. For this purpose, it will be convenient to recall Pontryagin duality for second countable locally compact abelian groups. For more details on this construction, see chapter V of [P39], particularly Sections 30 through 35.

Definition 2.5. Let $G$ be a locally compact abelian group. Its Pontryagin dual $G^*$ is the group of continuous group homomorphisms from $G$ to $S^1$.

Theorem 2.6. *The assignment $G \mapsto G^*$ is an exact contravariant endofunctor in the category of second countable locally compact abelian groups and $(G^*)^*$ is naturally isomorphic to $G$.*

*Proof.* Theorem 31 in Chapter V of [P39] shows that this is an endofunctor. Theorem 32 in Chapter V of [P39] shows that $(G^*)^*$ is isomorphic to $G$. This isomorphism is clearly natural. The exactness follows from this duality. □

Every continuous homomorphism $\mathbb{Z}/p^\infty \to S^1$ factors through $\mathbb{Z}/p^\infty$, and it is well known that the endomorphism group of $\mathbb{Z}/p^\infty$ is $\mathbb{Z}_p^\wedge$. Therefore the dual of $\mathbb{Z}/p^\infty$ is $\mathbb{Z}_p^\wedge$. By Theorem 36 in Chapter V of [P39], the dual of $(\mathbb{Z}/p^\infty)^m$ is $(\mathbb{Z}_p^\wedge)^m$. Equip $(\mathbb{Z}/p^\infty)^m$ and $(\mathbb{Z}_p^\wedge)^m$ with the coordinate-wise actions of $\mathbb{Z}_p^\wedge$. Group homomorphisms between discrete $p$-tori are equivariant with respect to this $\mathbb{Z}_p^\wedge$-action. It is easy to check that if $f$ is a group homomorphism between discrete $p$-tori, then $f^*$ is $\mathbb{Z}_p^\wedge$-equivariant, that is, $f^*$ is a map of $\mathbb{Z}_p^\wedge$-modules.

We use additive notation for $\mathbb{Z}/p^\infty$ and multiplicative notation for the action of $\mathbb{Z}_p^\wedge$ on $(\mathbb{Z}/p^\infty)^m$. So if $a \in \mathbb{Z}_p^\wedge$ and $x \in (\mathbb{Z}/p^\infty)^m$, we denote by $ax$ the result of the coordinate-wise action of $a$ on $x$. Given $b=(b_1,b_2,\dots,b_m)$ in $(\mathbb{Z}_p^\wedge)^m$, we denote by $\rho_b$ the homomorphism from $(\mathbb{Z}/p^\infty)^m$ to $\mathbb{Z}/p^\infty$ that takes an element $(x_1,\dots,x_r)$ to $b_1x_1+b_2x_2+\dots+b_rx_r$. And we denote by $\phi_b$ the composition of $\rho_b$ with the standard inclusion $\mathbb{Z}/p^\infty \to S^1$.

Lemma 2.7. *Let $T$ be a discrete $p$-torus of rank $n$ and $K$ a discrete $p$-subtorus of $T$ of rank $m<n$. Then there exist $b^1,\dots,b^{n-m}$ in $(\mathbb{Z}_p^\wedge)^n$ such that an element $x$ in $T$ belongs to $K$ if and only if $\rho_{b^i}(x)=0$ for all $i=1,\dots,n-m$.*

*Proof.* Consider the Pontryagin dual $j^*$ of the inclusion $j: K\to T$. Since the duality functor is exact, $j^*$ is onto and as $\mathbb{Z}_p^\wedge$ is a principal ideal domain, the kernel of $j^*$ is also a free $\mathbb{Z}_p^\wedge$-module. Therefore we get a short exact sequence of $\mathbb{Z}_p^\wedge$-modules.

$$0 \longrightarrow (\mathbb{Z}_p^\wedge)^{n-m} \xrightarrow{g} (\mathbb{Z}_p^\wedge)^n \xrightarrow{j^*} (\mathbb{Z}_p^\wedge)^m \longrightarrow 0$$

The map $g$ is represented by an $n\times(n-m)$-matrix $B$ with entries in $\mathbb{Z}_p^\wedge$. Applying the duality functor again we obtain a short exact sequence

$$0 \longrightarrow (\mathbb{Z}/p^\infty)^m \xrightarrow{j} (\mathbb{Z}/p^\infty)^n \xrightarrow{g^*} (\mathbb{Z}/p^\infty)^{n-m} \longrightarrow 0$$

where $g^*$ is represented by the transpose of $B$. It is now clear that the rows $b^1,\dots,b^{n-m}$ of $B^T$ satisfy the conditions of the statement. $\square$

Lemma 2.8. *Let $P\leqslant S$. There is a representation $\rho_P: T\to U(m_p)$ such that $\chi_{\rho_P}(x)=0$ if $x\in P\cap T-P_0$*

*Proof.* Note that this immediate if $T\cap P=P_0$, in which case we can take any representation of $T$. Since $T$ satisfies this condition, we can assume that $P_0$ is a strict subgroup of $T$ and therefore $\operatorname{rank}(P)=\operatorname{rank}(P_0)=m<n=\operatorname{rank}(T)$.

Consider the inclusion $j: P_0\to T$. By Lemma 2.7, there exist $b^1,\dots,b^{n-m}$ in $(\mathbb{Z}_p^\wedge)^n$ such that an element $x$ in $T$ belongs to $j(P_0)$ if and only if $\rho_{b^i}(x)=0$ for all $i=1,\dots,n-m$. Note that $P\cap T=P_0\times Z$ for some finite abelian $p$-group $Z$. We define now

$$\rho_P=\bigotimes_{i=1}^{n-m}(\epsilon\oplus\varphi_{b^i}\oplus\varphi_{b^i}^2\oplus\dots\oplus\varphi_{b^i}^{r-1}),$$

where $r$ is the exponent of $Z$ and $\epsilon$ is the trivial 1-dimensional representation of $T$. Now let $y\in P\cap T-P_0$, which can be written as $y=xz$ with $x\in P_0$ and $z\in Z-\{1\}$. Since the order of $z$ divides $r$, so does the order of $\varphi(z)$ for any 1-dimensional representation $\varphi$ of $Z$, and so $\varphi(z)^r=1$. But note that $\varphi_{b^i}(z)\neq 1$ for some $i$ because $z$ does not belong to $P_0$, that is, $\varphi_{b^i}(z)$ is an $r$th root of unity different from 1 and so it is a solution of the equation $1+x+\dots+x^{r-1}=0$. Then we have:

$$\begin{aligned}\chi_{\rho_P}(y)&=\prod_{i=1}^{k}(1+\varphi_{b^i}(y)+\varphi_{b^i}^2(y)+\dots+\varphi_{b^i}^{r-1}(y))\\&=\prod_{i=1}^{k}(1+\varphi_{b^i}(z)+\varphi_{b^i}^2(z)+\dots+\varphi_{b^i}^{r-1}(z))=0\end{aligned}$$

where the second equality follows because $\varphi_{b^i}(x)=1$ for all $i$. And so $\rho_P(y)=0$ if $y$ belongs to $P\cap T-P_0$. $\square$

Theorem 2.9. *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$. There exists a faithful unitary representation of $S$ which is fusion-preserving.*

*Proof.* By Lemma 2.8, for each $P$ in $\mathcal{F}^{cr}$ there is a representation $\rho_P : T \to U(m_p)$ such that $\chi_{\rho_P}(x) = 0$ if $x \in P \cap T - P_0$. Recall that $\mathrm{Aut}_{\mathcal{F}}(T)$ is a finite group and consider

$$\phi_P = \bigotimes_{w \in \mathrm{Aut}_{\mathcal{F}}(T)} w^* \rho_P,$$

where $w^* \rho_P = \rho_P \circ w$. This representation is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$ and if $x \in P \cap T - P_0$, then:

$$\chi_{\phi_P}(x) = \prod_{w \in \mathrm{Aut}_{\mathcal{F}}(T)} \chi_{w^*\rho_P}(x) = \chi_{\rho_P}(x) \cdot \prod_{1_T \neq w \in \mathrm{Aut}_{\mathcal{F}}(T)} \chi_{w^*\rho_P}(x) = 0$$

If $P$ and $Q$ are conjugate by an element $s \in S$, the conjugation $c_s$ takes $P \cap T$ to $Q \cap T$ and $P_0$ to $Q_0$. If $y \in Q \cap T - Q_0$, then $y = c_s(x)$ for some $x \in P \cap T - P_0$ and

$$\chi_{\phi_P}(y) = \chi_{\phi_P}(c_s(x)) = \chi_{\phi_P \circ c_s}(x) = \chi_{\phi_P}(x) = 0$$

since $c_s \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\phi_P$ is $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant. Therefore $\phi_P$ is such that $\chi_{\phi_P}(y) = 0$ if $y \in Q \cap T - Q_0$ for any $Q$ that is $S$-isomorphic to $P$.

Since there are a finite number of $S$-conjugacy classes of subgroups in $\mathcal{F}^{cr}$ (Corollary 3.5 in [BLO07]), we can construct

$$\psi = \bigotimes_{[P] \in \mathcal{F}^{cr}} \phi_P$$

where $[P]$ denotes the $S$-conjugacy class of $P$ in $\mathcal{F}^{cr}$. It is clear that $\chi_\psi(x) = 0$ if $x \in Q \cap T - Q_0$ for any $Q$ in $\mathcal{F}^{cr}$. By Proposition 2.4, the existence of a fusion-preserving representation of $S$ follows.

