Fine asymptotics of profiles and relaxation to equilibrium for growth-fragmentation equations with variable drift rates

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Abstract

We are concerned with the long-time behavior of the growth-fragmentation equation. We prove fine estimates on the principal eigenfunctions of the growth-fragmentation operator, giving their first-order behavior close to 0 and $+\infty$. Using these estimates we prove a spectral gap result by following the technique in [1], which implies that solutions decay to the equilibrium exponentially fast. The growth and fragmentation coefficients we consider are quite general, essentially only assumed to behave asymptotically like power laws.

Keywords. Fragmentation, growth, eigenvalue problem, entropy, exponential convergence, long-time behavior.

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1 Introduction

In this paper we are interested in the long-time behavior of the growth-fragmentation equation, commonly used as a model for cell growth and division and other phenomena involving fragmentation [10, 6, 4]. There are a number of works which study the existence and other properties of the first eigenfunctions (also called *profiles*) of the growth-fragmentation operator and its dual [7, 2, 3] and the convergence of solutions to equilibrium [5, 11, 9, 8, 1, 12]. These eigenfunctions are fundamental since they give the asymptotic shape of solutions (i.e., the stationary solution of the rescaled equation) and a conserved quantity of the time evolution. However, precise estimates on their behavior close to 0 and $+\infty$ are usually not given, are very rough, or are restricted to a particular kind of growth or fragmentation coefficients. Our first objective is to give accurate estimates on the first eigenfunctions, valid for a wide range of growth and fragmentation coefficients which include most cases in which they behave like power laws. We give, in most cases, the first-order behavior of both first eigenfunctions (of the growth-fragmentation operator and its dual); detailed results are given later in this introduction.

Our second objective is to use these estimates to show that the growth-fragmentation operator has a spectral gap (in a certain natural Hilbert space) for a wide choice of the coefficients, which is interesting because it readily implies exponential convergence to equilibrium of solutions. For this we follow the techniques in [1], which require careful estimates on the profiles which were previously available only for particular growth rates (constant and linear). Our results on exponential convergence to equilibrium are valid for general coefficients behaving like power laws, improving or complementing known results applicable to constant or linear total fragmentation rates [5, 11, 1]. However, our results still impose some restrictions on the fragment distribution (which must be bounded below) and the decay of the total fragmentation rate for small sizes.

Let us introduce the equation under study more precisely and state our main results. The *growth-fragmentation equation* is given by

$$\partial_t g_t(x) + \partial_x(\tau(x) g_t(x)) + \lambda g_t(x) = \mathcal{L}[g_t](x), \tag{1a}$$

$$(\tau g_t)(0) = 0 \qquad (t \geq 0), \tag{1b}$$

$$g_0(x) = g_{\text{in}}(x) \qquad (x > 0). \tag{1c}$$

The unknown is a function $g_t(x)$ which depends on the time $t \geq 0$ and on $x > 0$, and for which an initial condition g_{in} is given at time $t = 0$. The positive function τ represents the *growth rate*. The symbol $\mathcal{L}$ stands for the fragmentation operator (see below), and λ is the largest eigenvalue of the operator $g \mapsto -\partial_x(\tau g) + \mathcal{L}g$, acting on a function $g = g(x)$ depending only on x. The main motivation for the study of equation (1) is the closely related

$$\partial_t n_t(x) + \partial_x(\tau(x) n_t(x)) = \mathcal{L}[n_t](x), \tag{2}$$

with the same initial and boundary conditions. Solutions of the two are related by $n_t(x) = e^{\lambda t} g_t(x)$, and n_t represents the size distribution at a given time t of a population of cells (or other objects) undergoing growth and division phenomena. The population grows at an exponential rate determined by $\lambda > 0$, called the *Malthus parameter*, and approaches an asymptotic shape for large times. Equation (1) has a stationary solution and is more convenient for studying its asymptotic behavior, which is why it is commonly considered. Of course, results about (1) are easily translated to results about (2) through the simple change $n_t(x) = e^{\lambda t} g_t(x)$.

The *fragmentation operator* $\mathcal{L}$ acts on a function $g = g(x)$ as

$$\mathcal{L}g(x) := \mathcal{L}_+ g(x) - B(x) g(x),$$

where the positive part $\mathcal{L}_+$ is given by

$$\mathcal{L}_+ g(x) := \int_x^\infty b(y, x) g(y)\, dy.$$

The coefficient $b(y, x)$, defined for $y \geq x \geq 0$, is the *fragmentation coefficient*, and $B(x)$ is the *total fragmentation rate* of cells of size $x > 0$. It is obtained from b through

$$B(x) := \int_0^x \frac{y}{x}\, b(x, y)\, dy \qquad (x > 0).$$

The *eigenproblem* associated to (1) is the problem of finding both a stationary solution and a stationary solution of the dual equation, this is, the first eigenfunction of the growth-fragmentation operator $g \mapsto -(\tau g)' + \mathcal{L}(g)$ and of its dual $\varphi \mapsto \tau\varphi' + \mathcal{L}^*(\varphi)$. If λ is the largest eigenvalue of the operator $g \mapsto -(\tau\, g)' + \mathcal{L}g$, the associated eigenvector G satisfies

$$(\tau(x)\, G(x))' + \lambda G(x) = \mathcal{L}(G)(x), \tag{3a}$$

$$\tau(x) G(x)\big|_{x=0} = 0, \tag{3b}$$

$$G \geq 0, \quad \int_0^\infty G(x)\, dx = 1. \tag{3c}$$

Of course, the eigenvector G is an equilibrium (i.e., a stationary solution) of equation (1). The associated dual eigenproblem reads

$$-\tau(x)\phi' + (B(x)+\lambda)\,\phi(x) = \mathcal{L}_+^*\phi(x), \tag{4a}$$

$$\phi \geq 0, \quad \int_0^\infty G(x)\phi(x)\,dx = 1, \tag{4b}$$

where

$$\mathcal{L}_+^*\phi(x) := \int_0^x b(x,y)\phi(y)\,dy.$$

This dual eigenproblem is interesting because ϕ gives a conservation law for (1):

$$\int_0^\infty \phi(x)\,g_t(x)\,dx = \int_0^\infty \phi(x)\,g_{\text{in}}(x)\,dx = \text{Cst} \qquad (t \geq 0).$$

In this paper we always denote by G, ϕ and λ the solution to (3) and (4).

In the rest of this introduction we describe the assumptions used throughout the paper and state our main results. In Section 2 we give the proof of our estimates on the stationary solution G, and Section 3 is devoted to estimates of the dual eigenfunction ϕ. Our results on the spectral gap of the growth-fragmentation operator are proved in Section 4, and we also include two appendices: one, Appendix A, on different approximation procedures that may be used for G and ϕ, and which are more convenient in some of our arguments; and Appendix B, which gives asymptotic estimates of some of the expressions involving the positive part $\mathcal{L}^+$ of the fragmentation operator, and are used for our large-x estimates of G.

1.1 Assumptions on the coefficients

For proving our results we need some or all of the following assumptions. First of all, we assume that the fragmentation coefficient b is of self-similar form, which is general enough to encompass most interesting examples and still allows us to obtain accurate results on the asymptotics of G and ϕ:

Hypothesis 1.1 (Self-similar fragmentation rate)**.** *The coefficient $b(x,y)$ is of the form*

$$b(x,y) = B(x)\frac{1}{x}p\left(\frac{y}{x}\right) \tag{5}$$

for some locally integrable $B : (0,+\infty) \to (0,+\infty)$, and some nonnegative finite measure p on $[0,1]$ satisfying the mass preserving condition

$$\int_0^1 z\,p(z)\,dz = 1,$$

and also the condition

$$\int_0^1 p(z)\,dz > 1.$$

(When writing the integral of a measure it is always understood that the integration limits are included in the integral.)

The measure p gives the distribution of fragments obtained when a particle of a certain size breaks.

Remark 1.1. Define, for $k \geq 0$, the moment

$$\pi_k := \int_0^1 z^k p(z) dz.$$

We have from Hypothesis 1.1 that $\pi_1 = 1$ and $\pi_0 > 1$. Physically, π_0 represents the mean quantity of fragments produced by the fragmentation of one particle. Because of the strict inequality $\pi_0 > \pi_1$, one can deduce that p is not concentrated at $z = 1$ (*i.e.* $p \neq \pi_0 \delta_1$). As a consequence we have that $\pi_k < 1$ if $k > 1$ and $\pi_k > 1$ if $k < 1$.

Hypothesis 1.2. *The growth rate is a continuous and strictly positive function* $\tau : (0, +\infty) \to (0, +\infty)$.

Our next assumption says that the growth rate and total fragmentation rate have a power-law behavior for large and small sizes:

Hypothesis 1.3 (Asymptotics of fragmentation and drift rates)**.** *Assume that for some constants* $B_0, B_\infty, \tau_0, \tau_\infty > 0$ *and* $\gamma_0, \gamma, \alpha_0, \alpha \in \mathbb{R}$

$$B(x) \sim B_0\, x^{\gamma_0} \quad \text{as } x \to 0, \tag{6}$$

$$B(x) \sim B_\infty\, x^{\gamma} \quad \text{as } x \to +\infty, \tag{7}$$

$$\tau(x) \sim \tau_0\, x^{\alpha_0} \quad \text{as } x \to 0, \tag{8}$$

$$\tau(x) \sim \tau_\infty\, x^{\alpha} \quad \text{as } x \to +\infty. \tag{9}$$

We also impose the conditions

$$\gamma_0 - \alpha_0 + 1 > 0, \tag{10}$$

$$\gamma - \alpha + 1 > 0, \tag{11}$$

to ensure the existence of a solution to the eigenproblem (see [2, 7]).

Likewise, we impose that the distribution of small fragments behave like a power law:

Hypothesis 1.4 (Behavior of p close to 0)**.** *There exist* $p_0 \geq 0$ *and* $\mu > 0$ *such that*

$$p(z) = p_0\, z^{\mu - 1} + o(z^{\mu - 1}) \quad \text{as } z \to 0, \tag{12}$$

with the condition

$$\mu - \alpha_0 + 1 > 0. \tag{13}$$

Remark 1.2. When $p_0 > 0$ condition (12) is the same as

$$p(z) \sim p_0\, z^{\mu - 1} \tag{14}$$

as $z \to 0$. We prefer to write it as given in order to allow for $p_0 = 0$, which is usually not allowed in the notation (14). For instance, if $p(z)$ is equal to 0 in a neighborhood of 0 (such as for the mitosis case, see below), then (12) holds with $p_0 = 0$, but (14) does not make sense.

To find the asymptotic behavior of the function G when $x \to \infty$ the following hypothesis will also be needed.

