ON EXTENDED CHEBYSHEV SYSTEMS WITH POSITIVE ACCURACY

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Abstract. A classical necessary condition for an ordered set of $n+1$ functions $\mathcal{F}$ to be an ECT-system in a closed interval is that all the Wronskians do not vanish. With this condition all the elements of Span($\mathcal{F}$) have at most n zeros taking into account the multiplicity. Here the problem of bounding the number of zeros of Span($\mathcal{F}$) is considered as well as the effectiveness of the upper bound when some Wronskians vanish. For this case we also study the possible configurations of zeros that can be realized by elements of Span($\mathcal{F}$). An application to count the number of isolated periodic orbits for a family of nonsmooth systems is performed.

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1. Introduction and statement of the main results

Let $\mathcal{F} = [u_0, \dots, u_n]$ be an ordered set of functions of class $\mathcal{C}^r$, $r \geq n$, on the closed interval $[a, b]$. We denote by $Z(\mathcal{F})$ the maximum number of zeros counting multiplicity that any nontrivial function $v \in \text{Span}(\mathcal{F})$ can have. Here $\text{Span}(\mathcal{F})$ is the set of functions generated by liner combinations of elements of $\mathcal{F}$, that is $v(s) = a_0 u_0(s) + a_1 u_1(s) + \cdots + a_n u_n(s)$ where a_i, for $i = 0, 1, \dots, n$, are real numbers.

The theory of Chebyshev systems is a classical tool to study the quantity $Z(\mathcal{F})$. In this theory, when $Z(\mathcal{F}) \leq n$, the set $\mathcal{F}$ is called an *extended Chebyshev system* or ET-system on $[a, b]$, see [7]. When the functions in $\mathcal{F}$ are linearly independent there always exists an element in $\text{Span}(\mathcal{F})$ with n zeros, see [10]. From this property $Z(\mathcal{F}) = n$, but in general we can not assure if they are simple or not. This problem will be addressed later. In [6], when $Z(\mathcal{F}) \leq n + k$, the set $\mathcal{F}$ is called an extended Chebyshev system with *accuracy* k on $[a, b]$. From this definition it is natural to consider the lowest possible k. Therefore there exists an element in $\text{Span}(\mathcal{F})$ with $n+k$ zeros and, consequently, $Z(\mathcal{F}) = n+k$. Indeed, this is the definition used in [4] and it is the one that we shall use throughout the present work.

In concrete problems this is not a useful definition in order to get if $\mathcal{F}$ is an ET-system, with accuracy or not. Following the book of Karlin and Studden [7] we can see that the condition $W(u_0, u_1, \dots, u_n)(t) \neq 0$ implies that $\mathcal{F}$ is an ET-system, the converse, in general, is not true. Here $W(u_0, u_1, \dots, u_k)(t)$ denotes the Wronskian of the ordered set of functions $[u_0, u_1, \dots, u_k]$ with respect to t. We recall the definition of the Wronskian of a set of functions:

$$W_k(t) = W_k(u_0, \dots, u_k)(t) = \det\big(M(u_0, \dots, u_k)(t)\big), \tag{1}$$

where

$$M(u_0, \dots, u_k)(t) = \begin{pmatrix} u_0(t) & \cdots & u_k(t) \\ u_0'(t) & \cdots & u_k'(t) \\ \vdots & \ddots & \vdots \\ u_0^{(k)}(t) & \cdots & u_k^{(k)}(t) \end{pmatrix}.$$

We say that $\mathcal{F}$ is an *Extended Complete Chebyshev* system or an ECT-system on a closed interval $[a, b]$ if and only if for any k, $0 \leq k \leq n$, $[u_0, u_1, \dots, u_k]$ is an ET-system. In order to prove that $\mathcal{F}$ is an ECT-system on $[a, b]$ it is sufficient and necessary to show that $W(u_0, u_1, \dots, u_k)(t) \neq 0$ on $[a, b]$ for $0 \leq k \leq n$, see [7]. Furthermore this property also provides that each configuration of $m \leq n$ zeros, taking into account their multiplicity, is realizable.

Initially Chebyshev systems were used in approximation theory in the study of spline functions and in the theory of fine moment, see [7] and [1] for more recent results on this field. Lately they where used in the theory of differential equations to study versal unfoldings of singularities of vector fields, see [12, 17]. Recently it has also been used to study the period function of centers of potential systems, see [11]. In the qualitative theory of differential equations, ECT-systems are used to study the number of isolated periodic orbits (limit cycles) bifurcating from a period annulus, see also [17]. More concretely, this technique is useful to get upper bounds for the number of zeros of the Poincaré–Pontryaguin–Melnikov function. In fact these studies provided lower bounds for the so called weak Hilbert 16th problem, see [2, 18].

In the context of Chebyshev systems, a possible issue to be considered consists in to obtain exactly the number $Z(\mathcal{F})$. Since this problem is very difficult to be solved in general, it can be split in three distinct related problems: the first one consists in estimating lower and upper bounds for $Z(\mathcal{F})$, the second one deals with the effectiveness of these bounds, and the third one studies the possible configurations of zeros that can be realized by the elements of $\mathrm{Span}(\mathcal{F})$. As we have commented above, these three problems are solved when the set of functions $\mathcal{F}$ is an ECT-system. Nevertheless when $\mathcal{F}$ is not an ECT-system, as far as we know, there are no well developed tools to deal with these problems. So, taking into account all the possible applications given above, our main goal in this paper consists in providing similar results for systems which are not ECT. Meaning, to give lower and upper bound for $Z(\mathcal{F})$, and to establish if these bounds are optimal when some of the Wronskians of $\mathcal{F}$ vanish. In particular Corollary 1.4 provides a complete answer for these problems when all the Wronskians are nonvanishing except the last one which has exactly one simple zero.

The upper bounds for the number of zeros of the elements in $\mathrm{Span}(\mathcal{F})$, either counting multiplicity or not, have been treated in several works. For instance in [16], for an ordered set $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ of sufficiently smooth functions, it was proved that the number of isolated zeros of any element of $\mathrm{Span}(\mathcal{F})$ does not exceed

$$n+\hat{\nu}_n+\hat{\nu}_{n-1}+2\hat{\nu}_{n-2}+2\hat{\nu}_{n-3}+3\hat{\nu}_{n-4}+4\hat{\nu}_{n-5}+\cdots+(n-j-1)\nu_j+\cdots+(n-1)\hat{\nu}_0, \quad (2)$$

where $\hat{\nu}_i$ is the number of zeros of the Wronskian W_i, for $i = 0, \ldots, n$. Later in [13], assuming the analyticity of the functions in $\mathcal{F}$, it was proved that

$$Z(\mathcal{F}) \leq n + \nu_n + 3\nu_{n-1} + 4(\nu_{n-2} + \nu_{n-3} + \cdots + \nu_0). \quad (3)$$

where ν_i is the number of zeros of the Wronskian W_i, for $i = 0, \ldots, n$, taking into account the multiplicity. Even in the case that the zeros of the Wronskians are simple ($\hat{\nu}_i = \nu_i$) one cannot decide in general, between (2) and (3), which is the best upper bound for $Z(\mathcal{F})$. Our first result provides a better estimation for it in this case.

Theorem 1.1. *Let $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ be an ordered set of analytic functions on $[a, b]$. Assume that all the ν_i zeros of the Wronskian W_i are simple for $i = 0, \ldots, n$. Then*

$$Z(\mathcal{F}) \leq n + \nu_n + \nu_{n-1} + 2(\nu_{n-2} + \cdots + \nu_0) + \mu_{n-1} + \cdots + \mu_3, \quad (4)$$

where $\mu_i = \min(2\nu_i\,,\, \nu_{i-3} + \cdots + \nu_0)$, for $i = 3, \ldots, n-1$.