If $\psi$ is faithful, Proposition 2.4 shows the existence of a faithful fusion-preserving representation. Otherwise, given a faithful representation $\alpha$ of $T$, such as the standard representation of $T$, we consider

$$\alpha_f = \bigoplus_{w \in \mathrm{Aut}_{\mathcal{F}}(T)} w^* \alpha$$

Note that $\rho = \psi \otimes \alpha_f$ is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$ and if $y \in P \cap T - P_0$ for some $P$ in $\mathcal{F}^{cr}$ then

$$\chi_\rho(y) = \chi_\psi(y)\chi_{\alpha_f}(y) = 0$$

Each $\rho_P$ contains the trivial 1-dimensional representation as a subrepresentation, and therefore so does $\psi$. The representation $\rho$ is then faithful because it has $\alpha$ as a subrepresentation, which is faithful. The result follows from Proposition 2.4. $\square$

Proposition 2.10. *Given a representation $\rho$ of $S$, there exists a fusion-invariant representation $\alpha$ of $S$ such that $\rho$ is a direct summand of $\alpha$.*

*Proof.* Let $\psi$ be the representation of $T$ constructed in the proof of Theorem 2.9. This representation is invariant under the action of $\mathrm{Aut}_{\mathcal{F}}(T)$ and such that $\chi_\psi(x) = 0$ if $x \in Q \cap T - Q_0$ for any $Q$ in $\mathcal{F}^{cr}$. Now consider the following representation of $T$

$$\beta = \psi \bigotimes \left( \bigoplus_{w \in \mathrm{Aut}_{\mathcal{F}}(T)} w^* \operatorname{res}_T^S(\rho) \right)$$

This representation still satisfies the same two conditions as $\psi$ and moreover, it contains $\operatorname{res}_T^S(\rho)$ as a direct summand because $\psi$ contains the trivial 1-dimensional representation of $T$ as a direct summand. By Proposition 2.4, the representation $\alpha = \mathrm{ind}_T^S(\beta)$ is fusion-invariant and it contains $\rho$ as a direct summand. $\square$

## 3. Embeddings of classifying spaces

The aim of this section is to find criteria for the existence of unitary embeddings of $p$-local compact groups. To be more precise, we make the following definitions:

Definition 3.1. A space $X$ is called quasi-finite at the prime $p$ if the pointed mapping space $\mathrm{Map}_*(B\mathbb{Z}/p, X)$ is contractible for all choices of basepoint in $X$. A map $f : X \to Y$ is called a homotopy monomorphism at $p$ if its homotopy fiber $F$ over any connected component of $Y$ is quasi-finite at $p$.

It is easy to check that the composition of homotopy monomorphisms at $p$ is again a homotopy monomorphism at $p$. The $p$-completion map $X \to X^\wedge_p$ is always a homotopy monomorphism for any connected space $X$ of finite type whose fundamental group is a finite $p$-group (see proof of Proposition 3.1 in [DZ87]).

Definition 3.2. Let $X$ be a topological space. A unitary embedding at $p$ of $X$ is a homotopy monomorphism $X \to BU(N)^\wedge_p$ for some $N > 0$. If $(S, \mathcal{F}, \mathcal{L})$ is a $p$-local compact group, a unitary embedding of $(S, \mathcal{F}, \mathcal{L})$ is a unitary embedding at $p$ of $|\mathcal{L}|^\wedge_p$.

Proposition 3.3. *Given a map $f : |\mathcal{L}|^\wedge_p \to X$, $f$ is a homotopy monomorphism at $p$ if and only if $f_{|BS}$ is so.*

*Proof.* Consider the diagram:

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & F \arrow[d] \\
G \arrow[r] \arrow[d] & BS \arrow[r, "Bi"] \arrow[d, "f_{|BS}"] & {|\mathcal{L}|^\wedge_p} \arrow[d, "f"] \\
* \arrow[r] & X \arrow[r] & X
\end{tikzcd}$$

From Theorem 6.3 in [BLO07], one can deduce that $Bi : BS \to |\mathcal{L}|^\wedge_p$ is a homotopy monomorphism and the composition of two homotopy monomorphisms is a homotopy monomorphism. Thus if $f$ is a homotopy monomorphism, so is $f_{|BS}$.

Conversely, if we assume that $f_{|BS}$ is a homotopy monomorphism, let us show that if we have $\alpha : B\mathbb{Z}/p \to |\mathcal{L}|^\wedge_p$ such that $f \circ \alpha \simeq *$, then $\alpha \simeq *$. In this case Theorem 6.3 of [BLO07] shows that there exists $\rho : \mathbb{Z}/p \to S$ such that $\alpha \simeq Bi \circ B\rho$. Then $f_{|BS} \circ B\rho \simeq f \circ Bi \circ B\rho \simeq f\alpha \simeq *$. Since $f_{|BS}$ is a homotopy monomorphism, $B\rho \simeq *$ and $\alpha \simeq Bi \circ B\rho \simeq *$. $\square$

Because of the homotopy decomposition of the classifying space of $(S, \mathcal{F}, \mathcal{L})$ in terms of the classifying spaces of its $\mathcal{F}$-centric subgroups given in Proposition 1.9, the restriction map $[|\mathcal{L}|^\wedge_p, BU(n)^\wedge_p] \to [BS, BU(n)^\wedge_p]$ factors through an inverse limit:

$$[|\mathcal{L}|^\wedge_p, BU(n)^\wedge_p] = [\operatorname*{hocolim}_{\overrightarrow{\mathcal{O}(\mathcal{F}^c)}} \widetilde{B}P, BU(n)^\wedge_p] \xrightarrow{\Psi_n} \lim_{\overleftarrow{\mathcal{O}(\mathcal{F}^c)}} [\widetilde{B}P, BU(n)^\wedge_p]$$

And

$$\lim_{\overleftarrow{\mathcal{O}(\mathcal{F}^c)}} [\widetilde{B}P, BU(n)^\wedge_p] \cong \lim_{\overleftarrow{\mathcal{O}(\mathcal{F}^c)}} \mathrm{Rep}(P, U(n)) \cong \mathrm{Rep}^{\mathcal{F}}(S, U(n))$$

where the first isomorphism holds by the argument in the proof of Theorem 1.1 (i) in [JMO95], which does not need the discrete $p$-toral group to be an approximation of a $p$-toral group. We are interested in finding elements in the image of $\Psi_n$ for some $n$ which are faithful representations, based on the following result:

Proposition 3.4. *If there exists a faithful fusion-invariant representation in the image of $\Psi_n$ for some $n$, then there exists a unitary embedding of $(S, \mathcal{F}, \mathcal{L})$.*

*Proof.* Let $f \in \mathrm{Rep}^{\mathcal{F}}(S, U(n))$ be a faithful $\mathcal{F}$-invariant representation of $S$ such that $f = \Psi_n(g)$ for some $g : |\mathcal{L}|^\wedge_p \to BU(n)^\wedge_p$. We claim that $g$ is a homotopy monomorphism at $p$. By Proposition 3.3, it is enough to show that $g_{|BS}$ is a homotopy monomorphism at $p$. But this follows from the fact that $g_{|BS}$ is the composition of $Bf$ with the $p$-completion map $BU(n) \to BU(n)^\wedge_p$, which are both homotopy monomorphisms at $p$. □

The behaviour of the maps $\Psi_n$ was considered in [W87] in more generality, where obstructions to the surjectivity and injectivity were found. In this case, given $\rho$ in $\mathrm{Rep}^{\mathcal{F}}(S, U(n))$, the obstructions for $\rho$ to be in the image or to have a unique preimage lie in higher limits of the functors:

$$\begin{aligned} F_i^\rho : \mathcal{O}(\mathcal{F}^c)^{op} &\to \mathbb{Z}_{(p)}\text{-Mod} \\ P &\mapsto \pi_i \left( \mathrm{Map}(\widetilde{B}P, BU(n)^\wedge_p)_{\widetilde{B}\rho_{|P}} \right). \end{aligned}$$

We offer next a precise and modified version of the results in [W87] which are suitable for our purpose.

Theorem 3.5. *Let $\rho \in \mathrm{Rep}^{\mathcal{F}}(S, U(n))$ be a representation which respects fusion and for each $i > 0$, let $F_i^\rho$ be the functor defined above restricted to the category $\mathcal{O}(\mathcal{F}^{cr})$. Then*

- *$\rho$ belongs to the image of $\Psi_n$ if an obstruction*
$$[E_i^\rho] \in \lim_{\substack{\longleftarrow \\ \mathcal{O}(\mathcal{F}^{cr})}}{}^{i+1} F_i^\rho$$
*is zero for all $i \geqslant 1$.*
- *$\rho$ has a unique preimage under $\Psi_n$ if an obstruction class in*
$$[U_i^\rho] \in \lim_{\substack{\longleftarrow \\ \mathcal{O}(\mathcal{F}^{cr})}}{}^{i} F_i^\rho$$
*is zero for all $i \geqslant 1$.*

*Proof.* Higher limits over $\mathcal{O}(\mathcal{F}^{cr})$ and $\mathcal{O}(\mathcal{F}^c)$ coincide by Proposition 5.4 in [BLO07] and Proposition 6.1 (ii) in [JMO92]. The theorem follows from Propositions 3 and 4 of [W87]. □

In this paper we are concerned with the surjectivity of the maps $\Psi_n$ up to stabilization. For this weaker property, we found that we can actually refine our obstructions. For that purpose, we start by recalling some definitions from [L89].

Definition 3.6. An EI-category $\mathcal{O}$ is finite if the set of isomorphism classes of objects is finite and the set of morphisms between any two given objects is finite.

Note that $\mathcal{O}(\mathcal{F}^{cr})$ is a finite EI-category by Lemma 2.5 and Corollary 3.5 in [BLO07].

Definition 3.7. Let $\mathcal{O}$ be a finite EI-category. The length $l(\mathcal{O})$ of $\mathcal{O}$ is the maximum integer $n$ such that there exist $n+1$ different objects $x_0, x_1, \ldots, x_n$ with $x_0 < x_1 < \ldots < x_n$.