Hypothesis 1.5 (Asymptotics to second order)**.** *Assume that τ is a $\mathcal{C}^1$ function and that, for some $\delta > 0$ and $\nu > -1$,*

$$B(x) = B_\infty\, x^\gamma + O(x^{\gamma-\delta}) \quad \textit{as } x \to +\infty, \tag{15}$$

$$\tau(x) = \tau_\infty\, x^\alpha + O(x^{\alpha-\delta}) \quad \textit{as } x \to +\infty, \tag{16}$$

$$p(z) = p_1(1-z)^\nu + O((z-1)^{\nu+\delta}) \quad \textit{as } z \to 1. \tag{17}$$

Finally, to prove the entropy-entropy dissipation inequality, we will need an additional restriction on the fragmentation coefficient. It essentially says that p is uniformly bounded below by some constant $\underline{p} > 0$, and that it behaves like a constant at the endpoints 0 and 1:

Hypothesis 1.6. *There exist positive constants $\underline{p}$, p_0, $p_1 > 0$ such that*

$$\forall\, z \in (0,1), \quad p(z) \geq \underline{p} \qquad \textit{(in the sense of measures)},$$

$$p(z) \xrightarrow[z\to 0]{} p_0, \qquad p(z) \xrightarrow[z\to 1]{} p_1,$$

and $\alpha_0 < 2$ (which is nothing but condition (13) in the case $\mu = 1$).

The reader may check that [2, Theorem 1], which gives existence and uniqueness of G, ϕ, λ satisfying (3) and (4) is applicable under Hypotheses 1.1–1.4. We assume at least these hypotheses throughout the paper in order to ensure the existence of profiles.

Let us give some common examples of coefficients satisfying the above assumptions:

Power coefficients. If we set

$$b(x,y) = 2x^{\gamma-1} \text{ for } x > y > 0, \quad \tau(x) = x^\alpha \text{ for } x > 0,$$

then all our hypotheses are satisfied when $\gamma - \alpha + 1 > 0$ and $\alpha < 2$. Observe that in this case $B(x) = x^\gamma$ and $p(z) \equiv 1$, which satisfies Hypotheses 1.1, 1.4 with $\mu = 1$ and $\nu = 0$, and also 1.6. Since $\tau(x)$ is a power, it satisfies Hypothesis 1.2. Hypotheses 1.3 and 1.5 are also satisfied.

Self-similar fragmentation. The previous case with $\tau(x) = x$ is referred to as the *self-similar fragmentation equation*. It is closely related to the fragmentation equation $\partial_t g_t = \mathcal{L}(g_t)$ (see [3, 1]).

Mitosis. Cellular division by equal mitosis is modeled by a distribution of fragments p concentrated at a size equal to one half:

$$p(z) = 2\delta_{z=\frac{1}{2}}.$$

This measure p satisfies Hypothesis 1.4 with $p_0 = p_1 = 0$ (the value of μ, ν being irrelevant). In order to make the theory work, one has to choose B and τ such that the rest of Hypotheses are satisfied. For instance, $B(x) = x^\gamma$ and $\tau(x) = x^\alpha$ with $\gamma - \alpha + 1 > 0$ (and then defining $b(x,y)$ through (5)) are valid choices for the same reasons as before.

1.2 Summary of main results

Estimates on the profiles. We describe the asymptotics of the profile G and give accurate bounds on the eigenvector ϕ. Define

$$\Lambda(x) := \int_1^x \frac{\lambda + B(y)}{\tau(y)} dy$$

and

$$\xi := \begin{cases} p_1 & \text{if } \gamma > 0 \text{ and } \nu = 0, \\ p_1 \frac{B_\infty}{\lambda + B_\infty} & \text{if } \gamma = \nu = 0, \\ 0 & \text{if } \gamma \geq 0 \text{ and } \nu > 0, \text{ or } \gamma < 0 \text{ and } \nu > -1 + \frac{\gamma+1-\alpha}{1-\alpha}, \end{cases}$$

where the parameters are the ones appearing in the previous hypotheses. In Section 2.2 we prove the following result, which improves previous estimates of the profile G given in [3, 1, 2]

Theorem 1.7. *Assume Hypotheses 1.1–1.5. There exists $C > 0$ such that*

$$G(x) \underset{x \to +\infty}{\sim} C e^{-\Lambda(x)} x^{\xi - \alpha}. \tag{18}$$

This result works for all the examples given before. For all of them, it shows that the profile G decays exponentially for large sizes, with a precise exponential rate given by $\Lambda(x)$. We observe that $\Lambda(x)$ behaves like $x^{\gamma_+ - \alpha + 1}$ (with $\gamma_+ := \max\{\gamma, 0\}$), which is always a positive power of x. There are some observations about this that match intuition: the equilibrium profile decays faster when the total fragmentation rate is stronger for large sizes, and it decays slower when the growth rate is larger for large sizes. Also, it is interesting to notice that Λ does not depend on the fragment distribution (this is, p), but only on the total fragmentation rate B.

The additional power $x^{\xi - \alpha}$ which gives a correction to the exponential behavior, in turn, depends only on the behavior of the distribution of fragments $p(z)$ close to $z = 1$, this is, on fragments of size close to the size of the particle that breaks. In the mitosis case, for example, $\xi = 0$ since we obtain no fragments of similar size when a particle breaks.

The behavior of $G(x)$ for x close to 0 depends on the power α_0 from Hypothesis 1.3 and the distribution of small fragments that result when a particle breaks. The following result is proven in Section 2.3:

Theorem 1.8. *Assume Hypotheses 1.1–1.4 with $p_0 > 0$. If $\alpha_0 < 1$, there exists $C > 0$ such that*

$$G(x) \underset{x \to 0}{\sim} C\, x^{\mu - \alpha_0}.$$

If $\alpha_0 \geq 1$, there exists $C > 0$ such that

$$G(x) \underset{x \to 0}{\sim} C\, x^{\mu - 1}.$$

This shows that G is (roughly) more concentrated close to 0 the weaker the growth is for smaller sizes; and is less concentrated when there are fewer smaller fragments resulting from breakage. This result includes cases in which $G(x)$ blows up as $x \to 0$, cases in which it behaves like a constant, and cases in which it tends to 0 like a power. We recall that the boundary condition is $\tau(x)G(x) \to 0$ as $x \to 0$, which is always ensured by $\mu > 0$ from Hypothesis 1.4.

For the profile ϕ we derive the following estimates, proved in section 3, by the use of a maximum principle (Lemma 3.2):

Theorem 1.9. *Assume Hypotheses 1.1–1.4. If $\gamma > 0$, there are two positive constants C_1 and C_2 such that*

$$C_1 x \leq \phi(x) \leq C_2 x, \qquad \forall x > 1. \tag{19}$$

If $\gamma < 0$ and under the additional assumption that $\mu = 1$ and $p_0 > 0$ in Hypothesis 1.4, there exist two positive constants C_1 and C_2 such that

$$C_1 x^{\gamma-1} \leq \phi(x) \leq C_2 x^{\gamma-1}, \qquad \forall x > 1. \tag{20}$$

Estimates of ϕ are significantly harder than those of G, and they have to be obtained through comparison arguments. To our knowledge, this is the first result in which ϕ can be bounded above and below by the same power (except for the cases in which ϕ can be found explicitly). This improves the results in [1] also in that it is valid for a general power-law behavior of τ.

We do not include the case $\gamma = 0$ in the above theorem (this is, $B(x)$ asymptotic to a constant as $x \to +\infty$), but we remark that in the case of $B(x)$ *equal* to a constant (and with the very mild condition that $\int b(x,y)\,dy$ is equal to a constant independent of x), then $\phi \equiv 1$. The case $\tau(x) = \tau_0 x$ is also explicit: in that case, $\lambda = \tau_0$ and $\phi(x) = Cx$ for some number $C > 0$.

As for the behavior at zero, we prove the following result:

Theorem 1.10. *Assume Hypotheses 1.1–1.4. Then there exists a constant $C > 0$ such that*

$$\phi(x) \underset{x \to 0}{\sim} C e^{\Lambda(x)}.$$

We remark that the behavior of $\Lambda(x)$ for small x is determined by whether $(B(x)+\lambda)/\tau(x)$ is integrable close to $x = 0$. Since $B(x)/\tau(x)$ is always integrable close to $x = 0$ by hypothesis (as $\gamma_0 - \alpha_0 > -1$), we deduce that:

1. If $\gamma_0 \leq 0$, then $\phi(x)$ tends to a positive constant as $x \to 0$.
2. If $\gamma_0 > 0$, then there are three possible cases:
   (a) If $\alpha_0 < 1$, then again $\phi(x)$ tends to a positive constant as $x \to 0$.
   (b) If $\alpha_0 = 1$, then $\phi(x)$ behaves like a positive power of x as $x \to 0$.
   (c) If $\alpha_0 > 1$, then $\phi(x)$ decays exponentially fast as $x \to 0$.

Spectral gap. The estimates of the previous theorems allow us to prove a spectral gap inequality. The *general relative entropy principle* [8, 9] applies here and we have

$$\frac{d}{dt}\int_0^\infty \phi(x)G(x)H(u(x))\,dx = \int_0^\infty\int_y^\infty \phi(y)b(x,y)G(x) \times (H(u(x)) - H(u(y)) + H'(u(x))(u(y)-u(x)))\,dx\,dy,$$

where H is any function and we denote

$$u(x) := \frac{g(x)}{G(x)} \quad (x > 0).$$

In the particular case of $H(x) := (x-1)^2$ we define

$$H[g|G] := \int_0^\infty \phi\, G(u-1)^2\,dx \tag{21}$$

$$D[g|G] := \int_0^\infty\int_x^\infty \phi(x)G(y)b(y,x)(u(x)-u(y))^2\,dydx, \tag{22}$$

and obtain that

$$\frac{d}{dt}H[g|G] = -D[g|G] \le 0.$$

The next result shows that H is in fact bounded by a constant times D:

Theorem 1.11. *Assume that the coefficients satisfy Hypotheses 1.1–1.6 with one of the following additional conditions on the exponents γ_0 and α_0:*

- *either $\alpha_0 = 1$ and $\gamma_0 \le 1 + \lambda/\tau_0$,*
- *or $\alpha_0 < 1$ and $\gamma_0 \le 2 - \alpha_0$.*

Consider also that we are in the case $\gamma \neq 0$. Then the following inequality holds

$$H[g|G] \le C D[g|G], \tag{23}$$

for some constant $C > 0$ and for any nonnegative measurable function $g : (0,\infty) \to \mathbb{R}$ such that $\int \phi g = 1$. Consequently, if g_t is a solution of problem (1) the speed of convergence to equilibrium is exponential in the L^2-weighted norm $\|\cdot\| = \|\cdot\|_{L^2(G^{-1}\phi dx)}$, i.e.,

$$H[g_t|G] \le H[g_0|G]\,e^{-Ct} \quad \textit{for } t \ge 0.$$

Remark that in general we do not know the value of the eigenvalue λ which appears in the assumption on γ_0 for the case $\alpha_0 = 1$. Nevertheless in the case of the self-similar fragmentation equation (*i.e.* $\tau(x) \equiv \tau_0 x$) we know by integration of equation (3a) multiplied by x that $\lambda = \tau_0$ and the condition on γ_0 becomes $\gamma_0 \le 2$. Thus Theorem 1.11 includes the result of the first part of [1, Theorem 1.9].