An improvement of both upper bounds is given, for example, when $n = 4$, $\nu_0 \neq 0$, and $2\nu_3 < \nu_0$, that is $\mu_3 = 2\nu_3$. In this case Theorem 1.1 provides the upper bound $Z(\mathcal{F}) \leq 4 + \nu_4 + 3\nu_3 + 2\nu_2 + 2\nu_1 + 2\nu_0$, which is strictly less than the upper bounds given by (2) and (3). As a second example consider $\hat{\nu}_i = \nu_i = 1$, for $i = 0, 1, \ldots, n \geq 3$. In this case the upper bounds (2) and

(3) are, respectively, $(n^2+n+6)/2$ and $5n$, and the upper bound provided by Theorem 1.1 is $4n-3$.

We remark that [13, 16] also deal with sets of periodic functions. In this case the upper bounds given in (2), (3), and (4) are decreased by n. However in this paper we shall concentrate our attention only in the nonperiodic case.

In the next result we establish the optimality of Theorem 1.1 when all the Wronskians are nonvanishing except $W_{n-1}(x)$ and $W_n(x)$, which have $\nu_{n-1}=k$ and $\nu_n=\ell$ zeros on (a,b), respectively.

Theorem 1.2. *Let n, k, and ℓ be nonnegative integers. There exists an ordered polynomial set, $\mathcal{F}=[u_0,u_1,\ldots,u_n]$, such that all the Wronskians are nonvanishing except $W_{n-1}(x)$ and $W_n(x)$, which have k and ℓ zeros on (a,b), respectively, and with an element in* $\mathrm{Span}(\mathcal{F})$ *having exactly $n+k+\ell$ simple zeros. In particular $Z(\mathcal{F})=n+k+\ell$.*

In [3] it is proved that for a family of $n+1$ linearly independent analytical functions, such that at least one of that has constant sign in its domain, there exists a linear combination of these functions having at least n simple zeros. The following theorem extends this result showing that for each configuration of $m\leq n$ zeros, taking into account their multiplicity, there exists a linear combination of those function having this configuration. Moreover, allowing the last Wronskian $W_n(x)$ to vanish, we provide sufficient conditions which assure the same result but now for each configuration of $m\leq n+1$ zeros. In Section 6 we give some examples showing that these lower bounds cannot be improved in general.

Theorem 1.3. *Let $\mathcal{F}=[u_0,u_1,\ldots,u_n]$ be an ordered set of real $\mathcal{C}^\infty$ functions on (a,b) such that there exists $\xi\in(a,b)$ with $W_{n-1}(\xi)\neq 0$. Then the next properties hold:*

- (a) *If $W_n(\xi)\neq 0$ then for each configuration of $m\leq n$ zeros, taking into account their multiplicity, there exists $F\in\mathrm{Span}(\mathcal{F})$ with this configuration of zeros.*
- (b) *If $W_n(\xi)=0$ and $W_n'(\xi)\neq 0$ then for each configuration of $m\leq n+1$ zeros, taking into account their multiplicity, there exists $F\in\mathrm{Span}(\mathcal{F})$ with this configuration of zeros.*

As an immediate consequence of Theorems 1.1 and 1.3 we have:

Corollary 1.4. *Let $\mathcal{F}=[u_0,u_1,\ldots,u_n]$ be an ordered set of $\mathcal{C}^\infty$ functions on $[a,b]$. Assume that all the Wronskians are nonvanishing except $W_n(x)$, which has exactly one simple zero on (a,b), then $Z(\mathcal{F})=n+1$. Moreover, for any configuration of $n+1$ zeros there exists an element in* $\mathrm{Span}(\mathcal{F})$ *realizing it.*

In [4] it is proved that the set $[1,x,\sqrt{x+1},x\sqrt{x+1},\sqrt{x},x\sqrt{x},x^2\sqrt{x}]$ is an ET-system with accuracy 1 on $(0,\infty)$. This fact can be obtained from Corollary 1.4 because all the ordered Wronskians are nonvanishing except the last one that has exactly one positive zero taking as the interval of definition any closed interval in $(0,\infty)$. Another set considered in [4] is $\mathcal{F}=[\cup_{i=0}^{2k-1}\{\sqrt{x+a_i}\}\cup_{i=k}^{2k-1}\{x\sqrt{x+a_i}\}]$. In that paper, taking $k=3$ for example, the authors prove that $Z(\mathcal{F})\leq 4k-1=11$. For the concrete values $a_0=0,a_1=1,a_2=3,a_3=5,a_4=$

$2, a_5 = 7$, this upper bound can be decreased by one from Theorem 1.1 because all the Wronskians are nonvanishing except W_7 and W_8 that vanish exactly once. Moreover, with a new order (permuting the last two elements) only the last Wronskian vanishes exactly once. So, from Corollary 1.4, this family is an ET-system with accuracy 1, and therefore $Z(\mathcal{F}) = 9$. Another application of Corollary 1.4 can be found in [8] where the considered set of functions is also an ET-system with accuracy 1.

This paper is organized as follows. In Section 2 we recall some useful definitions and results about Wronskians. In Section 3 we prove Theorem 1.1 which provides an upper bound for $Z(\mathcal{F})$. The effectiveness of these upper bounds, see Theorem 1.2, are studied in Section 4. The lower bounds for $Z(\mathcal{F})$ given by Theorem 1.3 are proved in Section 5. In Section 6 we study the optimality of the bounds given by Theorem 1.3. Finally in Section 7, as a nontrival application of the above results, we improve the results of [9] where the maximum number of limit cycles for a class of nonsmooth systems is studied. Here we prove that this maximum is three.

2. Preliminaries

In this section we recall some relations between a set of functions and their Wronskians (see [7, sec. 2 chap. XI]). In particular, we link them with the Division-Derivation algorithm (see [17, p. 119]).

Let $w_0, w_1, \ldots, w_n$ be nonidentically zero functions such that w_i is of class $\mathcal{C}^{n-i}$ on $[a, b]$. If we define

$$\begin{aligned} u_0(x) &= w_0(x), \\ u_1(x) &= w_0(x)\int_a^x w_1(s_1)ds_1, \\ u_2(x) &= w_0(x)\int_a^x w_1(s_1)\int_a^{s_1} w_2(s_2)ds_2ds_1, \\ &\vdots \\ u_n(x) &= w_0(x)\int_a^x w_1(s_1)\int_a^{s_1} w_2(s_2)\cdots w_{n-1}(s_{n-1})\int_a^{s_{n-1}} w_n(s_n)ds_n\cdots ds_1, \end{aligned} \tag{5}$$

then straightforward calculations establishes a first relation between the ordered set $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ with their Wronskians,

$$W_k(x) = W(u_0, u_1, \ldots, u_k)(x) = (w_0(x))^{k+1}(w_1(x))^k \cdots (w_{k-1}(x))^2(w_k(x)) \tag{6}$$

for $k = 0, 1, \ldots, n$. These expressions write, recurrently, as

$$w_0(x) = W_0(x), \quad w_1(x) = \frac{W_1(x)}{(W_0(x))^2}, \quad \text{and} \quad w_k(x) = \frac{W_{k-2}(x)W_k(x)}{(W_{k-1}(x))^2}, \tag{7}$$

for $k = 2, 3, \ldots, n$. We remark that the functions $\{u_0, u_1, \ldots, u_n\}$ are linearly independent. By introducing the differential operators

$$D_j v = \partial \frac{v}{w_j}, \quad \text{for} \quad j = 0, 1, \ldots, n, \tag{8}$$

the Division-Derivation algorithm for the functions u_i, defined in (5), give us the second relation

$$w_{j+1} = D_j D_{j-1} \cdots D_0\, u_{j+1}, \quad \text{for} \quad j = 0, 1, \ldots, n-1. \tag{9}$$

Here the operator ∂ denotes the derivative in the variable x. Furthermore for any set of $\mathcal{C}^n$ linearly independent functions $u_0, u_1, \ldots, u_n$, if we take (9) as the definition of the functions w_i, for $i = 0, 1, \ldots, n$, then the equality (6) holds.