Proposition 3.8. *Let $\Phi : \mathcal{O}(\mathcal{F}^{cr}) \to \mathbb{Z}_{(p)}$-Mod be a functor. Then*

$$\lim_{\overleftarrow{\mathcal{O}(\mathcal{F}^{cr})}}{}^{i}(\Phi) = 0$$

*for all $i > l(\mathcal{O}(\mathcal{F}^{cr}))$.*

*Proof.* Corollary 5.5 in [BLO07] is equivalent to saying that the constant functor $\mathbb{Z}_{(p)}$ has finite homological dimension as a $\mathbb{Z}_{(p)}\mathcal{O}(\mathcal{F}^{cr})$-module. By Proposition 17.31 in [L89], this homological dimension is bounded by $l(\mathcal{O}(\mathcal{F}^{cr}))$. □

Let $\rho : S \to U(n)$ be an $\mathcal{F}$-invariant representation. For any $P \leqslant S$, let $\mathrm{Irr}(P, \rho)$ be the set of isomorphism classes of irreducible subrepresentations of $\rho_{|P}$ and let $R(P, \rho)$ be the subgroup of $R(P)$ generated by $\mathrm{Irr}(P, \rho)$. Note that for any map $f : P \to Q$ in $\mathcal{F}$, the representations$\rho_{|Q} \circ f$ and $\rho_{|P}$ are isomorphic, and so the induced homomorphism $R(Q) \to R(P)$ restricts to a group homomorphism $R(Q, \rho) \to R(P, \rho)$. We will consider the following contravariant functor

$$R(-, \rho)^{\wedge}_{p} : \mathcal{O}(\mathcal{F}^{cr}) \to \mathbb{Z}_{(p)} - \mathrm{Mod}$$

$$P \mapsto R(P, \rho) \otimes_{\mathbb{Z}} \mathbb{Z}^{\wedge}_{p} \cong \bigoplus_{\mathrm{Irr}(P,\rho)} \mathbb{Z}^{\wedge}_{p}.$$

Proposition 3.10 will show that this functor is naturally isomorphic to $F^{\rho}_{2j}$ if the representation $\rho$ is faithful and the centralizers of the subrepresentations of $\rho$ satisfy a stability condition.

First we need an auxiliary lemma that extends Proposition 4.4 of [DZ87] to discrete $p$-toral groups (see also Theorem 1.1 (ii) of [JMO95] and Theorem 5.1 of [N91] for analogous results). We introduce some notation inspired by this article. For a compact Lie group $G$, let $\mathcal{B}G$ be the topological category with one object and $G$ as the space of automorphism of this object. Given two categories $C$ and $D$ and a functor $F : C \to D$, let $\mathrm{Hom}(C, D)_F$ be the category whose only object is $F$ and whose morphisms are natural transformation from $F$ to $F$. Note that if $G$ and $H$ are topological groups and $f : H \to G$ is a group homomorphism there is an equivalence of topological categories

$$\mathcal{B}C_G(f(H)) \to \mathrm{Hom}(\mathcal{B}H, \mathcal{B}G)_{\mathcal{B}f}$$

that takes the object of $\mathcal{B}C_G(f(H))$ to the functor $\mathcal{B}f$, and the morphism $g \in C_G(f(H))$ to the natural transformation given by $g \in G$.

Lemma 3.9. *If $P$ is a discrete $p$-toral group, $G$ is a compact Lie group such that $\pi_0(G)$ is a finite $p$-group, and $\rho : P \to G$ is a homomorphism, then the map induced by taking classifying spaces*

$$[B\,\mathrm{Hom}(\mathcal{B}P, \mathcal{B}G)_{\mathcal{B}\rho}]^{\wedge}_{p} \to [\mathrm{Map}(BP, BG)_{B\rho}]^{\wedge}_{p}$$

*is a homotopy equivalence.*

*Proof.* Let $P$ be the union of an increasing sequence $P_1 \leqslant P_2 \leqslant \ldots$ of finite subgroups, as in Lemma 1.9 of [BLO07]. Since $G$ is artinian with respect to closed subgroups, there is an integer $k$ such that if $n \geqslant k$, then $C_G(\rho(P)) = C_G(\rho(P_n))$.

Since $\pi_0(G)$ is a finite $p$-group, by Proposition 6.22 of [DW94], there is an integer $m$ such that if $n \geqslant m$, then the restriction map $\mathrm{Map}(BP, BG^{\wedge}_{p})_{B\rho} \to \mathrm{Map}(BP_n, BG^{\wedge}_{p})_{B\rho}$ is a homotopy equivalence.

Let $N$ be the maximum of $k$ and $m$ and consider the following commutative diagram

$$\begin{tikzcd}
{[B\operatorname{Hom}(\mathcal{B}P,\mathcal{B}G)_{\mathcal{B}\rho}]^\wedge_p} \arrow[r] \arrow[d] & \operatorname{Map}(BP,BG)_{B\rho} \arrow[r] \arrow[d] & \operatorname{Map}(BP,BG^\wedge_p)_{B\rho} \arrow[d] \\
{[B\operatorname{Hom}(\mathcal{B}P_N,\mathcal{B}G)_{\mathcal{B}\rho}]^\wedge_p} \arrow[r] & {[\operatorname{Map}(BP_N,BG)_{B\rho}]^\wedge_p} \arrow[r] & \operatorname{Map}(BP_N,BG^\wedge_p)_{B\rho}
\end{tikzcd}$$

where the vertical maps are induced by the inclusion $i : P_N \to P$ and the horizontal maps by taking classifying spaces. We have already shown that the first and last vertical maps are homotopy equivalences and composite of the lower horizontal maps is a homotopy equivalence by [JMO92, Theorem 3.2]. This proves the lemma. □

Proposition 3.10. *Let $\rho : S \to U(V)$ be a faithful $\mathcal{F}$-invariant representation. Assume the decomposition $\rho = \oplus n_i\mu_i$ in irreducible representations is such that the standard inclusion $U(n_i) \to U(M)$ induces an isomorphism in $\pi_{k-1}$ for all $M \geqslant n_i$ for each $i$. Then if $k > 0$ is even, then the functors $F^\rho_k$ and $R(-,\rho)^\wedge_p$ are naturally isomorphic, and if $k$ is odd, then $F^\rho_k$ is the zero functor.*

*Proof.* There is an equivalence of topological categories $\mathcal{B}C_{U(V)}(\rho(P)) \to \operatorname{Hom}(\mathcal{B}P,\mathcal{B}U(V))_{\mathcal{B}\rho}$ for each $P \leqslant S$, as we mentioned above. Taking a functor to its geometric realization induces a homotopy equivalence

$$[B\operatorname{Hom}(\mathcal{B}P,\mathcal{B}U(V))_{\mathcal{B}\rho}]^\wedge_p \to \left[\operatorname{Map}(BP,BU(V)^\wedge_p)_{B\rho}\right]$$

by Lemma 3.9. The composition $BC_{U(V)}(\rho(P))^\wedge_p \to [\operatorname{Map}(BP,BU(V)^\wedge_p)_{B\rho}]$ is a homotopy equivalence, which we claim it is natural as functors $\mathcal{O}(\mathcal{F}^{cr}) \to \mathrm{HoTop}$. It is clear that this composition is a natural transformation of functors $\mathcal{F}^{cr} \to \mathrm{HoTop}$, so we only need to check both functors descend to $\mathcal{O}(\mathcal{F}^{cr})$. Let $a$ be an element of $\operatorname{Rep}_\mathcal{F}(Q,P)$ represented by $f : Q \to P$. This map is determined by $a$ up to conjugation by an element of $P$ so the map $Bf : BQ \to BP$ is determined up to homotopy. By precomposition we get a map

$$\left[\operatorname{Map}(BP,BU(V)^\wedge_p)_{B\rho}\right] \to \left[\operatorname{Map}(BQ,BU(V)^\wedge_p)_{B(f\rho)}\right] = \left[\operatorname{Map}(BQ,BU(V)^\wedge_p)_{B\rho}\right]$$

determined up to homotopy by $a$. On the other hand, $\rho f \rho^{-1} : \rho(Q) \to \rho(P)$ is given by conjugation by an element of $U(V)$ and determined by $a$ up to conjugation by an element of $\rho(P)$. The induced map $C_{U(V)}(\rho(P)) \to C_{U(V)}(\rho(Q))$ is uniquely determined, since conjugation by an element of $\rho(P)$ is the identity. Therefore the map $BC_{U(V)}(\rho(P))^\wedge_p \to BC_{U(V)}(\rho(Q))^\wedge_p$ is determined by $a$.

By definition, the rigidification $\tilde{B} : \mathcal{O}(\mathcal{F}^{cr}) \to \mathrm{Top}$ is such that there is a natural homotopy equivalence of functors from $B$ to $ho \circ \tilde{B} : \mathcal{O}(\mathcal{F}^{cr}) \to \mathrm{HoTop}$. Therefore there is a homotopy equivalence, natural as functors $\mathcal{O}(\mathcal{F}^{cr}) \to \mathrm{HoTop}$

$$\operatorname{Map}(\tilde{B}P,BU(V)^\wedge_p)_{\tilde{B}\rho} \to \operatorname{Map}(BP,BU(V)^\wedge_p)_{B\rho}$$

In particular, for any $k \geqslant 1$, there is a natural isomorphism of functors $\mathcal{O}(\mathcal{F}^{cr}) \to \mathbb{Z}_{(p)} - \mathrm{Mod}$

$$\pi_k\left(BC_{U(V)}(\rho(P))^\wedge_p\right) \to \pi_k\left(\operatorname{Map}(\tilde{B}P,BU(V)^\wedge_p)_{\tilde{B}\rho}\right)$$