The main restrictions on the coefficients needed for Theorem 1.11 to hold are the following. First, we require Hypothesis 1.6, which says that the fragment distribution p should be bounded below. Consequently, this does not include the mitosis case and other cases in which the fragment distribution has "gaps"; we refer to [5] for a proof that exponential decay does hold in that case, at least for a constant total fragmentation rate. Second, the exponents α_0 and γ_0 cannot be too large, since we need the profile G to be bounded (for which $\alpha_0 \leq 1$ is required) and $B(x)$ not to be too small close to $x = 0$ (in order to ensure that the term $b(x, y)$ which appears in the entropy dissipation is not too small and can be bounded below by our methods). The latter restriction (on γ_0) is probably a shortcoming of the arguments we are using, and we do not know whether the first one $\alpha_0 \leq 1$ is a more fundamental one.

On the other hand, it is remarkable that Theorem 1.11 does not place any restrictions on the behavior of the fragmentation or growth coefficients for large sizes. This is a significant improvement over [1], where the behavior at 0 and $+\infty$ of the coefficients was taken to be the same power of x, and results were restricted to the cases in which τ is constant or linear.

2 Estimates of the profile G

2.1 Estimates of the moments of G

When Hypothesis 1.3 is satisfied, we define

$$\zeta := \lim_{x \to +\infty} x^{\alpha - \gamma_+} \frac{B(x) + \lambda}{\tau(x)} = \frac{\lambda \mathbb{1}_{\gamma \leq 0} + B_\infty \mathbb{1}_{\gamma \geq 0}}{\tau_\infty} = \begin{cases} \frac{B_\infty}{\tau_\infty} & \text{if } \gamma > 0, \\ \frac{\lambda + B_\infty}{\tau_\infty} & \text{if } \gamma = 0, \\ \frac{\lambda}{\tau_\infty} & \text{if } \gamma < 0, \end{cases} \tag{24}$$

where $\gamma_+ = \max\{0, \gamma\}$. Remark that, for $\gamma \geq 0$, we have the relation

$$\xi = p_1 \frac{B_\infty}{\tau_\infty} \zeta^{-1}. \tag{25}$$

Lemma 2.1. *Assume Hypotheses 1.1–1.4. For any $m > 1 + \xi$ it holds that*

$$\int_1^\infty G(x)\, e^{\Lambda(x)}\, x^{\alpha - m}\, dx < +\infty.$$

Proof. As usual, we carry out a priori estimates which can be rigorously justified by an approximation procedure (such as the truncated equation (49)). As G is integrable, it is enough to prove the convergence of the above integral on $(x_0, +\infty)$ for a sufficiently large $x_0 > 0$. Hence, take any $x_0 > 0$, multiply (3a) by $x^{1-m} e^{\Lambda(x)}$ with $m > 1 + \xi$ and integrate on $(x_0, +\infty)$ to obtain

$$\begin{aligned} - G(x_0) e^{\Lambda(x_0)} \tau(x_0) x_0^{1-m} + (m-1) \int_{x_0}^\infty G(x) e^{\Lambda(x)} \tau(x) x^{-m}\, dx \\ = \int_{x_0}^\infty G(y) \int_{x_0}^y e^{\Lambda(x)} x^{1-m} b(y, x)\, dx\, dy \end{aligned} \tag{26}$$

where we have done an integration by parts on the last term.

We first consider the case $\xi > 0$ (this is, $\gamma \geq 0$ and $\nu = 0$). From Equation (9) we have that for any $\epsilon > 0$ there exists $x_0 > 0$ such that

$$(m-1)\int_{x_0}^{\infty} G(x)e^{\Lambda(x)}\tau(x)x^{-m}\,dx \\ \geq (m-1)(1-\epsilon)\tau_\infty \int_{x_0}^{\infty} G(x)e^{\Lambda(x)}x^{\alpha-m}\,dx, \quad (27)$$

and, applying Lemma B.3, also such that

$$\int_{x_0}^{\infty} G(y)\int_{x_0}^{y} e^{\Lambda(x)}x^{1-m}b(y,x)\,dx\,dy \\ \leq (1+\epsilon)B_\infty p_1 \zeta^{-1}\int_{x_0}^{\infty} G(y)e^{\Lambda(y)}y^{\alpha-m}\,dy \quad (28)$$

(observe that we have used $\gamma \geq 0$ and $\nu = 0$ here). Using (27) and (28) we obtain from (26) that

$$\Big((m-1)(1-\epsilon)\tau_\infty - (1+\epsilon)B_\infty p_1\zeta^{-1}\Big)\int_{x_0}^{\infty} G(x)e^{\Lambda(x)}x^{\alpha-m}\,dx \\ \leq G(x_0)e^{\Lambda(x_0)}\tau(x_0)x_0^{1-m}.$$

When $(m-1)(1-\epsilon)\tau_\infty - (1+\epsilon)B_\infty p_1\zeta^{-1} > 0$ this gives a bound for the integral on the left hand side. If $m > 1+\xi$ we can always choose ϵ small enough for this to be true, because of relation (25), and it proves the result.

The remaining case is $\xi = 0$, this is, $\nu > -1 + \frac{\gamma+1-\alpha}{\gamma_+ + 1-\alpha}$. In this case we have to substitute (28) by the following, according to Lemma B.3:

$$\int_{x_0}^{\infty} G(y)\int_{x_0}^{y} e^{\Lambda(x)}x^{1-m}b(y,x)\,dx\,dy \\ \leq (1+\epsilon)B_\infty p_1\zeta^{-1-\nu}\Gamma(1+\nu)\int_{x_0}^{\infty} G(y)e^{\Lambda(y)}y^{1-m+\gamma-(\gamma_+-\alpha+1)(1+\nu)}\,dy. \quad (29)$$

Since $\nu > -1 + \frac{\gamma+1-\alpha}{\gamma_+ + 1-\alpha}$, we have

$$1-m+\gamma-(\gamma_+-\alpha+1)(1+\nu) < -m+\alpha.$$

Thus the exponent of y on the right hand side of (29) is strictly smaller than $\alpha - m$, so we can always find x_0 large enough so that

$$\int_{x_0}^{\infty} G(y)\int_{x_0}^{y} e^{\Lambda(x)}x^{1-m}b(y,x)\,dx\,dy \leq \epsilon\int_{x_0}^{\infty} G(y)e^{\Lambda(y)}y^{\alpha-m}\,dy.$$

Using this and (27) in (26) we may follow a similar reasoning as before to obtain the result. □

2.2 Asymptotic estimates of G as $x \to +\infty$

In this section we prove Theorem 1.7 by using the moment estimates in Section 2.1.

Proof of Theorem 1.7. We divide the proof in two steps:

Step 1: proof that the limit is finite. Again, we carry out a priori estimates on the solution which can be fully justified by using the approximation (49). Let us first prove that $x^{\alpha-\xi} G(x) e^{\Lambda(x)}$ has a finite limit $C \geq 0$ as $x \to +\infty$, and later we will show that $C > 0$. We use equation (3a) to obtain

$$(x^{-\xi}\tau(x)G(x)e^{\Lambda(x)})' = -\xi x^{-\xi-1}\tau(x)G(x)e^{\Lambda(x)} + x^{-\xi}e^{\Lambda(x)}\int_x^\infty b(y,x)G(y)\,\mathrm{d}y.$$

Let us show that the right hand side of this last expression is integrable on $(x_0, +\infty)$ for some $x_0 > 0$. Once we have this the result is proved, since then $x^{\alpha-\xi}G(x)e^{\Lambda(x)}$ must have a limit as $x \to +\infty$. Integrating the right hand side we obtain:

$$\begin{aligned} - \xi \int_{x_0}^\infty x^{-\xi-1}\tau(x)G(x)e^{\Lambda(x)}\,\mathrm{d}x + \int_{x_0}^\infty x^{-\xi}e^{\Lambda(x)} \int_x^\infty b(y,x)G(y)\,\mathrm{d}y\,\mathrm{d}x \\ = \int_{x_0}^\infty G(x) \left(\int_{x_0}^x y^{-\xi}b(x,y)e^{\Lambda(y)}\,\mathrm{d}y - \xi x^{-\xi-1}\tau(x)e^{\Lambda(x)} \right) dx. \end{aligned} \tag{30}$$

We just need to show that the parenthesis is of the order of $e^{\Lambda(x)}x^{\alpha-\xi-1-\epsilon}$ for some $\epsilon > 0$, and then Lemma 2.1 shows that the above integral is finite.

The case $\xi > 0$. Let us start considering the case $\xi > 0$ (this is, $\gamma \geq 0$ and $\nu = 0$). Using Lemma B.3

$$\int_{x_0}^x y^{-\xi}b(x,y)e^{\Lambda(y)}\,\mathrm{d}y = p_1 B_\infty \zeta^{-1}\, x^{\alpha-\xi-1}e^{\Lambda(x)} + O(x^{-\eta+\alpha-1-\epsilon})e^{\Lambda(x)}, \tag{31}$$

for some $\epsilon > 0$. From (16) we also have

$$\xi x^{-\xi-1}\tau(x)e^{\Lambda(x)} = \tau_\infty\, \xi\, x^{\alpha-\xi-1}e^{\Lambda(x)} + O(x^{\alpha-\xi-1-\delta})e^{\Lambda(x)}. \tag{32}$$

Using (31)-(32) and the relation (25), the parenthesis in (30) is, in absolute value, less than $Cx^{\alpha-\xi-1-\delta}e^{\Lambda(x)}$ for some constant $C > 0$. Hence by Lemma 2.1 the integral in (30) is finite, and we conclude that $x^{\alpha-\xi}G(x)e^{\Lambda(x)}$ has a finite limit as $x \to +\infty$ when $\xi > 0$.