We recall that when all the Wronskians W_k are nonvanishing the set $\mathcal{F}$ is an ECT-system. The proof follows because at each step of the Division-Derivation algorithm, see (9), we have $j+1$ derivatives. Hence, from Rolle's Theorem, the function u_{j+1} have at most j zeros more than w_{j+1} for $j = 0, 1, \ldots, n-1$. When some of the Wronskians vanish this problem can be also studied with the same procedure but we need to take into account the appearance of vertical asymptotes in some of the steps.

3. Upper bounds of the number of zeros

This whole section is devoted to prove Theorem 1.1. The key point of the proof consists in mixing the ideas of the works [13, 16] to obtain a best upper bound for $Z(\mathcal{F})$. The proofs of the upper bounds (2) and (3) are both based on the fact that each element $f \in \operatorname{Span}(\mathcal{F})$ satisfies the Frobenius formula (see [15] for a proof of it):

$$\frac{W_{n-1}^2}{W_n W_{n-2}} \cdot \partial \cdots \partial \cdot \frac{W_1^2}{W_2 W_0} \cdot \partial \cdot \frac{W_0^2}{W_1} \cdot \partial \cdot \frac{1}{W_0}\ f = k,$$

which can be transformed (see again [16]) into

$$\Delta_{n-1} \cdot \frac{1}{W_{n-3}} \cdot \Delta_{n-2} \cdots \Delta_2 \cdot \frac{1}{W_0} \cdot \Delta_1 \cdot \Delta_0 \cdot f = k W_{n-2} W_n. \tag{10}$$

Here $k \neq 0$ is a real constant and $\Delta_j = W_j^2 \cdot \partial \cdot (W_j)^{-1}$, for $i = 0, 1, \ldots, n$.

Following [16], we take $F_0 = f$, $F_1 = \Delta_0 F_0$, and $F_2 = \Delta_1 F_1$, and then we define recurrently $F_j = \Delta_{j-1} W_{j-3}^{-1} F_{j-1}$, for $j = 3, \ldots, n$. From (10) we have that $F_n = k W_{n-2} W_n$. We note that the function F_j, for $j \geq 3$, may admits poles, and that the number of poles does not exceed $\nu_{n-3} + \nu_{n-4} + \ldots + \nu_0$. We recall that, as the functions are real, the poles are the vertical asymptotes. Let $N(\cdot)$ denote the sum of the number of zeros and poles of a function. An adaptation of the Holder inequality, in order to allow poles, has been stated in [16]. It says that, if $g : I \to \mathbb{R}$ is smooth except for a finite number p of poles, then $N(g) \leq N(\Delta_j g) + \nu_j + p + 1$. This result provides the following list of inequalities:

$$\begin{aligned}
&N(F_1) \geq N(F_0) - 1 - \nu_0, \\
&N(F_2) \geq N(F_1) - 1 - \nu_1, \\
&N(F_3) \geq N(F_2) - 1 - \nu_2 - \nu_0, \\
&N(F_j) \geq N(F_{j-1}) - 1 - \nu_{j-1} - \nu_{j-3} - (\nu_{j-4} + \cdots + \nu_0), \ \text{for } j = 4, \ldots, n.
\end{aligned} \tag{11}$$

The upper bound (2) can be obtained by adding up all the above inequalities and using that $N(F_n) = N(W_n W_{n-2}) \leq \nu_n + \nu_{n-2}$.

In [13] it was used the Rolle–Voohoeve inequalities in order to establish the upper bound (3). Given two functions $g, h : I \to \mathbb{R}$, which are analytic except for a finite number of poles, the Rolle–Voohoeve inequalities state that $N(g) \leq N(\partial g) + 1$, $|N(g) - N(h)| \leq N(g\,h) \leq N(g) + N(h)$, and $N(1/g) = N(g)$. We may use them to provide another list of inequalities, similar to (11).

$$\begin{array}{l} N(F_1) \geq N(F_0) - 1 - 3\nu_0, \\ N(F_2) \geq N(F_1) - 1 - 3\nu_1, \\ N(F_3) \geq N(F_2) - 1 - \nu_2 - \nu_0 - 2\nu_2, \\ N(F_j) \geq N(F_{j-1}) - 1 - \nu_{j-1} - \nu_{j-3} - 2\nu_{j-1}, \text{ for } j = 4, \ldots, n. \end{array} \tag{12}$$

Analogously, the upper bound (3) can be obtained by adding up all the above inequalities and using again that $N(F_n) = N(W_n W_{n-2}) \leq \nu_n + \nu_{n-2}$.

Joining the two lists, (11) and (12), we are able to produce a third list by taking the best lower bound of each one:

$$\begin{array}{l} N(F_1) \geq N(F_0) - 1 - \nu_0, \\ N(F_2) \geq N(F_1) - 1 - \nu_1, \\ N(F_3) \geq N(F_2) - 1 - \nu_2 - \nu_0, \\ N(F_j) \geq N(F_{j-1}) - 1 - \nu_{j-1} - \nu_{j-3} - \mu_{j-1}, \text{ for } j = 4, \ldots, n, \end{array} \tag{13}$$

where $\mu_i = \min(2\nu_i\,,\, \nu_{i-3} + \cdots + \nu_0)$, for $i = 3, \ldots, n$. The proof finishes adding up all the above inequalities and using that $N(F_n) = N(W_n W_{n-2}) \leq \nu_n + \nu_{n-2}$.

4. Effectiveness of some upper bounds

In this section the proof of Theorem 1.2 will follow directly from Propositions 4.1, 4.2, and 4.3.

Proposition 4.1. *Let n and ℓ be nonnegative integers. There exists an ordered polynomial set, $\mathcal{F} = [u_0, u_1, \ldots, u_n]$, such that all the Wronskians are nonvanishing except $W_n(x)$, which has exactly ℓ simple zeros and with an element in* Span$(\mathcal{F})$ *having exactly $n + \ell$ simple zeros. In particular $Z(\mathcal{F}) = n + \ell$.*

Proof. Let α_i, $i = 0, \ldots, \ell - 1$, be real numbers and take $w_i(x) = 1$ for $i = 0, \ldots, n - 1$, and

$$w_n(x) = x^\ell + \sum_{i=0}^{\ell-1} \alpha_i x^i.$$

Consequently, from (6) and (5) we get $W_i(x) = w_i(x)$, for $i = 0, \ldots, n$, and

$$u_i(x) = x^i, \quad \text{for} \quad i = 0, \ldots, n-1, \quad \text{and} \quad u_n(x) = \sum_{i=0}^{\ell} \frac{i!}{(n+i)!} \alpha_i x^{i+n},$$

respectively. These functions define the ordered set $\mathcal{F} = [u_0, u_1, \ldots, u_n]$. Therefore, the function

$$v = \sum_{j=0}^{n-1} a_j u_j(x) + u_n(x) = \sum_{j=0}^{n+\ell} b_j\, x^j, \tag{14}$$

is in Span($\mathcal{F}$), where

$$b_j = \begin{cases} a_j, & \text{for} \quad j = 0, 1, \ldots, n-1, \\ \dfrac{(j-n)!}{j!}\alpha_{j-n}, & \text{for} \quad j = n, \ldots, n+\ell. \end{cases}$$

We note that (14) is a full polynomial of degree $n+\ell$ in x for which the parameters b_j can be chosen in an arbitrary way. Clearly there exists a function v with exactly $n+\ell$ simple zeros. It remains to prove that the function W_n has exactly ℓ zeros. If W_n has less distinct zeros, $\kappa < \ell$, applying the upper bound (2) we have that the function v has at most $n+\kappa < n+\ell$ isolated zeros, which contradicts the conclusion about the number of zeros of v. Therefore W_n has ℓ disctinc zeros, and from its degree we conclude that the zeros are simple. □

Proposition 4.2. *Let n and k be nonnegative integers. There exists an ordered polynomial set, $\mathcal{F} = [u_0, u_1, \ldots, u_n]$, such that all the Wronskians are nonvanishing except $W_{n-1}(x)$, which has exactly k simple zeros, and with an element in* Span($\mathcal{F}$) *having exactly $n+k$ simple zeros. In particular $Z(\mathcal{F}) = n+k$.*

Proof. The proofs follows by adding a new polynomial to the family given in Proposition 4.1 in such a way the last Wronskian is nonvanishing. Then we shall show that there is an element in the span of this family with the prescribed zeros of the statement.