For each $P \leqslant S$, we decompose

$$\rho_{|P} = \bigoplus_{\operatorname{Irr}(P,\rho)} b_i\alpha_i$$

into irreducible representations, and if $n$ is the minimum of the $n_i$'s, then clearly $b_i \geqslant n$. If $\alpha_i : P \to U(W_i)$, then the decomposition can be expressed by Proposition 2.3.15 of [Z05] as the isomorphism induced by the evaluation map:

$$\bigoplus_{W_i \in \mathrm{Irr}(P)} W_i \otimes \mathrm{Hom}_P(W_i, V) \to V,$$

and so $C_{U(V)}(\rho(P)) \cong \hat{\prod} C_{U(b_iW_i)}(b_i\alpha_i(P)) \cong \hat{\prod}\,\mathrm{Aut}(\mathrm{Hom}_P(W_i,V)) \cong \hat{\prod} U(b_i)$, where $\hat{\prod}$ denotes the restricted product. Hence we have a natural equivalence

$$BC_{U(V)}(\rho(P))^\wedge_p \simeq \hat{\prod}_{\mathrm{Irr}(P,\rho)} BC_{U(b_iW_i)}(b_i\alpha_i(P))^\wedge_p.$$

We will now understand how maps between centralizers translate into maps of these unitary groups. We have an analogous decomposition

$$\rho_{|Q} = \bigoplus_{\mathrm{Irr}(Q,\rho)} d_i\beta_i$$

with $\beta_i : Q \to U(Z_i)$ and $d_i \geqslant n$. Given a map $f : Q \to P$, since $\rho$ is $\mathcal{F}$-invariant, we have $\rho_{|P} \circ f \cong \rho_{|Q}$. Via $f$, the representations $W_i$ decompose as a sum of $Z_k$ and so again have an isomorphism

$$\bigoplus_{W_i \in \mathrm{Irr}(P)} \left( \bigoplus_{Z_k \in \mathrm{Irr}(Q)} Z_k \otimes \mathrm{Hom}_Q(Z_k, W_i) \right) \otimes \mathrm{Hom}_P(W_i, V) \to V$$

Note that the map $f$ induces a homomorphism $C_{U(b_iW_i)}(b_i\alpha_i(P)) \to C_{U(V)}(\rho \circ f(Q))$ which takes $\mathrm{Aut}(\mathrm{Hom}_P(W_i,V))$ to $\mathrm{Aut}(\bigoplus_{Z_k \in \mathrm{Irr}(Q)} \mathrm{Hom}_Q(Z_k,W_i) \otimes \mathrm{Hom}_P(W_i,V))$. It is clear that the component $\mathrm{Aut}(\mathrm{Hom}_P(W_i,V)) \to \mathrm{Aut}(\mathrm{Hom}_Q(Z_k,W_i) \otimes \mathrm{Hom}_P(W_i,V))$ is given by the diagonal inclusion in blocks.

Let $k = 2j$ and fix a generator $\iota$ of $\pi_{2j-1}(U)$. Let $r$ be the smallest integer such that the standard inclusions $U(r) \to U(s)$ induce an isomorphism on $\pi_{2j-1}$ for all $s > r$. For each $s \geqslant r$, we let $\iota_s$ be the generator of $\pi_{2j-1}(U(s))$ which maps to $\iota$ under the standard inclusion $\pi_{2j-1}(U(s)) \to \pi_{2j-1}(U)$, which is an isomorphism . Consider the isomorphisms:

$$\begin{aligned} F^\rho_{2j} = \pi_{2j}\left(\mathrm{Map}(\widetilde{B}P, BU(V)^\wedge_p)_{\widetilde{B}\rho_{|P}}\right) &\cong \pi_{2j}\left(BC_{U(V)}(\rho(P))^\wedge_p\right) \\ &\cong \pi_{2j}\left(\hat{\prod}_{\mathrm{Irr}(P,\rho)} BC_{U(b_iW_i)}(b_i\alpha_i(P))^\wedge_p\right) \\ &\cong \pi_{2j}\left(\hat{\prod}_{\mathrm{Irr}(P,\rho} BU(b_i)^\wedge_p\right) \\ &\cong \pi_{2j-1}\left(\hat{\prod}_{\mathrm{Irr}(P,\rho)} U(b_i)^\wedge_p\right) \\ &\cong \bigoplus_{\mathrm{Irr}(P,\rho)} \mathbb{Z}^\wedge_p \\ &\cong R(P,\rho) \otimes_{\mathbb{Z}} \mathbb{Z}^\wedge_p \end{aligned}$$

where the fifth isomorphism follows since $b_i \geqslant n$ and $\pi_{2j-1}(U) \cong \mathbb{Z}$. We claim that this composition of isomorphisms is a natural isomorphism. We have already shown that the first two are natural isomorphisms, so we only need to show naturality for the composition of the last four isomorphisms. Indeed, a morphism $f : Q \to P$ induces the diagonal inclusion of blocks

$U(b_i) \to U(m_k b_i)$ for each decomposition of $W_i$ into $Z_k$. And this inclusion induces multiplication by $m_k$ on the homotopy groups. If we let $\mu_i$ be the image of $\iota_{b_i}$ in $R(P, \rho)$ under the composition of the last two isomorphisms, then this is mapped to its decomposition into irreducible representations of $Q$ via $f : Q \to P$. If $k$ is odd, the same chain of isomorphisms gives the zero functor since then $\pi_{k-1}(U) = 0$. □

Lemma 3.11. *Let $\rho : S \to U(n)$ be a faithful $\mathcal{F}$-invariant representation. Then there exists a positive integer $M_0$ such that for each $M \geqslant M_0$, the obstructions $[E^{M\rho}_{2i+1}]$ are zero. The representation $M\rho$ belongs to the image of $\Psi_{Mn}$ if the obstructions $[E^{M\rho}_{2i}]$ vanish for all $i$ with $3 \leqslant 2i+1 \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$. Moreover, these obstructions lie in the $\mathbb{Z}_{(p)}$-modules:*

$$\varprojlim_{\mathcal{O}(\mathcal{F}^{cr})}{}^{2i+1} R(-, \rho)^\wedge_p$$

*Proof.* By Proposition 3.8, given any $\mathcal{F}$-invariant representation $\tau$ of $S$, we only need to deal with a finite number of obstructions $[E^\tau_i]$, namely, when $i+1 \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$. Let $\rho_{|S} = n_1\mu_1 \oplus \ldots \oplus n_s\mu_s$ be a decomposition as a direct sum of irreducible representations. Since the homotopy groups of unitary groups stabilize, we can choose a positive integer $M_0$ such that the standard inclusions induce isomorphisms $\pi_j(BU(M_0 n_i)) \cong \pi_j(BU(N))$ for all $1 \leqslant j \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$, for all $i$ and for all $N \geqslant M_0 n_i$.

If $M \geqslant M_0$, then the representation $M\rho$ satisfies the hypothesis of Proposition 3.10 and so $F^{M\rho}_k = 0$ if $k$ is odd, and $F^{M\rho}_k$ is naturally isomorphic to $R(-, \rho)^\wedge_p$ if $k$ is even. Recall that by Theorem 3.5, we have obstructions classes

$$\left[E^{M\rho}_i\right] \in \varprojlim_{\mathcal{O}(\mathcal{F}^{cr})}{}^{i+1} F^{M\rho}_i$$

to the existence of a map $f : |\mathcal{L}|^\wedge_p \to BU(Mn)^\wedge_p$ such that $f \circ Bi = B(M\rho)$. Hence all the obstructions $[E^{M\rho}_{2i+1}]$ vanish and the only obstructions remaining are the classes

$$\left[E^{M\rho}_{2i}\right] \in \varprojlim_{\mathcal{O}(\mathcal{F}^{cr})}{}^{2i+1} F^{M\rho}_{2i} \cong \varprojlim_{\mathcal{O}(\mathcal{F}^{cr})}{}^{2i+1} R(-, \rho)^\wedge_p$$

for $2 \leqslant 2i < l(\mathcal{O}(\mathcal{F}^{cr}))$. □

Lemma 3.12. *Let $\rho : S \to U(n)$ be an $\mathcal{F}$-invariant representation such that $[E^\rho_i]$ is torsion for $1 \leqslant i \leqslant k$. Then there is $N \in \mathbb{N}$ such that $N\rho : S \to U(Nn)$ satisfies $[E^{N\rho}_i] = 0$ for $i \leqslant k$.*

*Proof.* Following [CL09], we use the notation $X \wr \Sigma_N = X^N \times_{\Sigma_N} E\Sigma_N$ for the wreath product of a space $X$ with $\Sigma_N$.

For each $P \leqslant S$, the space $\mathrm{Map}(\tilde{B}P, [BU(n)^\wedge_p \wr \Sigma_N]^\wedge_p)_{\tilde{B}\rho_{|P} \times \ldots \times \tilde{B}\rho_{|P}}$ is $p$-complete. Proposition 4.3 in [CL09] shows that the corresponding diagonal maps induce maps which fit into a commutative diagram:

$$\begin{tikzcd}
 & \left[\mathrm{Map}(\tilde{B}P, BU(n)^\wedge_p)_{\tilde{B}\rho_{|P}} \wr \Sigma_N\right]^\wedge_p \arrow[d] \\
\mathrm{Map}(\tilde{B}P, BU(n)^\wedge_p)_{\tilde{B}\rho_{|P}} \arrow[ur] \arrow[r] & \mathrm{Map}(\tilde{B}P, [BU(n)^\wedge_p \wr \Sigma_N]^\wedge_p)_{\tilde{B}\rho_{|P} \times \ldots \times \tilde{B}\rho_{|P}}
\end{tikzcd}$$

for a certain vertical map. Note that $BU(n) \wr \Sigma_N$ is the classifying space of the group $U(n) \wr \Sigma_N$. There is a homomorphism $U(n) \wr \Sigma_N \to U(Nn)$ that takes the element $\big((A_1, \ldots, A_n), \sigma\big)$ to the matrix that has the matrices $A_i$ as its diagonal blocks in such a way that the $j$th block is $A_{\sigma(j)}$. The composition of the homomorphism $U(n) \to U(n) \wr \Sigma_N$ induced by the diagonal map with the latter $U(n) \wr \Sigma_N \to U(Nn)$ corresponds to the map that takes a matrix $A$ to the matrix with $N$ diagonal blocks equal to $A$, which we will denote by $NA$. Consider the induced map:

$$\phi_P : \mathrm{Map}(\tilde{B}P, BU(n)^\wedge_p)_{\tilde{B}\rho_{|P}} \to \mathrm{Map}(\tilde{B}P, BU(Nn)^\wedge_p)_{\tilde{B}N\rho_{|P}}$$

By the previous commutative diagram, we see that the map $\phi_P$ factors through the space $[\mathrm{Map}(\tilde{B}P, BU(n)^\wedge_p)_{\tilde{B}\rho_{|P}} \wr \Sigma_N]^\wedge_p$.