The case $\xi = 0$. In this case we have from Lemma B.3

$$\int_{x_0}^{x} b(x,y)e^{\Lambda(y)}\,\mathrm{d}y \sim\ p_1 B_\infty \zeta^{-1-\nu}\Gamma(1+\nu)x^{\gamma-(\gamma_+-\alpha+1)(1+\nu)}e^{\Lambda(x)}.$$

Using the same reasoning as at the end of the proof of Lemma 2.1 we have that, when $\xi = 0$,

$$\gamma - (\gamma_+ - \alpha + 1)(1+\nu) < \alpha - 1,$$

which then shows that the right hand side of (30) is finite due to Lemma 2.1.

Step 2: proof that $C > 0$. In order to show that $C > 0$ in (18) set $F(x) := \tau(x)G(x)e^{\Lambda(x)}$ and obtain the following from (3a):

$$F'(x) = e^{\Lambda(x)}\int_x^\infty b(y,x)G(y)\,\mathrm{d}y. \tag{33}$$

In particular, F is nondecreasing, and this is enough to conclude in the case $\xi = 0$ (since then $\tau(x)G(x)e^{\Lambda(x)}$ must converge to a positive quantity, so the same must be true of $x^\alpha G(x)e^{\Lambda(x)}$). In the case $\xi > 0$ we may bound

$$\begin{aligned} F'(x) = e^{\Lambda(x)}\int_x^\infty b(y,x)\frac{1}{\tau(y)}e^{-\Lambda(y)}F(y)\,\mathrm{d}y \\ \geq F(x)e^{\Lambda(x)}\int_x^\infty b(y,x)\frac{1}{\tau(y)}e^{-\Lambda(y)}\,\mathrm{d}y, \end{aligned}$$

which implies that

$$F(x) \geq F(x_0)\exp\left(\int_{x_0}^x S(w)\,\mathrm{d}w,\right)$$

with

$$S(w) := e^{\Lambda(w)}\int_w^\infty b(y,w)\frac{1}{\tau(y)}e^{-\Lambda(y)}\,\mathrm{d}y.$$

Due to equation (16) we have

$$\frac{1}{\tau(x)} = \frac{1}{\tau_\infty x^\alpha} + R_1(x),$$

with $R_1(x) = O(x^{-\alpha-\delta})$. Using this, and due to Lemma B.3,

$$\begin{aligned} \int_{x_0}^x S(w)\,\mathrm{d}w &\geq \int_{x_0}^x e^{\Lambda(w)}\int_w^x b(y,w)\frac{1}{\tau(y)}e^{-\Lambda(y)}\,\mathrm{d}y\,\mathrm{d}w \\ &= \int_{x_0}^x \frac{1}{\tau(y)}e^{-\Lambda(y)}\int_{x_0}^y e^{\Lambda(w)}b(y,w)\,\mathrm{d}w\,\mathrm{d}y \\ &= p_1 B_\infty \zeta^{-1}\int_{x_0}^x \frac{1}{\tau(y)}(y^{\alpha-1} + R_2(y))\,\mathrm{d}y \\ &= \xi\int_{x_0}^x \frac{1}{y}\,\mathrm{d}y + \int_{x_0}^x R_3(y)\,\mathrm{d}y \geq \xi\log(y) + C_1, \end{aligned}$$

with $R_2(y) = O(\alpha - 1 - \epsilon)$, $R_3(y) = O(-1 - \epsilon)$, and $C_1 \in \mathbb{R}$ some real number. As a consequence,

$$F(x) \geq F(x_0) x^{\xi} e^{C_1},$$

which shows that $\lim_{x \to +\infty} F(x) x^{-\xi}$ (which we know exists) must be strictly positive. This finishes the proof. □

2.3 Asymptotic estimates of G as $x \to 0$

Proof of Theorem 1.8. Define

$$F(x) := \tau(x) G(x) e^{\Lambda(x)}.$$

We know from [2] that $F(x) \to 0$ when $x \to 0$ and more precisely that $F(x) \leq C\, x^{\mu}$. The derivative of F, as noted in (33), is

$$F'(x) = e^{\Lambda(x)} \int_x^{\infty} b(y, x) G(y) \, \mathrm{d}y > 0$$

so F is increasing.

Case $\alpha_0 < 1$. In this case, $\Lambda(x) \to \Lambda(0) < 0$. Choose $\epsilon > 0$ such that p is a function on $[0, \epsilon)$ (the fact that this can be done for small enough ϵ is implicit in Hypothesis 1.4), and call $p_* = p \mathbb{1}_{[0,\epsilon]}$. Then, from Hypothesis 1.4,

$$x^{1-\mu} p_* \Big(\frac{x}{y} \Big) \to p_0 \, y^{1-\mu} \qquad \text{as } x \to 0, \tag{34}$$

with the above convergence being pointwise in y. We may additionally choose $\epsilon \in (0, 1)$ and $C > 0$ such that

$$p(z) \leq C z^{\mu - 1} \qquad \text{for all } z \in (0, \epsilon). \tag{35}$$

Now we write

$$\begin{aligned} x^{1-\mu} & \int_x^{\infty} b(y, x) G(y) \, dy \\ &= x^{1-\mu} \int_x^{\frac{x}{\epsilon}} \frac{B(y)}{y} G(y) p\Big(\frac{x}{y} \Big) \, dy + x^{1-\mu} \int_{\frac{x}{\epsilon}}^{\infty} \frac{B(y)}{y} G(y) p\Big(\frac{x}{y} \Big) \, dy \\ &= x^{1-\mu} \int_{\epsilon}^{1} B\Big(\frac{x}{z} \Big) G\Big(\frac{x}{z} \Big) p(z) \frac{dz}{z} + x^{1-\mu} \int_{\frac{x}{\epsilon}}^{\infty} \frac{B(y)}{y} G(y) p_*\Big(\frac{x}{y} \Big) \, dy. \end{aligned}$$

For the first term in the r.h.s., we use that $B(y) \underset{y \to 0}{\sim} B_0 y^{\gamma_0}$ and $G(y) \leq C y^{\mu - \alpha_0}$ (see [2]) to write

$$x^{1-\mu} \int_{\epsilon}^{1} B\Big(\frac{x}{z} \Big) G\Big(\frac{x}{z} \Big) p(z) \frac{dz}{z} \leq C x^{\gamma_0 + 1 - \alpha_0} \int_{\epsilon}^{1} z^{\alpha_0 - \mu - \gamma_0 - 1} p(z) \, dz$$

and conclude that it tends to zero when $x \to 0$ since $\gamma_0 + 1 - \alpha_0 > 0$. For the second term, we use (34) and (35) to obtain by dominated convergence

$$x^{1-\mu} \int_{\frac{x}{\epsilon}}^{\infty} \frac{B(y)}{y} G(y) p_* \left(\frac{x}{y}\right) dy \xrightarrow[x \to 0]{} p_0 \int_0^{\infty} B(y) y^{-\mu} G(y)\, dy.$$

This limit is strictly positive and finite, since $G(y) \le C y^{\mu - \alpha_0}$ and $\gamma_0 - \alpha_0 > -1$. Finally, we have deduced that there is a positive constant $C > 0$ such that

$$F'(x) \underset{x \to 0}{\sim} C\, x^{\mu - 1},$$

which by integration gives

$$\tau(x) G(x) \sim C\, x^{\mu}$$

and so

$$G(x) \underset{x \to 0}{\sim} C\, \frac{x^{\mu}}{\tau(x)} \underset{x \to 0}{\sim} C\, x^{\mu - \alpha_0}.$$

Case $\alpha_0 \ge 1$. In this case we necessarily have $\gamma > 0$ and

$$\Lambda(x) \sim -C\, x^{1-\alpha_0}.$$

As a consequence, following a similar reasoning as for the previous case, we have

$$F'(x) \sim C_1\, x^{\mu - 1} e^{-C_2\, x^{1-\alpha_0}}$$

and consequently

$$F(x) \sim C_1 \int_0^x y^{\mu-1} e^{-C_2\, y^{1-\alpha_0}} dy \sim C_3\, x^{\alpha_0 + \mu - 1} e^{-C_2\, x^{1-\alpha_0}}$$

due to the l'Hôpital's rule. This finally gives $G(x) \underset{x \to 0}{\sim} C\, x^{\mu - 1}$. □

3 Estimates of the dual eigenfunction ϕ

3.1 Asymptotic estimates of ϕ as $x \to 0$

We first give the proof of Theorem 1.10, which is rather direct:

Proof of Theorem 1.10. Define

$$\psi(x) := \phi(x) e^{-\Lambda(x)}.$$

This function is decreasing since it satisfies

$$\psi'(x) = -\frac{1}{\tau(x)} \int_0^x b(x,y) \phi(y)\, dy\, e^{-\Lambda(x)} < 0.$$

Moreover it is a positive function, so to prove Theorem 1.10 we only have to prove that ψ is bounded at $x = 0$. Consider, for $\eta > 0$, τ_η as defined in the approximation procedure (see (48) in Appendix A). Then denote by ϕ_η, Λ_η and ψ_η the corresponding functions. First we know from [2] that ϕ_η converges locally uniformly to ϕ when $\eta \to 0$. We have, for $\eta > 0$, that $-\Lambda_\eta(x) = \int_x^1 \frac{\lambda + B(y)}{\tau_\eta(y)} dy$ is bounded at $x = 0$ and this is why it is useful to consider this regularization. We have for any $x_0 > 0$,

$$\begin{aligned}
\sup_{\mathbb{R}^+} \psi_\eta = \psi_\eta(0) &= \psi_\eta(x_0) + \int_0^{x_0} \frac{1}{\tau_\eta(y)} \int_0^y b(y,z)\phi_\eta(z)\, dz\, e^{-\Lambda_\eta(y)}\, dy \\
&\le \psi_\eta(x_0) + \int_0^{x_0} \frac{1}{\tau(y)} \int_0^y b(y,z)\phi_\eta(z)\, e^{-\Lambda_\eta(z)}\, dz\, dy \\
&\le \psi_\eta(x_0) + \sup \psi_\eta \int_0^{x_0} \frac{1}{\tau_\eta(y)} \int_0^y b(y,z)\, dz\, dy \\
&= \psi_\eta(x_0) + \sup \psi_\eta \int_0^{x_0} \frac{B(y)}{\tau_\eta(y)} \int_0^y p\left(\frac{z}{y}\right) \frac{dz}{y}\, dy \\
&= \psi_\eta(x_0) + \pi_0 \sup \psi_\eta \int_0^{x_0} \frac{B(y)}{\tau(y)}\, dy.
\end{aligned}$$

Now, because $\frac{B}{\tau}$ is integrable at $x = 0$, we can choose $x_0 > 0$ such that $\pi_0 \int_0^{x_0} \frac{B(y)}{\tau(y)}\, dy = \rho < 1$ and we obtain

$$(1-\rho) \sup \psi_\eta(x) \le \psi_\eta(x_0) \xrightarrow[\eta \to 0]{} \psi(x_0).$$