Consider the ordered set of polynomials $[u_0, \ldots, u_{n-1}]$ given in the proof of Proposition 4.1. So $W_i(x) = w_i(x) = 1$, for $i = 0, \ldots, n-2$, and

$$W_{n-1}(x) = w_{n-1}(x) = x^k + \sum_{i=0}^{k-1} \alpha_i x^i = \prod_{j=1}^{k} (x - \xi_j), \tag{15}$$

where ξ_j, for $j = 1, \ldots k$, are the simple zeros of W_{n-1}. Now, we obtain the function u_n by taking the last nonvanishing Wronskian

$$W_n(x) = \prod_{j=1}^{k} (x-\xi_j)^2 \left(1 + \sum_{i=1}^{k} \frac{1}{(x-\xi_i)^2}\right) = \prod_{j=1}^{k} (x-\xi_j)^2 + \sum_{i=1}^{k} \prod_{j\neq i}^{k} (x-\xi_j)^2. \tag{16}$$

Then using (5) and (7), we compute

$$u_n(x) = \frac{(k+1)!}{(n+k)!} x^{n+k} + \alpha_{k-1} \frac{k!}{(n+k-1)!} x^{n+k-1} + \sum_{j=1}^{k-1} (\alpha_{j-1} - B_j) x^{j+n-1} - B_0 x^{n-1},$$

where $B_j = \sum_{i=1}^{k} \beta_j^i$ and $\prod_{j\neq i}^{k} (x-\xi_j) = \sum_{j=0}^{k-1} \beta_j^i x^j$.

Let $\mathcal{F} = [u_0, u_1, \ldots, u_n]$ and let $v_\varepsilon \in$ Span($\mathcal{F}$) be given by

$$v_\varepsilon(x) = v_0(x) - \varepsilon\, u_n(x), \tag{17}$$

where v_0 is a monic polynomial of degree $n+k-1$ with $n+k-1$ simple zeros provided by Proposition 4.1. Hence, for $\varepsilon > 0$ small enough, we conclude that v_ε has $n+k-1$ simple zeros close to the zeros of v_0 and, because of the degree of u_n and the sign of its coefficient, another zero which bifurcates from infinity. Moreover this zero is simple because the degree of v_ε coincides with the number of zeros. □

Proposition 4.3. *Let n, k, and ℓ be nonnegative integers. There exists an ordered polynomial set, $\mathcal{F} = [u_0, u_1, \ldots, u_n]$, such that all the Wronskians are nonvanishing except $W_{n-1}(x)$ and $W_n(x)$, which have k and ℓ simple zeros on (a, b), respectively, and with an element in* $\mathrm{Span}(\mathcal{F})$ *having exactly $n + k + \ell$ simple zeros. In particular $Z(\mathcal{F}) = n + k + \ell$.*

Proof. The proof follows by perturbing the family given in Proposition 4.2 in such a way the last Wronskian has ℓ simple zeros. Then we shall show that there is an element in the span of this family with the prescribed zeros of the statement.

Firstly, consider the ordered set of polynomials $\mathcal{F}_0 = [u_0, \ldots, u_n]$ given in the proof of Proposition 4.2. Denote W_n^0 the last wronskian of the family $\mathcal{F}_0$ given in (16), which has $2k$ complex zeros away from the real line in the complex plane. Let $\epsilon > 0$ small and $u(x)$ a polynomial. Using (6), (8), and (9), the last Wronskian of $\mathcal{G}_\epsilon = [u_0, \ldots, u_n + \epsilon u]$, which is a perturbation of $\mathcal{F}_0$, writes

$$W_n(x) = W_n^0 + \epsilon\big(u^{(n)}(x)w_{n-1}(x) - u^{(n-1)}(x)w'_{n-1}(x)\big), \tag{18}$$

where $w_{n-1}(x)$ is given in (15).

Now, let $v_\varepsilon \in \mathrm{Span}(\mathcal{F}_0)$ be the function (17) satisfying that for some fixed $\varepsilon > 0$ it has $n+k$ simple zeros. We know that $-\varepsilon(k+1)!/(n+k)! < 0$ is the coefficient of the leading term x^{n+k} of v_ε. Now we define $\mathcal{F}_1 = [u_0, \ldots, u_n - \epsilon_1 x^{n+k+1}]$. Clearly, $v^1_{\epsilon_1} = v_\varepsilon + \varepsilon\, \epsilon_1\, x^{n+k+1} \in \mathrm{Span}(\mathcal{F}_1)$. Hence, analogously to the proof of Proposition 4.2, we conclude that, for $\epsilon_1 > 0$ small enough, $v^1_{\epsilon_1}$ has $n+k$ simple zeros close to the simple zeros of v_ε and another zero, which bifurcates from the infinity. Moreover this additional zero is simple because of the degree of $v^1_{\epsilon_1}$. Let us see that the last Wronskian W_n of $\mathcal{F}_1$ has exactly one simple zero. From (18) the Wronskian W_n reads

$$W_n(x) = W_n^0 + \epsilon_1 \left(-2\frac{(n+k+1)!}{(k+2)!}x^{2k+1} + \mathcal{P}_1(x)\right),$$

where $\mathcal{P}_1$ is a polynomial of degree less than $2k + 1$. Since 1 is the coefficient of leading term x^{2k} of W_n^0 we conclude, just like before, that for $\epsilon_1 > 0$ small enough the Wronskian W_n has $2k$ complex zeros away from the real line and close to the complex zeros of W_n^0 and another zero, which bifurcates from the infinity. Again this additional zero is simple, now because of the degree of W_n.

The above procedure can be repeated in order to construct the ordered set of polynomials

$$\mathcal{F}_i = [u_0, \ldots, u_n - \epsilon_1 x^{n+k+1} + \epsilon_2 x^{n+k+2} - \cdots + (-1)^i \epsilon_i x^{n+k+i}],$$

and an element $v^i_{\epsilon_i} = v^{i-1}_{\epsilon_{i-1}} - (-1)^i\, \varepsilon\, \epsilon_i x^{n+k+i}$ in $\mathrm{Span}(\mathcal{F}_i)$, for $i = 2, 3, \ldots, \ell$, having exactly $n + k + i$ simple zeros in such way that the last Wronskian W_n of $\mathcal{F}_i$ has exactly i simple zeros. This proof ends by taking $\mathcal{F} = \mathcal{F}_\ell$. □

5. Lower bounds of the number of zeros

This section is completely devoted to prove Theorem 1.3. Firstly, we will look for an element in $\mathrm{Span}(\mathcal{F})$ with a zero of the highest multiplicity. Secondly, we will perturb it inside $\mathrm{Span}(\mathcal{F})$ in order to have the prescribed configuration of zeros. We point out that the first part is common for both statements but the second is not.

As $W_{n-1}(\xi)\neq 0$ there exists a unique function $F_0(x)=\sum_{i=0}^n a_i u_i(x)$ in $\mathrm{Span}(\mathcal{F})$ such that $F_0(\xi)=0$, $F_0^{(i)}(\xi)=0$, for $i=1,\dots,n-1$, and $a_n=1$. The coefficients a_i's can be obtained from the linear system of equations

$$\begin{pmatrix} u_0(\xi) & \cdots & u_{n-1}(\xi)\\ u_0'(\xi) & \cdots & u_{n-1}'(\xi)\\ \vdots & \ddots & \vdots\\ u_0^{(n-1)}(\xi) & \cdots & u_{n-1}^{(n-1)}(\xi)\end{pmatrix}\begin{pmatrix} a_0\\ a_1\\ \vdots\\ a_{n-1}\end{pmatrix}=-\begin{pmatrix} u_n(\xi)\\ u_n'(\xi)\\ \vdots\\ u_n^{(n-1)}(\xi)\end{pmatrix}.$$

Using the Cramer's rule we get

$$\begin{aligned} a_i &= -\frac{W_{n-1}(u_0,\dots,\overset{(i)}{u_n},\dots,u_{n-1})(\xi)}{W_{n-1}(\xi)}\\ &= (-1)^{n-i}\frac{W_{n-1}(u_0,\dots,\widehat{u}_i,\dots,u_n)(\xi)}{W_{n-1}(\xi)}.\end{aligned} \tag{19}$$

The notation $(u_0,\dots,\widehat{u}_i,\dots,u_n)$ means that the element u_i is removed. This concludes the first part of the proof.