Let $K$ be the least common multiple of the orders of the elements $[E_i^\rho]$ for $1 \leqslant i \leqslant k$ and

$$N = \begin{cases} K & \text{if } K \geqslant 3 \\ 3K & \text{if } K < 3. \end{cases}$$

Since $N \geqslant 3$, we can use Lemma 4.2 in [CL09] and so maps induced by $\phi_P$ on the homotopy groups have all the elements of exponent $N$ in their kernel. By Proposition 3.8, the category $\mathcal{O}(\mathcal{F}^{cr})$ has bounded limits at $p$. Now we note that the conclusion (a) of Theorem 6.2 in [CL09] still holds if we replace the assumption (i) by the weaker assumption that the induced map between the higher limits takes the obstructions $[E_i^\rho]$ for $1 \leqslant i \leqslant k$ to zero. This is the case because the obstructions are natural and so the induced maps take $[E_i^\rho]$ to $[E_i^{N\rho}]$. Therefore we conclude that $[E_i^{N\rho}] = 0$ for $1 \leqslant i \leqslant k$. $\square$

Theorem 3.13. *Let $\rho : S \to U(n)$ be a faithful $\mathcal{F}$-invariant representation such that the groups $\varprojlim_{\mathcal{O}(\mathcal{F}^{cr})}{}^{2i+1} R(-,\rho)^\wedge_p$ are torsion for $3 \leqslant 2i+1 \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$. Then there is an integer $K_0$ such that $M\rho \in \mathrm{Im}(\Psi_{Mn})$ if $M \geqslant K_0$.*

*Proof.* By Lemma 3.11, there exists a positive integer $M_0$ such that the obstructions $[E_{2i+1}^{M_0\rho}]$ are zero and $M_0\rho$ is in the image of $\Psi_{M_0 n}$ if the obstructions

$$\left[E_{2i}^{M_0\rho}\right] \in \varprojlim_{\mathcal{O}(\mathcal{F}^{cr})}{}^{2i+1} R(-,\rho)^\wedge_p$$

vanish for all $i$ with $3 \leqslant 2i+1 \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$. By assumption, the classes $[E_{2i}^{M_0\rho}]$ are torsion for $3 \leqslant 2i+1 \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$. We can use now Lemma 3.12, from where there is an integer $M_1$ such that $[E_i^{M_1M_0\rho}] = 0$ for all $1 \leqslant i \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$. Since all limits vanish for any $p$-local functor when $i > l(\mathcal{O}(\mathcal{F}^{cr}))$ by Proposition 3.8, we have $[E_i^{M_1M_0\rho}] = 0$ for all $i > 0$. By Theorem 3.5 and the naturality of obstructions, if $M \geqslant M_1M_0$, then $M\rho$ is in the image of $\Psi_{Mn}$. $\square$

*Remark* 3.14. The obstructions $[U_{2i}^\tau]$ to uniqueness of a preimage of $\tau$ under the maps $\psi_n$ can be treated analogously. Therefore given two maps $f, g : |\mathcal{L}|^\wedge_p \to BU(n)^\wedge_p$ such that $\psi_n(f) = \psi_n(g) = B\rho$, if the groups

$$\varprojlim_{\mathcal{O}(\mathcal{F}^{cr})}{}^{2i} R(-,\rho)^\wedge_p$$

are torsion for $3 \leqslant 2i+1 \leqslant l(\mathcal{O}(\mathcal{F}^{cr}))$, then there is an integer $M$ such that $Mf \simeq Mg$. For a map $k : |\mathcal{L}|^\wedge_p \to BU(n)^\wedge_p$, we are denoting by $Mk$ the composition of $k$ with the map $BU(n)^\wedge_p \to BU(Mn)^\wedge_p$ induced by the diagonal inclusion in blocks.

*Remark* 3.15. Observe that Theorem 3.13 only depends on the vanishing of obstructions coming from higher limits of the algebraic functor $R(-,\rho)^\wedge_p$. As a corollary, we can conclude that if $\mu : S \to U(m)$ is an $\mathcal{F}$-invariant representation such that $\mathrm{Irr}(S,\mu)$ is a subset of $\mathrm{Irr}(S,\rho)$, where

$\rho$ is a faithful representation such that all $[E_j^\rho]$ vanish, then there is an integer $M$ such that

$$\left[E_{2k}^{M\rho}\right] = \left[E_{2k}^{M\rho\oplus\mu}\right] \in \lim_{\overleftarrow{\mathcal{O}(\mathcal{F}^{cr})}}{}^{2k+1} R(-,\rho)_p^\wedge$$

and $[E_{2k+1}^{M\rho\oplus\mu}] = 0$. And therefore $\mu \oplus M\rho$ extends to a unitary embedding of $(S, \mathcal{F}, \mathcal{L})$.

The same observation holds for Remark 3.14. That is, if $\mu$ and $\rho$ are as in the previous paragraph and moreover $\rho$ is such that all $[U_j^\rho]$ vanish, then there is an integer $N$ such that $N\rho$ and $N\rho \oplus \mu$ extend to a unitary embedding and such that

$$\left[U_{2k}^{N\rho}\right] = \left[U_{2k}^{M\rho\oplus\mu}\right] \in \lim_{\overleftarrow{\mathcal{O}(\mathcal{F}^{cr})}}{}^{2k} R(-,\rho)_p^\wedge$$

and $[U_{2k+1}^{M\rho\oplus\mu}] = 0$. Hence $\mu \oplus N\rho$ has a unique extension to a unitary embedding of $(S, \mathcal{F}, \mathcal{L})$.

Now Theorem 3.13 and Proposition 3.4 imply:

Theorem 3.16. *A $p$-local compact group $(S, \mathcal{F}, \mathcal{L})$ with a faithful $\mathcal{F}$-invariant representation that satisfies the hypothesis of Theorem 3.13 has a unitary embedding.*

Every $p$-local compact group $(S, \mathcal{F}, \mathcal{L})$ has a faithful $\mathcal{F}$-invariant unitary representation by Theorem 2.9 and so the following corollary follows:

Corollary 3.17. *Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group with $l(\mathcal{O}(\mathcal{F}^{cr})) < 3$. Then it has a unitary embedding.*

## 4. Unitary embeddings of $p$-compact groups

In this section we apply the results of Section 3 to show the existence of unitary embeddings of $p$-compact groups whose orbit categories of centric radical subgroups have small length, namely for the Clark-Ewing and the Aguadé-Zabrodsky spaces. The existence of unitary embeddings of connected $p$-compact groups, also known as the Peter-Weyl theorem for connected $p$-compact groups ([AGMV08], Theorem 1.6) was proved by using the classification of $p$-compact groups [AGMV08], [AG09], and showing the existence of such embeddings for the irreducible sporadic $p$-compact groups. This was done for the generalized Grassmanians in [C06], for the Aguadé-Zabrodsky and Clark-Ewing spaces in [C00] and for $DI(4)$ in [Z05], [Z09]. The existence for the Aguadé-Zabrodsky and Clark-Ewing spaces spaces has not been published and we fill this void in the literature here.

Recall that a $p$-compact group is a triple $(X, BX, e)$ where $X$ is a space, $BX$ is a $p$-complete connected pointed space, $H^*(X;\mathbb{F}_p)$ is finite, and $e : X \to \Omega BX$ is a homotopy equivalence from $X$. When there is no danger of confusion we will use $X$ to denote $(X, BX, e)$.

The $p$-completion $\hat{T} = \Omega(BT_p^\wedge)$ of $T = (S^1)^r$ is called a $p$-compact torus of rank $r$. A homomorphism $f : X \to Y$ of $p$-compact groups is a pointed map $Bf : BX \to BY$. A maximal torus of a $p$-compact group $X$ is a monomorphism $T \to X$ from a $p$-compact torus of maximal dimension into $X$. The Weyl group of $X$ is the group of homotopy classes of maps $BT \to BT$ that commute up to homotopy with the inclusion $BT \to BX$.

All the irreducible $p$-compact groups are of the form $BG_p^\wedge$, where $G$ is an irreducible compact connected Lie group, or an irreducible sporadic $p$-compact groups. Sporadic $p$-compact groups belong to one of the following four families: Clark-Ewing spaces, Aguadé-Zabrodsky spaces, generalized Grassmanians and $DI(4)$.

For more information on the foundations of $p$-compact groups from the point of view of homotopy theory, see [DW94]. We now give some of the details from Section 10 of [BLO07], where it is shown that every $p$-compact group is modeled by a $p$-local compact group.

Given a discrete $p$-toral group $P$, consider the $p$-compact toral group $\hat{P} = \Omega(BP^\wedge_p)$. Let $X$ be a $p$-compact group. A discrete $p$-toral subgroup of $X$ is a pair $(P, u)$, where $P$ is a discrete $p$-toral group and $u : \hat{P} \to X$ is a homotopy monomorphism. Then $X$ has a maximal discrete $p$-toral subgroup $(S, f)$ and given any discrete $p$-toral subgroup $(P, u)$, it is maximal if and only if $p$ does not divide $\chi(X/u(\hat{P}))$. The Euler characteristic is taken with respect to homology with coefficientes in $\mathbb{F}_p$.