So ψ_η is uniformly bounded when $\eta \to 0$ and thus the limit $\psi(x)$ is bounded. □

3.2 A maximum principle

For finding the bounds on the dual eigenfunction at $x \to +\infty$ we use comparison arguments, valid for each truncated problem on $[0, L]$ (see Appendix A for details on the truncation). Then we pass to the limit, as the bounds we obtain are independent of L. The function $\phi_L(x)$ satisfies the equation

$$\mathcal{S}\phi_L(x) = 0 \quad (x \in (0, L)),$$

where $\mathcal{S}$ is the operator given by

$$\mathcal{S}w(x) := -\tau(x)w'(x) + (B(x) + \lambda_L)w(x) - \int_0^x b(x,y)w(y)dy,$$

defined for all functions $w \in W^{1,\infty}(0, L)$ and for $x \in (0, L)$. This operator satisfies

$$\forall w \in W^{1,\infty}(0, L) \ s.t.\ w(L) = 0, \qquad \int_0^L \mathcal{S}w(x)\, G_L(x)\, dx = 0 \qquad (36)$$

where G_L is the eigenfunction of the truncated growth-fragmentation operator. We recall the concept of *supersolution*:

Definition 3.1. We say that $w \in W^{1,\infty}(0,L)$ is a *supersolution* of $\mathcal{S}$ on the interval $I \subseteq (0,L)$ when

$$\mathcal{S}w(x) > 0 \ (x \in I).$$

Maximum principles were a powerful tool for proving the existence of sub and supersolutions for the growth-fragmentation models as in [7, 2]. For our case, we recall the maximum principle given in [2].

Lemma 3.2 (Maximum principle for $\mathcal{S}$)**.** *Assume Hypotheses 1.1-1.3. There exists $A > 0$, independent of L, such that if w is a supersolution of $\mathcal{S}$ on (A,L), $w \geq 0$ on $[0,A]$ and $w(L) \geq 0$ then $w \geq 0$ on $[A,L]$.*

Proof. We start from the fact w is a supersolution on (A,L)

$$-\tau(x)w'(x) + (B(x) + \lambda_L)w(x) - \int_0^x b(x,y)w(y)\,dy =: f(x) > 0.$$

Testing this equation against $\mathbb{1}_{w\leq 0}$ we obtain on (A,L)

$$\begin{aligned}-\tau(x)w_-'(x) + (B(x) + \lambda_L)w_-(x) &= \mathbb{1}_{w(x)<0}\int_0^x b(x,y)w(y)\,dy + f(x)\mathbb{1}_{w(x)\leq 0}\\ &\geq \int_0^x b(x,y)w_-(y)\,dy + f(x)\mathbb{1}_{w(x)\leq 0}.\end{aligned}$$

Extend f by zero on $[0,A]$. Since $w_-(x) = 0$ on $[0,A]$ by assumption, the latter inequality holds true on $(0,L)$ and it writes

$$\forall x \in (0,L), \quad \mathcal{S}w_-(x) \geq f(x)\mathbb{1}_{w(x)\leq 0}.$$

Testing this last inequality against G_L, we obtain using (36)

$$0 \geq \int_0^L f(x)\mathbb{1}_{w(x)\leq 0}G_L(x)\,dx = \int_A^L f(x)\mathbb{1}_{w(x)\leq 0}G_L(x)\,dx.$$

Because f and G_L are positive on (A,L), this is possible only if $\mathbb{1}_{w\leq 0} \equiv 0$ on (A,L) and it ends the proof. □

3.3 Asymptotic estimates of ϕ as $x \to +\infty$

Now we prove the results concerning the asymptotic behavior of $\phi(x)$ when $x \to +\infty$, Theorem 1.9. For these results, we still assume that Hypotheses 1.1-1.4 are satisfied and, in the case $\gamma < 0$, we additionally assume that $\mu = 1$ and $p_0 > 0$ (so that $p(z) \xrightarrow[z\to 0]{} p_0 > 0$). We recall that Hypothesis 1.3 says that $B(x)$ behaves like a γ-power of x and $\tau(x)$ like an α-power of x, with $\gamma + 1 - \alpha > 0$.

Proof of Theorem 1.9. The proof is done in two cases, and each case is proved in two steps. In the first step we give particular supersolutions and prove the upper bound, and in the second one we do the corresponding for lower bounds.

Case 1: $\gamma > 0$**.**

Step 1: Upper bounds. We claim that for any $C > 0$, there exists $A > 0$ and $L_* > 0$ such that

$$v(x) := Cx + 1 - x^k$$

is a supersolution on $[A, L]$ for any $L > L_*$, provided that $\max(0, \alpha-\gamma) < k < 1$. First we recall that $\gamma + 1 - \alpha > 0$ by assumption, so $\alpha - \gamma < 1$ and we can find $k \in (\alpha - \gamma, 1)$. Then

$$\mathcal{S}v(x) = -\tau(x)(C - kx^{k-1}) + \lambda(Cx - x^k + 1) + (\pi_k - 1)B(x)x^k - (\pi_0 - 1)B(x)$$

and the right hand side is positive for x large enough because the dominant term is $C\lambda x + (\pi_k - 1)B(x)x^k \sim C\lambda x + (\pi_k - 1)B_\infty x^{\gamma+k}$. Indeed $\pi_k > 1$ because $k < 1$ (see Remark 1.1) and the dominant power is $\gamma + k$ because $k > 0$ and $\gamma + k > \alpha$.

Now we prove that there exists $C > 0$ such that

$$\forall\, x > 0, \qquad \phi(x) \leq C(1 + x).$$

First we can choose C such that $v(x) = Cx + 1 - x^k$ is bounded below by a positive constant. Moreover we take an approximation ϕ_L of ϕ such that $\phi_L(L) = 0$. Then, choosing $K > 0$ large enough, we have that $Kv(x) > \phi(x)$ on $[0, A]$ because ϕ is bounded uniformly in L on $[0, A]$, and $Kv(L) = KCL + K - KL^k > 0$ for L large enough. So, using the maximum principle and the previous lemma, we obtain that

$$\forall\, x > 0, \qquad \phi(x) \leq Kv(x) \leq C(1 + x).$$

Step 2: Lower bounds. For the lower bounds we first prove that $v(x) := x + x^k - 1$ is a subsolution for $\max(0, 1 - \gamma) < k < 1$.
The idea is to use x^k to transform x which is a supersolution into a subsolution.

$$\mathcal{S}v(x) = -\tau(x)(1 + kx^{k-1}) + \lambda(x + x^k - 1) - (\pi_k - 1)B(x)x^k + (\pi_0 - 1)B(x)$$

where $\pi_k > 1$ since $k < 1$. Due to Assumption (7), $B(x)x^k \sim B_\infty x^{\gamma+k}$ and $v(x)$ is a subsolution for x large because $k > 0$ and $\gamma + k > 1$.

For $\gamma > 0$, there exists $C > 0$ such that

$$\forall\, x > 0, \qquad \phi(x) \geq C(x - 1)_+.$$

We know that ϕ is positive, so for C small enough, $C(x + x^k - 1) - \phi(x) < 0$ on $[0, A]$. Moreover, taking an approximation ϕ_L of ϕ such that $\phi_L(L) = L$, we have $Cv(L) - \phi(L) < 0$ for $C < 1$ and L large enough. Finally we use the lemmas on the maximum principle and the subsolution to conclude that there exists $C > 0$ such that

$$\forall x > 0, \qquad \phi(x) > C(x + x^k - 1)$$

and the result follows.

Case 2: $\gamma < 0$.

Step 1: Upper bounds. We start by proving that for any $\eta > \left(\frac{-\gamma\lambda}{B_\infty p_0}\right)^{\frac{1}{\gamma}}$, $v(x) = (\eta + x)^{\gamma-1}$ is a supersolution. We compute

$$\mathcal{S}v(x) = (1-\gamma)\tau(x)(\eta+x)^{\gamma-2} + (\lambda + B(x))(\eta+x)^{\gamma-1} \\ - \int_0^x b(x,y)(\eta+y)^{\gamma-1}\,dy$$

and to estimate the last term in the r.h.s. we proceed similarly as in the proof of Theorem 1.8. We write, for $\epsilon \in (0,1)$,

$$\int_0^x b(x,y)(\eta+y)^{\gamma-1}\,dy = \frac{B(x)}{x}\int_0^{\epsilon x}(\eta+y)^{\gamma-1}p\Big(\frac{y}{x}\Big)\,dy \\ + B(x)\int_\epsilon^1 (\eta+zx)^{\gamma-1}p(z)\,dz.$$

Then, choosing ϵ such that (35) is satisfied (for this we use Hypothesis 1.4), we obtain by dominated convergence from (34) that

$$\frac{B(x)}{x}\int_0^{\epsilon x}(\eta+y)^{\gamma-1}p\Big(\frac{y}{x}\Big)\,dy \underset{x\to+\infty}{\sim} \frac{B(x)}{x}\frac{p_0\eta^\gamma}{-\gamma}.$$

On the other hand we have

$$B(x)\int_\epsilon^1(\eta+zx)^{\gamma-1}p(z)\,dz \underset{x\to+\infty}{\sim} x^{\gamma-1}B(x)\int_\epsilon^1 z^{\gamma-1}p(z)\,dz.$$

Since $\gamma < 0$, we obtain

$$\int_0^x b(x,y)(\eta+y)^{\gamma-1}\,dy \underset{x\to+\infty}{\sim} \frac{B_\infty p_0 \eta^\gamma}{-\gamma}x^{\gamma-1}$$

and finally

$$\mathcal{S}v(x) \underset{x\to+\infty}{\sim} \left(\lambda - \frac{B_\infty p_0\eta^\gamma}{-\gamma}\right)x^{\gamma-1}$$

because $\tau(x) \sim \tau_\infty x^\alpha$ and $\alpha - 1 < \gamma < 0$. So $v(x)$ is a supersolution for x large when $\eta > \left(\frac{-\gamma\lambda}{B_\infty p_0}\right)^{\frac{1}{\gamma}}$.

Now, we claim that there exist $C > 0$ and $\epsilon > 0$ such that

$$\forall\, x > 0, \qquad \phi(x) \le C(\eta+x)^{\gamma-1}.$$

The proof of this fact follows from the maximum principle and taking the an approximation ϕ_L of ϕ such that $\phi_L(L) = 0$ and that $v(x)$ is a supersolution.