From the Leibniz formula for determinants we can express the Wronskian $W_n(\xi)$ and its derivative as

$$\begin{aligned} W_n(\xi) = &\sum_{i=0}^{n}(-1)^{n-i}u_i^{(n)}(\xi)W_{n-1}(u_0,\dots,\widehat{u}_i,\dots,u_n)(\xi),\\ W_n'(\xi) = &\begin{vmatrix} u_0(\xi) & \cdots & u_n(\xi)\\ \vdots & \ddots & \vdots\\ u_0^{(n-1)}(\xi) & \cdots & u_n^{(n-1)}(\xi)\\ u_0^{(n+1)}(\xi) & \cdots & u_n^{(n+1)}(\xi)\end{vmatrix}\\ = &\sum_{i=0}^{n}(-1)^{n-i}u_i^{(n+1)}(\xi)W_{n-1}(u_0,\dots,\widehat{u}_i,\dots,u_n)(\xi).\end{aligned} \tag{20}$$

More details on the derivatives of the Wronskians can be found in [5]. From (19) and (20) we write

$$\begin{aligned} F_0^{(n+q)}(\xi) = &\sum_{i=0}^{n-1}a_i u_i^{(n+q)}(\xi)+u_n^{(n+q)}(\xi)\\ = &\frac{1}{W_{n-1}(\xi)}\sum_{i=0}^{n}(-1)^{n-i}u_i^{(n+q)}(\xi)W_{n-1}(u_0,\dots,\widehat{u}_i,\dots,u_n)(\xi)\\ = &\frac{W_n^{(q)}(\xi)}{W_{n-1}(\xi)}\end{aligned} \tag{21}$$

where $W_n^{(0)}=W_n$ and $q\in\{0,1\}$.

Now we prove statement (a), so $q=0$ and $W_n(\xi)\neq 0$. Consequently, from (21), the Taylor series of $F_0(x)$ in $x=\xi$ writes

$$F_0(x)=\sum_{i=0}^{n}\frac{F_0^{(i)}(\xi)}{i!}(x-\xi)^i+O_{n+1}(x-\xi)=\frac{W_n(\xi)}{n!W_{n-1}(\xi)}(x-\xi)^n+O_{n+1}(x-\xi).$$

Consider the perturbation

$$F(x,\varepsilon)=F_\varepsilon(x)=\sum_{i=0}^{n-1}(a_i+\varepsilon_i)u_i(x)+u_n(x), \tag{22}$$

which is in Span($\mathcal{F}$). Here $\varepsilon=(\varepsilon_0,\ldots,\varepsilon_{n-1})\in\mathbb{R}^n$. Clearly n is the first positive integer such that

$$\frac{\partial^n F}{\partial x^n}(\xi,0)=F_0^{(n)}(\xi)=\frac{W_n(\xi)}{W_{n-1}(\xi)}\neq 0. \tag{23}$$

From Malgrange Preparation Theorem, see [14], there exists a $\mathcal{C}^\infty$ function $h(x,\varepsilon)$ defined in a neighbourhood of the origin of $\mathbb{R}\times\mathbb{R}^n$ such that

$$F(x,\varepsilon)=\mathcal{P}_\varepsilon(x)h(x,\varepsilon)=\left(\sum_{i=0}^{n-1}\delta_i(\varepsilon)(x-\xi)^i+(x-\xi)^n\right)h(x,\varepsilon),$$

with $h(\xi,0)\neq 0$. The proof of this statement will follow by proving that the function $\Delta(\varepsilon)=\big(\delta_0(\varepsilon),\ldots,\delta_{n-1}(\varepsilon)\big)$ is invertible in a neighbourhood $U\subset\mathbb{R}^n$ of $\varepsilon=0$, being $V=\Delta(U)\subset\mathbb{R}^n$ also a neighbourhood of 0. From (23) it is easy to see that

$$\delta_i(\varepsilon)=\frac{1}{i!}\frac{\partial^i}{\partial x^i}\big(g(x,\varepsilon)F(x,\varepsilon)\big)\Big|_{x=\xi}, \tag{24}$$

for $i=0,1,\ldots,n-1$, where $g(x,\varepsilon)=1/h(x,\varepsilon)$. We shall obtain that $\det(J\Delta(0)) = g(\xi,0)^nW_{n-1}(\xi)\neq 0$, being $J\Delta$ the Jacobian matrix of Δ. Consequently, the parameters δ_i will be chosen freely in V to give all the configurations of n zeros of the polynomial $\mathcal{P}_\varepsilon(x)$. To do that, denote by $\pi_k\nabla\delta_i(0)$ the $(k+1)$-th component of the vector $\nabla\delta_i(0)\in\mathbb{R}^n$ where $\nabla\delta_i$ is the gradient of the function δ_i. Thus, for $i=0,1,\ldots,n-1$, (22), (23), and (24) implies that

$$\begin{aligned}\pi_k\nabla\delta_i(0)=&\ \frac{\partial\delta_i}{\partial\varepsilon_k}(0)=\frac{1}{i!}\frac{\partial^i}{\partial x^i}\left(\frac{\partial}{\partial\varepsilon_k}\big(g(x,\varepsilon)F(x,\varepsilon)\big)\Big|_{\varepsilon=0}\right)\Bigg|_{x=\xi}\\ =&\ \frac{1}{i!}\frac{\partial^i}{\partial x^i}\left(F_0(x)\frac{\partial g}{\partial\varepsilon_k}(x,0)+g(x,0)\frac{\partial F}{\partial\varepsilon_k}(x,0)\right)\Bigg|_{x=\xi}\\ =&\ \frac{1}{i!}\frac{\partial^i}{\partial x^i}\big(g(x,0)u_k(x)\big)\big|_{x=\xi}.\end{aligned} \tag{25}$$

After straightforward computations, (25) leads to

$$\pi_k\nabla\delta_0(0)=c_0u_k(\xi),\quad\text{and}\quad \pi_k\nabla\delta_i(0)=c_0u_k^{(i)}(\xi)+\sum_{\ell=0}^{i-1}\binom{i}{\ell}c_{i-\ell}u_k^{(\ell)}(\xi), \tag{26}$$

for $i=1,\ldots,n-1$, where $c_j=\partial^j g/\partial x^j(\xi,0)$. Now for each $i=1,\ldots,n-1$ we claim that there exists $\alpha^i=\big(\alpha^i_0,\alpha^i_1,\ldots,\alpha^i_{i-1}\big)\in\mathbb{R}^i$ such that

$$\pi_k\nabla\delta_i(0)=c_0u_k^{(i)}(\xi)+\sum_{s=0}^{i-1}\alpha^i_s\pi_k\nabla\delta_s(0). \tag{27}$$

Indeed, substituting (26) in both sides of the equality (27) and then comparing the coefficients of $u_k^{(\ell)}$, for $\ell=0,1,\ldots i-1$, we get

$$\sum_{s=\ell}^{i-1}\binom{s}{\ell}c_{s-\ell}\alpha^i_s=\binom{i}{\ell}c_{i-\ell}.$$

Therefore we see that the vector α^i is given as a solution of a linear system $C_i\cdot\alpha^i=b_i$, where C_i is an upper triangular $i\times i$ matrix having only $c_0=g(\xi,0)\neq0$ in every entry of its diagonal, and $b_i\in\mathbb{R}^i$. Since $\det(C_i)\neq0$ there exists such a vector α^i, for each $i=0,1,\ldots,n-1$, and the claim is proved. Observe that $\big(\pi_k\nabla\delta_0(0),\ldots,\pi_k\nabla\delta_{n-1}(0)\big)$ is the $(k+1)$-th column of the matrix $J\Delta(0)$. Therefore, by the n-linearity property of the determinant, it follows from (27) that $\det(J\Delta(0))=g(\xi,0)^nW_{n-1}(\xi)\neq0$. This completes the proof of statement (a).