Given such a maximal subgroup $(S, f)$, the fusion system $\mathcal{F}_{S,f}(X)$ is the category with discrete $p$-toral subgroups of $X$ as objects and morphisms

$$\mathrm{Hom}_{\mathcal{F}_{S,f}(X)}(P, Q) = \{\phi \in \mathrm{Hom}(P, Q) \mid Bf_{|BQ} \circ B\phi \simeq Bf_{|BP}\}.$$

This is a saturated fusion system over $S$ and the centric subgroups correspond to the centric subgroups as defined in [CLN07]. There is a unique centric linking $\mathcal{L}^c_{S,f}(X)$ associated to $\mathcal{F}_{S,f}(X)$ such that $|\mathcal{L}^c_{S,f}(X)|^\wedge_p \simeq BX$. The maximal torus of $X$ corresponds to the maximal torus or connected component $T$ of $S$ and so the Weyl group of $X$ is isomorphic to $\mathrm{Aut}_{\mathcal{F}}(T)$. Since the torus of a connected $p$-compact group is self-centralizing, we have $C_S(T) = T$ and in particular $S/T$ is the $p$-Sylow subgroup of $\mathrm{Aut}_{\mathcal{F}}(T)$.

We now focus on the families of Clark-Ewing and Aguadé-Zabrodsky spaces.

### 4.1 Clark-Ewing spaces

A Clark-Ewing space is a $p$-compact group $X$ such that the order of the Weyl group is prime to $p$. Let $\hat{T}$ be a maximal torus of $X$ and $W \leqslant GL(\pi_1(\hat{T}))$ the Weyl group endowed with a faithful $p$-adic representation. $W$ acts on $B\hat{T}$ via the above representation. The Clark-Ewing spaces are homotopy equivalent to the Borel construction $X = (B\hat{T} \times_W EW)^\wedge_p$.

Since the Weyl group has order prime to $p$, a discrete approximation $T$ to $\hat{T}$ is a maximal discrete $p$-toral subgroup of $X$. Since $T$ is abelian, the fusion system is determined by $\mathrm{Aut}_{\mathcal{F}}(T)$, which equals the Weyl group $W$ of $X$. It follows that the only $\mathcal{F}$-centric radical subgroup of $T$ is itself and so the depth of $\mathcal{O}(\mathcal{F}^{cr})$ is zero. By Corollary 3.17, there is a unitary embedding of $BX$.

### 4.2 Aguadé-Zabrodsky spaces

Let $G_i$ be one of the groups $G_{12}$, $G_{29}$, $G_{31}$, $G_{34}$ from [A89]. In this article it is shown that these groups have a subgroup isomorphic to $\Sigma_{p_i}$, where $p_{12} = 3$, $p_{29} = p_{31} = 5$ and $p_{34} = 7$. Let $\mathbb{I}_i$ be the category with two objects 0 and 1 and morphisms $\mathrm{Hom}_{\mathbb{I}_i}(0, 0) = G_i$, $\mathrm{Hom}_{\mathbb{I}_i}(1, 0) = G_i/\Sigma_{p_i}$, $\mathrm{Hom}_{\mathbb{I}_i}(1, 1) = Z(G_i)$ and $\mathrm{Hom}_{\mathbb{I}_i}(0, 1) = \emptyset$. Consider the functors $F'_i : \mathbb{I}_i \to \mathrm{HoTop}$ defined by $F'_i(0) = BSU(p_i)$, $F'_i(1) = BT^{p_i-1}$ and $Z(G_i)$ acts via unstable Adams' operations. These functors lift to a functor $F_i : \mathbb{I}_i \to \mathrm{Top}$ and the Aguadé-Zabrodsky $p$-compact groups $X_{12}$, $X_{29}$, $X_{31}$ and $X_{34}$ are given by:

$$BX_i = (\operatorname*{hocolim}_{\overrightarrow{\mathbb{I}_i}} F_i)^\wedge_p.$$

Note that $X_{12}$ and $X_{31}$ were first described in [Z84] using other techniques.

The Weyl group of $X_i$ is $G_i$, which has $\mathbb{Z}/p_i$ as its $p_i$-Sylow subgroup. Therefore a maximal discrete $p$-toral subgroup of $X_i$ is the semidirect product $S_i$ of $T_i = (\mathbb{Z}/p_i^\infty)^{p_i-1}$ and $\mathbb{Z}/p_i$, where

$\mathbb{Z}/p_i$ acts on $(\mathbb{Z}/p_i^\infty)^{p_i-1}$ via the inclusion $\mathbb{Z}/p_i \to \Sigma_{p_i}$ that sends 1 to the $p_i$-cycle $(1\dots p_i)$. Let $\sigma$ be a generator of $\mathbb{Z}/p_i$. Let us denote by $\mathcal{F}_i$ and $\mathcal{L}_i$ the fusion system and centric linking systems corresponding to $X_i$.

The subgroup $S_i$ is $\mathcal{F}_i$-centric radical and since $T_i$ is self-centralized, it is $\mathcal{F}_i$-centric. Now $\mathrm{Out}_{\mathcal{F}_i}(T_i) = \mathrm{Aut}_{\mathcal{F}_i}(T_i) = G_i$ has a $p_i$-Sylow subgroup of rank 1 which is not normal in $G_i$ and so $T_i$ is $\mathcal{F}_i$-centric radical. The center of $S_i$ is the subgroup generated by $(\xi_i,\dots,\xi_i) \in T$ where $\xi_i$ is a $p_i$th root of unity.

In order to apply the results of Section 3, we need to find the centric radical subgroups of the Aguadé-Zabrodsky spaces. We will do so by showing that they coincide with the centric radical subgroups of the centralizer fusion system of $Z(S_i)$ and identifying this centralizer with the fusion system of $SU(p_i)$, for which they are known.

Lemma 4.1. *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$, and $Q$ a fully centralized subgroup of $S$. Then a subgroup $P \leqslant C_S(Q)$ is $C_{\mathcal{F}}(Q)$-centric if and only if $P \geqslant Z(Q)$ and $PQ$ is $\mathcal{F}$-centric.*

*Proof.* The proof of the analogous result for fusion systems over finite $p$-groups, that is, Proposition 2.5 (a) in [BLO03], is still valid in this context. □

Recall that if $X$ is a $p$-compact group and $(E,i)$ is a discrete $p$-toral subgroup of $X$, then $\mathrm{Map}(BE,BX)_i$ is the classifying space of a $p$-compact group $C_X(E)$ (see Proposition 5.1, Theorem 6.1 and Proposition 6.8 in [DW94]). If $E$ is a fully centralized subgroup of a $p$-local compact group $(S,\mathcal{F},\mathcal{L})$, it is not known in general whether $|C_{\mathcal{L}}(E)|^\wedge_p \simeq \mathrm{Map}(BE, |\mathcal{L}|^\wedge_p)_{Bi}$, but next proposition shows a particular case when this holds at least at the level of fusion systems.

Proposition 4.2. *Let $X$ be a $p$-compact group, $S$ a maximal discrete $p$-toral subgroup and $\mathcal{F}$ the associated fusion system over $S$. Let $E$ be a fully centralized subgroup of $Z(S)$. Then the fusion system $C_{\mathcal{F}}(E)$ coincides with the fusion system $\mathcal{G}$ of the $p$-compact group $C_X(E)$ over $C_S(E)$.*

*Proof.* Note that both fusion systems are defined over the same Sylow $C_S(E)$. We first show that they have the same centric subgroups. Let $P$ be centric in $C_{\mathcal{F}}(E)$, then by Lemma 4.1 we have that $P \geqslant E$ and $P$ is $\mathcal{F}$-centric. Since $P$ is $\mathcal{F}$-centric, there is a homotopy equivalence $\mathrm{Map}(BP,BX)_j \simeq BZ(P)$, where $j : BP \to BX$ is the standard inclusion. And then

$$\mathrm{Map}(BP, \mathrm{Map}(BE,BX)_i)_k \simeq \mathrm{Map}(BE, \mathrm{Map}(BP,BX)_j)_l \simeq \mathrm{Map}(BE, BZ(P))_m \simeq BZ(P)$$

where $i$, $k$, $l$ and $m$ are the standard inclusions. Therefore $P$ is centric in $\mathcal{G}$. Conversely, if $P$ is centric in $\mathcal{G}$, then it contains $Z(C_S(E))$ and so it contains $E$. There is a homotopy equivalence $\mathrm{Map}(BP, \mathrm{Map}(BE,BX)_i)_k \simeq BZ(P)$. Now note that

$$\mathrm{Map}(BP, \mathrm{Map}(BE,BX)_i)_k \simeq BC_{C_X(E)}(P)$$

and $E \leqslant P$ gives us a map $BC_X(P) \to BC_X(E)$, which in turn induces $BC_X(P) \simeq BC_{C_X(P)}(P) \to BC_{C_X(E)}(P)$. The inclusion $BC_X(E) \to BX$ gives rise to a map $BC_{C_X(E)}(P) \to BC_X(P)$, which is the homotopy inverse. Therefore

$$BZ(P) \simeq \mathrm{Map}(BP, \mathrm{Map}(BE,BX)_i)_k \simeq BC_{C_X(E)}(P) \simeq BC_X(P)$$

and so $P$ is centric in $X$.