Step 2: Lower bounds. For the lower bounds we define

$$v(x) := \begin{cases} 0 & \text{for } 0 < x < \epsilon, \\ (x-\epsilon)x^{\gamma-2} & \text{for } x > \epsilon. \end{cases}$$

Then for $\epsilon < \left(\frac{\lambda\gamma(\gamma-1)}{B_\infty p_0}\right)^{\frac{1}{\gamma}}$, v is a subsolution. Indeed we have for $x > \epsilon$

$$\mathcal{S}v(x) = \tau(x)(x^{\gamma-2} + (\gamma-2)(x-\epsilon)x^{\gamma-3}) + (\lambda + B(x))(x-\epsilon)x^{\gamma-2} \\ - \int_\epsilon^x b(x,y)(y-\epsilon)y^{\gamma-2}\,dy$$

and, reasoning as in Step 1, we obtain that

$$\mathcal{S}v(x) \underset{x\to+\infty}{\sim} (\lambda - B_\infty p_0 C_\epsilon)x^{\gamma-1}.$$

Finally, there exist $C > 0$ and $\epsilon > 0$ such that

$$\forall\, x > 0, \qquad \phi(x) \geq Cx^{\gamma-2}(x-\epsilon)_+.$$

Again, choosing an approximation ϕ_L of ϕ such that $\phi_L(L) = L$, the proof uses the maximum principle and the fact that $v(x)$ is a subsolution. □

4 Entropy dissipation inequality

As it was seen in [8, 9, 5, 1] the *general relative entropy principle* applies to solutions of (1). We remind that we use the entropy $H[g|G]$ defined in (21), with dissipation $D[g|G]$ given by (22). We recall that

$$\frac{d}{dt}H[g|G] = -D[g|G] \leq 0.$$

For the proof of the entropy inequality we will use [1, Lemma 2.2] with $\zeta(x) \equiv 1$. We need to check its hypotheses.

Lemma 4.1. *Assume that Hypotheses 1.1-1.3 and 1.6 are satisfied with $\gamma \neq 0$ and $\alpha_0 \leq 1$. Given $M > 1$ there exists $K > 0$ and $R > 1$ such that the profiles ϕ and G satisfy the relations*

$$0 \leq G(x) \leq K \quad (x > 0), \tag{37}$$

$$\int_{Rx}^\infty G(y)\phi(y)\,dy \leq KG(x) \quad (x > M), \tag{38}$$

$$\phi(y) \leq K\phi(z) \quad (\max\{2RM, Rz\} < y < 2Rz). \tag{39}$$

Proof. Relation (37) is true because of Theorem 1.7 and Theorem 1.8 with $\mu = 1$ and $\alpha_0 \leq 1$.

For the second one, we have due to the l'Hôpital's rule and using Theorem 1.7

$$\int_{Rx}^{\infty} G(y)\phi(y)\,dy \le K\int_{Rx}^{\infty} y^{1+\xi-\alpha}e^{-\Lambda(y)}\,dy \sim Kx^{1+\xi-\gamma}e^{-\Lambda(Rx)}$$
$$\le Kx^{\xi-\alpha}e^{-\Lambda(x)} \sim KG(x).$$

Finally, (39) is a consequence of Theorem 1.9. □

Moreover, for proving the entropy-entropy dissipation inequality, we will need the following bounds coming from [1, Theorem 2.4].

Lemma 4.2. *Suppose that the coefficients satisfy Hypotheses 1.1-1.3 and 1.6 with one of the following additional conditions on the exponents γ_0 and α_0:*

- *either $\alpha_0 = 1$ and $\gamma_0 \le 1 + \lambda/\tau_0$,*
- *or $\alpha_0 < 1$ and $\gamma_0 \le 2 - \alpha_0$.*

Let G and ϕ be the stationary profiles for the problems (3) and (4). Then we can choose constants $K, M > 0$ and $R > 1$ such that the profiles ϕ and G satisfy

- *If $\gamma > 0$ then*

$$G(x)\phi(y) \le Kb(y,x) \qquad (0 < x < y < \max\{2Rx, 2RM\}), \tag{40}$$
$$y^{-1} \le Kb(y,x) \qquad (y > M,\ y > x > 0). \tag{41}$$

- *If $\gamma < 0$ then*

$$G(x)\phi(y) \le Kb(y,x) \qquad (0 < x < y). \tag{42}$$

Observe that under Hypothesis 1.6, there is no chance for $\alpha_0 > 1$ to bound $G(x)\phi(y)$ by $b(y,x)$ when $y > 0$ is fixed and $x \to 0$. Indeed Hypothesis 1.6 imposes $b(y,x) \ge C_1\frac{B(y)}{y}$ and $\mu = 1$, and Theorem 1.8 ensures that $G(x) \sim Cx^{1-\alpha_0}$.

Proof. **Step 1: $y \le 2RM$ and $x < y$.** We need to estimate $G(x)\phi(y)$ at the limit $x < y \to 0$. Using Theorem 1.8 ($G(x) \sim Cx^{1-\alpha_0}$, notice that due to Hypothesis 1.6 one has $\mu = 1$) and Theorem 1.10 to bound $G(x)$ and $\phi(y)$ respectively, we have

$$G(x)\phi(y) \le Cx^{1-\alpha_0}e^{\Lambda(y)} \le C'\,y^{1-\alpha_0}$$

since $\alpha_0 \le 1$. Then under the condition $\gamma_0 \le 2 - \alpha_0$ and from Hypothesis 1.6, we get

$$G(x)\phi(y) \le Cy^{\gamma_0 - 1} \le Kb(y,x).$$

When $\alpha_0 = 1$, we can do better since in this case we have necessarily $\gamma_0 > 0$ and

$$\Lambda(y) \underset{y\to 0}{\sim} \frac{\lambda}{\tau_0}\ln(y).$$

Thus we can write

$$G(x)\phi(y) \leq Cy^{\lambda/\tau_0}$$

and we obtain the bound $G(x)\phi(y) \leq Kb(y,x)$ from Hypothesis 1.6 as soon as $\gamma_0 \leq 1 - \frac{\lambda}{\tau_0}$.

Step 2: $2RM < y \leq 2Rx$. We need to estimate $G(x)\phi(y)$ at the limit $2Rx \geq y \to +\infty$. Using (19) and (18) we have

$$\begin{aligned} G(x)\phi(y) &\leq C(1+y)x^{\xi-\alpha}e^{-\Lambda(x)} \\ &\leq C(1+y)y^{\xi-\alpha}e^{-\Lambda(y/2R)} \\ &\leq C''y^{\gamma-1} \end{aligned}$$

where C'' depends on α, γ and R. We conclude by using Hypothesis 1.6.

Step 3: $y > M$ **and** $y > x > 0$. Since $\alpha_0 \leq 1$ by assumption, we know from Theorems 1.8 and 1.7 that $G(x)$ is bounded. When $\gamma > 0$, we observe first that $y^{-1} \leq Cy^{\gamma-1}$ and we conclude that (41) holds true by using Hypothesis 1.6. When $\gamma < 0$, we have from Theorem 1.9 and Hypothesis 1.6 that $\phi(y) \leq Cy^{\gamma-1} \leq Kb(y,x)$. □

At this point, we have all the tools to prove the entropy - entropy dissipation inequality.

Proof of Theorem 1.11. From [1, Lemma 2.1] one can rewrite the entropy as follows

$$D_2[g|G] := \int_0^\infty \int_x^\infty \phi(x)G(x)\phi(y)G(y)(u(x)-u(y))^2 \,\mathrm{d}y\,\mathrm{d}x = H[g|G]. \quad (43)$$

If one looks at the integrand, one realizes that D and D_2 have both $\phi(x)$ and $G(y)$ as a common terms. So we would like to compare and check that

$$G(x)\phi(y) \leq Kb(y,x). \quad (44)$$

We will denote by C any constant depending on G, ϕ, K, M, or R, but not on g. We now distinguish two cases.

Case $\gamma < 0$**.** The relation (44) is satisfied due to (42). So we can compare pointwise the integrands of $D_2[g|G]$ with $D[g|G]$ and the inequality (23) holds.

Case $\gamma > 0$**.** For proving the case $\gamma > 0$ we follow the same argument as in [1, Theorem 2.4]. We start by rewriting $D_2[g|G]$ as follows:

$$D_2[g|G] = D_{2,1}[g|G] + D_{2,2}[g|G],$$

where

$$D_{2,i} := \iint_{A_i} \phi(x)G(x)\phi(y)G(y)(u(x)-u(y))^2 \,\mathrm{d}y\,\mathrm{d}x$$

with $A_1 := \{(x,y) \in \mathbb{R}^2_+ \,:\, y > x \,,\, y \le RM \text{ or } y < Rx\}$ and $A_2 = A_1^c$. For the first term and thanks to (40) one has

$$D_{2,1}[g|G] \le \int_0^\infty \int_x^{\max\{2Rx, 2RM\}} Kb(y,x)\phi(x)G(y)(u(x)-u(y))^2 \,\mathrm{d}y\,\mathrm{d}x$$
$$\le KD[g|G]. \tag{45}$$

For the other term, what we have is

$$\begin{aligned} D_{2,2}[g|G] \le& C\int_0^\infty \int_{\max\{x,M\}}^\infty y^{-1}\phi(x)G(y)(u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \\ \le& C\,K\int_0^\infty \int_{\max\{x,M\}}^\infty b(y,x)\phi(x)G(y)(u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \\ \le& C\,K\,D[g|G], \end{aligned} \tag{46}$$

where in the first inequality we applied [1, Lemma 2.2] with the bounds given in Lemma 4.2 and for the second one we used (41). The proof concludes by gathering (45) and (46). □

Appendix

A Approximation procedures

To prove the estimates on the dual eigenfunction ϕ, we use a truncated problem. More precisely, we use alternatively one of the following ones, which differ only in their boundary condition

$$\begin{cases} -\tau(x)\partial_x\phi_L(x) + (B(x)+\lambda_L)\phi_L(x) = \mathcal{L}^*_+(\phi_L)(x) & \text{for } x \in (0,L), \\ \phi_L(L) = 0 \quad \text{or} \quad \phi_L(L) = \delta > 0 \quad \text{or} \quad \phi_L(L) = \delta L, & \\ \phi_L \ge 0, \qquad \int_0^L G(x)\phi_L(x)\,dx = 1. & \end{cases} \tag{47}$$

The following lemma ensures that these truncations converge to the accurate limit when $L \to +\infty$.