Now we prove statement (b), so $q=1$, $W_n(\xi)=0$, and $W_n'(\xi)\neq0$. Consequently, from (21), the Taylor series of $F_0(x)$ in $x=\xi$ writes

$$\begin{aligned}F_0(x)=&\sum_{i=0}^{n+1}\frac{F_0^{(i)}(\xi)}{i!}(x-\xi)^i+O_{n+2}(x-\xi)\\ =&\frac{W_n'(\xi)}{(n+1)!W_{n-1}(\xi)}(x-\xi)^{n+1}+O_{n+2}(x-\xi).\end{aligned}$$

Consider a second perturbation $\widetilde{F}(x,\widetilde{\varepsilon})=F_\varepsilon(x+\varepsilon_n)$, where $\widetilde{\varepsilon}=(\varepsilon_0,\ldots,\varepsilon_n)\in\mathbb{R}^{n+1}$. Clearly $n+1$ is the first positive integer such that

$$\frac{\partial^{n+1}\widetilde{F}}{\partial x^{n+1}}(\xi,0)=F_0^{n+1}(\xi)=\frac{W_n'(\xi)}{W_{n-1}(\xi)}\neq0.$$

From Malgrange Preparation Theorem there exists a $\mathcal{C}^\infty$ function $\widetilde{h}(x,\varepsilon)$ defined in a neighbourhood of the origin of $\mathbb{R}\times\mathbb{R}^{n+1}$ such that

$$\widetilde{F}(x,\widetilde{\varepsilon})=\widetilde{\mathcal{P}}_{\widetilde{\varepsilon}}(x)\widetilde{h}(x,\widetilde{\varepsilon})=\left(\sum_{i=0}^{n}\lambda_i(\varepsilon)(x-\xi)^i+(x-\xi)^{n+1}\right)\widetilde{h}(x,\varepsilon),$$

with $\widetilde{h}(\xi,0)\neq0$. Analogously to the previous case we shall prove that the function $\Lambda(\widetilde{\varepsilon})=\big(\lambda_0(\widetilde{\varepsilon}),\ldots,\lambda_n(\widetilde{\varepsilon})\big)$ is invertible in a neighbourhood $U\subset\mathbb{R}^{n+1}$ of $\widetilde{\varepsilon}=0$ by showing that $\det(J\Lambda(0))=\widetilde{g}(\xi,0)^{n+1}W_n'(\xi)\neq0$, where $\widetilde{g}(x,\varepsilon)=1/\widetilde{h}(x,\varepsilon)$. So let $\widetilde{c}_0=\widetilde{g}(\xi,0)\neq0$ and denote by $\pi_k\nabla\lambda_i(0)$ the $(k+1)$-th component of the vector $\nabla\lambda_i(0)\in\mathbb{R}^{n+1}$. Proceeding as above, for each $i=1,\ldots,n$, straightforward computations lead to the existence of a vector $\widetilde{\alpha}^i=\big(\widetilde{\alpha}^i_0,\ldots,\widetilde{\alpha}^i_{i-1}\big)\in\mathbb{R}^i$

such that

$$\pi_k \nabla \lambda_0(0) = \widetilde{c}_0 u_k(\xi), \quad \text{and} \quad \pi_k \nabla \lambda_i(0) = \widetilde{c}_0 u_k^{(i)} + \sum_{s=0}^{i-1} \widetilde{\alpha}_s^i \pi_k \nabla \lambda_s(0), \tag{28}$$

for $k = 0, \ldots, n-1$, and for $k = n$

$$\pi_n \nabla \lambda_i(0) = \begin{cases} 0, & i = 1, \ldots, n-1, \\ \widetilde{c}_0 F^{(n+1)}(\xi), & i = n. \end{cases} \tag{29}$$

Observe that $\big(\pi_k \nabla \lambda_0(0), \ldots, \pi_k \nabla \lambda_n(0)\big)$ is the $(k+1)$-th column of the matrix $J\Lambda(0)$. Therefore, by the $(n+1)$-linearity property of determinant, it follows from (28) and (29) that

$$\begin{aligned} \det(J\Lambda(0)) = & \ \widetilde{g}(\xi,0)^{n+1} F^{(n+1)}(\xi) W_{n-1}(\xi) \\ = & \ \widetilde{g}(\xi,0)^{n+1} \frac{W_n'(\xi)}{W_{n-1}(\xi)} W_{n-1}(\xi) = \widetilde{g}(\xi,0)^{n+1} W_n'(\xi) \neq 0. \end{aligned}$$

This completes the proof of statement (b).

6. Optimality of the lower bounds

The results of this section provide examples of families assuring the optimality of the lower bound given in Theorem 1.3. Indeed for these families $Z(\mathcal{F})$ coincides with that.

Proposition 6.1. *The ordered set of functions $\mathcal{F} = [1, t\,, \cos t]$ is an ET-system with accuracy 1 in $[-\pi, \pi]$.*

Proof. Let $f(t) = a + b\,t + c\cos t$ be an element of Span$(\mathcal{F})$. Clearly, when $b = 0$, $f(t)$ has at most 2 zeros in $[-\pi, \pi]$. For $b \neq 0$, the derivative $f'(t) = b - c\sin t$ has at most 2 zeros in $[-\pi, \pi]$ which implies that $f(t)$ has at most 3 zeros in $[-\pi, \pi]$. Moreover, we can easily find an example in Span$(\mathcal{F})$ having 3 zeros, so we conclude that $Z(\mathcal{F}) = 3$. □

For the above family $W_0(t) = W_1(t) = 1$ and $W_2(t) = -\cos(t)$ which has two zeros in the interval $[-\pi, \pi]$. We remark that for an arbitrary set of three functions ($n = 2$) such that all the Wronskians are nonvanishing except the last one (W_2) which has exactly two zeros ($k = 0$ and $\ell = 2$), applying Theorems 1.1 and 1.3, we obtain that $3 \leq Z(\mathcal{F}) \leq 4$. But for the set given in the above proposition we have $Z(\mathcal{F}) = 3$.

Proposition 6.2 (See [20])**.** *The ordered set of functions $\mathcal{F} = [1,\, t\,\cos\, t,\, t\,\sin\, t]$ is an ET-system in $[0, \pi]$ which is not an ECT-system.*

For the above family $W_0(t) = 1$, $W_1(t) = \cos t - t\sin t$, and $W_2(t) = t^2 + 2$ which W_1 has one zero in the interval $[0, \pi]$. We note that for an arbitrary set of three functions ($n = 2$) such that all the Wronskians are nonvanishing except $W_1(t)$ which has exactly one zero ($k = 1$ and $\ell = 0$), applying Theorems 1.1 and 1.3, we obtain that $2 \leq Z(\mathcal{F}) \leq 3$. But Zielke in [20] (example (3) of page 363) proves the last proposition, which implies $Z(\mathcal{F}) = 2$.

7. Application to nonsmooth systems

In [9], the number of limit cycles bifurcating from a period annulus of some classes of nonsmooth systems is studied. Each of these classes has associated a Poincaré–Pontryaguin–Melnikov function. The simple zeros of each function provide limit cycles for the respective perturbed class. In two of that classes these functions define the sets $\mathrm{Span}(\mathcal{G}^1)$ and $\mathrm{Span}(\mathcal{G}^3)$, where $\mathcal{G}^i = \{g_1, g_2, g_3^i\}$ for $i = 1, 3$, being

$$
\begin{aligned}
g_1(y) &= 1,\\
g_2(y) &= \frac{(ay^2 - b\,c^2)}{y} \log\left(\frac{\sqrt{b}\,c + \sqrt{a}\,y}{\sqrt{b}\,c - \sqrt{a}\,y}\right),\\
g_3^1(y) &= \frac{(d^2 + y^2)}{y}\left(3\pi + 2\arctan\left(\frac{d^2 - y^2}{2dy}\right)\right) \text{ with } d > 0,\\
g_3^3(y) &= \frac{(d^2 + y^2)}{y}\left(\pi + 2\arctan\left(\frac{d^2 - y^2}{2dy}\right)\right) \text{ with } d < 0,
\end{aligned}
\tag{30}
$$

for $a, b, c > 0$.