Now we will show that the centric subgroups in $C_{\mathcal{F}}(E)$ and $\mathcal{G}$ have the same automorphism groups. Let $f$ be a $C_{\mathcal{F}}(E)$-automorphism of a centric subgroup $P$. Since $P$ contains $E$, this is an $\mathcal{F}$-automorphism of $P$ which restricts to the identity on $E$. And since $E$ is central in $P$, we have a diagram which is commutative up to homotopy:

$$\begin{tikzcd} BP \times BE \arrow[r] \arrow[d, "Bf\times\mathrm{id}"'] & BX \\ BP \times BE \arrow[ur] & \end{tikzcd}$$

Taking adjoints, we get another diagram, commutative up to homotopy:

$$\begin{tikzcd} BP \arrow[r] \arrow[d, "Bf"'] & \mathrm{Map}(BE, BX)_i \\ BP \arrow[ur] & \end{tikzcd}$$

which means that $f$ is a $\mathcal{G}$-automorphism of $P$. Conversely, a $\mathcal{G}$-automorphism $g$ of $P$ defines a homotopy commutative diagram as above whose adjoint shows that $g$ is a $C_{\mathcal{F}}(E)$-automorphism of $P$. By Alperin's fusion theorem, morphisms in saturated fusion systems are generated by automorphisms of centric subgroups, thus $\mathcal{G} = C_{\mathcal{F}}(E)$. □

The following lemma is a more general version of Lemma 3.8 in [CLN07] for $p$-local compact groups.

Lemma 4.3. *Let $\mathcal{F}$ be a saturated fusion system over a discrete $p$-toral group $S$, and $P \leqslant C_S(E)$, where $E$ be a fully centralized abelian subgroup of $S$. Then $P$ is centric in $C_{\mathcal{F}}(E)$ if and only if it is centric in $\mathcal{F}$.*

*Proof.* It is clear that centric subgroups in $\mathcal{F}$ are centric in $C_{\mathcal{F}}(E)$. Let $P$ be centric in $C_{\mathcal{F}}(E)$. By Lemma 4.1, we have $P \geqslant E$ and $PE = P$ is $\mathcal{F}$-centric. □

Lemma 4.4. *Let $\mathcal{F}_i$ be the fusion system over $S_i$ associated to the Aguadé-Zabrodsky space $X_i$. Let $Q \neq T_i$ be an $\mathcal{F}_i$-centric subgroup of $S_i$. Then $Q$ is radical in $C_{\mathcal{F}_i}(Z(S_i))$ if and only if it is radical in $\mathcal{F}_i$.*

*Proof.* If $Q$ is centric, $Q$ can not be contained in $T_i$. Therefore, $Q$ must be a semidirect product of $Q \cap T_i \leqslant T_i$ and $\mathbb{Z}/p_i$. Since $Q$ is centric, we must have $Z(S_i) \leqslant Q$ and so $Z(S_i) \leqslant Q \cap T_i$. Recall that $Z(S_i) \cong \mathbb{Z}/p_i$. If we had $Z(S_i) = Q \cap T_i$, then $Q = \mathbb{Z}/p_i \times \mathbb{Z}/p_i$, but in this case $Q$ is $\mathcal{F}$-subconjugate to $T_i$ and it can not be centric. Therefore $Z(S_i) \lneq Q \cap T_i$ and so $Z(Q) = (Q \cap T_i)^{\mathbb{Z}/p_i} = Z(S_i)$.

Restriction induces a map $\mathrm{Aut}_{\mathcal{F}_i}(Q) \to \mathrm{Aut}(Z(Q)) = \mathrm{Aut}(Z(S_i)) = \mathrm{Out}(Z(S_i))$, whose kernel is $\mathrm{Aut}_{C_{\mathcal{F}_i}(Z(S_i))}(Q)$. Since $\mathrm{Inn}(Q)$ is in the kernel of this map, there is an induced map $\mathrm{Out}_{\mathcal{F}_i}(Q) \to \mathrm{Out}(Z(S_i))$ with kernel $\mathrm{Out}_{C_{\mathcal{F}_i}(Z(S_i))}(Q)$. By Lemma 3.12 in [CLN07], if $Q$ is radical in $\mathcal{F}_i$, then it is radical in $C_{\mathcal{F}_i}(Z(S_i))$. On the other hand, if $Q$ is not radical in $\mathcal{F}_i$, let $B$ be a nontrivial normal $p$-subgroup of $\mathrm{Out}_{\mathcal{F}_i}(Q)$. Its restriction to $\mathrm{Out}(Z(S_i))$ must be trivial because $\mathrm{Out}(Z(S_i)) \cong \mathbb{Z}/(p_i - 1)$, therefore $B$ is a nontrivial normal $p$-subgroup of $\mathrm{Out}_{C_{\mathcal{F}_i}(Z(S_i))}(Q)$. □

Now we use Lemma 4.3 with $E = Z(S_i)$ and Lemma 4.4, and so the only other centric radical subgroups of $\mathcal{F}_i$ must be centric radical in the fusion system $C_{\mathcal{F}_i}(Z(S_i))$. This fusion system coincides with the fusion system of the $p$-compact group $C_{X_i}(Z(S_i))$ by Proposition 4.2.

By Example 4.14.(1) in [M99], $C_{N(T_i)}(Z(S_i))$ is the normalizer of the maximal torus of $C_{X_i}(Z(S_i))$. Since $p_i$ is odd, $N(T_i)$ is a semi-direct product and so $C_{N(T_i)}(Z(S_i))$ must be $T_i \rtimes W(SU(p_i)) = N(SU(p_i))$. Corollary 7.21 in [M02] says that polynomial $p$-compact groups are determined by their maximal torus normalizers and so the $p$-compact group $C_{X_i}(Z(S_i))$ is $SU(p_i)$.

From [O94], we know that the only other stubborn subgroup of $SU(p_i)$ is the subgroup $\Gamma_i \leqslant S_i$ generated by $Z(S_i)$, $(1, \xi_i, \xi_i^2, \ldots, \xi_i^{p_i-1})$ and $\sigma$. Therefore, the only centric radical subgroups for any of the Aguadé-Zabrodsky $p$-compact groups are $T_i$, $S_i$ and $\Gamma_i$. Hence the length of $\mathcal{O}(\mathcal{F}_i^{cr})$ is one. By Corollary 3.17, there is a unitary embedding of $BX_i$.

## 5. Unitary embeddings of $p$-local compact groups

A recent preprint [BLO13] shows that the $p$-completion of finite loop spaces are also modeled by $p$-local compact groups. More generally, if $f : X \to Y$ is a finite regular covering space, where $X$ is the classifying space of a $p$-local compact group, then $Y_p^\wedge$ is the classifying space of a $p$-local compact group.

Proposition 5.1. *Let $f : X \to Y$ be a finite regular covering space of a $p$-good space $Y$, and assume that $X$ has a unitary embedding at $p$. Then $Y_p^\wedge$ has a unitary embedding at $p$.*

*Proof.* Let $\pi$ be the group of deck transformations of the covering $f$ and let $n$ be the order of $\pi$. Let $h : X \to BU(N)_p^\wedge$ be a homotopy monomorphism at $p$. Consider the pretransfer $Y \to X^n \times_{\Sigma_n} E\Sigma_n$ (see Section 1 in [KP72]). We claim that the composition

$$\begin{array}{ccc}
Y & \longrightarrow & X^n \times_{\Sigma_n} E\Sigma_n \\
& & \big\downarrow {\scriptstyle h^n \times \mathrm{id}} \\
& & (BU(N)_p^\wedge)^n \times_{\Sigma_n} E\Sigma_n = (BU(N)^n)_p^\wedge \times_{\Sigma_n} E\Sigma_n \\
& & \big\downarrow \\
& & [(BU(N)^n)_p^\wedge \times_{\Sigma_n} E\Sigma_n]_p^\wedge \\
& & \big\downarrow {\scriptstyle \simeq} \\
& & B(U(N) \wr \Sigma_n)_p^\wedge \\
& & \big\downarrow \\
& & BU(Nn)_p^\wedge
\end{array}$$

is a homotopy monomorphism at $p$. This holds because all of the maps in the diagram are homotopy monomorphisms at $p$. To see that the horizontal map is a homotopy monomorphism at $p$ let us consider the following diagram which is commutative up to homotopy and where the rows are homotopy fibrations

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d] & X \arrow[r, "\Delta"] \arrow[d] & X^n \arrow[d] \\
F_2 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X^n \times_{\Sigma_n} E\Sigma_n \arrow[d] \\
\Sigma_n/\pi \arrow[r] & B\pi \arrow[r] & B\Sigma_n
\end{tikzcd}$$

The map $\Delta$ is a homotopy monomorphism at $p$ because the homotopy fiber is $(\Omega X)^{n-1}$ and $\Omega X$

is quasi-finite at $p$, which follows from the fibration sequence

$$\Omega X \to U(N)^\wedge_p \to F \to X \to BU(N)^\wedge_p$$

Therefore $F_1$ is quasi-finite at $p$ and since $\Sigma_n/\pi$ is finite, $F_2$ is also quasi-finite at $p$. Since $h$ is a homotopy monomorphism at $p$, so is the map $h^n \times \mathrm{id}$. The next two vertical maps are homotopy monomorphism because they are the $p$-completion of a simply-connected space of finite type and a homotopy equivalence, respectively. Finally the last map is a homotopy monomorphism at $p$ because it is induced by the monomorphism of groups $U(N) \wr \Sigma_n \to U(Nn)$.

The map $Y \to BU(Nn)^\wedge_p$ factors through $Y^\wedge_p$ and since $Y$ is $p$-good, this map is a homotopy monomorphism at $p$. □

Note that the following theorem depends on the existence of unitary embeddings for any $p$-compact group.