Lemma A.1. *There exists $L_0 > 0$ such that for each $L \ge L_0$ the problem (47) has a unique solution (λ_L, ϕ_L) with $\lambda_L > 0$ and $\phi_L \in W^{1,\infty}_{loc}(\mathbb{R}_+)$. Moreover we have*

$$\lambda_L \xrightarrow[L\to+\infty]{} \lambda,$$

$$\forall A > 0, \quad \phi_L \xrightarrow[L\to+\infty]{} \phi \quad \textit{uniformly on } [0,A).$$

Proof. We start with the case $\phi_L(L) = 0$ by following the method in [2]. Define for $\eta > 0$

$$\tau_\eta(x) := \begin{cases} \eta & \text{for } 0 < x < \eta, \\ \tau(x) & \text{for } x > \eta. \end{cases} \tag{48}$$

Then consider for $\epsilon > 0$ and $L > 0$ the truncated (and regularized) eigenvalue problem on $[0, L]$

$$\begin{cases} \dfrac{\partial}{\partial x}(\tau_\eta(x)G_L(x)) + (B(x) + \lambda_L)G_L(x) = \displaystyle\int_0^L b(y,x)G_L(y)\,dy, \\ \tau_\eta G_L(0) = \epsilon \int_0^L G_L(y)\,dy, \qquad G_L(x) > 0, \qquad \int_0^L G_L(x)dx = 1, \\ -\tau_\eta(x)\dfrac{\partial}{\partial x}\phi_L(x) + (B(x) + \lambda_L)\phi_L(x) - \displaystyle\int_0^L b(x,y)\phi_L(y)\,dy = \tau_\eta(0)\epsilon\phi_L(0), \\ \phi_L(L) = 0, \qquad \phi_L(x) > 0, \qquad \int_0^L \phi_L(x)G_L(x)dx = 1. \end{cases} \tag{49}$$

Notice that in this problem, the eigenelements (λ_L, G_L, ϕ_L) depend on η, and ϵ and should be denoted $(\lambda_L^{\eta,\epsilon}, G_L^{\eta,\epsilon}, \phi_L^{\eta,\epsilon})$. We forget here the superscripts for the sake of clarity.

The existence of a solution to Problem (49) is proved in the Appendix of [2] by using the Krein-Rutman theorem. Then we need to pass to the limit η, $\epsilon \to 0$. The uniform estimates in [2] allow to do so, provided that $\lambda_L^{\eta,\epsilon}$ is positive for all η, ϵ. In [2] this condition is ensured for L large enough under the constraint that ϵL is a fixed constant, which means that $L = L(\epsilon)$ tends to $+\infty$ as $\epsilon \to 0$. Here we want to pass to the limit $\epsilon \to 0$ for a fixed positive value of L. For this we prove the existence of a constant $L_0 > 0$ such that $\lambda_L^{\eta,\epsilon} > 0$ for all η, $\epsilon \geq 0$ and all $L \geq L_0$.

Assume by contradiction that $\lambda_L \leq 0$. Then we have by integration of the direct eigenequation between 0 and $x < L$

$$\begin{aligned} 0 &\geq \lambda \int_0^x G(y)\,dy \\ &= -\tau(x)G(x) - \int_0^x B(y)G(y)\,dy + \int_0^x \int_z^L b(y,z)G(y)\,dy\,dz \\ &= -\tau(x)G(x) + (\pi_0 - 1)\int_0^x B(y)G(y)\,dy + \int_x^L \left(\int_0^x b(y,z)\,dz\right) G(y)\,dz. \end{aligned}$$

We assume that $b(y,x) = \frac{B(y)}{y} p\big(\frac{x}{y}\big)$ with $\int_0^1 p(h)\,dh = \pi_0 > 1$. Thus, for p bounded, there exists $s \in (0,1)$ such that $\int_0^s p(h)\,dh \geq \pi_0 - 1$. For $L \geq y \geq x \geq sL$, we have

$$\begin{aligned} \int_0^x b(y,z)\,dz = B(y) \int_0^{\frac{x}{y}} p(h)\,dh \\ \geq B(y) \int_0^{\frac{sL}{y}} p(h)\,dh \geq B(y) \int_0^s p(h)\,dh \geq (\pi_0 - 1)B(y), \end{aligned}$$

so for all $x \geq sL$

$$0 \geq -\tau(x)G(x) + (\pi_0 - 1)\int_0^L B(y)G(y)\,dy$$

which leads to

$$B(x)G(x) \geq (\pi_0 - 1)\frac{B(x)}{\tau(x)}\int_0^L B(y)G(y)\,dy \geq (\pi_0 - 1)\frac{B(x)}{\tau(x)}\int_{sL}^L B(y)G(y)\,dy$$

and finally, by integration on $[sL, L]$,

$$(\pi_0 - 1)\int_{sL}^L \frac{B(y)}{\tau(y)}\,dy \leq 1. \tag{50}$$

We have from Hypothesis 1.3 that

$$\exists\, A > 0, \qquad \forall x \geq A, \quad \frac{xB(x)}{\tau(x)} > \frac{1}{(\pi_0 - 1)|\ln(s)|}$$

so, for $L \geq \frac{A}{s}$, we obtain

$$(\pi_0 - 1)\int_{sL}^L \frac{B(y)}{\tau(y)}\,dy > \frac{1}{|\ln(s)|}\int_{sL}^L \frac{1}{y}\,dy = 1$$

which contradicts (50). Finally, $\lambda_L > 0$ for all $L \geq L_0 := \frac{A}{s}$.

We have proved the existence of solution for Problem (47) in the case $\phi_L(L) = 0$ and we know that

$$\lambda_L \xrightarrow[L\to+\infty]{} \lambda,$$

$$G_L \xrightarrow[L\to+\infty]{} G \qquad \text{in } L^1(\mathbb{R}_+),$$

$$\forall A > 0, \quad \phi_L \xrightarrow[L\to+\infty]{} \phi \qquad \text{uniformly on } [0, A).$$

Now we use this result to treat the cases $\phi_L(L) = \delta > 0$ and $\phi_L(L) = \delta L$. Since $\delta > 0$, we can prove by using the Krein-Rutman theorem the existence of a solution to Problem (47). To prove the convergence of λ_L to λ, we integrate the equation on ϕ_L multiplied by G and we obtain

$$\lambda - \lambda_L = \tau(L)G(L)\phi_L(L).$$

We know from estimates on G that $\tau(L)G(L)L \to 0$ when $L \to +\infty$, which ensures the convergence of λ. Because $\lambda > 0$, it also ensures the existence of L_0 such that $\lambda_L > 0$ for $L \geq L_0$, which allows to prove the convergence of ϕ_L to ϕ locally uniformly (see [2] for details).

□

B Laplace's method

Laplace's method (see [13, II.1, Theorem 1] for example) gives a way to calculate the asymptotic behavior of integrals which contain an exponential term with a large factor in the exponent. We give here a result of this kind, with conditions which are adapted to the situation encountered in Section 2.

Lemma B.1. *Take $x_0, D_0 \in \mathbb{R}$. Assume that $g : [x_0, +\infty) \to \mathbb{R}$ is a measure and $h : [x_0, +\infty) \times [D_0, +\infty) \to \mathbb{R}$ a measurable function satisfying*

$$g(x) \sim g_0(x - x_0)^\sigma \qquad \text{as } x \to x_0\text{, for some } g_0 \neq 0 \text{ and } \sigma > -1, \tag{51}$$

$$h(x, D) - h(x_0, D) \sim h_0(x - x_0)^\omega \qquad \text{as } x \to x_0 \text{ and } D \to +\infty\text{, for some } h_0, \omega > 0, \tag{52}$$

$$\int_{x_0}^{\infty} |g(x)| e^{-D_0 h(x,D)}\, dx < C_0 \qquad \text{for some } C_0 \geq 0 \text{ and all } D \geq D_0. \tag{53}$$

Assume also that for all $D \geq D_0$, the function $x \mapsto h(x, D)$ (with D fixed) attains its unique global minimum at $x = x_0$, in the following strong sense: there exists a nondecreasing strictly positive function $\theta : (0, +\infty) \to (0, +\infty)$ such that

$$h(x, D) - h(x_0, D) \geq \theta(x - x_0) \qquad \text{for all } x > x_0 \text{ and all } D \geq D_0. \tag{54}$$

Then, as $D \to +\infty$,

$$\int_{x_0}^{\infty} e^{-Dh(x,D)} g(x)\, dx \sim g_0 D^{\frac{-1-\sigma}{\omega}} e^{-Dh(x_0,D)} \int_0^\infty x^\sigma e^{-h_0 x^\omega}\, dx. \tag{55}$$

The constants implicit in (55) depend only on the constants implicit or explicit in (51)–(54).

Some remarks on the conventions used above are in order. Although g is a measure we denote it as a function in the expressions in which it appears. For example, integrals in which g appears should be considered as integrals with respect to the measure g. Also, in equation (51), it is understood that close to x_0 the measure g is equal to a function, and the asymptotic approximation (51) holds.

Proof. First of all, by translating g and h we may consider always that $x_0 = 0$. We may also assume that $h(x_0, D) = 0$ for all $D \geq D_0$, as otherwise one obviously obtains the additional factor $e^{-Dh(x_0,D)}$.

An important part of the argument is based on the observation that if one excludes a small region close to 0, then the rest of the integral decreases fast as $D \to +\infty$: from (52) and (54) we deduce that for some $\rho > 0$

$$h(x, D) \geq \rho \min\{1, x^\omega\} \qquad \text{for all } x \geq 0,\ D \geq D_0. \tag{56}$$

Then, for $D \geq D_0$ and $0 < \epsilon < 1$ we have from (56):

$$\left| \int_\epsilon^\infty g(x) e^{-Dh(x,D)} \, dx \right| \\ \leq e^{-(D-D_0)\rho\epsilon^\omega} \int_\epsilon^\infty |g(x)| e^{-D_0 h(x,D)} \, dx \leq C_0 e^{-(D-D_0)\rho\epsilon^\omega},$$

due to (53). If we take $\epsilon := D^{-\frac{1}{2\omega}}$ then for all $D > D_0$ we have

$$\left| \int_{D^{-\frac{1}{2\omega}}}^\infty g(x) e^{-Dh(x,D)} \, dx \right| \leq C_0 e^{-\left(\sqrt{D} - \frac{D_0}{\sqrt{D}}\right)\rho}.$$

This quantity decreases faster than any power of D as $D \to +\infty$.