The initial problems are reduced to study the maximum number of zeros that the elements of $\mathrm{Span}(\mathcal{G}^1)$ and $\mathrm{Span}(\mathcal{G}^3)$ can have in $(0, \sqrt{b}c/\sqrt{a})$. We denote these maxima by $Z(\mathcal{G}^1)$ and $Z(\mathcal{G}^3)$, respectively. The authors show that $Z(\mathcal{G}^1), Z(\mathcal{G}^3) \geq 2$. As an application of our theorems we shall prove that $Z(\mathcal{G}^3) = 2$, and $Z(\mathcal{G}^1) = 2$ or $Z(\mathcal{G}^1) = 3$ depending on the parameters a, b, c, and d.

We can unify the notation considering the functions

$$
\begin{aligned}
u_0(t) &= 1,\\
u_1^\alpha(t) &= -\frac{\alpha^2 - t^2}{t} \log\left(\frac{\alpha + t}{\alpha - t}\right),\\
u_2^\beta(t) &= \frac{t^2 + 1}{t}\left(\beta\pi - 2(\beta - 2)\arctan\left(\frac{t^2 - 1}{2t}\right)\right).
\end{aligned}
\tag{31}
$$

More concretely, we can embed the sets $\mathcal{G}^1$ and $\mathcal{G}^3$ in the ordered set of functions $\mathcal{F}(\alpha, \beta) = [u_0, u_1^\alpha, u_2^\beta]$ for $\alpha > 0$ and $\beta > 0$. In particular $u_1^\alpha(t) = g_2(|d|\,t)/(a|d|)$ when $\alpha = |\sqrt{b}c/(\sqrt{a}d)|$, $u_2^3(t) = g_3^1(|d|\,t)/|d|$, and $u_2^1(t) = g_3^3(|d|\,t)/|d|$. So the numbers $Z(\mathcal{G}^1)$ and $Z(\mathcal{G}^3)$ are now equivalent to $Z(\mathcal{F}(\alpha, 3))$ and $Z(\mathcal{F}(\alpha, 1))$ in $(0, \alpha)$, respectively. The next result provides them.

Theorem 7.1. *Let $\mathcal{F}(\alpha, \beta) = [u_0, u_1^\alpha, u_2^\beta]$ be the ordered set of the functions* (31)*. Then*

$$
Z\left(\mathcal{F}(\alpha, 1)\right) = 2 \quad \text{and} \quad Z\left(\mathcal{F}(\alpha, 3)\right) = \begin{cases} 2 & \text{if } 0 < \alpha \leq \alpha^*,\\ 3 & \text{if } \alpha > \alpha^*. \end{cases}
$$

Where α^* is the unique positive solution of $2(\alpha^2 - 1)\arctan\left((\alpha^2 - 1)/(2\alpha)\right) - 3\pi\alpha^2 + 3\pi + 4\alpha = 0$. Moreover $\alpha^* \approx 1.24701$.

Before proving the above theorem we shall see that, in general, $Z\left(\mathcal{F}(\alpha,\beta)\right) \leq 3$. To do that we proceed with an easier approach which consist in an embedding of this set of three functions (31) into a set of four functions $\{u_0, u_1^\alpha, u_2, u_3\}$, where

$$\begin{aligned}
u_0(t) &= 1,\\
u_1^\alpha(t) &= -\frac{\alpha^2-t^2}{t}\log\left(\frac{\alpha+t}{\alpha-t}\right),\\
u_2(t) &= -\frac{t^2+1}{t}\arctan\left(\frac{t^2-1}{2t}\right).\\
u_3(t) &= \frac{t^2+1}{t}.
\end{aligned} \tag{32}$$

This kind of approach has been used previously in [19].

Proposition 7.2. *For $\alpha > 0$, the ordered set of functions $\mathcal{G}(\alpha) = [u_0, u_2, u_3, u_1^\alpha]$ is an ECT-system in the interval $(0,\alpha)$. Consequently $Z(\mathcal{G}(\alpha)) = 3$.*

Proof. Straightforward computations give us the Wronskians of the ordered set $\mathcal{G}(\alpha)$:

$$\begin{aligned}
W_0(t) &= 1,\\
W_1(t) &= \frac{-(t^2-1)}{t^2}\arctan\left(\frac{t^2-1}{2t}\right) - \frac{2}{t},\\
W_2(t) &= \frac{-8}{t^2(t^2+1)},\\
W_3(t) &= \frac{16(\alpha^2+1)}{t^4(t^2+1)^2}\log\left(\frac{\alpha+t}{\alpha-t}\right) - \frac{32\alpha(\alpha^2+1)(\alpha^2+t^2)}{t^3(\alpha-t)^2(\alpha+t)^2(t^2+1)^2}.
\end{aligned}$$

Clearly, $W_0(t) > 0$, $W_1(t) < 0$ because both summands are negative for $t > 0$, and $W_2(t) < 0$ for $t \neq 0$. Moreover $W_3(t) = P(t)Q(t)$ where

$$P(t) = \frac{-16(\alpha^2-t^2)^2(\alpha^2+1)}{t^4(t^2+1)^2(\alpha-t)^2(\alpha+t)^2} \quad \text{and} \quad Q(t) = \frac{2\alpha t(\alpha^2+t^2)}{(\alpha^2-t^2)^2} - \log\left(\frac{\alpha+t}{\alpha-t}\right).$$

We note that $P(t) < 0$ and $Q(t) > 0$ for $t \in (0,\alpha)$, because $Q'(t) = 16\alpha^2 t^2/(\alpha^2 - t^2)^3 > 0$ and $Q(0) = 0$. Consequently, $W_3(t) < 0$ in $(0,\alpha)$. Hence we conclude that $\mathcal{G}(\alpha)$ is an ECT-system in $(0,\alpha)$. □

We stress that Proposition 7.2 represents, by itself, an improvement of the results obtained in [9] implying that $Z(\mathcal{G}^1), Z(\mathcal{G}^3) \leq 3$. Nevertheless this approach do not use the new theorems developed in this paper. In what follows we show, by proving Theorem 7.1, how to use our main results to give still better estimations for these upper bounds.

Proof of Theorem 7.1. Straightforward computations give us the Wronskians of the ordered set $\mathcal{F}(\alpha,\beta)$:

$$\begin{aligned} W_0(t) =&\ 1,\\ W_1(t) =&\ \frac{\alpha^2+t^2}{t^2}\log\left(\frac{\alpha+t}{\alpha-t}\right)-\frac{2\alpha}{t} := \frac{\alpha^2+t^2}{t^2}P(t),\\ W_2(t) =&\ \frac{Q(t)}{t^3}, \end{aligned}$$

where

$$\begin{aligned} Q(t) =&\ 4\alpha t\left(\frac{4t(\alpha^2+1)(\beta-2)}{(t^2+1)(\alpha^2-t^2)}-\frac{\beta\pi(\alpha^2-1)}{\alpha^2-t^2}\right)\\ &+\frac{8\alpha(\alpha^2-1)(\beta-2)t}{\alpha^2-t^2}\arctan\left(\frac{t^2-1}{2t}\right)\\ &-2\left(\frac{4t(\alpha^2-1)(\beta-2)}{t^2+1}-\beta\pi(\alpha^2+1)\right)\log\left(\frac{\alpha+t}{\alpha-t}\right)\\ &-4(\alpha^2+1)(\beta-2)\arctan\left(\frac{t^2-1}{2t}\right)\log\left(\frac{\alpha+t}{\alpha-t}\right). \end{aligned}$$

Clearly $W_0(t)>0$, and since $P'(t)=8\alpha^3t^2/((\alpha^2-t^2)(\alpha^2+t^2)^2)>0$ and $P(0)=0$, we get that $W_1(t)$ has no zeros in $(0,\alpha)$.