Theorem 5.2. *Let $BX$ be any path connected space such that $\Omega BX$ is $\mathbb{F}_p$-finite. Then $BX^\wedge_p$ has a unitary embedding at $p$. In particular, if $(X, BX, e)$ is a finite loop space, then $BX$ has a unitary embedding at $p$.*

*Proof.* Let $X = \Omega BX$. Since $X$ is $\mathbb{F}_p$-finite, $H_0(X;\mathbb{Z})$ is finite and therefore the group $\pi = \pi_1(BX) \cong \pi_0(X)$ is finite. So $BX$ is $p$-good by [BK72][VII.5]. Consider the universal cover $BZ$ of $BX$ which fits into a homotopy fibration sequence

$$BZ \to BX \to B\pi$$

Note that $Z = \Omega BZ$ is a connected component of $X$ and thus it is $\mathbb{F}_p$-finite. Applying fiberwise $p$-completion, we obtain a map of fibrations

$$\begin{tikzcd} BZ \arrow[r, "f"] \arrow[d] & BZ^\wedge_p \arrow[d, "j"] \\ BX \arrow[r, "g"] \arrow[d] & \overline{BX} \arrow[d, "q"] \\ B\pi \arrow[r] & B\pi \end{tikzcd}$$

But now $BZ^\wedge_p$ is $p$-complete and $\Omega BZ^\wedge_p$ is $\mathbb{F}_p$-finite. So $BZ^\wedge_p$ is the classifying space of a $p$-compact group. Since $\overline{BX}$ has a unitary embedding at $p$, so does $\overline{BX}^\wedge_p = BX^\wedge_p$ by Proposition 5.1. □

Now we study the two exotic 3-local compact groups constructed in [G10] from the exotic 3-local finite groups of [DRV07]. We will use the results in Section 3 to show the existence of unitary embeddings.

Consider the finite groups

$$S_k =< s, s_1, s_2 \mid s^3 = s_1^{3^k} = s_2^{3^k} = 1, [s_1, s_2] = 1, [s, s_1] = s_2, [s, s_2] = (s_1 s_2)^{-3} >$$

for each $k \geqslant 1$ and form the union $S = \cup_{k\geqslant 1} S_k$ with respect to the inclusions $S_k \to S_{k+1}$ that take $s$ to $t$ and $s_i$ to $t_i^3$, where $t$ and $t_i$ are the generators for the analogous presentation of $S_k$. The group $S$ is a discrete 3-toral group, and it is an extension of $T = (\mathbb{Z}/3^\infty)^2$ by $\mathbb{Z}/3$. Let $z_1$ and $z_2$ be the elements of order 3 in $T$ corresponding to $s_1$ and $s_2$ via the inclusion $S_1 \to S$. For what follows it will be convenient to consider the following two subgroups of $S$

$$E_0 =< z_2, s >$$

$$V_0 = < z_1, z_2, s >$$

Section 5.4 of [G10] shows the existence of two saturated fusion systems $\mathcal{F}(2)$ and $\mathcal{F}(3)$ over $S$, and respective centric linking systems $\mathcal{L}(2)$ and $\mathcal{L}(3)$. These fusion systems have three isomorphism classes of centric radical subgroups, two of which are given by $T$ and $S$. The third is given by the subgroup $E_0$ in the case of $\mathcal{F}(2)$ and $V_0$ in the case of $\mathcal{F}(3)$. Note that neither $E_0$ nor $V_0$ are subgroups of $T$, therefore the length of $\mathcal{O}(\mathcal{F}(i)^{cr})$ is one.

Corollary 5.3. *There exist unitary embeddings of* $(S, \mathcal{F}(2), \mathcal{L}(2))$ *and* $(S, \mathcal{F}(3), \mathcal{L}(3))$.

There is work in progress of A. González and A. Ruiz to construct generalizations of these 3-local compact groups for any other prime $p$, which by construction also satisfy that the length of the corresponding orbit categories is one.

## 6. Homological consequences

The existence of unitary embeddings of a $p$-local compact group $(S, \mathcal{F}, \mathcal{L})$ has consequences on the $p$-local cohomology and the Grothendieck ring of vector bundles of $|\mathcal{L}|^\wedge_p$, which we discuss in this short section.

Proposition 6.1. *If* $f : X \to BU(n)^\wedge_p$ *is a unitary embedding at* $p$ *such that the homotopy fiber* $F$ *is* $\mathbb{F}_p$*-finite, then* $H^*(X; \mathbb{Z}^\wedge_p)$ *is Noetherian.*

*Proof.* $H^*(X; \mathbb{F}_p)$ becomes a finitely generated $H^*(BU(M)^\wedge_p; \mathbb{F}_p)$-module via $f$ by a Serre spectral sequence argument. By the Nakayama lemma, since $H^j(F; \mathbb{Z}^\wedge_p) \otimes_{\mathbb{Z}^\wedge_p} \mathbb{F}_p$ is zero for $j$ large enough, the same happens for $H^j(F; \mathbb{Z}^\wedge_p)$. Using again the same Serre spectral sequence argument, $H^*(X; \mathbb{Z}^\wedge_p)$ becomes a finitely generated $H^*(BU(M)^\wedge_p; \mathbb{Z}^\wedge_p)$-module via $f$. Since $H^*(BU(M)^\wedge_p; \mathbb{Z}^\wedge_p)$ is Noetherian, so is $H^*(X; \mathbb{Z}^\wedge_p)$. $\square$

Corollary 6.2. *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a* $p$*-local compact group with a faithful* $\mathcal{F}$*-invariant representation that satisfies the hypothesis of Theorem 3.13. Then* $H^*(|\mathcal{L}|^\wedge_p; \mathbb{Z}^\wedge_p)$ *is Noetherian.*

*Proof.* By previous proposition, we only need to check that the fiber of a unitary embedding $f : |\mathcal{L}|^\wedge_p \to BU(n)^\wedge_p$ is $\mathbb{F}_p$-finite. Note that the restriction of $f$ to $\tilde{B}P$ for each $P \leqslant S$ is a monomorphism of $p$-compact groups and so the homotopy fiber $F_P$ is $\mathbb{F}_p$-finite. By Puppe's theorem [Pu74], the homotopy fiber $F$ of $f$ is the homotopy colimit of the spaces $F_P$ over the orbit category. This category has bounded limits at $p$ by Proposition 3.8, and so the Bousfield-Kan spectral sequence shows that $F$ is $\mathbb{F}_p$-finite. $\square$

Corollary 6.3. *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a* $p$*-local compact group such that at least one the following conditions hold:*

(i) $l(\mathcal{O}(\mathcal{F}^{cr})) < 3$.

(ii) $(S, \mathcal{F}, \mathcal{L})$ *models a finite loop space or a* $p$*-compact group.*

(iii) $(S, \mathcal{F}, \mathcal{L})$ *is one of the exotic* 3*-local compact groups of [G10].*

*Then* $H^*(|\mathcal{L}|^\wedge_p; \mathbb{Z}^\wedge_p)$ *is Noetherian.*

For a $p$-compact group, this is the main result in [DW94] (combined with [AC13]).

Let $\mathbb{K}(X)$ be the Grothendieck ring of vector bundles over $X$. We make now the observation that the maps

$$\Psi_n : [|\mathcal{L}|_p^\wedge, BU(n)_p^\wedge] \to \varprojlim_{\mathcal{O}(\mathcal{F}^c)} \mathrm{Rep}(P, U(n)) \cong \mathrm{Rep}^{\mathcal{F}}(S, U(n))$$

from Section 3 assemble to form a map $\Psi : \mathbb{K}(|\mathcal{L}|_p^\wedge) \to \varprojlim_{\mathcal{O}(\mathcal{F}^c)} R(P)$. We denote this limit by $R^{\mathcal{F}}(S)$ because it coincides with the Grothendieck ring of $\mathcal{F}$-invariant representations, as we show now. Given $\chi = \alpha_1 - \alpha_2$ in $R^{\mathcal{F}}(S)$, by Proposition 2.10, there is a faithful $\mathcal{F}$-invariant representation $\rho$ such that $\rho = \alpha_1 \oplus \beta$. But then

$$\chi = \alpha_1 - \alpha_2 = (\alpha_1 + \beta) - (\alpha_2 + \beta) = \rho - (\alpha_2 + \beta)$$

Since $\rho$ and $\chi$ are $\mathcal{F}$-invariant, so is $\alpha_2 + \beta$, and this proves our claim.

Theorem 6.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local compact group. If $l(\mathcal{O}(\mathcal{F}^{cr})) < 3$, then $\Psi$ is surjective. If $l(\mathcal{O}(\mathcal{F}^{cr})) < 2$, then $\Psi$ is an isomorphism.*

*Proof.* Let us assume $l(\mathcal{O}(\mathcal{F}^{cr})) < 3$ and let $\chi \in R^{\mathcal{F}}(S)$. Then $\chi = \alpha_1 - \alpha_2$, where $\alpha_i$ is an $\mathcal{F}$-invariant. By Proposition 2.10, there exist faithful $\mathcal{F}$-invariant representations $\rho_i$ such that $\alpha_i$ is a subrepresentation of $\rho_i$. Let $\rho = \rho_1 \oplus \rho_2$. By Theorem 3.13 and Remark 3.15, there is an integer $M_0$ such that $\alpha_i \oplus M\rho$ and $M\rho$ belong to the image of $\Psi$ if $M \geqslant M_0$. Then

$$\alpha_1 - \alpha_2 = (\alpha_1 + M_0\rho) - (\alpha_2 + 2M_0\rho) + M_0\rho$$

belongs to the image of $\psi$.

Now let $l(\mathcal{O}(\mathcal{F}^{cr})) < 2$ and consider $f, g : |\mathcal{L}|_p^\wedge \to BU(n)_p^\wedge$ such that $f_{|BS} \simeq g_{|BS} \simeq B\alpha$, where $\alpha$ is some $\mathcal{F}$-invariant representation of $S$. By Proposition 2.10, there is a faithful $\mathcal{F}$-invariant representation $\rho$ such that $\alpha$ is a subrepresentation of $\rho$. By Remark 3.15, there is an integer $N$ such $\alpha \oplus N\rho$ and $N\rho$ belong to the image of $\Psi$ and have a unique preimage. Let $N\rho = \Psi(h)$. Then $\Psi(f + h) = \Psi(g + h) = \alpha \oplus N\rho$, from where $f + h \simeq g + h$ and so $f = g$ in $\mathbb{K}(|\mathcal{L}|_p^\wedge)$. □

*Remark* 6.5. This theorem applies to the $p$-local compact groups which model the Clark-Ewing or the Aguadé-Zabrodsky $p$-compact groups and to the exotic 3-local compact groups of [G10].

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