For the remaining part of the integral, since we are integrating in a region which is closer and closer to 0 it is easy to see due to (51) and (52) that for all $\epsilon > 0$ there exists $D_\epsilon > 0$ such that

$$\int_0^{D^{-\frac{1}{2\omega}}} (1-\epsilon) g_0 x^\sigma e^{-D(1+\epsilon)h_0 x^\omega} \, dx \leq \int_0^{D^{-\frac{1}{2\omega}}} g(x) e^{-Dh(x,D)} \, dx \\ \leq \int_0^{D^{-\frac{1}{2\omega}}} (1+\epsilon) g_0 x^\sigma e^{-D(1-\epsilon)h_0 x^\omega} \, dx \quad (57)$$

for all $D > D_\epsilon$. Through the change of variables $z = xD^{1/\omega}$ we see that

$$\int_0^{D^{-\frac{1}{2\omega}}} (1-\epsilon) g_0 x^\alpha e^{-D(1+\epsilon)h_0 x^\omega} \, dx \\ = (1-\epsilon) g_0 D^{\frac{-1-\sigma}{\omega}} \int_0^{D^{\frac{1}{2\omega}}} z^\sigma e^{-(1+\epsilon)h_0 z^\omega} \, dz \\ \sim (1-\epsilon) g_0 D^{\frac{-1-\sigma}{\omega}} \int_0^\infty z^\sigma e^{-(1+\epsilon)h_0 z^\omega} \, dz,$$

where the '$\sim$' sign denotes asymptotics as $D \to +\infty$. Carrying out a similar calculation for the last integral in (57) and letting $\epsilon \to 0$ we deduce our result. □

For the next result we recall that $\gamma_+ = \max\{0, \gamma\}$ and ζ is defined by (24).

Lemma B.2. *Assume Hypotheses 1.1–1.5. There is $\epsilon > 0$ such that*

$$\int_{x_0}^x e^{\Lambda(y)} y^k \, dy = \zeta^{-1} x^{k-\gamma_+ + \alpha} e^{\Lambda(x)} + \underset{x \to +\infty}{O} (x^{k-\gamma_+ + \alpha - \epsilon}) e^{\Lambda(x)}.$$

Proof. We use l'Hôpital's rule to calculate the limit as $x \to +\infty$ of

$$\frac{\int_{x_0}^x e^{\Lambda(y)} y^k \, dy - \zeta^{-1} x^{k-\gamma_+ + \alpha} e^{\Lambda(x)}}{x^m e^{\Lambda(x)}}.$$

Differentiating both the numerator and denominator we find that this limit is the same as the limit as $x \to +\infty$ of

$$\frac{x^k - \zeta^{-1}(k-\gamma_+ + \alpha)x^{k-\gamma_+ + \alpha - 1} - \zeta^{-1} x^{k-\gamma_+ + \alpha}\frac{B(x)+\lambda}{\tau(x)}}{m x^{m-1} + x^m \frac{B(x)+\lambda}{\tau(x)}} =: \frac{T_N(x)}{T_D(x)}. \tag{58}$$

Using (15) and (16), we obtain that

$$x^k - \zeta^{-1} x^{k-\gamma_+ + \alpha}\frac{B(x)+\lambda}{\tau(x)} = O(x^{k-\delta})$$

for some $\delta > 0$. Observing that $\gamma_+ - \alpha + 1 > 0$ and calling $\epsilon := \min\{\delta, \gamma_+ - \alpha + 1\} > 0$ we have

$$T_N(x) = O(x^{k-\epsilon}).$$

In a similar way,

$$T_D(x) = x^{m+\gamma_+ - \alpha} + O(x^{m+\gamma_+ - \alpha - \delta}),$$

so from (58) we obtain that the limit is 0 whenever

$$m + \gamma_+ - \alpha > k - \epsilon, \qquad \text{i.e., } m > k - \gamma_+ + \alpha - \epsilon.$$

This shows the result. □

We now use this to prove an estimate which is needed in Section 2:

Lemma B.3. *Assume Hypotheses 1.1–1.4 with $p_1 > 0$, and take $k \in \mathbb{R}$. Then,*

$$\int_{x_0}^{x} e^{\Lambda(y)} y^k b(x,y)\, dy \underset{x\to+\infty}{\sim} p_1 B_\infty \zeta^{-1-\nu}\Gamma(\nu+1) x^{k+\gamma-(\gamma_+-\alpha+1)(1+\nu)} e^{\Lambda(x)}. \tag{59}$$

If we also assume Hypothesis 1.5 and $\nu = 0$ (and now we allow any $p_1 \geq 0$) then there is $\epsilon > 0$ such that

$$\begin{aligned}\int_{x_0}^{x} e^{\Lambda(y)} y^k b(x,y)\, dy \\ &= p_1 B_\infty \zeta^{-1}\, x^{k+\gamma-\gamma_+ + \alpha - 1} e^{\Lambda(x)} + \underset{x\to+\infty}{O}(x^{k+\gamma-\gamma_+ + \alpha - 1 - \epsilon}) e^{\Lambda(x)}.\end{aligned} \tag{60}$$

Proof. We call p_* and p^*, respectively, the parts of the measure p on the intervals $[0,1/2)$ and $[1/2,1]$, i.e., $p_* := p\,\mathbb{1}_{[0,1/2)}$ and $p^* := p\,\mathbb{1}_{[1/2,1]}$. With this we break the integral we want to estimate in two parts:

$$\begin{aligned} I(x) &:= \int_{x_0}^{x} e^{\Lambda(y)} y^k p\left(\frac{y}{x}\right) dy \\ &= \int_{x_0}^{x} e^{\Lambda(y)} y^k p_*\left(\frac{y}{x}\right) dy + \int_{x_0}^{x} e^{\Lambda(y)} y^k p^*\left(\frac{y}{x}\right) dy =: I_*(x) + I^*(x). \end{aligned}$$

The first part, I_*, can be estimated by

$$
\begin{aligned}
|I_*(x)| = \int_{x_0}^{x} e^{\Lambda(y)} y^k p_* \left(\frac{y}{x}\right)\, dy &\le e^{\Lambda(x/2)} \int_{x_0}^{x} y^k p_* \left(\frac{y}{x}\right)\, dy \\
&\le e^{\Lambda(x/2)} \max\{x^k, x_0^k\} \int_{x_0}^{x} p_* \left(\frac{y}{x}\right)\, dy \le \pi_0\, x\, e^{\Lambda(x/2)} \max\{x^k, x_0^k\}.
\end{aligned}
$$

Since we will show that $I^*(x)$ behaves as given in the statement, this term is of lower order (since $\Lambda(x)$ is asymptotic to a positive power of x as $x \to +\infty$) and can be disregarded.

For I^* we make the change of variables $z = y/x$ and denote $D := x^{\gamma_+ - \alpha + 1}$ to obtain

$$
\begin{aligned}
I^*(x) := \int_{x_0}^{x} e^{\Lambda(y)} y^k p^* \left(\frac{y}{x}\right)\, dy &= x^{k+1} \int_{\max\left\{\frac{x_0}{x}, \frac{1}{2}\right\}}^{1} e^{\Lambda(xz)} z^k p\,(z)\; dz \\
&= x^{k+1} \int_{\max\left\{\frac{x_0}{x}, \frac{1}{2}\right\}}^{1} \exp\left(-Dh(z, D)\right) g(z)\, dz \qquad (61)
\end{aligned}
$$

with

$$
h(z, D) := -\frac{1}{D} \Lambda\,(xz) = -\frac{1}{D} \Lambda\left(D^{\frac{1}{\gamma_+ - \alpha + 1}} z\right), \qquad g(z) := z^k p\,(z)\,.
$$

Now, the asymptotics in D of the integral in (61) can be obtained from Lemma B.1 with $x_0 = 1$. Let us see that h and g indeed satisfy the needed hypotheses. The property (51) is satisfied with $g_0 = p_1$ and $\sigma = \nu$ due to Hypothesis 1.4, and to show (52) we write (with asymptotics notation understood to be for $z \to 1$ and $D \to +\infty$)

$$
\begin{aligned}
&h(z, D) - h(1, D) = \frac{1}{D} \left(\Lambda(x) - \Lambda\,(xz)\right) = \frac{1}{D} \int_{xz}^{x} \frac{\lambda + B(u)}{\tau(u)}\, du \\
&\sim \frac{\zeta}{\gamma_+ - \alpha + 1} \frac{1}{D} x^{\gamma_+ - \alpha + 1} (1 - z^{\gamma_+ - \alpha + 1}) = \frac{\zeta}{\gamma_+ - \alpha + 1} (1 - z^{\gamma_+ - \alpha + 1}) \sim \zeta(1 - z),
\end{aligned}
\qquad (62)
$$

which corresponds to $h_0 = \zeta$, $\omega = 1$ in Lemma B.1. For (53) we write

$$
\begin{aligned}
&\int_{\max\left\{\frac{x_0}{x}, \frac{1}{2}\right\}}^{1} \exp\left(-D_0 h(z, D)\right) g(z)\, dz \\
&\qquad \le \int_{\frac{1}{2}}^{1} \exp\left(\frac{D_0}{D} \Lambda(xz)\right) z^k p(z)\, dz \\
&\qquad\qquad \le \exp\left(\frac{D_0}{D} \Lambda(x)\right) \int_{\frac{1}{2}}^{1} z^k p(z)\, dz \le C_0
\end{aligned}
$$

for some $C_0 > 0$ (which in particular depends on k), since $x \mapsto \Lambda(x)/D = \Lambda(x)/x^{\gamma_+ - \alpha + 1}$ is bounded for $x > 1$. This gives (53). Obviously $z \mapsto h(z, D)$

attains its minimum at $z = 1$, and (54) is a consequence of (62) and the fact that $h(z, D) - h(1, D)$ is decreasing in z for all D.

We may then apply Lemma B.1 to obtain

$$I^*(x) \sim p_1 x^{k+1} D^{-1-\nu} e^{\Lambda(x)} \int_0^\infty z^\nu e^{-\zeta z}\, dz \\ = p_1 \zeta^{-1-\nu} \Gamma(1+\nu) x^{k+1-(\gamma_+-\alpha+1)(1+\nu)} e^{\Lambda(x)}.$$

Since $I_*(x)$ was shown to be of lower order, this is enough to show (59).

Finally, in order to show (60), we have

$$\int_{x_0}^x e^{\Lambda(y)} y^k p\left(\frac{y}{x}\right) dy = \int_{x_0}^x e^{\Lambda(y)} y^k \left(p\left(\frac{y}{x}\right) - p_1\right) dy + p_1 \int_{x_0}^x e^{\Lambda(y)} y^k\, dy.$$

For the first term, using (17) and (59) we have

$$\int_{x_0}^x e^{\Lambda(y)} y^k \left(p\left(\frac{y}{x}\right) - p_1\right) dy = O(x^{k+1-(\gamma_+-\alpha+1)(1+\delta)}) e^{\Lambda(x)},$$

and for the second term, Lemma B.2 gives

$$\int_{x_0}^x e^{\Lambda(y)} y^k\, dy = \zeta^{-1} x^{k-\gamma_++\alpha} e^{\Lambda(x)} + O(x^{k-\gamma_++\alpha-\delta}) e^{\Lambda(x)}.$$

Since $\gamma_+ - \alpha + 1 > 0$, this shows the result. □

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