The proof follows studying the zeros of $W_2(t)$, which coincide with the zeros of $Q(t)$ for $t\in(0,\alpha)$. As we are interested only in the values $\beta=1$ and $\beta=3$, for both we analyse the limits of the function Q at the boundary of the interval $(0,\alpha)$ and its monotonicity properties in the full interval. The function Q close to the origin and close to α writes, respectively, as

$$\begin{aligned} Q(t) =&\ 16\,\frac{\pi\,(\beta-1)}{\alpha}t-\frac{16\,\pi\,(\alpha^2-2)\,(\beta-1)}{3\,\alpha^3}t^3\\ &-\frac{16\,\pi\,(2\,\alpha^2-3)\,(\beta-1)}{5\,\alpha^5}t^5+\frac{512\,(\alpha^2+1)\,(\beta-2)}{45\,\alpha^3}t^6+O\left(t^7\right) \end{aligned} \tag{33}$$

and

$$Q(t)=\frac{\alpha(\alpha^2-1)L_1(\alpha,\beta)}{\alpha-t}+(\alpha^2+1)L_2(\alpha,\beta)\log(\alpha-t)+O(1), \tag{34}$$

where

$$\begin{aligned} L_1(\alpha,\beta) =&4(\beta-2)\arctan\left(\frac{\alpha^2-1}{2\alpha}\right)-2\beta\pi+\frac{8\alpha(\beta-2)}{\alpha^2-1},\\ L_2(\alpha,\beta) =&4(\beta-2)\arctan\left(\frac{\alpha^2-1}{2\alpha}\right)-2\beta\pi+\frac{8\alpha(\alpha^2-1)(\beta-2)}{(\alpha^2+1)^2}. \end{aligned}$$

For $\beta=1$, we will show that Q, in $(0,\alpha)$, is a decreasing negative function. Applying Theorems 1.1 and 1.3 (ECT-system case) we have that $Z\left(\mathcal{F}(\alpha,\pi)\right)=2$.

For $\beta=3$, we will show that Q, in $(0,\alpha)$, is an increasing positive function for $\alpha<\alpha^*$ and a unimodal function for $\alpha\geq\alpha^*$ with a simple zero. Moreover it is increasing and positive close to the origin, see (33). The graphs of Q for both cases are depicted in Figure 1. Applying Theorem 1.3 and Proposition 7.2 we conclude that $Z(\mathcal{F}(\alpha, 3\pi)) = 3$ for $\alpha > \alpha^*$. Otherwise, applying Theorems 1.1 and 1.3 (ECT-system case), we obtain $Z(\mathcal{F}(\alpha, 3\pi)) = 2$.

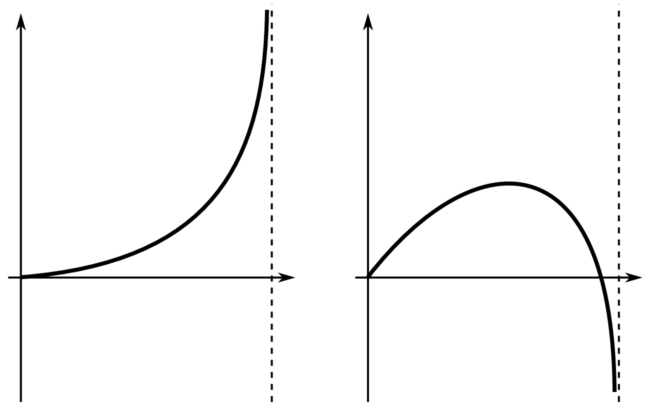

FIGURE 1. Graphs of Q for $\beta = 3$ and for $\alpha \leq \alpha^*$ and $\alpha > \alpha^*$, respectively

In what follows we prove the properties of the function Q described above distinguishing the cases $\beta = 1$ and $\beta = 3$.

For simplicity we define $A(t) = -\arctan((t^2-1)/(2t))$ and $L(t) = \log((\alpha + t)/(\alpha - t))$. The derivative of Q is $Q_1 = q_0(t) + q_1(t)A(t) + q_2(t)L(t)$, the derivative of Q_1/q_2 is $Q_2 = \tilde{q}_0(t) + \tilde{q}_1(t)A(t)$, and the derivative of $Q_2/\tilde{q}_1$ is

$$Q_3 = -\frac{32t^2(\alpha^2+t^4)(\alpha^2+t^2)}{(t^2+1)^2(t^4+3(\alpha^2-1)t^2-\alpha^2)^2}.$$

Here all the functions q_i and $\tilde{q}_i$, are rational, and

$$q_2(t) = -\frac{16(\beta-2)(\alpha^2+t^2)}{(t^2+1)^2},$$
$$\tilde{q}_1(t) = -\frac{2\alpha^3 t(t^2+1)(\alpha^2+1)(t^4+3(\alpha^2-1)t^2-\alpha^2)}{(\alpha^2+t^2)^2(\alpha^2-t^2)^3}.$$

The function $t^4 + 3(\alpha^2-1)t^2 - \alpha^2$, in $(0, \alpha)$ is negative when $\alpha \leq 1$ and has a unique zero at $t^* \in (1, \alpha)$ when $\alpha > 1$. Consequently Q_3 has an asymptote and $\tilde{q}_1$ has a zero, both at t^*. Additionally, the limit of the functions Q_1/q_2 and $Q_2/\tilde{q}_1$ are $-\pi(\beta-1)/((\beta-2)\alpha^3)$ and $-\pi(\beta-1)/(\beta-2)$, respectively.

For $\beta = 1$ and $\alpha \leq 1$, the functions Q_3, q_2, and $\tilde{q}_1$ in the interval $(0, \alpha)$ are negative, positive, and positive, respectively. Moreover the limit of the functions Q_1, Q_1/q_2 and $Q_2/\tilde{q}_1$ vanish at $t = 0$. Going back through the derivation/division procedure detailed above we can conclude that the function Q is decreasing and, as it vanishes at $t = 0$, negative. The same argument can be done to prove that Q is increasing and positive for $\beta = 3$ and $\alpha \leq 1$.

Let $\beta = 1$ and $\alpha > 1$. The function Q_3 is negative and has an asymptote at $t = t^*$. The function q_2 is positive and the function $\tilde{q}_1$ is positive in $(0, t^*)$ and negative in (t^*, α). As above, the limit of the functions Q_1, Q_1/q_2 and $Q_2/\tilde{q}_1$

vanish at $t = 0$. Moreover, the limit of $Q_2/\tilde{q}_1$ at $t = \alpha$ is $-L_1(\alpha, 1)/4 > 0$ and Q_2 is continuous in $(0, \alpha)$. Again going back through the derivation/division procedure detailed above we can conclude that the function Q is decreasing and negative.

Let $\beta = 3$ and $\alpha > 1$. The differences between this case with the latter one are that now the function q_2 is negative and the limit of $Q_2/\tilde{q}_1$ at $t = \alpha$ is $L_1(\alpha, 3)/4$ which has a unique zero at α^* and it is positive in $(0, \alpha^*)$ and negative in $\alpha > \alpha^*$. This follows because the $L_1(\alpha, 3)$ is decreasing, $\lim_{\alpha \to 1^+} L_1(\alpha, 3) = \infty$ and $\lim_{\alpha \to \infty} L_1(\alpha, 3) = -2\pi$. Therefore, for $\alpha \leq \alpha^*$, repeating the same argument of the latter case we prove that Q is increasing and positive. Finally, for $\alpha > \alpha^*$, going back through the derivation/division procedure we can conclude that the function Q is unimodal and increasing close to the origin. From (34) the sign of the limit of $Q(t)$ at $t = \alpha$ coincides with the sign of $L_1(\alpha, 3)$ when $\alpha \neq \alpha^*$. The proof follows because $L_1(\alpha, 3) < 0$ when $\alpha > \alpha^*$. □

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